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Free Electron Laser Supermodes for Bunches with Longitudinal Energy-Position Correlation*

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Summary. — In this note the theory of Free Electron Laser (FEL) Supermodes is extended to cover particle distributions with a correlation between energy and longitudinal position in a gaussian bunch. This extension is important for FEL's in storage rings, because wake forces stemming from discontinuities in the vacuum chamber as well as the FEL process itself give rise to such correlations.

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1. — Introduction

The longitudinal phase space of an electron bunch in a storage ring is usually described by the bunch length σ_z and the energy spread σ_ε. The equilibrium values of these quantities are determined by a balance of damping and quantum excitation stemming from the emission of synchrotron radiation. However, the presence of longitudinal wake fields lead to a correlation-between energy and position in the bunch, because the trailing particles loose energy due to the wake field the leading particles produce.

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In order to set up a self-consistent simulation of the coupled system of oscillator FEL and storage ring, an extension of the theory of FEL supermodes to encompass electron bunches with energy-position correlation is needed. This extension is presented in this paper. The description of the full coupled system and the simulation results are the subject of a forthcoming paper.

This paper is organized as follows. First, after a brief overview over the nature of the oscillator FEL process we will set up an eigenvalue equation that describes the spatial profiles of the supermodes. The next section is devoted to the solution of this equation employing a new technique using an ansatz of a distorted harmonic oscillator eigenfunctions and solving for a set of coefficients. In the final section physically relevant quantities, such as the gain, the length of the light pulse and the bandwidth are calculated. The proof of the biorthogonality of the supermodes with correlation is deferred to the appendix.

2. — The Eigenvalue Equation

In a FEL a bunched electron beam is passed through an undulator magnet which forces the electrons on a transversely sinusoidal path. In this way they can exchange energy with the fields of a copropagating electromagnetic wave. In an amplifier configuration this wave is externally provided, typically by a laser. In an oscillator configuration the spontaneously emitted synchrotron radiation in the undulator is fed back by mirrors. Thus it can interact with subsequent electron bunches or with the same bunch in a circular accelerator.

The optical resonator admits a large number of longitudinal modes. Their spacing is determined by the distance between the mirrors, typically of a few meters, leading to a frequency spacing of a few 10's of MHz. The FEL process couples the modes by an active mode locking mechanism and the modes interfere in such a way as to create light pulses that bounce between the mirrors. These light pulses are usually called Supermodes.

The number of coupled modes is mainly determined by the bunch length of the electron beam, because the bunch length determines the frequency content of the bunch spectrum. For a cm long bunch the frequency spectrum extends to a few GHz. The frequencies in the bunch spectrum are then used to bridge the gap between different resonator modes and couple them. An order of magnitude estimate of the number of coupled modes thus leads to 10^2 to 10^3.

The evolution equations for this large number of modes for a small signal-small gain oscillator FEL were derived in ref. [1]. Under the assumption of narrow mode spacing, the spatial fourier transform of these evolution equations led to a partial integro-differential equation for the longitudinal electric field profile $E(z,t)$ derived in ref. [2] as

$$(1)\, 2T_c \frac{\partial E(z,t)}{\partial t} + \left[\gamma_T + i g_0 \Theta(\nu_0 - 2\pi N)\right] E(z,t) + \Delta\Theta g_0 \frac{\partial E(z,t)}{\partial z}$$
$$= -i g_0 \frac{(2\pi)^{3/2}}{\mu_c \Delta^2} \int_0^\Delta dy\; y\; E(z+y,t) \int_{z+y}^{z+\Delta} dz_0 \int_{-\infty}^{\infty} d\varepsilon\; e^{4\pi i N \varepsilon y/\Delta} f(z_0,\varepsilon)\ .$$

where the temporal evolution is given on a time scale long compared to the cavity round trip time T_c. In eq. (1) the following abbreviations, consistent with ref. [2], are introduced

$$\Theta = \frac{\omega_0 \delta T}{\pi N g_0}$$

(2)

$$g_0 = 2\pi \sqrt{\frac{2\lambda}{l_w}}\ \frac{I_p}{I_0}\ \frac{L_w \lambda}{\Sigma_L}\ \frac{K'^2}{(1+K'^2)^{3/2}}\ (2N)^2\ .$$

Here Θ describes the detuning between the cavity round trip time and the recurrence time of the electron bunches and g_0 is the gain coefficient (here written for a helical wiggler). The other symbols are explained in Table 1. $f(z_0,\varepsilon)$ is the gaussian electron distribution function. It can be parametrized by its correlation matrix σ_{ij} with i and j being z or ε and can be written as

$$f(z_0,\varepsilon) = \frac{1}{2\pi\sqrt{\det\sigma}} \exp\left[-\frac{\sigma_\varepsilon^2}{2\det\sigma}\, z_0^2 + \frac{\sigma_{z\varepsilon}}{\det\sigma}\, z_0(\varepsilon-\varepsilon_0) - \frac{\sigma_z^2}{2\det\sigma}\,(\varepsilon-\varepsilon_0)^2\right]$$

(3)

with $\det\sigma = \sigma_\varepsilon^2\sigma_z^2 - \sigma_{z\varepsilon}^2$. Here σ_ε is the relative energy spread, σ_z is the bunch length and $\sigma_{z\varepsilon}$ describes the correlation between energy and position in the bunch. ε_0 describes the relative energy offset between the resonance energy of the FEL and the electron energy.

