

The time-oriented boundary states and the Lorentzian-spinfoam correlation functions

Eugenio Bianchi^{1,2} and You Ding^{1,3}

¹ Centre de Physique Théorique de Luminy, Case 907, F-13288 Marseille, France

E-mail: ebianchi@perimeterinstitute.ca, dingy@bjtu.edu.cn

Abstract. A time-oriented semiclassical boundary state is introduced to calculate the correlation function in the Lorentzian Engle-Pereira-Rovelli-Livine spinfoam model. The resulting semiclassical correlation function is shown to match with the one in Regge calculus in a proper limit.

To test the semiclassical behavior of the Engle-Pereira-Rovelli-Livine (EPRL) spinfoam amplitude [1], we compute the two-point correlation functions for the Penrose metric operator in the Lorentzian signature. The setting is the one introduced in [2] and developed in [3, 4]. In particular, the correlation functions are calculated on a *time-oriented* semiclassical state [4], which is peaked on the space-like boundary geometry of a Lorentzian 4-simplex.

1. The time-oriented boundary states

The boundary of a Lorentzian 4-simplex consists of five space-like tetrahedra a which meet at ten triangles (ab) ($a, b = 1\dots 5$ and $a < b$). Suppose all the time-like normals to the tetrahedra are outward-pointing, then the tetrahedra can be divided into two types: the time-like normals to the tetrahedra are either future-pointing or past-pointing. Each triangle, with the two adjoining tetrahedra, defines a *wedge*. These wedges are then classified into two types: it is called in [5, 6] *thick wedge* if the incident tetrahedra are of same pointing type, which means both future-pointing or both past-pointing, otherwise called *thin wedge*. Let us assign a quantity to the triangle (ab) to denote this classification:

$$\Pi_{ab} = \begin{cases} 0, & \text{thick wedge;} \\ \pi, & \text{thin wedge.} \end{cases} \quad (1)$$

Consider the complete graph Γ dual to the boundary of the 4-simplex, with five nodes a dual to tetrahedra and ten links (ab) dual to the corresponding meeting triangles. Consider the group $SU(2)$, the *spin network states* $|\Gamma, j_{ab}, i_a\rangle$ supported on this graph are given by coloring each link (ab) with an irreducible representation j_{ab} (a spin), and coloring each node a with an $SU(2)$ intertwiner i_a . These spin network states span the truncated $SU(2)$ Hilbert space \mathcal{H}_Γ of loop quantum gravity (LQG). There is another (overcomplete) basis of \mathcal{H}_Γ , which

² Present address: Perimeter Institute, 31 Caroline St N, Waterloo ON, Canada N2L 2Y

³ Present address: Department of Physics, Beijing Jiaotong University, 100044 Beijing, China

is *Lorentzian coherent spin network states* with nodes labeled by *Lorentzian Livine-Speziale coherent intertwiners* [4]:

$$|\Gamma, j_{ab}, \Upsilon_a(\vec{n})\rangle = \exp\left(-i \sum_{(a < b)} \Pi_{ab} j_{ab}\right) |\Gamma, j_{ab}, \Phi_a(\vec{n})\rangle. \quad (2)$$

Notation is as follows. On the right hand side, $\Phi_a(\vec{n})$ denotes the Euclidean coherent intertwiner between the representations j_{ab} ($b \neq a$) [7],

$$\Phi_a(\vec{n}) = \int_{SU(2)} dh \prod_{b \neq a} \langle j_{ab}, m_{ab} | h | j_{ab}, \vec{n}_{ab} \rangle \quad (3)$$

