Inverse problem for gravitational waves by three-body system in Lagrange's orbit

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Abstract

We study an inverse problem for gravitational waves by three-body systems in Lagrange's orbit. We present a method of determining the parameters such as each mass, source distance and orbital inclination angle from gravitational waves observations alone. A binary source test is also discussed.

1 Introduction

"*Can one hear the shape of a drum?*" is a well-known question posed by Mark Kac in 1966 [1], where to "hear" the shape of a drum is to infer information about the shape of the drumhead from the sound it makes. This question can be traced back to Hermann Weyl [2, 3]. Now, it is interesting to pose a gravitational-wave *inverse* problem for the forthcoming gravitational-wave astronomy by ground-based or space-borne detectors. To "hear" a source through gravitational-wave observations is to extract the information about the source from the gravitational waves it makes.

It is of general interest to ask "*can one tell how many apples are falling in the dark of night?*" One simpler question is how and whether two-body and three-body gravitating systems can be distinguished through observations of gravitational waves that are made by these sources.

Continuing work initiated in an earlier publication [Torigoe et al. Phys. Rev. Lett. **102**, 251101 (2009)], gravitational wave forms for a three-body system in Lagrange's orbit are considered especially in an analytic method. We derive an expression of the three-body wave forms at the mass quadrupole, octupole and current quadrupole orders. By using the expressions, we solve a gravitational-wave *inverse* problem of determining the source parameters to this particular configuration (three masses, a distance of the source to an observer, and the orbital inclination angle to the line of sight) through observations of the gravitational wave forms alone. We discuss also whether and how a binary source can be distinguished from a three-body system in Lagrange's orbit or others.

2 Three-body system and gravitational waves

We consider a three-body system in Lagrange's orbit, where $m_{\rm tot}$ denotes the total mass, a denotes the length of an equilateral triangle, ν_i means mass ratios, ω is the orbital angular frequency, r is a source distance from an observer, and i defines the orbital inclination angle [5]. We obtain the quadrupolar part of gravitational waves as [5]

$$
\begin{aligned}
r \times h_{\rm Q}^{+} &= -m_{\rm tot} a^2 \omega^2 (1+\cos^2 i) \\
&\quad \times \left[\Big(\nu_1(\nu_2+\nu_3) - 2\nu_2\nu_3 \Big) \cos 2\omega t + \sqrt{3}\nu_1(\nu_2-\nu_3) \sin 2\omega t \right], \qquad (1)
\end{aligned}
$$

$$
\begin{aligned}
r \times h_{\rm Q}^{\times} &= -2 m_{\rm tot} a^2 \omega^2 \cos i \\
&\quad \times \left[\Big(\nu_1(\nu_2+\nu_3) - 2\nu_2\nu_3 \Big) \sin 2\omega t - \sqrt{3}\nu_1(\nu_2-\nu_3) \cos 2\omega t \right]. \qquad (2)
\end{aligned}
$$

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We also obtain the plus and cross modes of the mass octupolar waves as

$$
\begin{aligned}
r \times h^{+}_{\rm Oct} &= -\frac{1}{12} m_{\rm tot}^2 \omega \sin i \\
&\times \Big[27(1+\cos^2 i) \left(3^{3/2} \nu_1 \nu_2 \nu_3 \cos 3\omega t + (\nu_1 - \nu_2)(\nu_2 - \nu_3)(\nu_3 - \nu_1) \sin 3\omega t \right) \\
&+ (1 - 3\cos^2 i) \left(\frac{\sqrt{3}}{2} \nu_1 \{ \nu_2(\nu_2 - \nu_1) + \nu_3(\nu_3 - \nu_1) \} \cos \omega t \right. \\
&\left. - \frac{1}{2} (\nu_2 - \nu_3) \{ (\nu_1 - \nu_2)(\nu_1 - \nu_3) - 3\nu_2\nu_3 \} \sin \omega t \right) \Big],
\end{aligned}
\tag{3}
$$

$$
\begin{aligned}
r \times h^{\times}_{\rm Oct} &= -\frac{1}{12} m_{\rm tot}^2 \omega \sin 2i \\
&\times \Big[27 \left(3^{3/2} \nu_1 \nu_2 \nu_3 \sin 3\omega t - (\nu_1 - \nu_2)(\nu_2 - \nu_3)(\nu_3 - \nu_1) \cos 3\omega t \right) \\
&- \left(\frac{\sqrt{3}}{2} \nu_1 \{ \nu_2(\nu_2 - \nu_1) + \nu_3(\nu_3 - \nu_1) \} \sin \omega t \right. \\
&\left. + \frac{1}{2} (\nu_2 - \nu_3) \{ (\nu_1 - \nu_2)(\nu_1 - \nu_3) - 3\nu_2\nu_3 \} \cos \omega t \right) \Big].
\end{aligned}
\tag{4}
$$

Here, we consider current quadrupolar waves.

$$
\begin{aligned}
r \times h^{+}_{\rm C} &= \frac{4}{3} m_{\rm tot}^2 \omega \sin i \\
&\times \Big[\frac{\sqrt{3}}{2} \nu_1 \{ \nu_2(\nu_2 - \nu_1) + \nu_3(\nu_3 - \nu_1) \} \cos \omega t \\
&- \frac{1}{2} (\nu_2 - \nu_3) \{ (\nu_1 - \nu_2)(\nu_1 - \nu_3) - 3\nu_2\nu_3 \} \sin \omega t \Big],
\end{aligned}
\tag{5}
$$

$$
\begin{aligned}
r \times h^{\times}_{\rm C} &= \frac{2}{3} m_{\rm tot}^2 \omega \sin 2i \\
&\times \Big[\frac{\sqrt{3}}{2} \nu_1 \{ \nu_2(\nu_2 - \nu_1) + \nu_3(\nu_3 - \nu_1) \} \sin \omega t \\
&+ \frac{1}{2} (\nu_2 - \nu_3) \{ (\nu_1 - \nu_2)(\nu_1 - \nu_3) - 3\nu_2\nu_3 \} \cos \omega t \Big].
\end{aligned}
\tag{6}
$$

