Theory of Gravitational Monopole

Y.M. Cho
Department of Physics, Seoul National University
Seoul 151-742, Korea

Abstract

Recently a generalization of Einstein's theory of gravitation has been proposed which can accommodate the gravitational monopoles. The generalization is made in such a way that the equations of motion become exactly symmetric under the dual transformation of the curvature tensor. We review the recent development on the subjecct.

Long time ago Dirac[1,2] has taught us that the electrodynamics can be made consistent with the existence of the magnetic monopoles if the celebrated charge quantization condition is imposed. Inspired by an apparent analogy between the electrodynamics and the general relativity in the post-Newtonian limit, Zee[3] has made a very interesting conjecture that the Einstein's theory of gravitation could be generalized to include the "gravitational monopoles". Recently it has been shown that[4] indeed the generalization is possible provided that one is willing to accept an energy quantization requirement. We review the recent progress on the theory of gravitational monopole.

The generalization can be made in exact analogy with the Dirac's generalization of the electrodynamics. All we need is to enlarge the space of the metric to allow the string singularities, under the condition that the strings should be *physically* unobservable. Since the classical dynamics of gravitation is governed by the geodesic equation, a necessary condition for the invisibility of the string is that it should not create a physical singularity in the geodesic equation. This invisibility of the string at the classical level, however, is not sufficient to guarantee the invisibility of the string at the quantum level. To assure that the string remains invisible at the quantum level, we need the quantization of energy, as we will show in the followings.

Let us first show that the general relativity admits a gravitational monopole[5,6], if the metric is allowed to have a string singularity. To keep the analogy between the Dirac's generalization and ours as far as possible, we start by writing the most general stationary metric $g_{\mu\nu}$ $(\mu,\nu = 1,2,3,4)$ as

$$ds^2 = -\phi(dt + B_i dx^i)^2 + g_{ij}dx^i dx^j \qquad (1)$$

where g_{ij} $(i,j = 1,2,3)$, B_i, and ϕ are functions of the spatial coordinates only. The fact that the most general stationary metric can always be put into this form can easily be understood if one regards the Einstein's theory as a $(3+1)$-dimensional Kaluza-Klein theory[7,8] and dimensionally reduces it to the 3-dimensional space, assuming that the metric admits a time-like Killing vector. Notice that in this Kaluza-Klein point of view the space-time can be viewed as a principal fibre bundle in which the time-axis becomes the "internal" fibre over the 3-dimensional space. Thus B_i may be interpreted as a "gauge potential" which has the following gauge degree of freedom

$$B_i \longrightarrow B_i' = B_i - \partial_i\Lambda. \qquad (2)$$

This gauge degrees of freedom is guaranteed by the general invariance of Einstein's theory under the coordinate transformation

$$t \longrightarrow t' = t + \Lambda. \qquad (3)$$

Now we choose the polar coordinates and let

$$\begin{aligned} \phi = \phi(r), \quad B_i &= \frac{\kappa}{4\pi}(1-\cos\theta)\partial_i\varphi \\ (ds^2)_3 &= g_{ij}dx^i dx^j \\ &= f(r)dr^2 + r^2(d\theta^2 + \sin^2\theta d\varphi^2), \end{aligned} \qquad (4)$$

where we have introduced a scale parameter κ to keep B_i dimensionless. Then one can easily show that the vacuum Einstein's equation admits the following asymptotically flat solution

$$\phi(r) = 1 - 2\frac{Gm\left(r^2 - \frac{\kappa^2}{64\pi^2}\right)^{1/2} + \frac{\kappa^2}{64\pi^2}}{r^2}$$

$$f(r)\phi(r) = \left[1 - \frac{\kappa^2}{64\pi^2 r^2}\right]^{-1}. \qquad (5)$$

Obviously the potential B_i describes a "monopole" to which a string singularity is attached, but now as a component of the metric. The solution is characterized by two parameters, the monopole strength κ and the inertial mass m, so that it actually describes a "gravitational dyon". Notice that one may call κ a "magnetic mass" since it has the dimension of a length.