where the integral measure dh is the $SU(2)$ Haar measure and $|j_{ab}, \vec{n}_{ab}\rangle$ denotes the $SU(2)$ coherent state in the spin- j representation to the link (ab) of the boundary graph Γ ; these coherent states $|j, \vec{n}\rangle$ are peaked on the geometry of a classical triangle: \vec{n} are associated to unit-normals to triangles of a tetrahedron, and j areas of the triangles. We assume all the normals outward to the tetrahedron, which satisfy $j_1 \vec{n}_1 + j_2 \vec{n}_2 + j_3 \vec{n}_3 + j_4 \vec{n}_4 = 0$, thus we associate to each triangle (ab) normals $-\vec{n}_{ab}$ when a is target of the triangle and respectively \vec{n}_{ba} when b is the source. Given a tetrahedron a , the *Lorentzian-geometry phase* $\exp(-i \sum_{b > a} \Pi_{ab} j_{ab})$ maps the Euclidean coherent intertwiners into the Lorentzian ones [4]:

$$\Upsilon_a(\vec{n}) = \exp\left(-i \sum_{b > a} \Pi_{ab} j_{ab}\right) \Phi_a(\vec{n}), \quad (4)$$

where the Lorentzian coherent intertwiner $\Upsilon(\vec{n})$ in the l.h.s. is defined by time reversing the past-pointing tetrahedra of the Euclidean coherent intertwiner $\Phi(\vec{n})$:

$$\Upsilon(\vec{n}) := \begin{cases} \Phi(\vec{n}), & \text{for future-pointing tetrahedra;} \\ T\Phi(\vec{n}), & \text{for past-pointing tetrahedra.} \end{cases} \quad (5)$$

Given a Euclidean coherent intertwiner in equation (3), the effect of time reversal T on the (past-pointing) coherent intertwiner is defined as

$$T\Phi_a(\vec{n}) := \int_{SU(2)} dh \prod_{b \neq a} \langle j_{ab}, m_{ab} | h \tilde{T}_{ab} | j_{ab}, \vec{n}_{ab} \rangle, \quad (6)$$

where “thickener” \tilde{T}_{ab} associated to triangle (ab) time-reverses thin wedges but unchanges thick wedges:

$$\tilde{T}_{ab} | j_{ab}, \vec{n}_{ab} \rangle := \begin{cases} | j_{ab}, \vec{n}_{ab} \rangle, & \text{for thick wedge;} \\ (-1)^{j_{ab}} | j_{ab}, \vec{n}_{ab} \rangle, & \text{for thin wedge.} \end{cases} \quad (7)$$

The action (7) of the *thickener* \tilde{T} can be reexpressed as

$$\tilde{T}_{ab} | j_{ab}, \vec{n}_{ab} \rangle = \exp(-i \Pi_{ab} j_{ab}) | j_{ab}, \vec{n}_{ab} \rangle, \quad (8)$$

with Π_{ab} related to the Lorentzian geometry by equation (1), and thus equation (4) can be obtained by combining equations (5)-(8). The Lorentzian coherent spin network state $|\Gamma, j_{ab}, \Upsilon_a(\vec{n})\rangle$ with nodes labeled by Lorentzian coherent intertwiners $\Upsilon_a(\vec{n})$ is thus given by equation (2).

Before going to use the Lorentzian coherent spin network states to construct the boundary semiclassical states, we want to make clear two points: (i) The Lorentzian coherent intertwiner is obtained by time-reversing the past-pointing tetrahedra and unchanging the future-pointing ones; the effect of the time reversal on tetrahedra is given by thickener T_{ab} on each triangle (ab) . The action (7) of thickener \tilde{T}_{ab} is defined by “time-reversing” thin wedges and unchanging thick wedges. The action of the time-reversal on the thin wedge is motivated from the time reversal $T|j, \vec{n}\rangle = (-1)^j |j, -\vec{n}\rangle$ in quantum mechanics. However, there is a slight difference between them of the minus sign for \vec{n} . This minus sign is supposed to keep normals outward-pointing after time reversal. (ii) The two phases from the intertwiner relation (4) and from the spin network relation (2) are different, in the sense that a is fixed in the former but free in the latter. In fact, the latter phase is product of the five phase in the former form. However, here, we call both of them *Lorentzian-geometry phase*, if no confusion arises.