In order to simplify this integro-differential equation we follow the strategy outlined in ref. [2] and note that the gaussian integral over ε can be done immediately. Assuming that the light pulse is only situated around the peak of the electron distribution, because the gain is maximum there, we can expand the integrand of the z_0-integral in z_0/σ_z and then evaluate the z_0-integral. Upon expanding the electric field up to second order in the small quantity $y = x\Delta$ and keeping only terms up to second order in $\mu_c = \Delta/\sigma_z$ and z/σ_z we can rewrite eq. (1) as an eigenvalue equation

$$\lambda\phi(\tilde{z}) = \left[\Omega_0\frac{1}{4}\left(1 + 2\tilde{z}\frac{\partial}{\partial\tilde{z}}\right) + \Omega_1\left(\frac{1}{2}\tilde{z}^2\right) + \Omega_2\left(\frac{1}{2}\frac{\partial^2}{\partial\tilde{z}^2}\right) + \Omega_3\tilde{z} + \Omega_4\frac{\partial}{\partial\tilde{z}} + \Omega_5\right]\phi(\tilde{z}) \tag{4}$$

with

$$\begin{aligned}
\Omega_0 &= 2i\mu_c\pi\mu_s G_4 \\
\Omega_1 &= -G_1 - \pi^2\mu_s^2 G_4 \\
\Omega_2 &= \mu_c^2 G_4 \\
\Omega_3 &= -\frac{\mu_c}{2}\left(G_2 + \pi^2\mu_s^2 G_5\right) + i\pi\mu_s G_3 \\
\Omega_4 &= \mu_c(G_3 - \Theta) + i\frac{\mu_c^2}{2}\pi\mu_s G_5 \\
\Omega_5 &= G_1 - \frac{\gamma_T}{g_0} + i\frac{\mu_c}{2}\pi\mu_s G_3\ .
\end{aligned}$$

Here we have introduced scaled variables $\tilde{z} = z/\sigma_z$, $\tau = g_0 t/2T_c$ and redefined the electric field as $E(z) = \phi(\tilde{z})\exp\left[\tau\left(\lambda - i\Theta(\nu_0 - 2\pi N)\right)\right]$ where λ is the complex gain coefficient. Its real part determines the growth rate of the electric field and its imaginary part the phase shift introduced by the FEL interaction. $\mu_\varepsilon = 4N\sigma_\varepsilon$ parametrizes the energy spread and $\mu_s = 4N\sigma_{z\varepsilon}/\sigma_z$ parametrizes the correlation between position and energy in the bunch.

$\mu_D = \sqrt{\mu_\varepsilon^2 - \mu_s^2}$ is introduced to facilitate the writing. The functions $G_i(\nu_0, \mu_D)$ are given by the following expressions.

$$
\begin{aligned}
G_0(\nu_0,\mu_D) &= 2\pi \int_0^1 dx\ e^{i\nu_0 x - \pi^2\mu_D^2 x^2/2} \\
&= \frac{\sqrt{2\pi}}{\mu_D}\left[w\left(\frac{\nu_0}{\sqrt{2\pi}\mu_D}\right) - e^{i\nu_0 - \pi^2\mu_D^2/2}\ w\left(\frac{\nu_0}{\sqrt{2\pi}\mu_D} + i\frac{\pi\mu_D}{\sqrt{2}}\right)\right] \\
G_1(\nu_0,\mu_D) &= -\frac{\partial}{\partial\nu_0}\left(1 + i\frac{\partial}{\partial\nu_0}\right)\ G_0(\nu_0,\mu_D) \\
(5)\ G_2(\nu_0,\mu_D) &= \left(1 - i\frac{\partial}{\partial\nu_0}\right)\ G_1(\nu_0,\mu_D) \\
G_3(\nu_0,\mu_D) &= \left(-i\frac{\partial}{\partial\nu_0}\right)\ G_1(\nu_0,\mu_D) \\
G_4(\nu_0,\mu_D) &= -\frac{\partial^2}{\partial\nu_0^2}\ G_1(\nu_0,\mu_D) \\
G_5(\nu_0,\mu_D) &= -\left(1 - i\frac{\partial}{\partial\nu_0}\right)\ \frac{\partial^2}{\partial\nu_0^2}\ G_1(\nu_0,\mu_D)
\end{aligned}
$$

where $w(z)$ is the complex error function [3]. Note that all functions G_i can be expressed in terms of complex error functions and derivatives thereof. This makes their numerical evaluation very fast. We have introduced the adimensional detuning $\nu_0 = 4\pi N\varepsilon_0$.

The results given in eq. (4) and eq. (5) differ from those in ref. [2] in three ways. First, terms proportional to μ_s appear in the definition of the Ω_i, in particular a new coefficient Ω_0 appears which was previously zero. Second, a new G–function, namely G_5 is introduced. A plot of the real and imaginary part of the new function $G_5(\nu_0, \mu_D)$ for $\mu_D = 0$ is shown in Fig. 1. Plots of $G_1, \ldots, G_4$ can be found in ref. [2]. Third, all G–functions now depend on μ_D rather than on μ_ε. Note that in the limit $\sigma_{z\varepsilon} \to 0$, or equivalently, $\mu_s \to 0$ eqs. (4) and (5) reduce to those in ref. [2].

Equation 4 describes the influence of the electron bunch represented by the quantities $\nu_0, \mu_c, \mu_\varepsilon$ and μ_s on the spatial profile of the supermodes. The inclusion of the correlation μ_s made it slightly more complicated but the general structure of the equation remains the same already encountered in ref. [2].

The next section is devoted to the solution of eq. (4) using a new technique.

