

Unitary quantum physics with time-space noncommutativity

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Abstract. In these lectures⁴ quantum physics in noncommutative spacetime is developed. It is based on the work of Doplicher et al. which allows for time-space noncommutativity. In the context of noncommutative quantum mechanics, some important points are explored, such as the formal construction of the theory, symmetries, causality, simultaneity and observables. The dynamics generated by a noncommutative Schrödinger equation is studied. The theory is further extended to certain noncommutative versions of the cylinder, $\mathbb{R}^3$ and $\mathbb{R} \times S^3$. In all these models, only discrete time translations are possible. One striking consequence of quantised time translations is that even though a time independent Hamiltonian is an observable, in scattering processes, it is conserved only modulo $2\pi/\theta$, where θ is the noncommutative parameter. Scattering theory is formulated and an approach to quantum field theory is outlined.

1. Introduction

Considerations based on quantum gravity and black hole physics led to the suggestion several years ago [1] that spacetime commutativity may be lost at the smallest scale, the commutators of time and space coordinates ($\hat{x}_0$ and $\hat{x}_i$) having the form

$$[\hat{x}_\mu, \hat{x}_\nu] = i\theta_{\mu\nu}\mathbb{I}, \quad (1)$$

with $\theta_{\mu\nu}$ being constants of the order of the square of Planck length. String theory also incorporates relations like (1).

Commutators such as (1) actually have a much more ancient origin. They seem to have first appeared in a letter from Heisenberg to Peierls in 1930 [2]. Spacetime noncommutativity was later revived by Snyder [3] who sought to use it to regularize quantum field theories (qft's), and then by Yang [4]. Madore [5] also attributes similar ideas to Dirac. Among the early works in noncommutative spacetime is that of Kempf et al. [6]. A subsequent related work is that of Lizzi et al. [7].

Conventional studies of (1) assume that $\theta_{0i} = 0$ so that the time coordinate commutes with the rest. There are even claims that qft's based on (1) are nonunitary if $\theta_{0i} \neq 0$.

⁴ Lectures delivered by A P Balachandran.

In contrast, in a series of fundamental papers, Doplicher et al. [1] have studied (1) in complete generality, without assuming that $\theta_{0i} \neq 0$ and developed unitary qft's which are ultraviolet finite to all orders.

These lectures are based on the work of Doplicher et al. Using their ideas, we systematically develop unitary quantum mechanics based on (1). It indicates where to look for phenomenological consequences of (1) and also easily leads to the considerations of Doplicher et al. [1] on qft's. References to these lectures are [23], [24], [25] and [26].

The relation (1) will be treated with θ being constant. Our focus is on time and its noncommutativity with spatial coordinates. For this purpose, it is enough to examine (1) on a $(1+1)$ -spacetime. We assume with no loss of generality that $\theta > 0$, as we can change its sign by flipping $\hat{x}_1$ to $-\hat{x}_1$. We denote by $\mathcal{A}_\theta(\mathbb{R}^2)$ the unital algebra generated by $\hat{x}_0, \hat{x}_1$ and $\mathbb{I}$.

2. Qualitative Remarks

2.1. Symmetries

If a group of transformations cannot be implemented on the algebra $\mathcal{A}_\theta(\mathbb{R}^2)$ generated by $\hat{x}_\mu$ with relation (1), then it is not likely to be a symmetry of any physical system based on (1) [25]. So let us check what are the automorphisms of (1).

2.1.1. Translations

First we readily see that spacetime translations $\mathcal{U}(\vec{a})$, $\vec{a} = (a_0, a_1)$, $a_\mu \in \mathbb{R}$, are automorphisms of $\mathcal{A}_\theta(\mathbb{R}^2)$: with

$$\mathcal{U}(\vec{a})\hat{x}_\mu = \hat{x}_\mu + a_\mu, \quad (2)$$

we see that

$$[\mathcal{U}(\vec{a})\hat{x}_\mu, \mathcal{U}(\vec{a})\hat{x}_\nu] = i\theta\varepsilon_{\mu\nu}. \quad (3)$$

The existence of these automorphisms allows the possibility of energy-momentum conservation. The time-translation automorphism

$$U(\tau) := \mathcal{U}((\tau, 0)) \quad (4)$$

is of particular importance. Without it, we cannot formulate conventional quantum physics.

The infinitesimal generators of $\mathcal{U}(\vec{a})$ can be defined by writing

$$\mathcal{U}(\vec{a}) = e^{-ia_0\hat{P}_0 + ia_1\hat{P}_1}. \quad (5)$$

Then we have

$$\hat{P}_0 = -\frac{1}{\theta}\text{ad } \hat{x}_1, \quad \hat{P}_1 = -\frac{1}{\theta}\text{ad } \hat{x}_0, \quad \text{ad } \hat{x}_\mu \hat{a} \equiv [\hat{x}_\mu, \hat{a}], \quad \hat{a} \in \mathcal{A}_\theta(\mathbb{R}^2). \quad (6)$$

The relations (6) show that the automorphisms $\mathcal{U}(\vec{a})$ are inner.

2.1.2. The Lorentz Group

It is a special feature of two dimensions that the $(2+1)$ connected Lorentz group is an inner automorphism group of (1). The above group is the two-dimensional projective symplectic group, the symplectic group quotiented by its center $\mathbb{Z}_2$. Its generators are $\text{ad } \hat{J}_3$ and $\text{ad } \hat{K}_a$, where

$$\hat{J}_3 = \frac{1}{4\theta}(\hat{x}_0^2 + \hat{x}_1^2), \quad \hat{K}_1 = \frac{1}{4\theta}(\hat{x}_0\hat{x}_1 + \hat{x}_1\hat{x}_0), \quad \hat{K}_2 = \frac{1}{4\theta}(\hat{x}_0^2 - \hat{x}_1^2), \quad (7)$$

with the ad notation explained by (6). Although this group generates inner automorphisms, it cannot be implemented on the quantum Hilbert space because, as we shall later see, $\hat{x}_0$ is not an operator on the physical Hilbert space. The algebra $\mathcal{A}_\theta(\mathbb{R}^2)$ is a $*$ -algebra with $\hat{x}_\mu^* = \hat{x}_\mu$.

2.1.3. P , T , C Symmetries

There are certain important transformations which are automorphisms for $\theta = 0$, but not for $\theta \neq 0$. One such is parity P :

$$P : \hat{x}_0 \rightarrow \hat{x}_0, \hat{x}_1 \rightarrow -\hat{x}_1, \mathbb{I} \rightarrow \mathbb{I}. \quad (8)$$

We want it furthermore to be linear. But that does not preserve (1) if $\theta \neq 0$:

$$P : [\hat{x}_0, \hat{x}_1] \rightarrow -[\hat{x}_0, \hat{x}_1], i\theta\mathbb{I} \rightarrow i\theta\mathbb{I}. \quad (9)$$

In contrast, time-reversal T ,

$$T : \hat{x}_0 \rightarrow -\hat{x}_0, \hat{x}_1 \rightarrow \hat{x}_1 \quad (10)$$

is anti-linear $T : i\theta\mathbb{I} \rightarrow -i\theta\mathbb{I}$, so that it is an automorphism of $\mathcal{A}_\theta(\mathbb{R}^2)$.

Hence any theory based on (1) violates P and PT . Superficially there seems to be no problem in writing charge conjugation invariant models based on (1). For such models, CPT will also fail to be a symmetry [8].

The symmetries P and PT are automorphisms of the algebra which is the direct sum of $\mathcal{A}_\theta(\mathbb{R}^2)$ and $\mathcal{A}_{-\theta}(\mathbb{R}^2)$. In that case, spacetime will have two leaves. The Doplicher et al. models are based on such algebras.

2.1.4. Further Automorphisms

As fully discussed in [9, 10], infinitesimal transformations $\hat{x}_\mu \rightarrow \hat{x}_\mu + \delta\hat{x}_\mu$ of the form $\delta\hat{x}_\mu = \hat{f}_\mu(\hat{x}_0, \hat{x}_1)$ generate automorphisms of $\mathcal{A}_\theta(\mathbb{R}^2)$ if the condition

$$[\hat{f}_\mu, \hat{x}_\nu] + [\hat{x}_\mu, \hat{f}_\nu] = 0 \quad (11)$$

is satisfied. The associated group of transformations exhausts the noncommutative version of the area-preserving transformations (in two dimensions and connected to the identity), and includes the Lorentz group as a particular case.

2.2. Causality

It is impossible to localize (the representation of) “coordinate” time $\hat{x}_0$ in $\mathcal{A}_\theta(\mathbb{R}^2)$ sharply. Any state will have a spread in the spectrum of $\hat{x}_0$. This leads to failure of causality as explained by Chaichian et al. [11].

The following important point was emphasised to us by Doplicher [12]. In quantum mechanics, if $\hat{p}$ is momentum, $\exp(i\xi\hat{p})$ is spatial translation by amount ξ . This ξ is not the eigenvalue of the position operator $\hat{x}$. In the same way, the amount τ of time translation in (4) is not “coordinate time”, the eigenvalue of $\hat{x}_0$ [1]. It makes sense to talk about a state and its translate by $U(\tau)$. For $\theta = 0$, it is possible to identify coordinate time with τ : the former is just a parameter we need for labelling time-slices of spacetime and increasing with τ . But for $\theta \neq 0$, $\hat{x}_0$ is an operator not commuting with $\hat{x}_1$, and cannot be interchanged with τ .

Concepts like duration of an experiment for $\theta = 0$ [13] are expressed using $U(\tau)$. They carry over to the noncommutative case too.

2.3. The Spin-Statistics Connection

With loss of causality, one loses local qft’s as well. As the best proofs of the spin-statistics connection require locality [14], we can anticipate the breakdown of this connection as well when $\mathcal{A}_\theta(\mathbb{R}^2)$ is generalised to $(3+1)$ dimensions. Precision experiments to test the spin-statistics connection are possible [15]. If signals for this violation due to $\theta \neq 0$ can be derived, good phenomenological bounds on $|\theta|$ should be possible.

3. Representation Theory

Observables, states and dynamics of quantum theory are to be based on the algebra $\mathcal{A}_\theta(\mathbb{R}^2)$ defined by (1). The formalism for their construction, using the methods of the GNS approach [16] in the commutative and non-commutative contexts, will be explored in the following.

Now to each $\hat{\alpha} \in \mathcal{A}_\theta(\mathbb{R}^2)$, we can canonically associate its left and right regular representations $\hat{\alpha}^L$ and $\hat{\alpha}^R$,

$$\hat{\alpha}^L \hat{\beta} = \hat{\alpha} \hat{\beta} , \quad \hat{\alpha}^R \hat{\beta} = \hat{\beta} \hat{\alpha} , \quad \hat{\beta} \in \mathcal{A}_\theta(\mathbb{R}^2) , \quad (12)$$

with $\hat{\alpha}^L \hat{\beta}^L = (\hat{\alpha} \hat{\beta})^L$ and $\hat{\alpha}^R \hat{\beta}^R = (\hat{\beta} \hat{\alpha})^R$. The carrier space of this representation is $\mathcal{A}_\theta(\mathbb{R}^2)$ itself.