Now let us come to the Lorentzian semiclassical state superposed by the Lorentzian coherent spin network states. Let ϕ_o label the simplicial extrinsic curvature, which is an angle associated to the triangle shared by the tetrahedra; a Lorentzian semiclassical state peaked both on intrinsic and extrinsic geometry can be given by a superposition of Lorentzian coherent spin network states:

$$|\Psi_o\rangle = \sum_{j_{ab}} \psi_{j_o, \phi_o}(j) |j, \Upsilon(\vec{n})\rangle, \quad (9)$$

with coefficients $\psi_{j_o, \phi_o}(j)$ given by a gaussian times a phase [2],

$$\psi_{j_o, \phi_o}(j) = \exp\left(-i \sum_{ab} \gamma \phi_o^{ab} (j_{ab} - (j_o)_{ab})\right) \times \exp\left(- \sum_{ab, cd} \gamma \alpha^{(ab)(cd)} \frac{j_{ab} - (j_o)_{ab}}{\sqrt{(j_o)_{ab}}} \frac{j_{cd} - (j_o)_{cd}}{\sqrt{(j_o)_{cd}}}\right), \quad (10)$$

where the 10×10 matrix $\alpha^{(ab)(cd)}$ is assumed to be complex with positive definite real part. In the following we will use the Lorentzian semiclassical state (9) to calculate the Lorentzian two-point correlation function.

2. The EPRL correlation functions

The connected two-point correlation function G_{nm}^{abcd} on a semiclassical boundary state $|\Psi_o\rangle$ is defined as

$$G_{nm}^{abcd} = \langle E_n^a \cdot E_n^b E_m^c \cdot E_m^d \rangle - \langle E_n^a \cdot E_n^b \rangle \langle E_m^c \cdot E_m^d \rangle, \quad (11)$$

where $(E_n^a)_i$ is a flux operator through a surface f_{an} dual to the triangle between the tetrahedra a and n , parallel transported in the tetrahedron n . Here the dynamical expectation value of an operator \mathcal{O} on the semiclassical state $|\Psi_o\rangle$ is defined via

$$\langle \mathcal{O} \rangle = \frac{\langle W | \mathcal{O} | \Psi_o \rangle}{\langle W | \Psi_o \rangle}, \quad (12)$$

where W is the EPRL spinfoam amplitude [1]. The correlation function (11) can be reexpressed in terms of group integral [4]:

$$G_{nm}^{abcd} = \frac{\sum_j \psi_j \int d^4 g d^{10} z q_n^{ab} q_m^{cd} e^S}{\sum_j \psi_j \int d^4 g d^{10} z e^S} - \frac{\sum_j \psi_j \int d^4 g d^{10} z q_n^{ab} e^S}{\sum_j \psi_j \int d^4 g d^{10} z e^S} \frac{\sum_j \psi_j \int d^4 g d^{10} z q_m^{cd} e^S}{\sum_j \psi_j \int d^4 g d^{10} z e^S}. \quad (13)$$

Notation is as follows. The coefficient ψ_j is short for $\psi_{j_o, \phi_o}(j)$ given in equation (10). The group integral $d^4 g$ is short for the Haar measure of $SL(2, \mathbb{C})^4$, one per each tetrahedron, and one redundant integral is removed to obtain the finite integral [8]. We consider an irreducible

representation of the Lorentz group $SL(2, \mathbb{C})$ on the space $\mathcal{H}_{(k,p)}$ of homogeneous functions $f_m^j(z)^{(k,p)}$ on complex projective line \mathcal{P} , and the integral $d^{10}z$ is over one \mathcal{P} per each triangle, up to a factor $-(\langle Z_{ab}, Z_{ab} \rangle \langle Z_{ba}, Z_{ba} \rangle)^{-1}$, with $Z_{ab} = g_a^\dagger z$ and $Z_{ba} = g_b^\dagger z$. The “action” S is given by

$$S(g, z) = \sum_{(a < b)} \left(j_{ab} \log \frac{\langle J\xi_{ab}, Z_{ab} \rangle^2 \langle Z_{ba}, \xi_{ba} \rangle^2}{\langle Z_{ab}, Z_{ab} \rangle \langle Z_{ba}, Z_{ba} \rangle} + i\gamma j_{ab} \log \frac{\langle Z_{ba}, Z_{ba} \rangle}{\langle Z_{ab}, Z_{ab} \rangle} - i\Pi_{ab} j_{ab} \right), \quad (14)$$

