Double soft graviton theorems and Bondi-Metzner-Sachs symmetries

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It is now well understood that Ward identities associated with the (extended) BMS algebra are equivalent to single soft graviton theorems. In this work, we show that if we consider nested Ward identities constructed out of two BMS charges, a class of double soft factorization theorems can be recovered. By making connections with earlier works in the literature, we argue that at the subleading order, these double soft graviton theorems are the so-called consecutive double soft graviton theorems. We also show how these nested Ward identities can be understood as Ward identities associated with BMS symmetries in scattering states defined around (non-Fock) vacua parametrized by supertranslations or superrotations.

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I. INTRODUCTION

Since the seminal work by Strominger [1], there has been a flurry of activity towards understanding the role of a class of symmetries known as "asymptotic symmetries" in gauge theories and gravity [2–19]. For theories containing massless particles of spin $1 \leq s \leq 2$, asymptotic symmetries are obtained by considering gauge transformations which do not fall off at infinity. Such large gauge transformations have nontrivial asymptotic charges and their conservation laws have nontrivial implications for the $\mathcal{S}$–matrix.

For example, it has now become clear that the "universal" soft theorems (i.e., those soft theorems whose structure is completely determined by gauge invariance [20,21]), such as the leading soft theorems in gauge theories and gravity, as well as the subleading soft theorem in gravity, are manifestations of Ward identities associated with a class of asymptotic symmetries (in four dimensions due to the infrared divergences in these theories, the cleanest statement can be made at the tree-level $\mathcal{S}$–matrix.). In the case of gravity, these symmetries are nothing but an infinite-dimensional extension of the famous Bondi, Metzner, Sachs (BMS) group. Next-to-soft gravity radiation was first considered by Gross and Jackiw [22]. For an wonderful insight on this from Wilson line operator perspective, we refer the reader to the recent work by White [23].

However, factorization theorems in gauge theories and in quantum gravity have a richer structure. In the case of gravity, in a recent paper by Chakrabarti *et al.* [24], it was shown that there exists a hierarchy of factorization theorems when arbitrary but finite number of gravitons are taken to be soft in a scattering process. Of particular interest is the so-called "double soft graviton theorem," which is a constraint on the scattering amplitude when two of the gravitons become soft. Such double soft theorems have a history in pion physics [25]. In the case of pions which are Goldstone modes of a spontaneously broken global non-Abelian symmetry, double soft pion limits have an interesting structure. As was shown in [25], if we consider a scattering amplitude in which two of the pions are taken to soft limit simultaneously, the scattering amplitude factorizes and the double soft theorem contains information about the structure of the (unbroken) symmetry generators. Due to the presence of an Adler zero, which ensures that single soft pion limit vanishes, it is easy to see that there is no nontrivial factorization theorem if two pions are taken soft consecutively as opposed to when they are done so at the same rate.

Double soft graviton theorems are distinct in this regard. Not only is the simultaneous soft limit nontrivial and highly intricate, unlike the case of soft pions even the consecutive

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soft limit does not vanish and gives rise to factorization constraints on the scattering amplitude which are called the "consecutive double soft theorems."

In this paper we try to find an interpretation of such consecutive double soft theorems as a consequence of Ward identities associated with the generalized BMS algebra.[1]

The outline of this paper is as follows. In Sec. II, we recall the equivalence between leading and subleading soft graviton theorems and Ward identities associated with asymptotic symmetries [1,12,26,27]. In Sec. III, we explain the consecutive double soft limit and how it gives rise to a leading and two subleading consecutive double soft theorems. In Sec. IV A, we propose asymptotic Ward identities, which, as we show in Appendix A, can be heuristically derived from Ward identities associated with Noether's charges [28]. In Appendix B, we discuss the conceptual subtleties associated with the domain of soft operators, which is an obstacle to the full rigorous derivation of one of the subleading consecutive double soft theorems from asymptotic symmetries. In Sec. V, we present a formal derivation of this subleading consecutive double soft theorem from asymptotic symmetries. We conclude with some remarks, which primarily focus on the key open question that pertaining to the study of the simultaneous double soft graviton theorem from the perspective of asymptotic symmetries.

II. SINGLE SOFT GRAVITON THEOREMS AND ASYMPTOTIC SYMMETRIES

We begin by reviewing the derivations of the single soft graviton theorems (both leading and subleading) from asymptotic symmetries [1,12]. In the process, we also define the notations that we use later.

According to present understanding, the asymptotic symmetry group of gravity, acting on the asymptotic phase space of gravity is the "generalized BMS" group—it is a semidirect product of supertranslations and $\mathrm{Diff}(S^2)$. They can be thought of as a local generalization of translations and the Lorentz group, respectively. While the original BMS group [29,30] is a semidirect product of supertranslations and $SL(2,\mathbb{C})$, in the generalized BMS group the $SL(2,\mathbb{C})$ symmetry is further extended to $\mathrm{Diff}(S^2)$. Each of the supertranslations and $\mathrm{Diff}(S^2)$ symmetry gives rise to conserved asymptotic charges, namely, the supertranslation charge (Q_f) and superrotation charge (Q_V), respectively. These charges are determined completely by the asymptotic "free data" and are parametrized by an arbitrary function $f(z,\bar{z})$ and an arbitrary vector field $V^A(z,\bar{z})$, respectively, both of which are defined on the conformal sphere at null infinity. By studying the algebra, one finds that supertranslations and superrotations form a closed algebra [16].

To define a symmetry of a gravitational scattering problem at the quantum level, these charges are elevated to a symmetry of the quantum gravity $\mathcal{S}$–matrix. Corresponding to each such symmetry one gets a Ward identity. In next two sections, we discuss that how the single soft graviton theorems are equivalent to Ward identities of generalized BMS charges.

A. Leading single soft graviton theorem and supertranslation symmetry

The leading supertranslation charge Q_f [1], which physically corresponds to the conservation of energy at each direction on the conformal sphere at null infinity.[2]

The supertranslation charge Q_f is given by [1]

$$Q_f = \int dud^2z f\gamma_{z\bar{z}}N_{zz}N^{zz} + 2\int dud^2z f\partial_u(\partial_z U_{\bar{z}} + \partial_{\bar{z}}U_z). \tag{2.1}$$

Here, $U_z = -\frac{1}{2}D^z C_{zz}$, and $N_{zz} = \partial_u C_{zz}$ is the Bondi news tensor, where C_{zz} is the "free data". The derivative D^z is the covariant derivative with respect to the 2-sphere metric.

It is important to note that, the supertranslation charge Q_f is characterized by the arbitrary function $f(z,\bar{z})$, where $(z,\ \bar{z})$ are coordinates on the conformal sphere at null infinity. Notice that, the first term in (2.1) is quadratic in C_{zz} while the second is linear in C_{zz}—these are conventionally referred as the "hard part" (Q_f^{hard}) and the "soft part" (Q_f^{soft}) of the supertranslation charge, respectively.

In order to establish the equivalence between the supertranslation Ward identity and the leading single soft graviton theorem, the asymptotic charge (2.1) is conjectured to be a symmetry of the quantum gravity $\mathcal{S}$–matrix [1]. As a result, one gets the Ward identity for supertranslation as

$$\begin{aligned}\langle \mathrm{out}|[Q_f,\mathcal{S}]|\mathrm{in}\rangle = 0 \Leftrightarrow \langle \mathrm{out}|[Q_f^{\mathrm{soft}},\mathcal{S}]|\mathrm{in}\rangle \\ = -\langle \mathrm{out}|[Q_f^{\mathrm{hard}},\mathcal{S}]|\mathrm{in}\rangle,\end{aligned} \tag{2.2}$$

where in writing the above, the classical charges have been promoted to quantum operators. This quantization is carried out using the asymptotic quantization of C_{zz} [1], which expresses them in terms of graviton creation and annihilation operators.

[1] There are two known extensions of the BMS group in the literature. One is the "extended BMS" [2], which is the semidirect product of supertranslations and the Virasoro group and the other is the "generalized BMS" [8], which is the semidirect product of supertranslations and $\mathrm{Diff}(S^2)$. Each of them give rise to the same asymptotic charges and hence the same Ward identities for the quantum gravity $\mathcal{S}$–matrix. Since, these Ward identities are the starting point of our analysis, this difference is irrelevant.

[2] The seminal work [1] was based on the case when external states contain only massless particles. Generalization to the case where external states can have massive particles was done in [14].

To evaluate (2.2), one needs to know the action of the hard and soft charges on the "in" and "out" states. Let's begin by discussing the soft charge. Note that, we are working with Christodoulou-Klainerman (CK) spaces (which satisfy $D_z^2 C_{\bar{z}\bar{z}}|_{\mathcal{I}^+_\pm} = D_{\bar{z}}^2 C_{zz}|_{\mathcal{I}^+_\pm}$) [1]. This, together with the crossing symmetry of the scattering amplitude, allows one to write the soft charge as

$$\begin{aligned} Q_f^{\text{soft}} &= \lim_{E_p\to 0} \frac{E_p}{2\pi}\int d^2w D_w^2 f(w,\bar{w}) a_-(E_p,w,\bar{w}) \\ &= \lim_{E_p\to 0} \frac{E_p}{2\pi}\int d^2w D_{\bar{w}}^2 f(w,\bar{w}) a_+(E_p,w,\bar{w}). \end{aligned} \tag{2.3}$$

Hence, $Q_f^{\text{soft}}|\text{in}\rangle = 0$. Here, E_p is the energy of the soft graviton and $(w,\bar{w})$ characterizes its direction on the conformal sphere.

The hard charge can also be evaluated in a similar procedure, finally giving the action on "in" and "out" states as

$$\begin{aligned} Q_f^{\text{hard}}|\text{in}\rangle &= \sum_{\text{in}} E_i f(\hat{k}_i)|\text{in}\rangle \\ \langle\text{out}|Q_f^{\text{hard}} &= \sum_{\text{out}} E_i f(\hat{k}_i)\langle\text{out}|. \end{aligned} \tag{2.4}$$

Here, the sum $\sum_{\text{in}}$ and $\sum_{\text{out}}$ is over all the hard particles in the "in" and "out" states, respectively, with energy $E_i = |\vec{k}_i|$ and the unit spatial vector $\hat{k}_i = \vec{k}_i/E_i$ characterizing the direction of ith particle.

