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FIELD THEORIES IN THE INFINITE-MOMENTUM FRAME

DAVISON E. SOPER

STANFORD LINEAR ACCELERATOR CENTER
STANFORD UNIVERSITY
Stanford, California 94305

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ABSTRACT

We examine the formal foundations of quantum electrodynamics and several other field theories in the infinite-momentum frame.

The infinite-momentum frame is interpreted as being given by the change of variables $\tau = 2^{-\frac{1}{2}}(t + z)$, $\mathfrak{z} = 2^{-\frac{1}{2}}(t - z)$. The variable τ plans the role of time. We discuss the Galilean subgroup of the Poincare group, which results in a non-relativistic structure of quantum mechanics in the infinite-momentum frame.

We derive a τ-ordered perturbation series for quantum electrodynamics and show how such a series arises from a canonical formulation of the field theory. We quantize the theory directly in the infinite-momentum frame by postulating equal-τ commutation relations among the fields.

We also discuss several other field theories: massive quantum electrodynamics; scalar meson with ϕ^N coupling; neutral pions coupled to protons with γ_5 coupling; scalar mesons coupled to protons with unit matrix coupling; and electrodynamics of a spin zero boson.

PREFACE

Most of the work described here was done in collaboration with my advisor, James D. Bjorken, and with John B. Kogut. I thank them both. I have also benefited from discussions with my colleagues at SLAC.

Much of the material in this report is based on previously published work. Chapter IV and parts of Chapter II are contained in "Quantum Electrodynamics in the Infinite Momentum Frame" (D. E. Soper and J. B. Kogut, Phys. Rev. D1, 2901 (1970)). Chapter V is based on "Massive Quantum Electrodynamics in the Infinite Momentum Frame" (D. E. Soper, SLAC-PUB-918, May 1971, to be published in Phys. Rev.).

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CHAPTER I

Introduction

The infinite-momentum frame first appeared in connection with current algebra[1] as the limit of a reference frame moving with almost the speed of light. Weinberg[2] asked whether this limit might be more generally useful. He considered the infinite-momentum limit of the old-fashioned perturbation diagrams for scalar meson theories and showed that the vacuum structure of these theories simplified in the limit. Later, Susskind[3,4] showed that the infinities which occur among the generators of the Poincaré group when they are boosted to a fast-moving reference frame can be scaled or subtracted out consistently. The result is essentially a change of variables. Susskind used the new variables to draw attention to the (two-dimensional) Galilean subgroup of the Pioncaré group. He pointed out that the simplified vacuum structure and the nonrelativistic kinematics of theories at infinite momentum might offer potential-theoretic intuition in relativistic quantum mechanics.

Bardakci and Halpern[5] further analyzed the structure of theories at infinite momentum. They viewed the infinite-momentum limit as a change of variables from the laboratory time and z coordinates to a new "time" $\tau = 2^{-\frac{1}{2}}(t+z)$ and a new space coordinate $\mathfrak{z} = 2^{-\frac{1}{2}}(t-z)$. Chang and Ma[6] considered the Feynman diagrams for a ϕ^3 theory and quantum electrodynamics from this point of view and were able to demonstrate the advantages of their approach in several illustrative calculations.

In this dissertation, we examine the formal foundations of quantum electrodynamics and several other field theories in the infinite-momentum frame. We interpret the infinite-momentum frame as being given by a change of variables

$\tau = 2^{-\frac{1}{2}}(t+z)$, $\mathscr{Y} = 2^{-\frac{1}{2}}(t-z)$, thus avoiding limiting procedures. We derive a τ-ordered perturbation series for quantum electrodynamics and show how such a series arises from a canonical formulation of the field theory.

The methods employed here do not involve any high energy approximations. However, we believe that the "exact" field theories in the infinite-momentum frame may be well adapted for high energy approximations.

This dissertation is divided into six chapters, of which this is the first. In the second chapter we discuss the infinite-momentum coordinate system, $(\tau, \underset{\sim}{x}, \mathscr{Y})$. By using these coordinates we obscure the rotational symmetry of the underlying physics. However, we will find that other Galilean symmetries more appropriate to the description of high energy processes are thereby made manifest. We will see, in fact, that the subgroup of the Poincaré group consisting of τ-translations together with those transformations which leave the planes "τ =constant" invariant is isomorphic with the symmetry group of nonrelativistic quantum mechanics in two dimensions. We will also give a nonrelativistic interpretation to the remaining Poincaré generators and to the parity and time reversal operators. Finally, we will discuss the single particle states most natural in the infinite-momentum frame. These are the infinite-momentum helicity states, which are eigenstates of helicity as measured by an observer moving in the $-z$ direction with almost the speed of light.

In the third chapter we examine fields and wave functions for free particles of arbitrary mass and spin as they appear in the infinite-momentum frame. The discussion for the most part follows the methods of Weinberg and others except for the use of the infinite-momentum helicity basis for the particle states. The main result is that the spinor wave functions for infinite-momentum helicity states have a remarkably simple form.

The fourth chapter forms the heart of this dissertation. It is devoted to the reformulation of conventional quantum electrodynamics in the infinite-momentum frame variables. We begin by considering the Feynman perturbation expansion for the S matrix, divorced from its field theoretical underpinnings. We write the covariant Feynman diagrams using the variables $(\tau, \underset{\sim}{x}, \mathfrak{z})$ and then decompose each covariant diagram into a sum of old fashioned τ-ordered diagrams. The results are similar to Weinberg's results concerning the $P_z \rightarrow \infty$ limit of t-ordered diagrams, but the appearance of spin results in the emergence of new types of vertices.[7] In the second part of the chapter we look at the field theoreticic underpinnings. We quantize the theory directly in the infinite-momentum frame by postulating equal-τ commutation relations among the fields. We find that these equal-τ commutation relations make the unquantized field theory into a formally consistant quantum field theory; in particular, the canonical Hamiltonian generates τ-translations of the fields according to their equations of motion. Finally, we find that the old-fashioned perturbation expansion for the S matrix derived using the canonical Hamiltonian agrees with the τ-ordered expansion derived directly from the covariant Feynman diagrams.

In the fifth chapter we extend the canonical formulation of quantum electrodynamics in the infinite-momentum frame by replacing the photons by massive vector mesons. The structure of the theory remains nearly the same as that of quantum electrodynamics except that a new term appears in the Hamiltonian describing the emission of helicity zero vector mesons with an amplitude proportional to the meson mass.

In the last chapter we make use of the familiarity gained with the two previous model field theories in order to apply the same methods to several other theories. These are: scalar mesons with ϕ^{μ} self-coupling; neutral pions coupled to protons with a γ_5 coupling; neutral scalar mesons coupled to protons with a $\mathbb{1}$ coupling; and electrodynamics of a spin zero boson. Each of these theories has the attractive feature that it is simpler than quantum electrodynamics.

References — Chapter I

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6. S. Chang and S. Ma, Phys. Rev. 180, 1506 (1969).
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CHAPTER II

The Poincaré Group in the Infinite-Momentum Frame

A. Choice of Variables

In low energy processes the trajectories of particles cluster about a single direction in space-time. It is sensible to describe such processes using coordinates t, x, y, z, with the t-axis chosen in the direction of the particle trajectories. This choice of coordinates emphasizes the rotational symmetry of the underlying physics.

In high energy collisions, the particle trajectories will generally lie near a plane, which we may take to be the t-z plane. However, the trajectories will not cluster about a single time-like line in space-time. Rather, the trajectories of energetic right-moving particles cluster about the light-like line $t - z = 0$ in the t-z plane, while the trajectories of energetic left-moving particles cluster about the light-like line $t + z = 0$. Thus it is sensible to describe such particles using coordinate axes lying along these lines. Hence we adopt the coordinates $\tau = 2^{-\frac{1}{2}}(t+z)$, $x, y, \mathfrak{z} = 2^{-\frac{1}{2}}(t-z)$, as shown in Figure II-1. We will let the variable τ play the role of "time" in the description of the dynamics of right-moving particles, since the trajectories of these particles cluster about the τ-axis. (Similarly, $\mathfrak{z}$ can play the role of "time" for left-moving particles.)

By using these "infinite-momentum frame" coordinates we obscure the rotational symmetry of the underlying physics. However, we will find that other "Galilean" symmetries more appropriate to the description of high energy processes are thereby made manifest.

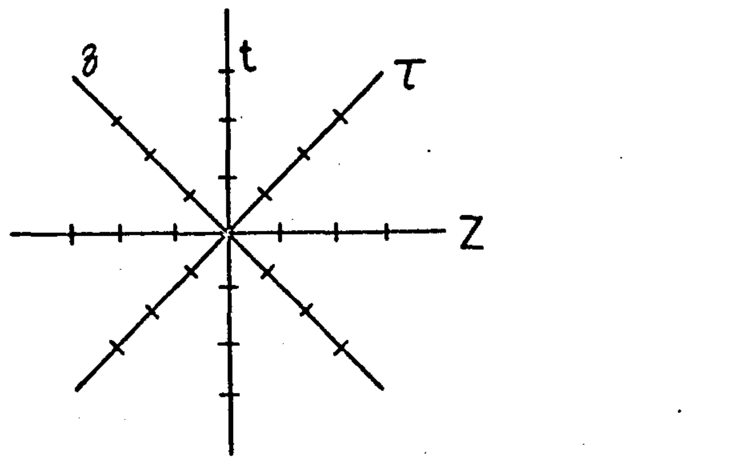

Fig. II-1

The coordinate axes of the infinite momentum frame.

It will be convenient to use the usual covariant tensor notation for quantities in the new coordinate system. Let $\hat{x}^\mu = (\hat{x}^0, \hat{x}^1. \hat{x}^2, \hat{x}^3) = (t, x, y, z)$ be the coordinates of a space-time point in the ordinary coordinate system, $x^\mu = (x^0, x^1, x^2, x^3) = (\tau, x, y, \mathfrak{z}) = (\tau, \mathbf{x}, \mathfrak{z})$ be the new coordinates of the same point.[1] Then

$$x^\mu = C^\mu{}_\nu \, \hat{x}^\nu \quad , \tag{II.1}$$

where

$$C^\mu{}_\nu = \begin{pmatrix} 2^{-1/2} & 0 & 0 & 2^{-1/2} \\ 0 & 1 & 0 & 0 \\ 0 & 0 & 1 & 0 \\ 2^{-1/2} & 0 & 0 & -2^{-1/2} \end{pmatrix} \quad . \tag{II.2}$$

In general, we shall use hatted symbols for vectors and tensors in the ordinary coordinate system, unhatted symbols for vectors and tensors in the new coordinate system. In particular, we shall use $g_{\mu\nu}$ for the metric tensor in the new coordinate system:

$$g_{\mu\nu} = (C^{-1})^\alpha{}_\mu \, (C^{-1})^\beta{}_\nu \, \hat{g}_{\alpha\beta} \quad . \tag{II.3}$$

We take for the ordinary metric tensor $\hat{g}_{00} = 1$, $\hat{g}_{11} = \hat{g}_{22} = \hat{g}_{33} = -1$. Then

$$g_{\mu\nu} = \begin{pmatrix} 0 & 0 & 0 & 1 \\ 0 & -1 & 0 & 0 \\ 0 & 0 & -1 & 0 \\ 1 & 0 & 0 & 0 \end{pmatrix} \quad . \tag{II.4}$$

We use $g_{\mu\nu}$ to lower indices, so that $a_0 = a^3$, $a_3 = a^0$; this may seem confusing, but it has important consequences. For instance, the wave operator $\partial_\mu \partial^\mu = 2\partial_0\partial_3 - \partial_1\partial_1 - \partial_2\partial_2$ is only first order in $\partial_0 = \partial/\partial\tau$.

B. Generators of the Poincaré Group

Let us consider the generators of the Poincaré group in the new notation. Our conventions for the Poincaré algebra in the ordinary notation are

$$[\hat{P}^\mu, \hat{P}^\nu] = 0 \qquad [\hat{M}_{\mu\nu}, \hat{P}_\rho] = i(\hat{g}_{\nu\rho}\hat{P}_\mu - \hat{g}_{\mu\rho}\hat{P}_\nu)$$
$$[\hat{M}_{\mu\nu}, \hat{M}_{\rho\sigma}] = i(\hat{g}_{\mu\sigma}\hat{M}_{\nu\rho} + \hat{g}_{\nu\rho}\hat{M}_{\mu\sigma} - \hat{g}_{\mu\rho}\hat{M}_{\nu\sigma} - \hat{g}_{\nu\sigma}\hat{M}_{\mu\rho}) \tag{II.5}$$

The generators of rotations and boosts are, respectively, $\hat{M}_{ij} = \epsilon_{ijk}J_k$ and $\hat{M}_{i0} = K_i$. Using the matrix C^μ_ν to transform from the usual notation to the new notation, we obtain

$$P^\mu = (P^0, P^1, P^2, P^3) = (\eta, P^1, P^2, H) \tag{II.6}$$

and

$$M_{\mu\nu} = \begin{pmatrix} 0 & -S^1 & -S^2 & K_3 \\ S^1 & 0 & J_3 & B^1 \\ S^2 & -J_3 & 0 & B^2 \\ -K_3 & -B^1 & -B^2 & 0 \end{pmatrix} \tag{II.7}$$

where

$$\eta = (\hat{P}^0 + \hat{P}^3)/\sqrt{2}$$

$$H = (\hat{P}^0 - \hat{P}^3)/\sqrt{2}$$

$$B^1 = (K_1 + J_2)/\sqrt{2}$$

$$B^2 = (K_2 - J_1)/\sqrt{2} \tag{II.8}$$

$$S^1 = (K_1 - J_2)/\sqrt{2}$$

$$S^2 = (K_2 + J_1)/\sqrt{2}$$

If we consider τ to play the role of "time", we will be particularly interested in the generator of τ-translations in space-time. Since $\exp(iP_\mu x^\mu) = \exp(i[H\tau - \underset{\sim}{P}\cdot\underset{\sim}{x} + \eta \mathfrak{z}])$, we see that it is H which generates τ-translations and thus contains the "dynamics" of quantum mechanics in the infinite-momentum frame. Similarly, it is easy to verify that the subgroup of the Poincaré group generated by η, $\underset{\sim}{P}$, J_3 and $\underset{\sim}{B}$ consists of those transformations which leave the planes τ = const. invariant. Thus these operators might be called "kinematical" symmetry operators in the infinite momentum frame.

The commutation relations among the Poincaré generators are, of course, given by (II.5) without the hats. The commutation relations among the operators $H, \eta, \underset{\sim}{P}, J_3, \underset{\sim}{B}$ are particularly interesting. They are the same as the commutation relations among the symmetry operators of nonrelativistic quantum mechanics in two dimensions with

$H \rightarrow$ hamiltonian,

$\eta \rightarrow$ mass ,

$\underset{\sim}{P} \rightarrow$ momentum ,

$J_3 \rightarrow$ angular momentum ,

B^1 and $B^2 \rightarrow$ generators of (Galilean) boosts in the x and y directions, respectively.

In fact, the subgroup of the Poincaré groups generated by η, $\underset{\sim}{P}$, H, J_3, $\underset{\sim}{B}$ is isomorphic to the Galilean symmetry groups of nonrelativistic quantum mechanics in two dimensions.[2,3] It is instructive to explicate this isomorphism in some detail.

The mass operator η commutes with all of the other generators, $\underset{\sim}{P}$, H, J_3, $\underset{\sim}{B}$. Also, $[B^1, B^2] = [P^1, P^2] = 0$. The Hamiltonian H is invariant under rotations and translations:

$$[H, J_3] = [H, \underset{\sim}{P}] = 0 \quad .$$

The momentum $\underset{\sim}{P}$ and the boost generator $\underset{\sim}{B}$ are vectors under rotations:

$$e^{i\varphi J_3} P^k e^{-i\varphi J_3} = M_{k\ell} P^\ell$$

$$e^{i\varphi J_3} B^k e^{-i\varphi J_3} = M_{k\ell} B^\ell$$

$$M_{k\ell} = \begin{pmatrix} \cos\varphi & -\sin\varphi \\ \sin\varphi & \cos\varphi \end{pmatrix} .$$

When a system is given a Galilean boost through velocity $\underset{\sim}{v}$, its total momentum changes by an amount equal to its mass times v:

$$e^{i\underset{\sim}{v}\cdot\underset{\sim}{B}}\ \underset{\sim}{P}\ e^{-i\underset{\sim}{v}\cdot\underset{\sim}{B}} = \underset{\sim}{P} + \eta\,\underset{\sim}{v} \quad .$$

At the same time the "internal energy" of the system is unaltered, but its kinetic energy, $\underset{\sim}{P}^2/2\eta$, is changed:

$$\begin{aligned} e^{i\underset{\sim}{v}\cdot\underset{\sim}{B}}\ H\ e^{-i\underset{\sim}{v}\cdot\underset{\sim}{B}} &= H + \frac{(\underset{\sim}{P}+\eta\underset{\sim}{v})^2}{2\eta} - \frac{\underset{\sim}{P}^2}{2\eta} \\ &= H + \underset{\sim}{P}\cdot\underset{\sim}{v} + \tfrac{1}{2}\eta\,\underset{\sim}{v}^2 \quad . \end{aligned}$$

It is also interesting to think of $\underset{\sim}{B}$ as a position operator. In nonrelativistic quantum mechanics, the generator of boosts is $\vec{B} = -\Sigma m_i \vec{r}_i = -$(total mass) $\times$ (position of the center of mass). Thus we are led to define the operator $\underset{\sim}{R} = \underset{\sim}{B}/\eta$ and interpret $\underset{\sim}{R}$ as the operator giving the position of the center of "mass" of the system. In support of this interpretation, we note that $[P^k, R^\ell] = -i\delta_{k\ell}$ and that the rate of change of R is equal to the velocity of the system, P/η: $\dot{\underset{\sim}{R}} \equiv i[H, \underset{\sim}{R}] = \underset{\sim}{P}/\eta$.

We can also use $\underset{\sim}{R}$ to decompose the total angular momentum $\underset{\sim}{J}$ into a part representing the orbital angular momentum of the center of mass about the origin,

$$J_{CM} = \underset{\sim}{R}\times\underset{\sim}{P} = R^1 P^2 - R^2 P^1 \quad ,$$

and the remaining "internal" angular momentum,

$$\mathcal{J}_3 = J_3 - \underset{\sim}{R}\times\underset{\sim}{P} \quad .$$

Just as in nonrelativistic quantum mechanics, the internal angular momentum is invariant under translations and boosts and is conserved:

$$e^{i\underset{\sim}{a}\cdot\underset{\sim}{P}}\,\mathcal{J}_3\,e^{-i\underset{\sim}{a}\cdot\underset{\sim}{P}} = e^{i\underset{\sim}{v}\cdot\underset{\sim}{B}}\,\mathcal{J}_3\,e^{-i\underset{\sim}{v}\cdot\underset{\sim}{B}} = \mathcal{J}_3$$

$$i[H, \mathcal{J}_3] = 0 \quad .$$

Finally, it may be worthwhile to note that the Hamiltonian for a free particle takes a simple nonrelativistic form. From the mass shell condition $P^{\mu}P_{\mu} = M^2$ we obtain

$$H = \frac{\underset{\sim}{P}^2}{2\eta} + V_0 \qquad ,$$

where $V_0 = M^2/2\eta$ is a constant potential.

The operator K_3, which generates Lorentz boosts in the z-direction, can also be given a simple interpretation in the analogy to nonrelativistic quantum mechanics. Suppose that we were to formulate ordinary nonrelativistic quantum mechanics using length and mass as the basic units, with the unit of time chosen so that $\hbar = 1$. (Thus 1 sec $= 1.05 \times 10^{-27}$ gm cm^2.) Then we would find that our theory was invariant under rescaling of the unit of mass. In the "nonrelativistic" interpretation of quantum mechanics in the infinite-momentum frame, this symmetry is built in as a consequence of Lorentz invariance. A simple calculation shows that $\exp(-i\omega K_3)$ simply rescales each of the other group generators according to the number of powers of "mass" each contains:

$$
\begin{aligned}
e^{+i\omega K_3}\, \eta\, e^{-i\omega K_3} &= e^{\omega}\, \eta \\
e^{i\omega K_3}\, \underset{\sim}{P}\, e^{-i\omega K_3} &= \underset{\sim}{P} \\
e^{+i\omega K_3}\, H\, e^{-i\omega K_3} &= e^{-\omega}\, H \\
e^{+i\omega K_3}\, \underset{\sim}{B}\, e^{-i\omega K_3} &= e^{\omega}\, \underset{\sim}{B} \\
e^{i\omega K_3}\, J_3\, e^{-i\omega K_3} &= J_3 \\
e^{+i\omega K_3}\, \underset{\sim}{S}\, e^{-i\omega K_3} &= e^{-\omega}\, \underset{\sim}{S}\,.
\end{aligned}
$$

The fact that the operators P^{μ} and $M_{\mu\nu}$ in the infinite momentum frame transform under z-boosts according to simple scaling laws suggests that the infinite momentum frame may be particularly adapted for high energy approximations.[4]

We come now to the final operators in our menagerie of Poincaré generators, S^1 and S^2. These operators commute with H, form a vector under rotations, and scale under z-boosts like η^{-1}. The commutation relations of S with η, $\underset{\sim}{P}$, and $\underset{\sim}{B}$ are

$$
\begin{aligned}
[S^k, \eta] &= -i\, P^k \\
[S^k, P^l] &= -i\, \delta_{kl}\, H \\
[S^k, B^l] &= -i\, \epsilon_{kl}\, J_3 + i\, \delta_{kl}\, K_3
\end{aligned}
$$

where $\epsilon_{12} = -\epsilon_{21} = 1$, $\epsilon_{11} = \epsilon_{22} = 0$. We can give these commutation relations an interpretation in the nonrelativistic analogy if we write $\underset{\sim}{S}$ as the sum of an "internal" part $\underset{\sim}{s}$ associated with the internal dynamics of the system and a remaining "external" part as follows[5]:

$$S^k = K_3 \frac{P^k}{\eta} + B^k \frac{H}{\eta} - j_3 \epsilon_{k\ell} \frac{P^\ell}{\eta} + s^k$$

where j_3 is the internal angular momentum discussed earlier. Simple computation shows that $\underset{\sim}{s}$ is indeed an "internal" operator in the sense that it is invariant under translations and Galilean boosts and commutes with the total "mass" operator η. Furthermore, $\underset{\sim}{s}$ commutes with the Hamiltonian H. Thus $\underset{\sim}{s}$ plays the same role as the "dynamical" symmetry operators sometimes encountered in nonrelativistic quantum mechanics.[6] In this interpretation, $\underset{\sim}{s}$-invariance is an extra symmetry of the Hamiltonian in addition to its Galilean invariance which is needed to insure the full Poincaré invariance of the theory.

It will come as no surprize that the "internal" operators j_3 and $\underset{\sim}{s}$ provide just another description of the spin of the system. The connection between these operators and more conventional spin operators can be clarified by means of a few simple observations.

First, we notice that j_3 measures the helicity of the system as viewed in a reference frame moving in the -z direction with (almost) the speed of light:

$$\lim_{\omega\to\infty} e^{i\omega K_3} \frac{\vec{J}\cdot\vec{P}}{|\vec{P}|} e^{-i\omega K_3}$$

$$= \lim_{\omega\to\infty} e^{i\omega K_3} \frac{2^{-1/2} J_3 (\eta - H) + 2^{-1/2}(B^\ell - S^\ell)\epsilon_{\ell k} P^k}{\left[\frac{1}{2}(\eta - H)^2 + \underset{\sim}{P}^2\right]^{1/2}} e^{-i\omega K_3}$$

$$= J_3 + \frac{1}{\eta} \underset{\sim}{B} \times \underset{\sim}{P} \qquad = J_3 - \underset{\sim}{R} \times \underset{\sim}{P}$$

$$= j_3$$

Second, we compute the covariant spin vector[7,8]

$$w^{\mu} = \frac{1}{2} \epsilon^{\mu\nu\rho\sigma} M_{\nu\rho} P_{\sigma} ,$$

and find

$$w^{0} = j_3 \eta$$
$$w^{k} = j_3 P^{k} + \epsilon_{k\ell} s^{\ell} \eta$$
$$w^{3} = j_3 \left(\frac{1}{\eta} \underset{\sim}{P}^{2} - H \right) - \underset{\sim}{s} \times \underset{\sim}{P} .$$

Then we can compute the Poincaré group Casimir operator $W = -w^{\mu} w_{\mu}$. (We recall that for a single particle with mass $M > 0$ and spin s, $W = M^2 S(S+1)$. For mass zero irreducible representations of the Poincaré group, W is zero except for the unphysical "continuous spin" representations.) We find

$$W = (2\eta H - \underset{\sim}{P}^{2}) (j_3)^{2} + \eta^{2} \underset{\sim}{s}^{2}$$

Finally, we compute the commutation relations among the "internal" operators:

$$[j_3 , s^{k}] = i \epsilon_{k\ell} s^{\ell}$$
$$[s^{1} , s^{2}] = i \eta^{-2} (2\eta H - \underset{\sim}{P}^{2}) j_3 .$$

Thus, as long as the spectrum of $M^2 = (2\eta H - \underset{\sim}{P}^2)$ is strictly positive, the operators $j_1 = \eta s^2/M$, $j_2 = -\eta s^1/M$, j_3 are well defined and obey the SU(2) algebra $[j_k, j_\ell] = i\epsilon_{k\ell n} j_n$. Therefore, it is quite plausible that $\vec{j} \equiv (j_1, j_2, j_3)$ measures the angular momentum of the system in its rest frame. To prove this, note first that $\vec{j}$ commutes with P^μ, $\underset{\sim}{B}$ and K_3. Let $|\psi_p\rangle$ be a state with definite momentum P^μ and U_p be a Lorentz transformation of the form $U_p = \exp(-i\underset{\sim}{B}\cdot\underset{\sim}{v})\exp(-i\omega K_3)$ constructed so that $|\psi_o\rangle = U_p|\psi_p\rangle$ is at rest. Then

$$\begin{aligned} \vec{j}\,|\psi_p\rangle &= U_p^{-1}\,(U_p\,\vec{j}\,U_p^{-1})\,|\psi_o\rangle \\ &= U_p^{-1}\,\vec{j}\,|\psi_o\rangle\,. \end{aligned}$$

But for a state at rest we have $\eta = H = 2^{-\frac{1}{2}}M$, $\underset{\sim}{P} = 0$, so

$$\begin{aligned} j_3\,|\psi_o\rangle &= J_3\,|\psi_o\rangle \\ j_1\,|\psi_o\rangle &= 2^{-1/2}\,s^2\,|\psi_o\rangle \\ &= 2^{-1/2}\,(S^2 - B^2)\,|\psi_o\rangle \\ &= J_1\,|\psi_o\rangle \\ j_2\,|\psi_o\rangle &= -2^{-1/2}\,s^1\,|\psi_o\rangle \\ &= 2^{-1/2}\,(B^1 - S^1)\,|\psi_o\rangle \\ &= J_2\,|\psi_o\rangle\,. \end{aligned}$$

Thus

$$\vec{j}\ |\psi_p\rangle = U_P^{-1}\ \vec{J}\ U_P\ |\psi_p\rangle\ ,$$

as we claimed. We will meet the operators $\vec{j}$ and U_p again when we construct free particle states appropriate to the infinite-momentum frame. But first we turn to a short discussion of the parity and time reversal operators.

C. Parity and Time Reversal in the Infinite-Momentum Frame

In the infinite-momentum coordinate system the ordinary parity transformation is

$$x^\mu = (\tau, \underset{\sim}{x}, z) \longrightarrow \Lambda_P{}^\mu{}_\nu\, x^\nu = (z, -\underset{\sim}{x}, \tau)\ .$$

If parity is a symmetry of the theory under discussion, there will be a unitary operator U_p with

$$U_P^{-1}\ P^\mu\ U_P = \Lambda_P{}^\mu{}_\nu\ P^\nu$$

$$U_P^{-1}\ M^{\mu\nu}\ U_P = \Lambda_P{}^\mu{}_\alpha\ \Lambda_P{}^\nu{}_\beta\ M^{\alpha\beta}\ .$$

The details of this transformation are presented in the first column of Table II-1. We see there that the parity transformation has the effect of interchanging the roles of η and H, $\underset{\sim}{B}$ and $\underset{\sim}{S}$, K_3 and $-K_3$ (as well as changing the sign of transverse vector operators). This transformation is useful for comparing the dynamics of left-moving and right-moving systems.

