

# Interval estimation of bounded parameters

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## Abstract.

We consider the construction of interval estimates for the parameters with one-sided constraints. We show that the so-called method of sensitivity limit yields a correct solution of the problem. Derived are the solutions for the cases of a continuous distribution with non-negative estimated parameter and a discrete distribution, specifically a Poisson process with background. For both cases, the best upper limit is constructed that accounts for the a priori information. Particular applications to the neutrino mass measurements, rare processes (neutrinoless double beta-decay etc.) searches and cosmic ray studies are discussed.

## 1. Introduction

The Neyman construction [1] of confidence intervals for estimated parameters is a basic element of experimental data processing. Often one also possesses a priori information about the estimated parameters, and it is important to include that information into the confidence intervals in a consistent way. A limited domain of the parameters is an example of such a priori information. The problem with the conventional confidence intervals is seen if the experimental estimate of the parameter falls out of the domain. For instance, in the Troitsk- $\nu$ -mass experiment on the direct measurement of the mass of neutrino in tritium  $\beta$ -decay [2],[3] the neutrino mass squared is non-negative while the formal fit yields a negative value. The construction of confidence intervals for Poisson distribution with Poisson-distributed background is another situation where one should take into account the a priori information about the background. The situation is usual for studying rare events (in experiments on neutrinoless double  $\beta$ -decay [4], and neutrino oscillations, for instance T2K, MINOS [5], etc.) and in astrophysical observations [6]. In Section 2 of this paper we briefly remind the concept of frequentist approach to interval estimation using the Neyman construction. In Section 3 we consider various solutions to the problem of estimating the bounded parameters, indicating their advantages and possible drawbacks. Sections 4 and 5 present three examples in neutrino physics and cosmic rays of using the sensitivity limit method for the continuous and discrete distributions correspondingly. Section 6 contains the concluding remarks on the use of the sensitivity limit method.

## 2. Neyman confidence intervals

Parameter estimation starts from choosing a function of experimental data  $X$  termed “estimator”. The estimator  $\hat{\theta} = \hat{\theta}(X)$  for a parameter  $\theta$  is a variable with distribution  $P_{\theta}(\hat{\theta})$  (see



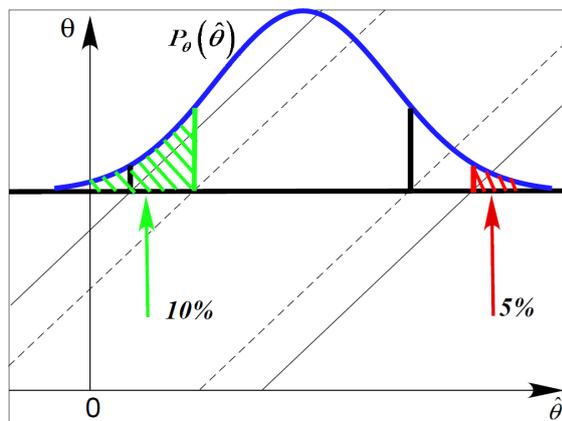
Fig.1). The probability density is considered known; it contains the whole information about the experiment, i.e. the set of control and apparatus parameters and the estimation method.

To construct an  $\alpha$ -level (c.f. 90%) left one-sided interval for the parameter  $\theta$  one defines for each value of  $\theta$  the corresponding value of  $\hat{\theta}_\alpha$  for which the following condition is satisfied:  $P_\theta(\hat{\theta} > \hat{\theta}_\alpha) = \alpha$ . All the values of  $\theta$  to the right of the  $\hat{\theta}_\alpha$  (upper arrow in Fig.2) form a so-called acceptance region. The construction for the symmetrical two-sided 90% interval gives the acceptance region shown in Fig.2 by the lower arrow. Constructing the acceptance region for all  $\theta$  usually yields a so-called confidence belt (between the solid lines in Fig.2).

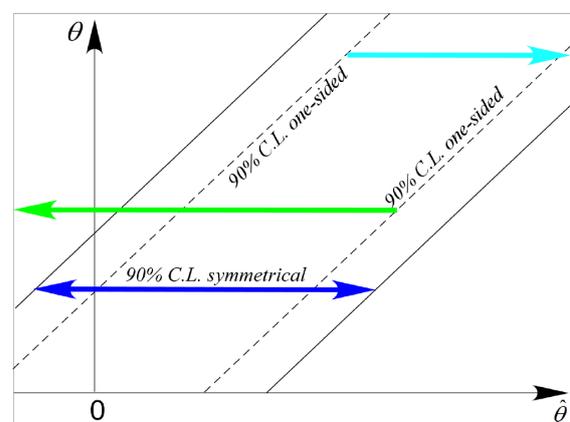
The last step in finding a confidence interval is calculating the value of  $\hat{\theta}^{exp}$  - the actual value of estimator on the measured experimental data. The two intersections of vertical line  $\hat{\theta} = \hat{\theta}^{exp}$  define the confidence interval.