But such representations are not enough for quantum physics. An “inner” product on $\mathcal{A}_\theta(\mathbb{R}^2)$ is needed for an eventual construction of a Hilbert space.

Doplicher et al. get this inner product using positive maps. Consider a map $\chi : \mathcal{A}_\theta(\mathbb{R}^2) \rightarrow \mathbb{C}$ with the usual properties of $\mathbb{C}$ -linearity and preservation of $*$: $\chi(\hat{\alpha}^*) = \overline{\chi(\hat{\alpha})}$ (bar meaning complex conjugation). It is a positive map if $\chi(\hat{\alpha}^* \hat{\alpha}) \geq 0$.

Given such a map, we can set $\langle \hat{\alpha}, \hat{\beta} \rangle = \chi(\hat{\alpha}^* \hat{\beta})$. It will be a scalar product if $\chi(\hat{\alpha}^* \hat{\alpha}) = 0$ implies $\hat{\alpha} = 0$. If that is not the case, it is necessary to eliminate nonzero vectors of zero norm (null vectors).

We illustrate these ideas first in the context of the commutative case, when $\theta = 0$. Then we generalise these ideas to (1) and discuss a positive map. There are actually several possible maps at our disposal, but they lead to equivalent physics [23].

3.1. The Commutative Case

3.1.1. The Positive Map

The algebra $\mathcal{C}$ in the commutative case is $\mathcal{A}_0(\mathbb{R}^2) = C^\infty(\mathbb{R} \times \mathbb{R})$, the product being point-wise multiplication, and $*$ being complex conjugation. If $\psi \in \mathcal{C}$, then $\psi(x_0, x_1) \in \mathbb{C}$, where (x_0, x_1) are coordinates of $\mathbb{R}^2$.

There is no distinction now between $\hat{\alpha}^L$ and $\hat{\alpha}^R$: $\hat{\alpha}^L = \hat{\alpha}^R$.

There is actually a family of positive maps χ_t of interest obtained by integrating ψ in x_1 at “time” t :

$$\chi_t(\psi) = \int dx_1 \psi(t, x_1) , \quad \chi_t(\psi^* \psi) \geq 0 . \quad (13)$$

This defines a family of spaces $\mathcal{C}_t$ with a positive-definite sesquilinear form (an “inner product”) $(\cdot, \cdot)_t$:

$$(\psi, \varphi)_t = \int dx_1 \psi^*(t, x_1) \varphi(t, x_1) . \quad (14)$$

(We associate χ_t with $\mathcal{C}$ to get $\mathcal{C}_t$.)

3.1.2. The Null Space $\mathcal{N}_t^0$

Every function $\hat{\alpha}$ which vanishes at time t is a two-sided ideal $\mathcal{I}_t^{\theta=0} := \mathcal{I}_t^0$ of $\mathcal{C}$. As elements of $\mathcal{C}_t$, they become null vectors $\mathcal{N}_t^0$ in the inner product (14). (We associate χ_t also to $\mathcal{I}_t^0$ to get $\mathcal{N}_t^0$.) Thus as in the GNS construction [16], we can quotient by these vectors and work with $\mathcal{C}_t/\mathcal{N}_t^0$. For elements $\psi + \mathcal{N}_t^0$ and $\chi + \mathcal{N}_t^0$ in $\mathcal{C}_t/\mathcal{N}_t^0$, the scalar product is

$$(\psi + \mathcal{N}_t^0, \chi + \mathcal{N}_t^0)_t = (\psi, \chi)_t . \quad (15)$$

There are no non-trivial vectors of zero norm now. The completion $\overline{\mathcal{C}_t/\mathcal{N}_t^0}$ of $\mathcal{C}_t/\mathcal{N}_t^0$ in this scalar product gives a Hilbert space $\widehat{\mathcal{H}}_t^0$. We have also that $\mathcal{C}_t/\mathcal{I}_t^0$ acts on it faithfully, preserving its $*$,

$$(\psi + \mathcal{I}_t^0)^* = \psi^* + (\mathcal{I}_t^0)^* = \psi^* + \mathcal{I}_t^0 . \quad (16)$$

In the expression above, S^* is the set obtained from S by taking the complex conjugate of each element. Hence $(\mathcal{I}_t^0)^* = \mathcal{I}_t^0$.

3.1.3. The Quantum Mechanical Hilbert Space $\mathcal{H}_t^0$

The quantum mechanical Hilbert space however is not $\widehat{\mathcal{H}}_t^0$. It is constructed in a different way, starting from a subspace $\tilde{\mathcal{H}}_{0,t} \subset \mathcal{C}_t$ which contains only $\{0\}$ as the null vector: $\tilde{\mathcal{H}}_{0,t} \cap \mathcal{N}_t^0 = \{0\}$. Then χ_t is a good scalar product on $\tilde{\mathcal{H}}_{0,t}$ and the quantum mechanical Hilbert space is given by $\mathcal{H}_t^0 = \overline{\tilde{\mathcal{H}}_{0,t}}$, the completion of $\tilde{\mathcal{H}}_{0,t}$.

The subspace $\tilde{\mathcal{H}}_{0,t}$ depends on the Hamiltonian H and is chosen as follows. Suppose first that H is a time-independent Hamiltonian on commutative spacetime, self-adjoint on the standard quantum mechanical Hilbert space $L^2(\mathbb{R})$. It acts on $\mathcal{C}_t$ and obeys $(\psi, H\chi)_t = (H\psi, \chi)_t$.

We now pick the subspace $\tilde{\mathcal{H}}_{0,t}$ of $\mathcal{C}_t$ by requiring that vectors in $\mathcal{C}_t$ obey the time-dependent Schrödinger equation:

$$\tilde{\mathcal{H}}_{0,t} = \{\psi \in \mathcal{C}_t : (i\partial_{x_0} - H)\psi(x_0, x_1) = 0\} . \quad (17)$$

The operator $i\partial_{x_0}$ is not “hermitian” on all vectors of $\mathcal{C}_t$:

$$(\psi, i\partial_{x_0}\chi)_t \neq (i\partial_{x_0}\psi, \chi)_t \text{ for generic } \psi, \chi \in \mathcal{C}_t , \quad (18)$$

but on $\tilde{\mathcal{H}}_{0,t}$, it equals H and does fulfill this property:

$$(\psi, i\partial_{x_0}\chi)_t = (i\partial_{x_0}\psi, \chi)_t \text{ for generic } \psi, \chi \in \tilde{\mathcal{H}}_{0,t} . \quad (19)$$

Since $[i\partial_{x_0}, H] = 0$, both $i\partial_{x_0}$ and H leave the subspace $\tilde{\mathcal{H}}_{0,t}$ invariant:

$$i\partial_{x_0}\tilde{\mathcal{H}}_{0,t} = H\tilde{\mathcal{H}}_{0,t} \subseteq \tilde{\mathcal{H}}_{0,t} . \quad (20)$$

We see also that since

$$\psi(x_0 + \tau, x_1) = \left(e^{-i\tau(i\partial_{x_0})} \psi \right) (x_0, x_1) = \left(e^{-i\tau H} \psi \right) (x_0, x_1) , \quad (21)$$

time evolution preserves the norm of $\psi \in \tilde{\mathcal{H}}_{0,t}$. Therefore if it vanishes at $x_0 = t$, it vanishes identically and is the zero element of $\tilde{\mathcal{H}}_{0,t}$: the only null vector in $\tilde{\mathcal{H}}_{0,t}$ is 0: $\mathcal{N}_t^0 \cap \tilde{\mathcal{H}}_{0,t} = \{0\}$. That means that χ_t gives a true scalar product on $\tilde{\mathcal{H}}_{0,t}$. The completion of $\tilde{\mathcal{H}}_{0,t}$ is the quantum Hilbert space $\mathcal{H}_t^0$.

We can find no convenient inclusion of $\mathcal{H}_t^0$ in $\widehat{\mathcal{H}}_t^0$. The reason is that $\mathcal{N}_t^0$ is not in the kernel of $(i\partial_{x_0} - H)$, only its zero vector is.

Elements of $\tilde{\mathcal{H}}_{0,t}$ are very conventional. Let $\hat{x}_\mu$ be coordinate functions ($\hat{x}_\mu(x_0, x_1) = x_\mu$) so that $i\partial_{x_0}\hat{x}_\mu = i\delta_{0\mu}$, and let ψ_0 be a constant function of x_0 so that $i\partial_{x_0}\psi_0 = 0$. Then

$$\psi = e^{-i\hat{x}_0 H} \psi_0 \in \tilde{\mathcal{H}}_{0,t} . \quad (22)$$

Under time evolution by amount τ , ψ becomes

$$e^{-i\tau H} \psi = e^{-i(\hat{x}_0 + \tau)H} \psi_0 \in \tilde{\mathcal{H}}_{0,t} . \quad (23)$$

The conceptual difference between coordinate time $\hat{x}_0$ and amount of time translation τ is apparent here. As one learns from Doplicher et al. [1], this difference cannot be ignored with spacetime noncommutativity.

As ψ_0 is constant in x_0 , its values may be written as $\psi_0(x_1)$.

3.1.4. On Observables

An observable $\hat{K}$ has to respect the Schrödinger constraint and leave $\tilde{\mathcal{H}}_{0,t}$ (and hence $\mathcal{H}_t^0$) invariant. This means that

$$\left[i\partial_{x_0} - H, \hat{K} \right] = 0 . \quad (24)$$

Let $\hat{L}$ be any operator with no explicit time dependence so that $\hat{L}$ is a function of $\hat{x}_1$ and momentum. Then

$$\hat{K} = e^{-i\hat{x}_0 H} \hat{L} e^{+i\hat{x}_0 H} \quad (25)$$

is an observable. We have also that $\hat{K}$ acts on ψ in a familiar manner:

$$\hat{K}\psi = \left(\hat{L}\psi_0 \right) e^{-i\hat{x}_0 H} . \quad (26)$$

Under time translation, $\hat{x}_0$ in $\hat{K}$ shifts to $\hat{x}_0 + \tau$ as it should:

$$e^{-i\tau H} \hat{K} e^{+i\tau H} = e^{-i(\hat{x}_0 + \tau)H} \hat{L} e^{+i(\hat{x}_0 + \tau)H} . \quad (27)$$

Response under time-translations is dynamics, it gives time-evolution. Just as in the conventional approach, here and elsewhere we should time-evolve either vector states (Schrödinger representation) or observables (Heisenberg representation). One can also formulate the interaction representation.

A final important point is the following. The observables have the expected reality properties. In particular, $\mathcal{C}$ is a $*$ -algebra, with star being complex conjugation, denoted here by a bar. So are the functions $\hat{L}$ on $\mathbb{R}^2$ which are constant in x_0 , that is, functions of position only. If $\hat{K}$ is its image on $\mathcal{H}_t^0$, as in (25), then $\bar{\hat{L}}$ has image $\hat{K}^\dagger$: we have a $*$ -representation of these functions. Momentum too is a self-adjoint operator on $\mathcal{H}_t^0$.