One might notice that the above solution is not exactly new, because one can easily show that it is locally identical to the well-known Taub-NUT solution[5,6]. However, we emphasize that *there is one big difference between the Taub-NUT solution and ours, and that is in the global topology of the space-time.* The asymptotic topology of the Taub-NUT space-time at the spatial infinity is well-known to be isomorphic to S^3, which makes it totally unphysical because it has a periodic time which violates the causality and admits no reasonable space-like hypersurface. But here we require the asymptotic topology of our space-time to be $R^1 \times S^2$ (where R^1 represents the time axis). How can this be possible? Notice that the reason why the S^3 topology of Taub-NUT space-time is forced upon us is because this guarantees the metric to be smooth everywhere on the space-time manifold. But obviously we can escape from this pathological S^3 topology if we are willing to accept the string singularity[9]. So we choose to accept the string as it is to retain the physically desirable space-time topology, but will get rid of the string by making it physically unobservable.

What should we do to make the string invisible? At the classical level the string is by itself invisible because the string does not produce any gravitational effect which can be detected by a classical neutral test particle[4]. To see this notice that the Riemannian curvature of the metric (4) becomes spherically symmetric and does *not* contain any string singularity. This suggests that the string is not a physical singularity but a simple coordinate singularity. To show that this is indeed the case, notice that the gravitational monopole can be described by any metric which may be related to (4) by a general coordinate transformation. So consider two metrics $g^{(\pm)}_{\mu\nu}$ which have potential $B^{(\pm)}_i$ given by

$$B^{(\pm)}_i = \frac{\kappa}{4\pi}(\pm 1 - \cos\theta)\partial_i\varphi.$$

Clearly both $g^{(+)}_{\mu\nu}$ and $g^{(-)}_{\mu\nu}$ describe the same monopole since they are related by the following coordinate transformation

$$t \longrightarrow t' = t + \frac{\kappa}{2\pi}\varphi. \qquad (6)$$

The coordinate transformation moves the string along the negative z-axis to the positive z-axis. So dividing the space-time into two cross sections[2] and sewing them together in the overlapping region with (6), one can remove the string completely from our space-time. This shows that *the string is not a physical singularity but indeed an artifact of the coordinates we have chosen, which is why it can not be detected by a classical test particle.*

To make the string invisible at the quantum level, however, we must impose the following quantization condition of energy E

$$\kappa E = 2\pi n \quad (n; \text{integer}). \qquad (7)$$

We present three independent arguments for this[3, 4]:

A) Consider the quantum wave function Ψ of a particle with energy E rotating around the string, and let

$$\Psi(t, \vec{x}) = e^{-iEt}R(\vec{x}).$$

Then under the transformation (8) the wave function should transform as

$$\Psi(t, \vec{x}) \longrightarrow \Psi(t', \vec{x}) = e^{-i\frac{\kappa E}{2\pi}\varphi}\Psi(t, \vec{x}). \qquad (8)$$

But since the wave function should remain single-valued under the coordinate transformation, one need to have the quantization condition.

B) We have emphasized that the string is not a physical singularity. Nevertheless let us try to treat the string *as if* it is real. Then asymptotically the string of the metric (4) becomes nothing but the spinning string[10] which can be described in the cylindrical coordinates by

$$ds^2 = -(dt + \frac{\kappa}{2\pi}d\varphi)^2 + d\rho^2 + \rho^2 d\varphi^2 + dz^2. \qquad (9)$$

In principle this string could be detected by a scattering of a test particle around it. The quantum scattering cross section of a scalar particle with energy E around the spinning string is known to be[11]

$$\frac{d^2\sigma}{d\varphi dz} = \frac{1}{2k\pi}\frac{\sin^2(\kappa E/2)}{\sin^2(\varphi/2)}, \qquad (10)$$

where k is the momentum of the particle. This shows that the string becomes invisible under the quantum scattering if and only if the quantization condition is satisfied. The quantum scattering of a spin 1/2 particle around the string[12] gives the same conclusion. This argument is particularly enlightening because it tells us that, even if one tries to treat the string as physical, the quantization condition will forbid us to detect it with a quantum scattering of a test particle.

C) Remember that our space-time may be viewed as a principal fibre bundle in which the time-axis becomes the fibre over the 3-dimensional space. In this picture the string singularity manifests itself when the bundle is regarded trivial. But we have already shown that we can always make the string disappear by introducing two cross sections and making the fibre bundle topologically non-trivial[2,13], provided that the quantization condition holds true. The coordinate transformation (6) tells us how one can remove the string with a mixing of the time coordinate with the azimuthal coordinate, just as Wu and Yang[2] have told us how one can remove the Dirac string with a mixing of the internal coordinate (i.e., the fifth-coordinate) with the azimuthal coordinate. Notice that, when the time is assumed to be periodic, this mixing is precisely what one need to obtain the S^3 topology of the Taub-NUT space-time. We emphasize, however, that our way of removing the string need not necessarily require the physical time to be periodic, in as much as the Dirac's theory does not necessarily require the existence of a 5-dimensional space-time. According to (6), the periodicity of the azimuthal coordinate requires us to identify (t', φ) with $(t' + \kappa n, \varphi + 2\pi n)$, but not with $(t' + \kappa n, \varphi)$. This assures us that our time must be helical[12], but not periodic.