The two important features should be emphasized. First, the confidence interval is stochastic by its nature, it depends on measured data and by its definition contains (or covers) the true value of the estimated parameter in  $\alpha$  percent of all measurements/experiments. Second, the described construction procedure has an inherited freedom: one can continuously modify the boundaries of the confidence belt in the horizontal direction, i.e. along the  $\hat{\theta}$  axis. For more details refer to [10],[11].

To construct a confidence interval for a parameter of a discrete distribution one has to weaken the condition  $P_\theta(\hat{\theta} > \hat{\theta}_\alpha) = \alpha$  substituting the equality with inequality  $P_\theta(\hat{\theta} > \hat{\theta}_\alpha) \geq \alpha$ . That leads to conservative intervals (the acceptance region can contain more probability than chosen value  $\alpha$ ).



**Figure 1.** Schematic presentation of distribution of estimator  $\hat{\theta}$  and its dependence on the real value of the parameter  $\theta$ .



**Figure 2.** Schematic presentation of constructing one-sided and symmetrical confidence intervals.

### 3. Estimation of bounded parameters

In many cases one possesses some additional information about the estimated parameter. An example of such a priori knowledge is a known boundary for the values of the parameter, for instance if the parameter is non-negative. Another example is a known background in a poissonian process. A priori information on the parameters should be used in construction of the confidence intervals to get a “better” parameter estimation.

There were several solutions suggested. They could be divided into two groups according to the way of incorporating the additional information. The first group of candidate solutions uses the information while constructing the acceptance region. Here belong the so called CCGV

constructions (or Power Constrained limits) [8] and the Feldman-Cousins recipe [7]. Including a priori information into the construction of the acceptance region could lead to undesirable features of the resulting confidence belts such as overcoverage and undercoverage (see [7], [11] for the details). The Feldman-Cousins recipe, though formally correct, yields a shrinking confidence interval for a non-negative parameter, so the more negative is the estimate, the narrower is the interval.

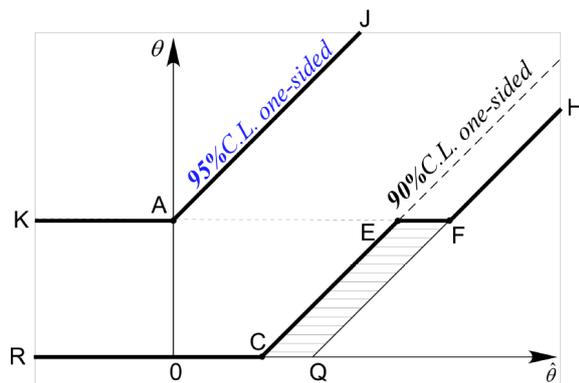
The second class of solutions incorporates the a priori information into the estimator itself. It was shown to be more consistent with respect to comparability of the resulting intervals [11]. The particular case of maximal likelihood estimation method was considered in [9]. The full solution (so-called *sensitivity limit method*), independent from the estimation method, was given in [10] and extended to the case of discrete distributions in [11].

#### 4. Sensitivity limit method

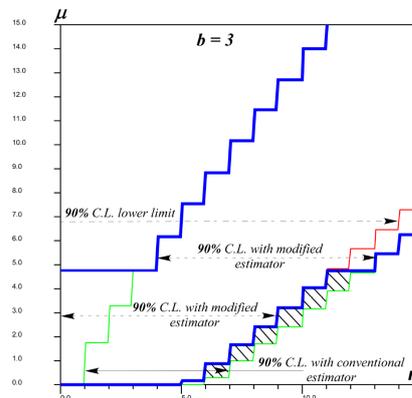
The comprehensive solution of the issue of constructing confidence intervals for a parameter of a continuous distribution with a priori information about the limited domain of the parameter was presented in [10]. The essential quality of an estimator of a parameter is that its value is as close to the true value of the parameter as possible. Therefore one defines a new estimator:

$$\tilde{\theta} = \max(\hat{\theta}, 0) \tag{1}$$

for the non-negative parameter  $\theta \geq 0$ . The corresponding confidence belt is presented in Fig.3.