3.1.5. Time-dependent H

We refer to [23] for the treatment of time-dependent H .

3.1.6. Is Time an Observable?

What we have described above leads to conventional physics. Just as in the latter, here too, $\hat{x}_0$ is not an observable as it does not commute with $i\partial_{x_0} - H$:

$$[\hat{x}_0, i\partial_{x_0} - H] = -i\mathbb{I} . \quad (28)$$

Transformations with $\exp(-i\hat{x}_0 H)$ or U does not affect $\hat{x}_0$. So we cannot construct an observable therefrom as we did to get $\hat{K}$ from $\hat{L}$.

3.1.7. On the Time-dependence of $\mathcal{H}_t^0$

In conventional quantum physics, the Hilbert space has no time-dependence, whereas $\mathcal{H}_t^0$ has a label t . This is puzzling.

But the puzzle is easy to resolve: $\mathcal{H}_t^0$ is independent of t . Thus the solutions ψ of the Schrödinger constraint do not depend on t and are elements of every $\mathcal{H}_t^0$. Their scalar products too are independent of t because of the unitarity of H . There is thus only one Hilbert space which we call $\mathcal{H}_0$ (0 standing for the value of θ). We also denote $\tilde{\mathcal{H}}_{0,t}$ by $\tilde{\mathcal{H}}_0$ henceforth. Further the observables have no explicit t -dependence and act on $\mathcal{H}_0$ as in standard quantum theory.

3.2. The Noncommutative Case

The above discussion shows that for quantum theory, what we need are: (1) a suitable inner product on $\mathcal{A}_\theta(\mathbb{R}^2)$; (2) a Schrödinger constraint on $\mathcal{A}_\theta(\mathbb{R}^2)$; and (3) a Hamiltonian $\hat{H}$ and observables which act on the constrained subspace of $\mathcal{A}_\theta(\mathbb{R}^2)$. We also require that (1) is compatible with the self-adjointness of $\hat{H}$ and classically real observables.

We now consider these items one by one.

3.2.1. The Inner Product

There are several suitable inner products at first sight. But it can be argued that they are all equivalent [23]. So we work with just the one described below.

This inner product is based on symbol calculus. If $\hat{\alpha} \in \mathcal{A}_\theta(\mathbb{R}^2)$, we write it as

$$\hat{\alpha} = \int d^2k \tilde{\alpha}(k) e^{ik_1 \hat{x}_1} e^{ik_0 \hat{x}_0}, \quad (29)$$

and associate the symbol α_S with $\hat{\alpha}$ where

$$\alpha_S(x_0, x_1) = \int d^2k \tilde{\alpha}(k) e^{ik_1 x_1} e^{ik_0 x_0}. \quad (30)$$

The symbol is a function on $\mathbb{R}^2$. It is not the Moyal symbol. For the latter, the exponentials in (29) must be written as $\exp(ik_1 \hat{x}_1 + ik_0 \hat{x}_0)$.

Using this symbol, we can define a positive map S_t by

$$S_t(\hat{\alpha}) = \int dx_1 \alpha_S(t, x_1). \quad (31)$$

Properties of S_t are similar to χ_t . In particular it gives the inner product $(\cdot, \cdot)_t$, where

$$(\hat{\alpha}, \hat{\beta})_t = S_t(\hat{\alpha}^* \hat{\beta}) = \int dx_1 \alpha_S^*(t, x_1) \beta_S(t, x_1). \quad (32)$$

This inner product has null vectors $\mathcal{N}_t^\theta : \hat{\alpha} \in \mathcal{N}_t^\theta$ if $\alpha_S(t, \cdot) = 0$. But that result is not important for what follows as the Hilbert space is obtained only after constraining the vector states by the noncommutative Schrödinger equation.

3.2.2. The Schrödinger Constraint

The noncommutative analogue “ $i \frac{\partial}{\partial x_0}$ ” of the corresponding commutative operator is

$$i \frac{\partial}{\partial x_0} \equiv \hat{P}_0 = -\frac{1}{\theta} \text{ad } \hat{x}_1, \quad (33)$$

since

$$-\frac{1}{\theta} \text{ad } \hat{x}_1 \hat{x}_\lambda = i \delta_{\lambda 0} \mathbb{I}. \quad (34)$$

If the Hamiltonian $\hat{H}$ is time-independent,

$$[i \partial_{x_0}, \hat{H}] = 0, \quad (35)$$

it depends on the momentum $\hat{P}_1$ in (6) and $\hat{x}_1^L$, and we can write it as

$$\hat{H} = \hat{H}(\hat{x}_1^L, \hat{P}_1). \quad (36)$$

It can depend on $\hat{x}_1^R$ as well if we rely just on (35). But since $\hat{x}_1^R = -\text{ad } \hat{x}_1 + \hat{x}_1^L$, that means $\hat{H}$ has dependence also on $i\partial_{x_0}$ and we can write

$$\hat{H} = \hat{H}(\hat{x}_1^L, \hat{P}_1, i\partial_{x_0}) . \quad (37)$$

This generalisation however seems unwarranted: there is never such dependence of H on $i\partial_{x_0}$ for $\theta = 0$, and we will generally obtain $\hat{H}$ from H in a manner that does not induce this dependence.

If $\hat{H}$ has time-dependence and (35) is not correct, it will have $\hat{x}_0^L$, $\hat{x}_0^R$ or both in its arguments. But $\hat{x}_0^L = \theta\hat{P}_1 + \hat{x}_1^R$, so in the time-dependent case we write

$$\hat{H} = \hat{H}(\hat{x}_0^R, \hat{x}_1^L, \hat{P}_1) , \quad (38)$$

ignoring a possible $i\partial_{x_0}$ dependence for reasons above.

The family of vector states constrained by the Schrödinger equation is

$$\tilde{\mathcal{H}}_\theta = \left\{ \hat{\psi} \in \mathcal{A}_\theta(\mathbb{R}^2) : (i\partial_{x_0} - \hat{H}) \hat{\psi} = 0 \right\} , \quad (39)$$

where arguments of $\hat{H}$ can be appropriately inserted.

The solutions of (39) are easy to come by. For the time-independent case,

$$\hat{\psi} \in \tilde{\mathcal{H}}_\theta \implies \hat{\psi} = e^{-i(\hat{x}_0^R - \tau_I)\hat{H}(\hat{P}_1, \hat{x}_1^L)} \hat{\chi}(\hat{x}_1) . \quad (40)$$

The product $\hat{x}_0^R \hat{H}$ has no ordering problem since $[\hat{x}_0^R, \hat{H}(\hat{x}_1^L, \hat{P}_1)] = 0$. Also τ_I is the initial time when $\hat{\psi} = \hat{\chi}$. Since $\hat{x}_0^R$, $\hat{x}_1^L$ occur in the first factor, we should read the R.H.S. as the exponential acting on the algebra element $\hat{\chi}(\hat{x}_1)$.

Suppose next that $\hat{H}$ depends on $\hat{x}_0^R$ as in (38). As $\hat{x}_0^R$ commutes with $\hat{P}_1$ and $\hat{x}_1^L$, we can easily generalise the formula (40) to write

$$\begin{aligned} \hat{\psi} \in \tilde{\mathcal{H}}_\theta \implies \hat{\psi} &= U(\hat{x}_0^R, \tau_I) \hat{\chi}(\hat{x}_1) , \\ U(\hat{x}_0^R, \tau_I) &= T \exp \left[-i \left(\int_{\tau_I}^{x_0} d\tau \hat{H}(\tau, \hat{x}_1^L, \hat{P}_1) \right) \right] \Big|_{x_0=\hat{x}_0^R} . \end{aligned} \quad (41)$$

Just as in (40), the dependence of U on $\hat{x}_0^R$ and τ_I has been displayed, while τ_I is the initial time when $\hat{\psi} = \hat{\chi}$.

Time translation by amount τ shifts $\hat{x}_0^R$ to $\hat{x}_0^R + \tau$ in both (40) and (41).

An alternative useful form for $\hat{\psi}$ in (41) is

$$\hat{\psi} = V(\hat{x}_0^R, -\infty) \hat{\chi}(\hat{x}_1) , \quad (42)$$

$$V(\hat{x}_0^R, -\infty) = T \exp \left[-i \int_{-\infty}^0 d\tau \hat{H}(\hat{x}_0^R + \tau, \hat{x}_1^L, \hat{P}_1) \right] , \quad (43)$$

where the integral can be defined at the lower limit using the usual adiabatic cut-off.

The Hilbert space $\mathcal{H}_\theta$ based on the scalar product $(\cdot, \cdot)_t$ is obtained from $\tilde{\mathcal{H}}_\theta$ by completion. Our basic assumption is that $\hat{H}$ is self-adjoint in the chosen scalar product. Then as before, the resultant Hilbert space $\mathcal{H}_\theta$ has no dependence on t .

Assuming that

$$\hat{H} = \frac{\hat{P}_1^2}{2m} + V(\hat{x}_1) \quad (44)$$

is a self-adjoint Hamiltonian for $\theta = 0$, then we note that its $\theta \neq 0$ version

$$\hat{H} = \frac{\hat{P}_1^2}{2m} + V(\hat{x}_1^L) \quad (45)$$

is self-adjoint on $\mathcal{H}_\theta$.

If $\hat{H}(\hat{x}_0, \hat{x}_1, \hat{P}_1)$ is time-dependent for $\theta = 0$, we can form its $\theta \neq 0$ version

$$\hat{H}(\hat{x}_0^L, \hat{x}_1^L, \hat{P}_1) = \hat{H}(-\theta\hat{P}_1 + \hat{x}_0^R, \hat{x}_1^L, \hat{P}_1) . \quad (46)$$

As $\hat{x}_0^L$ and $\hat{P}_1$ do not commute with $\hat{x}_1^L$, we should check this $\hat{H}$ for factor-ordering problems. But for this potential trouble, $\hat{H}$ is self-adjoint if H is.

3.2.3. Remarks on Time for $\theta \neq 0$

In the passage from H to $\hat{H}$, there is an apparent ambiguity. Above we replaced x_0 by $\hat{x}_0^L$, but we may be tempted to replace x_0 by $\hat{x}_0^R$. In that case the passage to $\theta \neq 0$ will involve no factor-ordering problem as $\hat{x}_0^R$ commutes with $\hat{x}_1^L$ and $\hat{P}_1$. At the same time, θ -dependent terms in $\hat{H}$ disappear.

But it is incorrect to replace x_0 by $\hat{x}_0^R$ and at the same time x_1 by $\hat{x}_1^L$. Time and space should fulfill the relation (1) when θ becomes nonzero whereas $\hat{x}_0^R$ and $\hat{x}_1^L$ commute.

Note that $\hat{x}_0^{L,R}$ do not preserve the Schrödinger constraint so that there is no time operator for $\theta \neq 0$ as well.