If, on the other hand, we are more interested in the internal dynamics of right-moving systems, the "parity" operator

$$U_a = \exp(-i\pi J_x)\, U_P$$

is more useful. We see from Table I that the operators η, H, and K_3 are "scalars" under this parity operator. The operators $\underset{\sim}{P}$, $\underset{\sim}{B}$ and $\underset{\sim}{S}$ are "vectors" and transform according to $(V^1, V^2) \rightarrow (-V^1, V^2)$. The rotation operator J_3 is a "pseudo-scalar" and changes sign under this parity transformation. In addition, "pseudo-vector" objects like $F^k = \epsilon_{k\ell} V^\ell$ (where V_ℓ is a vector) somestimes occur; these transform according to $(F_1, F^2) \rightarrow (F^1, -F^2)$.

We come now to time reversal. The ordinary time reversal operator U_T does not seem to be very useful in the infinite momentum frame. A much more natural operator is

$$U_\tau = \exp(-i\pi J_3)\, U_P U_T \quad ,$$

which we might call the "τ-reversal" operator. The corresponding Lorentz transformation matrix Λ_τ is given by

$$\Lambda_\tau{}^\mu{}_\nu\, x^\nu = x'^\mu = (-\tau, \underset{\sim}{x}, -z) \quad .$$

However, since U_τ is antiunitary, the transformation law for the Poincaré generators under τ-reversal is

$$U_\tau^{-1} P^\mu U_\tau = -\Lambda_\tau{}^\mu{}_\nu\, P^\nu$$

$$U_\tau^{-1} M^{\mu\nu} U_\tau = -\Lambda_\tau{}^\mu{}_\alpha\, \Lambda_\tau{}^\nu{}_\beta\, M^{\alpha\beta} \quad .$$

This leads to the last column of Table II-1. We see there that U_τ acts just like the time-reversal operator of nonrelativistic quantum mechanics. The mass η and energy H of a system are unchanged under τ-reversal; the momentum $\underset{\sim}{P}$ is reversed; and the boost operator $\underset{\sim}{B}$ (and the position operator $\underset{\sim}{R} = -\underset{\sim}{B}/\eta$) is unchanged.

The final discrete symmetry operator which we will find useful later is the PCT operator U_{PCT}. If we write U_C for the charge conjugation operator which interchanges particles with antiparticles but commutes with the Poincare generators, then

$$U_{PCT} \equiv U_P U_T U_C = e^{i\pi J_3} U_\tau U_C \ .$$

Since the Lorentz transformation matrix $\Lambda_{PT}{}^{\mu}{}_{\nu}$ is simply $(-1)\delta^{\mu}{}_{\nu}$ and U_{PCT} is antiunitary, the transformation law for the poincare generators under U_{PCT} is simply

$$U_{PCT}^{-1} \ P^\mu \ U_{PCT} = P^\mu$$
$$U_{PCT}^{-1} \ M^{\mu\nu} \ U_{PCT} = - M^{\mu\nu} \ .$$

TABLE II-1

Behavior of the Poincaré Generators under
Parity and Time-Reversal Transformations

O	$U_P^{-1} O U_P$	$U_a^{-1} O U_a$	$U_\tau^{-1} O U_\tau$
η	H	η	η
H	$-\eta$	H	H
$\mathbf{P}$	$-\mathbf{P}$	$(-P^1, P^2)$	$-\mathbf{P}$
$\mathbf{B}$	$-\mathbf{S}$	$(-B^1, B^2)$	$\mathbf{B}$
J_3	J_3	$-J_3$	$-J_3$
K_3	$-K_3$	K_3	$-K_3$
$\mathbf{S}$	$-\mathbf{B}$	$(-S^1, S^2)$	$\mathbf{S}$
$\mathbf{R}$		$(-R^1, R^2)$	$\mathbf{R}$
j_3		$-j_3$	$-j_3$
$\mathbf{s}$		$(-s^1, s^2)$	$\mathbf{s}$

D. Single Particle States

The states of a single particle with mass M, spin S are generally represented by state vectors $| P, \lambda >$, where P^{μ} is the momentum of the particle and the discrete index λ labels its spin state. Many definitions of "spin state" are available: helicity, z-component of spin in the particle rest frame, etc. Unfortunately, the familiar kinds of spin states are ill adapted for use in the infinite momentum frame; hence we devote this section to yet another variation of the description of particle spin.

We will use an informal version of the famous Wigner construction[9] to define the states $| P, \lambda >$. Consider first the case $M > 0$. We let P_0^{μ} be the momentum of a particle at rest,

$$P_0^{\mu} = 2^{-\frac{1}{2}} M \ (1, 0, 0, 1) \ .$$

Then for any other momentum on the particle's mass shell, we choose a standard transformation $\beta(P)$ in SL(2,c)[10] which transform P_0 into P:

$$\Lambda(\beta(P))^{\mu}{}_{\nu} \, P_0^{\nu} = P^{\mu} \ .$$

We define the states $| P, \lambda >$ for $P \neq P_0$ by

$$| P, \lambda \rangle = U(\beta(P)) \, | P_0, \lambda \rangle \ .$$

Then we will know how the states $| P, \lambda >$ transform under all Lorentz transformations when we give the transformation law of the rest states $| P_0, \lambda >$ under rotations (which leave P_0 invariant). If we want states with spin S, we have only to require that the states $| P_0, \lambda >$ transform under rotations according to the spin S representation of SU(2):

$$U(A)\,|P_0,\lambda\rangle = \mathcal{D}^{(s)}(A)_{\lambda'\lambda}\;|P_0,\lambda'\rangle$$

$$\text{for } A \in SU(2)$$

When we combine these two equations, we get the general transformation law for the states $|P,\lambda\rangle$. If A is in SL(2,C) and we denote $\hat{P} = \Lambda(A)P$, we find

$$U(A)\,|P,\lambda\rangle = U(\beta(\hat{P}))\,U(\beta(\hat{P})^{-1} A\, \beta(P))\;|P_0,\lambda\rangle$$

$$= \mathcal{D}^{(s)}(\beta(\hat{P})^{-1} A\,\beta(P))_{\lambda'\lambda}\;|\hat{P},\lambda'\rangle\ .$$

Note that the Lorentz transformation $\Lambda(\beta(\hat{P})^{-1}A\beta(P))$ is a rotation, since it maps P_0^{μ} back into itself; thus $\mathcal{D}^{(S)}(\beta(\hat{P})^{-1}A>(P))$ is well defined.

This is the Wigner construction. All that remains for us to do is to specify the "standard transformation" $\beta(P)$ which carries P_0 into P. The natural appearance of an internal angular momentum operator, $j_3 = J_3 - \underset{\sim}{R} \times \underset{\sim}{P}$, in the infinite-momentum frame suggests that we should choose $\beta(P)$ so that $j_3\,|P,\lambda\rangle =$ $\lambda\,|P,\lambda\rangle$. We already have $j_3\,|P_0,\lambda\rangle = \lambda\,|P_0,\lambda\rangle$, so our requirement will be met if $U(\beta(P))$ commutes with j_3. Since j_3 commutes with K_3 and $\underset{\sim}{B}$, we choose

$$\beta(P) = e^{-i\,\underset{\sim}{v}\cdot\underset{\sim}{B}}\quad e^{-i\omega K_3}$$

with $\underset{\sim}{v} = \underset{\sim}{P}/\eta$ and $e^{\omega} = 2^{\frac{1}{2}}\,\eta/M$.

Finally, we note that the states $|P,\lambda\rangle$ must be covariantly normalized if the operators U(A) are to be unitary. Thus we take

$$\langle P', \lambda' | P, \lambda \rangle = \delta_{\lambda'\lambda} \, (2\pi)^3 \, 2\eta \, \delta(\eta' - \eta) \, \delta^2(\underset{\sim}{P}' - \underset{\sim}{P}) .$$

This gives the covariant phase space integral

$$\langle \Phi | \Psi \rangle = (2\pi)^{-3} \int_0^\infty \frac{d\eta}{2\eta} \int d\underset{\sim}{P} \sum_\lambda \langle \Phi | P, \lambda \rangle \langle P, \lambda | \Psi \rangle$$

$$= (2\pi)^{-3} \int \frac{d\vec{P}}{2\sqrt{\vec{P}^2 + M^2}} \sum_\lambda \langle \Phi | P, \lambda \rangle \langle P, \lambda | \Psi \rangle .$$

Let us turn to the description of mass zero particles. Since we are now unable to consider the states of a particle at rest, we must choose another "standard momentum" P_0. A convenient choice is

$$P_0^\mu = (1, 0, 0, 0) .$$

We choose for the standard transformation $\alpha(P)$ which transforms P_0 into P

$$\alpha(P) = \exp(-i \underset{\sim}{v} \cdot \underset{\sim}{B}) \exp(-i \omega K_3) ,$$

where now $\underset{\sim}{v} = \underset{\sim}{P}/\eta$ and $e^\omega = \eta$. As before, we define

$$|P, \lambda\rangle = U(\alpha(P)) \, |P_0, \lambda\rangle .$$

We are now left with the well-known problem of deciding how the states $| P_0, \lambda >$ should transform under the groups of transformations which leave P_0 unchanged —

the so-called "little group" of P_0. It is easy to see that this little group is the subgroup of SL(2,C) generated by J_3 and $\underset{\sim}{S}$. We demand that the states $| P_0, \lambda >$ transform under this group according to one of its unitary finite dimensional irreducible representations. But the only such representations are the one-dimensional representations

$$J_z \, | P_0, \lambda \rangle = \lambda \, | P_0, \lambda \rangle$$

$$\underset{\sim}{S} \, | P_0, \lambda \rangle = 0 \quad ,$$

where λ (the helicity) can be $0, \pm 1, \pm 2, \ldots$.

Let us for the moment call the helicity λ representation $\mathbb{D}_\lambda$:

$$\mathbb{D}_\lambda(R) = e^{-i\varphi\lambda}$$

$$\text{if} \quad R = e^{-i\varphi J_3} \, e^{-i \underset{\sim}{u} \cdot \underset{\sim}{S}}$$

Then the complete transformation law for the states $| P, \lambda >$ is

$$U(R) \, | P, \lambda \rangle = \mathbb{D}_\lambda\left(\alpha(\hat{P})^{-1} \, R \, \alpha(P) \right) | \hat{P}, \lambda \rangle$$

where $\hat{P}^\mu = \Lambda(A)^\mu{}_\nu P^\nu$.

Finally, we normalize the states $| P, \lambda >$ for mass zero just as for $M > 0$:

$$\langle P', \lambda \, | P, \lambda \rangle = (2\pi)^3 \; 2\eta \, \delta(\eta' - \eta) \, \delta(\underset{\sim}{P}' - \underset{\sim}{P}) \, .$$

At this point in his trek through the group theoretic jungle, the reader may wish to pause to ask whether the spin states described above have any special properties which made them well suited to a description of physics in the infinite-momentum frame. Two closely related properties can be named. First, when the transformations generated by $\eta, \underset{\sim}{P}, H, \underset{\sim}{B}, J_3, K_3$, act on the states $|\ \eta, \underset{\sim}{P}, \lambda >$, these states transform just like states with mass η, momentum $\underset{\sim}{P}$, and spin λ in nonrelativistic quantum mechanics in two dimensions. All of these operators except J_3 act only on the variables η, $\underset{\sim}{P}$ and leave λ unaffected; under a rotation $\exp(-i\phi J_3)$ the momentum P is rotated and the state receives an extra phase $\exp(-i\phi\lambda)$. Secondly, the "internal" Lorentz generators j_3 and s discussed in Section B act in a simple way on these states. For either M > 0 or M = 0 we find

$$
\begin{aligned}
j_3\ |P,\lambda\rangle &= \lambda\ |P,\lambda\rangle \\
s^k\ |P,\lambda\rangle &= -\frac{M}{\eta}\ \epsilon_{k\ell}\ (\mathcal{J}_\ell)_{\lambda'\lambda}\ |P,\lambda'\rangle\ ,
\end{aligned}
$$

where the matrices $\mathcal{J}_\ell$ are the standard angular momentum matrices for the appropriate spin.

E. Transformation of Single Particle States under Parity and Time Reversal

It is sometimes necessary to know how the states $|\ P, \lambda >$ transform under the parity and time reversal operators discussed in Section C. Let us consider first the states $|\ P, \lambda >$ describing massive spin-s particles.

How do the states $|\ P, \lambda >$ transform under the infinite-momentum frame "parity" operator $U_a = \exp(-i\pi J_x) U_P$? It is easy to give the rule if we start from the transformation law of the states at rest, $|\ P_0, \lambda >$, under the ordinary parity operator, namely

$$
U_P\ |P_0,\lambda\rangle = c_P\ |P_0,\lambda\rangle
$$

Here C_P is the intrinsic parity of the particle, with $|C_P| = 1$. To derive the desired transformation law for a general state $|P, \lambda>$ we write $U_a |P, \lambda>$ in the form

$$U_a |P,\lambda\rangle = U_a U(\beta(P)) |P_0,\lambda\rangle$$
$$= U_a U(\beta(P)) U_a^{-1} \; e^{-i\pi J_x} \; U_p |P_0,\lambda\rangle$$
$$= C_P \; U_a U(\beta(P)) U_a^{-1} \; e^{-i\pi J_x} \; |P_0,\lambda\rangle \; .$$

By definition, the states $|P_0, \lambda>$ transform under rotations according to the spin-s representation of SU(2); thus

$$e^{-i\pi J_x} |P_0,\lambda\rangle = \mathcal{D}^{(s)}\left(e^{-i\pi J_x}\right)_{\sigma\lambda} |P_0,\sigma\rangle$$
$$= (-i)^{2s} \, \delta_{\sigma,-\lambda} \; |P_0, \sigma\rangle \; .$$

If we use Table II-1 in Section C and the definition of β (P) we can write

$$U_a \; U(\beta(P)) \, U_a^{-1}$$
$$= U_a \exp(-i \, \underset{\sim}{B}\cdot\underset{\sim}{P}/\eta) \exp(-iK_3 \ln[\sqrt{2}\,\eta/M]) \, U_a^{-1}$$
$$= \exp(-i \, \underset{\sim}{B}\cdot\hat{\underset{\sim}{P}}/\hat{\eta}) \exp(-iK_3 \ln[\sqrt{2}\,\hat{\eta}/M])$$
$$= U(\beta(\hat{P})) \quad ,$$

where $\hat{\eta} = \eta$ and $\hat{p} = (-p^1, p^2)$. Putting these results together, we have

$$U_a \,|\eta, p^1, p^2; \lambda\rangle = C_P (-i)^{2s} \,|\eta, -p^1, p^2; -\lambda\rangle .$$

Thus the action of U_a on the single particle states is very simple: it changes the transverse momentum from p to $\hat{p} = (-p^1, p^2)$ and flips the spin. In addition, the state vectors are multiplied by an overall phase factor $C_a = C_p(-i)^{2S}$.

We can similarly derive the transformation law for the states $|P, \lambda\rangle$ under the τ-reversal operator $U_\tau = \exp(-i\pi J_3)U_P U_T$. We begin with the transformation law for states at rest under the ordinary time-reversal operator U_T,

$$U_T \,|P_0, \lambda\rangle = C_T \,\mathcal{D}^{(s)}\left(e^{-i\pi J_y}\right)_{\sigma\lambda} \,|P_0, \sigma\rangle .$$

Then if we also use the formula $U_P |P_0, \lambda\rangle = C_P |P_0, \lambda\rangle$ and proceed in the same manner as before, we obtain

$$\begin{aligned}
U_\tau |P, \lambda\rangle &= U_\tau \, U(\beta(P)) \, U_\tau^{-1} \;\; e^{-i\pi J_3} \, U_P U_T \,|P_0, \lambda\rangle \\
&= U(\beta(\eta, -p)) \; C_P C_T \;\; \mathcal{D}^{(s)}\left(e^{-i\pi J_3}\right)_{\rho\sigma} \\
&\qquad\qquad \mathcal{D}^{(s)}\left(e^{-i\pi J_y}\right)_{\sigma\lambda} \,|P_0, \rho\rangle \\
&= C_P C_T \;\, \mathcal{D}^{(s)}\left(e^{+i\pi J_x}\right)_{\rho\lambda} \,|\eta, -p; \rho\rangle \\
&= C_P C_T \;\, (+i)^{2s} \;\delta_{\rho, -\lambda} \,|\eta, -p; \rho\rangle .
\end{aligned}$$

Thus finally

$$U_\tau \,|\eta, \not{p}; \lambda\rangle = C_\tau \,|\eta, -\not{p}; -\lambda\rangle \quad ,$$

where $C_\tau = C_P C_T (i)^{2S}$ is a phase factor. This is just what we would expect for a nonrelativistic time reversal operator — U_τ simply reverses the particle momentum and flips its spin.

The action of the PCT operator on the states $|\, P, \lambda >$ is also quite simple. If we define the charge conjugation operator U_C by

$$U_C \,|P, \lambda; \text{particle}\rangle = C_C \,|P, \lambda; \text{antiparticle}\rangle \quad ,$$

then we find

$$\begin{aligned} U_{PCT} \,&|P, \lambda; \text{particle}\rangle \\ &= C_{PCT}\, \mathcal{D}^{(S)}\left(e^{-i\pi J_y}\right)_{\sigma\lambda} |P, \sigma; \text{antiparticle}\rangle \\ &= C_{PCT}\, (-1)^{S-\lambda}\, |P, -\lambda; \text{antiparticle}\rangle \;. \end{aligned}$$

So far, we have discussed the parity and τ-reversal properties of the states for massive particles only. If we want to extend the discussion to cover massless particles, we must assume that the massless spin S particle can be found with both possible helicities, $\lambda = \pm S$. This is because the inversion operators U_a and U_τ both change the sign of the spin operator j_3.

Let us therefore assume that the space of single particle states contains states $|p, s>$ and states $|p, -s>$. Since the states $U_a | p_0, s>$ and $U_\tau | p_0, s>$ have helicity $-s$ and momentum $p^\mu = p_0^\mu$, we further assume that[11]

$$U_a | p_0, \lambda \rangle = C_a | p_0, -\lambda \rangle$$
$$U_\tau | p_0, \lambda \rangle = C_\tau | p_0, -\lambda \rangle$$

Using this assumption and a short calculation similar to that for massive particles, it is easy to show that

$$U_a | p, \lambda \rangle = C_a | \eta, -p^1, p^2 ; -\lambda \rangle$$
$$U_\tau | p, \lambda \rangle = C_\tau | \eta, -p ; -\lambda \rangle .$$

Since $U_{PCT} = e^{i\pi J_3} U_\tau U_C$, we also obtain

$$U_{PCT} | p, \lambda ; \text{particle} \rangle = C_{PCT} (-1)^{s-\lambda} | p, -\lambda , \text{antiparticle} \rangle$$

Note that all of these results are exactly the same as the corresponding results for massive particles.

References — Chapter II

1. We adopt the convention that transverse vectors (a^1, a^2) are denoted by boldface $\mathbf{a}$.
2. Cf. references 4 and 5 of Chapter I.
3. We should note, however, that in ordinary nonrelativistic quantum mechanics, the spectrum of the mass operator consists of a single point, whereas the operator η will generally have a continuous spectrum $0 \leq \eta < \infty$.
4. Cf. J. D. Bjorken, J. B. Kogut, and D. E. Soper, Phys. Rev. D3, 1382 (1971); J. B. Kogut, Ph.D. Thesis, Stanford University, 1971 (unpublished).
5. This form is suggested by the work of Bardakci and Halpern (reference 5 of Chapter I) on the construction of Hamiltonians in the infinite-momentum frame.
6. Cf. L. I. Schiff, Quantum Mechanics, 3rd Ed. (McGraw-Hill, New York, 1968), p. 234 ff.
7. Cf. S. Schweber, An Introduction to Relativistic Quantum Field Theory (Harper and Row, New York, 1961), p. 45 ff.
8. Note that the completely antisymmetric matrix $\epsilon^{\mu\nu\rho\sigma}$ is a pseudo tensor. If in the ordinary coordinate system $\hat{\epsilon}_{0123} = +1$, then in the infinite momentum system $\epsilon^{0123} = -\epsilon_{0123} = +1$ since the determinant of the transformation matrix between the two systems, $C^{\mu}{}_{\nu}$, is -1.
9. E. P. Wigner, Annals of Math. 40, 139 (1939).
10. We regard SL(2, C), the group of 2×2 complex matrices with unit determinant, as being nearly synonymous with the Lorentz group. To be precise, the matrices $\Lambda^{\mu}{}_{\nu}$ of the proper Lorentz group provide a representation

$A \rightarrow \Lambda(A)$ of SL(2, C). The representation is two to one: $\Lambda(A) = \Lambda(B)$ if and only if $A = \pm B$. The group SU(2) of 2×2 unitary matrices for a subgroup of SL(2, C), and the corresponding $\Lambda^{\mu}{}_{\nu}$ matrices are just the rotation matrices. Cf. R. F. Streater and A. S. Wightman, PCT, Spin and Statistics, and All that (Benjamin, New York, 1964); V. Bargman, Annals of Math. 59, 1 (1954).

11. It should be pointed out that this is an assumption; it does not follow from the previous discussion, although it is consistant with the previous discussion.

CHAPTER III

Wave Functions and Free Fields

In subsequent chapters, we will develop several interacting field theories using a canonical quantization procedure in the infinite-momentum frame. We prepare ourselves here by investigating free field theories. Fortunately, it is not necessary to use the canonical procedure to discuss free fields; one can write down an exact free field for particles with any mass and spin once one knows how to write wave functions describing the particles[1]. (For example, the free Dirac field has the form

$$\psi_\alpha(x) = (2\pi)^{-3} \int d\underline{p} \int_0^\infty \frac{d\eta}{2\eta} \sum_{\lambda=\pm 1/2} \Big\{ U_\alpha(p,\lambda)\, e^{-ip\cdot x}\, b(p,\lambda) + V_\alpha(p,\lambda)\, e^{+ip\cdot x}\, d^\dagger(p,\lambda) \Big\},$$

where, for instance, $U_\alpha(p, \lambda)\exp(-ip\cdot x)$ is the wave function in coordinate space for the electron state $|\, p, \lambda >$ destroyed by $b(p, \lambda)$.)

The free fields constructed here will be useful for checking the results of the canonical quantization procedure of later chapters when the interactions between the fields are turned off. In addition, free fields and wave functions are useful by themselves for discussing the general form of scattering amplitudes.

A. Finite Dimensional Representations of SL(2,C)

We pause to recall the finite-dimensional representations of SL(2,C).[2] These are named $\mathscr{D}^{(S_1, S_2)}$, where S_1 and S_2 are $0, \frac{1}{2}, 1, \ldots$; the individual matrices representing an SL(2,C) transformation A in the representation $\mathscr{D}^{(S_1, S_2)}$ are written $\mathscr{D}^{(S_1, S_2)}(A)_{\alpha\beta}$.

The representations $\mathcal{D}^{(S,0)}$ can be specified as follows: the rotation generators $\vec{J}$ are the standard $(2S+1) \times (2S+1)$-component angular momentum matrices $\vec{M}^{(S)}_{\alpha\beta}$ defined in quantum mechanics textbooks; the generators of Lorentz boosts and simply i times these angular momentum matrices, $\vec{K} = i\,\vec{M}$. Note that according to this definition, $\mathcal{D}^{(\frac{1}{2},0)}(A)_{\alpha\beta} = A_{\alpha\beta}$. It is also useful to note that $\mathcal{D}^{(S,0)}(A^{\dagger}) = \mathcal{D}^{(S,0)}(A)^{\dagger}$ and $\mathcal{D}^{(S,0)}(A^*) = \mathcal{D}^{(S,0)}(A)^*$.

The representations $\mathcal{D}^{(0,S)}$ can be specified by defining $\mathcal{D}^{(0,S)}(A) = \mathcal{D}^{(S,0)}(A^{\dagger-1})$. (Thus note that $\mathcal{D}^{(0,S)}(A) = \mathcal{D}^{(S,0)}(A)$ if A represents a rotation, $A^{\dagger} = A^{-1}$.) The infinitesimal generators in the (0, S) representation are $\vec{J} = \vec{M}^{(S)}$, $\vec{K} = -i\vec{M}$.

Finally, the representation $\mathcal{D}^{(S_1,S_2)}$ can be obtained by forming the Kronecker product $\mathcal{D}^{(S_1,0)} \times \mathcal{D}^{(0,S_2)}$:

$$\mathcal{D}^{(S_1,S_2)}(A)_{\alpha\beta,\alpha'\beta'} = \mathcal{D}^{(S_1,0)}(A)_{\alpha\alpha'}\ \mathcal{D}^{(0,S_2)}(A)_{\beta\beta'}$$

B. (2S + 1) - Component Wave Functions for Mass M > 0, Spin S

Consider the space of states $|\psi\rangle$ of a single particle with mass $M > 0$, spin S. The amplitude $\langle p,\lambda|\psi\rangle$ for the particle to have momentum p^{μ} and infinite-momentum helicity λ might be considered to be a wave function representing the state $|\psi\rangle$. However, such amplitudes have a very messy transformation law under SL(2, C): the "wave function" representing the transformed state $U(A)|\psi\rangle$ is

$$\begin{aligned}&\langle p,\lambda|U(A)|\psi\rangle \\ &\quad = \mathcal{D}^{(S)}\left(\beta(p)^{-1} A\, \beta(p')\right)_{\lambda\sigma}\langle p',\sigma|\psi\rangle\end{aligned} \tag{III.1}$$

where $p'^{\mu} = \Lambda(A^{-1})^{\mu}{}_{\nu} p^{\nu}$, $\mathscr{D}^{(S)}$ is the spin S representation of the rotation group, SU(2), and $\beta(p)$ is the "standard SL(2, C) transformation" which carries the rest momentum p_0^{μ} into p^{μ}.

In order to define wave functions which have a simple transformation law, we need only recall that the representation $\mathscr{D}^{(S)}$ of SU(2) can be extended to give the representation $\mathscr{D}^{(S,0)}$ of SL(2,C). Thus we can write the matrix in (III.1) as

$$\mathscr{D}^{(s)}(\beta(p)^{-1} A\, \beta(p')) = \mathscr{D}^{(s,0)}(\beta(p)^{-1} A\, \beta(p'))$$
$$= \mathscr{D}^{(s,0)}(\beta(p)^{-1})\, \mathscr{D}^{(s,0)}(A)\, \mathscr{D}^{(s,0)}(\beta(p')) .$$

Now the ugly momentum dependent matrices $\mathscr{D}(\beta(p)^{-1})$ can be absorbed into the definition of the wave functions, leaving only the matrix $\mathscr{D}(A)$ in the transformation law.