3. — The Solution

In ref. [2] an equation similar to eq. (4) was solved using algebraic techniques. Here we will employ a more heuristic approach that nevertheless reproduces the results from ref. [2] in the limit $\mu_s \to 0$. The new approach is based on the observation that the right hand side of eq. (4) looks like the Hamilton operator of a "torn and twisted, scaled and shifted" harmonic oscillator. The eigenfunctions of a harmonic oscillator are known to be gaussians with Hermite polynomials as forefactors [3]. Therefore, we use the ansatz

$$\phi_n(\tilde{z}) = H_n\left(a(\tilde{z} + b)\right)\ e^{-c(\tilde{z}+d)^2}\ . \tag{6}$$

We substitute this ansatz in eq. (4), and use the following relations among the Hermite polynomials in order to express derivatives and powers of Hermite polynomials by sums of Hermite polynomials.

$$\begin{aligned}
\dot{H}_n(x) &= 2n\, H_{n-1}(x) \\
\dot{H}_n(x) &= 2x\, H_n(x) - H_{n+1}(x) \\
\ddot{H}_n(x) &= 4n(n-1)\ H_{n-2}(x) \\
\ddot{H}_n(x) &= (4x^2 - 2n)\ H_n(x) - 2x\, H_{n+1}(x) \\
xH_n(x) &= \frac{1}{2}\ H_{n+1}(x) + nH_{n-1}(x) \\
x^2 H_n(x) &= \frac{1}{4}\ H_{n+2}(x) + \left(n + \frac{1}{2}\right)\ H_n(x) + n(n-1)\ H_{n-2}(x)
\end{aligned} \tag{7}$$

where dots denote differentiation with respect to x. Finally, we compare coefficients before the Hermite polynomials to obtain a coupled set of equations for the unknowns a, b, c, d, λ_n. The solution of this set of equations leads after some tedious algebra to the result [4]

$$
\begin{aligned}
a^2 &= \frac{1}{2\Omega_2}\sqrt{\Omega_0^2 - 4\Omega_1\Omega_2} = \frac{1}{\mu_c}\sqrt{\frac{G_1}{G_4}} \\
b &= -2\frac{2\Omega_2\Omega_3 - \Omega_0\Omega_4}{\Omega_0^2 - 4\Omega_1\Omega_2} = \frac{\mu_c G_2 - i2\pi\mu_s\Theta}{2G_1} \\
c &= \frac{1}{4\Omega_2}\left[\Omega_0 + \sqrt{\Omega_0^2 - 4\Omega_1\Omega_2}\right] = \frac{1}{2\mu_c}\left[\sqrt{\frac{G_1}{G_4}} + i\pi\ \mu_s\right] \\
d &= -2\frac{2\Omega_2\Omega_3 - \Omega_4(\Omega_0 + \sqrt{\Omega_0^2 - 4\Omega_1\Omega_2})}{\sqrt{\Omega_0^2 - 4\Omega_1\Omega_2}\left[\Omega_0 + \sqrt{\Omega_0^2 - 4\Omega_1\Omega_2}\right]} \\
&= \frac{\mu_c G_2 - 2i\pi\mu_s\Theta + 2\sqrt{G_1/G_4}\left[(G_3 - \Theta) + i\,\frac{\mu_c}{2}\,\pi\mu_s G_5\right]}{2(G_1 + i\pi\mu_s\sqrt{G_1G_4})} \\
\lambda_n &= \Omega_5 - \left(n + \frac{1}{2}\right)\frac{1}{2}\sqrt{\Omega_0^2 - 4\Omega_1\Omega_2} + 2\,\frac{\Omega_2\Omega_3^2 - \Omega_0\Omega_3\Omega_4 + \Omega_1\Omega_4^2}{\Omega_0^2 - 4\Omega_1\Omega_2} \\
&= G_1 - \frac{\gamma_T}{g_0} - \frac{(G_3 - \Theta)^2}{2G_4} - \pi^2\mu_s^2\,\frac{\Theta^2}{2G_1} - \mu_c\left(n + \frac{1}{2}\right)\sqrt{G_1G_4} \\
&\quad + i\frac{\mu_c}{2}\pi\mu_s\left[\Theta\left(\frac{G_5}{G_4} - \frac{G_2}{G_1}\right) + G_3\left(1 - G_5\frac{G_1 + \pi^2\mu_s^2 G_4}{G_1G_4}\right)\right] \\
&\quad + \mu_c^2\left[\frac{G_2^2}{8G_1} + \pi^2\mu_s^2\frac{G_5^2}{8G_4}\right]\ .
\end{aligned}
\tag{8}
$$

Here, a, b, c and d describe the eigenfunctions of eq. (4) by virtue of eq. (6) and λ_n is the complex gain coefficient that determines the growth rate of the nth supermode. The influence of the electron distribution function on the supermodes is buried in the dependence of eqs. (8) on the parameters ν_0, μ_s, μ_D and μ_c. In particular, note that the functions G_i depend explicitely on ν_0 and μ_D.

The growth rate Re$[\lambda_n]$ shows a characteristic quadratic dependence on the cavity detuning Θ, already reported in refs. [2]. This is shown in Fig. 2 where Re$[\lambda_0]$ is plotted as a function of Θ in the range $0 \leq \Theta \leq 1$ for $\mu_\varepsilon = 0.5$, $\mu_s = 0.2$ and $\mu_c = 5 \times 10^{-3}$. Re$[\lambda_0]$ assumes its maximum value for a non zero Θ in order to compensate for the lethargic effect [2] of the FEL interaction.

The dependence of the growth rate on the index n is weak, because n enters only to order μ_c in λ_n.

The maximum growth rate can be found analytically by differentiating the last of equations 8 with respect to Θ. For the maximum we obtain

$$\Theta_{max} = \mathrm{Re}\left[\frac{G_3 + i\mu_c\pi\mu_s(G_5 - G_2G_4/G_1)/2}{1+\pi^2\mu_s^2 G_4/G_1}\right] . \tag{9}$$

This optimum resonator detuning is related to the symmetry properties of the supermodes. In fact, from eq. (6) we see that we have $\phi_n(-(\tilde{z}+b)) = (-1)^n\phi_n(\tilde{z}+b)$ if we require $b = d$. By virtue of eqs. (8) this can be rewritten as $2\Omega_1\Omega_4 = \Omega_0\Omega_3$, which reduces after some manipulations to eq. (9). This fact leads us to the conclusion that the symmetric supermodes have the largest growth rate.