with γ the Barbero-Immirzi parameter; the spinor $\xi \in \mathbb{C}^2$ is related to the coherent state $|j, \vec{n}\rangle$ in the sense that the corresponding $SU(2)$ group element $n(\xi)$ rotate the direction $(0, 0, 1)$ in sphere S^2 into the \vec{n} direction. The antipodal vector $-\vec{n}(\xi)$ can be associated to $J\xi$, i.e. $-\vec{n}(\xi) = \vec{n}(J\xi)$. The insertion q_n^{ab} is given by

$$q_n^{ab} = \gamma^2 j_{an} j_{bn} \frac{\langle \vec{\sigma} Z_{an}, \xi_{an} \rangle}{\langle Z_{an}, \xi_{an} \rangle} \cdot \frac{\langle \vec{\sigma} Z_{bn}, \xi_{bn} \rangle}{\langle Z_{bn}, \xi_{bn} \rangle}. \quad (15)$$

The large-spin asymptotics of the correlation function (13) can be obtained via stationary phase approximation [3, 6], which is given by (for details, see [4]):

$$G_{nm}^{abcd}(\alpha) = (\gamma j_o)^3 (R_{nm}^{abcd}(\alpha) + O(\gamma)) \quad (16)$$

with

$$R_{nm}^{abcd} = \frac{1}{\gamma^3 j_o^3} \sum_{p < q, r < s} Q_{(pq)(rs)}^{-1} \frac{\partial q_n^{ab}}{\partial j_{pq}} \frac{\partial q_m^{cd}}{\partial j_{rs}} \quad (17)$$

and

$$Q_{(ab)(cd)} = -\frac{\gamma \alpha^{(ab)(cd)}}{\sqrt{(j_o)_{ab}} \sqrt{(j_o)_{cd}}} + S''_{\text{Regge}}. \quad (18)$$

If we take the classical limit, introduced in [3], where the Barbero-Immirzi parameter is taken to zero $\gamma \rightarrow 0$, and the spin of the boundary state is taken to infinity $j \rightarrow \infty$, keeping the size of the quantum geometry $A \sim \gamma j$ finite and fixed, the two-point function (16) we obtain exactly matches the one obtained from Lorentzian Regge calculus [9].

Deriving the LQG correlation function at the level of a single spin foam vertex is certainly only a first step. Within the setting of a vertex expansion, an analysis of the LQG correlation function for an arbitrary number of spinfoam vertices is needed. It would be interesting to investigate the contribution of the γ -term to correlation functions when more than a single spin foam vertex is considered.

References

- [1] Engle J, Livine E, Pereira R and Rovelli C 2008 *Nucl. Phys.* **B799** 136–149 (*Preprint* 0711.0146)
- [2] Rovelli C 2006 *Phys. Rev. Lett.* **97** 151301 (*Preprint* gr-qc/0508124)
- [3] Bianchi E, Magliaro E and Perini C 2009 *Nucl. Phys.* **B822** 245–269 (*Preprint* 0905.4082)
- [4] Bianchi E and Ding Y 2011 (*Preprint* 1109.6538)
- [5] Barrett J W and Foxon T J 1994 *Class. Quant. Grav.* **11** 543–556 (*Preprint* gr-qc/9310016)
- [6] Barrett J W, Dowdall R J, Fairbairn W J, Hellmann F and Pereira R 2010 *Class. Quant. Grav.* **27** 165009 (*Preprint* 0907.2440)
- [7] Livine E R and Speziale S 2007 *Phys. Rev.* **D76** 084028 (*Preprint* 0705.0674)
- [8] Engle J and Pereira R 2009 *Phys. Rev.* **D79** 084034 (*Preprint* 0805.4696)
- [9] Regge T 1961 *Nuovo Cim.* **19** 558–571