We are therefore led to define a (2S+1)-component wave function $\psi_{\alpha}(p)$ representing the state $|\psi\rangle$ by

$$\psi_{\alpha}(p) = \mathscr{D}^{(s,0)}(\beta(p))_{\alpha\beta} \langle p, \beta | \psi \rangle . \tag{III.2}$$

Then under a Lorentz transformation $|\psi\rangle \to |\hat{\psi}\rangle = U(A)|\psi\rangle$, the wave function transforms simply according to the $\mathscr{D}^{(S,0)}$ representation of SL(2,C):

$$\psi_{\alpha}(p) \to \hat{\psi}_{\alpha}(p) = \mathscr{D}^{(s,0)}(A)_{\alpha\beta}\, \psi_{\beta}(p') . \tag{III.3}$$

C. (2S+1)-Component Free Fields for Mass M > 0, Spin S

Now that we know how to write wave functions, it is easy to construct a free field. We note that the wave function in coordinate space corresponding to the state $|\psi\rangle = |k, \lambda\rangle$ is $\psi_\alpha(x) = u_\alpha(k,\lambda)\, e^{-ik\cdot x}$ where

$$u_\alpha(k,\lambda) = M^S\, \mathcal{D}^{(S,0)}(\beta(k))_{\alpha\lambda} \quad . \tag{III.4}$$

A particle destruction field can be formed by multiplying the particle destruction operators $b(p,\lambda)$ by the corresponding wave functions, then summing over the complete set of single particles states:

$$\psi_\alpha^{(+)}(x) = (2\pi)^{-3}\int d\underset{\sim}{p}\int_0^\infty \frac{d\eta}{2\eta}\sum_\lambda u_\alpha(p,\lambda)\, e^{-ip\cdot x}\, b(p,\lambda) \quad . \tag{III.5}$$

It is easy to see that the field so constructed transforms according to the representation $\mathcal{D}^{(S,0)}$ of SL(2,C):

$$U(A)^{-1}\psi_\alpha^{(+)}(x)\,U(A) = \mathcal{D}^{(S,0)}(A)_{\alpha\beta}\,\psi_\beta^{(+)}(\Lambda(A)^{-1}x) \quad . \tag{III.6}$$

An antiparticle creation field $\psi^{(-)}(x)$ can be constructed in a similar fashion:

$$\psi_\alpha^{(-)}(x) = (2\pi)^{-3}\int d\underset{\sim}{p}\int_0^\infty \frac{d\eta}{2\eta}\sum_\lambda u_\alpha(p,\lambda)\, e^{+ip\cdot x}\, b^\dagger_{PCT}(p,\lambda) \quad , \tag{III.7}$$

where the creation operator $b^\dagger_{PCT}(p,\lambda)$ creates the antiparticle state

$$b^\dagger_{PCT}(p,\lambda)|0\rangle = \frac{1}{C_{PCT}}\, U_{PCT}\, |p,\lambda;\text{particle}\rangle \quad . \tag{III.8}$$

Here U_{PCT} is the PCT operator discussed in Chapter II and C_{CPT} is the "PCT phase" of the particle (which is normally arbitrary unless the particle is its own antiparticle). If we recall that U_{PCT} is antiunitary and commutes with Lorentz transformations U(A), we can show quite simply that $\psi_\alpha^{(-)}(x)$ also transforms according to the representation $\mathscr{D}^{(S,0)}$ of SL(2,C).

The creation field $\psi_\alpha^{(-)}(x)$ can be written without the explicit appearance of the PCT operator if desired. We recall from Chapter II that

$$\begin{aligned} U_{PCT} & |p, \lambda; \text{particle}\rangle \\ &= C_{PCT}\, \mathscr{D}^{(S,0)}(-i\gamma_y)_{\sigma\lambda}\, |p, \sigma; \text{antiparticle}\rangle . \end{aligned}$$

Thus if we write $d^\dagger(p, \sigma)$ for the creation operator which creates the state $|\, p, \sigma$; antiparticle >, we have

$$u_\alpha(p,\lambda)\, b^\dagger_{PCT}(p,\lambda) = \sum_\sigma u_\alpha(p,\lambda)\, \mathscr{D}(-i\gamma_y)_{\sigma\lambda}\, d^\dagger(p,\sigma) .$$

Therefore, an alternate form of $\psi^{(-)}(x)$ is

$$\psi_\alpha^{(-)}(x) = (2\pi)^{-3} \int d\mathbf{p} \int_0^\infty \frac{d\eta}{2\pi} \sum_\lambda v_\alpha(p,\lambda)\, e^{+ip\cdot x}\, d^\dagger(p,\lambda) \tag{III.9}$$

where

$$v_\alpha(p,\lambda) = \sum_\beta u_\alpha(p,\beta)\, \mathscr{D}^{(S,0)}(-i\gamma_y)_{\lambda\beta} . \tag{III.10}$$

The complete (2S+1)-component field $\psi(x)$ is the sum of $\psi^{(+)}(x)$ and $\psi^{(-)}(x)$:

$$\psi(x) = (2\pi)^{-3} \int d\underset{\sim}{p} \int_0^\infty \frac{d\eta}{2\eta} \sum_{\lambda=\pm 1/2} \times \left\{ u(p,\lambda)\, e^{-ip\cdot x}\, b(p,\lambda) + v(p,\lambda)\, e^{+ip\cdot x}\, d^\dagger(p,\lambda) \right\} \tag{III.11}$$

It is an instructive (although by no means novel) exercise to compute the free field commutator or anticommutator $\left[\psi(x),\ \psi(0)^\dagger\right]_\pm$. Simple calculation gives

$$\left[\psi_\alpha(x),\ \psi_\beta^\dagger(0)\right]_\pm = (2\pi)^{-3} \int d\underset{\sim}{p} \int_0^\infty \frac{d\eta}{2\eta} \left[e^{-ip\cdot x} \pm e^{+ip\cdot x}\right] \times \sum_\lambda u_\alpha(p,\lambda)\, u_\beta^*(p,\lambda) . \tag{III.12}$$

The spinor sum is $\Sigma_\lambda U_\alpha(p,\lambda) U_\beta^*(p,\lambda) = M^{2S}\, \mathscr{D}^{(S,0)}(\beta(p)\beta(p)^\dagger)_{\alpha\beta}$. We can see that $\beta(p)\beta(p)^\dagger$ is quite a simple matrix if we recall the formula defining the relationship between a matrix $A \in SL(2,C)$ and the corresponding Lorentz transformation $\Lambda(A)^\mu{}_\nu$: [3]

$$A\, \underset{\sim}{p}\, A^\dagger = \underset{\sim}{\Lambda(A)p}$$

where

$$\underset{\sim}{p} = \sqrt{2} \begin{pmatrix} \eta & p_- \\ p_+ & H \end{pmatrix} , \quad p_\pm = 2^{-1/2}(p^1 \pm i p^2) .$$

Since $\Lambda(\beta(p))^{\mu}{}_{\nu} p_0^{\nu} = p^{\mu}$, where $p_0^{\mu} = 2^{-\frac{1}{2}}$ (M, 0, 0, M), this relation gives

$$\beta(p)\beta(p)^{\dagger} = \frac{1}{M} \not{p} = \frac{\sqrt{2}}{M} \begin{pmatrix} \eta & p_- \\ p_+ & H \end{pmatrix} .$$

Finally, we recall that the matrix elements $\mathcal{D}^{(S,0)}(A)_{\alpha\beta}$ are polynomials of degree 2S in the matrix elements of A. Thus the matrix elements

$$\sum_{\lambda} u_{\alpha}(p,\lambda)\, u^{*}_{\beta}(p,\lambda) = M^{2S} \mathcal{D}^{(S,0)}\left(\frac{1}{M} \not{p}\right)_{\alpha\beta}$$

are polynomials of degree 2S in the momentum components p^{μ}. Integration by parts in (III.12) then gives

$$[\psi_{\alpha}(x), \psi_{\beta}^{\dagger}(0)]_{\pm} = \mathcal{D}^{(S,0)}\left(\underset{\sim}{i \frac{\partial}{\partial x^{\nu}}}\right)_{\alpha\beta} \Delta(X, M) , \tag{III.13}$$

where $\Delta(X, M)$ is the ordinary scalar commutator function — provided we made the right association between spin and statistics.

D. <u>2 × (2S+1) - Component Wave Functions and Free Fields</u>

In the last two sections, we chose to use the (S, 0)-representation of SL(2,C), but we could just as well have used the (0, S)-representation. Had we done so, we would have defined wave functions

$$\psi^{\nu}_{\alpha}(p) = \mathcal{D}^{(0,S)}(\beta(p))_{\alpha\beta} \langle p, \beta \mid \psi \rangle . \tag{III.14}$$

The relation between the (S, 0)-wave functions ψ (p) and the (0, S)-wave functions ψ^{ν} (p) is

$$\psi_\alpha(p) = \mathcal{D}^{(s,0)}(\beta(p))_{\alpha\beta}\, \mathcal{D}^{(0,s)}(\beta(p)^{-1})_{\beta\gamma}\, \psi'_\gamma(p)$$

This relation can be simplified if we recall that $\mathscr{D}^{(0,S)}(A) = \mathscr{D}^{(S,0)}(A^{\dagger -1})$ and that $\beta(p)\beta(p)^\dagger = \frac{1}{M}\underset{\sim}{p}$:

$$\psi_\alpha(p) = \mathcal{D}^{(s,0)}(\tfrac{1}{M}\underset{\sim}{p})_{\alpha\beta}\, \psi'_\beta(p) \quad \text{(III.15)}$$

The inverse if this is

$$\psi'_\alpha(p) = \mathcal{D}^{(s,0)}(\tfrac{1}{M}\tilde{p})_{\alpha\beta}\, \psi_\beta(p) \quad \text{(III.16)}$$

where

$$\tilde{p} = \sqrt{2}\begin{pmatrix} H & -p_- \\ -p_+ & \eta \end{pmatrix} . \quad \text{(III.17)}$$

One can then use the wave functions $\psi'(p)$ to form a free field $\psi'(x)$ which transforms according to the (0, S)-representation of SL(2,C):

$$\psi'(x) = (2\pi)^{-3}\int d\underset{\sim}{p}\int_0^\infty \frac{d\eta}{2\eta}\sum_\lambda \times\left\{ u'(p,\lambda)\, e^{-ip\cdot x}\, b(p,\lambda) + (-1)^{2s}\, u'(p,\lambda)\, e^{ip\cdot x} \times b^\dagger_{PCT}(p,\lambda)\right\} \quad \text{(III.18)}$$

(The factor $(-1)^{2S}$ is superfluous of $\psi'(x)$ is being considered by itself, but is needed to insure that $\psi'(x)$ has a causal commutator with $\psi(x)$.) Apparently the fields $\psi(x)$ and $\psi'(x)$ are related to one another by

$$M^{2s}\,\psi(x) = \mathcal{D}^{(s,0)}(i\underset{\sim}{\partial}_\nu)\,\psi'(x)$$

$$M^{2s}\,\psi'(x) = \mathcal{D}^{(s,0)}(i\widetilde{\partial}_\nu)\,\psi(x) \quad .$$

An elementary calculation will show that the two fields ψ, ψ' are parity transforms of one another: $U_P^{-1}\,\psi(x)\,U_P = C_P\,\psi'(\Lambda_P x)$. Thus, if we want to discuss a theory invariant under the parity transformation, it makes sense to use a combined $2 \times (2S+1)$ component field

$$\Psi(x) = \begin{pmatrix} \psi(x) \\ \psi'(x) \end{pmatrix} \quad . \tag{III.19}$$

Since $\psi(x)$ and $\psi'(x)$ are related, $\Psi(x)$ satisfies an equation of motion in addition to $\left[\partial_\mu\partial^\mu - M^2\right]\psi = 0$:

$$\left\{ \begin{bmatrix} 0 & \mathcal{D}(i\underset{\sim}{\partial}) \\ \mathcal{D}(i\widetilde{\partial}) & 0 \end{bmatrix} - M^{2s}\,\mathbb{1} \right\} \Psi(x) = 0 \quad . \tag{III.20}$$

We recall that the matrix elements $\mathcal{D}(\underset{\sim}{p})_{\alpha\beta}$, $\mathcal{D}(\widetilde{p})_{\alpha\beta}$ are polynomials of degree 2S in the momentum components p_μ. Thus (III.19) is a differential equation of the form .

$$\left\{ (i\partial_\mu)(i\partial_\nu)\cdots(i\partial_\sigma)\,\Gamma^{\mu\nu\cdots\sigma} - M^{2S} \right\} \Psi = 0$$

where the $\Gamma^{\mu\nu\cdots\sigma}$ are certain matrices which can be easily computed. For instance, for $S = \frac{1}{2}$ the Γ^{μ} form a certain representation of the Dirac γ-matrix algebra, and Eq. (III.20) is just the Dirac equation.

Using our previous results, we can write out the Fourier expansion of the $2(2S+1)$-component field $\psi(x)$:

$$\Psi(x) = (2\pi)^3 \int d\mathbf{p} \int_0^\infty \frac{d\eta}{2\eta} \sum_{\lambda=-s}^{s} \times \left\{ \mathcal{U}(p,\lambda)\, e^{-ip\cdot x}\, b(p,\lambda) + \mathcal{V}(p,\lambda)\, e^{+ip\cdot x}\, d^\dagger(p,\lambda) \right\} \tag{III.21}$$

where the spinors $U(p, \lambda)$ are

$$\mathcal{U}(p,\lambda) = \begin{pmatrix} u_\alpha(p,\lambda) \\ u'_\alpha(p,\lambda) \end{pmatrix} = M^s \begin{pmatrix} \mathcal{D}^{(s,0)}(\beta(p))_{\alpha\lambda} \\ \mathcal{D}^{(0,s)}(\beta(p))_{\alpha\lambda} \end{pmatrix} \tag{III.22}$$

and the charge conjugate spinors $V(p, \lambda)$ are

$$\mathcal{V}(p,\lambda) = \begin{pmatrix} \mathbb{1} & 0 \\ 0 & (-1)^{2s}\mathbb{1} \end{pmatrix} \mathcal{U}(p,\sigma)\, \mathcal{D}^{(s,0)}(-i\gamma_J)_{\lambda\sigma} \,. \tag{III.23}$$

When the free fields $\psi(x)$ are used in applications, either as in- and out-fields or as "bare fields" in perturbation theory, matrix elements of covariant operators will consist of the spinors $U(p, \lambda)$, $V(p, \lambda)$ tied together with such covariant objects as the matrices $\Gamma^{\mu\nu\ldots\sigma}$, momenta p_μ, and scalar form factors. Thus it is quite generally useful to have explicit expressions for these spinors. Such expressions will be derived in the next section.

E. Evaluation of the Spinors U(p, λ).

The observant reader will have noticed that the well-known formalism which we have briefly outlined in this chapter can be used with any choice of the type of spin states. In this section, we will specialize to the choice of states useful in the infinite-momentum frame — namely the infinite-momentum helicity states $|\, p, \lambda >$. We will obtain explicit expressions for the spinors $u_\alpha(p, \lambda)$ and $u'_\alpha(p, \lambda)$.

We begin by constructing the "standard Lorentz transformation" $\beta(p)$,

$$\beta(p) = e^{-i\, \underset{\sim}{v} \cdot \underset{\sim}{B}}\ e^{-i \omega K_3}$$

where $\underset{\sim}{v} = \underset{\sim}{p}/\eta$ and $e^{\omega} = \sqrt{2}\,\eta/M$. We recall that the generators of rotations in SL(2,C) are the Pauli spin matrices, $\vec{J} = \frac{1}{2}\vec{\tau}$, and that the generators of Lorentz boosts are $\vec{K} = \frac{i}{2}\vec{\tau}$. Thus the generators $\underset{\sim}{B}$ and K_3 are

$$\begin{aligned} B^1 &= 2^{-\frac{1}{2}}(K_1 + J_2) = 2^{-\frac{1}{2}}\begin{pmatrix} 0 & 0 \\ i & 0 \end{pmatrix} \\ B^2 &= 2^{-\frac{1}{2}}(K_2 - J_1) = 2^{-\frac{1}{2}}\begin{pmatrix} 0 & 0 \\ -1 & 0 \end{pmatrix} \\ K_3 &= \frac{i}{2}\begin{pmatrix} 1 & 0 \\ 0 & -1 \end{pmatrix} . \end{aligned} \tag{III.24}$$

The exponentials $\exp(-i\underset{\sim}{v}\cdot\underset{\sim}{B})$ and $\exp(-i\omega K_3)$ can be easily worked out, giving

$$\beta(p) = \left(\frac{M}{\sqrt{2}}\eta\right)^{-\frac{1}{2}} \begin{pmatrix} \eta & 0 \\ p_+ & \frac{M}{\sqrt{2}} \end{pmatrix} , \tag{III.25}$$

$$\beta(p)^{\dagger -1} = \left(\frac{M}{\sqrt{2}}\eta\right)^{-\frac{1}{2}} \begin{pmatrix} \frac{M}{\sqrt{2}} & -p_- \\ 0 & \eta \end{pmatrix} . \tag{III.26}$$

Now we need an expression for the matrix elements of $\mathscr{D}^{(S,0)}(A)$ in terms of the matrix elements of A. To obtain such an expression we consider the reducible representation $\mathscr{D}^{(\frac{1}{2},0)} \times \mathscr{D}^{(\frac{1}{2},0)} \times \ldots \times \mathscr{D}^{(\frac{1}{2},0)}$ of SL(2,C) with 2S factors of $\mathscr{D}^{(\frac{1}{2},0)}$. This representation acts on the space of spinors $\xi_{\alpha_1 \alpha_2 \ldots \alpha_{2S}}$, where each index α takes the values $\pm \frac{1}{2}$, according to the rule

$$\xi_{\alpha_1 \ldots \alpha_{2S}} \xrightarrow[A]{} \sum_{(\beta)} A_{\alpha_1 \beta_1} \cdots A_{\alpha_{2S} \beta_{2S}} \, \xi_{\beta_1 \cdots \beta_{2S}} . \tag{III.27}$$

It is not difficult to see that the subspace of totally symmetric spinors is left invariant under these operators, and that the representation of SL(2,C) defined in the symmetric subspace by (III.27) is $\mathscr{D}^{(S,0)}$. A suitable[4] orthonormal basis for the symmetric subspace consists of 2S+1 vectors $\xi(\lambda)$, $\lambda = -S, \ldots, S$, defined by

$$\xi(\lambda)_{\alpha_1 \cdots \alpha_{2S}} = \left[(2S)!\,(S+\lambda)!\,(S-\lambda)!\right]^{-1/2} \sum_{\substack{\text{permutations} \\ \text{of the } \alpha\text{'s}}} \times \underbrace{\delta_{\alpha_1, \frac{1}{2}} \cdots \delta_{\alpha_{S+\lambda}, \frac{1}{2}}}_{\substack{(S+\lambda) \text{ factors} \\ \text{of } \delta_{\alpha, +1/2}}} \underbrace{\delta_{\alpha_{S+\lambda+1}, -\frac{1}{2}} \cdots \delta_{\alpha_{2S}, -\frac{1}{2}}}_{\substack{(S-\lambda) \text{ factors} \\ \text{of } \delta_{\alpha, -1/2}}} \tag{III.28}$$

The desired matrix elements of $\mathscr{D}^{(S,0)}(A)$ are simply

$$\mathscr{D}^{(s,0)}(A)_{\lambda'\lambda} = \xi(\lambda')^*_{\alpha_1\cdots\alpha_{2s}} A_{\alpha_1\beta_1}\cdots A_{\alpha_{2s}\beta_{2s}}\, \xi(\lambda)_{\beta_1\cdots\beta_{2s}} \qquad \text{(III.29)}$$

Thus the matrix elements $\mathscr{D}^{(S,0)}(A)_{\lambda'\lambda}$ are polynomials in the matrix elements A_{++}, A_{+-}, A_{-+}, A_{--} of A. It is not difficult to compute the coefficient of the general term $(A_{++})^a(A_{+-})^b(A_{-+})^c(A_{--})^d$ in this polynomial by a simple counting argument. The result is

$$\mathscr{D}^{(s,0)}(A)_{\lambda'\lambda} = \left[(s+\lambda')!\,(s-\lambda')!\,(s+\lambda)!\,(s-\lambda)!\right]^{\frac{1}{2}}$$

$$\times \sum_{a,b,c,d} (a!\,b!\,c!\,d!)^{-1}\,(A_{++})^a\,(A_{+-})^b\,(A_{-+})^c\,(A_{--})^d, \qquad \text{(III.30)}$$

where the sumincludes all those values of a, b, c, d in the range 0, 1, ..., 2S which satisfy

$$\begin{aligned} a+b+c+d &= 2s \\ a+b-c-d &= 2\lambda' \\ a-b+c-d &= 2\lambda \end{aligned} \qquad \text{(III.31)}$$

Now we are ready to evaluate $\mathscr{D}^{(S,0)}(\beta(p))_{\lambda'\lambda}$, where $\beta(p)$ is given by (III.25). Since the component $\beta(p)_{+-}$ is zero, the only non-zero terms in (III.30) are those with $b = 0$; but there is only one solution of (III.31) with $b = 0$, namely $a = S + \lambda'$, $b = 0$, $c = \lambda - \lambda'$, $d = s - \lambda$. Since the sum in (III.30) includes only non-negative values of the exponents a, b, c, d, this solution leads to a non-zero matric element $\mathscr{D}(A)_{\lambda'\lambda}$ only if $c = \lambda - \lambda' \geq 0$.

Thus we obtain for the spinor $u_\alpha(p,\lambda) = M^S \mathscr{D}(\beta(p))_{\alpha\lambda}$

$$u_\alpha(p,\lambda) = \sqrt{\frac{(S-\alpha)!\,(S+\lambda)!}{(S+\alpha)!\,(S-\lambda)!}}\ \frac{2^{\lambda/2}}{(\lambda-\alpha)!}\ \Theta(\alpha\leq\lambda) \times \eta^S \left(\frac{M}{\eta}\right)^{S-\lambda}\left(\frac{p_+}{\eta}\right)^{\lambda-\alpha} \tag{III.32}$$

where $\Theta(\alpha \leq \lambda)$ is 1 for $\alpha \leq \lambda$, zero if $\alpha > \lambda$. We find in a similar fashion that the spinor $u'_\alpha(p,\lambda) = M^S \mathscr{D}(\beta(p)^{\dagger-1})_{\alpha\lambda}$ is

$$u'_\alpha(p,\lambda) = \sqrt{\frac{(S+\alpha)!\,(S-\lambda)!}{(S-\alpha)!\,(S+\lambda)!}}\ \frac{2^{-\lambda/2}}{(\alpha-\lambda)!}\ \Theta(\lambda\leq\alpha) \times \eta^S \left(\frac{M}{\eta}\right)^{S+\lambda}\left(\frac{-p_-}{\eta}\right)^{\alpha-\lambda}. \tag{III.33}$$

It is remarkable that these infinite-momentum helicity spinors are so simple. The spinors for Jacob and Wick helicity states, by way of contrast, have the form

$$u_\alpha(p,\lambda) \sim e^{\omega\lambda}\, e^{-i\varphi(\alpha-\lambda)}\, [\sin\Theta]^S \sum_{m=-S}^{S} c(m)_{\alpha\lambda}\, [\tan\tfrac{\Theta}{2}]^m$$

where (Θ, ϕ) are the polar angles of $\vec{p}$ and $\cosh\omega = \sqrt{\vec{p}^{\,2} + M^2}/M$. One specific effect of this general simplicity of spinors for infinite-momentum helicity states may be seen in calculations of scattering amplitudes in quantum electrodynamics using "old-fashioned" perturbation theory in the infinite-momentum frame.[5] In such practical calculations, it is generally easy to sum over the intermediate spin states directly to product the scattering amplitude; one is not forced to avoid the spinors by casting the problem in such a way as to make the answer proportional to a trace.

In this chapter we have not discussed wave functions or fields for massless particles. Such a discussion would be quite simple and would run along the same lines as the present discussion of massive particles. However we will be content merely to state the result: the spinors and fields for massless particles can be obtained from those for massive particles by simply setting $M = 0$. In particular, note that the massless spin-S field $\psi(x)$ transforming according to the $\mathscr{D}^{(S,0)}$ representation of SL(2,C) destroys only parricles with helicity $\lambda = S$. Similarly, the field $\psi'(x)$ transforming according to the $\mathscr{D}^{(0,S)}$ representation destroys only particles with helicity $\lambda = -S$.

F. Fields Transforming under Other Representations of SL(2,C).

We have described fields for massive particles with spin S which transform according to the representations $\mathscr{D}^{(S,0)}$ and $\mathscr{D}^{(0,S)}$ of SL(2, C). It is, of course, also possible to describe such particles using a field which transforms under any of the representations $\mathscr{D}^{(a,b)}$ with $S = |a-b|, |a-b|+1, \ldots$ or $|a+b|$. For example, spin 1 particles can be described by a 4-vector field (which transforms according to $\mathscr{D}^{(\frac{1}{2},\frac{1}{2})}$). For the sake of completeness — and since we will make use of 4-vector fields in later chapters — we will recall here how such fields can be constructed.

We note that the irreducible representation $\mathscr{D}^{(a,b)} = \mathscr{D}^{(a,0)} \times \mathscr{D}^{(0,b)}$ of SL(2,C) is reducible when it is considered as a representation of SU(2). If $S = |a-b|, |a-b|+1, \ldots,$ or $|a+b|$ then the spin-S representation of SU(2) will be contained in this representation. We let the $(2S+1)$ vectors $\omega(\lambda)$ be the basis vectors in the representation space of $\mathscr{D}^{(a,b)}$ for the spin-S representation of SU(2). Thus for A ϵ SU(2) ,

$$\mathcal{D}^{(a,b)}(A)_{\alpha\beta}\, w_\beta(\lambda) = w_\alpha(\lambda')\, \mathcal{D}^{(s)}(A)_{\lambda'\lambda} \quad . \tag{III.34}$$

For example, if we are using the $(\frac{1}{2},\frac{1}{2})$-representation of SL(2,C) to describe massive spin one particles, the $\omega(\lambda)$ are 4-vectors (in infinite-momentum coordinates) :

$$\begin{aligned} w^\mu(+1) &= -\tfrac{1}{\sqrt{2}}\,(0,1,i,0) \\ w^\mu(0) &= \tfrac{1}{\sqrt{2}}\,(1,0,0,-1) \\ w^\mu(-1) &= \tfrac{1}{\sqrt{2}}\,(0,1,-i,0) \quad . \end{aligned} \tag{III.35}$$

Spinors to describe a state with momentum p, helicity λ can now be defined by

$$u_\alpha(p,\lambda) = \mathcal{D}^{(a,b)}(\beta(p))_{\alpha\beta}\, w_\beta(\lambda) \quad . \tag{III.36}$$

These spinors can be used to construct the field

$$\begin{aligned} \Psi_\alpha(x) = (2\pi)^3 \int d\underset{\sim}{p} \int_0^\infty \frac{d\eta}{2\eta} \sum_{\lambda=-s}^{s} \\ \times \Big\{ u_\alpha(p,\lambda)\, e^{-ip\cdot x}\, b(p,\lambda) \\ + (-1)^{2b}\, u_\alpha(p,\lambda)\, e^{+ip\cdot x}\, b^\dagger_{PCT}(p,\lambda) \Big\} . \end{aligned} \tag{III.37}$$

It is a simple exercise to use (III.37) and the transformation laws for the states $| p, \lambda >$ to show that this field does indeed transform according to the (a, b)-representation of SL(2,C).[6]

In our example of a massive vector field, the vectors $U(p, \lambda)$ are $U^{\nu}(p, \lambda) = \Lambda(\beta(p))^{\nu}{}_{\mu}\, \omega^{\mu}(\lambda)$:

$$
\begin{aligned}
\mathcal{U}^{\nu}(p,+1) &= -\frac{1}{\sqrt{2}} \left(0,\ 1,\ i,\ [p^1 + i p^2]/\eta\right) \\
\mathcal{U}^{\nu}(p,0) &= \frac{1}{M} \left(\eta,\ p^1,\ p^2,\ [p^2 - M^2]/2\eta\right) \\
\mathcal{U}^{\nu}(p,-1) &= \frac{1}{\sqrt{2}} \left(0,\ 1,\ -i,\ [p^1 - i p^2]/\eta\right) \quad .
\end{aligned}
\tag{III.38}
$$

It is instructive to notice that the massive spin one field constructed using the $(\frac{1}{2}, \frac{1}{2})$-representation of SL(2,C) does not have a nice limit as $M \rightarrow 0$ since the polarization vector for helicity zero blows up instead of vanishing as $M \rightarrow 0$. We will see in the next two chapters that this problem can be overcome in massive and massless quantum electrodynamics by the use of gauge invariance.

References — Chapter III

1. Cf. S. Weinberg, Phys. Rev. 133, B1318 (1964).
2. See, for example, R. F. Streater and A. S. Wightman, PCT, Spin and Statistics, and All That (Benjamin, New York, 1964) and references quoted therein; our notation differs slightly from that of Streater and Wightman.
3. Cf. Streater and Wightman, op. cit.
4. This choice of basis bakes the angular momentum magrices $\vec{J}$ in the representation $\mathscr{D}^{(S,0)}$ obey the standard phase conventions (i. e. the matrix J_z is diagonal and the matrix elements of $J_x \pm iJ_y$ are real and positive).
5. Cf. J. D. Bjorken, J. B. Kogut, and D. E. Soper, Phys. Rev. D3, 1382 (1971).
6. The factor $(-1)^b$ in (III.37) insures that all of the free fields which might be used to describe the same particle have causal commutation relations with one another.