**Figure 3.** 90% C.L. confidence interval for a non-negative parameter  $\theta$  constructed via the sensitivity limit method with the estimator in the form Eq.1.



**Figure 4.** Constructing the 90% C.L. confidence interval for a parameter of the Poisson distribution with known background  $b = 3$  using the sensitivity limit estimator in the form of Eq.2.

One can extend the above method to the case of discrete distributions. For instance, the experiments on detecting the rare processes (neutrinoless double  $\beta$ -decay etc.) usually deal with a poissonian signal and poissonian background that is measured or estimated separately and can be considered as known. The number of measured events  $n$  in such a case is governed by a Poisson distribution:  $P_{\mu}(n) = \frac{(\mu+b)^n}{n!} e^{-(\mu+b)}$ , where  $\mu$  is the signal and  $b$  is the known background value. Following [11] one chooses a proper estimator for the parameter  $\mu + b$ :

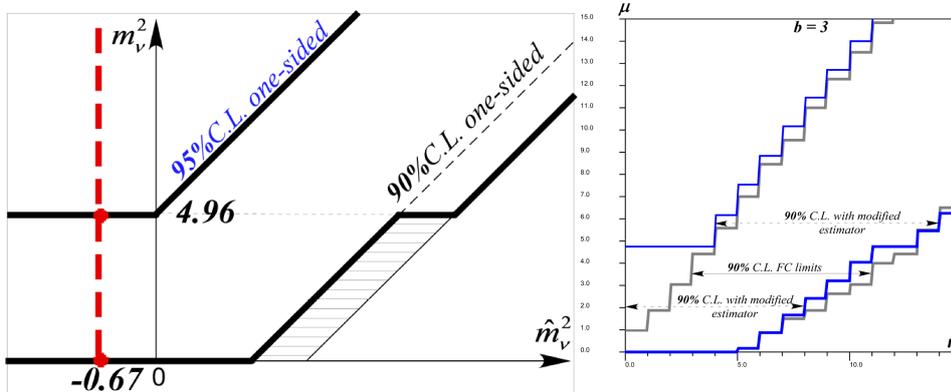
$$(\widetilde{\mu + b}) = \max(n, b), \tag{2}$$

that prevents the estimated value of  $\mu$  to go below 0. The corresponding scheme of the confidence belt for the estimator Eq.2 is given in Fig.4.

Both sensitivity limit constructions (see Fig.3 and 4) treat the unphysical values of the original estimators in the following way: for all the values of estimator  $\hat{\theta}$  below 0 one receives the same interval, these values are not differentiated. Similarly, if the measured number of events fall below the expected/known background, the sensitivity limit estimator yields the same value that corresponds to the sensitivity of the experiment. That feature of the sensitivity limit intervals allows one to directly compare the results of different experiments which is the goal of producing the confidence intervals in the first place. In the following examples it is shown that other recipes (like Feldman-Cousins and CCGV) do not always produce comparable results.

### 5. Neutrino mass estimation via sensitivity limit method

The first example of the sensitivity limit method application comes from the direct neutrino mass measurements performed in the Troitsk- $\nu$ -mass experiment [2]. The estimated parameter - neutrino mass squared - is non-negative, but its estimator was constructed in a way to include also negative values. As a result of the measurement and fitting one obtains [2]  $m_\nu^2 = -0.67 \pm 2.53 \text{ eV}^2$ . With the estimator 1 one constructs the confidence belt as presented in Fig.5. Using the plot and the estimated value of  $m_\nu^2 = -0.67 \text{ eV}^2$  one obtains the confidence interval upper and lower limits as intersections of a vertical line  $m_\nu^2 = -0.67 \text{ eV}^2$  with the confidence belt boundaries. For the given value of the error the interval is  $0 \leq m_\nu^2 \leq 4.96 \text{ eV}^2$ . Finally, taking a square root one arrive at the estimate of  $0 \leq m_\nu \leq 2.2 \text{ eV}$  at 95% C.L. The value could be directly compared to the estimate of the neutrino mass from the Mainz experiment [3] ( $0 \leq m_\nu \leq 2.4 \text{ eV}$  at 95% C.L., obtain via the same steps as above).