3.2.4. Time-dependence for $\theta = 0 \implies$ Spatial nonlocality for $\theta \neq 0$

We noted above that $\hat{x}_0^L = -\theta\hat{P}_1 + \hat{x}_0^R$ and that $\hat{x}_0^R$ behaves much like the $\theta = 0$ time x_0 . Thus if H has time-dependence, its effect on $\hat{H}$ is to induce new momentum-dependent terms. The x_0 -dependence in H need not to be polynomial so that in $\hat{H}$ they induce nonpolynomial interactions in momentum, that is, instantaneous spatially nonlocal (“acausal”) interactions.

3.2.5. Observables

We can construct observables as in (25) or its version for time-dependent Hamiltonians. No complications are encountered.

4. Examples

4.1. Plane Waves

Let

$$\hat{H}_0 = \frac{\hat{P}_1^2}{2m} \quad (47)$$

be the free Hamiltonian. Its eigenstates are

$$\hat{\psi}_k = e^{ik\hat{x}_1} e^{-i\omega(k)\hat{x}_0} , \quad \omega(k) = \frac{k^2}{2m} , \quad k \in \mathbb{R} . \quad (48)$$

The eigenvalues are $k^2/2m$:

$$\hat{H}_0 \hat{\psi}_k = \left(\hat{H}_0 e^{ik\hat{x}_1} \right) e^{-i\omega(k)\hat{x}_0} = \omega(k) \hat{\psi}_k . \quad (49)$$

The second factor in $\hat{\psi}_k$ is dictated by the Schrödinger constraint:

$$\hat{P}_0 \hat{\psi}_k = e^{ik\hat{x}_1} \hat{P}_0 e^{-i\omega(k)\hat{x}_0} = \omega(k) \hat{\psi}_k \implies \left(\hat{P}_0 - \hat{H} \right) \hat{\psi}_k = 0 . \quad (50)$$

The spectrum of $\hat{H}_0$ is completely conventional while the noncommutative plane waves too resemble the ordinary plane waves. But phenomena like beats and interference show new features [26].

The coincidence of spectra of the free Hamiltonians in commutative and noncommutative cases is an illustration of a more general result which we now establish.

4.2. A Spectral Map

For $\theta = 0$ consider the Hamiltonian

$$H = -\frac{1}{2m} \frac{\partial^2}{\partial x_1^2} + V(\hat{x}_1) \quad (51)$$

with eigenstates ψ_E fulfilling the Schrödinger constraint:

$$\psi_E(\hat{x}_0, \hat{x}_1) = \varphi_E(\hat{x}_1) e^{-iE\hat{x}_0}, \quad (52)$$

$$H\varphi_E = E\varphi_E. \quad (53)$$

The Hamiltonian $\hat{H}$ associated to H for $\theta \neq 0$ is

$$\hat{H} = \frac{\hat{P}_1^2}{2m} + V(\hat{x}_1). \quad (54)$$

Then $\hat{H}$ has exactly the same spectrum as H while its eigenstates $\hat{\psi}_E$ are obtained from ψ_E just by regarding $\hat{x}_0$ and $\hat{x}_1$ as fulfilling (1):

$$\hat{\psi}_E = \varphi_E(\hat{x}_1) e^{-iE\hat{x}_0}, \quad (55)$$

$$\hat{H}\varphi_E(\hat{x}_1) = E\varphi_E(\hat{x}_1). \quad (56)$$

The proof of (56) follows from (54) as it involves no feature associated with spacetime noncommutativity. Since

$$\hat{P}_0\hat{\psi}_E = \varphi_E(\hat{x}_1)\hat{P}_0 e^{-iE\hat{x}_0} = E\hat{\psi}_E, \quad (57)$$

we see that $\hat{\psi}_E$ fulfills the Schrödinger constraint as well.

When the spatial slice for a commutative spacetime $\mathbb{R}^d$ is of dimension two or larger, one can introduce space-space noncommutativity as well. That would change the noncommutative Hamiltonian. The spectral map may not then exist.

5. Conserved Current

The existence of a current j_λ which fulfills the continuity equation has a particular importance when $\theta = 0$. It is this current which after second quantization couples to electromagnetism [17].

There is such a conserved current also for $\theta \neq 0$. It follows in the usual way from (39) and its $*$:

$$\left(\hat{P}_0\hat{\psi}\right)^* - \hat{\psi}^*\hat{H} = -\hat{P}_0\hat{\psi}^* - \hat{\psi}^*\hat{H} = 0. \quad (58)$$

Here we assumed that $\hat{V}^* = \hat{V}$.

Multiplying the Schrödinger constraint in (39) on left by $\hat{\psi}^*$ and (58) on right by $\hat{\psi}$ and subtracting,

$$\hat{P}_0(\hat{\psi}^*\hat{\psi}) = \hat{\psi}^*\left(\frac{\hat{P}_1^2}{2m}\hat{\psi}\right) - \left(\frac{\hat{P}_1^2}{2m}\hat{\psi}^*\right)\hat{\psi} = \frac{\hat{P}_1}{2m} \left[\hat{\psi}^*(\hat{P}_1\hat{\psi}) - (\hat{P}_1\hat{\psi}^*)\hat{\psi} \right]. \quad (59)$$

With

$$\hat{\rho} = \hat{\psi}^*\hat{\psi}, \quad \hat{j} = \frac{1}{2m} \left[\hat{\psi}^*(\hat{P}_1\hat{\psi}) - (\hat{P}_1\hat{\psi}^*)\hat{\psi} \right] \quad (60)$$

as the noncommutative charge and current densities, (59) can be interpreted as the noncommutative continuity equation.

6. Towards Quantum Field Theory

Perturbative quantum field theories (qft's) based on algebras like (1) have been treated with depth by Doplicher et al. [1]. We can also see how to do perturbative qft's, our approach can be inferred from the work of Doplicher et al.

In the interaction representation, an operator U_I determines the S -matrix. It is in turn determined by the interaction Hamiltonian $\hat{H}_I$. The latter is based on “free fields” which are solutions of the Klein-Gordon equation (We assume zero spin for simplicity). Examples of $\hat{H}_I$ can be based on interaction Hamiltonians H_I such as $\lambda \int dx_1 \Phi(x_0, x_1)^4$ (with $\Phi^\dagger = \Phi$ being a free field) for $\theta = 0$. For this particular H_I , $\hat{H}_I$ can be something like $\lambda S_{x_0} [\hat{\Phi}(\hat{x}_0, \hat{x}_1)^4]$ (cf. (31)), where $\hat{\Phi}$ is the self-adjoint free field for $\theta \neq 0$. We make this expression more precise below.

We require of $\hat{\Phi}$ that it is a solution of the massive Klein-Gordon equation:

$$\left(\text{ad} \hat{P}_0^2 - \text{ad} \hat{P}_1^2 + \mu^2 \right) \hat{\Phi} = 0 . \quad (61)$$

The plane wave solutions of (61) are

$$\hat{\phi}_k = e^{ik\hat{x}_1} e^{-i\omega(k)\hat{x}_0} , \quad \omega(k)^2 - k^2 = \mu^2 . \quad (62)$$

So for $\hat{\Phi}$, we write [11]

$$\hat{\Phi} = \int \frac{dk}{2\omega(k)} \left[a_k \hat{\phi}_k + a_k^\dagger \hat{\phi}_k^\dagger \right] , \quad (63)$$

where a_k and $a_k^\dagger$ commute with $\hat{x}_\mu$ and define harmonic oscillators: $[a_k, a_k^\dagger] = 2\omega(k)\delta(k - k')$.

The expression (63) is the “free” field “coinciding with the Heisenberg field initially”. After time translation by amount τ using the free Schrödinger Hamiltonian

$$\hat{H}_0 = \int \frac{dk}{2\omega(k)} a_k^\dagger a_k , \quad (64)$$

it becomes

$$U_0(\tau) \left(\hat{\Phi} \right) = e^{i\tau \hat{H}_0} \hat{\Phi} e^{-i\tau \hat{H}_0} , \quad (65)$$

The interaction Hamiltonian is accordingly

$$\hat{H}_I(x_0) = \lambda : S_{x_0} \left(U_0(\tau) \left(\hat{\Phi} \right)^4 \right) : = \lambda : S_{x_0+\tau} \left(\hat{\Phi}^4 \right) : , \quad \lambda > 0 , \quad (66)$$

where $: :$ denotes the normal ordering of a_k and $a_k^\dagger$.

The S -matrix S can be worked out as usual:

$$S = T \exp \left[-i \int_{-\infty}^{+\infty} d\tau \lambda : S_\tau \left(\hat{\Phi}^4 \right) : \right] . \quad (67)$$

It is important to recognise, as is clear from Doplicher et al. [1], that time-ordering is with respect to the time-translation parameter τ and not the spectrum of the operator $\hat{x}_0^L$. Its perturbation series can be developed since we understand the relevant properties of $\hat{\Phi}$.

Scattering amplitudes can be calculated from (67). There is no obvious reason why they are not compatible with perturbative unitarity [18].

7. Quantised Evolutions

We now study three algebras where time evolution becomes discrete because of noncommutativity. Their physics have striking features.

7.1. The Noncommutative Cylinder $\mathcal{A}_\theta (\mathbb{R} \times S^1)$

It is generated by $\hat{x}_0$ and $e^{-i\hat{x}_1}$ with the relation

$$[\hat{x}_0, e^{-i\hat{x}_1}] = \theta e^{-i\hat{x}_1}. \quad (68)$$

7.2. Noncommutative $\mathbb{R}^3$

The algebra $\hat{e}_2$ in this case is the enveloping algebra of e_2 , the Lie algebra of the Euclidean group E_2 . Spacetime coordinates $\hat{x}_\mu$ form a basis of e_2 and fulfill the commutation relations

$$[\hat{x}_i, \hat{x}_j] = 0, \quad [\hat{x}_0, \hat{x}_i] = i\theta \epsilon_{ij} \hat{x}_j, \quad \epsilon_{ij} = -\epsilon_{ji}, \quad \epsilon_{12} = 1. \quad (69)$$

Thus $\hat{x}_i$ are identified with translations and $\hat{x}_0/\theta$ is the canonically normalised angular momentum J :

$$e^{i2\pi J} \hat{x}_\mu e^{-i2\pi J} = \hat{x}_\mu. \quad (70)$$

The Lie algebra e_2 is a contraction of $so(2, 1)$, the Lie algebra of $SO(2, 1)$. The latter and its enveloping algebra have occurred as spacetime algebras in 2+1 gravity [19]. (See also [20]-[22] and also [27] in this connection.)