Using (2.4), (2.3) and (2.2) then, one obtains a factorization of the form

$$\begin{aligned} &\lim_{E_p\to 0}\frac{E_p}{2\pi}\int d^2w D_{\bar{w}}^2 f(w,\bar{w})\langle\text{out}|a_+(E_p,w,\bar{w})\mathcal{S}|\text{in}\rangle \\ &\quad = -\left[\sum_{\text{out}} E_i f(\hat{k}_i) - \sum_{\text{in}} E_i f(\hat{k}_i)\right]\langle\text{out}|\mathcal{S}|\text{in}\rangle. \end{aligned} \tag{2.5}$$

Structure of the terms in (2.5) encourages one to ask whether this can be related to Weinberg's soft graviton theorem [26]. This reads

$$\begin{aligned} &\lim_{E_p\to 0} E_p\langle\text{out}|a_+(E_p,w,\bar{w})\mathcal{S}|\text{in}\rangle \\ &\quad = \sum_i \frac{(\epsilon^+(w,\bar{w})\cdot k_i)^2}{(p/E_p)\cdot k_i}\langle\text{out}|\mathcal{S}|\text{in}\rangle, \end{aligned} \tag{2.6}$$

where the soft graviton has energy E_p and momentum p. Its direction is parametrized by $(w,\bar{w})$ and its polarization is given by $\epsilon^+(w,\bar{w}) = 1/\sqrt{2}(\bar{w},1,-i,-\bar{w})$. We adopt the notation

$$\hat{S}^{(0)}(p;k_i) \equiv \frac{1}{E_{k_i}}\frac{(\epsilon^+(w,\bar{w})\cdot k_i)^2}{(p/E_p)\cdot k_i}, \tag{2.7}$$

with which, the leading soft factor in the rhs of [26] can be written as

$$\begin{aligned} \sum_i \frac{(\epsilon^+(w,\bar{w})\cdot k_i)^2}{(p/E_p)\cdot k_i} &\equiv S^{(0)}(p;\{k_i\}) \equiv \sum_i S^{(0)}(p;k_i) \\ &\equiv \sum_i E_{k_i}\hat{S}^{(0)}(p;k_i). \end{aligned} \tag{2.8}$$

It is important to notice that the contribution to the soft factor $S^{(0)}(p;\{k_i\})$ from the i^{th} hard particle with momentum k_i and energy E_{k_i}, namely $S^{(0)}(p;k_i)$, depends on the energy of the hard particle. But, $\hat{S}^{(0)}(p;k_i)$ does not depend on E_{k_i}; as written in (2.8), the energy dependence has been separated out.

Now, consider a hard particle of momentum k parametrized by $(E,z,\bar{z})$. If one chooses

$$f(z,\bar{z}) = s(z,\bar{z};w,\bar{w}) \equiv \frac{1+w\bar{w}}{1+z\bar{z}}\cdot\frac{\bar{w}-\bar{z}}{w-z} \tag{2.9}$$

in (2.5), then the rhs of the soft theorem (2.6) and the Ward identity (2.5) match, since

$$\frac{(\epsilon^+(w,\bar{w})\cdot k)^2}{(p/E_p)\cdot k} = -E_k s(z,\bar{z};w,\bar{w}). \tag{2.10}$$

Further, the lhs of the soft theorem (2.6) and the Ward identity (2.5) match because of the identity,

$$D_{\bar{z}}^2 s(z,\bar{z};w,\bar{w}) = 2\pi\delta^2(w-z). \tag{2.11}$$

It is also possible to go from the soft theorem (2.6) to the Ward identity (2.5) by acting $(2\pi)^{-1}\int d^2w f(w,\bar{w})D_{\bar{w}}^2$ on both sides of (2.6). In this case, the rhs matches because of the identity

$$D_{\bar{w}}^2 s(z,\bar{z};w,\bar{w}) = 2\pi\delta^2(w-z). \tag{2.12}$$

Hence, the equivalence of the soft theorem and Ward identity is established. It should also be noted that Weinberg's soft theorem for the negative helicity graviton is not an independent soft theorem and can be obtained through a similar derivation.

B. Subleading single soft graviton theorem and superrotation symmetry

The subleading single soft graviton theorem follows from the Ward identity of the superrotation charge Q_V [12], which physically corresponds to the conservation of angular momentum at each angle in a gravitational scattering process. This charge is given by:

$$Q_V = \frac{1}{4}\int dud^2z\sqrt{\gamma}\partial_u C^{AB}(\mathcal{L}_V C_{AB} - \alpha C_{AB} + \alpha u \partial_u C_{AB}) + \frac{1}{2}\int dud^2z\sqrt{\gamma}(C^{zz}D_z^3V^z + C^{\bar{z}\bar{z}}D_{\bar{z}}^3V^{\bar{z}}), \quad (2.13)$$

where $\alpha = \frac{1}{2}(D_zV^z + D_{\bar{z}}V^{\bar{z}})$ and $V^A(z,\bar{z})$ is an arbitrary vector field on the conformal sphere at null infinity. As usual, the covariant derivatives are with respect to the 2–sphere metric. As before, the first term is the "hard part" Q_V^{hard} and the second is the "soft part" Q_V^{soft} of the superrotation charge.

Proceeding in a manner similar to the case of supertranslation, the Ward identity for superrotations can be written as

$$\langle\mathrm{out}|[Q_V,\mathcal{S}]|\mathrm{in}\rangle = 0 \Leftrightarrow \langle\mathrm{out}|[Q_V^{\mathrm{soft}},\mathcal{S}]|\mathrm{in}\rangle = -\langle\mathrm{out}|[Q_V^{\mathrm{hard}},\mathcal{S}]|\mathrm{in}\rangle. \quad (2.14)$$

Now, using the asymptotic quantization of the "free data" and crossing symmetry one can write the soft superrotation charge as

$$Q_V^{\mathrm{soft}} = \frac{1}{4\pi i}\lim_{E_p\to 0}(1+E_p\partial_{E_p}) \times \int d^2w[V^{\bar{w}}\partial_{\bar{w}}^3 a_+(E_p,w,\bar{w}) + V^w\partial_w^3 a_-(E_p,w,\bar{w})]. \quad (2.15)$$

Hence, $Q_V^{\mathrm{soft}}|\mathrm{in}\rangle = 0$. Note that, unlike the previous case, due to the absence of a CK–like condition, the action of Q_V^{soft} on the "out" state gives gravitons of both helicities. Also, the action of the hard superrotation charge gives

$$\langle\mathrm{out}|Q_V^{\mathrm{hard}} = i\sum_{\mathrm{out}} J_{V_i}^{h_i}\langle\mathrm{out}|$$
$$Q_V^{\mathrm{hard}}|\mathrm{in}\rangle = i\sum_{\mathrm{in}} J_{V_i}^{-h_i}|\mathrm{in}\rangle. \quad (2.16)$$

Again, the sum $\sum_{\mathrm{in}}$ and $\sum_{\mathrm{out}}$ is over all the hard particles in the "in" and "out" states, respectively, with the i^{th} particle having energy $E_i = |\vec{k}_i|$ and direction characterized by the vector $\hat{k}_i = \vec{k}_i/E_i$. A detailed expression of $J_{V_i}^{h_i}$ can be found in [12].

As a result, one can write the Ward identity for superrotations (2.14) as

$$-\frac{1}{4\pi}\lim_{E_p\to 0}(1+E_p\partial_{E_p})\int d^2w[V^{\bar{w}}\partial_{\bar{w}}^3\langle\mathrm{out}|a_+(E_p,w,\bar{w})\mathcal{S}|\mathrm{in}\rangle + V^w\partial_w^3\langle\mathrm{out}|a_-(E_p,w,\bar{w})\mathcal{S}|\mathrm{in}\rangle] = \left[\sum_{\mathrm{out}} J_{V_i}^{h_i} - \sum_{\mathrm{in}} J_{V_i}^{-h_i}\right]\langle\mathrm{out}|\mathcal{S}|\mathrm{in}\rangle. \quad (2.17)$$

Now, the Cachazo-Strominger (CS) subleading soft theorem reads [27]

$$\lim_{E_p\to 0}(1+E_p\partial_{E_p})\langle\mathrm{out}|a_+(E_p,w,\bar{w})\mathcal{S}|\mathrm{in}\rangle = \sum_i \frac{\epsilon^+(w,\bar{w})\cdot k_i}{p\cdot k_i}\epsilon_\mu^+(w,\bar{w})p_\nu J_i^{\mu\nu}\langle\mathrm{out}|\mathcal{S}|\mathrm{in}\rangle, \quad (2.18)$$

where, $J_i^{\mu\nu}$ is the angular momentum operator acting on the ith hard particle. For further use, we adopt the notation

$$S^{(1)}(p;k_i) = \frac{\epsilon^+(w,\bar{w})\cdot k_i}{p\cdot k_i}\epsilon_\mu^+(w,\bar{w})p_\nu J_i^{\mu\nu}. \quad (2.19)$$

Using this, the subleading soft factor in the rhs of (2.18) can be written as

$$\sum_i \frac{\epsilon^+(w,\bar{w})\cdot k_i}{p\cdot k_i}\epsilon_\mu^+(w,\bar{w})p_\nu J_i^{\mu\nu} = \sum_i S^{(1)}(p;k_i) = S^{(1)}(p;\{k_i\}). \quad (2.20)$$