Fig. 3 shows the effect of μ_s on the growth rate $\mathrm{Re}[\lambda_0]$ of the fundamental supermode. Here ν_0 and the cavity detuning Θ are always adjusted to obtain maximum growth rate. $\mathrm{Re}[\lambda_0]$ is almost independent of μ_s, however, it shows a minimum for $\mu_s = 0$ and slightly increases for increasing $|\mu_s|$, because $\mu_D = \sqrt{\mu_\varepsilon^2 - \mu_s^2}$ is reduced and μ_D is proportional to the longitudinal emittance $\sqrt{\det\sigma}$. Increasing $|\mu_s|$ therefore reduces the randomness of the electron distribution and consquently increases the gain.

Having determined the spatial field profile and the gain of the supermodes we will calculate the physically relevant quantities, namely the length and the spectral bandwidth of the light pulse in the next section.

4. — Spatial and Spectral Characteristics of the Fundamental Supermode

The spatial and spectral characteristics of the fundamental supermode ϕ_0 which has, as is apparent from the last of eqs. (8), the largest growth rate can be determined from the spatial and spectral energy distribution functions. The spatial energy distribution for the fundamental supermode is given by

$$S(\tilde{z}) = \frac{\phi_0(\tilde{z})\ \phi_0^*(\tilde{z})}{\int_{-\infty}^{\infty} d\tilde{z}\ \phi_0(\tilde{z})\ \phi_0^*(\tilde{z})} \tag{10}$$

where the asterisk denotes complex conjugate. The physical relevant quantities are $< \tilde{z} >$ and $\tilde{\sigma}_L^2 =< \tilde{z}^2 > - < \tilde{z} >^2$ where the acute brackets denote averaging with respect to $S(\tilde{z})$. All needed integrals can be calculated from the generating function

$$(11) \qquad \begin{aligned} I_z(B) &= \int_{-\infty}^{\infty} d\tilde{z}\ \phi(\tilde{z})\ \phi^*(\tilde{z})\ e^{B\tilde{z}} \\ &= \sqrt{\frac{\pi}{2\mathrm{Re}\ [c]}}\ \exp\left[+\frac{2(\mathrm{Im}[d])^2}{\mathrm{Re}\ [1/c]} + \frac{B^2}{8\mathrm{Re}\ [c]} - \frac{\mathrm{Re}\ [cd]}{\mathrm{Re}\ [c]}B\right] \end{aligned}$$

by partial differentiation with respect to B. This leads to the longitudinal displacement of the light pulse relative to the maximum of the electron distribution of $< \tilde{z} >= -\mathrm{Re}[cd]/\mathrm{Re}[c]$ and the length of the lightpulse (both quantities are given in units of σ_z)

$$(12) \qquad \tilde{\sigma}_L = \frac{1}{2}\sqrt{\frac{1}{\mathrm{Re}\ [c]}} = \frac{\sqrt{\mu_c}}{\sqrt{2\mathrm{Re}\ [G_1/G_4]}}\ .$$

Upon fourier transforming the spatial profile of the electric field and utilizing the fact that the longitudinal resonator modes are densely spaced we can construct the spectral density function. From this the bandwidth $\tilde{\sigma}_\nu^2$ is calculated. We obtain

$$(13) \qquad \tilde{\sigma}_\nu^2 = \frac{\mu_c^2}{\mathrm{Re}\ [1/c]} = \mu_c \frac{|G_1/G_4| + 2\pi\mu_s \mathrm{Im}[\sqrt{G_1/G_4}] + \pi^2\mu_s^2}{2\mathrm{Re}\ [\sqrt{G_1/G_4}]}\ .$$

$\tilde{\sigma}_\nu$ is the bandwidth in the normalized units and is related to the physical bandwidth $\Delta\omega/\omega$ by $\Delta\omega/\omega = \tilde{\sigma}_\nu/2\pi N$.

Fig. 4 shows the length of the light pulse in units of the electron bunch length σ_z as a function of μ_s. Clearly the effect of μ_s is weak and symmetric. The dependence of the bandwidth $\tilde{\sigma}_\nu$ is shown in Fig. 5. Here the influence of μ_s is also weak but an asymmetry shows up which can be accounted for by the term linear in μ_s in the expression for $\tilde{\sigma}_\nu$.

5. — Conclusions

In this note we calculated the influence of a gaussian bunch with energy-position correlation on FEL supermodes using a new technique. The technique is based on the observation that the right-hand side of eq. (4) is quadratic in $\tilde{z}$ and $\partial/\partial\tilde{z}$ and therefore suggests an ansatz using harmonic oscillator eigenfunctions. The general result was then used to determine the gain, spatial and spectral characteristics of the fundamental supermode. The influence of the correlation, parametrized by $\mu_s = 4N\sigma_{z\varepsilon}/\sigma_z$ was shown to be weak.

These results developed in this paper can now be used to investigate the output characteristics of a FEL in a storage ring. For a full self-consistent simulation of the coupled system of FEL and storage ring the effect of the light pulse on the electron bunch has to be investigated. This undertaking will be the topic of a forthcoming paper.

In ref. [5] it was shown that the eigenfunctions of a "Hamilton" operator similar to the right hand side of eq. (4) are orthogonal to those of the adjoint operator $H^\dagger$ and thus form a biorthogonal set. We will complement the above analysis and prove in the appendix that this is still true for the general case given by eq. (4).