Both the mass octupolar and current quadrupolar parts are proportional to $m_{\rm tot}^2 \omega$. Hence they can be combined as

$$
\begin{aligned}
r \times h^{+}_{\rm Oct+C} &= -\frac{1}{4} m_{\rm tot}^2 \omega \sin i \\
&\times \Big[9(1+\cos^2 i) \left(3^{3/2} \nu_1 \nu_2 \nu_3 \cos 3\omega t + (\nu_1 - \nu_2)(\nu_2 - \nu_3)(\nu_3 - \nu_1) \sin 3\omega t \right) \\
&- (5 + \cos^2 i) \left(\frac{\sqrt{3}}{2} \nu_1 \{ \nu_2(\nu_2 - \nu_1) + \nu_3(\nu_3 - \nu_1) \} \cos \omega t \right. \\
&\left. - \frac{1}{2} (\nu_2 - \nu_3) \{ (\nu_1 - \nu_2)(\nu_1 - \nu_3) - 3\nu_2\nu_3 \} \sin \omega t \right) \Big],
\end{aligned}
\tag{7}
$$

and

$$
\begin{aligned}
r \times h^{\times}_{\mathrm{Oct+C}} \quad = \quad & -\frac{1}{4} m_{\mathrm{tot}}^2 \omega \sin 2i \\
& \times \Big[9 \Big(3^{3/2} \nu_1 \nu_2 \nu_3 \sin 3\omega t - (\nu_1 - \nu_2)(\nu_2 - \nu_3)(\nu_3 - \nu_1) \cos 3\omega t \Big) \\
& -3 \left(\frac{\sqrt{3}}{2} \nu_1 \{ \nu_2 (\nu_2 - \nu_1) + \nu_3 (\nu_3 - \nu_1) \} \sin \omega t \right. \\
& \left. + \frac{1}{2} (\nu_2 - \nu_3) \{ (\nu_1 - \nu_2)(\nu_1 - \nu_3) - 3 \nu_2 \nu_3 \} \cos \omega t \right) \Big] . \qquad (8)
\end{aligned}
$$

Here, we can find (in principle) measurable quantities: Phase differences and amplitudes of each wave mode [5].

3 Conclusion

Figure shows a schematic flowchart toward the parameter determinations and binary source test. Further investigations are needed in order to extend to arbitrary number of particles, other orbits or alternative theories of gravity more interestingly.

References

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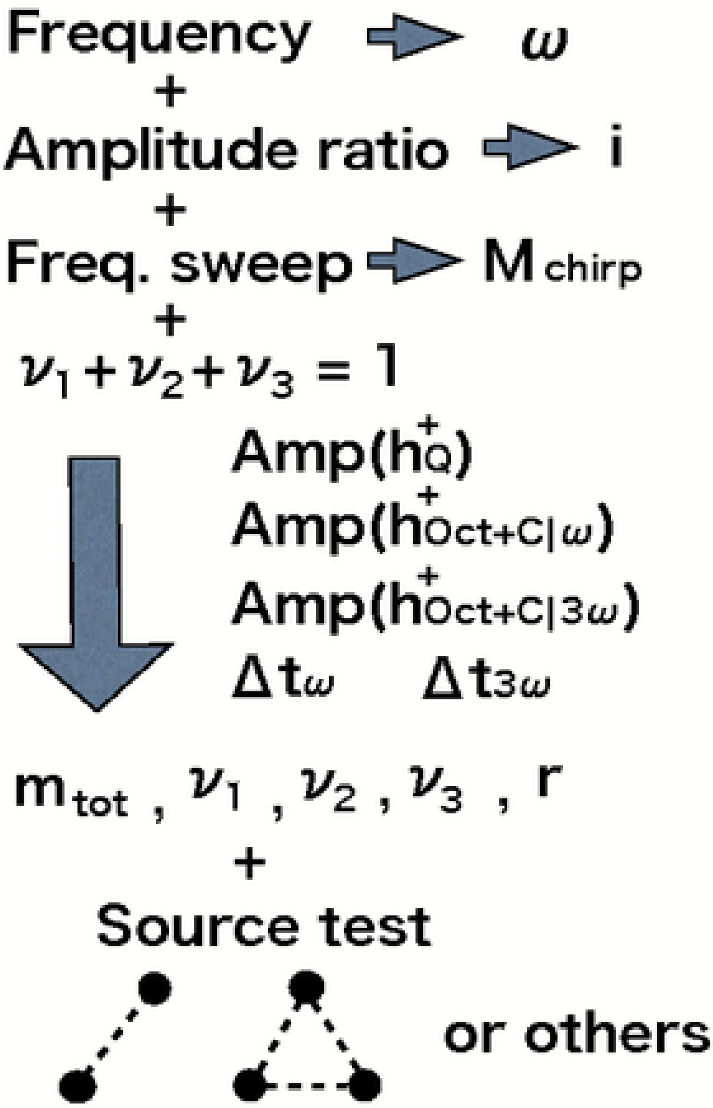

Figure 1: Flow chart of the parameter determinations and source tests. By using equations derived in [5], the source parameters can be determined through gravitational-wave observations alone. In addition, a binary source can be distinguished from a three-body system in Lagrange's orbit or others.