Now, in the Ward identity (2.17), if one chooses the vector field V^A as

$$V^A = K^+_{(w,\bar{w})} \equiv \frac{(\bar{z}-\bar{w})^2}{(z-w)}\partial_{\bar{z}}, \quad (2.21)$$

the rhs of the soft theorem (2.18) and the Ward Idenity (2.17) match since

$$\frac{\epsilon^+(w,\bar{w})\cdot k_i}{p\cdot k_i}\epsilon_\mu^+(w,\bar{w})p_\nu J_i^{\mu\nu} = J^i_{K^+_{(w,\bar{w})}}. \quad (2.22)$$

The lhs of the soft theorem (2.18) and the Ward identity (2.17) also match due to the identity:

$$\partial_{\bar{z}}^3\frac{(\bar{z}-\bar{w})^2}{(z-w)} = 4\pi\delta^2(w-z). \quad (2.23)$$

To go from the CS soft theorem(2.18) to the superrotation Ward identity (2.17) one acts the operator $-(4\pi)^{-1}\int d^2wV^{\bar{w}}\partial_{\bar{w}}^3$ on both sides of (2.18). Then, using the linearity of J_V in vector field V,

$$-(4\pi)^{-1}\int d^2wV^{\bar{w}}\partial_{\bar{w}}^3 J^i_{K^+_{(w,\bar{w})}} = -(4\pi)^{-1}J_W, \quad (2.24)$$

and the identity,

$$\partial_{\bar{w}}^3\frac{(\bar{z}-\bar{w})^2}{(z-w)} = -4\pi\delta^2(w-z), \quad (2.25)$$

one recovers Ward identity (2.17) with the vector field $V^{\bar{w}}\partial_{\bar{w}}$. The vector field W in above expression is given by

$$W = \int V^{\bar{w}}\partial_{\bar{w}}^3 K^+_{(w,\bar{w}).} \quad (2.26)$$

Here, unlike the Ward identity for the leading case (2.5), it is important to note that the Ward identity for the subleading case (2.17), contains both negative and positive helicity soft graviton amplitudes. To get a clear factorization, one of the components of vector field V^A is chosen to be zero, depending upon which soft graviton helicity we want in the soft theorem.

III. CONSECUTIVE DOUBLE SOFT GRAVITON THEOREMS (CDST)

Having reviewed the relationship between asymptotic symmetries and the single soft theorem, the next natural question is to ask if such a relationship holds between the generalized BMS algebra and double soft graviton theorems. These theorems (and its generalization to the multiple soft graviton case) have been studied previously using various methods including BCFW recursions [31], CHY amplitudes [32–35] and Feynman diagram techniques [24]. In a recent work [36], the authors have studied the symmetry foundations of the double soft theorems of certain classes of theories like the dilaton, DBI, and special Galileon.

As has been analyzed in the literature, there are two kinds of double soft graviton theorems depending upon the relative energy scale of the soft gravitons. The simultaneous soft limit is the one where soft limit is taken on both the gravitons at the same rate. It was shown in [24], that simultaneous soft limit yields a universal factorization theorem. However, as we argue in Appendix A, from the perspective of Ward identities, it is the consecutive soft limits which arise rather naturally. Consecutive double soft graviton theorems (CDST) elucidate the factorization property of scattering amplitudes when the soft limit is taken on one of the gravitons at a faster rate than the other [31]. We now review this factorization property when such soft limits are taken and show that they give rise to three CDSTs. The first one, we refer to as the leading CDST which is the case where the leading soft limit is taken on both the soft gravitons. The remaining two theorems refer to the case where the leading soft limit is taken with respect to one of the gravitons and the subleading soft limit is taken with respect to the other.

We begin with a $(n+2)$ particle scattering amplitude denoted by $\mathcal{A}_{n+2}(q,p,\{k_m\})$ where p,q are the momenta of the two gravitons which will be taken to be soft and $\{k_m\}$ is the set of momenta of the n hard particles. Consider the consecutive limit where the soft limit is first taken on graviton with momentum q, keeping all the other particles momenta unchanged and then a soft limit is taken on the graviton with momentum p.

Using the single soft factorization, the scattering amplitude $\mathcal{A}_{n+2}(q,p,\{k_m\})$ can be written as

$$\mathcal{A}_{n+2}(q,p,\{k_m\}) = \left[\sum_i \frac{E_{k_i}}{E_q}\hat{S}^{(0)}(q;k_i) + \frac{E_p}{E_q}\hat{S}^{(0)}(q;p) + \sum_i S^{(1)}(q;k_i) + S^{(1)}(q;p)\right]\mathcal{A}_{n+1}(p,\{k_m\}) + \mathcal{O}(E_q), \tag{3.1}$$

where $\mathcal{A}_{n+1}(p,\{k_m\})$ is the $n+1$ particle scattering amplitude. It is important to recall the notations used here, which we explained in Sec. II [(2.8) and (2.19)]. As mentioned, $S^{(1)}(q;k_i)$ is the contribution to the subleading soft factor with soft momentum q with k_i being the i^{th} hard particle. Similarly $\hat{S}^{(0)}(q;k_i)$ denotes the contribution to the subleading soft factor with soft momentum q with k_i being the i^{th} hard particle, with energy dependences with respect to both the soft and hard particles separated out. $\hat{S}^{(0)}(q;p)$ and $S^{(1)}(q;p)$ denote similar contributions to the soft factor where the graviton with momentum p is treated as hard with respect to the graviton with momentum q.

Now, the amplitude $\mathcal{A}_{n+1}(p,\{k_m\})$ further factorizes as

$$\mathcal{A}_{n+1}(p,\{k_m\}) = \left[\sum_i \frac{E_{k_i}}{E_p}\hat{S}^{(0)}(p;k_i) + \sum_i S^{(1)}(p;k_i)\right] \times \mathcal{A}_n(\{k_m\}) + \mathcal{O}(E_p) \tag{3.2}$$

Note that, according to our notation, $S^{(1)}(p;k_i)$ is the contribution to the subleading soft factor with soft momentum p and k_i is the i^{th} hard particle. Again, $\hat{S}^{(0)}(p;k_i)$ denotes the contribution to the subleading soft factor with soft momentum p and k_i the i^{th} hard particle, with energy dependences with respect to both the soft and the hard particles separated out.

Substituting (3.2) in (3.1), we get the factorization of the $(n+2)$ particle amplitude containing two soft gravitons in terms of the amplitude of the n hard particles (up to subleading order in energy of the individual soft particles),

$$\mathcal{A}_{n+2}(q,p,\{k_m\}) = \left[\frac{1}{E_pE_q}\sum_{i,j}E_{k_i}E_{k_j}\hat{S}^{(0)}(q;k_i)\hat{S}^{(0)}(p;k_j) + \sum_{i,j}\frac{E_{k_i}}{E_q}\hat{S}^{(0)}(q;k_i)S^{(1)}(p;k_j) + \sum_i \frac{E_{k_i}}{E_q}\hat{S}^{(0)}(q;p)\hat{S}^{(0)}(p;k_i) + \sum_{i,j}S^{(1)}(q;k_i)\frac{E_{k_j}}{E_p}\hat{S}^{(0)}(p;k_j) + S^{(1)}(q;p)\sum_i\frac{E_{k_i}}{E_p}\hat{S}^{(0)}(p;k_i)\right]\mathcal{A}_n(\{k_m\}) + \mathcal{O}(E_p) + \mathcal{O}(E_q). \tag{3.3}$$

This expansion contains three types of terms. The first type scales as $1/(E_pE_q)$ (and hence gives rise to a pole in both the soft graviton energies), giving the leading contribution to the factorization. The second and the third type of terms scale as E_q^0/E_p and E_p^0/E_q, respectively, both contributing to the subleading order of the factorization.

The leading order contribution, described above, is

$$\left[\frac{1}{E_pE_q}\sum_{i,j}E_{k_i}E_{k_j}\hat{S}^{(0)}(q;k_i)\hat{S}^{(0)}(p;k_j)\right]\mathcal{A}_n(\{k_m\}). \quad (3.4)$$

This gives the leading CDST as

$$\lim_{E_p\to 0}E_p\lim_{E_q\to 0}E_q\mathcal{A}_{n+2}(q,p,\{k_m\}) = [S^{(0)}(q;\{k_i\})S^{(0)}(p;\{k_j\})]\mathcal{A}_n(\{k_m\}). \quad (3.5)$$

As is evident, the leading double soft factor is just the product of the individual leading soft factors. One obtains this same theorem in the case of the simultaneous double soft limit as well [24,31,33,34]. In Sec. IV B, we show that this soft theorem matches with the result derived from the Ward identity of two supertranslation charges (4.10).

Let us now consider the subleading soft limit. At this order of factorization we have four terms:

$$\left[\sum_{i,j}\frac{E_{k_i}}{E_q}\hat{S}^{(0)}(q;k_i)S^{(1)}(p;k_j)+\sum_i\frac{E_{k_i}}{E_q}\hat{S}^{(0)}(q;p)\hat{S}^{(0)}(p;k_i) + \sum_i S^{(1)}(q;k_i)\sum_j\frac{E_{k_j}}{E_p}\hat{S}^{(0)}(p;k_j) + S^{(1)}(q;p)\sum_i\frac{E_{k_i}}{E_p}\hat{S}^{(0)}(p;k_i)\right]\mathcal{A}_n(\{k_m\}). \quad (3.6)$$

Notice that the first two terms in (3.6) scale with soft graviton energies as E_p^0/E_q and the second two terms scale as E_q^0/E_p.