7.3. The Noncommutative $\mathbb{R} \times S^3$, $\mathcal{A}_\theta (\mathbb{R} \times S^3)$

We can represent $S^3 = \langle x \in \mathbb{R}^4 : \sum_\lambda x_\lambda^2 = 1 \rangle$ by $SU(2)$ matrices:

$$x_0 \mathbb{I} + i\vec{\tau} \cdot \vec{x} \in SU(2), \quad (71)$$

where $\mathbb{I}$ is the 2×2 unit matrix and τ_i are Pauli matrices. In this way we identify S^3 and $SU(2)$. Left- and right- regular representations of $SU(2)$ act on functions $\mathcal{C}^\infty(SU(2))$ on $SU(2)$.

Let $su(2)$ be the Lie algebra of $SU(2)$ with conventional angular momentum operators J_i . Then in particular, J_3 has a right action J_3^R on $\mathcal{C}^\infty(SU(2))$:

$$(e^{i\theta J_3^R} \hat{f})(g) = \hat{f}(g e^{i\theta J_3/2}) \quad (72)$$

for $\hat{f} \in \mathcal{C}^\infty(SU(2))$ and $g \in SU(2)$.

In the algebra $\mathcal{A}_\theta (\mathbb{R} \times S^3)$, the spatial slice S^3 is represented by the commutative algebra $\mathcal{C}^\infty(SU(2))$, and time $\hat{x}_0$ is identified with $2\theta J_3^R$ in the following way:

$$(e^{i\omega \hat{x}_0} \hat{f} e^{-i\omega \hat{x}_0})(s) := (e^{i\omega 2\theta J_3^R} \hat{f})(s). \quad (73)$$

Cases 1) and 3) are actually very similar. In case 1), the spatial slice has algebra $\mathcal{C}^\infty(S^1)$ and $J = \hat{x}_0/\theta$ is the canonically normalised generator of rotations: $e^{i2\pi J} \hat{\alpha} e^{-i2\pi J} = \hat{\alpha}$, for $\hat{\alpha} \in \mathcal{A}_\theta (\mathbb{R} \times S^1)$.

In all these cases, time translations get quantised in units of θ in quantum physics. This result is known for cases 1) and 2) [11], [19]-[22]. It comes from the fact that the spectrum $\text{spec } J$ or $\text{spec } J_3$ of J or J_3 in an irreducible representation of the associated algebra is spaced in units of θ . We will prove it fully as we go along.

Using a different approach, a model with quantised evolution was also constructed in [27].

There are generalisations of these constructions to manifolds $\mathbb{R} \times M$ where $\mathbb{R}$ accounts for time and M is the spatial slice, provided M admits a $U(1)$ action. If J is its generator on $C^\infty(M)$, we can set $\hat{x}_0 = \theta J$ and get an algebra $\mathcal{A}_\theta(\mathbb{R} \times M)$ with quantised evolution.

The mathematical approach to noncommutativity in this paper is similar to that of Rieffel, Connes, Landi and others [28]-[29]. We have drawn much inspiration from their work.

After reviewing [23] in the next section, we will study the three preceding examples in the subsequent sections. Issues related to energy nonconservation and also scattering and quantum field theory are taken up after that.

8. The Noncommutative Cylinder

The noncommutative cylinder $\mathcal{A}_\theta(\mathbb{R} \times S^1)$ has been considered in great detail by Chaichian et al. [11], especially as regards its quantum field theory aspects. They have pointed out and emphasised that time gets quantised on $\mathcal{A}_\theta(\mathbb{R} \times S^1)$ (see also [27]) and studied the impact of this quantisation on causality and unitarity. Below, we review how this quantisation comes about and develop quantum physics on $\mathcal{A}_\theta(\mathbb{R} \times S^1)$. We do not encounter problems with unitarity.

For $\theta = 0$, there is a close relation between $C^\infty(\mathbb{R} \times \mathbb{R})$ and the functions $C^\infty(\mathbb{R} \times S^1)$ on a cylinder. The former is generated by coordinate functions $\hat{x}_0$ and $\hat{x}_1$, and the latter by $\hat{x}_0$ and $e^{i\hat{x}_1}$, $e^{i\hat{x}_1}$ being invariant under the 2π -shifts $\hat{x}_1 \rightarrow \hat{x}_1 \pm 2\pi$. Following this idea, we can regard the noncommutative $\mathbb{R} \times S^1$ algebra $\mathcal{A}_\theta(\mathbb{R} \times S^1)$ as generated by $\hat{x}_0$ and $e^{i\hat{x}_1}$ with the defining relation

$$e^{i\hat{x}_1}\hat{x}_0 = \hat{x}_0e^{i\hat{x}_1} + \theta e^{i\hat{x}_1}. \quad (74)$$

For $C^\infty(\mathbb{R} \times S^1)$, the momentum $\hat{p}_1$ is the differential operator defined by

$$[\hat{p}_1, e^{i\hat{x}_1}] = e^{i\hat{x}_1}. \quad (75)$$

By evaluating (75) at x_1 , we can write it in the usual way: $[-i\frac{\partial}{\partial x_1}, e^{ix_1}] = e^{ix_1}$.

It follows from (75) that

$$e^{i2\pi\hat{p}_1}e^{i\hat{x}_1}e^{-i2\pi\hat{p}_1} = e^{i\hat{x}_1}. \quad (76)$$

So $e^{i2\pi\hat{p}_1}$ is in the center of the algebra generated by $\hat{p}_1$, $e^{i\hat{x}_1}$ with the relation (75). In an irreducible representation (IRR), it is a phase $e^{i\varphi}$ times $\mathbb{I}$. The spectrum of $\hat{p}_1$ in an IRR is hence

$$\text{spec } \hat{p}_1 = \mathbb{Z} + \frac{\varphi}{2\pi} \equiv \left\{ n + \frac{\varphi}{2\pi} : n \in \mathbb{Z} \right\}. \quad (77)$$

Its domain $\mathcal{D}_\varphi(\hat{p}_1)$ in such an IRR is spanned by quasi-periodic functions χ_n :

$$\chi_n = e^{i(n + \frac{\varphi}{2\pi})\hat{x}_1}, \quad n \in \mathbb{Z}, \quad \chi_n(\hat{x}_1 + 2\pi) = e^{i\varphi}\chi_n(\hat{x}_1). \quad (78)$$

If for example

$$H = \frac{\hat{p}_1^2}{2m} \quad (79)$$

is the Hamiltonian, its domain $\mathcal{D}_\varphi(H)$ fulfilling the Schrödinger constraint as well is spanned by

$$\psi_n = \chi_n e^{-iE_n\hat{x}_0}, \quad (80)$$

with ψ_n being eigenstates of H :

$$H\psi_n = E_n\psi_n, \quad (81)$$

$$E_n = \frac{1}{2m} \left(n + \frac{\varphi}{2\pi} \right)^2. \quad (82)$$

The quantity φ is generally interpreted as the flux through the circle.

For the noncommutative cylinder, (76) generalises in a striking manner:

$$e^{-i\frac{2\pi}{\theta}\hat{x}_0}e^{i\hat{x}_1}e^{i\frac{2\pi}{\theta}\hat{x}_0} = e^{i\hat{x}_1}. \quad (83)$$

Hence in an IRR,

$$e^{-i\frac{2\pi}{\theta}\hat{x}_0} = e^{-i\varphi}\mathbb{I}, \quad (84)$$

so that for the spectrum $\text{spec } \hat{x}_0$ of $\hat{x}_0$ in an IRR, we have,

$$\text{spec } \hat{x}_0 = \theta\mathbb{Z} + \frac{\theta\varphi}{2\pi} = \theta\left(\mathbb{Z} + \frac{\varphi}{2\pi}\right) \equiv \left\{\theta\left(n + \frac{\varphi}{2\pi}\right) : n \in \mathbb{Z}\right\}. \quad (85)$$

We can realise $\mathcal{A}_\theta(\mathbb{R} \times S^1)$ irreducibly in the auxiliary Hilbert space $L^2(S^1, dx_1)$. It has the scalar product given by

$$(\alpha, \beta) = \int_0^{2\pi} dx_1 \alpha^*(e^{ix_1}) \beta(e^{ix_1}), \quad \alpha, \beta \in L^2(S^1, dx_1). \quad (86)$$

On this space, $e^{i\hat{x}_1}$ acts by evaluation map,

$$(e^{i\hat{x}_1}\alpha)(e^{ix_1}) = e^{ix_1}\alpha(e^{ix_1}), \quad (87)$$

while $\hat{x}_0/\theta$ acts like the $\theta = 0$ momentum with domain $D_\varphi(\hat{p}_1)$.

We denote this particular representation of $\mathcal{A}_\theta(\mathbb{R} \times S^1)$ as $\mathcal{A}_\theta(\mathbb{R} \times S^1, e^{i\frac{\varphi}{2\pi}})$.

Let us examine $\mathcal{A}_\theta(\mathbb{R} \times S^1, e^{i\frac{\varphi}{2\pi}})$ more closely. We can regard it as generated by $e^{i\hat{x}_1}$ and $e^{i\omega\hat{x}_0}$ where ω is real. Now because of the spectral result (85),

$$e^{i(\omega + \frac{2\pi}{\theta})\hat{x}_0} = e^{i\varphi}e^{i\omega\hat{x}_0}. \quad (88)$$

Thus elements of $\mathcal{A}_\theta(\mathbb{R} \times S^1, e^{i\frac{\varphi}{2\pi}})$ are quasiperiodic in ω just as χ_n is quasiperiodic in $\hat{x}_1$, and we can restrict ω to its fundamental domain: $\omega \in [-\frac{\pi}{\theta}, \frac{\pi}{\theta}]$. The general element of $\mathcal{A}_\theta(\mathbb{R} \times S^1, e^{i\frac{\varphi}{2\pi}})$ is thus

$$\hat{\alpha} = \sum_{n \in \mathbb{Z}} \int_{-\frac{\pi}{\theta}}^{+\frac{\pi}{\theta}} d\omega \tilde{\alpha}_n(\omega) e^{in\hat{x}_1} e^{i\omega\hat{x}_0}, \quad (89)$$

as first discussed by Chaichian et al. [11].

8.1. Positive Maps and Inner Products

A positive map on $\mathcal{A}_\theta(\mathbb{R} \times S^1, e^{i\frac{\varphi}{2\pi}})$ can be found from symbol calculus. Since the spectrum of $\hat{x}_0$ is $\theta(\mathbb{Z} + \frac{\varphi}{2\pi})$ and the spectrum of $e^{i\hat{x}_1}$ is S^1 , the symbol of $\hat{\alpha}$ is a function α on $\theta(\mathbb{Z} + \frac{\varphi}{2\pi}) \times S^1$:

$$\alpha : \theta\left(\mathbb{Z} + \frac{\varphi}{2\pi}\right) \times S^1 \rightarrow \mathbb{C}. \quad (90)$$

It is defined by

$$\alpha\left(\theta\left(m + \frac{\varphi}{2\pi}\right), e^{ix_1}\right) = \sum_{n \in \mathbb{Z}} \int_{-\frac{\pi}{\theta}}^{+\frac{\pi}{\theta}} d\omega \tilde{\alpha}_n(\omega) e^{inx_1} e^{i\omega\theta(m + \frac{\varphi}{2\pi})}. \quad (91)$$

Before proceeding, we show that $\hat{\alpha}$ determines $\tilde{\alpha}_n$ and hence α uniquely, so that the map $\hat{\alpha} \rightarrow \alpha$ is well-defined. We show also the converse, that α determines $\tilde{\alpha}_n$ and hence $\hat{\alpha}$ uniquely, so that the map $\hat{\alpha} \rightarrow \alpha$ is bijective.