CHAPTER IV

Quantum Electrodynamics in the Infinite-Momentum Frame

This chapter is devoted to a reformulation of conventional quantum electrodynamics in the infinite-momentum frame variables. The chapter is divided into two parts. In the first part, we consider the Feynman perturbation expansion for the S matrix, divorced from its field theoretic underpinnings. We write the covariant Feynman diagrams using the variables $(\tau, \underset{\sim}{x}, \mathfrak{z})$ and then decompose each covariant diagram into a sum of "old-fashioned" τ-ordered diagrams. The results are similar to Weinberg's results concerning the infinite-momentum limit of t-ordered diagrams, but the appearance of spin results in the emergence of new types of vertices. In the second part, we look at the field theoretic underpinnings. We quantize the theory directly in the infinite-momentum frame by postulating equal-τ commutation relations among the fields. We find that these equal-τ commutation relations make the unquantized field theory into a formally consistent quantum field theory; in particular, the canonical Hamiltonian generates τ-translations of the fields according to their equations of motion. Finally, we find that the old-fashioned perturbation expansion for the S-matrix derived using the canonical Hamiltonian agrees with the τ-ordered expansion derived directly from the covariant Feynman diagrams in Part I.

In subsequent chapters, we will discuss several other field theories in the infinite-momentum frame. However, none of these theories will present any new difficulties not already present in quantum electrodynamics. Thus the experience gained from a detailed discussion of quantum electrodynamics will enable us to develop the other theories in a more compact fashion.

Part 1: Scattering Theory

In this section, we regard the theory of quantum electrodynamics as being defined by the usual perturbation expansion of the S-matrix in Feynman diagrams. We rewrite the theory in the infinite momentum frame by systematically decomposing each covariant Feynman diagram into a sum of non-covariant τ-ordered diagrams. We consider the Feynman expansion as a formal expansion; thus we shall not be concerned in this paper with the convergance of the perturbation series, or convergence and regularization of the integrals.

A. Proparators

If we wanted to derive t-ordered diagrams from the Feynman diagrams we w ould begin by writing the Feynman electron propagator in the form

$$S_F(x) = \Theta(t)\, S^{(+)}(x) + \Theta(-t)\, S^{(-)}(x) \quad . \tag{IV.1}$$

We will try to do the same thing using $\Theta(\tau)$ instead of $\Theta(t)$.

We start by considering the Klein-Gordon propagator

$$\begin{aligned}\Delta_F(x) &= (2\pi)^{-4} \int d^4p \; e^{-ip_\mu x^\mu} \left[p^\nu p_\nu - m^2 + i\varepsilon\right]^{-1} \\ &= (2\pi)^{-4} \int d\underset{\sim}{p} \int d\eta \; e^{-i(\eta z - \underset{\sim}{p}\cdot\underset{\sim}{x})} \\ &\quad \times \int dH \; e^{-iH\tau} \left[2\eta H - \underset{\sim}{p}^2 - m^2 + i\varepsilon\right]^{-1} .\end{aligned}$$

We can do the H-integral by contour integration. If $\tau > 0$ we close the contour in the lower half H-plane. The integrand has one pole at $H = (\vec{p}_T^{\,2} + m^2 - i\epsilon)/2\eta$, which is in the lower (upper) half plane if η is positive (negative). Thus we get

$$\Delta_F(x) = \frac{-i}{(2\pi)^3} \int d\mathbf{p} \int_0^\infty \frac{d\eta}{2\eta}\; e^{-i\left[\frac{\mathbf{p}^2+m^2}{2\eta}\tau + \eta z - \mathbf{p}\cdot\mathbf{x}\right]}$$

Similarly, if $\tau < 0$ we get

$$\Delta_F(x) = \frac{+i}{(2\pi)^3} \int d\mathbf{p} \int_{-\infty}^0 \frac{d\eta}{2\eta}\; e^{-i\left[\frac{\mathbf{p}^2+m^2}{2\eta}\tau + \eta z - \mathbf{p}\cdot\mathbf{x}\right]}$$

Thus (with the change of variable $\vec{p}_T \to -\vec{p}_T$ and $\eta \to -\eta$ for $\tau < 0$) we have the required decomposition for $\Delta_F(x)$:[1]

$$\Delta_F(x) = \frac{-i}{(2\pi)^3} \int d\mathbf{p} \int_0^\infty \frac{d\eta}{2\eta} \left\{ \theta(\tau)\, e^{-ip_\mu x^\mu} + \theta(-\tau)\, e^{+ip_\mu x^\mu} \right\} \tag{IV.2}$$

where

$$p_0 = H(\eta, \mathbf{p}) = \frac{\mathbf{p}^2 + m^2}{2\eta}$$

is the free particle hamiltonian. Notice that

$$d\mathbf{p}\, \frac{d\eta}{\eta} = \frac{d\vec{p}}{\sqrt{\vec{p}^{\,2} + m^2}}$$

is the invariant differential sufrace element on the mass shell.

We can use the decomposition (IV.2) of $\Delta_F(x)$ to derive a decomposition for the electron propagator,

$$S_F(x) = \left(i\partial_\mu \gamma^\mu + m\right) \Delta_F(x) \quad . \tag{IV.3}$$

(In keeping with our convention, the γ^μ are the γ-matrices in the new notation. We shall use $\hat{\gamma}^\mu$ for the γ-matrices in the ordinary notation; thus $\gamma^0 = 2^{-\frac{1}{2}}(\hat{\gamma}^0 + \hat{\gamma}^3)$ etc. Table I in Part 2 contains some useful identities for the new γ-matrices.) When we differentiate $\Delta_F(x)$ in (IV.2) we get a term proportional to $\Theta(\tau)$, a term proportional to $\Theta(-\tau)$, a third term proportional to $\delta(\tau) = \partial_0\Theta(\tau)$. As we will see, this third term results in an extra term in the infinite momentum frame Hamiltonian. Doing the differentiation we get

$$\begin{aligned} S_F(x) = \frac{-i}{(2\pi)^3}\int d\underset{\sim}{p}\int_0^\infty \frac{d\eta}{2\eta}\Big\{ &\Theta(\tau)\left[\not{p} + m\right] e^{-ip_\nu x^\nu} \\ &+ \Theta(-\tau)\left[-\not{p} + m\right] e^{+ip_\nu x^\nu} \Big\} \\ + \frac{1}{(2\pi)^3}\,\delta(\tau)\,\gamma^0 \int d\underset{\sim}{p}\int_0^\infty &\frac{d\eta}{2\eta}\, e^{-i(\eta z - \underset{\sim}{p}\cdot\underset{\sim}{x})} \quad . \end{aligned} \tag{IV.4}$$

We will also need a decomposition for the photon propagator. We start with

$$D_F(x)^{\mu\nu} = (2\pi)^{-4}\int d^4p\; e^{-ip_\nu x^\nu}\,\frac{-g^{\mu\nu}}{p_\nu p^\nu + i\varepsilon} \tag{IV.5}$$

As we will see, a great simplification in the theory will result if we choose the gauge $A^0 = 0$, which might be called the infinite momentum gauge. To write the propagator in this gauge we define the polarization vectors

$$e(p,+1)^{\mu} = -\tfrac{1}{\sqrt{2}}\left(0, 1, i, [p^1 + i p^2]/\eta\right)$$
$$e(p,-1)^{\mu} = +\tfrac{1}{\sqrt{2}}\left(0, 1, -i, [p^1 - i p^2]/\eta\right) \qquad \text{(IV.6)}$$

(Cf. Eq. (III.37) in Chapter III.) These polarization vectors satisfy the orthogonality conditions $e(p,\lambda)^{*\mu} e(p,\lambda')_{\mu} = -\delta_{\lambda\lambda'}$, $p_{\mu} e(\lambda, p)^{\mu} = 0$. By direct calculation, we find

$$-g^{\mu\nu} = \sum_{\lambda=\pm 1} e(p,\lambda)^{\mu}\, e(p,\lambda)^{*\nu}$$
$$- \frac{1}{\eta}\delta^{\mu}_{3} p^{\nu} - \frac{1}{\eta} p^{\mu}\delta^{\nu}_{3} + \frac{2\eta H - p^2}{\eta^2}\delta^{\mu}_{3}\delta^{\nu}_{3} \qquad \text{(IV.7)}$$

Let us make the replacement (IV.7) in our integral for $D_F(x)^{\mu\nu}$. We note that the gauge terms $\eta^{-1}\delta^{\mu}_{3}p^{\nu}$ and $\eta^{-1}p^{\mu}\delta^{\nu}_{3}$ will not contribute to any physical process because of current conservation. Thus we may drop these terms without changing the theory. This leaves us with

$$D_F(x)^{\mu\nu} = (2\pi)^{-4}\int d^4p\, e^{-ip_{\nu}x^{\nu}}\, \frac{\sum_{\lambda} e(p,\lambda)^{\mu}\, e(p,\lambda)^{*\nu}}{p_{\nu}p^{\nu} + i\varepsilon}$$

$$+ (2\pi)^{-4}\delta^{\mu}_{3}\delta^{\nu}_{3}\int d^4p\, e^{-ip_{\nu}x^{\nu}}\, \frac{1}{\eta^2}\, \frac{p_{\nu}p^{\nu}}{p_{\nu}p^{\nu} + i\varepsilon}.$$

We can do the H-integration in the first term by contour integration, just as we did for $\Delta_F(x)$. The result is

$$\text{first term} = \frac{-i}{(2\pi)^3}\int d\underline{p}\int_0^\infty \frac{d\eta}{2\eta}\left(\sum_\lambda e(\underline{p},\lambda)^\mu\, e(\underline{p},\lambda)^{*\nu}\right)$$
$$\times\left\{\Theta(\tau)\, e^{-ip_\nu x^\nu} + \Theta(-\tau)\, e^{+ip_\nu x^\nu}\right\} .$$

In the second term $p_\mu p^\mu/(p_\mu p^\mu + i\epsilon) \rightarrow 1$ as $\epsilon \rightarrow 0^+$ so that the H-integral is

$$\int_{-\infty}^{\infty} dH\, e^{-iH\tau} = 2\pi\,\delta(\tau) .$$

Thus the second term is

$$\delta(\tau)\,(2\pi)^{-3}\,\delta^\mu_3\,\delta^\nu_3 \int d\underline{p}\int_{-\infty}^{\infty}\frac{d\eta}{\eta^2}\; e^{-i(\eta\zeta - \underline{p}\cdot\underline{x})} .$$

This term will result in an extra term in the hamiltonian which is analogous to the Coulomb force term which appears in quantum electrodynamics in the Coulomb gauge.

In sum, then, our photon propagator takes the form

$$D_F(x)^{\mu\nu} = \frac{-i}{(2\pi)^3}\int d\underline{p}\int_0^\infty\frac{d\eta}{2\eta}\left(\sum_\lambda e(\underline{p},\lambda)^\mu\, e(\underline{p},\lambda)^{*\nu}\right)$$
$$\times\left\{\Theta(\tau)\, e^{-ip_\nu x^\nu} + \Theta(-\tau)\, e^{+ip_\nu x^\nu}\right\} \qquad \text{(IV.8)}$$
$$+ \frac{1}{(2\pi)^3}\,\delta(\tau)\,\delta^\mu_3\,\delta^\nu_3\int d\underline{p}\int_{-\infty}^{\infty}\frac{d\eta}{\eta^2}\; e^{-i(\eta\zeta - \underline{p}\cdot\underline{x})} ,$$

where

$$p_0 = H(\eta, \mathbf{p}) = \mathbf{p}^2/2\eta \quad .$$

B. Diagrams

We start with the usual Feynman rules in coordinate space. For definiteness let us consider a particular diagram, say the one shown in Figure IV-1a. We fix our conventions by writing out the contribution of this diagram to the S-matrix:

$$\begin{aligned} M = (-ie)^3 \int d^4x_1\, d^4x_2\, d^4x_3 \, & \left[\overline{\psi_1(x_1)_c}\, \gamma_\mu\, \psi_3(x_1)_c \right] \\ \times & \left[\overline{\psi_4(x_2)}\, \gamma_\nu\, i S_F(x_2 - x_3)\, \gamma_\sigma\, \psi_2(x_3) \right] \\ \times & \; i D_F(x_2 - x_1)^{\mu\nu} \quad \epsilon^\sigma(x_3)^* \quad . \end{aligned} \tag{IV.9}$$

The electron wave functions used here are

$$\psi(x) = e^{-ip_\nu x^\nu}\, u(p, \lambda) \quad .$$

where p and λ are the momentum and spin of the electron and the spinors $U(p, \lambda)$ are normalized to $\bar{u}u = 2m$. For positrons we use the charge conjugate wave functions

$$\psi(x)_c = e^{+ip_\nu x^\nu}\, u(p, \lambda)_c$$

where p and λ are the physical momentum and spin of the positron. The photon wave function is

$$\epsilon^\mu(x) = e^{-ip_\nu x^\nu}\, e(p, \lambda)^\mu \quad ,$$

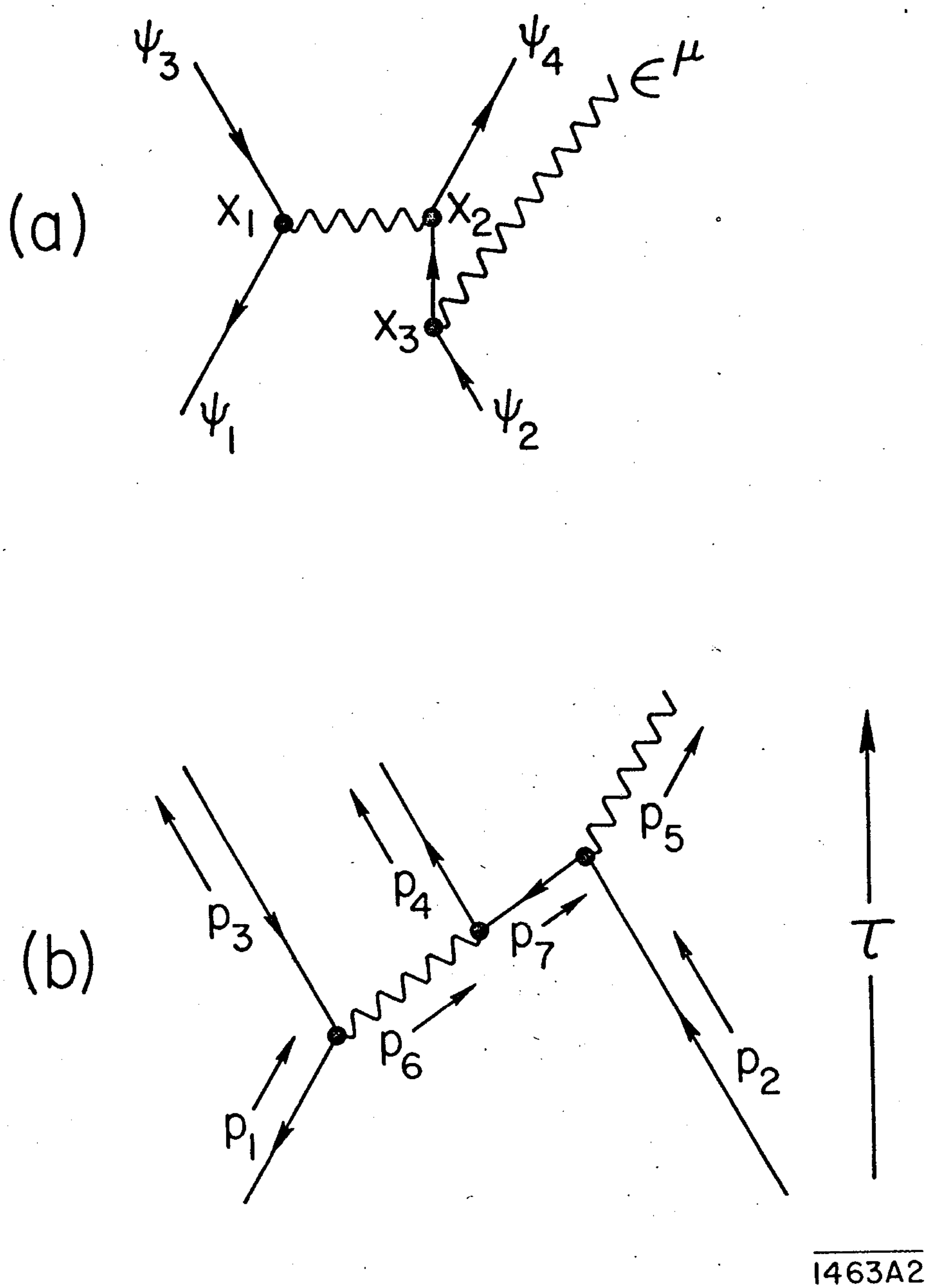

Fig. IV-1

Typical Feynman diagram in coordinate space (a), and in momentum space after τ-ordering (b).

where $e(p,\lambda)^{\mu}$ is one of our infinite-momentum gauge polarization vectors. Finally, it may be useful to note that although the γ-matrices appearing explicitly in Eq. (IV.9) are, as always, the "new" γ-matrices, the old $\hat{\gamma}^0$ still plays a role in $\bar{\psi} = \psi^{\dagger}\hat{\gamma}^0$.

We begin the program of deriving the rules for τ-ordered diagrams by inserting the comentum expansions (IV.4) and (IV.8) for the propagators into (IV.9). Let us, for the moment, ignore the contributions to S_F and $D_F^{\mu\nu}$ proportional to $\delta(\tau)$. Then each of the 3! possible τ-orderings of the vertices determines a τ-ordered diagram; let us consider, say, the ordering $\tau_1 < \tau_2 < \tau_3$. For this diagram we draw the picture in Figure IV-1b. The corresponding contribution to the S-matrix is obtained by inserting $\Theta(\tau_3-\tau_2)\Theta(\tau_2-\tau_1)$ into (IV.9). Thus only one of the $\Theta(\tau)$ or $\Theta(-\tau)$ terms survives from each propagator. We can do the $\vec{x}_T$- and $\mathcal{J}$-integrations to give $(2\pi)^3\delta^2(\vec{p}_{T\,in}-\vec{p}_{T\,out})\delta(\eta_{in}-\eta_{out})$ at each vertex. The τ-integrals in this example are

$$\int d\tau_1\, d\tau_2\, d\tau_3\; \Theta(\tau_3-\tau_2)\;\Theta(\tau_2-\tau_1)$$
$$\times \exp\left(-i\left[(H_1-H_3-H_6)\tau_1 + (H_6-H_4-H_7)\tau_2 + (H_7+H_2-H_5)\tau_3\right]\right).$$

With the change of variables

$$\begin{aligned} T_0 &= \tau_1 & \qquad \tau_1 &= T_0 \\ T_1 &= \tau_2-\tau_1 & \tau_2 &= T_0+T_1 \\ T_2 &= \tau_3-\tau_2 & \tau_3 &= T_0+T_1+T_2 \quad , \end{aligned}$$

The τ- integrals become

$$\int d\tau_0 \, d\tau_1 \, d\tau_2 \; \Theta(\tau_1)\,\Theta(\tau_2)\; e^{-i\left[(\mathcal{H}_i-\mathcal{H}_f)\tau_0+(\mathcal{H}_1-\mathcal{H}_f)\tau_1+(\mathcal{H}_2-\mathcal{H}_f)\tau_2\right]},$$

where $\mathcal{H}_1 = H_1 + H_2$ is the total "energy" of the initial state, $\mathcal{H}_1 = H_3 + H_6 + H_2$ is the total "energy" of the first intermediate state, $\mathcal{H}_2 = H_3 + H_4 + H_7 + H_2$ is the total "energy" of the second intermediate state, and $\mathcal{H}_f = H_3 + H_4 + H_5$ is the total "energy" of the final state. The integrals can now be done using

$$\int_{-\infty}^{\infty} d\tau \; e^{-i\mathcal{H}\tau} = 2\pi\,\delta(\tau)$$

$$\int_{0}^{\infty} d\tau \; e^{i\mathcal{H}\tau} = \frac{i}{\mathcal{H}+i\epsilon}\,.$$

Thus we get an overall factor of $(2\pi)\delta(\mathcal{H}_f - \mathcal{H}_i)$ and a factor of $i(\mathcal{H}_f - \mathcal{H} + i\epsilon)^{-1}$ for each intermediate state. With a little thought, one can convince himself that this result is completely general.

We now have to consider the effect of the $\delta(\tau)$ terms in the propagators, which we have so far omitted. The contributions to the S-matrix from a particular Feynman diagram so far obtained, we should add the contributions obtained by replacing the $\tau \neq 0$ parts of $S_F(x)$ and $D_F(x)^{\mu\nu}$ with the $\delta(\tau)$ part in any of the internal lines. We will use the pictures in Figure IV-2 for the $\delta(\tau)$ parts of $S_F(x_2-x_1)$ and $D_F(x_2-x_1)^{\mu\nu}$. Diagrams containing one or more of these $\delta(\tau)$ internal lines are then treated as before except that we consider structures such as those shown in Figure IV-3 as single vertices when we do the τ-ordering. Thus we get $(2\pi)^3\delta^2(\vec{p}_{T\,in}-\vec{p}_{T\,out})\delta(\eta_{in}-\eta_{out})$ at each end of a $\delta(\tau)$ internal line, an overall $(2\pi)\delta(\mathcal{H}_f - \mathcal{H}_i)$, and a factor $i(\mathcal{H}_f - \mathcal{H} + i\epsilon)^{-1}$ for each intermediate state between two different "times".

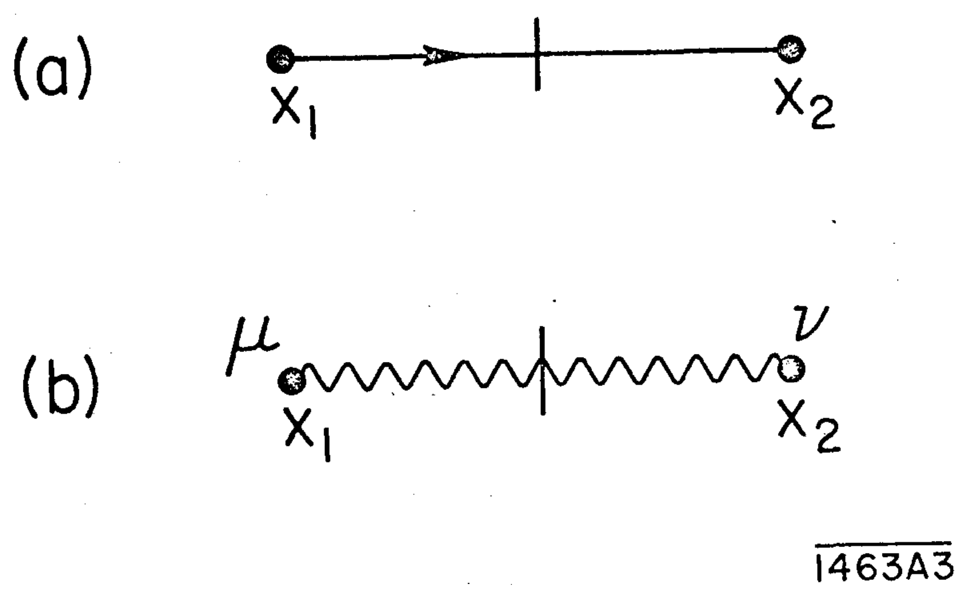

Fig. IV-2

Pictures used for the $\delta(\tau)$ terms in the electron propagator (a) and the photon propagator (b).

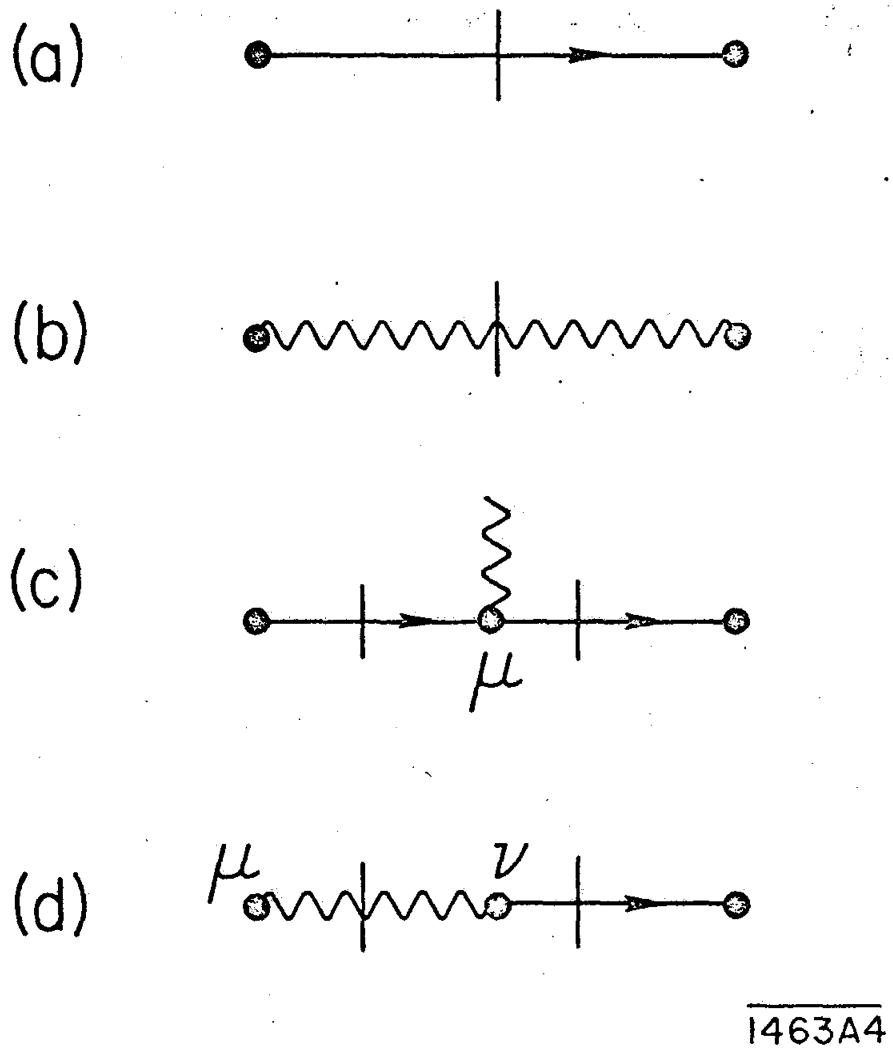

Fig. IV-3

Structures considered as single vertices. Structures like (c) and (d) give zero.

At this point let us notice that diagrams in which two or more $\delta(\tau)$ parts of propagators are linked together give a zero contribution to the S-matrix. Indeed, consider a diagram containing a part like that shown in Figure IV.3c. The corresponding contribution to the S-matrix contains $\gamma^o \gamma_\mu \gamma^o$ times $D_F^{\mu\nu}$ or e^μ. Because of our choice of gauge, only $\mu = 1, 2, 3$ occurs; but, since $\gamma^o \gamma^o = g^{oo} = 0$, we have $\gamma^o \gamma_1 \gamma^o = -\gamma^o \gamma^o \gamma_1 = 0$, $\gamma^o \gamma_2 \gamma^o = -\gamma^o \gamma^o \gamma_2 = 0$, and $\gamma^o \gamma_3 \gamma^o = \gamma^o \gamma^o \gamma^o = 0$. Hence $\gamma^o \gamma_\mu \gamma^o e^\mu = \gamma^o \gamma_\mu \gamma^o D_F^{\mu\nu} = 0$. Now consider a diagram in which the structure shown in Figure IV-3d occurs. The corresponding contribution to the S-matrix contains a factor $\delta^\mu_3 \delta^\nu_3 (\ldots \gamma_\nu \gamma^o \ldots) = \delta^\mu_3 (\ldots \gamma_3 \gamma^o \ldots) = \delta^\mu_3 (\ldots \gamma^o \gamma^o \ldots) = 0$.