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Appendix: Supermode Biorthogonality

In this appendix we will investigate the mathematical properties of the Hamiltonian for the supermodes, defined by the right-hand side of eq. (4) more closely. To exhibit its group-theoretical properties it can be written as

$$H = \Omega_0 \hat{k}_0 + \Omega_1 \hat{k}_+ + \Omega_2 \hat{k}_- + \Omega_3 \hat{a}_+ + \Omega_4 \hat{a}_- + \Omega_5 \hat{1}\,. \tag{14}$$

Here the $\hat{k}$'s and $\hat{a}$'s are defined by

$$\begin{aligned} \hat{a}_+ &= \tilde{z} \\ \hat{a}_- &= \frac{\partial}{\partial \tilde{z}} \\ \hat{k}_0 &= \frac{1}{4}\left(1 + 2\tilde{z}\frac{\partial}{\partial \tilde{z}}\right) = \frac{1}{4}\left(\hat{a}_+\hat{a}_- + \hat{a}_-\hat{a}_+\right) \\ \hat{k}_+ &= \frac{1}{2}\,\tilde{z}^2 = \frac{1}{2}\,\hat{a}_+\hat{a}_+ \\ \hat{k}_- &= \frac{1}{2}\,\frac{\partial^2}{\partial \tilde{z}^2} = \frac{1}{2}\,\hat{a}_-\hat{a}_-\,. \end{aligned} \tag{15}$$

The $\hat{k}$'s obey the commutation relations of SU(1,1) [6]

$$\begin{aligned} \left[\hat{k}_+, \hat{k}_-\right] &= -2\hat{k}_0 \\ \left[\hat{k}_0, \hat{k}_+\right] &= \hat{k}_+ \\ \left[\hat{k}_0, \hat{k}_-\right] &= -\hat{k}_- \end{aligned} \tag{16}$$

and the $\hat{a}$'s obey the usual harmonic oscillator commutation relation

$$[\hat{a}_-, \hat{a}_+] = \hat{1}\;. \tag{17}$$

From eqs. (15) and (16) follows in a straight forward fashion that

$$
\begin{aligned}
\left[\hat{k}_0, \hat{a}_+\right] &= \frac{1}{2}\,\hat{a}_+ \\
\left[\hat{k}_0, \hat{a}_-\right] &= -\frac{1}{2}\,\hat{a}_- \\
\left[\hat{k}_+, \hat{a}_+\right] &= 0 \\
\left[\hat{k}_+, \hat{a}_-\right] &= -\hat{a}_+ \\
\left[\hat{k}_-, \hat{a}_-\right] &= 0 \\
\left[\hat{k}_-, \hat{a}_+\right] &= \hat{a}_-
\end{aligned}
\tag{18}
$$

To investigate the spectrum of H we have to determine the adjoint operators of the $\hat{k}$'s and $\hat{a}$'s with respect to the scalar product

$$(g, f) = \int_{-\infty}^{\infty} d\tilde{z}\; g(\tilde{z})\; f^*(\tilde{z}) \tag{19}$$

where the asterisk denotes the complex conjugate. With the aid of some partial integrations the following relations immediately follow

$$
\begin{aligned}
\hat{a}_-^\dagger &= -\hat{a}_- \;, &\quad \hat{a}_+^\dagger &= \hat{a}_+ \\
\hat{k}_0^\dagger &= -\hat{k}_0 \;, &\quad \hat{k}_+^\dagger &= \hat{k}_+ \;, &\quad \hat{k}_-^\dagger &= \hat{k}_-
\end{aligned}
\tag{20}
$$

where the dagger denotes adjoint operators. Now we are in a position to calculate the adjoint of the Hamiltonian H

$$H^\dagger = -\Omega_0^*\hat{k}_0 + \Omega_1^*\hat{k}_+ + \Omega_2^*\hat{k}_- + \Omega_3^*\hat{a}_+ - \Omega_4^*\hat{a}_- + \Omega_5^*\hat{1}\;. \tag{21}$$

If we require that H is hermitian it has to fulfill $H = H^\dagger$. This poses the following constraints on the coefficients Ω_i

$$
\begin{aligned}
\Omega_0 &= -\Omega_0^*\;, &\quad \Omega_1 &= \Omega_1^*\;, &\quad \Omega_2 &= \Omega_2^* \\
\Omega_3 &= \Omega_3^*\;, &\quad \Omega_4 &= -\Omega_4^*\;, &\quad \Omega_5 &= \Omega_5^*\;.
\end{aligned}
\tag{22}
$$

Obviously we have to require that Ω_0 and Ω_4 are purely imaginary and the others are real. Reexpressing the Ω_i by the G_j–functions we can approximately fulfill these requirements if we choose $\nu_0 = 2.6$ where the G–functions are almost real [2] and setting the resonator detuning Θ to the value given by eq. (9). Only under these circumstance can we expect "almost orthogonal" eigenfunctions from the "almost hermitian" operator.

If we require that H is normal, i.e. it commutes with its adjoint we have to check under what conditions the commutator $[H, H^\dagger]$ vanishes. Using the definition of the adjoint and the commutation relations we obtain

$$\begin{aligned}
\left[H, H^\dagger\right] &= (\Omega_0\Omega_1^* + \Omega_0^*\Omega_1)\,\hat{k}_+ - (\Omega_0\Omega_2^* + \Omega_0^*\Omega_2)\,\hat{k}_- - 2(\Omega_1\Omega_2^* - \Omega_1^*\Omega_2)\,\hat{k}_0 \\
&+ \left[\frac{1}{2}\,(\Omega_0\Omega_3^* + \Omega_0^*\Omega_3) + (\Omega_1\Omega_4^* + \Omega_1^*\Omega_4)\right]\hat{a}_+ \\
&+ \left[\frac{1}{2}\,(\Omega_0\Omega_4^* - \Omega_0^*\Omega_4) + (\Omega_2\Omega_3^* - \Omega_2^*\Omega_3)\right]\hat{a}_- \\
&+ (\Omega_3\Omega_4^* + \Omega_3^*\Omega_4)\,\hat{1}
\end{aligned} \tag{23}$$