Let $|n\rangle$ be the normalised eigenstates of $\hat{x}_0$:

$$\hat{x}_0 |n\rangle = \theta \left(n + \frac{\varphi}{2\pi} \right) |n\rangle, \quad \langle m|n\rangle = \delta_{mn}, \quad n \in \mathbb{Z}. \quad (92)$$

Then

$$e^{i\hat{x}_1} |n\rangle = |n-1\rangle. \quad (93)$$

Therefore

$$\langle m|\hat{\alpha}|n\rangle = \int_{-\frac{\pi}{\theta}}^{+\frac{\pi}{\theta}} d\omega \tilde{\alpha}_{n-m}(\omega) e^{i\omega\theta(n+\frac{\varphi}{2\pi})}, \quad (94)$$

and since

$$\frac{\theta}{2\pi} \sum_n e^{i(\omega-\omega')\theta n} = \delta(\omega-\omega'), \quad (95)$$

we find

$$\frac{\theta}{2\pi} \sum_n e^{-i\omega\theta(n+\frac{\varphi}{2\pi})} \langle n-m|\hat{\alpha}|n\rangle = \tilde{\alpha}_m(\omega). \quad (96)$$

The inverse map follows similarly:

$$\tilde{\alpha}_n(\omega) = \frac{\theta}{(2\pi)^2} \sum_m e^{-i\omega\theta(m+\frac{\varphi}{2\pi})} \int_0^{2\pi} dx_1 e^{-inx_1} \alpha\left(\theta\left(m+\frac{\varphi}{2\pi}\right), e^{ix_1}\right). \quad (97)$$

Our positive map is $S_{\theta(m+\frac{\varphi}{2\pi})}$:

$$S_{\theta(m+\frac{\varphi}{2\pi})}(\hat{\alpha}) = \int_0^{2\pi} dx_1 \alpha\left(\theta\left(m+\frac{\varphi}{2\pi}\right), e^{ix_1}\right). \quad (98)$$

Just as in (32), we then have, for inner product,

$$\begin{aligned} (\hat{\alpha}, \hat{\beta})_{\theta(m+\frac{\varphi}{2\pi})} &= S_{\theta(m+\frac{\varphi}{2\pi})}(\hat{\alpha}^* \hat{\beta}) \\ &= \int_0^{2\pi} dx_1 \alpha^*\left(\theta\left(m+\frac{\varphi}{2\pi}\right), e^{ix_1}\right) \beta\left(\theta\left(m+\frac{\varphi}{2\pi}\right), e^{ix_1}\right). \end{aligned} \quad (99)$$

There are other possibilities for inner product such as the one based on coherent states. The equivalence of theories based on different inner products is discussed in [23].

8.2. Spectrum of Momentum

We can infer the spectrum of the momentum operator $\hat{P}_1$ when it acts on $\mathcal{A}_\theta(\mathbb{R} \times S^1, e^{i\frac{\varphi}{2\pi}})$. Since this algebra allows for only integral powers of $e^{i\hat{x}_1}$, and

$$\hat{P}_1 e^{in\hat{x}_1} = n e^{in\hat{x}_1}, \quad (100)$$

we have $\text{spec } \hat{P}_1 = \mathbb{Z}$. The flux term is 0 in this spectrum.

For the construction of a Hilbert space, we do not need this algebra. It is enough to have an $\mathcal{A}_\theta(\mathbb{R} \times S^1, e^{i\frac{\varphi}{2\pi}})$ -module which can be consistently treated. Such a module is

$$\mathcal{A}_\theta(\mathbb{R} \times S^1, e^{i\frac{\varphi}{2\pi}}, e^{i\frac{\psi}{2\pi}}) = \left\langle \hat{\gamma} = e^{i\frac{\psi}{2\pi}\hat{x}_1} \sum_{n \in \mathbb{Z}} \int_{-\frac{\pi}{\theta}}^{\frac{\pi}{\theta}} d\omega \tilde{\gamma}_n(\omega) e^{in\hat{x}_1} e^{i\omega\hat{x}_0} \right\rangle. \quad (101)$$

The eigenvalues of $\hat{P}_1$ are now shifted by $\frac{\psi}{2\pi}$:

$$\hat{P}_1 e^{i\frac{\psi}{2\pi}\hat{x}_1} e^{in\hat{x}_1} = \left(n + \frac{\psi}{2\pi}\right) e^{i\frac{\psi}{2\pi}\hat{x}_1} e^{in\hat{x}_1}, \quad n \in \mathbb{Z}. \quad (102)$$

So we now have a flux term $\frac{\psi}{2\pi}$.

We have to check that $\mathcal{A}_\theta(\mathbb{R} \times S^1, e^{i\frac{\varphi}{2\pi}}, e^{i\frac{\psi}{2\pi}})$ also has an inner product. That is so because if

$$\hat{\gamma}, \hat{\delta} \in \mathcal{A}_\theta(\mathbb{R} \times S^1, e^{i\frac{\varphi}{2\pi}}, e^{i\frac{\psi}{2\pi}}), \quad (103)$$

then

$$\hat{\gamma}^* \hat{\delta} \in \mathcal{A}_\theta(\mathbb{R} \times S^1, e^{i\frac{\varphi}{2\pi}}), \quad (104)$$

(the ψ -dependent factors $e^{i\frac{\psi}{2\pi}\hat{x}_1}$ cancelling out), so that the inner product is still like (99):

$$(\hat{\gamma}, \hat{\delta})_{\theta(m+\frac{\varphi}{2\pi})} = S_{\theta(m+\frac{\varphi}{2\pi})}(\hat{\gamma}^* \hat{\delta}). \quad (105)$$

It is interesting that the flux terms in time and momentum can be different.

We remark that the Schrödinger constraint below does not alter the spectrum of $\hat{P}_1$.

8.3. The Schrödinger Constraint

8.3.1. The Time-Independent Hamiltonian

Since

$$i\partial_{x_0} e^{i\omega\hat{x}_0} = -\omega e^{i\omega\hat{x}_0} \quad (106)$$

is not quasiperiodic in ω , continuous time translations and the Schrödinger constraint in the original form cannot be defined on $\mathcal{A}_\theta(\mathbb{R} \times S^1)$.

But translation of $\hat{x}_0$ by $\pm\theta$ leaves its spectrum intact. Hence the operator

$$e^{-i\theta(i\partial_{x_0})} = e^{i\theta\hat{P}_1}, \quad (107)$$

and its integral powers act on $\mathcal{A}_\theta(\mathbb{R} \times S^1)$. The conventional Schrödinger constraint is thus changed to a discrete Schrödinger constraint. In the time-independent case when the Hamiltonian can be written as $\hat{H}(e^{i\hat{x}_1^L}, \hat{P}_1)$, the family of vector states constrained by the discrete Schrödinger equation is

$$\tilde{\mathcal{H}}_\theta(e^{i\frac{\varphi}{2\pi}}, e^{i\frac{\psi}{2\pi}}) = \left\{ \hat{\psi} \in \mathcal{A}_\theta(\mathbb{R} \times S^1, e^{i\frac{\varphi}{2\pi}}, e^{i\frac{\psi}{2\pi}}) : e^{-i\theta(i\partial_{x_0})} \hat{\psi} = e^{-i\theta\hat{H}} \hat{\psi} \right\}. \quad (108)$$

It has solutions

$$\hat{\psi} = e^{-i\hat{x}_0^R \hat{H}(e^{i\hat{x}_1^L}, \hat{P}_1)} e^{i\frac{\psi}{2\pi}\hat{x}_1} \hat{\chi}(e^{i\hat{x}_1}), \quad (109)$$

just as in (40).

8.3.2. The Time-Dependent Hamiltonian

Reference [24] contains its treatment.

8.4. Remarks

We point out that we can see the absence of nontrivial null states in $\tilde{\mathcal{H}}_\theta \left(e^{i\frac{\varphi}{2\pi}}, e^{i\frac{\psi}{2\pi}} \right)$ as before so that the inner product becomes a true scalar product for $\tilde{\mathcal{H}}_\theta \left(e^{i\frac{\varphi}{2\pi}}, e^{i\frac{\psi}{2\pi}} \right)$. Also, the Hilbert space $\mathcal{H}_\theta \left(e^{i\frac{\varphi}{2\pi}}, e^{i\frac{\psi}{2\pi}} \right)$ obtained by completion of $\tilde{\mathcal{H}}_\theta \left(e^{i\frac{\varphi}{2\pi}}, e^{i\frac{\psi}{2\pi}} \right)$ is independent of m in (105) while $\hat{x}_0^{L,R}$ do not act on $\mathcal{H}_\theta \left(e^{i\frac{\varphi}{2\pi}}, e^{i\frac{\psi}{2\pi}} \right)$.

Note that while $e^{-i\frac{2\pi}{\theta}\hat{x}_0^R}$ acts on $\mathcal{H}_\theta \left(e^{i\frac{\varphi}{2\pi}}, e^{i\frac{\psi}{2\pi}} \right)$, it is $e^{-i\varphi}\mathbb{I}$. So it cannot be the starting point to define a time operator.

These remarks generalise to the other examples of discrete evolution considered below.

9. Noncommutative $\mathbb{R}^3$

Here we show that the algebra $\hat{e}_2$ admits a positive map. With that, one can proceed to develop quantum physics.

If $\hat{x}_0, \hat{x}_a$ ($a = 1, 2$) are time and space coordinate functions in commutative spacetime, we call their noncommutative analogues also by $\hat{x}_0, \hat{x}_a$. They fulfill the relations

$$[\hat{x}_a, \hat{x}_b] = 0, \quad a, b = 1, 2, \quad [\hat{x}_0, \hat{x}_a] = i\theta\varepsilon_{ab}\hat{x}_b, \quad \varepsilon_{12} = -\varepsilon_{21} = 1, \quad \theta > 0. \quad (110)$$

(110) defines the Lie algebra of the two-dimensional Euclidean group, and admits a $*$ -operation: $\hat{x}_\mu^* = \hat{x}_\mu$. Equally important, it admits the time-translation automorphism $U(\tau) : U(\tau)\hat{x}_0 = \hat{x}_0 + \tau$. But it is not an inner automorphism, $\hat{x}_0$ having no conjugate operator.

Spatial translations are not automorphisms of (110). That means that momenta, free Hamiltonian or plane waves do not exist for (110).