We are now in a position to summarize the rules for τ-ordered diagrams. With our choice of gauge there are three types of interactions as shown in Figure IV-4. These interactions are to be τ-ordered in all possible ways. We then associate the following factors with the parts of the diagram:[2]

i) spinors $U(p,\lambda)$, $\overline{U}(p,\lambda)$, $\overline{U}_c(p,\lambda)$, $U_c(p,\lambda)$, and $e(p,\lambda)^\mu$, $e(p,\lambda)^{*\mu}$ for external lines;

ii) $(\not{p} + m) = \Sigma_\lambda U(p,\lambda) \overline{U}(p,\lambda)$ for electron propagators; $(-\not{p} + m) = -\Sigma_\lambda U(p,\lambda) \overline{U}(p,\lambda)$ for positron propagators; $\Sigma_\lambda e(p,\lambda)^\mu e(p,\lambda)^{*\nu}$ for photon propagators.

iii) $e\gamma_\mu (2\pi)^3 \delta(\eta_{out} - \eta_{in}) \delta^2(\underset{\sim}{p}_{out} - \underset{\sim}{p}_{in})$ for each vertex as shown in Figure IV.4a;

iv) $e\gamma_\mu$ for each ordinary vertex as shown in Figure IV-4a ;

$$\gamma_\mu \cdots \delta^\mu_3 \frac{e^2}{\eta_o^2} \delta^\nu_3 \cdots \gamma_\nu$$

for each "Coulomb force" vertex as shown in Figure IV-4b, where η_0 is the total η transferred across the vertex;

$$\frac{e^2}{2} \gamma_\nu \gamma^o \gamma_\mu \left(\frac{1}{\eta_o}\right)$$

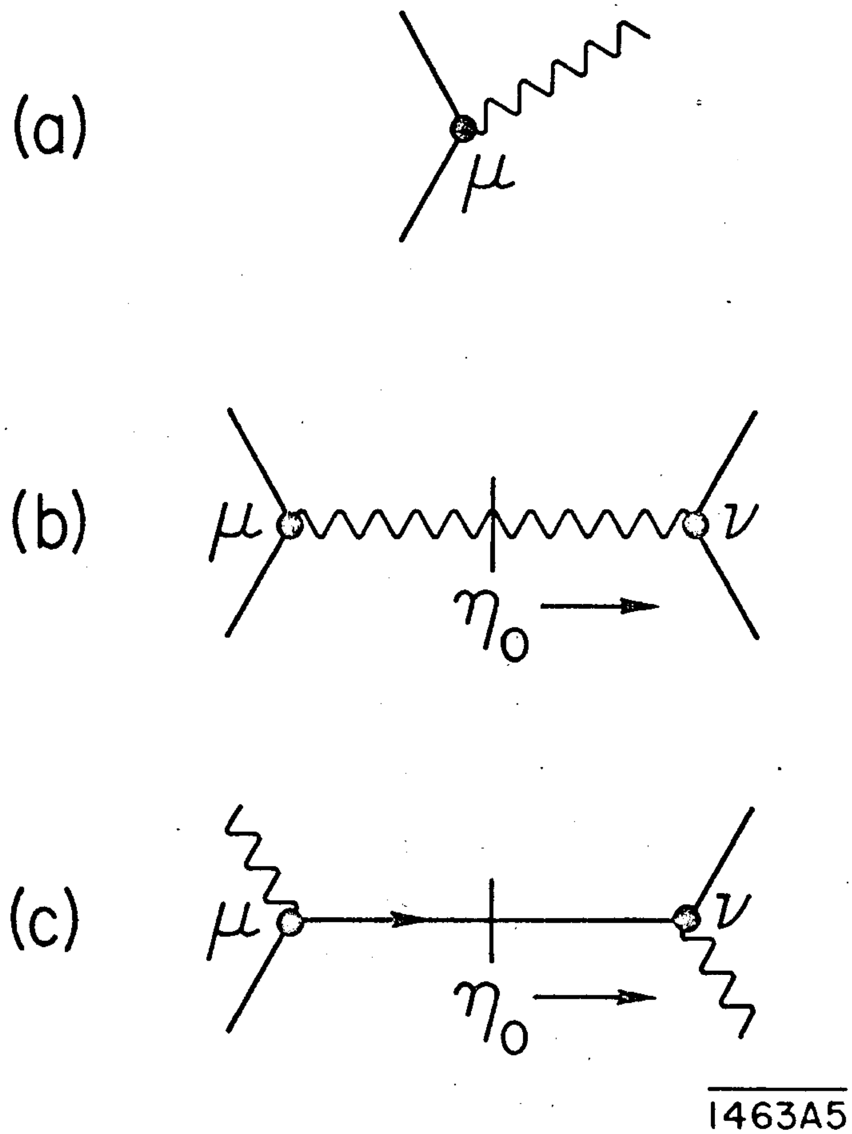

Fig. IV-4

Vertices in the infinite momentum frame.

for each "instantaneous electron exchange" vertex as shown in Figure IV-4c;

v) $(2\pi)^2 \delta(\eta_{out} - \eta_{in}) \delta^2 (\underset{\sim}{p}_{out} - \underset{\sim}{p}_{in})$ for each vertex;

vi) an overall factor of $(-2\pi i)\delta(\mathscr{H}_f - \mathscr{H}_i)$, and a factor of $(\mathscr{H}_f - \mathscr{H} + i\epsilon)^{-1}$ for each intermediate state;

vii) the usual overall sign from the Wick reduction, determined by the structure of the origional Feynman diagram;

viii) an integration $(2\pi)^{-3} \; d\underset{\sim}{p} \int_0^\infty \frac{d\eta}{2\eta}$ for each internal line.

Note that since each line carried positive η and η is conserved in each interaction, vacuum diagrams like those shown in Figure IV-5 cannot occur.

In the next section we shall develop the canonical field theory for quantum electrodynamics in the infinite momentum frame. As we will see, the hamiltonian we will obtain reproduces the scattering theory we have developed here.

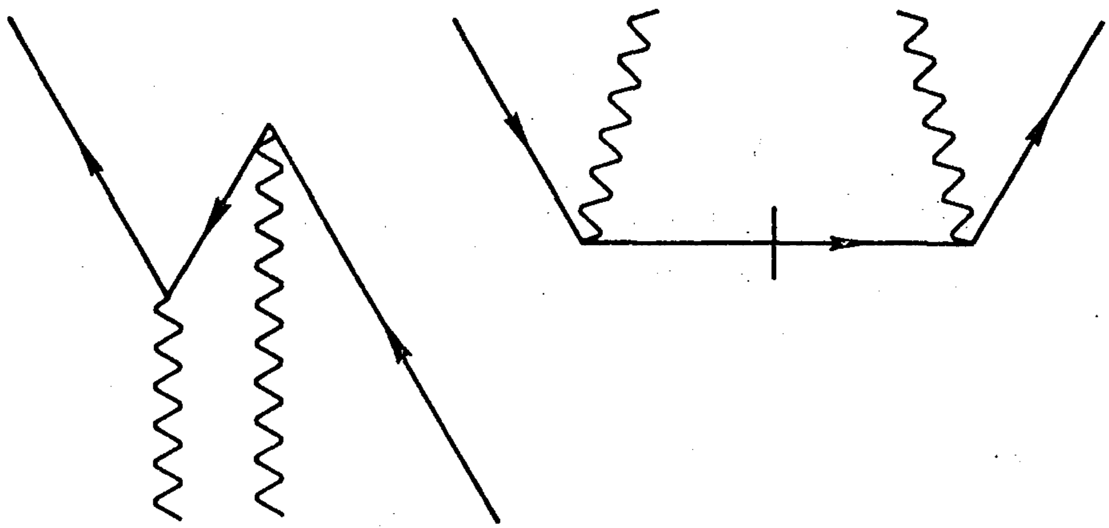

Fig. IV-5

Typical diagrams that vanish because of η - conservation.

Part 2: Canonical Field Theory

A. Equations of Motion

We base our field theory on the usual lagrangian density[3]

$$\mathcal{L}(x) = \overline{\Psi}\left\{\left(\tfrac{i}{2}\overleftrightarrow{\partial}_\mu - eA_\mu\right)\gamma^\mu - m\right\}\Psi - \tfrac{1}{4}F^{\mu\nu}F_{\mu\nu} \tag{IV.10}$$

where the electromagnetic field tensor $F^{\mu\nu}$ is related to the potential A^μ by $F^{\mu\nu} = \partial^\nu A^\mu - \partial^\mu A^\nu$. Variation of the fields Ψ, $\bar{\Psi}$, and A^μ give the Dirac equation and Maxwell's equations:

$$\left\{\left(i\partial_\mu - eA_\mu\right)\gamma^\mu - m\right\}\Psi = 0 \tag{IV.11}$$

$$\partial_\lambda F^{\mu\lambda} = e\overline{\Psi}\gamma^\mu\Psi = J^\mu \quad , \tag{IV.12}$$

It will be convenient to work in the infinite momentum gauge, $A^0(x) = 0$. In this gauge the field tensor is related to the potential by

$$F^{0\mu} = -\partial^0 A^\mu = -\partial_3 A^\mu \quad . \tag{IV.13}$$

In order to completely specify the gauge, we must choose boundary conditions for $A^\mu(x)$. For reasons of symmetry, we will require that $A^\mu(x^0, x^1, x^2, +\infty) = -A^\mu(x^0, x^1, x^2, -\infty)$. With these boundary conditions, the solution of (IV.13) is

$$A^\mu(x) = -\tfrac{1}{2}\int d\xi\, \epsilon(x^3-\xi)\, F^{0\mu}(x^0, x^1, x^2, \xi) \quad , \tag{IV.14}$$

where
$$\epsilon(x) = \begin{cases} 1 & x > 0 \\ -1 & x < 0 \end{cases}$$

It is perhaps not obvious that the gauge conditions we have imposed are consistent with Maxwell's equations. Thus it is reassuring to note that the definition (IV.14) of $A^\mu(x)$ works for the classical electromagnetic field. If the field $F^{\mu\nu}(x)$ is produced by a current which, say, is non-zero only in a bounded space-time region, then the components $F^{0\mu}(x)$ go to zero like $(x^3)^{-2}$ as $|x^3| \to \infty$. Thus the integral (IV.14) is well defined. Using the homogeneous Maxwell's equations, $\partial^\mu F^{\nu\lambda} + \partial^\nu F^{\lambda\mu} + \partial^\lambda F^{\mu\nu} = 0$, one can easily show that the potential A^μ defined by (IV.14) satisfies $\partial^\nu A^\mu - \partial^\mu A^\nu$ for all indices μ, ν.

We have eliminated one component of $A^\mu(x)$ by our choice of gauge. Only two of the remaining three components can be independent dynamical variables, since the three components of $A^\mu(x)$ are related at any "time" x^0 by the differential equation

$$\begin{aligned} \partial_3 \left(\partial_1 A^1 + \partial_2 A^2 + \partial_3 A^3 \right) &= -\partial_\mu F^{0\mu} \\ &= -J^0 \end{aligned} \tag{IV.15}$$

It will be convenient to regard A^1 and A^2 as the independent components. Then A^3 satisfies

$$\partial_3 \partial_3 A^3 = -\partial_3 \partial_j A^j - J^0 .$$

(We adopt the convention that Latic indices are to be summed from 1 to 2.) The solution of this equation which equals A^3 as defined by (IV.14) is

$$A^3(x) = -\frac{1}{2}\int d\xi \; |x^3 - \xi| \left\{ \partial_3 \partial_j A^j(x^0, \underset{\sim}{x}, \xi) + J^0(x^0, \underset{\sim}{x}, \xi) \right\} \quad . \tag{IV.16}$$

To see that this equation reproduces our definition of A^3 in terms of F^{o3}, write it as[4]

$$\begin{aligned} A^3(x) &= -\frac{1}{2}\int d\xi \; |x^3 - \xi| \; \partial_3 F^{o3}(x^0, \underset{\sim}{x}, \xi) \\ &= -\frac{1}{2}\int d\xi \left(\frac{\partial}{\partial x^3} |x^3 - \xi| \right) F^{o3}(x^0, \underset{\sim}{x}, \xi) \\ &= -\frac{1}{2}\int d\xi \; \epsilon(x^3 - \xi) \;\; F^{o3}(x^0, \underset{\sim}{x}, \xi) \quad . \end{aligned} \tag{IV.17}$$

Thus only two components, $A^1(x)$ and $A^2(x)$, of $A^\mu(x)$ are dynamical variables. $A^o(x)$ is identically zero, and $A^3(x)$ is determined at any "time" x^2 by $A^1(x)$, $A^2(x)$, and $\Psi(x)$ at that x^o by means of Eq. (IV.16). This reduction in the number of independent components of A^μ is a familiar feature of quantum electrodynamics in any reference frame.

In the infinite momentum frame, we find that the number of independent components of the electron field $\Psi(x)$ is also reduced from four to two. In order to show this we pause briefly to examine the proporties of the infinite momentum γ-matrices, $\gamma^\mu = C^\mu_\nu \hat{\gamma}^\mu$. The "ordinary" γ-matrices $\hat{\gamma}^\mu$ are chosen to satisfy $\left\{\hat{\gamma}^\mu, \hat{\gamma}^\nu\right\} = 2\hat{g}^{\mu\nu}$ and $\hat{\gamma}^{\mu\dagger} = \gamma_\mu$. Thus the infinite momentum γ-matrices satisfy $\left\{\gamma^\mu, \gamma^\nu\right\} = 2\,g^{\mu\nu}$, $\gamma^{\mu\dagger} = \gamma_\mu$. From this it follows easily that $P_+ \equiv \frac{1}{2}\gamma^3\gamma^o$ and $P_- \equiv \frac{1}{2}\gamma^o\gamma^3$ are hermitial projection operators with $P_+P_- = 0$ and $P_+ + P_- = 1$.

These facts, as well as some others that we will need later, are listed for convenient reference in Table IV-1.

It will be helpful to have a specific representation of the γ-matrices in mind. We will consistently use

$$\hat{\gamma}^0 = \begin{pmatrix} 0 & \mathbb{1} \\ \mathbb{1} & 0 \end{pmatrix} \quad , \quad \hat{\gamma}^\alpha = \begin{pmatrix} 0 & -\sigma^\alpha \\ \sigma^\alpha & 0 \end{pmatrix} \quad \alpha = 1,2,3 \quad , \tag{IV.18}$$

where σ^1, σ^2, σ^3 are the usual 2×2 Pauli matrices. With this choice for the γ^μ, we find that

$$P_+ = \begin{pmatrix} 1 & 0 & 0 & 0 \\ 0 & 0 & 0 & 0 \\ 0 & 0 & 0 & 0 \\ 0 & 0 & 0 & 1 \end{pmatrix} \qquad P_- = \begin{pmatrix} 0 & 0 & 0 & 0 \\ 0 & 1 & 0 & 0 \\ 0 & 0 & 1 & 0 \\ 0 & 0 & 0 & 0 \end{pmatrix} \quad . \tag{IV.19}$$

By applying the projection matrices $P_\pm$ to the electron field $\Psi(x)$ we obtain two two-component fields which we call $\Psi_+(x)$ and $\Psi_-(x)$:

$$\Psi_+ = P_+ \Psi = \begin{pmatrix} \psi_1 \\ 0 \\ 0 \\ \psi_4 \end{pmatrix} \qquad \Psi_- = P_- \Psi = \begin{pmatrix} 0 \\ \psi_2 \\ \psi_3 \\ 0 \end{pmatrix} \tag{IV.20}$$

With this preparation completed, we are ready to examine the dynamics of the electron field $\Psi(x)$. If we multiply the Dirac equation by γ^0 and recall that $\gamma^0\gamma^0 = 0$, we obtain

$$(i\partial_3 - eA_3)\gamma^0\gamma^3\Psi = \gamma^0[-(i\partial_i - eA_i)\gamma^i + m]\Psi \, .$$

TABLE IV-1

γ–Matrix Identities

$$\left\{\gamma^{\mu}, \gamma^{\nu}\right\} = 2\, g^{\mu\nu} \qquad \gamma^{\mu\dagger} = \gamma_{\mu}$$

$$P_+ \equiv \tfrac{1}{2}\gamma^3\gamma^0 \qquad P_- \equiv \tfrac{1}{2}\gamma^0\gamma^3$$

$$P_{\pm}^{\dagger} = (P_{\pm})^2 = P_{\pm}$$

$$P_+ + P_- = 1 \qquad P_+P_- = P_-P_+ = 0$$

$$\gamma^3 P_+ = P_-\gamma^3 = 0 \qquad \gamma^3 P_- = P_+\gamma^3 = \gamma^3$$

$$\gamma^0 P_- = P_+\gamma^0 = 0 \qquad \gamma^0 P_+ = P_-\gamma^0 = \gamma^0$$

$$\hat{\gamma}^0 = \frac{1}{\sqrt{2}}(\gamma^0 + \gamma^3) = \frac{1}{\sqrt{2}}(P_-\gamma^0 P_+ + P_+\gamma^3 P_-)$$

$$\hat{\gamma}^0\gamma^0 = \sqrt{2}\, P_+ \qquad \hat{\gamma}^0\gamma^3 = \sqrt{2}\, P_-$$

Using our γ-matrix identities, this becomes

$$\left(i\,\partial_3 - eA_3\right)\Psi_- = \tfrac{1}{2}\left[(i\partial_j - eA_j)\gamma^j + m\right]\gamma^0\,\Psi_+$$

This differential equation is considerably simplified because of our choice of gauge, $A_3 = A^0 = 0$. Thus

$$\partial_3\,\Psi_- = -\tfrac{i}{2}\left[(i\,\partial_j - eA_j)\gamma^j + m\right]\gamma^0\,\Psi_+ \quad . \tag{IV.21}$$

For reasons of symmetry, we write the solution of Eq. (IV.21) as

$$\Psi_-(x) = -\tfrac{i}{4}\int d\xi\;\epsilon(x^3-\xi)\left\{\left[i\,\partial_j - eA_j(x^0,\underline{x},\xi)\right]\gamma^j + m\right\} \times \gamma^0\,\Psi_+(x^0,\underline{x},\xi) \quad . \tag{IV.22}$$

Thus the two components of $\Psi_-(x)$ are dependent variables in the infinite momentum frame. They are determined at any "time" x^0 by the independent fields $\Psi_+(x)$ and $A^j(x)$ at the same x^0. We recall that the dependent variable $A^3(x)$ is determined at any x^0 by A^j and J^0 at that x^0. It is reassuring to note that the dependence of $J^0(x)$ on the independent fields Ψ_+, A^j is very simple:

$$\tfrac{1}{e}J^0 = \overline{\Psi}\,\gamma^0\,\Psi = \Psi^\dagger\,\hat{\gamma}^0\gamma^0\,\Psi = \sqrt{2}\,\Psi_+^\dagger\,\Psi_+ \quad . \tag{IV.23}$$

What are the equations of motion for our independent fields $A^j(x)$ and $\Psi_+(x)$? For $A^j(x)$ we have the Maxwell's equations

$$\partial_\nu\left(\partial^\nu A^j - \partial^j A^\nu\right) = J^j \quad ,$$

or

$$
\begin{aligned}
2\partial_0\partial_3 A^j &= J^j + \partial^j \partial_\nu A^\nu - \partial_i \partial^i A^j \\
&= J^j + \partial^j \partial_3 A^3 + \partial^j \partial_i A^i - \partial_i \partial^i A^j \\
&= J^j + \partial^j \partial_3 A^3 + \partial_i F^{ij}
\end{aligned}
\tag{IV.24}
$$

Using the definition (IV.14) of A^j in terms of F^{oj}, we have

$$\partial_0 A^j(x) = \frac{1}{2}\int d\xi \, \epsilon(x^3-\xi)\, \partial_0\partial_3 A^j(x^0, \underset{\sim}{x}, \xi) \, . \tag{IV.25}$$

Substituting into (IV.25) from (IV.24), we obtain

$$
\begin{aligned}
\partial_0 A^j(x) &= \frac{1}{4}\partial^j \int d\xi \, \epsilon(x^3-\xi)\, \partial_3 A^3(x^0, \underset{\sim}{x}, \xi) \\
&\quad + \frac{1}{4}\int d\xi \, \epsilon(x^3-\xi) \Big\{ J^j(x^0, \underset{\sim}{x}, \xi) \\
&\qquad + \partial_i F^{ij}(x^0, \underset{\sim}{x}, \xi) \Big\}
\end{aligned}
$$

Since the integral in the first term is just $2A^3(x)$ because of Eq. (IV.14), we have, finally,

$$
\begin{aligned}
\partial_0 A^j(x) &= \frac{1}{2}\partial^j A^3(x) \\
&\quad + \int d\xi \, \frac{1}{4}\epsilon(x^3-\xi) \Big\{ J^j(x^0, \underset{\sim}{x}, \xi) + \partial_i F^{ij}(x^0, \underset{\sim}{x}, \xi) \Big\} .
\end{aligned}
\tag{IV.26}
$$

We can obtain the equation of motion for $\Psi_+(x)$ by multiplying the Dirac equation by γ^3. After making use of some of our γ-matrix identities, we obtain

$$
\begin{aligned}
\partial_0 \Psi_+ &= -ie A^3(x)\Psi_+(x) \\
&\quad - \frac{i}{2}\Big[(i\partial_j - eA_j(x))\gamma^j + m \Big] \gamma^3 \Psi_-(x) \, .
\end{aligned}
\tag{IV.27}
$$

B. Momentum and Angular Momentum

The invariance of the Lagrangian under the Poincare group provides us, using Noether's theorem, with a conserved momentum tensor $T_{\mu}^{\lambda}(x)$ and a conserved angular momentum tensor $J_{\mu\nu}{}^{\lambda}(x)$:

$$T_{\mu}{}^{\lambda} = \bar{\Psi}\frac{i}{2}\overleftrightarrow{\partial}_{\mu}\gamma^{\lambda}\Psi + (\partial_{\mu}A_{\alpha})F^{\lambda\alpha} - g_{\mu}{}^{\lambda}\mathcal{L} \tag{IV.28}$$

$$J_{\mu\nu}{}^{\lambda} = x_{\mu}T_{\nu}{}^{\lambda} - x_{\nu}T_{\mu}{}^{\lambda} + S_{\mu\nu}{}^{\lambda} \tag{IV.29}$$

where

$$S_{\mu\nu}{}^{\lambda} = \frac{i}{8}\bar{\Psi}\left\{\gamma^{\lambda}[\gamma_{\mu},\gamma_{\nu}] + [\gamma_{\mu},\gamma_{\nu}]\gamma^{\lambda}\right\}\Psi + F^{\lambda}{}_{\mu}A_{\nu} - F^{\lambda}{}_{\nu}A_{\mu} \quad . \tag{IV.30}$$

If the fields satisfy the equations of motion, then T_{μ}^{λ} and $J_{\mu\nu}{}^{\lambda}$ are conserved:

$$\partial_{\lambda}T_{\mu}{}^{\lambda} = 0 \qquad \partial_{\lambda}J_{\mu\nu}{}^{\lambda} = 0 \, . \tag{IV.31}$$

Thus the total momentum,

$$P_{\mu} = \int d\underset{\sim}{x}\, dz\; T_{\mu}{}^{0} \quad , \tag{IV.32}$$

and the total angular momentum,

$$M_{\mu\nu} = \int d\underset{\sim}{x}\, dz\; J_{\mu\nu}{}^{0} \quad , \tag{IV.33}$$

are constants of the motion. In our quantum theory, P_μ and $M_{\mu\nu}$ are the generators of the Poincaré group.[5]

We recall from our discussion of the Poincare group in Chapter II that the operators $P_1, P_2, P_3, M_{12}, M_{13}$, and M_{23} are "kinematical" symmetry operators in that the subgroups of the Poincaré group which they generate leaves the planes τ-constant invariant. Thus we might expect that they take a particularly simple form. Indeed, we find that

$$T_\alpha{}^0 = \sqrt{2}\,\Psi_+^\dagger \tfrac{i}{2}\overleftrightarrow{\partial}_\alpha \Psi_+ - (\partial_\alpha A^i)(\partial_3 A_i) \qquad \alpha = 1,2,3 \tag{IV.34}$$

$$J_{12}{}^0 = x_1 T_2{}^0 - x_2 T_1{}^0 + \sqrt{2}\,\Psi_+^\dagger \tfrac{i}{2}\gamma_1\gamma_2 \Psi_+ + A^1(\partial_3 A^2) - A^2(\partial_3 A^1) \tag{IV.35}$$

$$J_{12}{}^0 = x_1 T_3{}^0 - x_3 T_1{}^0 \tag{IV.36}$$

$$J_{23}{}^0 = x_2 T_3{}^0 - x_3 T_2{}^0 \quad . \tag{IV.37}$$

Note that these operators involve only the independent fields Ψ_+ and A^i, and thus do not depend on the coupling constant e.