The constraints for the Ω's are then given by

$$\begin{aligned}
\mathrm{Re}\,[\Omega_0\Omega_1^*] &= 0 \\
\mathrm{Re}\,[\Omega_0\Omega_2^*] &= 0 \\
\mathrm{Im}\,[\Omega_1\Omega_2^*] &= 0 \\
\mathrm{Re}\,[2\Omega_1\Omega_4^* + \Omega_0\Omega_3^*] &= 0 \\
\mathrm{Im}\,[2\Omega_2\Omega_3^* + \Omega_0\Omega_4^*] &= 0 \\
\mathrm{Re}\,[\Omega_3\Omega_4^*] &= 0\,.
\end{aligned} \tag{24}$$

These conditions are not generally fulfilled but can approximately be fulfilled under the same conditions stated above. We conclude that the Hamiltonian is neither normal nor hermitian and in general we cannot expect anything special about the eigenfunctions of H or $H^\dagger$. However, in ref. 6 it is shown that the eigenfunctions of a reduced Hamiltonian H (the terms with $\hat{k}_0$ and $\hat{a}_+$ are missing) are biorthogonal to an adjoint set of eigenfunctions of the adjoint Hamiltonian $H^\dagger$. We will prove this statement for the full Hamiltonian H. The eigenfunctions of H are given by eq. (6). Using the same technique we obtain for the eigenfunctions of the adjoint operator $H^\dagger$

$$\psi_m(\tilde{z}) = H_m(\bar{a}(\tilde{z}+\bar{b}))\; e^{-\bar{c}(\tilde{z}+\bar{d})^2} \tag{25}$$

where $\bar{a}, \bar{b}, \bar{c}, \bar{d}$ are now given by

$$\bar{a}^2 = \frac{1}{2\Omega_2^*}\sqrt{\Omega_0^{*2} - 4\Omega_1^*\Omega_2^*} \tag{26}$$

$$\bar{b} = -2\,\frac{2\Omega_2^*\Omega_3^* - \Omega_0^*\Omega_4^*}{\Omega_0^{*2} - 4\Omega_1^*\Omega_2^*} \tag{27}$$

$$\bar{c} = \frac{1}{4\Omega_2^*}\left[-\Omega_0^* + \sqrt{\Omega_0^{*2} - 4\Omega_1^*\Omega_2^*}\right] \tag{28}$$

$$\bar{d} = -2\,\frac{2\Omega_2^*\Omega_3^* + \Omega_4^*\left(-\Omega_0^* + \sqrt{\Omega_0^{*2} - 4\Omega_1^*\Omega_2^*}\right)}{\sqrt{\Omega_0^{*2} - 4\Omega_1^*\Omega_2^*}\left(-\Omega_0^* + \sqrt{\Omega_0^{*2} - 4\Omega_1^*\Omega_2^*}\right)}\,. \tag{29}$$

For completeness we add that the eigenvalues of the adjoint operator $\bar{\lambda}_m$ are given by $\bar{\lambda}_m = \lambda_m^*$. To prove the biorthogonality we have to show

$$\int_{-\infty}^{\infty} d\tilde{z}\; \phi_n(\tilde{z})\; \psi_m^*(\tilde{z}) \;\propto\; \delta_{nm}\,. \tag{30}$$

Using the generating function for the Hermite polynomials [3]

$$e^{-s^2+2sx} = \sum_{n=0}^{\infty} H_n(x)\,\frac{s^n}{n!} \tag{31}$$

we calculate the expression

$$\sum_{n=0}^{\infty}\sum_{m=0}^{\infty} \int_{-\infty}^{\infty} d\tilde{z}\ \phi_n(\tilde{z})\psi_m^*(\tilde{z})\ \frac{s^n}{n!}\frac{t^m}{m!}$$

$$= \int_{-\infty}^{\infty} d\tilde{z}\ e^{-s^2+2sa(\tilde{z}+b)}\ e^{-t^2+2ta(\tilde{z}+b)}\ e^{-c(\tilde{z}+d)^2-\bar{c}^*(\tilde{z}+\bar{d}^*)^2}$$

$$= \frac{\sqrt{\pi}}{a}\ \exp\left\{2st - \frac{c\bar{c}^*}{a^2}\,(d-\bar{d}^*)^2 + 2a(s+t)\left[b - \frac{cd+\bar{c}^*\bar{d}^*}{a^2}\right]\right\}$$

$$= \frac{\sqrt{\pi}}{a}\ \exp\left\{4a^2\ \frac{\Omega_2}{\Omega_1}\left(\frac{\Omega_0\Omega_3-2\Omega_1\Omega_4}{\Omega_0^2-4\Omega_1\Omega_2}\right)^2\right\} e^{2st} \tag{32}$$

$$= \frac{\sqrt{\pi}}{a}\exp\left\{4a^2\frac{\Omega_2}{\Omega_1}\left(\frac{\Omega_0\Omega_3-2\Omega_1\Omega_4}{\Omega_0^2-4\Omega_1\Omega_2}\right)^2\right\}\sum_{n=0}^{\infty}\frac{(2st)^n}{n!}$$

$$= \frac{\sqrt{\pi}}{a}\exp\left\{4a^2\frac{\Omega_2}{\Omega_1}\left(\frac{\Omega_0\Omega_3-2\Omega_1\Omega_4}{\Omega_0^2-4\Omega_1\Omega_2}\right)^2\right\}\sum_{n=0}^{\infty}\sum_{m=0}^{\infty}2^n n!\ \frac{s^n}{n!}\frac{t^m}{m!}\ \delta_{nm}$$

where we had calculated $b-(cd+\bar{c}^*\bar{d}^*)/a^2=0$. Comparing the coefficients in front of $(s^n/n!)(t^m/m!)$ we obtain our final result

$$\int_{-\infty}^{\infty} d\tilde{z}\ \phi_n(\tilde{z})\psi_m^*(\tilde{z}) = \delta_{nm}\frac{\sqrt{\pi}}{a}2^n n!\ \exp\left\{4a^2\frac{\Omega_2}{\Omega_1}\left(\frac{\Omega_0\Omega_3-2\Omega_1\Omega_4}{\Omega_0^2-4\Omega_1\Omega_2}\right)^2\right\}\ . \tag{33}$$