The algebra $\hat{e}_2$ with relations (110) has been treated in detail by Chaichian et al. [11]. As they observe, the operator

$$\rho^2 = \sum \hat{x}_a \hat{x}_a \quad (111)$$

is in the center of $\hat{e}_2$. We can fix its value to be r^2 in an IRR just as we fixed the value of $e^{-i\frac{2\pi}{\theta}\hat{x}_0} \in \mathcal{A}_\theta(\mathbb{R} \times S^1)$. For $r^2 > 0$, we have the polar decomposition

$$\hat{x}_1 \pm i\hat{x}_2 = re^{\mp i\hat{x}}. \quad (112)$$

Now

$$e^{i\hat{x}}\hat{x}_0 = \hat{x}_0 e^{i\hat{x}} + \theta e^{i\hat{x}}, \quad (113)$$

and $\hat{x}_0, e^{i\hat{x}}$ generate $\mathcal{A}_\theta(\mathbb{R} \times S^1)$, the algebra treated before. Hence we can borrow ideas from the treatment of $\mathcal{A}_\theta(\mathbb{R} \times S^1)$.

We briefly treat (110) regarding $\hat{x}_a$ as generators of $C^\infty(\mathbb{R}^2)$ and $\hat{x}_0/\theta$ as the generator of rotations in the 1 – 2 plane. The algebra will be realised by operators on the auxiliary Hilbert space $L^2(\mathbb{R}^2, d^2x)$ with its standard scalar product $(\cdot, \cdot)$ where

$$(\alpha, \beta) = \int d^2x \alpha^*(x) \beta(x). \quad (114)$$

On this space, $\hat{x}_a$ acts by evaluation map,

$$\hat{x}_a \alpha(x) = x_a \alpha(x), \quad (115)$$

while $\hat{x}_0/\theta$ acts like angular momentum with

$$e^{i2\pi\hat{x}_0/\theta} = \mathbb{I}. \quad (116)$$

Then for the spectrum of $\hat{x}_0$,

$$\text{spec } \hat{x}_0 = \theta\mathbb{Z}. \quad (117)$$

Time is quantised in units of θ as for $\mathcal{A}_\theta(\mathbb{R} \times S^1)$, but there is no shift from $\theta\mathbb{Z}$ by a flux term $\theta\varphi/2\pi$.

There are also ray representations of the Euclidean group which are representations of (110), where the spectrum $\theta\mathbb{Z}$ is shifted by a flux term $\frac{\theta\varphi}{2\pi}$. Our discussion can be adapted to this case as well.

We now give the positive map and inner product for $\hat{e}_2$. The algebra $\hat{e}_2$ is generated by

$$e^{i\omega\hat{x}_0}, \quad e^{i\vec{p}\cdot\hat{x}}, \quad \vec{p}\cdot\hat{x} = p_1\hat{x}_1 + p_2\hat{x}_2, \quad \omega, p_a \in \mathbb{R}, \quad (118)$$

where because of the spectral condition (117),

$$e^{i(\omega + \frac{2\pi}{\theta})\hat{x}_0} = e^{i\omega\hat{x}_0}. \quad (119)$$

Thus we restrict ω according to $|\omega| \leq \pi/\theta$.

The general element of the algebra is

$$\hat{\alpha} = \int d^2p \int_{-\frac{\pi}{\theta}}^{+\frac{\pi}{\theta}} d\omega \tilde{\alpha}(\omega, \vec{p}) e^{i\vec{p}\cdot\hat{x}} e^{i\omega\hat{x}_0}. \quad (120)$$

The symbol we associate to $\hat{\alpha}$ is the function

$$\begin{aligned} \alpha : \theta\mathbb{Z} \times \mathbb{R}^2 &\rightarrow \mathbb{C}, \\ \alpha(\theta n, x) &= \int d^2p \int_{-\frac{\pi}{\theta}}^{+\frac{\pi}{\theta}} d\omega \tilde{\alpha}(\omega, \vec{p}) e^{i\vec{p}\cdot\vec{x}} e^{i\omega\theta n}, \quad n \in \mathbb{Z}. \end{aligned} \quad (121)$$

This gives the map

$$S_{\theta n}(\hat{\alpha}) = \int d^2x \alpha(\theta n, x). \quad (122)$$

We can show that (122) is a positive map. We have the identity

$$e^{-i\omega\hat{x}_0}\hat{x}_a e^{i\omega\hat{x}_0} = R_{ab}(\theta\omega)\hat{x}_b, \quad R(\theta\omega) = \begin{pmatrix} \cos(\theta\omega) & \sin(\theta\omega) \\ -\sin(\theta\omega) & \cos(\theta\omega) \end{pmatrix}. \quad (123)$$

A short calculation which uses this identity shows, in an obvious manner, that

$$S_{\theta n}(\hat{\alpha}^* \hat{\alpha}) = (2\pi)^2 \int d^2p \left| \int d\omega \tilde{\alpha}(\omega, \vec{p}) e^{i\omega\theta n} \right|^2 \geq 0. \quad (124)$$

Thus an inner product for $\hat{e}_2$ is

$$(\hat{\beta}, \hat{\alpha}) = S_{\theta n}(\hat{\beta}^* \hat{\alpha}). \quad (125)$$

10. The Noncommutative $\mathbb{R} \times S^3$

The noncommutative $\mathbb{R} \times S^3 \simeq \mathbb{R} \times SU(2)$ is denoted by $\mathcal{A}_\theta(\mathbb{R} \times S^3)$. Section I indicates its construction: we set the time operator $\hat{x}_0$ equal to $2\theta J_3^R$ where θ is the noncommutativity parameter. With $\mathcal{C}^\infty(SU(2))$ denoting the commutative algebra of functions on $SU(2)$, $\mathcal{A}_\theta(\mathbb{R} \times S^3)$ is generated by $\mathcal{C}^\infty(SU(2))$ and $\hat{x}_0$ with relation (73).

Let $L^2(SU(2), d\mu)$ denote the Hilbert space of functions on $SU(2)$ with scalar product $(\cdot, \cdot)$ given by the Haar measure $d\mu$:

$$(\hat{a}, \hat{b}) = \int d\mu(s) \hat{a}^*(s) \hat{b}(s). \quad (126)$$

Then $\mathcal{A}_\theta(\mathbb{R} \times S^3)$ acts naturally on this Hilbert space, $\mathcal{C}^\infty(SU(2))$ acting by point-wise multiplication and $\hat{x}_0$ as the differential operator $2\theta J_3^R$.

The spectrum $\text{spec } J_3^R$ of J_3^R is $\mathbb{Z}/2$. Hence $\text{spec } \hat{x}_0 = \theta\mathbb{Z}$. Therefore

$$e^{i2\pi\hat{x}_0/\theta} = \mathbb{I}. \quad (127)$$

It follows that time evolution is quantised in units of θ . Furthermore

$$e^{i(\omega + \frac{2\pi}{\theta})\hat{x}_0} = e^{i\omega\hat{x}_0}. \quad (128)$$

Hence we can restrict ω to $[-\frac{\pi}{\theta}, \frac{\pi}{\theta}]$ and represent an element $\hat{\psi}$ of $\mathcal{A}_\theta(\mathbb{R} \times S^3)$ as

$$\hat{\psi} = \int_{-\pi/\theta}^{\pi/\theta} d\omega \hat{\psi}_\omega e^{i\omega\hat{x}_0}, \quad \hat{\psi}_\omega \in \mathcal{C}^\infty(SU(2)). \quad (129)$$

The symbol of $\hat{\psi}$ is the function $\psi : (\text{spec } \hat{x}_0 = \theta\mathbb{Z}) \times SU(2) \longrightarrow \mathbb{C}$ defined by

$$\psi(\theta n, s) = \int_{-\pi/\theta}^{\pi/\theta} d\omega \hat{\psi}_\omega(s) e^{i\omega\theta n}, \quad n \in \mathbb{Z}. \quad (130)$$

The inner product can be obtained from an associated map $S_{\theta n}$:

$$S_{\theta n}(\hat{\psi}) = \int d\mu(s) \psi(\theta n, s). \quad (131)$$

We can check using the right-invariance of the Haar measure that

$$S_{\theta n}(\hat{\psi}^* \hat{\varphi}) = \int d\mu(s) \psi^*(\theta n, s) \varphi(\theta n, s), \quad (132)$$

where φ is the symbol of $\hat{\varphi}$. Hence $S_{\theta n}$ is a positive map. The rest of the treatment involving the Schrödinger constraint follows previous sections.

11. On Energy Conservation

We focus on time-independent Hamiltonians $\hat{H}$. In that case, the Schrödinger constraint such as (108) is preserved by $\hat{H}$,

$$\hat{\psi} \in \tilde{\mathcal{H}}_\theta \left(e^{i\frac{\varphi}{2\pi}}, e^{i\frac{\psi}{2\pi}} \right) \implies \hat{H}\hat{\psi} \in \tilde{\mathcal{H}}_\theta \left(e^{i\frac{\varphi}{2\pi}}, e^{i\frac{\psi}{2\pi}} \right), \quad (133)$$

and consequently $\hat{H}$ is an observable for $\mathcal{A}_\theta(\mathbb{R} \times S^1)$. The same is true for $\hat{e}_2$ and $\mathcal{A}_\theta(\mathbb{R} \times S^3)$.

However time evolution involves

$$U(\theta) = e^{-i\theta\hat{H}}, \quad (134)$$

its inverse and powers. It is the same for $\hat{H}$ and $\hat{H} + \frac{2\pi}{\theta}$. Hence time evolution need conserve energy only mod $\frac{2\pi}{\theta}$.

This energy nonconservation should show up in scattering and decay processes. In either case, if E_i and E_f are initial and final energies, then for $\theta = 0$, energy conservation is enforced by the factor

$$\int_{-\infty}^{\infty} d\tau e^{-i\tau(E_f - E_i)} = 2\pi\delta(E_f - E_i) \quad (135)$$

in the scattering matrix element. For quantised evolutions such as ours, the factor becomes

$$\sum_{n \in \mathbb{Z}} e^{-in\theta(E_f - E_i)} = 2\pi\delta_{S^1}[\theta(E_f - E_i)] \quad (136)$$

where δ_{S^1} is the δ -function on S^1 : $\delta_{S^1}(\theta + 2\pi) = \delta_{S^1}(\theta)$. Thus from an initial state of energy E_i , there can be transitions to energies $E_f = E_i + \frac{2\pi}{\theta}n$, $n \in \mathbb{Z}$.

In specific models, the probability $P_n(E)$ for transitions from $E_i = E$ to $E_f = E + \frac{2\pi}{\theta}n$ can be calculated. We initiate the theory for this purpose in the next section. We are looking for a manageable model for a specific calculation.

Suppose that we start with a state of sharp energy E and let it undergo multiple scattering. Let the probability for finding energy $E + \frac{2\pi}{\theta}n$ after k scatterings be $P_n(E, k)$. Then

$$P_n(E, k+1) = \sum_m P_{n-m} \left(E + \frac{2\pi}{\theta}m, 1 \right) P_m(E, k) \quad (137)$$

where $P_n(E, 1) = P_n(E)$. Equation (137) defines a Markov process with $P_n(E, 1)$ giving the rule for updating at each step. It is of considerable interest to study $P_n(E, k)$ and its limit $k \rightarrow \infty$.