The most important operator in the theory is, of course, the Hamiltonian $H = P_0$. From the definition (IV.28) we have

$$T_0{}^0 = \bar{\Psi}\tfrac{i}{2}\overleftrightarrow{\partial}_0 \gamma^0 \Psi + (\partial_0 A_\lambda)F^{0\lambda} - \bar{\Psi}\left\{(\tfrac{i}{2}\partial_\mu - eA_\mu)\gamma^\mu - m\right\}\Psi + \tfrac{1}{4}F^{\mu\nu}F_{\mu\nu} \; .$$

The first two terms cancel the terms in the Lagrangian containing ∂_0, and we are left with

$$T_0{}^0 = -\bar{\Psi}\left(\frac{i}{2}\sum_{\nu=1}^{3}\overleftrightarrow{\partial}_\nu \gamma^\nu - m\right)\Psi + eA_\mu \bar{\Psi}\gamma^\mu \Psi$$
$$+ \frac{1}{2}F^{12}F_{12} - \frac{1}{2}(\partial_3 A^3)(\partial_3 A^3) - (\partial_j A^3)(\partial_3 A^j) \,. \quad \text{(IV.38)}$$

C. Momentum Space Expansions of the Fields; Commutation Relations

Let $\Psi_+(\eta, \underset{\sim}{p}, \tau)$ be the Fourier transform, at the "time" τ, of $\Psi_+(x)$, so that

$$\Psi_+(\tau, \underset{\sim}{x}, \zeta) = (2\pi)^{-3}\int d\underset{\sim}{p}\, d\eta \; e^{-i(\eta\zeta - \underset{\sim}{p}\cdot\underset{\sim}{x})} \; \Psi_+(\eta, \underset{\sim}{p}, \tau) \quad \text{(IV.39)}$$

It will be useful to define operators $b(\eta, \underset{\sim}{p}, \lambda; \tau)$ and $d(\eta, \underset{\sim}{p}, \lambda; \tau)$, where λ takes the values $\pm\frac{1}{2}$, by

$$\begin{aligned}
2^{-1/4}(2\eta)^{-1/2}\, b(\eta, \underset{\sim}{p}, +\tfrac{1}{2}; \tau) &= \Psi_{+1}(\eta, \underset{\sim}{p}, \tau) && \text{for } \eta > 0 \\
2^{-1/4}(2\eta)^{-1/2}\, b(\eta, \underset{\sim}{p}, -\tfrac{1}{2}; \tau) &= \Psi_{+4}(\eta, \underset{\sim}{p}, \tau) && \text{for } \eta > 0 \\
2^{-1/4}(2\eta)^{-1/2}\, d^\dagger(\eta, \underset{\sim}{p}, +\tfrac{1}{2}; \tau) &= \Psi_{+4}(-\eta, -\underset{\sim}{p}, \tau) && \text{for } \eta > 0 \\
2^{-1/4}(2\eta)^{-1/2}\, d^\dagger(\eta, \underset{\sim}{p}, -\tfrac{1}{2}; \tau) &= \Psi_{+1}(-\eta, -\underset{\sim}{p}, \tau) && \text{for } \eta > 0 \,.
\end{aligned} \quad \text{(IV.40)}$$

Then the Fourier expansion of $\Psi_+(x)$ takes the form

$$2^{1/4}\,\Psi_+(x) = (2\pi)^{-3}\int d\underset{\sim}{p}\int_0^\infty \frac{d\eta}{2\eta} \sum_{\lambda=\pm 1/2} \sqrt{2\eta} \times\Big\{ w(\lambda)\, e^{-i(\eta z - \underset{\sim}{p}\cdot \underset{\sim}{x})}\, b(\underset{\sim}{p},\lambda;\tau) + w(-s)\, e^{+i(\eta z - \underset{\sim}{p}\cdot\underset{\sim}{x})}\, d^\dagger(\underset{\sim}{p},\lambda;\tau)\Big\}\,, \tag{IV.41}$$

where the spinors $w(\lambda)$ are

$$w(+\tfrac{1}{2}) = \begin{pmatrix}1\\0\\0\\0\end{pmatrix} \qquad w(-\tfrac{1}{2}) = \begin{pmatrix}0\\0\\0\\1\end{pmatrix} \;. \tag{IV.42}$$

Let us see what the electron parts of the momentum operators P_1, P_2, P_3 look like in momentum space. Taking the operators P_α from (IV.32) and (IV.34), and doing a little algebra, we get

$$\begin{aligned} P_{\alpha\,(\text{electron})} &= \int d\underset{\sim}{x}\, dz\; \sqrt{2}\, \Psi_+^\dagger(x)\, \frac{i}{2}\overset{\leftrightarrow}{\partial}_\alpha\, \Psi_+(x) \qquad \alpha = 1,2,3 \\ &= (2\pi)^{-3}\int d\underset{\sim}{p}\int_0^\infty \frac{d\eta}{2\eta} \sum_{\lambda=\pm 1/2} \times p_\alpha \big\{ b^\dagger(\underset{\sim}{p},\lambda;\tau)\, b(\underset{\sim}{p},\lambda;\tau) - d(\underset{\sim}{p},\lambda;\tau)\, d^\dagger(\underset{\sim}{p},\lambda;\tau)\big\}. \end{aligned} \tag{IV.43}$$

Up until now we have not mentioned the commutation relations of our independent fields. The form of (IV.44) makes a very clear suggestion as to what commutation relations to choose. We are led to interpret $b(p;s;\tau)$ and $d(p;s;\tau)$ as destruction operators for electrons and positrons, respectively. (The minus sign in (IV.45) can then be disposed by a normal ordering.) We thus postulate the covariant anti-commutation relations

$$\left\{ b(p,\lambda;\tau),\, b^\dagger(p',\lambda';\tau) \right\} = \left\{ d(p,\lambda;\tau),\, d^\dagger(p',\lambda',\tau) \right\}$$
$$= \delta_{\lambda\lambda'}\, (2\pi)^3\, 2\eta\; \delta(\eta-\eta')\, \delta^2(\underset{\sim}{p}-\underset{\sim}{p}') \tag{IV.44}$$

with all other anticommutators vanishing. Transforming back to coordinate space, we obtain the following equal-τ anticommutation relations:

$$\left\{ \Psi_+(\tau,\underset{\sim}{x},z),\, \Psi_+^\dagger(\tau,\underset{\sim}{x}',z') \right\} = \frac{1}{\sqrt{2}}\, P_+\, \delta(z-z')\, \delta^2(\underset{\sim}{x}-\underset{\sim}{x}')$$
$$\left\{ \Psi_+(\tau,\underset{\sim}{x},z),\, \Psi_+(\tau,x',z') \right\} = 0 \tag{IV.45}$$

We will use the same procedure to find commutation rules for the field $A^j(x)$. Since $A^j(x)$ is to be Hermitian field, we write its Fourier expansion as

$$A^j(x) = (2\pi)^{-3} \int d\underset{\sim}{p} \int_0^\infty \frac{d\eta}{2\eta} \sum_{\lambda=\pm 1}$$
$$\times \left\{ e(\lambda)^j\, e^{-i(\eta z - \underset{\sim}{p}\cdot\underset{\sim}{x})}\, a(\eta,\underset{\sim}{p},\lambda;\tau) \right. \tag{IV.46}$$
$$\left. + e(\lambda)^{*j}\, e^{+i(\eta z - \underset{\sim}{p}\cdot\underset{\sim}{x})}\, a^\dagger(\eta,\underset{\sim}{p},\lambda;\tau) \right\}$$

where
$$e(\lambda)^j = \mp 2^{-1/2}\,(1,\pm i) \quad \text{for } \lambda = \pm 1 .$$

In terms of the operators $a(p;\lambda;\tau)$, the photon part of the momentum P_α is

$$P_{\alpha\,(\mathrm{Photon})} = -\int d\underset{\sim}{x}\, dz\, \left(\partial_\alpha A^i(x)\right)\left(\partial_3 A_i(x)\right) \qquad \alpha=1,2,3$$
$$= (2\pi)^{-3} \int d\underset{\sim}{p} \int_0^\infty \frac{d\eta}{2\eta} \sum_{\lambda=\pm 1} \tag{IV.47}$$
$$\times \frac{1}{2} p_\alpha \left\{ a(p,\lambda;\tau)\, a^\dagger(p,\lambda;\tau) \right.$$
$$\left. + a^\dagger(p,\lambda;\tau)\, a(p,\lambda;\tau) \right\} .$$

The interpretation of (IV.47) is clear if we let the operators $a(p;\lambda;\tau)$ be destruction operators for photons and normal order the expression for P_α. Thus we are led to postulate the covariant commutation relations

$$\begin{aligned} [a(p,\lambda;\tau), a^\dagger(p',\lambda';\tau)] &= \delta_{\lambda\lambda'} (2\pi)^3 2\eta \,\delta(\eta-\eta')\,\delta^2(p-p') \\ [a(p,\lambda;\tau), a(p,\lambda;\tau)] &= 0 \;. \end{aligned} \qquad \text{(IV.48)}$$

Transforming back to coordinate space, we obtain easily the equal-τ commutation relations

$$\left[\partial_3 A^i(\tau,x,z), A^j(\tau,x',z')\right] = -\frac{i}{2}\,\delta_{ij}\,\delta(z-z')\,\delta^2(x-x') \;. \qquad \text{(IV.49)}$$

Utilizing the relation (IV.14) between A^i and $\partial_3 A^i = -F^{oi}$, we obtain

$$\left[A^i(\tau,x,z), A^j(\tau,x',z')\right] = -\frac{i}{4}\,\delta_{ij}\,\epsilon(z-z')\,\delta^2(x-x') \;. \qquad \text{(IV.50)}$$

We also assume, of course, that the photon creation and destruction operators commute (at equal τ) with the fermion creation and destruction operators. Thus

$$\left[A^i(\tau,x,z), \Psi_+(\tau,x',z')\right] = 0 \qquad \text{(IV.51)}$$

Our field theory in the infinite momentum frame is based on the equal-τ commutation relations (IV.45), (IV.50), and (IV.51). We would expect, a priori, that dynamical effects could propagate from one point to another in a plane τ = constant along a line $\vec{x}_T$ = constant (i. e. along a light cone). Thus we might expect that the commutation relations would depend on the coupling constant e. The commutation relations among the independent fields of the theory are in fact independent of e. However, the electrodynamic interaction does affect in the equal-τ commutation relations among the components of the complete fields $A^\mu(x)$ and $\Psi(x)$, since the charge e appears in the definition of the "auxiliary" components, A^3 and Ψ_-, of the fields. We find, for instance, that

$$\left[A^3(\tau,\underset{\sim}{x},z), \Psi_+(\tau,\underset{\sim}{x}',z')\right] = \frac{e}{2}\,|z-z'|\,\delta^2(\underset{\sim}{x}-\underset{\sim}{x}')\,\Psi_+(\tau,\underset{\sim}{x}',z').$$

We can gain further confidence in the equal-τ commutation relations by using them to show that the operators P_μ and $M_{\mu\nu}$ actually generate translations and Lorentz transformations when commuted with the independent fields of the theory. The verification for the "kinematical" operators is particularly simply because these operators involve only the independent fields. One finds

$$
\begin{aligned}
& i\,[P_j, A^i(x)] = \partial_j A^i(x) \qquad i\,[P_j, \Psi_+(x)] = \partial_j \Psi_+(x) \\
& i\,[\eta, A^i(x)] = \partial_3 A^i(x) \qquad i\,[\eta, \Psi_+(x)] = \partial_3 \Psi_+(x) \\
& i\,[J_3, A^i(x)] = (x_1\partial_2 - x_2\partial_1) A^i(x) - \epsilon_{ij} A^j(x) \\
& i\,[J_3, \Psi_+(x)] = (x_1\partial_2 - x_2\partial_1)\Psi_+(x) + \tfrac{1}{2}\gamma_1\gamma_2 \Psi_+(x) \\
& i\,[B_j, A^i(x)] = (x_3\partial_j - x_j\partial_3) A^i(x) \\
& i\,[B_j, \Psi_+(x)] = (x_3\partial_j - x_j\partial_3)\Psi_+(x) .
\end{aligned}
\tag{IV.52}
$$

It is considerably more tedious to show that the operators H, S_1, S_2, and K_3 have the proper commutation relations with the fields. We present in the Appendix at the end of this chapter some details of the calculation which verifies the crucial assertion

$$
\begin{aligned}
i\left[H, A^i(x)\right] &= \partial_0 A^i(x) \\
i\left[H, \Psi_+(x)\right] &= \partial_0 \Psi_+(x) \quad .
\end{aligned}
\tag{IV.53}
$$

Similar but lengthier algebra gives

$$
\begin{aligned}
i\left[K_3, A^i(x)\right] &= (x_0\partial_3 - x_3\partial_0)\, A^i(x) \\
i\left[K_3, \Psi_+(x)\right] &= (x_0\partial_3 - x_3\partial_0)\, \Psi_+(x) \\
i\left[S_i, A^j(x)\right] &= (x_i\partial_0 - x_0\partial_i)\, A^j(x) \\
&\quad - g_i{}^j\, A_0(x) + \partial^j \Lambda_i(x) \\
i\left[S_i, \Psi_+(x)\right] &= (x_i\partial_0 - x_0\partial_i)\, \Psi_+(x) \\
&\quad + \tfrac{1}{2}\gamma_i\gamma_0\, \Psi_+(x) - ie\Lambda_i(x)\, \Psi_+(x),
\end{aligned}
\tag{IV.54}
$$

where $\Lambda_i(x) = \frac{1}{2}\int d\xi\, \epsilon(x^3 - \xi)\, A_i(x^0, \vec{x}_T, \xi)$ is that function which preserves the gauge during the Lorentz transformation.[6]

D. Free Fields

Let us see how the methods of the preceeding sections work if the interaction is turned off. Consider first the electron field $\Psi(x)$. With no interaction, each component of $\Psi(x)$ satisfies the Klein-Gordon equation

$$\left(2\partial_0\partial_3 + \partial_i\partial^i + m^2 \right) \Psi(x) = 0 \tag{IV.55}$$

Using this in the Fourier expansion (IV.41) of $\Psi_+(x)$, we find that the operators $b(p;s;\tau)$, $d^\dagger(p;s;\tau)$ satisfy the differential equations

$$\left(-2i\eta \frac{\partial}{\partial\tau} + p^2 + m^2 \right) b(p,\lambda;\tau) = 0$$
$$\left(+2i\eta \frac{\partial}{\partial\tau} + p^2 + m^2 \right) d^\dagger(p,\lambda;\tau) = 0 \quad .$$

Solving these equations, we get

$$b(p,\lambda;\tau) = e^{-ip_0\tau}\, b(p,\lambda;0)$$
$$d^\dagger(p,\lambda;\tau) = e^{+ip_0\tau}\, d^\dagger(p,\lambda;0) \quad , \tag{IV.56}$$

where $p_0 = (p^2 + m^2)/2\eta$ is the free particle Hamiltonian. Thus the Fourier expansion of $\Psi_+(x)$ takes the form

$$\Psi_+(x) = (2\pi)^3 \int dp \int_0^\infty \frac{d\eta}{2\eta} \sum_{\lambda=\pm 1}$$
$$\left\{ 2^{1/4}\eta^{\frac{1}{2}}\, w(\lambda)\, e^{-ip,x^\nu}\, b(p,\lambda;0) + 2^{1/4}\eta^{1/2}\, w(-\lambda)\, e^{+ip,x^\nu}\, d^\dagger(p,\lambda;0) \right\} . \tag{IV.57}$$

The auxiliary field $\Psi_-(x)$ is given in terms of $\Psi_+(x)$ by Eq. (IV.22):

$$\Psi_-(x) = -\frac{i}{4}\int d\xi \, \epsilon(x^3-\xi)\,[i\partial_j\gamma^j + m]\,\gamma^0\, \Psi_+(x^0, \underline{x}, \xi) .$$

Substituting the Fourier expansion of $\Psi_+(x)$ into this expression and doing the ξ - integration we obtain the Fourier expansion of $\Psi_-(x)$. We have now only to add $\Psi_+(x)$ and $\Psi_-(x)$ to obtain the complete field $\Psi(x)$:

$$\begin{aligned}\Psi(x) = (2\pi)^{3}\int d\underline{p}\int_0^\infty \frac{dp_+}{2p_+} \sum_{\lambda=\pm 1/2} \\ \times\Big\{ U(p,\lambda)\, e^{-ip_\nu x^\nu}\, b(p,\lambda;0) \\ + V(p,\lambda)\, e^{+ip_\nu x^\nu}\, d^\dagger(p,\lambda;0)\Big\} ,\end{aligned} \tag{IV.58}$$

where

$$\begin{aligned} U(p,\lambda) &= 2^{1/4}\, p_+^{1/2}\left(1 + \frac{p_j\gamma^j + m}{2p_+}\gamma^0\right) w(\lambda) \\ V(p,\lambda) &= 2^{1/4}\, p_+^{1/2}\left(1 + \frac{p_j\gamma^j - m}{2p_+}\gamma^0\right) w(-\lambda) . \end{aligned} \tag{IV.59}$$

Recalling the definition of the spinors w(s) from Eq. (IV.42), we can calculate $U(p,\lambda)$ and $V(p,\lambda)$. We find

$$u(p,+\tfrac{1}{2}) = 2^{-1/4}\eta_1^{-1/2}\begin{pmatrix}\sqrt{2}\,\eta_1\\ p^1+ip^2\\ m\\ 0\end{pmatrix} \qquad u(p,-\tfrac{1}{2}) = 2^{-1/4}\eta_1^{-1/2}\begin{pmatrix}0\\ m\\ -p^1+ip^2\\ \sqrt{2}\,\eta_1\end{pmatrix}$$

$$v(p,+\tfrac{1}{2}) = 2^{-1/4}\eta_1^{-1/2}\begin{pmatrix}0\\ -m\\ -p^1+ip^2\\ \sqrt{2}\,\eta_1\end{pmatrix} \qquad v(p,-\tfrac{1}{2}) = 2^{-1/4}\eta_1^{-1/2}\begin{pmatrix}\sqrt{2}\,\eta_1\\ p^1+ip^2\\ -m\\ 0\end{pmatrix} \tag{IV.60}$$

If the field $\Psi(x)$ which we have obtained in the infinite momentum frame is to be equal to the usual free Dirac field, they the spinors u(p, s) should be solutions of the Dirac equation normalized to $\bar{u}(p,s)u(p,s') = 2m\,\delta_{ss'}$, and the spinors v(p, s) should be related to u(p, s) by charge conjugation. A quick check shows that this is indeed the case. In fact, the spinors u and v which arose here from a canonical formalism in the infinite-momentum frame are exactly equal to the infinite-momentum helicity spinors derived in Chapter III.

Apparently the destruction operator $b(p, s, \tau)$ destroys an electron with momentum p and infinite-momentum helicity λ.

We can also check to see that, with the interaction turned off, our field $A^\mu(x)$ is just the usual free photon field (in the appropriate gauge). The calculation is completely analogous to the calculation for $\Psi(x)$, so we just state the result. With $e = 0$, we find

$$\begin{aligned} A^\mu(x) = {} & (2\pi)^{-3}\int d\underline{p}\int_0^\infty \frac{d\eta_+}{2\eta_1}\sum_{\lambda=\pm 1} \\ & \times\Big\{ e(p,\lambda)^\mu\, e^{-ip\cdot x}\, a(p,\lambda;0) \\ & + e(p,\lambda)^{*\mu}\, e^{+ip\cdot x}\, a^\dagger(p,\lambda;0)\Big\}, \end{aligned} \tag{IV.61}$$

where the $e(p,\lambda)^\mu$ are just the infinite momentum gauge polarization vectors defined in Eq. (IV.6).

E. Scattering Theory

We have seen that infinite momentum quantum electrodynamics is the same as ordinary quantum electrodynamics in the trivial case e = 0. The two theories can be compared for $e \neq 0$, at least formally, by constructing the S-matrix in old-fashioned perturbation theory in the infinite momentum frame and comparing it with the S-matrix given by the τ-ordered diagrams of Part 1.

The perturbation expansion of the S-matrix takes a familiar form once we have divided the Hamiltonian into a free part and an interaction part. To make this division, we start with the Hamiltonian density $T_0^{\,0}(x)$:

$$\begin{aligned} T_0^{\,0} = &- \bar{\Psi}\left(\left[\tfrac{i}{2}\overleftrightarrow{\partial_j} - eA_j\right]\gamma^j - m\right)\Psi - \bar{\Psi}\tfrac{i}{2}\overleftrightarrow{\partial_3}\gamma^3\Psi \\ &+ eA^3\bar{\Psi}\gamma^3\Psi + \tfrac{1}{2}F^{12}F_{12} \\ &- \tfrac{1}{2}(\partial_3 A^3)(\partial_3 A^3) - (\partial_j A^3)(\partial_3 A^j) \; . \end{aligned} \qquad \text{(IV.62)}$$

The integrated Hamiltonian can be somewhat simplified if we realize that the first term is equal to -2 times the second term after an integration by parts in the transverse variables x^1, x^2. To see this, write -2 times the second term as

$$\bar{\Psi}\, i\,\partial_3\gamma^3\Psi = \Psi^\dagger\gamma^0\gamma^3\, i\overleftrightarrow{\partial_3}\Psi$$

Using Eq. (IV.21) for $\partial_3\Psi_-$, this is

$$-\frac{1}{\sqrt{2}}\Psi_-^{\dagger}\gamma^0[(i\overrightarrow{\partial}_j - eA_j)\gamma^j - m]\Psi_+$$
$$-\frac{1}{\sqrt{2}}\Psi_+^{\dagger}\gamma^3[(-i\overleftarrow{\partial}_j - eA_j)\gamma^j - m]\Psi_- \ .$$

With an integration by parts in the transverse variables, we can replace $i\overrightarrow{\partial}_j$ and $-i\overleftarrow{\partial}_j$ by $\frac{i}{2}\overleftrightarrow{\partial}_j$ and obtain

$$-\frac{1}{\sqrt{2}}\Psi^{\dagger}(P_-\gamma^0P_+ + P_+\gamma^3P_-)\left[(\tfrac{i}{2}\overleftrightarrow{\partial}_j - eA_j)\gamma^j - m\right]\Psi \ .$$

But $P_-\gamma^0P_+ + P_+\gamma^3P_- = \gamma^0 + \gamma^3 = \sqrt{2}\,\hat{\gamma}^0$, so this is just

$$-\overline{\Psi}\left[(\tfrac{i}{2}\overleftrightarrow{\partial}_j - eA_j)\gamma^j - m\right]\Psi \ .$$

Thus the Hamiltonian density can be rewritten as

$$T_0{}^0 = \overline{\Psi}\,\tfrac{i}{2}\overleftrightarrow{\partial}_3\gamma^3\Psi + eA^3\overline{\Psi}\gamma^0\Psi + \tfrac{1}{2}F^{12}F_{12} - \tfrac{1}{2}(\partial_3A^3)(\partial_3A^3) - (\partial_jA^3)(\partial_3A^j) \ . \tag{IV.63}$$

At this point we realize that part of the interaction is buried in the dependence of Ψ_- and A^3 on e. In order to bring out this dependence we write Ψ_- as the sum of a "free" part ψ_- and an "interaction part" Υ, where

$$\psi_-(x) = -\frac{i}{4}\int d\xi\ \epsilon(x^3-\xi)\left\{i\partial_j\gamma^j + m\right\}\gamma^0\,\Psi_+(x^0,\underset{\sim}{x},\xi) \tag{IV.64}$$

$$\Upsilon(x) = \frac{ie}{4}\int d\xi\ \epsilon(x^3-\xi)\,A_j(x^0,\underset{\sim}{x},\xi)\,\gamma^j\gamma^0 \times \Psi_+(x^0,\underset{\sim}{x},\xi) \ . \tag{IV.65}$$

We also define $\psi_+ = \Psi_+$ and $\psi = \psi_+ + \psi_-$. Similarly, we write $A^3 = \mathscr{A}^3 + \phi$, where

$$a^3(x) = -\frac{1}{2}\int d\xi \; |x^3 - \xi| \; \partial_3 \partial_j A^j(x^0, \underset{\sim}{x}, \xi) \tag{IV.66}$$

$$\phi(x) = -\frac{1}{2}\int d\xi \; |x^3 - \xi| \; J^0(x^0, \underset{\sim}{x}, \xi) \tag{IV.67}$$

and we put $\mathscr{A}^j = A^j$, $\mathscr{A}^0 = 0$. Let us insert $\Psi = \psi + \Upsilon$ and $A^\mu = \mathscr{A}^\mu + \delta^\mu_3 \phi$ into our Hamiltonian density (IV.63) and simplify the result.

From the first term in T^0_0 we get four terms

$$\begin{aligned} \overline{\Psi} \tfrac{i}{2} \overleftrightarrow{\partial_3} \gamma^3 \Psi &= \sqrt{2}\, \Psi_-^\dagger \tfrac{i}{2} \overleftrightarrow{\partial_3} \Psi_- \\ &= \sqrt{2}\, \psi_-^\dagger \tfrac{i}{2} \overleftrightarrow{\partial_3} \psi_- + \sqrt{2}\, \Upsilon^\dagger \tfrac{i}{2} \overleftrightarrow{\partial_3} \Upsilon + \sqrt{2}\, \psi_-^\dagger \tfrac{i}{2} \overleftrightarrow{\partial_3} \Upsilon + \sqrt{2}\, \Upsilon^\dagger \tfrac{i}{2} \overleftrightarrow{\partial_3} \psi_- . \end{aligned} \tag{IV.68}$$

The first two terms can be left as they stand. The integrated form of the third term can be integrated by parts to that $\frac{i}{2}\overleftrightarrow{\partial_3}$ is replaced by $i\overrightarrow{\partial_3}$. This integration by parts can be justified simply by using the definitions (IV.64) and (IV.65) to write

$$\begin{aligned} -\int dz \left(\partial_3 \psi_-^\dagger(z)\right) \Upsilon(z) &= -\frac{1}{2}\int dz\, d\xi \left(\partial_3 \psi_-^\dagger(z)\right) \epsilon(z-\xi) \left(\partial_3 \Upsilon(\xi)\right) \\ &= +\frac{1}{2}\int dz\, d\xi \left(\partial_3 \psi_-^\dagger(z)\right) \epsilon(\xi - z) \left(\partial_3 \Upsilon(\xi)\right) \\ &= +\int d\xi\, \psi_-^\dagger(\xi) \left(\partial_3 \Upsilon(\xi)\right) . \end{aligned}$$

Similarly, we can replace $\frac{i}{2} \overleftrightarrow{\partial}_3$ by $-\frac{i}{2} \overleftarrow{\partial}_3$ in the fourth term. Then, making use of the definition (IV.65) of Υ, we obtain for the sum of the third and fourth terms of (IV.68)

$$\frac{e}{\sqrt{2}} \psi^\dagger [P_- \gamma^0 P_+ + P_+ \gamma^3 P_-] a_j \gamma^j \psi = e a_j \bar{\psi} \gamma^j \psi . \tag{IV.69}$$

Turning now to the second term in T^0_0, we write simply

$$\begin{aligned} e A^3 \bar{\Psi} \gamma^0 \Psi &= e A^3 \bar{\Psi}_+ \gamma^0 \Psi_+ \\ &= e a^3 \bar{\psi} \gamma^0 \psi + e \phi \bar{\psi} \gamma^0 \psi . \end{aligned} \tag{IV.70}$$

The third term in T^0_0 can be left unchanged since it involves only $A^j = \mathscr{A}^j$. The fourth term requires some work. With an integration by parts we can make the replacement[7]

$$-\frac{1}{2} (\partial_3 A^3)(\partial_3 A^3) \rightarrow +\frac{1}{2} A^3 \partial_3 \partial_3 A^3 .$$

Writing $A^3 = \mathscr{A}^3 + \phi$, we obtain the sum

$$\frac{1}{2} a^3 \partial_3 \partial_3 a^3 + \frac{1}{2} \phi \partial_3 \partial_3 \phi + \frac{1}{2} \phi \partial_3 \partial_3 a^3 + \frac{1}{2} a^3 \partial_3 \partial_3 \phi . \tag{IV.71}$$

We write the first and second terms simply as

$$\frac{1}{2} a^3 \partial_3 \partial_3 a^3 = -\frac{1}{2} a^3 \partial_3 \partial_j a^j \tag{IV.72}$$

and

$$\begin{aligned} \frac{1}{2} \phi \partial_3 \partial_3 \phi &= -\frac{1}{2} \phi J^0 \\ &= -\frac{e}{2} \phi \bar{\psi} \gamma^0 \psi . \end{aligned} \tag{IV.73}$$

We see, with the use of the definitions (IV.66) and (IV.67), that the integrated forms of the third and fourth terms in (IV.71) are equal. Indeed,

$$\int d\xi \, \left(\partial_3\partial_3 \phi(\xi)\right) a^3(\xi) = \frac{1}{2}\int d\xi \, dz \left(\partial_3\partial_3 \phi(\xi)\right) |z-\xi| \left(\partial_3\partial_3 a^3(z)\right)$$
$$= \int dz \, \phi(z) \left(\partial_3\partial_3 a^3(z)\right) .$$

Thus we can write for the sum of the last wee terms in (IV.71)

$$\frac{1}{2} \phi \, \partial_3\partial_3 a^3 + \frac{1}{2} a^3 \partial_3\partial_3 \phi \rightarrow \phi \, \partial_3\partial_3 a^3 \qquad \text{(IV.74)}$$
$$= -\phi \, \partial_3 \partial_j a^j .$$

Finally, we consider the fifth term of T^0_0, which we write, using an integration by parts of the variables x^1, x^2, as

$$-(\partial_j A^3)(\partial_3 A^j) \rightarrow A^3 \partial_3\partial_j A^j \qquad \text{(IV.75)}$$
$$= \phi \, \partial_3\partial_j a^j + a^3 \partial_3\partial_j a^j .$$

The integrated Hamiltonian is now in the form we wanted. Adding up the pieces, we have

$$H = H_0 + V \qquad \text{(IV.76)}$$

where

$$H_0 = \int d_\perp \, dz \left\{ \sqrt{2} \, \psi^\dagger_- \frac{i}{2} \overleftrightarrow{\partial_3} \psi_- + \frac{1}{2} F^{12} F_{12} \right. \qquad \text{(IV.77)}$$
$$\left. + \frac{1}{2} a^3 \partial_3 \partial_j a^j \right\} ,$$

$$V = \int d\underset{\sim}{x}\, dz \left\{ e\, a_\mu\, \bar{\psi}\gamma^\mu\psi + \sqrt{2}\, \Upsilon^\dagger \tfrac{i}{2} \overleftrightarrow{\partial}_3 \Upsilon + \tfrac{1}{2} e\, \phi\, \bar{\psi}\gamma^0\psi \right\} \tag{IV.78}$$

If we work in the Schroedinger picture, we can evaluate all Heisenberg operators at "time" $\tau = 0$. We note that the Fourier expansion of the fields $\psi(x)$ and $\mathscr{A}^\mu(x)$ at $\tau = 0$ in terms of creation and destruction operators are the same as the expansions (IV.58) and (IV.61) for free fields. Thus the free Hamiltonian H_0 generates the free motion of the quanta created by $a^\dagger(p;\lambda;0)$, $b^\dagger(p;\lambda;0)$, $d^\dagger(p;\lambda;0)$. The remaining part of the Hamiltonian, V, gives rise to the scattering of these quanta.

We can formally calculate the scattering matrix with the aid of the "old-fashioned" perturbation theory expansion

$$S_{fi} = \mathbb{1} - 2\pi i\, \delta(\mathcal{H}_f - \mathcal{H}_i) \times \left\{ V + V \frac{1}{\mathcal{H}_i - H_0 + i\epsilon} V + \cdots \right\} . \tag{IV.79}$$

In a field theory in an ordinary Lorentz frame, this formula leads to a set of rules for calculating scattering matrix elements using time ordered diagrams. In the present case, we are led in the same way[8] to rules for τ-ordered diagrams.