This completes the proof. Note, that the exponential on the right hand side vanishes if we have $\Omega_0\Omega_3 = 2\Omega_1\Omega_4$. This is exactly the condition for optimum detuning eq. (9) or highest symmetry of the supermodes. The relation among these conditions has to be investigated more closely in the future.

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Figure Captions

1. $G_5(\nu_0, \mu_D = 0)$ (real=solid, imaginary=dashed).
2. Growth rate $\mathrm{Re}[\lambda_0]$ as a function of the resonator detuning Θ in the range $0 \leq \Theta \leq 1$ for $\mu_\varepsilon = 0.5$, $\mu_s = 0.2$ and $\mu_c = 5 \times 10^{-3}$.
3. Growth rate $\mathrm{Re}[\lambda_0]$ as a function of μ_s in the range $-\mu_\varepsilon \leq \mu_s \leq \mu_\varepsilon$ for $\mu_\varepsilon = 0.5$ and $\mu_c = 5 \times 10^{-3}$.
4. The length of the laser pulse $\tilde{\sigma}_L$ in units of the electron bunch length as a function of μ_s in the range $-\mu_\varepsilon \leq \mu_s \leq \mu_\varepsilon$ for $\mu_\varepsilon = 0.5$ and $\mu_c = 5 \times 10^{-3}$.
5. The bandwidth of the laser pulse in units of $1/2\pi N$ as a function of μ_s in the range $-\mu_\varepsilon \leq \mu_s \leq \mu_\varepsilon$ for $\mu_\varepsilon = 0.5$ and $\mu_c = 5 \times 10^{-3}$.

TABLE I. – *List of symbols used*

N	number of wiggles
l_w	wiggle period
$L_w = N\, l_w$	length of wiggler
T_c	optical cavity round trip time
$L_c = T_c/2\,c$	distance between mirrors
γ_T	losses in the optical cavity
ε_0	relative energy offset
σ_ε	relative energy spread
σ_z	bunch length
$\sigma_{z\varepsilon}$	energy-position correlation
δT	difference between cavity round trip time and recurrence time of the electron bunch
K'	rms wiggler parameter
$\Delta = N\lambda$	slippage between the light and the electrons
λ	laser wavelength
$\omega_0 = 2\pi c/\lambda$	laser frequency
$\nu_0 = 4\pi N \varepsilon_0$	adimensional detuning
$\mu_L = \Delta/L_c$	mode spacing parameter
$\mu_c = \Delta/\sigma_z$	mode coupling parameter
$\mu_\varepsilon = 4N\sigma_\varepsilon$	normalized energy spread
$\mu_s = 4N\sigma_{z\varepsilon}/\sigma_z$	normalized energy position correlation
$\mu_D = \sqrt{\mu_\varepsilon^2 - \mu_s^2}$	measure of the longitudinal emittance
Σ_L	transverse cross section of the laser mode
I_p	peak electron current
I_0	Alvén current