We remark that the limiting distribution $P_n(E, \infty)$ may be of use to provide bounds on θ in conjunction with cosmological data. Presumably distant star or quasar signals arrive at us after a large number of scattering processes. We can imagine estimating their frequency dispersion after accounting for energy loss by standard $\theta = 0$ effects, and getting information on θ therefrom.

12. Scattering Theory

We consider only a situation where the Hamiltonian $\hat{H}$ is time-independent. The transition amplitude from the in state vector $|+, \alpha\rangle$ with label α to an out state vector $|-, \beta\rangle$ with label β defines the matrix element $\mathcal{S}_{\beta\alpha}$ of the S -matrix $\mathcal{S}$:

$$\mathcal{S}_{\beta\alpha} = \langle -, \beta | +, \alpha \rangle. \quad (138)$$

Let $\hat{H}_0$ be the “free” or “comparison” Hamiltonian. Then $|+, \alpha\rangle$ has the property

$$U(\theta)^N |+, \alpha\rangle = U_0(\theta)^N |\alpha\rangle \quad \text{as } N \rightarrow -\infty, \quad \text{with } N \in \mathbb{Z} \quad (139)$$

where

$$U_0(\theta) = e^{-i\theta\hat{H}_0}. \quad (140)$$

The meaning of (139) is that in the distant past, $|+, \alpha\rangle$ evolves like the free evolution of the vector $|\alpha\rangle$.

The label α can be given a meaning in terms of observables of the free system such as energy.

The limit involved requires care. It is to be understood in the strong sense. It defines the Møller operator

$$\Omega^+ = \lim_{\substack{N \rightarrow -\infty, \\ N \in \mathbb{Z}}} U(\theta)^{-N} U_0(\theta)^N \quad (141)$$

with the properties

$$\Omega^+ |\alpha\rangle = |+, \alpha\rangle, \quad (142)$$

$$\Omega^+ e^{-i\theta \hat{H}_0} = e^{-i\theta \hat{H}} \Omega^+. \quad (143)$$

Equation (142) follows from (139) while the proof of (143) is as follows:

$$\Omega^+ e^{-i\theta \hat{H}_0} = \lim_{\substack{N \rightarrow -\infty, \\ N \in \mathbb{Z}}} U(\theta)^{-N} U_0(\theta)^{N+1} = \lim_{\substack{N' \rightarrow -\infty, \\ N' \in \mathbb{Z}}} U(\theta)^{-(N'-1)} U_0(\theta)^{N'} = e^{-i\theta \hat{H}} \Omega^+. \quad (144)$$

Thus Ω^+ intertwines the quantised evolutions due to $\hat{H}_0$ and $\hat{H}$.

For $\theta = 0$, time t is continuous. In that case, (143) is replaced by

$$\Omega^+ e^{-it \hat{H}_0} = e^{-it \hat{H}} \Omega^+. \quad (145)$$

So for $\theta = 0$, by differentiating in t , we get the stronger result

$$\Omega^+ \hat{H}_0 = \hat{H} \Omega^+. \quad (146)$$

But we cannot get such a stronger equation from (143) for $\theta \neq 0$. This is yet another indication that for $\theta \neq 0$, energy is conserved only mod $\frac{2\pi}{\theta}$.

Just as $|+, \alpha\rangle$ fulfills the Schrödinger constraint involving $\hat{H}$, $|\alpha\rangle$ fulfills the Schrödinger constraint involving $\hat{H}_0$ as follows from (143):

$$e^{-i\theta \hat{P}_0} |\alpha\rangle = e^{-i\theta \hat{H}_0} |\alpha\rangle. \quad (147)$$

So scalar products involving $|\alpha\rangle$'s are also time-independent and admit a general solution of a form such as (109).

In a similar way, if

$$\Omega^- = \lim_{\substack{M \rightarrow \infty, \\ M \in \mathbb{Z}}} U(\theta)^{-M} U_0(\theta)^M, \quad (148)$$

then

$$\Omega^- |\beta\rangle = |-, \beta\rangle, \quad \Omega^- e^{-i\theta \hat{H}_0} = e^{-i\theta \hat{H}} \Omega^-. \quad (149)$$

Hence

$$\mathcal{S}_{\beta\alpha} = \lim_{\substack{M \rightarrow \infty, \\ N \rightarrow -\infty, \\ M, N \in \mathbb{Z}}} \langle \beta | U_0(\theta)^{-M} U(\theta)^{M-N} U_0(\theta)^N | \alpha \rangle := \lim_{\substack{M \rightarrow \infty, \\ N \rightarrow -\infty, \\ M, N \in \mathbb{Z}}} \langle \beta | U_I(\theta, M, N) | \alpha \rangle, \quad (150)$$

$$U_I(\theta, M, N) = U_0(\theta)^{-M} U(\theta)^{M-N} U_0(\theta)^N = e^{iM\theta \hat{H}_0} e^{-i(M-N)\theta \hat{H}} e^{-iN\theta \hat{H}_0}. \quad (151)$$

In commutative physics, where $\theta = 0$, the corresponding expression $U_I(t, t')$ is

$$U_I(t, t') = e^{it\hat{H}_0} e^{-i(t-t')\hat{H}} e^{-it'\hat{H}_0} = T \exp \left\{ -i \int_{t'}^t d\tau \hat{H}_I(\tau) \right\}, \quad (152)$$

$$\hat{H}_I(\tau) = e^{i\hat{H}_0\tau} (\hat{H} - \hat{H}_0) e^{-i\hat{H}_0\tau}, \quad (153)$$

T denoting time-ordering, the interaction representation S -matrix being $U_I(\infty, -\infty)$.

Comparison of (151) and (152) shows that

$$U_I(\theta, M, N) = T \exp \left\{ -i \int_{N\theta}^{M\theta} d\tau \hat{H}_I(\tau) \right\}, \quad (154)$$

$$\hat{H}_I(\tau) = e^{i\hat{H}_0\tau} (\hat{H} - \hat{H}_0) e^{-i\hat{H}_0\tau}. \quad (155)$$

For $\theta = 0$, (152) has a power series expansion in $\hat{H}_I$. But there is a problem with such an expansion of (151): $U(\theta)$, $U_0(\theta)$ and $U_I(\theta, M, N)$ are invariant under *separate* shifts of $\hat{H}$ and $\hat{H}_0$ by $\pm \frac{2\pi}{\theta}$, however $\hat{H}_I(\tau)$ and hence the terms of the perturbation series are invariant only under the *joint* shift of both by the same amount, the joint shift leaving $\hat{H}_I(\tau)$ invariant. Thus perturbative approximation disturbs an essential feature of quantised evolution.

It remains to find a substitute for perturbation theory. Perhaps an approximation based on the K -matrix formalism and effective range expansion [30], [31] or separable potentials [32] may be acceptable.

13. On Quantum Fields

As the spacetime algebras of our interest admit only quantised time evolutions as automorphisms, a field cannot be the solution of a Klein-Gordon or Dirac equation. We need another approach to quantising spacetime fields for purposes of constructing quantum fields.

One way is to define the quantum field $\hat{\Phi}$ by expanding it in a basis of orthonormal solutions of the Schrödinger constraint. The coefficients of the expansion would be annihilation operators. This is a common approach in condensed matter theory.

For specificity consider $\mathcal{A}_\theta(\mathbb{R} \times S^1)$ and the “free” Hamiltonian

$$\hat{H}_0 = \frac{\hat{P}_1^2}{2M}. \quad (156)$$

In that case, $\tilde{\mathcal{H}}_\theta \left(e^{i\frac{\varphi}{2\pi}}, e^{i\frac{\psi}{2\pi}} \right)$ of (108) is spanned by

$$\hat{\psi}_n = \frac{1}{\sqrt{2\pi}} e^{i(n + \frac{\psi}{2\pi})\hat{x}_1} e^{-i\omega_n \hat{x}_0}, \quad (157)$$

where

$$\omega_n = \frac{1}{2M} \left(n + \frac{\psi}{2\pi} \right)^2, \quad (158)$$

$$\hat{H}_0 \hat{\psi}_n = \frac{1}{2M} \left(n + \frac{\psi}{2\pi} \right)^2 \hat{\psi}_n, \quad (159)$$

$$(\hat{\psi}_m, \hat{\psi}_n)_{\theta(m + \frac{\varphi}{2\pi})} = \delta_{mn}. \quad (160)$$

We can now write

$$\hat{\Phi} = \sum_n a_n \hat{\psi}_n, \quad [a_n, a_m^\dagger] = \delta_{nm} \quad (161)$$

where $\hat{\Phi}$ describes a free “nonrelativistic” spin-zero field.

The second-quantised free Hamiltonian associated with $\hat{\Phi}$ is

$$\hat{H}_0 = \sum_n \omega_n a_n^\dagger a_n. \quad (162)$$

$\hat{\Phi}$ fulfills the second-quantised Schrödinger constraint:

$$e^{-i\theta \hat{P}_0} \hat{\Phi} = U_0(\theta)^{-1} \hat{\Phi} U_0(\theta), \quad (163)$$

$$U_0(\theta) \equiv e^{-i\theta \hat{H}_0}. \quad (164)$$

The next step is to introduce an interaction Hamiltonian. We follow earlier works [1], [23] in this regard. An example of an interaction Hamiltonian in interaction representation is

$$\hat{H}_I(\tau) = : e^{i\tau \hat{H}_0} \lambda S_{\theta(m+\frac{\varphi}{2\pi})} \left(\hat{\Phi}^\dagger \hat{\Phi} \hat{\Phi}^\dagger \hat{\Phi} \right) e^{-i\tau \hat{H}_0} : \quad (165)$$

where $: \cdot :$ denotes normal ordering of $a_n, a_n^\dagger$.

The expression for $U_I(\theta, M, N)$ follows from (154):

$$U_I(\theta, M, N) = T \exp \left\{ -i \int_{N\theta}^{M\theta} d\tau \hat{H}_I(\tau) \right\}, \quad M, N \in \mathbb{Z}, \quad (166)$$

the S -matrix being

$$\mathcal{S} = \lim_{\substack{M \rightarrow \infty, \\ N \rightarrow -\infty, \\ M, N \in \mathbb{Z}}} U_I(\theta, M, N). \quad (167)$$

As before, perturbation series, term by term, is not invariant under the shifts of $\hat{H}_I(\tau)$ by $\pm \frac{2\pi}{\theta}$, whereas (166) is. That leaves us with a problem. It is also important to know if and how $\mathcal{S}$ depends on $(m + \frac{\varphi}{2\pi})$.

14. Final Remarks

There exist several models of noncommutative spacetimes with time-space noncommutativity which admit consistent formulations of quantum physics. We have discussed many such models in these lectures. Some are striking in their novel features, admitting only quantised evolutions and predicting energy nonconservation in a controlled manner. Phenomenological consequences of these models are yet to be explored.

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