**Figure 5.** The use of sensitivity limit estimator and confidence belt for the Troitsk- $\nu$ -mass neutrino mass estimation.

**Figure 6.** 90% C.L. confidence intervals for a parameter of the Poisson distribution with known background  $b = 3$  with the sensitivity limit estimator in the form of Eq.2 and in the Feldman-Cousins procedure [12].

### 6. Problem of lower then the background number of events

The problem of measuring less events than the expected/known background can be considered in light of the recent results of GERDA collaboration [4]. The search for the neutrinoless double  $\beta$ -decay entered a new era of background free measurements. The new generation of experiment aims at achieving the level of background that is less than one event in the lifetime of the experiment. The expected background in GERDA is  $b \simeq 0.97 \text{ count} \leq 1 \text{ count}$ . The number of measured events is 0.

In that case it is worth comparing the confidence intervals, given by the Feldman-Cousins recipe [7], [12] and the sensitivity limit method (see also Fig.6). For the given values of  $b$  and

$n$  the implementation of Feldman-Cousins recipe in a special software [12] yields the 90% *C.L.* limit on the signal rate as  $\mu \leq 1.6$  events. The sensitivity limit method yields the limit of  $\mu \leq 2.0$  events at the same *C.L.* The considerable difference comes from the unphysical shrinking of the Feldman-Cousins intervals in the region  $n < b$  while the sensitivity limit method provides a robust estimate.

The shrinking intervals of Feldman-Cousins recipe do not allow a direct comparison of the limits and can lead to rather confusing results. Consider the derivation of the upper limits on the integral diffuse gamma-ray fluxes estimated in Table I of [6]. For instance, for the photon energies above  $E_{min} = 6 \cdot 10^{16}$  eV the measured number of events is  $n_{obs} = 29$  while the expected background is  $b = 42.6$ . The Feldman-Cousins recipe yields an extremely strong limit on the signal events  $n_{\gamma}^{FC} = 2.46$  which does not seem reliable since it, probably, exceeds the sensitivity of the measurement. The method of sensitivity limit [11],[12] provides a more reliable estimate of the number of signal events:  $n_{\gamma}^{SensLim} = 11.7$  at 90% *C.L.*. The estimate  $n_{\gamma}^{SensLim}$  also answers the given background rate of 42.6 events.

## 7. Conclusions

We considered a problem of constructing the confidence intervals for bounded parameters (limited to non-negative values, poissonian process with a priori known value of background etc.). It was shown that the sensitivity limit method consistently incorporates the a priori information into the estimator of the parameter. In comparison to the other proposed solutions the method: a) provides the explicit formula for the estimator; b) yields the so-called sensitivity limit for the unphysical values of the estimator; c) provides a confidence belt constructed within the frequentist approach via usual Neyman procedure.

The main feature of the sensitivity limit method is the comparability of results of different experiments. The estimation is robust; it treats the non-physical values of the estimator correctly. It solves the problem of less than the background number of events in the case of discrete distributions. The presented examples show how the method is applied to various problems in particle physics and illustrate the comparability of the resulting intervals which is, possibly, the main advantage and the main goal of the presenting the experimental results in form of confidence intervals.

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## References

- [1] Neyman 1937 *Phil. Trans. R. Soc. Lond. A* **236** 333-380
- [2] V.N. Aseev et al. 2011 *Phys. Rev. D* **84** 112003
- [3] Ch. Kraus et al. 2005 *Eur.Phys.J. C* **40** 447
- [4] GERDA Collaboration 2017 *Nature* **544** 7648 pp5-132
- [5] K. Abe et al. 2011 *Phys. Rev. Lett.* **107** 041801, P. Adamson et al. 2011 *Phys. Rev. Lett* **107** 181802
- [6] Yu.A. Fomin et al. 2017 *Phys. Rev. D* **95** 123011
- [7] G. J. Feldman and R. D. Cousins 1998 *Phys. Rev. D* **57** 3873
- [8] G. Cowan, K. Cranmer, E. Gross, O. Vitells 2011 *arXiv:1105.3166*
- [9] M. Mandelkern, J. Schultz 2000 *J.Math.Phys.* **41** 5701-5709
- [10] F. Tkachov 2009 *arXiv: 0911.4271; arXiv:0912.1555*
- [11] A.V. Likhov, F.V. Tkachov 2015 *Phys.Part.Nucl.* **46** 347-365
- [12] <http://www.inr.ac.ru/blackbox/stat/intervals/>