In all the above arguments the striking similarity between the Dirac string and ours is unmistakable. But *it should be made absolutely clear that the quantization of energy has nothing to do with the periodic time coordinate* as has been repeatedly suggested in the literature[5,6]. In retrospect the quantization condition (7) could easily have been understood from the following simple dimensional argument. The existence of a gravitational monopole necessarily requires the existence of a fundamental length scale κ which in turn implies the existence of a fundamental energy scale, and hence the quantization of energy.

We now show how Einstein's equation should be generalized in the presence of a gravitational monopole[4]. To do this let us first define the dual curvature tensors as follows

$$R^*_{ABCD} = \frac{1}{2}\epsilon_{AB}{}^{MN}R_{MNCD}$$

$$R^*_{AB} = g^{CD}R^*_{ACBD} = -\frac{1}{2}\epsilon_A{}^{PQR}R_{PQRB} \qquad (11)$$

$$R^* = g^{AB}R^*_{AB} = \frac{1}{2}\epsilon^{ABCD}R_{ABCD}.$$

With this the first and second Bianchi identities can be expressed as

$$R^*_{AB} = 0$$

$$\nabla^A R^*_{ABCD} = 0. \qquad (12)$$

This should be contrasted with the following expressions of Einstein's equation

$$R_{AB} = -8\pi G(T_{AB} - \frac{1}{2}Tg_{AB})$$

$$\nabla^A R_{ABCD} = -8\pi G[\nabla_C(T_{DB} - \frac{1}{2}Tg_{DB}) - \nabla_D(T_{CB} - \frac{1}{2}Tg_{CB})], \qquad (13)$$

where the second equation follows from the first one with the help of the Bianchi identities. This shows that Einstein's theory of gravitation is "maximally asymmetric" under the dual transformation. With this clarification it now becomes clear what we should do to accommodate the gravitational monopoles. *We need to have two ("magnetic" as well as "electric") energy-momentum tensors which must couple to the curvature tensor in such a way that the generalized theory becomes symmetric under the dual transformation.* But obviously this can be made possible only if the Bianchi identities are violated. So we only require the curvature to be metric-compatible,

$$R_{ABCD} = R_{[AB][CD]} \qquad (14)$$

and replace the first Bianchi identity by

$$R_{[ABC]D} = -\frac{1}{3}\epsilon_{ABC}{}^{M}R^*_{MD}$$

$$R^*_{[ABC]D} = \frac{1}{3}\epsilon_{ABC}{}^{M}R_{MD}. \qquad (15)$$

Notice that now the Ricci tensors are no longer symmetric,

$$R_{[AB]} = -\frac{1}{2}\epsilon_{AB}{}^{CD}R^*_{CD}$$

$$R^*_{[AB]} = \frac{1}{2}\epsilon_{AB}{}^{CD}R_{CD}. \qquad (16)$$

With this we propose the following generalization of Einstein's equation

$$R_{(AB)} - \frac{1}{2}Rg_{AB} = -8\pi GT_{AB}$$

$$R^*_{(AB)} - \frac{1}{2}R^*g_{AB} = -8\pi G'S_{AB}, \qquad (17)$$

where G' and S_{AB} are the gravitational constant and the energy-momentum tensor of the magnetic matter. To ensure the conservation of the two energy-momentum tensors, we now need to generalize the second Bianchi identity. To see how, let

$$\nabla^A R_{ABCD} = J_{BCD}$$
$$\nabla^A R^*_{ABCD} = K_{BCD}$$

and find

$$\nabla^B[R_{(AB)} - \frac{1}{2}Rg_{AB}] = \frac{1}{2}\epsilon_A{}^{BCD}(K_{BCD} + \nabla_B R^*_{CD})$$
$$\nabla^B[R^*_{(AB)} - \frac{1}{2}R^* g_{AB}] = -\frac{1}{2}\epsilon_A{}^{BCD}(J_{BCD} + \nabla_B R_{CD}). \tag{18}$$

This tells that the second Bianchi identity should be modified in such a way that the following equalities hold

$$\nabla^A R_{A[BCD]} + \nabla_{[B} R_{CD]} = 0$$
$$\nabla^A R^*_{A[BCD]} + \nabla_{[B} R^*_{CD]} = 0. \tag{19}$$

This completes the desired generalization of Einstein's theory.