From the first two terms of (3.6), one gets a subleading CDST.

$$\lim_{E_p\to 0}(1+E_p\partial_{E_p})\lim_{E_q\to 0}E_q\mathcal{A}_{n+2}(q,p,\{k_m\}) = [S^{(0)}(q;\{k_i\})S^{(1)}(p;\{k_j\})+\mathcal{N}(q;p;\{k_i\})]\mathcal{A}_n(\{k_m\}). \quad (3.7)$$

Here, the first term is the product of single soft factors (2.8), (2.20), appearing in the leading and subleading single soft theorems, respectively. The second term in the r.h.s of (3.7) contains a single sum over the set of hard particles as opposed to the first term which is the product of single soft factors and contains two sums over the set of hard particles. Such terms are usually referred to as "contact terms" in the literature. One can evaluate this contact term as

$$\mathcal{N}(q;p;\{k_i\}) = \hat{S}^{(0)}(q;p)S^{(0)}(p;\{k_i\}) = \sum_i\frac{(\epsilon_q\cdot\tilde{p})^2}{\tilde{q}\cdot\tilde{p}}\cdot\frac{(\epsilon_p\cdot k_i)^2}{\tilde{p}\cdot k_i}, \quad (3.8)$$

where $\tilde{p}=p/E_p=(1,\hat{p})$ and similarly, $\tilde{q}=q/E_q=(1,\hat{q})$. ϵ_p and ϵ_q refer to the polarizations of soft gravitons with momentum p and q, respectively. This is the well-known consecutive double soft graviton theorem [31].

A. A different consecutive limit

We now take a different limit in Eq. (3.6) and show how it leads to a distinct factorization theorem. From the last two terms in (3.6) one gets

$$\lim_{E_p\to 0}E_p\lim_{E_q\to 0}(1+E_q\partial_{E_q})\mathcal{A}_{n+2}(q,p,\{k_m\}) = \left[\sum_i S^{(1)}(q;k_i)\sum_j E_{k_j}\hat{S}^{(0)}(p;k_j) + \lim_{E_p\to 0}E_pS^{(1)}(q;p)\sum_i\frac{E_{k_i}}{E_p}\hat{S}^{(0)}(p;k_i)\right]\mathcal{A}_n(\{k_m\}). \quad (3.9)$$

Now, $S^{(1)}(q;k_i)$ contains the angular momentum operator of the ith hard particle, and thus acts on $E_{k_j}\hat{S}^{(0)}(p;k_j)$, as well as the n particle amplitude $\mathcal{A}_n(\{k_m\})$. However, $S^{(1)}(q;p)$ does not depend on the set of hard particles labeled by momentum $\{k_m\}$. Hence, $S^{(1)}(q;p)$ acts only on the soft factor, and one can finally write the subleading CDST as

$$\lim_{E_p\to 0}E_p\lim_{E_q\to 0}(1+E_q\partial_{E_q})\mathcal{A}_{n+2}(q,p,\{k_m\}) = [S^{(0)}(p;\{k_i\})S^{(1)}(q;\{k_j\}) + \mathcal{M}_1(q;p;\{k_i\}) + \mathcal{M}_2(q;p;\{k_i\})]\mathcal{A}_n(\{k_m\}). \quad (3.10)$$

Similar to the other subleading CDST (3.7), the first term in the rhs of (3.10) is product of single soft factors. However, the important difference is that the role of the soft gravitons with momentum p and q is interchanged in the first term of (3.10) and the first term of (3.7). Here, $\mathcal{M}_1(q;p;\{k_i\})$ and $\mathcal{M}_2(q;p;\{k_i\})$ are contact terms which can be expressed as follows:

$$\begin{aligned}\mathcal{M}_1(q;p;\{k_i\}) &= \sum_i S^{(1)}(q;k_i)(E_{k_i}\hat{S}^{(0)}(p;k_i)) \\ &= \sum_i S^{(1)}(q;k_i)(S^{(0)}(p;k_i)) \\ &= \sum_i\Bigg[-\frac{(\epsilon_q\cdot k_i)^2(\epsilon_p\cdot k_i)^2(\tilde{p}\cdot q)}{(q\cdot k_i)(\tilde{p}\cdot k_i)^2} + \frac{(\epsilon_q\cdot k_i)(\epsilon_q\cdot\tilde{p})(\epsilon_p\cdot k_i)^2}{(\tilde{p}\cdot k_i)^2} \\ &\quad + 2\frac{(\epsilon_q\cdot k_i)^2(\epsilon_p\cdot k_i)(\epsilon_p\cdot q)}{(\tilde{p}\cdot k_i)(q\cdot k_i)} - 2\frac{(\epsilon_q\cdot k_i)(\epsilon_p\cdot\epsilon_q)(\epsilon_p\cdot k_i)}{(\tilde{p}\cdot k_i)}\Bigg] \end{aligned} \quad (3.11)$$

and

$$\mathcal{M}_2(q;p;\{k_i\}) = \sum_i \lim_{E_p\to 0} E_p S^{(1)}(q;p)\left(\frac{E_{k_i}}{E_p}\hat{S}^{(0)}(p;k_i)\right)$$
$$= \sum_i \left[\frac{(\epsilon_q\cdot\tilde{p})(\epsilon_q\cdot k_i)(\epsilon_p\cdot k_i)^2}{(\tilde{p}\cdot k_i)^2} - \frac{(\epsilon_q\cdot\tilde{p})^2(\epsilon_p\cdot k_i)^2(q\cdot k_i)}{(\tilde{p}\cdot k_i)^2(\tilde{p}.q)} - 2\frac{(\epsilon_q\cdot\tilde{p})(\epsilon_q\cdot k_i)(\epsilon_p\cdot q)(\epsilon_p\cdot k_i)}{(\tilde{p}.q)(\tilde{p}.k_i)} + 2\frac{(\epsilon_q\cdot\tilde{p})(\epsilon_q\cdot\epsilon_p)(\epsilon_p\cdot k_i)(q.k_i)}{(\tilde{p}.q)(\tilde{p}.k_i)}\right]. \quad (3.12)$$

Again, $\tilde{p} = p/E_p = (1,\hat{p})$ and ϵ_p and ϵ_q refer to the polarization of soft gravitons with momentum p and q, respectively.

In [31], the authors have considered similar consecutive limits for the double soft graviton and gluon amplitudes. There, they have imposed a gauge condition $\epsilon_p\cdot q = 0$ and $\epsilon_q\cdot p = 0$. However, our analysis proceeded without imposing any particular gauge condition. With the specific gauge condition used in [31], a few of the terms like $\hat{S}^{(0)}(q;p)$ and $S^{(1)}(q;p)$ drop out from the CDST result that we have obtained at the subleading level and we recover their result. This serves as a consistency check for our calculation.

One can also verify the consistency of both the consecutive limits with the general result which was given in [24]. That is, both the CDST (3.7) and (3.10) are special cases of the double soft limit in [24]. The CDST (3.7) can be recovered by imposing the condition $E_p \gg E_q$ on the result of [24] and taking the leading limit in E_q and subleading limit in E_p. Similarly, the CDST (3.10) can be obtained by imposing the same $E_p \gg E_q$ condition, but taking the leading limit in E_p and subleading limit in E_q.

In the subsequent sections, we will argue that these soft theorems are equivalent to Ward identities of asymptotic symmetries when the scattering states are defined with respect to supertranslated or superrotated vacua.

IV. CDST AND ASYMPTOTIC SYMMETRIES

A. Introduction

Having reviewed the relationship between Ward identities associated with the asymptotic symmetries and single soft graviton theorems, we now ask if there are Ward identities in the theory which are equivalent to the double soft graviton theorems at the leading and subleading order. In particular, we look for Ward identities that will lead us to the consecutive double soft theorems (CDST). Let us consider the family of Ward identities whose general structure is

$$\langle\text{out}|[Q_1,[Q_2,\mathcal{S}]]|\text{in}\rangle = 0, \quad (4.1)$$

where both Q_1 and Q_2 are either both supertranslation charges or Q_1 is a supertranslation charge and Q_2 is a superrotation charge.[3]

Following [28], we present a derivation of this proposed Ward identity in Appendix A. In the following sections, we show that such a proposal leads to the consecutive double soft theorems discussed in Sec. III. Depending on the choice of charges one gets the leading as well as the subleading consecutive double soft theorems.

B. Leading CDST and asymptotic symmetries

1. Ward identity from asymptotic symmetries

Following the discussion in Sec. IV A, we explore the factorization arising from two supertranslation charges, Q_f and Q_g characterized by arbitrary functions $f(z,\bar{z})$ and $g(z,\bar{z})$, on the conformal sphere. We start with

$$\langle\text{out}|[Q_f,[Q_g,\mathcal{S}]]|\text{in}\rangle = 0. \quad (4.2)$$

Proceeding in a manner similar to the single soft case in Sec. II, we can write Q_f and Q_g as sum of hard and soft charges as

$$Q_f = Q_f^{\text{hard}} + Q_f^{\text{soft}}, \qquad Q_g = Q_g^{\text{hard}} + Q_g^{\text{soft}}. \quad (4.3)$$

Thus, the Ward identity (4.2) becomes

$$\langle\text{out}|[Q_f^{\text{hard}},[Q_g^{\text{hard}},\mathcal{S}]]|\text{in}\rangle + \langle\text{out}|[Q_f^{\text{hard}},[Q_g^{\text{soft}},\mathcal{S}]]|\text{in}\rangle + \langle\text{out}|[Q_f^{\text{soft}},[Q_g^{\text{hard}},\mathcal{S}]]|\text{in}\rangle + \langle\text{out}|[Q_f^{\text{soft}},[Q_g^{\text{soft}},\mathcal{S}]]|\text{in}\rangle = 0. \quad (4.4)$$

Now using the Ward identity of supertranslation, namely $[Q_g^{\text{soft}},S] = -[Q_g^{\text{hard}},S]$, the first and the second terms cancel each other. One may be tempted to cancel the third and fourth terms, on similar lines. However, we contend that this isn't quite correct as the action of Q_f^{soft} maps ordinary the Fock vaccuum to a supertranslated vaccuum state parametrized by f. As a result, we are really looking at the following Ward identity,

$$\langle\text{out}, f|[Q_g,S]|\text{in}\rangle = 0, \quad (4.5)$$

where $|\text{out}, f\rangle$ is a finite energy state defined with respect to the supertranslated vacuum. The "in" state is defined with respect to standard Fock Vacuum because of our prescription $Q_f^{\text{soft}}|\text{in}\rangle = 0$. We can rewrite the above identity as

$$\langle\text{out}|[Q_f^{\text{soft}},[Q_g^{\text{soft}},\mathcal{S}]]|\text{in}\rangle = -\langle\text{out}|[Q_f^{\text{soft}},[Q_g^{\text{hard}},\mathcal{S}]]|\text{in}\rangle. \quad (4.6)$$

[3]The alternate case where Q_1 is superrotation charge and Q_2 is supertranslation charge is riddled with conceptual subtleties which remain unresolved—we return to this in Appendix B.