These rules are the same as the rules developed directly from the covariant Feynman rules in Part 1. This can be seen by calculating a few matrix elements of the interaction Hamiltonian V. One finds that the interaction term

$$V_1 = \int d\underset{\sim}{x}\, dz\;\; e\, a_\mu\, \bar{\psi}\gamma^\mu\psi \tag{IV.80}$$

gives the "ordinary" vertices of Figure IV-4a. The second term in V, when written out in full using the definition of Υ, is

$$V_2 = -\frac{ie^2}{4}\int d\underset{\sim}{x}\, dz\, d\xi \;\; \bar{\psi}(0,\underset{\sim}{x},z)\, \gamma^\mu a_\mu(0,\underset{\sim}{x},z)$$
$$\times\, \epsilon(z-\xi)\gamma^0 \gamma^\nu a_\nu(0,\underset{\sim}{x},\xi)\;\; \psi(0,\underset{\sim}{x},\xi) \quad . \tag{IV.81}$$

Using

$$\int dz\; e^{i\eta z}\, \epsilon(z) \;=\; \frac{2i}{\eta} \qquad ,$$

one finds that the interaction V_2 gives the vertices of Figure IV.4c.

The third term in V, written out in full, is

$$V_3 \;=\; -\frac{e^2}{4}\int d\underset{\sim}{x}\; dz\; d\xi \;\; \bar{\psi}(0,\underset{\sim}{x},z)\, \gamma^0\, \psi(0,\underset{\sim}{x},z)$$
$$\times\; |z-\xi|\;\; \bar{\psi}(0,\underset{\sim}{x},\xi)\, \gamma^0\;\; \psi(0,\underset{\sim}{x},\xi) \;\; . \tag{IV.82}$$

Using

$$\int dz\;\; e^{i\eta z}\; |z| \;\;=\; -\frac{2}{\eta^2} \qquad ,$$

it is easily shown that the interaction V_3 gives the "Coulomb" vertices of Figure IV-4b.

Thus when we formally calculate the S-matrix from canonical field theory developed in the infinite momentum frame, we get the same results as when we directly transform the S-matrix for ordinary quantum electrodynamics to the infinite momentum frame.

APPENDIX

In this appendix we will show that the canonical hamiltonian presented in section II-B generates the correct equations of motion for the independent field operators $A^i(y)$. We begin with expression (IV.63) for the hamiltonian,

$$H = \int d\vec{x} \left\{ i\, 2^{-1/2}\, \Psi_-^{\dagger} \overleftrightarrow{\partial}_3 \Psi_- + A^3 J^o - \frac{1}{2}\, \partial_3 A^3 \partial_3 A^3 + \frac{1}{2} F^{12} F_{12} - \partial_j A^3 \partial_3 A^j \right\}. \qquad \text{(A1)}$$

In order to compute $\left[H, A^i(y)\right]$ we need to first compute two rather complicated equal-τ commutators which we list here,

$$\left[\Psi_-(x), A^i(y)\right]\Big|_{x^o=y^o} = -\frac{e}{16} \int_{-\infty}^{\infty} d\xi\ \epsilon(x^3-\xi)\ \epsilon(\xi - y^3)\delta^2(\vec{x}_T - \vec{y}_T)\, \gamma^i \gamma^o\, \Psi_+(y^o, \vec{y}_T, \xi)$$

$$\left[A^3(x), A^i(y)\right]\Big|_{x^o=y^o} = \frac{1}{4i}\, |x^3 - y^3|\, \partial^i \delta^2(\vec{x}_T - \vec{y}_T) \quad . \qquad \text{(A2)}$$

These relations follow from the definitions of the auxiliary fields, $\Psi_-(y)$ and $A^3(y)$, and the basic equal-τ commutators of the independent fields.

With these preliminaries done, we can compute

$$\begin{aligned}
\left[H, A^i(y)\right] = {} & 2^{1/2} \int d\vec{x} \left[\Psi_-^{\dagger}(x)\, \frac{i}{2}\, \overleftrightarrow{\partial}_3\, \Psi_-(x), A^i(y)\right]_{x^o=y^o} \\
& + \int d\vec{x}\, J^o(x) \left[A^3(x), A^i(y)\right]_{x^o=y^o} \\
& - \frac{1}{2} \int d\vec{x} \left[\partial_3 A^3(x)\ \partial_3 A^3(x), A^i(y)\right]_{x^o=y^o} \\
& + \frac{1}{2} \int d\vec{x} \left[F^{12}(x)\ F_{12}(x), A^i(y)\right]_{x^o=y^o} \\
& - \int d\vec{x} \left[\partial_j A^3(x)\ \partial_3 A^j(x), A^i(y)\right]_{x^o=y^o} \quad . \qquad \text{(A3)}
\end{aligned}$$

For convenience we label these five terms (I.), (II.), (III.), (IV.), (V.), and compute each in its turn:

$$(\mathrm{I.}) = 2^{1/2}\int d\vec{x}\left[\Psi_-^\dagger(x)\,\frac{i}{2}\overleftrightarrow{\partial}_3\Psi_-(x), A^i(y)\right]_{x^o=y^o} \tag{A4}$$

$$= i\,2^{-1/2}\int d\vec{x}\Big\{\Psi_-^\dagger(x)\left[\partial_3\Psi_-(x), A^i(y)\right]_{x^o=y^o} + \left[\Psi_-^\dagger(x), A^i(y)\right]_{x^o=y^o}\partial_3\Psi_-(x)$$

$$- \partial_3\Psi_-^\dagger(x)\left[\Psi_-(x), A^i(y)\right]_{x^o=y^o} - \left[\partial_3\Psi_-^\dagger(x), A^i(y)\right]_{x^o=y^o}\Psi_-(x)\Big\}$$

$$= i\,2^{-1/2}\int d\vec{x}\Big\{-\frac{e}{8}\,\epsilon(x^3-y^3)\,\delta^2(\vec{x}_T-\vec{y}_T)\,\Psi_-^\dagger(x)\,\gamma^i\gamma^o\,\Psi_+(x)$$

$$-\frac{e}{16}\int d\xi\;\epsilon(x^3-\xi)\,\epsilon(\xi-y^3)\,\delta^2(\vec{x}_T-\vec{y}_T)\,\Psi_+^\dagger(y^o,\vec{x}_T,\xi)\;\gamma^3\gamma^i\partial_3\Psi_-(x)$$

$$+\frac{e}{16}\int d\xi\;\epsilon(x^3-\xi)\,\epsilon(\xi-y^3)\,\delta^2(\vec{x}_T-\vec{y}_T)\,\partial_3\Psi_-^\dagger(x)\;\gamma^i\gamma^o\,\Psi_+(y^o,\vec{x}_T,\xi)$$

$$+\frac{e}{8}\,\epsilon(x^3-y^3)\,\delta^2(\vec{x}_T-\vec{y}_T)\,\Psi_+^\dagger(x)\;\gamma^3\gamma^i\,\Psi_-(x)\Big\}$$

$$= ie\,2^{-7/2}\int dx^3\;\epsilon(x^3-y^3)\Big\{\Psi_-^\dagger(y^o,\vec{y}_T,x^3)\gamma^o\gamma^i\Psi_+(y^o,\vec{y}_T,x^3) + \Psi_+^\dagger(y^o,\vec{y}_T,x^3)\gamma^3\gamma^i\,\Psi_-(y^o,\vec{y}_T,x^3)\Big\}$$

$$-ie\,2^{-9/2}\int dx^3\int d\xi\;\epsilon(x^3-\xi)\,\epsilon(\xi-y^3)\Big\{\Psi_+^\dagger(y^o,\vec{y}_T,\xi)\gamma^3\gamma^i\partial_3\Psi_-(y^o,\vec{y}_T,x^3)+\partial_3\Psi_-^\dagger(y^o,\vec{y}_T,x^3)\gamma^o\gamma^i\Psi_+(y^o,\vec{y}_T,\xi)\Big\}$$

$$= \frac{1}{4i}\int d\xi\;\epsilon(y^3-\xi)\,J^i(y^o,\vec{y}_T,\xi)\quad . \tag{A5}$$

We have observed in this calculation that

$$\Psi_-(y) = \frac{1}{2}\int d\xi\;\epsilon(y^3-\xi)\;\partial_3\,\Psi_-(y^o,\vec{y}_T,\xi)$$

and

$$J^i(y) = 2^{-1/2}\, e\left\{\Psi_-^\dagger(y)\,\gamma^o\gamma^i\,\Psi_+(y) + \Psi_+^\dagger(y)\,\gamma^3\gamma^i\,\Psi_-(y)\right\} .$$

Continuing,

$$(\text{II.}) = \int d\vec{x}\; J^o(x)\left[A^3(x),\; A^i(y)\right]_{x^o=y^o} \tag{A6}$$

$$= \frac{1}{4i}\int d\vec{x}\; J^o(x)\left|x^3-y^3\right|\partial^i\delta^2(\vec{x}_T-\vec{y}_T)$$

$$= -\frac{1}{4i}\int dx^3\left|x^3-y^3\right|\partial^i J^o(y^o,\vec{y}_T,x^3) \quad . \tag{A7}$$

Next,

$$(\text{III.}) = -\frac{1}{2}\int d\vec{x}\left[\partial_3A^3(x)\;\partial_3A^3(x),\; A^i(y)\right]_{x^o=y^o} \tag{A8}$$

$$= -\int d\vec{x}\;\partial_3A^3(x)\left[\partial_3A^3(x),\; A^i(y)\right]_{x^o=y^o}$$

$$= \frac{1}{4i}\int d\vec{x}\;\partial_3A^3(x)\;\partial_i\,\delta^2(\vec{x}_T-\vec{y}_T)\;\epsilon(x^3-y^3)$$

$$= -\frac{1}{4i}\;\partial^i\int dx^3\;\epsilon(y^3-x^3)\;\partial_3A^3(y^o,\vec{y}_1,x^3)$$

$$= -\frac{1}{2i}\,\partial^iA^3(y) \quad . \tag{A9}$$

We have applied here the definition (IV.14)of $A^3(x)$.

The fourth term becomes

$$(\text{IV.}) = \frac{1}{2}\int d\vec{x}\left[F^{12}(x)\;F_{12}(x),\; A^i(y)\right]_{x^o=y^o} \tag{A10}$$

$$= \int d\vec{x}\;F^{12}(x)\left[\partial^2A^1(x) - \partial^1A^2(x),\; A^i(y)\right]_{x^o=y^o}$$

$$= \frac{1}{4i}\int d\vec{x}\, F^{12}(x)\left\{\delta_{1i}\,\partial^2\delta^2(\vec{x}_T-\vec{y}_T)\;\epsilon(x^3-y^3) - \delta_{2i}\,\partial^1\delta^2(\vec{x}_T-\vec{y}_T)\;\epsilon(x^3-y^3)\right\}$$

$$= \frac{1}{4i}\left\{\delta_{1i}\int dx^3\;\epsilon(x^3-y^3)\,\partial_2 F^{12}(y^o,\vec{y}_T,x^3) + \delta_{2i}\int dx^3\;\epsilon(x^3-y^3)\,\partial_1 F^{21}(y^o,\vec{y}_T,x^3)\right\}$$

$$= -\frac{1}{4i}\int dx^3\;\epsilon(y^3-x^3)\,\partial_j F^{ij}(y^o,\vec{y}_T,x^3)\quad . \tag{A11}$$

Finally,

$$(\mathrm{V.}) = -\int d\vec{x}\left[\partial_j A^3(x)\;\partial_3 A^j(x),\;A^i(y)\right]_{x^o=y^o} \tag{A12}$$

$$= -\int d\vec{x}\left\{\partial_j A^3(x)\;\partial_3\left[A^j(x),\;A^i(y)\right]_{x^o=y^o} + \partial_j\left[A^3(x),\;A^i(y)\right]_{x^o=y^o}\partial_3 A^j(x)\right\}$$

$$= -\frac{1}{4i}\int d\vec{x}\left\{\partial_j A^3(x)\delta_{ij}\,\delta^2(\vec{x}_T-\vec{y}_T)\,\partial_3\,\epsilon(x^3-y^3) + \partial_j\partial^i\delta^2(\vec{x}_T-\vec{y}_T)\,|x^3-y^3|\,\partial_3 A^j(x)\right\}$$

$$= -\frac{1}{4i}\int d\vec{x}\left\{2\partial_i A^3(x)\delta^3(\vec{x}-\vec{y}) + |x^3-y^3|\,\partial^i\partial_3\partial_j A^j(x)\delta^2(\vec{x}_T-\vec{y}_T)\right\}$$

$$= -\frac{1}{2i}\,\partial_i A^3(y) - \frac{1}{4i}\partial^i\int dx^3\,|x^3-y^3|\,\partial_3\partial_j A^j(y^o,\vec{y}_T,x^3)\quad . \tag{A13}$$

Collecting these five terms, we have the result

$$\left[H, A^i(y)\right] = \frac{1}{4i}\int dx^3\;\epsilon(y^3-x^3)\left\{J^i(y^o,\vec{y}_T,x^3) + \partial_j F^{ji}(y^o,\vec{y}_T,x^3)\right\}$$

$$- \frac{1}{4i}\,\partial^i\int dx^3\,|x^3-y^3|\left\{\partial_3\partial_j A^j(y^o,\vec{y}_T,x^3) + J^o(y^o,\vec{y}_T,x^3)\right\}\quad . \tag{A14}$$

Recalling the relation (IV.16) for $A^3(x)$, we have, more simply,

$$\left[H, A^i(y)\right] = \frac{1}{4i}\int dx^3\;\epsilon(y^3-x^3)\left\{J^i(y^o,\vec{y}_T,x^3) + \partial_j F^{ji}(y^o,\vec{y}_T,x^3)\right\} + \frac{1}{2i}\,\partial^i A^3(y)\quad . \tag{A15}$$

Referring to (IV.26), we see that we have indeed verified our claim,

$$\left[H, A^i(y)\right] = \frac{1}{i}\,\partial_0 A^i(y) \quad . \tag{A16}$$

The verification of the Heisenberg relation

$$\left[H, \Psi_+(y)\right] = \frac{1}{i}\,\partial_0 \Psi_+(y) \tag{A17}$$

is also tedious but straight-forward.

References - Chapter IV

1. Here and elsewhere, we encounter a singularity at $\eta = 0$. In this paper it will not be necessary to specify the precise nature of these singularities.

2. We have done the momentum integrations over the $\delta(\tau)$ lines and rearranged the factors of π, i, etc.

3. We use the notation $a \overleftrightarrow{\partial}_\mu b$ for $a(\partial_\mu b) - (\partial_\mu a)b$.

4. For classical fields, the integral (IV.17) converges because $\partial_3 F^{03}$ goes to zero like $(x^3)^{-3}$ as $x^3 \to \infty$. Furthermore, no surface term arises in the integration by parts since F^{03} falls off like $(x^3)^{-2}$ as $x^3 \to \infty$. Note, however, that it is not permissible to integrate by parts of Eq. (IV.16).

5. Of course, this remains to be verified using the commutation relations of the fields, which we discuss in Part C.

6. Cf. J. Bjorken and S. Drell, Relativistic Quantum Fields, (McGraw Hill Inc., New York, 1965); p. 88 ff.

7. We may find some reassurance about this in the fact that, in classical electrodynamics, the surface term $A^3 \partial_3 A^3$ vanishes like $\mathscr{x}^{-2}$ as $\mathscr{x} \to \infty$.

8. Of course, we encounter most of the usual problems too. Cf. W. Heitler, The Quantum Theory of Radiation, (Oxford University Press, New York, 1966); p. 276 ff, Third Edition.

CHAPTER V

Massive Quantum Electrodynamics

In this chapter we extend the canonical formulation of quantum electrodynamics in the infinite-momentum frame by replacing the photons by massive vector mesons. The resulting theory is interesting in its own right, and also has useful applications to the work of Cornwall and Jackiw[1], of Dicus, Jackiw and Teplitz[2], and of Gross and Trieman[3] on current commutators on the light cone in a quark-vector gluon model.

We find that the required generalization is quite simple if we consider, in addition to the vector field A^{μ}, a scalar field B in the manner of Stückelberg's 1938 paper on gluons.[4,5] The results confirm the belief of Cornwall and Jackiw that terms in the vector meson propagator which might cause trouble in the infinite-momentum frame can be eliminated because of current conservation. The structure of the theory remains nearly the same as that of quantum electrodynamics except that a new term appears in the Hamiltonian describing the emission of helicity zero vector mesons with an amplitude proportional to the meson mass.

We will make free use of the results of the last chapter and devote most of our attention to the changes made necessary by going from massless to massive vector mesons.

A. Equations of Motion

The canonical theory of quantum electrodynamics in the infinite-momentum frame was based on the Lagrangian

$$\mathcal{L}(x)_{QED} = \bar{\psi}\left[\left(\tfrac{i}{2}\overleftrightarrow{\partial}_{\mu} - eA_{\mu}\right)\gamma^{\mu} - m\right]\psi - \tfrac{1}{4}\left(\partial^{\nu}A^{\mu} - \partial^{\mu}A^{\nu}\right)\left(\partial_{\nu}A_{\mu} - \partial_{\mu}A_{\nu}\right) \, ,$$

where $A^{\mu}(x)$ is the real vector field of the massless vector meson and ψ is a four-component Dirac field. In order to introduce a meson mass $\kappa > 0$ and allow for meson with helicity zero while maintaining gauge invariance, we introduce a real scalar field B(x) in addition to A_{μ} and Ψ. Then we begin with the modified Lagrangian

$$\begin{aligned}\mathcal{L}(x) = \overline{\Psi}[(i\tfrac{1}{2}\partial_{\mu} - eA_{\mu})\gamma^{\mu} - m]\Psi \\ - \tfrac{1}{4}(\partial^{\nu}A^{\mu} - \partial^{\mu}A^{\nu})(\partial_{\nu}A_{\mu} - \partial_{\mu}A_{\nu}) \\ + \tfrac{1}{2}(\kappa A^{\mu} - \partial^{\mu}B)(\kappa A_{\mu} - \partial_{\mu}B)\,.\end{aligned} \tag{V.1}$$

Variation of the fields Ψ, $\overline{\Psi}$, A_{μ}, and B give the equations of motion

$$[\partial_{\nu}\partial^{\nu} + \kappa^2]A^{\mu} - \partial^{\mu}[\partial_{\nu}A^{\nu} + \kappa B] = J^{\mu} \tag{V.2}$$

$$\kappa\,\partial_{\mu}A^{\mu} - \partial_{\mu}\partial^{\mu}B = 0 \tag{V.3}$$

$$[(i\partial_{\mu} - eA_{\mu})\gamma^{\mu} - m]\Psi = 0 \tag{V.4}$$

where we have defined $J^{\mu} = e\overline{\Psi}\gamma^{\mu}\Psi$. (Notice that $\partial_{\mu}J^{\mu} = 0$ as a consequence of the Dirac equation (V.4), and thus that equation (V.3) is merely the divergence of equation (V.2).)

The reason for introduction of the seemingly superfluous scalar field B is that the gauge invariance of quantum electrodynamics is thereby preserved. Indeed, the Lagrangian, and hence the equations of motion, is left invariant by the gauge transformation

$$A_\mu(x) \to A_\mu(x) + \partial_\mu \Lambda(x)$$
$$B(x) \to B(x) + \kappa \Lambda(x) \qquad \text{(V.5)}$$
$$\psi(x) \to e^{-ie\Lambda(x)}\,\psi(x) \; .$$

We could, if we wanted, use this gauge invariance to choose the "Lorentz gauge" B = 0. In this gauge the equations of motion would take the familiar form (after some simplifications) ,

$$[\partial_\nu \partial^\nu + \kappa^2]\, A^\mu = J^\mu$$
$$\partial_\mu A^\mu = 0$$
$$[(i\partial_\mu - e A_\mu)\gamma^\mu - m]\,\psi = 0 \; .$$

However, it turns out that it is very difficult to quantize the theory in the infinite-momentum frame in this gauge.

Instead, we choose the "infinite-momentum gauge",

$$A^0(x) = 0 \; . \qquad \text{(V.6)}$$

Then the $\mu = 0$ component of the equation of motion (V.2) reads

$$\partial_3 [\partial_3 A^3 + \partial_j A^j + \kappa B] = -J^0 \; .$$

This equation can be solved for A^3 as follows:

$$A^3 = -\frac{i}{\eta}\left[\partial_k A^k + \kappa B\right] + \frac{1}{\eta^2} J^0 \quad , \tag{V.7}$$

where $(1/\eta)$ and $(1/\eta^2)$ are the integral operators[6]

$$\left[\frac{1}{\eta} f\right](x) = -\frac{i}{2}\int d\xi \; \epsilon(x^3-\xi)\, f(x^0, \underset{\sim}{x}, \xi)$$

$$\left[\frac{1}{\eta^2} f\right](x) = -\frac{1}{2}\int d\xi \; |x^3-\xi|\, f(x^0, \underset{\sim}{x}, \xi) \quad .$$

Thus if we regard A^1, A^2, and B as independent dynamical variables, then A^3 is reduced to the status of a dependent field since it is determined at any "time" x^0 by the other fields at that x^0 according to the constraint equation (V.7).

The equations of motion for the independent fields A^k and B can now be simplified by substituting the expression (V.7) for A^3 back into the equations of motion (V.2) and (V.3). From (V.7) we have

$$\partial_\nu A^\nu = -\kappa B - \frac{i}{\eta} J^0 \quad . \tag{V.8}$$

If we substitute this into (V.3) and remember that $\partial_\mu J^\mu = 0$ we get the equation of motion for B,

$$\left[\partial_\mu \partial^\mu + \kappa^2\right] B = -i\kappa \frac{1}{\eta} J^0 \quad . \tag{V.9}$$

If we substitute (V.8) into equation (V.2) with $\mu = 1$ or 2 we get the equation of motion for A,

$$[\partial_\nu \partial^\nu + \kappa^2] A^k = J^k - \frac{i}{\eta} \partial^k J^0 \quad . \tag{V.10}$$

The equations for the Dirac field are changed very little from those developed in Chapter IV for quantum electrodynamics. The two components $\Psi_+ = \frac{1}{2}\gamma^3\gamma^0\Psi$, are independent dynamical variables. The two components $\Psi_- = \frac{1}{2}\gamma^0\gamma^3\Psi$ are dependent variables, to be determined by the constraint equation

$$\Psi_- = \frac{1}{2\eta}\gamma^0\left[-(i\partial_k - eA_k)\gamma^k + m\right]\Psi_+ \quad , \tag{V.11}$$

which follows from the Dirac equation. The equation of motion for Ψ_+ is

$$i\partial_0 \Psi_+ = eA^3\Psi_+ + \frac{1}{2}\left[(i\partial_k - eA_k)\gamma^k + m\right]\gamma^3\Psi_- \quad . \tag{V.12}$$

The only difference between this equation of motion and the corresponding equation in quantum electrodynamics is that A^3 depends on B through the constraint equation (V.7).

B. <u>Equal-τ Commutation Relations and Fourier Expansions of the Fields</u>

In order to make quantum fields out of the independent fields Ψ_+, $\underset{\sim}{A}$, B we must specify their commutation relations at equal τ. By analogy with Chpater IV we choose

$$
\begin{aligned}
\sqrt{2}\left\{\psi_+(x)_\alpha\,,\,\psi_+^\dagger(0)\right\}_{\tau=0} &= \delta_{\alpha\beta}\,\delta(z)\,\delta^2(\underset{\sim}{x}) \\
\left[A^i(x)\,,\,A^j(0)\right]_{\tau=0} &= -\tfrac{i}{4}\,\delta_{ij}\,\epsilon(z)\,\delta^2(\underset{\sim}{x}) \\
\left[B(x)\,,\,B(0)\right]_{\tau=0} &= -\tfrac{i}{4}\,\epsilon(z)\,\delta^2(\underset{\sim}{x}) \\
\left[\underset{\sim}{A}(x)\,,\,B(0)\right]_{\tau=0} &= \left[A(x)\,,\,\psi_+(0)\right]_{\tau=0} = \left[B(x)\,,\,\psi_+(0)\right]_{\tau=0} \\
&= \left\{\psi_+(x)\,,\,\psi_+(0)\right\}_{\tau=0} = 0 \ .
\end{aligned} \tag{V.13}
$$

Using these commutation relations we can derive the commutation relations among the creation and destruction operators appearing in the Fourier expansion of the fields. Furthermore, the transformation properties of the fields under space translations in the transverse and z-directions and under rotation in the (x^1, x^2)-plane determine the momentum and "infinite-momentum helicity" of the states created and destroyed by these operators. Since the calculation is elementary, we only state the results. Let $b^\dagger(\eta, \underset{\sim}{p}, \lambda)$, $[d^\dagger(\eta, \underset{\sim}{p}, \lambda)]$ be creation operators for electrons, [positrons] with momentum $(\eta, \underset{\sim}{p})$ and helicity $\lambda(\lambda = \pm\frac{1}{2})$. Let $a^\dagger(\eta, \underset{\sim}{p}, \lambda)$ be creation operators for mesons with momentum $(\eta, \underset{\sim}{p})$ and helicity $\lambda\,(\lambda = -1,\ 0,\ +1)$. These operators have covariant commutation relations

$$
\begin{aligned}
\left\{b(p,\lambda)\,,\,b^\dagger(p',\lambda')\right\}_+ &= \left\{d(p,\lambda)\,,\,d^\dagger(p',\lambda')\right\} \\
&= \delta_{\lambda\lambda'}\,(2\pi)^3\,2\eta\,\delta(\eta-\eta')\,\delta^2(\underset{\sim}{p}-\underset{\sim}{p}') \\
\left[a(p,\lambda), a^\dagger(p',\lambda')\right] &= \delta_{\lambda\lambda'}\,(2\pi)^3\,2\eta\,\delta(\eta-\eta')\,\delta^2(\underset{\sim}{p}-\underset{\sim}{p}') \ .
\end{aligned} \tag{V.14}
$$

The expansion of $\Psi_+(x)$ at $\tau = 0$ in terms of b(p, s) and $d^\dagger$(p, s) is

$$2^{1/4}\,\psi_+(x) = (2\pi)^{-3}\int d\underset{\sim}{p}\int_0^\infty \frac{d\eta}{2\eta} \sum_{\lambda=\pm 1/2} \times\left\{\sqrt{2\eta}\, w(s)\, e^{-ip\cdot x}\, b(p,\lambda) + \sqrt{2\eta}\, w(-s)\, e^{+ip\cdot x}\, d^\dagger(p,\lambda)\right\} \tag{V.15}$$

where the spinors w(λ) are

$$w(+\tfrac{1}{2}) = \begin{pmatrix}1\\0\\0\\0\end{pmatrix} \qquad w(-\tfrac{1}{2}) = \begin{pmatrix}0\\0\\0\\-1\end{pmatrix} \quad . \tag{V.16}$$

The expansion of $\underset{\sim}{A}(x)$ at $\tau = 0$ contains creation and destruction operators for mesons with helicity + 1 and -1; the expansion of B(x) at $\tau = 0$ contains creation and destruction operators for mesons with helicity zero:

$$\underset{\sim}{A}(x) = (2\pi)^{-3}\int d\underset{\sim}{p}\int_0^\infty \frac{d\eta}{2\eta} \sum_{\lambda=\pm 1} \times\left\{\underset{\sim}{\epsilon}(\lambda)\, e^{-ip\cdot x}\, a(p,\lambda) + \underset{\sim}{\epsilon}(\lambda)^*\, e^{+ip\cdot x}\, a^\dagger(p,\lambda)\right\}, \tag{V.17}$$

$$B(x) = (2\pi)^{-3}\int d\underset{\sim}{p}\int_0^\infty \frac{d\eta}{2\eta}\left\{-i\, e^{-ip\cdot x}\, a(p,0) + i\, e^{+ip\cdot x}\, a^\dagger(p,0)\right\} \tag{V.18}$$

The vectors $\underset{\sim}{\epsilon}(\lambda)$ appearing in (V.17) are

$$\underset{\sim}{\epsilon}(+1) = -\tfrac{1}{\sqrt{2}}(1, i) \qquad \underset{\sim}{\epsilon}(-1) = +\tfrac{1}{\sqrt{2}}(1, -i) \quad . \tag{V.19}$$

C. Hamiltonian

The invariance of the Lagrangian under τ - translations provides us, using Noether's theorem, with a conserved canonical Hamiltonian

$$H = \int d\underset{\sim}{x}\, dz \;\; \mathcal{H}(\tau, \underset{\sim}{x}, z) \tag{V.20}$$

where

$$\mathcal{H} = \bar{\psi} \frac{i}{2} \overleftrightarrow{\partial}_0 \gamma^0 \psi - (\partial_0 A_\alpha)(\partial_3 A^\alpha) + (\partial_0 B)(\partial_3 B) - \mathcal{L} \; . \tag{V.21}$$

The first three terms in (V.21) cancel the terms in the Lagrangian containing ∂_0, and we are left with

$$\begin{aligned} \mathcal{H} = & - \bar{\psi} \left[\left(\frac{i}{2} \partial_k - e A_k \right) \gamma^k - m \right] \psi \\ & - \bar{\psi} \frac{i}{2} \overleftrightarrow{\partial}_3 \gamma^3 \psi + e A^3 \bar{\psi} \gamma^0 \psi - \frac{1}{2} (\partial_3 A^3)(\partial_3 A^3) \\ & - (\partial_3 A^k)(\partial_k A^3) + \frac{1}{2} (\partial^k A^l)(\partial_k A_l) \\ & - \frac{1}{2} (\partial^k A^l)(\partial_l A_k) - \frac{1}{2} \kappa^2 A^k A_k - \frac{1}{2} (\partial^k B)(\partial_k B) \\ & + \kappa A^k (\partial_k B) + \kappa A^3 (\partial_3 B) \; . \end{aligned} \tag{V.22}$$

It is apparent that this form for the Hamiltonian is not very useful. However, if we substitute the expressions for A^3 and Ψ_- given by the constraint equations (V.7) and (V.11) into (V.22), then integrate the resulting expression to form H, and finally integrate by parts freely, we obtain a useful expression:

$$
\begin{aligned}
H = \int d\underset{\sim}{x}\, dz \Big\{ & \frac{e^2}{2} \sqrt{2}\, \psi_+^\dagger \psi_+ \frac{1}{\eta^2} \sqrt{2}\, \psi_+^\dagger \psi_+ \\
& + e \sqrt{2}\, \psi_+^\dagger \psi_+ \frac{1}{\eta} [\underset{\sim}{p} \cdot \underset{\sim}{A} - i\kappa B] \\
& + \sqrt{2}\, \psi_+^\dagger [m - (\underset{\sim}{p} - e\underset{\sim}{A}) \cdot \underset{\sim}{\gamma}] \frac{1}{2\eta} [m + (\underset{\sim}{p} - e\underset{\sim}{A}) \cdot \underset{\sim}{\gamma}] \psi_+ \\
& + \frac{1}{2} \sum_{k=1}^{2} A^k (\underset{\sim}{p}^2 + \kappa^2) A^k + \frac{1}{2} B (\underset{\sim}{p}^2 + \kappa^2) B \Big\} .
\end{aligned}
\tag{V.23}
$$

Here p is the differential operator $p^k = i\partial^k$ and $\underset{\sim}{\gamma} = (\gamma^1, \gamma^2)$. For the sake of variety, we have not made use of the fields ψ, γ, a^μ, ϕ used in Chapter IV, but instead have written H directly in terms of the independent fields Ψ_+, A, B.