A slightly different generalization is possible if one is willing to accept non-symmetric energy-momentum tensors. Notice that (16) with (18) can be written as

$$\nabla^B[R_{AB} - \frac{1}{2}Rg_{AB}] = \frac{1}{2}\epsilon_A{}^{BCD}K_{BCD}$$
$$\nabla^B[R^*_{AB} - \frac{1}{2}R^* g_{AB}] = -\frac{1}{2}\epsilon_A{}^{BCD}J_{BCD}.$$

So one may have

$$R_{AB} - \frac{1}{2}Rg_{AB} = -8\pi G T_{AB}$$
$$R^*_{AB} - \frac{1}{2}R^* g_{AB} = -8\pi G' S_{AB} \tag{20}$$

together with the following modification of the second Bianchi identity

$$\nabla^A R_{A[BCD]} = 0$$
$$\nabla^A R^*_{A[BCD]} = 0. \tag{21}$$

In this generalization T_{AB} and S_{AB} become non-symmetric (but conserved) energy-momentum tensors.

This shows that Einstein's theory of gravitation could be generalized to accommodate the gravitational monopoles, exactly as the Maxwell's theory could be generalized to accommodate the magnetic monopoles. But obviously this generalization opens up much more questions, fundamental as well as phenomenological, than we have tried to answer in this paper. A more detailed discussion on the subject will be published elsewhere[14].

Acknowledgement

The work is supported in part by the Ministry of Education and the Korea Science and Engineering Foundation.

References

[1] P. A. M. Dirac, Proc. Roy. Soc. *A133*, 60(1931).

[2] T. T. Wu and C. N. Yang, Phys. Rev. *D12*, 3845(1975).

[3] A. Zee, Phys. Rev. Lett. *55*, 2379(1985).

[4] Y. M. Cho, SNUTP 90-07, submitted to Phys. Rev. Lett.

[5] E. T. Newman, L. Tamburino, and T. Unti, J. Math. Phys. *4*, 915(1963);
C. W. Misner, J. Math. Phys. *4*, 924(1963);
M. Demianski and E. T. Newman, Bull. Acad. Pol. Sci. *14*, 653(1966);
J. S. Dowker, Gen. Rel. Grav. *5*, 603(1974).

[6] S. Ramaswamy and A. Sen, J. Math. Phys. *22*, 2612(1981);
P. O. Mazur, Phys. Rev. Lett. *57*, 929(1986).

[7] Y. M. Cho, J. Math. Phys. *16*, 2029(1975);
Y. M. Cho and P. G. O. Freund, Phys. Rev. *D12*, 1711(1975).

[8] Y. M. Cho, Phys. Lett. *186B*, 38(1987);
Y. M. Cho, Phys. Lett. *199B*, 358(1987);
Y. M. Cho and D. S. Kimm, J. Math. Phys. *30*, 1570(1989).

[9] A. Ashtekar and A. Sen, J. Math. Phys. *23*, 2168(1982).

[10] A. Staruszkiewicz, Acta. Phys. Pol. *24*, 734(1963);
J. R. Gott III and M. Alpert, Gen. Rel. Grav. *16*, 243(1984);
D. Deser, R. Jackiw, and G. 't Hooft, Ann. Phys. (NY) *152*, 220(1984).

[11] G. 't Hooft, Comm. Math. Phys. *117*, 685(1988);
S. Deser and R. Jackiw, Comm. Math. Phys. *118*, 495(1988);
Y. Aharonov and D. Bohm, Phys. Rev. *115*, 485(1959).

[12] P. de Sousa Gebert and R. Jackiw, Comm. Math. Phys. *124*, 229(1989);
M. Alford and F. Wilczek, Phys. Rev. Lett. *62*, 1071(1989).

[13] Y. M. Cho, Phys. Rev. Lett. *44*, 1115(1980);
Y. M. Cho, Phys. Rev. *D23*, 2415(1981).

[14] Y. M. Cho and D. H. Park, to be published.