Now using the Jacobi identity among $Q_f^{\rm soft}$, $Q_g^{\rm hard}$ and $\mathcal{S}$, the commutation relation $[Q_f^{\rm soft}, Q_g^{\rm hard}] = 0$, and the single soft Ward identity, we can finally write

$$\langle {\rm out}|[Q_f^{\rm soft},[Q_g^{\rm soft},\mathcal{S}]]|{\rm in}\rangle = \langle {\rm out}|[Q_g^{\rm hard},[Q_f^{\rm hard},\mathcal{S}]]|{\rm in}\rangle. \tag{4.7}$$

Using the (known) action of charges on external states in (4.7) we finally arrive at the Ward identity:

$$\begin{aligned}&\lim_{E_p\to 0}\frac{E_p}{2\pi}\lim_{E_q\to 0}\frac{E_q}{2\pi}\int d^2w_1 d^2w_2 D^2_{\bar{w}_1} f(w_1,\bar{w}_1) D^2_{\bar{w}_2} g(w_2,\bar{w}_2)\\ &\quad\times\langle {\rm out}|a_+(E_p,w_1,\bar{w}_1)a_+(E_q,w_2,\bar{w}_2)\mathcal{S}|{\rm in}\rangle\\ &\quad=\left[\sum_{\rm out} f(\hat{k}_i)E_i - \sum_{\rm in} f(\hat{k}_i)E_i\right]\\ &\quad\times\left[\sum_{\rm out} g(\hat{k}_j)E_j - \sum_{\rm in} g(\hat{k}_j)E_j\right]\langle {\rm out}|\mathcal{S}|{\rm in}\rangle.\end{aligned} \tag{4.8}$$

The factorization above is just the product of two factors of the type obtained from the Ward identity for supertranslation (2.5). It is natural therefore to expect that the soft theorem we obtain from (4.8) will also be the product of two leading single soft factors. In the next section, we show that this is indeed true.

2. From Ward identity to soft theorem

From the factorization obtained in (4.8) from the Ward identity with two supertranslation charges, we try to understand what soft theorem follows from it. Motivated from the single soft case, we make the choices for arbitrary function f and g on the conformal sphere as

$$\begin{aligned} f(w_1,\bar{w}_1) &= s(w_1,\bar{w}_1;w_p,\bar{w}_p),\\ g(w_2,\bar{w}_2) &= s(w_2,\bar{w}_2;w_q,\bar{w}_q),\end{aligned} \tag{4.9}$$

where the definition of the functions $s(w_1,\bar{w}_1;w_p,\bar{w}_p)$ and $s(w_2,\bar{w}_2;w_q,\bar{w}_q)$ can be read from (2.9). Substituting these choices in (4.8), we finally get

$$\begin{aligned}&\lim_{E_p\to 0}E_p\lim_{E_q\to 0}E_q\langle {\rm out}|a_+(E_p,w_p,\bar{w}_p)a_+(E_q,w_q,\bar{w}_q)\mathcal{S}|{\rm in}\rangle\\ &\quad=[S^{(0)}(q;\{k_i\})S^{(0)}(p;\{k_j\})]\langle {\rm out}|\mathcal{S}|{\rm in}\rangle.\end{aligned} \tag{4.10}$$

This is the same as the leading double soft theorem (3.5) for the case of two positive helicity soft gravitons with momenta p and q, localized at $(w_p,\bar{w}_p)$ and $(w_q,\bar{w}_q)$, respectively, on the conformal sphere. Although we have chosen both the soft graviton helicities to be positive in the above, one can do a similar analysis for both the helicities being negative or one positive and one negative, and a similar result holds. This provides the equivalence of the leading CDST and the Ward identity (4.2).

We have thus shown that the leading order double soft graviton theorem is equivalent to the supertranslation Ward identity when this identity is evaluated in a Hilbert space built out of a supertranslated vacuum that containing a single soft graviton.

C. Subleading CDST and asymptotic symmetries

1. Ward identity from asymptotic symmetries

As motivated in Sec. IV A and derived in Appendix A, we now analyze with the Ward identity corresponding to one supertranslation charge (characterized by arbitrary function f) and one superrotation charge (characterized by vector field V^A):

$$\langle {\rm out}|[Q_f,[Q_V,\mathcal{S}]]|{\rm in}\rangle = 0. \tag{4.11}$$

We begin by writing the charges as the sum of hard and soft charges:

$$\begin{aligned}&\langle {\rm out}|[Q_f^{\rm hard},[Q_V^{\rm hard},\mathcal{S}]]|{\rm in}\rangle + \langle {\rm out}|[Q_f^{\rm hard},[Q_V^{\rm soft},\mathcal{S}]]|{\rm in}\rangle\\ &\quad+\langle {\rm out}|[Q_f^{\rm soft},[Q_V^{\rm hard},\mathcal{S}]]|{\rm in}\rangle + \langle {\rm out}|[Q_f^{\rm soft},[Q_V^{\rm soft},\mathcal{S}]]|{\rm in}\rangle = 0.\end{aligned} \tag{4.12}$$

Now, using the Ward identity for superrotation, namely $[Q_V^{\rm soft},\mathcal{S}] = -[Q_V^{\rm hard},\mathcal{S}]$, the first and the second term of (4.12) cancel each other. Again, one may be tempted to cancel the third and the fourth term of (4.12) instead, using the same superrotation Ward identity. However, if we do not cancel them, we are led to

$$\begin{aligned}\langle {\rm out}|Q_f^{\rm soft}[Q_V,S]|{\rm in}\rangle &= 0\\ \langle {\rm out},f|[Q_V,S]|{\rm in}\rangle &= 0.\end{aligned} \tag{4.13}$$

Thus, not cancelling the third and forth terms in (4.12) is tantamount to considering superrotation Ward identity in scattering states which are excitations around supertranslated vacuua. As we show below, it is precisely the Ward identity $\langle {\rm out},f|[Q_V,\mathcal{S}]|{\rm in}\rangle = 0$ that leads to a specific double soft graviton theorem.

Hence the above identity (4.12) reduces to

$$\begin{aligned}&\langle {\rm out}|[Q_f^{\rm soft},[Q_V^{\rm soft},\mathcal{S}]]|{\rm in}\rangle\\ &\quad= -\langle {\rm out}|[Q_f^{\rm soft},[Q_V^{\rm hard},\mathcal{S}]]|{\rm in}\rangle\\ &\quad= -\langle {\rm out}|Q_f^{\rm soft}Q_V^{\rm hard}\mathcal{S}|{\rm in}\rangle + \langle {\rm out}|Q_f^{\rm soft}\mathcal{S}Q_V^{\rm hard}|{\rm in}\rangle.\end{aligned} \tag{4.14}$$

Using the known action of the soft and hard charges, first term in the rhs of (4.14) can be written as

$$\begin{aligned}&\langle \text{out}|Q_f^{\text{soft}} Q_V^{\text{hard}}\mathcal{S}|\text{in}\rangle\\&=\frac{1}{2\pi}\lim_{E_p\to 0}\int d^2w_1 D_{\bar{w}_1}^2 f E_p \langle \text{out}|a_+(E_p\hat{x})Q_V^{\text{hard}}\mathcal{S}|\text{in}\rangle\\&=\frac{i}{2\pi}\lim_{E_p\to 0}\int d^2w_1 D_{\bar{w}_1}^2 f E_p\left(\sum_{\text{out}} J_V^{h_i}+J_V^+\right)\\&\quad\times\langle \text{out}|a_+(E_p\hat{x})\mathcal{S}|\text{in}\rangle,\end{aligned}\tag{4.15}$$

where $\hat{x}$ denotes the direction of the soft graviton parametrized by $(w_1,\bar{w}_1)$ on the conformal sphere. J_V^+ represents the action of Q_V^{hard} on the soft graviton with energy E_p.

Similarly, the second term in (4.14) can be evaluated to

$$\begin{aligned}&\langle \text{out}|Q_f^{\text{soft}}\mathcal{S}Q_V^{\text{hard}}|\text{in}\rangle\\&=\frac{i}{2\pi}\lim_{E_p\to 0}\int d^2w_1 D_{\bar{w}_1}^2 f\left(\sum_{\text{in}} J_V^{-h_i}\right)E_p\langle \text{out}|a_+(E_p\hat{x})\mathcal{S}|\text{in}\rangle.\end{aligned}\tag{4.16}$$

Hence, the Ward identity (4.14) simplifies to

$$\begin{aligned}&\langle \text{out}|Q_f^{\text{soft}} Q_V^{\text{soft}}\mathcal{S}|\text{in}\rangle\\&=-\frac{i}{2\pi}\lim_{E_p\to 0}\int d^2w_1 D_{\bar{w}_1}^2 f\left(\sum_{\text{out}} J_V^{h_i}-\sum_{\text{in}} J_V^{-h_i}\right)\\&\quad\times[E_p\langle \text{out}|a_+(E_p\hat{x})\mathcal{S}|\text{in}\rangle]\\&\quad-\frac{i}{2\pi}\lim_{E_p\to 0}\int d^2w_1 D_{\bar{w}_1}^2 f E_p (J_V^+)[\langle \text{out}|a_+(E_p\hat{x})\mathcal{S}|\text{in}\rangle],\end{aligned}\tag{4.17}$$

Note that, the lhs of (4.17) can be written as[4]

$$\begin{aligned}&\lim_{E_p\to 0}\frac{1}{2\pi}E_p\lim_{E_q\to 0}\frac{1}{4\pi i}(1+E_q\partial_{E_q})\\&\quad\times\int d^2w_1 d^2w_2 D_{\bar{w}_1}^2 f\partial_{\bar{w}_2}^3 V^{\bar{w}_2}\langle \text{out}|a_+(E_p\hat{x})a_+(E_q\hat{y})\mathcal{S}|\text{in}\rangle.\end{aligned}\tag{4.18}$$

It is important to note that the soft limits taken in the above equation do not follow any particular order in the energies of the soft gravitons. However, as we show in the next section, the right-hand side of the Ward identity is equivalent to the right-hand side of one of the CDSTs.