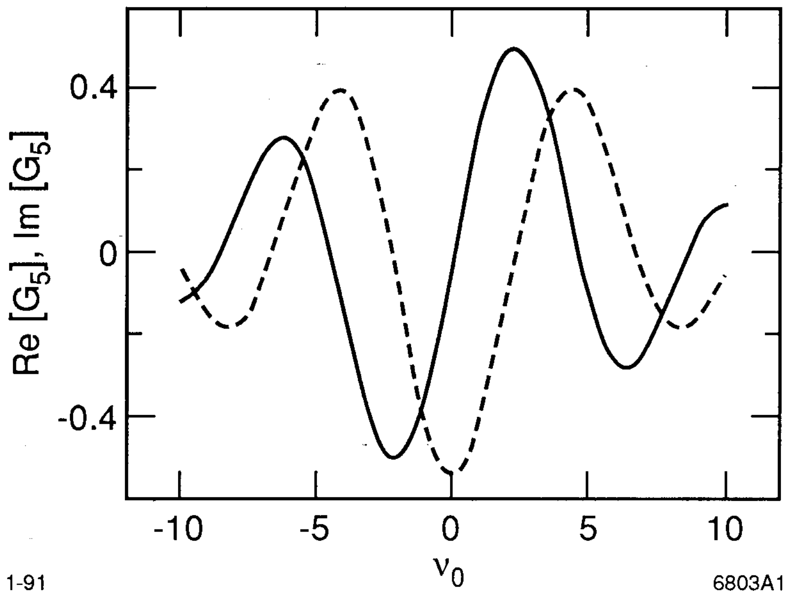

Fig. 1

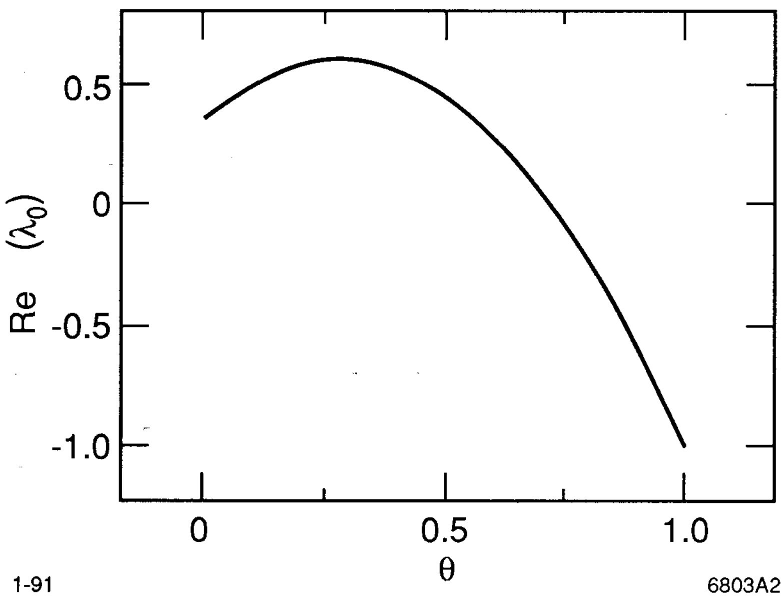

Fig. 2

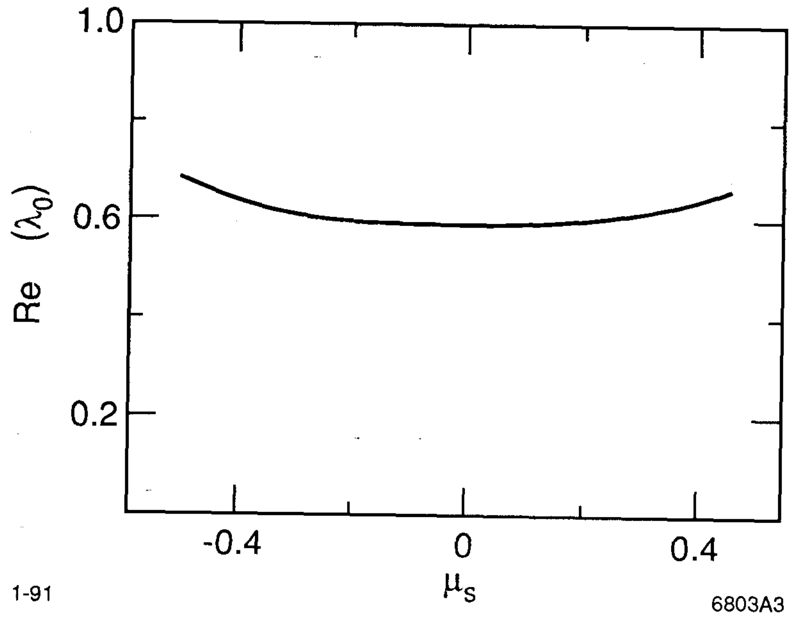

Fig. 3

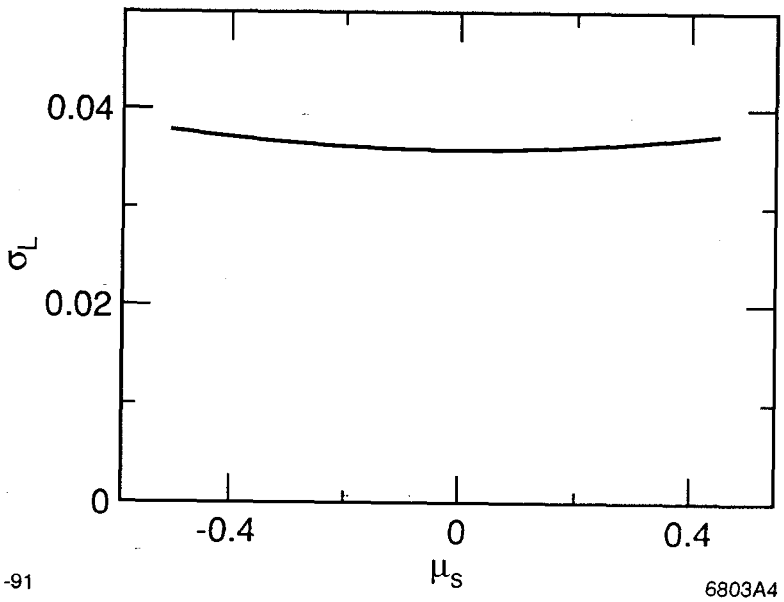

Fig. 4

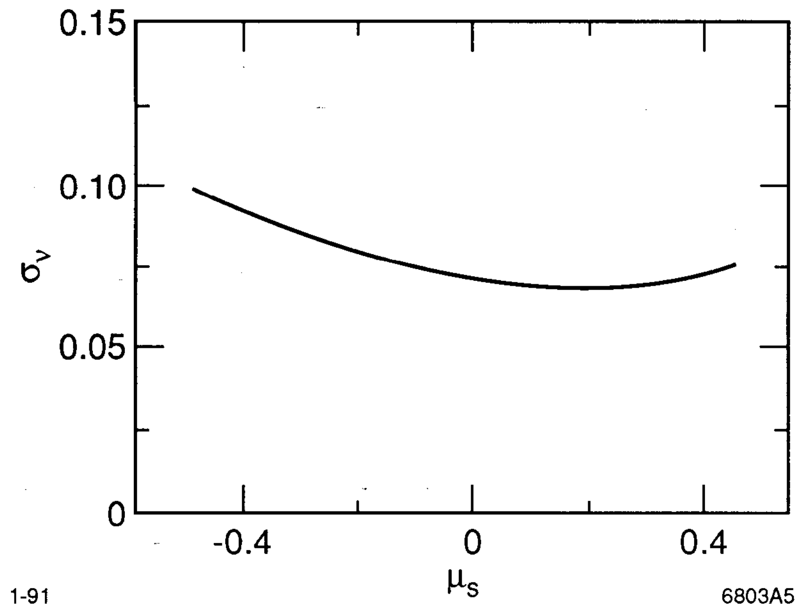

Fig. 5