2. From Ward identity to soft theorem

Having derived the Ward identity (4.17), we now ask whether it can be interpreted as a soft theorem. Motivated by the single soft graviton case, we make the following choices for function f and vector field V:

[4]More precise definition of lhs is given in Appendix B.

$$\begin{aligned}f(w_1,\bar{w}_1)&=s(w_1,\bar{w}_1;w_p,\bar{w}_p)\\V^{\bar{w}_2}&=K^+_{(w_q,\bar{w}_q)},\end{aligned}\tag{4.19}$$

where $s(w_1,\bar{w}_1;w_p,\bar{w}_p)$ and $K^+_{(w_q,\bar{w}_q)}$ follow the definitions in Sec. II. Using this, (4.18) becomes

$$\lim_{E_p\to 0}E_p\lim_{E_q\to 0}(1+E_q\partial_{E_q})\langle \text{out}|a_+(E_p\hat{x})a_+(E_q\hat{y})\mathcal{S}|\text{in}\rangle,\tag{4.20}$$

where the unit vectors $\hat{x}$ and $\hat{y}$ denote the coordinates $(w_p,\bar{w}_p)$ and $(w_q,\bar{w}_q)$ on the conformal sphere.

Further, for the rhs of (4.17), we have

$$\begin{aligned}&\lim_{E_p\to 0}\sum_i S^{(1)}(q;k_i)[E_p\langle \text{out}|a_+(E_p\hat{x})\mathcal{S}|\text{in}\rangle]\\&\quad+\lim_{E_p\to 0}E_p S^{(1)}(q;p)[\langle \text{out}|a_+(E_p\hat{x})\mathcal{S}|\text{in}\rangle].\end{aligned}\tag{4.21}$$

In the above expression, notice that in both the subleading factors $S^{(1)}(q;k_i)$ and $S^{(1)}(q;p)$, the soft graviton with momentum q is localized at $\hat{y}$ on the conformal sphere. However, the first one contains an angular momentum operator acting on the i^{th} hard particle and the latter contains an angular momentum operator acting on the soft graviton with momentum p.

Now, using the leading single soft theorem, the first term in (4.21) can be written as

$$\sum_i S^{(1)}(q;k_i)\left[\sum_j E_{k_j}\hat{S}^{(0)}(p;k_j)\langle \text{out}|\mathcal{S}|\text{in}\rangle\right].\tag{4.22}$$

For the second term in (4.21), we use the expansion of the $(n+1)$ particle amplitude (3.2) and we get a factorization of the form

$$\begin{aligned}\langle \text{out}|a_+(E_p\hat{x})\mathcal{S}|\text{in}\rangle&=\left[\sum_i\frac{E_{k_i}}{E_p}\hat{S}^{(0)}(p;k_i)+\sum_i S^{(1)}(p;k_i)\right]\\&\quad\times\langle \text{out}|\mathcal{S}|\text{in}\rangle+\mathcal{O}(E_p).\end{aligned}\tag{4.23}$$

The second term of (4.23) is at a higher order in soft graviton energy, and so does not contribute to (4.21). Thus, (4.21) finally becomes

$$\begin{aligned}&\sum_i S^{(1)}(q;k_i)\left[\sum_j E_{k_j}\hat{S}^{(0)}(p;k_j)\langle \text{out}|\mathcal{S}|\text{in}\rangle\right]\\&\quad+\lim_{E_p\to 0}E_p S^{(1)}(q;p)\left[\sum_j\frac{E_{k_j}}{E_p}\hat{S}^{(0)}(p;k_j)\right]\langle \text{out}|\mathcal{S}|\text{in}\rangle.\end{aligned}\tag{4.24}$$

Lastly, since $S^{(1)}(q;k_i)$ is a linear differential operator and $S^{(1)}(q;p)$ acts only on the soft coordinates, we can further simplify (4.24) as

$$\left[\sum_{i,j} E_{k_i}\hat{S}^{(0)}(p;k_i)S^{(1)}(q;k_j) + \sum_i S^{(1)}(q;k_i)(E_{k_i}\hat{S}^{(0)}(p;k_i)) + \lim_{E_p\to 0} E_p S^{(1)}(q;p)\left(\sum_j \frac{E_{k_j}}{E_p}\hat{S}^{(0)}(p;k_j)\right)\right]\langle \text{out}|\mathcal{S}|\text{in}\rangle. \tag{4.25}$$

Finally, putting this all together, we get a subleading double soft theorem:

$$\begin{aligned}\lim_{E_p\to 0} E_p \lim_{E_q\to 0}(1+E_q\partial_{E_q})\mathcal{A}_{n+2}(q,p,\{k_m\}) &= [S^{(0)}(p;\{k_i\})S^{(1)}(q;\{k_j\}) + \mathcal{M}_1(q;p;\{k_i\}) \\ &\quad + \mathcal{M}_2(q;p;\{k_i\})]\mathcal{A}_n(\{k_m\}),\end{aligned} \tag{4.26}$$

where $\mathcal{M}_1(q;p;\{k_i\})$ and $\mathcal{M}_2(q;p;\{k_i\})$ are the same contact terms obtained in subleading CDST (3.10), whose expressions can be read off from (3.11), (3.12), respectively. This is the same subleading consecutive double soft theorem (3.10), that we studied in the Sec. III. Note, however, that in (4.18), there is no particular ordering in the limits of the soft graviton energy obtained from the successive action of the soft charges. Hence, the lhs of the double soft theorem (4.26) contains independent limits as opposed to (3.10), where the limits have definite ordering. Although we believe this point needs to be better understood, what we have shown here is that the Ward identity of superrotation charges in a supertranslated vacuum leads to a particular CDST. It is also important to emphasize that there is a definite time ordering in $[Q_f,[Q_V,\mathcal{S}]]=0$. This is clear from the derivation of the Ward identity $\langle \text{out}|[Q_f,[Q_V,\mathcal{S}]]|\text{in}\rangle = 0$, which is presented in Appendix A.

V. RELATING THE STANDARD CDST TO A WARD IDENTITY

As we saw above, the Ward identity $[Q_f,[Q_V,\mathcal{S}]]=0$, gave rise to a double soft theorem whose rhs matched with the consecutive soft theorem, where we considered the subleading limit of the graviton which was taken soft first. This is in contrast to the more standard consecutive soft limit where we consider the leading soft limit of the graviton which is taken soft first and subleading soft limit of the graviton which is taken soft second. We will argue how this CDST could potentially arise out of the Ward identity

$$\langle \text{out}|[Q_V,[Q_f,\mathcal{S}]]|\text{in}\rangle = 0. \tag{5.1}$$

Expressing the charges in (5.1) as the sum of hard and soft charges, we get

$$\begin{aligned}&\langle \text{out}|[Q_V^{\text{hard}},[Q_f^{\text{hard}},\mathcal{S}]]|\text{in}\rangle + \langle \text{out}|[Q_V^{\text{soft}},[Q_f^{\text{hard}},\mathcal{S}]]|\text{in}\rangle \\ &+ \langle \text{out}|[Q_V^{\text{hard}},[Q_f^{\text{soft}},\mathcal{S}]]|\text{in}\rangle + \langle \text{out}|[Q_V^{\text{soft}},[Q_f^{\text{soft}},\mathcal{S}]]|\text{in}\rangle = 0.\end{aligned} \tag{5.2}$$

Using the Ward identity for supertranslation, namely $[Q_f^{\text{soft}},\mathcal{S}] = -[Q_f^{\text{hard}},\mathcal{S}]$, the first and the third terms cancel each other. Once again, this leads us to the following supertranslation Ward identity evaluated in states defined with respect to "superrotated vacuum"

$$\begin{aligned}\langle \text{out}|Q_V^{\text{soft}}[Q_f,\mathcal{S}]|\text{in}\rangle &= 0 \\ \langle \text{out},V|[Q_f,\mathcal{S}]|\text{in}\rangle &= 0.\end{aligned} \tag{5.3}$$

where by $|\text{out},V\rangle$ we mean a finite energy scattering state defined with respect to a vacuum which contains a subleading soft graviton mode.[5] However, as we explain in Appendix B, unlike the action of Q_f^{soft}, the action of Q_V^{soft} is not well understood thus far.[6] Consequently, the proposed Ward identity remains rather formal at this point. We will still proceed further and show that this proposed Ward identity, if well defined is equivalent to the standard CDST. We can rewrite the Ward identity as

$$\begin{aligned}&\langle \text{out}|Q_V^{\text{soft}}Q_f^{\text{soft}}\mathcal{S}|\text{in}\rangle \\ &= -\langle \text{out}|[Q_V^{\text{soft}},[Q_f^{\text{hard}},\mathcal{S}]]|\text{in}\rangle \\ &= \langle \text{out}|Q_V^{\text{soft}}\mathcal{S}Q_f^{\text{hard}} - Q_f^{\text{hard}}Q_V^{\text{soft}}\mathcal{S}|\text{in}\rangle \\ &\quad + \langle \text{out}|[Q_f^{\text{hard}},Q_V^{\text{soft}}]\mathcal{S}|\text{in}\rangle.\end{aligned} \tag{5.4}$$

We evaluate the two terms in the rhs of (5.4) one by one. The first term can be written as

$$\begin{aligned}&\langle \text{out}|Q_V^{\text{soft}}\mathcal{S}Q_f^{\text{hard}} - Q_f^{\text{hard}}Q_V^{\text{soft}}\mathcal{S}|\text{in}\rangle \\ &= -\langle \text{out}|[Q_f^{\text{hard}},Q_V^{\text{soft}}\mathcal{S}]|\text{in}\rangle = -\langle \text{out}|[Q_f^{\text{hard}},[Q_V^{\text{soft}},\mathcal{S}]]|\text{in}\rangle \\ &= \langle \text{out}|[Q_f^{\text{hard}},[Q_V^{\text{hard}},\mathcal{S}]]|\text{in}\rangle.\end{aligned} \tag{5.5}$$

Then, using the action of Q_f^{hard} and Q_V^{hard} on the external states, we can write the rhs of (5.5) as

$$\begin{aligned}&\langle \text{out}|[Q_f^{\text{hard}},[Q_V^{\text{hard}},\mathcal{S}]]|\text{in}\rangle \\ &= i\left[\sum_{\text{out}} f(\hat{k}_i)E_i - \sum_{\text{in}} f(\hat{k}_i)E_i\right] \\ &\quad\times\left[\sum_{\text{out}} J_{V_i}^{h_i} - \sum_{\text{in}} J_{V_i}^{-h_i}\right]\langle \text{out}|\mathcal{S}|\text{in}\rangle.\end{aligned} \tag{5.6}$$

To evaluate the second term in (5.4), note that for a single particle state $|k\rangle$,

[5] It was shown in [16] how Q_V^{soft} maps the vacuum to a different vacuum.

[6] We are indebted to Prahar Mitra for emphasizing this point.

$$\langle k|[Q_f^{\mathrm{hard}}, Q_V^{\mathrm{soft}}] = -\frac{1}{4\pi i}\lim_{E_p\to 0}(1+E_p\partial_{E_p})\int d^2w_2\partial_{\bar{w}_2}^3 V^{\bar{w}_2}E_p f(w_2,\bar{w}_2)\langle k|a_+(E_p,w_2,\bar{w}_2)$$
$$= -\frac{1}{4\pi i}\lim_{E_p\to 0}\int d^2w_2\partial_{\bar{w}_2}^3 V^{\bar{w}_2}E_p f(w_2,\bar{w}_2)\langle k|a_+(E_p,w_2,\bar{w}_2). \tag{5.7}$$

Where, in going from the first line to the second, we have used the fact that $a_+(E_p,w_2,\bar{w}_2)\sim\frac{1}{E_p}$.[7] Therefore,

$$-\frac{1}{4\pi i}\lim_{E_p\to 0}E_p\partial_{E_p}\int d^2w_2\partial_{\bar{w}_2}^3 V^{\bar{w}_2}E_p f(w_2,\bar{w}_2)\langle k|a_+(E_p,w_2,\bar{w}_2)=0. \tag{5.8}$$

Using the above expression (5.7), we can evaluate the second term of (5.4) as

$$\langle \mathrm{out}|[Q_f^{\mathrm{hard}}, Q_V^{\mathrm{soft}}]\mathcal{S}|\mathrm{in}\rangle = -\frac{1}{4\pi i}\lim_{E_p\to 0}\int d^2w_2\partial_{\bar{w}_2}^3 V^{\bar{w}_2}E_p\times f(w_2,\bar{w}_2)\langle \mathrm{out}|a_+(E_p,w_2,\bar{w}_2)\mathcal{S}|\mathrm{in}\rangle. \tag{5.9}$$

Lastly, using the single soft graviton theorem (with energy E_p), (5.9) simplifies to

$$\langle \mathrm{out}|[Q_f^{\mathrm{hard}}, Q_V^{\mathrm{soft}}]\mathcal{S}|\mathrm{in}\rangle = -\frac{1}{4\pi i}\sum_i\int d^2w_2\partial_{\bar{w}_2}^3 V^{\bar{w}_2}f(w_2,\bar{w}_2)E_{k_i}\hat{S}^{(0)}(p;k_i)\langle \mathrm{out}|\mathcal{S}|\mathrm{in}\rangle. \tag{5.10}$$

Finally, substituting (5.6) and (5.10) in (5.4), we arrive at the Ward identity:

$$\langle \mathrm{out}|Q_V^{\mathrm{soft}}Q_f^{\mathrm{soft}}\mathcal{S}|\mathrm{in}\rangle = i\left[\sum_{\mathrm{out}}f(\hat{k}_i)E_i-\sum_{\mathrm{in}}f(\hat{k}_i)E_i\right]\left[\sum_{\mathrm{out}}J_{V_i}^{h_i}-\sum_{\mathrm{in}}J_{V_i}^{-h_i}\right]\langle \mathrm{out}|\mathcal{S}|\mathrm{in}\rangle$$
$$-\frac{1}{4\pi i}\sum_{\mathrm{hard}}\int d^2w_2\partial_{\bar{w}_2}^3 V^{\bar{w}_2}f(w_2,\bar{w}_2)E_{k_i}S^{(0)}(w_2,\bar{w}_2;k_i)\langle \mathrm{out}|\mathcal{S}|\mathrm{in}\rangle, \tag{5.11}$$

where the lhs can be expressed as

$$\frac{1}{4\pi i}\lim_{E_p\to 0}(1+E_p\partial_{E_p})\frac{1}{2\pi}\lim_{E_q\to 0}E_q\int d^2w_1 d^2w_2 D_{\bar{w}_1}^2 f(w_1,\bar{w}_1)\partial_{\bar{w}_2}^3 V^{\bar{w}_2}\times\langle \mathrm{out}|a_+(E_q,w_1,\bar{w}_1)a_+(E_p,w_2,\bar{w}_2)\mathcal{S}|\mathrm{in}\rangle. \tag{5.12}$$

In order to proceed from the Ward identity (5.11) to a soft theorem, we make the following choices for f and V:

$$f(w_1,\bar{w}_1)=s(w_1,\bar{w}_1;w_q,\bar{w}_q),\qquad V^{\bar{w}_2}=K^+_{(w_p,\bar{w}_p)}. \tag{5.13}$$

Substituting these in (5.11), we formally get the subleading CDST for positive helicity gravitons as

$$\lim_{E_p\to 0}(1+E_p\partial_{E_p})\lim_{E_q\to 0}E_q\langle \mathrm{out}|a_+(E_q\hat{y})a_+(E_p\hat{x})\mathcal{S}|\mathrm{in}\rangle = [S^{(0)}(q;\{k_i\})S^{(1)}(p;\{k_j\})+\hat{S}^{(0)}(q;p)S^{(0)}(p;\{k_i\})]\langle \mathrm{out}|\mathcal{S}|\mathrm{in}\rangle. \tag{5.14}$$

Again, $\hat{x}$ and $\hat{y}$ denote the points $(w_p,\bar{w}_p)$, $(w_q,\bar{w}_q)$ on the conformal sphere. This is the same consecutive double soft theorem (3.7) discussed in Sec. III.

However, as discussed in Appendix B, there are some important subtleties in the definition of soft operators, especially the soft superrotation charge Q_V^{soft}. Due to this, in the evaluation of the Ward identity $\langle \mathrm{out}|[Q_V,[Q_f,\mathcal{S}]]|\mathrm{in}\rangle=0$, the steps which involve the operation of charge Q_V^{soft} first on the "out" state before the other charge are not mathematically rigorous. However, we present this calculation here, in the hope that this might give some hint to the structure of a more mathematically sound proof of this soft theorem as well as a more rigorous understanding of the operation of the soft superrotation charge.

VI. DISCUSSION AND CONCLUSION

It has now been well established in the literature that the supertranslation soft charge Q_f^{soft} shifts the Fock Vacuum to a vacuum parametrized by a soft graviton. If we consider

[7] This can be seen by writing the mode functions of the News tensor (N_{zz}^{ω}), in terms of graviton annihilation operators as in [1].

Ward identities associated with superrotation charges Q_V in this supertranslated vacuum, we are led to one of the two consecutive subleading double soft graviton theorems. In fact, as was argued in [16], the space of vacua of (perturbative) Quantum Gravity are parametrized by leading as well as subleading soft gravitons. Although we do not have a precise definition of a vacuum which is labelled by a subleading soft graviton, assuming such a definition exists, we can ask what the Ward identity of the supertranslation charge is in such a state. The answer appears to be related to the other consecutive double soft theorem at the subleading level.

Many questions remain open. A precise formulation of these Ward identities will require a careful definition of Q_V^{soft} which is lacking thus far. It is also not entirely clear why Ward identity associated with Q_V "in" states perturbed around the supertranslated vacuum leads to a specific CDST.

It will also be interesting to extend the analysis to the case where the finite energy scattering states are massive. This will require a detailed understanding of the BMS algebra at time–like infinity. Finally, the problem of relating the subleading simultaneous double soft theorem to Ward identities associated with asymptotic symmetries remain completely open. Based on our analysis above, we expect that this will require a detailed analysis of the moduli space of the vacuua (parametrized by leading and subleading soft gravitons) which is complicated by the non-Abelian nature of the BMS symmetries.

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APPENDIX A: WARD IDENTITIES FROM THE AVERY-SCHWAB METHOD

In this appendix, we derive the asymptotic Ward identity $\langle \text{out}|[Q_f,[Q_V,\mathcal{S}]]|\text{in}\rangle = 0$, based on a method that was proposed in [28]. The basic idea is to use Noether's second theorem and path integral techniques to derive Ward identities for asymptotic symmetries.

As shown in [28], given a asymptotic symmetry or large gauge transformation with a gauge parameter λ, at the level of correlation functions one obtains the following Ward identity.

$$\begin{aligned} &-i\langle 0|\delta_\lambda T(\Phi(x_1)\ldots\Phi(x_n))|0\rangle \\ &\quad = \langle 0|T((Q_{\mathcal{I}^+}[\lambda]-Q_{\mathcal{I}^-}[\lambda])\Phi(x_1)\ldots\Phi(x_n))|0\rangle \end{aligned} \tag{A1}$$

Here we use a generic label Φ to label the quantum field associated with scattering particles. $Q_{\mathcal{I}^\pm}[\lambda]$ are the asymptotic charges associated with large gauge transformations λ at future and past null infinity, respectively.

Before deriving the identity associated with the insertion of two charge operators, we first revisit the supertranslation Ward identity $\langle \text{out}|[Q_f,\mathcal{S}]|\text{in}\rangle = 0$. Let Φ be any massless field that interacts with gravity and $\delta_\lambda = \delta_f$ be the generator of supertranslation on the fields.

We begin by noting that through LSZ reduction we have the following[8]

$$\prod_{i=1}^{m} p_i^2 \int d^4x_i e^{-ip_i\cdot x_i} \prod_{j=m+1}^{n} p_j^2 \int d^4x_j e^{ip_j\cdot x_j}\langle 0|\delta_f T(\Phi(x_1)\ldots\Phi(x_n))|0\rangle = -i\langle p_1,\ldots,p_m|Q_f^{\text{hard}+}\mathcal{S}-\mathcal{S}Q_f^{\text{hard}-}|p_{m+1},\ldots,p_n\rangle \tag{A2}$$

We can schematically represent this step as,

$$\langle 0|\delta_\lambda T(\Phi(x_1)\ldots\Phi(x_n))|0\rangle \underset{[\text{LSZ}]}{\to} \langle p_1,\ldots,p_m|[Q_f^{\text{hard}},\mathcal{S}]|p_{m+1},\ldots,p_n\rangle \tag{A3}$$

where we have used the fact that

[8]These arguments are formal because they are tied to the fact that the usual Dyson $\mathcal{S}$–matrix with massless particles is only formally defined. However, as we are only analyzing symmetries of the tree–level $\mathcal{S}$–matrix, we will not worry about the issue of infra–red divergence.

$$\delta_f \Phi(p) = -i[Q_f, \Phi(p)] \tag{A4}$$

On the other hand, once again via LSZ and the fact that

$$\begin{aligned} Q_f^{\rm hard}|0\rangle &= 0 \\ Q_f^{\rm soft}|0\rangle &= 0 \\ \langle 0|Q_f^{\rm soft} &\neq 0 \end{aligned} \tag{A5}$$

we see that

$$\begin{aligned} &\langle 0|T((Q_{\mathcal{I}^+}[\lambda] - Q_{\mathcal{I}^-}[\lambda])\Phi(x_1)\ldots\Phi(x_n))|0\rangle \\ &\underset{[\rm LSZ]}{\to} \langle p_1,\ldots,p_m|[Q_f^{\rm soft},\mathcal{S}]|p_{m+1},\ldots,p_n\rangle \end{aligned} \tag{A6}$$

Substituting Eqs. (A3) and (A6) in Eq. (A1), we recover the supertranslation Ward identity,

$$\langle {\rm out}|[Q_f,\mathcal{S}]|{\rm in}\rangle = 0 \tag{A7}$$

We note that an identical derivation for Ward identity associated with large $U(1)$ gauge transformations was already given in [37].

We will now derive the Ward identities $[Q_f,[Q_V,\mathcal{S}]] = 0$ using this method. That is, we begin with the Ward identity where the superrotation δ_V is applied after the supertranslation δ_f. The starting point for the derivation is (45) in [28], which in the present context can be written as

$$\begin{aligned} &-\langle 0|T((Q_{\mathcal{I}^+}[f] - Q_{\mathcal{I}^-}[f])(Q_{\mathcal{I}^+}[V] \\ &- Q_{\mathcal{I}^-}\Big[V])\Phi(x_1)\ldots\Phi(x_n))|0\rangle \\ &= \langle 0|\delta_f\delta_V T(\Phi(x_1)\ldots\Phi(x_n))|0\rangle \end{aligned} \tag{A8}$$

With our prescription that the soft charges annihilate the "in" vacuum, the lhs of (A8) reduces to

$$\begin{aligned} &-\langle 0|T((Q_{\mathcal{I}^+}[f] - Q_{\mathcal{I}^-}[f])(Q_{\mathcal{I}^+}[V] \\ &- Q_{\mathcal{I}^-}[V])\Phi(x_1)\ldots\Phi(x_n))|0\rangle \\ &= -\langle 0|Q_{\mathcal{I}^+}^{\rm soft}[f](Q_{\mathcal{I}^+}^{\rm soft}[V] + Q_{\mathcal{I}^+}^{\rm hard}[V])T(\Phi(x_1)\ldots\Phi(x_n))|0\rangle \end{aligned} \tag{A9}$$

On the other hand, using (A4), it is easy to see that the rhs of (A8) is given by

$$\begin{aligned} &\langle 0|\delta_f\delta_V T(\Phi(x_1)\ldots\Phi(x_n))|0\rangle \\ &= -\langle 0|\sum_{i,j} T(\Phi(x_1)\ldots[Q_f,\Phi(x_i)] \\ &\ldots[Q_V,\Phi(x_j)]\ldots\Phi(x_n))|0\rangle \\ &\underset{[\rm LSZ]}{\to} -\langle {\rm out}|[Q_f^{\rm hard},[Q_V^{\rm hard},\mathcal{S}]]|{\rm in}\rangle \end{aligned} \tag{A10}$$

Thus the path integral identity and the LSZ formula lead to [equating the rhs of (A9) with rhs of (A10)],

$$\begin{aligned} &\langle {\rm out}|Q_f^{\rm soft}Q_V^{\rm soft}\mathcal{S}|{\rm in}\rangle \\ &= -\langle {\rm out}|Q_f^{\rm soft}Q_V^{\rm hard}\mathcal{S}|{\rm in}\rangle + \langle {\rm out}|[Q_f^{\rm hard},[Q_V^{\rm hard},\mathcal{S}]]|{\rm in}\rangle \end{aligned} \tag{A11}$$

A straightforward manipulation shows that above equation is equivalent to

$$\langle {\rm out}|[Q_f,[Q_V,\mathcal{S}]]|{\rm in}\rangle = 0 \tag{A12}$$

This is one of the Ward identities used in the main text of the paper. The remaining identities can be derived similarly.

APPENDIX B: SUBTLETIES ASSOCIATED WITH THE DOMAIN OF SOFT OPERATORS

We will now comment on the assumption that was implicitly used in previous section, and which has been used frequently in relating single soft theorems to BMS Ward identities.[9]

From the expressions of the supertranslation and superrotation soft charges, we can see that these are singular limits of single graviton annihilation operators,

$$\begin{aligned} Q_f^{\rm soft} &\sim \lim_{E\to 0} E a_+(E,w,\bar{w}) \\ Q_V^{\rm soft} &\sim \lim_{E\to 0}(1+E\partial_E)a_+(E,w,\bar{w}). \end{aligned} \tag{B1}$$

For simplicity we have just considered the expression of the soft charges for positive helicity graviton creation operators only. In the case of Ward identities associated with the single soft theorems, it has been implicitly assumed that the supertranslation soft charge can be defined as (apart from the extra factors),

$$\langle {\rm out}|\lim_{E\to 0} E a_+(E,w,\bar{w})\mathcal{S}|{\rm in}\rangle = \lim_{E\to 0} E\langle {\rm out}|a_+(E,w,\bar{w})\mathcal{S}|{\rm in}\rangle. \tag{B2}$$

A similar assumption is also made for the superrotation soft charge $Q_V^{\rm soft}$.

However, this does not take into account the fact that the supertranslation soft charge shifts the vacuum. This subtlety is now well understood for supertranslations. It was shown in [40–43] that the action of the supertranslation soft charge maps a standard Fock vaccuum to a supertranslated state which can be thought of as being labelled by

[9]The authors would like to thank Abhay Ashtekar and Miguel Campiglia for explaining this subtlety to us in detail in the context of supertranslations, and Prahar Mitra for patiently explaining to us why this subtlety cannot be avoided when we look at Ward identities associated with double soft theorems [38,39].

a single soft graviton. With this is in mind the precise definition of $\langle \mathrm{out}|Q_f^{\mathrm{soft}} Q_V^{\mathrm{soft}} \mathcal{S}|\mathrm{in}\rangle$ would be

$$\langle \mathrm{out}|Q_f^{\mathrm{soft}} Q_V^{\mathrm{soft}} \mathcal{S}|\mathrm{in}\rangle : \approx \int d^2w D_{\bar{w}}^3 V^{\bar{w}} \langle \mathrm{out}, f| \lim_{E\to 0}(1+E\partial_E)a_+(E,w,\bar{w})\mathcal{S}|\mathrm{in}\rangle, \tag{B3}$$

where $\langle \mathrm{out}, f|$ is the "out" state defined over the shifted vaccuum parametrized by f, generated by the action of supertranslation charge (Q_f^{soft}) on the Fock vaccuum.

In going from (4.17) to (4.18) we have made the same assumption for defining Q_V^{soft} on the shifted vacuum as has been made in the literature for defining it on the Fock vacuum, namely,

$$\langle \mathrm{out}, f| \lim_{E\to 0}(1+E\partial_E)a_+(E,w,\bar{w}) \coloneqq \lim_{E\to 0}(1+E\partial_E)\langle \mathrm{out}, f|a_+(E,w,\bar{w}). \tag{B4}$$

However, for reasons which can be traced back to the classical theory, it is still not clear what the precise definition of Q_V^{soft} is. That is, just as a rigorous definition of Q_f^{soft} being defined as an operator which maps the ordinary Fock vacuum to a supertranslated state [41,42], no corresponding definition is available for Q_V^{soft} as yet. Consequently, operator insertions like $\langle \mathrm{out}|Q_V^{\mathrm{soft}} Q_f^{\mathrm{soft}} \mathcal{S}|\mathrm{in}\rangle$ are not mathematically well defined, and we do not know how to make sense of them.

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