By using the equal-τ commutation relations (V.13), one can verify that the canonical Hamilton (V.23) actually generates τ-translations in the theory. One finds, indeed, that $[iH, \underset{\sim}{A}] = \partial_0 \underset{\sim}{A}$, $[iH, b] = \partial_0 B$ and $[iH, \Psi_+] = \partial_0 \Psi_+$, where the τ-derivatives of $\underset{\sim}{A}$, B and Ψ_+ are given by the equations of motion (V.9), (V.10) and (V.12).

An examination of the Hamiltonian (V.23) shows that the theory is changed very little when the vector meson mass is changed from $\kappa = 0$ to $\kappa > 0$. One must, of course, introduce a helicity zero meson into the theory and adjust the free meson Hamiltonian from $\underset{\sim}{p}^2/2\eta$ to $(\underset{\sim}{p}^2 + \kappa^2)/2\eta$. But the interactions among the electrons and helicity $\pm$ 1 mesons are unchanged, and the helicity zero mesons interact with the electrons only through a very simple coupling $-ie\kappa\sqrt{2}\, \Psi_+^\dagger \Psi_+ (1/\eta) B$. As $\kappa \to 0$ this coupling vanishes — so that the helicity zero mesons are never produced.

We can illustrate the dynamics more vividly by writing out the rules for old-fashioned (τ- ordered) diagrams using the Hamiltonian (V.23). (These rules are written in a form which facilitates practical calculations: the matrix elements of the Hamiltonian between infinite-momentum helicity states have been evaluated explicitly so that one can sum directly over the helicities of particles in intermediate states rather than write strings of γ-matrices and perform a trace.) If κ is set equal to zero in these rules, they are equivalent to the rules for quantum electrodynamics given in Chapter IV in an alternate form.

(1) A factor $(H_f - H + i\epsilon)^{-1}$ for each intermediate state.

(2) An overall factor $-2\pi\delta(H_f - H_i)$.

(3) For each internal line, a sum over spins and an integration

$$(2\pi)^{-3} \int d\underset{\sim}{p} \int_0^\infty \frac{d\eta}{2\eta}$$

(4) For each vertex

(a) a factor $(2\pi)^3 \delta(\eta_{out} - \eta_{in}) \delta^2(\underset{\sim}{p}_{out} - \underset{\sim}{p}_{in})$,

(b) a factor $[2\eta]^{\frac{1}{2}}$ for each fermion line entering or leaving the vertex. (The factors $[2\eta]^{\frac{1}{2}}$ associated with each internal fermion line have the effect of removing the factor $1/2\eta$ from the phase space integral.)

(5) Finally, a simple matrix element is associated with each vertex as a factor. There are three types of vertices, as shown in Figure V-1. The corresponding factors are

(a) for single meson emission (Figure V-1a), a factor eM, where M is given by Table V-1.

(b) for instantaneous electron exchange as shown in Figure V-1b, a factor e^2/η_0 if all the particles are right handed or if all the particles are left handed (otherwise, a factor zero);

(c) for the "Coulomb force" vertex as shown in Figure V-1c, a factor $e^2(\eta_0)^{-2}\delta_{s_1 s_2}\delta_{s_3 s_4}$.

D. Free Fields

In this section and the next we will examine the question of whether the infinite-momentum formalism presented here is equivalent to the usual formalism for massive quantum electrodynamics developed in an ordinary reference frame. We begin with a short discussion of the free fields.

If the coupling constant e is zero, the equations of motion for the meson fields $\underset{\sim}{A}$ and B are simply

$$\begin{aligned} \left[\partial_\nu \partial^\nu + \kappa^2\right] \underset{\sim}{A}(x) &= 0 \\ \left[\partial_\nu \partial^\nu + \kappa^2\right] B(x) &= 0 \end{aligned} \tag{V.24}$$

These equations can be solved exactly, given initial conditions at $\tau = 0$. If (V.17) and (V.18) are the Fourier expansions of $\underset{\sim}{A}(x)$ and B(x) at time $\tau = 0$, then these same expansions will give $\underset{\sim}{A}(x)$ and B(x) for all τ if we put

$$p_0 = H(\eta, \underset{\sim}{p}) = (\underset{\sim}{p}^2 + \kappa^2)/2\eta$$

in the exponentials $\exp(\pm i p_\mu x^\mu)$ inside the integrals.

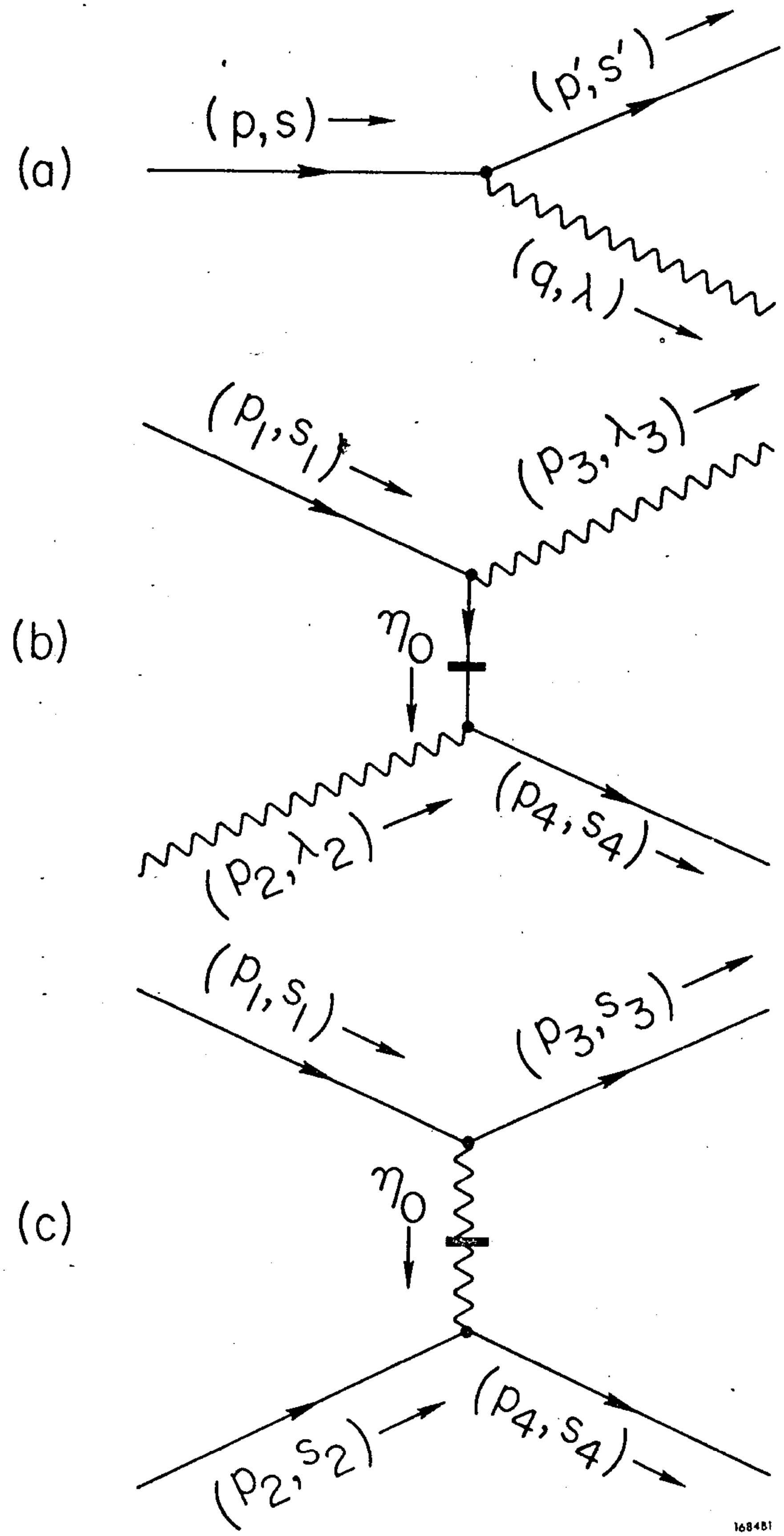

FIGURE V-1

Electron-Meson Vertices

TABLE V

Matrix Elements for Meson Emission

$$p_{\pm} = 2^{-\frac{1}{2}}(p^1 \pm ip^2)$$

S	S'	λ	M
$\frac{1}{2}$	$\frac{1}{2}$	1	$-q_-/\eta_q + p'_-/\eta'$
$\frac{1}{2}$	$\frac{1}{2}$	0	$-\kappa/\eta_q$
$\frac{1}{2}$	$\frac{1}{2}$	-1	$+q_+/\eta_q - p_+/\eta$
$\frac{1}{2}$	$-\frac{1}{2}$	1	$2^{-\frac{1}{2}} m\ \eta_q/\eta\eta'$
$\frac{1}{2}$	$-\frac{1}{2}$	0	0
$\frac{1}{2}$	$-\frac{1}{2}$	-1	0
$-\frac{1}{2}$	$\frac{1}{2}$	1	0
$-\frac{1}{2}$	$\frac{1}{2}$	0	0
$-\frac{1}{2}$	$\frac{1}{2}$	-1	$2^{-\frac{1}{2}} m\ \eta_q/\eta\eta'$
$-\frac{1}{2}$	$-\frac{1}{2}$	1	$-q_-/\eta_q + p_-/\eta$
$-\frac{1}{2}$	$-\frac{1}{2}$	0	$-\kappa/\eta_q$
$-\frac{1}{2}$	$-\frac{1}{2}$	-1	$+q_+/\eta_q - p'_+/\eta'$

With the solutions for $\underset{\sim}{A}(x)$ and B(x) in hand, we can write down $A^3(x)$ using the constraint equation (V.7). Finally, we recall that $A^0(x) = 0$. Thus we have the complete solution $(A^\mu(x), B(x))$ for the free vector meson field in the infinite-momentum gauge. We can use the gauge transformation (V.5) to transform this solution back to the more familiar Lorentz gauge. To do this, we let

$$\begin{aligned} A'_\mu(x) &= A_\mu(x) + \partial_\mu \Lambda(x) \\ B'(x) &= B(x) + \kappa \Lambda(x) \end{aligned} \tag{V.25}$$

be the fields in the new gauge, and require that $B'(x) = 0$. Then

$$A'_\mu(x) = A_\mu(x) - \kappa^{-1} \partial_\mu B(x) \quad . \tag{V.26}$$

(Note that this gauge transformation becomes singular in the limit $\kappa \to 0$.)

The free field $A'^\mu(x)$ which results from these operations can be written as

$$\begin{aligned} A'(x)^\mu = {} & (2\pi)^{-3} \int d\underset{\sim}{p} \int_0^\infty \frac{d\eta}{2\eta} \sum_{\lambda=-1}^{1} \\ & \times \left\{ e^\mu(p,\lambda)\, e^{-ip\cdot x}\, a(p,\lambda) + e^\mu(p,\lambda)^*\, e^{+ip\cdot x}\, a^\dagger(p,\lambda) \right\} \end{aligned} \tag{V.27}$$

where the polarization vectors $e^\mu(p, \lambda)$ are

$$\begin{aligned} e^\mu(p,+1) &= -2^{-1/2}\left(0, 1, i, [p^1 + ip^2]/\eta\right) \\ e^\mu(p,-1) &= +2^{-1/2}\left(0, 1, -i, [p^1 - ip^2]/\eta\right) \\ e^\mu(p,0) &= \kappa^{-1}\left(\eta, p^1, p^2, H - \kappa^2/\eta\right) \\ &= \kappa^{-1} p^\mu - \delta^\mu_3 (\kappa/\eta) \quad . \end{aligned} \tag{V.28}$$

This is exactly the form for the free vector meson field developed in Chapter III, Section F.

One can also show, just as in Chapter IV, that the free Dirac field obtained in the infinite-momentum frame is equal to the usual Dirac field. We will not comments on this proof here except to note that the gauge change discussed above does not affect the Dirac field if e = 0.

E. Scattering Theories Compared

We have seen that massive quantum electrodynamics in the infinite-momentum frame is the same as ordinary massive quantum electrodynamics in the trivial case e = 0. We cannot demonstrate that the two theories are the same for $e \neq 0$ since we are unable to solve for the exact interacting Heisenberg fields in either theory. However, it is possible to show that the perturbation expansions of the S matrix in the two theories are formally identical.

What we have to show is that the ordinary Feynman rules for massive quantum electrodynamics lead to the same expression for scattering amplitudes as the rules for old-fashioned diagrams given in Section C. Since the same demonstration has been given for quantum electrodynamics in Chapter IV, we will indicate here only how the argument can be modified to account for a non-zero meson mass and the contributions from helicity zero mesons.

To that end, we examine the Feynman propagator for massive vector mesons

$$D_F(x)^{\mu\nu} = (2\pi)^{-4}\int d^4p \; e^{-ip\cdot x}\,\frac{-g^{\mu\nu} + p^\mu p^\nu/\kappa^2}{p^2 - \kappa^2 + i\varepsilon} \quad . \qquad \text{(V.29)}$$

One can show (by simple computation if necessary) that

$$
\begin{aligned}
-g^{\mu\nu} + p^{\mu}p^{\nu}/\kappa^2 = {} & \sum e(p,\lambda)^{\mu}\, e(p,\lambda)^{*\nu} \\
& + \delta^{\mu}_{3}\delta^{\nu}_{3}\,\kappa^2/\eta \;+ \delta^{\mu}_{3}\delta^{\nu}_{3}\,(p^2-\kappa^2)/\eta^2 \\
& - (1/\eta)\,\delta^{\mu}_{3}\,p^{\nu} - (1/\eta)\,p^{\mu}\delta^{\nu}_{3} \;+ p^{\mu}p^{\nu}/\kappa^2
\end{aligned}
\tag{V.30}
$$

where the vectors $e(\eta,\underline{p},\lambda)$ are the polarization vectors for helicity ± 1 defined in Eq. (V.28). If one uses this expression in the numerator of the meson propagator, the last three terms will not contribute to any scattering process because of current conservation. Thus one is left with an effective propagator

$$
\begin{aligned}
D_F(x)^{\mu\nu} = {} & (2\pi)^{-4}\int d^4p\; e^{-ip\cdot x}\,(p^2-\kappa^2+i\varepsilon)^{-1} \\
& \times\Big[\sum_{\lambda=\pm1} e(p,\lambda)^{\mu}\, e^{*}(p,\lambda)^{\nu} + \delta^{\mu}_{3}\delta^{\nu}_{3}\,\kappa^2/\eta^2\Big] \\
& + \delta^{\mu}_{3}\delta^{\nu}_{3}\,(2\pi)^{-4}\int d^4p\; e^{-ip\cdot x}\,\frac{1}{\eta^2}\,\frac{p^2-\kappa^2}{p^2-\kappa^2+i\varepsilon} \;.
\end{aligned}
\tag{V.31}
$$

The H integral in the first term can be done by contour integration as in Chapter IV. In the second term, $(p^2-\kappa^2)(p^2-\kappa^2+i\epsilon)^{-1} \to 1$ as $\epsilon \to 0$ so that the H integral gives a factor $\delta(\tau)$. Thus the meson propagator takes the form

$$
\begin{aligned}
D_F(x)^{\mu\nu} = {} & -\frac{i}{(2\pi)^3}\int d\underline{p}\int_0^{\infty}\frac{d\eta}{2\eta}\Big[\sum_{\lambda=\pm1} e(p,\lambda)^{\mu}\, e(p,\lambda)^{*\nu} \\
& \qquad + \delta^{\mu}_{3}\delta^{\nu}_{3}\,\frac{\kappa^2}{\eta^2}\Big] \\
& \qquad \times\Big[\Theta(\tau)\,e^{-ip_\sigma x^\sigma} + \Theta(-\tau)\,e^{+ip_\sigma x^\sigma}\Big] \\
& + \frac{1}{(2\pi)^3}\,\delta(\tau)\,\delta^{\mu}_{3}\delta^{\nu}_{3}\int d\underline{p}\int_{-\infty}^{\infty}\frac{d\eta}{\eta^2}\,e^{-i(\eta\zeta-\underline{p}\cdot\underline{x})}
\end{aligned}
\tag{V.32}
$$

where

$$
p_0 = H(\eta,\underline{p}) = (\underline{p}^2+\kappa^2)/2\eta \;.
$$

Note that this expression for the vector meson propagator is nearly identical to the corresponding expression for the photon propagator derived in Chapter IV. In particular, the "Coulomb force" term proportional to $\delta(\tau)$ remains unchanged.

There are only two changes in $D_F^{\mu\nu}$, which account for the corresponding changes in the perturbation theory rules of Section C between $\kappa = 0$ and $\kappa > 0$. First, the free meson Hamiltonian is changed from $H = \underset{\sim}{p}^2/2\eta$ to $H = (\underset{\sim}{p}^2 + \kappa^2)/2\eta$. Second, a new term describing the propagation of helicity zero mesons is added to $D_F^{\mu\nu}$; namely

$$-i(2\pi)^{-3}\int d\underset{\sim}{p}\int_0^\infty \frac{d\eta}{2\eta}\; e_{eff}(p,0)^\mu\; e_{eff}(p,0)^{*\nu}$$
$$\times\left[\Theta(\tau)\, e^{-ip\cdot x} + \Theta(-\tau)\, e^{+ip\cdot x}\right]$$

where the "effective polarization vector" for helicity zero mesons is

$$e_{eff}(p,0)^\mu = -\frac{\kappa}{\eta}\,\delta^\mu_3 .$$

This is also the effective polarization vector for helicity zero mesons in the initial and final states, since $e(p,0)^\mu = \kappa^{-1}p^\mu - (\kappa/\eta)\delta^\mu_3$, and the term $\kappa^{-1}p^\mu$ does not contribute to scattering amplitudes because of current conservation.

From here on, one can continue the argument just as in Chapter IV to show that the covariant Feynman rules are equivalent to the rules for old-fashioned perturbation theory in the infinite-momentum frame given in Section C.

References — Chapter V

1. J. M. Cornwall and R. Jackiw, UCLA Preprint, 1971; with Addendum, MIT Preprint, 1971.

2. D. A. Dicus, R. Jackiw, and V. L. Toplitz, MIT Preprint, 1971.

3. D. J. Gross and S. B. Treiman, Princeton Preprint, 1971.

4. I am indebted to R. Jackiw for pointing this out.

5. E. C. G. Stückelberg, Helv. Phys. Acta 11, 299 (1938).

6. The observant reader may notice that in Chapter IV, Eq. (V.7) was written as $A^3 = (-i/\eta)\left[\partial_k A^k + \kappa B + (1/\eta) J^o\right]$ and arguments were given for preferring this form. In this paper, in contrast to Chapter IV, we will not try to make such nice distinctions, nor will we worry about possible surface terms arising from integrations by parts.

CHAPTER VI

Some Other Field Theories

Now that we have gained some familiarity with two model field theories in the infinite-momentum frame, it is easy to apply the same methods to other model theories. In this final chapter we will outline four such theories: scalar mesons with ϕ^N self-coupling; neutral pions coupled to protons with a γ_5 coupling; neutral scalar mesons coupled to protons with a $\mathbb{1}$ coupling; and electrodynamics of a spin zero boson. Each of these theories has the attractive feature that it is simpler than quantum electrodynamics. Since no new difficulties arise, we will be content with a very brief discription of each theory.

A. Neutral Scalar Mesons with ϕ^N Self-Coupling

This is the theory first discussed by Weinberg in 1966 using the $P \to \infty$ frame.[1] We begin with the Lagrangian

$$\mathcal{L}(x) = \frac{1}{2}\left\{(\partial_\mu \varphi)(\partial^\mu \varphi) - \kappa^2 \varphi^2\right\} - g\,\varphi^N \quad , \tag{VI.1}$$

which leads to the equation of motion

$$\left[\partial_\mu \partial^\mu + \kappa^2\right]\varphi(x) = -N g\left[\varphi(x)\right]^{N-1} \quad . \tag{VI.2}$$

The canonical momentum

$$P_\alpha = \int d\underline{x}\, dz\, (\partial_3 \varphi)(\partial_\alpha \varphi) \qquad \alpha = 1, 2, 3 \quad , \tag{VI.3}$$

will generate translations in the $\underset{\sim}{x}$- or η-directions if we choose the commutation relations

$$[\varphi(x), \varphi(0)]_{\tau=0} = -\frac{i}{4} \epsilon(\eta) \delta^2(\underset{\sim}{x}) . \tag{VI.4}$$

The Fourier expansion of the field at $\tau = 0$ is

$$\varphi(x) = (2\pi)^{-3} \int d\underset{\sim}{p} \int_0^\infty \frac{d\eta}{2\eta} \left\{ e^{-ip\cdot x} a(p) + e^{+ip\cdot x} a^\dagger(p) \right\} . \tag{VI.5}$$

where the destruction operators a(p) obey the usual commutation relations, $[a(\hat{p}), a(p)^\dagger] = (2\pi)^3 2\eta\, \delta(\hat{\eta} - \eta) \delta^2(\underset{\sim}{\hat{p}} - \underset{\sim}{p})$. The canonical Hamiltonian is

$$\mathcal{H}(x) = \frac{1}{2}\left\{ -(\partial_k \varphi)(\partial^k \varphi) + \kappa^2 \varphi^2 \right\} + g\, \varphi^N . \tag{VI.6}$$

If one uses this Hamiltonian and the expansion (VI.5) of the field, one can obtain the rules for old-fashioned perturbation theory. These are the same as the general rules (1), (2), (3), (4) given in Chapter V-C, together with a new rule for the simple matrix element to be associated with each vertex. In this case, there is only one kind of vertex — an N-meson vertex with an associated simple matrix element g.

These are exactly the rules obtained by Weinberg by starting with time-ordered diagrams and boosting each time-ordered diagram to infinite momentum.

B. Pions and Nucleons with γ_5 Coupling

We use a Dirac field $\psi(x)$ and a real pseudoscalar field $\phi(x)$. The Lagrangian is

$$\mathcal{L}(x) = \bar{\psi}\left[\tfrac{i}{2}\overleftrightarrow{\partial}_\mu \gamma^\mu - m\right]\psi + \tfrac{1}{2}\left[(\partial_\mu\varphi)(\partial^\mu\varphi) - \kappa^2\varphi^2\right] - ig\varphi\bar{\psi}\gamma_5\psi \quad . \tag{VI.7}$$

Thus the equations of motion are

$$[i\partial_\mu\gamma^\mu - m]\psi = ig\varphi\gamma_5\psi \tag{VI.8}$$

$$[\partial_\mu\partial^\mu + \kappa^2]\varphi = -ig\bar{\psi}\gamma_5\psi \quad . \tag{VI.9}$$

Just as in quantum electrodynamics, we find that two components of the Dirac field are dependent variables. Indeed, we find when we multiply (VI.8) by γ^0 that

$$\psi_- = \frac{1}{2\eta}\gamma^0\left[\underset{\sim}{p}\cdot\underset{\sim}{\gamma} + m + ig\varphi\gamma_5\right]\psi_+ \quad . \tag{VI.10}$$

(Note that the projection matrices $P_\pm$ commute with γ_5.)

The canonical momentum

$$P_\alpha = \int d\underset{\sim}{x}\, dz \left\{\sqrt{2}\,\psi_+^\dagger \tfrac{i}{2}\overleftrightarrow{\partial}_\alpha \psi_+ + (\partial_3\varphi)(\partial_\alpha\varphi)\right\}$$

$\alpha = 1, 2, 3$

will generate translations in the $\underset{\sim}{x}$- and y-directions if we choose the commutation relations

$$
\begin{aligned}
\left[\varphi(x),\, \varphi(0)\right]_{\tau=0} &= -\frac{i}{4}\,\epsilon(z)\;\delta^2(\underset{\sim}{x}) \\
\sqrt{2}\left\{\psi_+(x)_\alpha,\, \psi_+^\dagger(x)_\beta\right\}_{\tau=0} &= \delta_{\alpha\beta}\;\delta(z)\;\delta^2(\underset{\sim}{x})\,.
\end{aligned}
\tag{VI.11}
$$

Thus we can use the Fourier expansions (VI.5) and (V.15) of the independent fields $\phi(x)$ and $\psi_+(x)$.

The canonical Hamiltonian is (after some integrations by parts)

$$
\begin{aligned}
H = \int d\underset{\sim}{x}\, dz \Big\{ & \frac{1}{2}\varphi\left[\underset{\sim}{\nabla}^2 + \kappa^2\right]\varphi \\
& + \sqrt{2}\,\psi_+^\dagger\left[-\underset{\sim}{\nabla}\cdot\underset{\sim}{\gamma} + m - ig\varphi\gamma_5\right]\frac{1}{2\eta} \\
& \qquad \left[\underset{\sim}{\nabla}\cdot\underset{\sim}{\gamma} + m + ig\varphi\gamma_5\right]\psi_+\,.
\end{aligned}
\tag{VI.12}
$$

Apparently, the old-fashioned perturbation theory rules derived from this Hamiltonian will have two types of vertices as shown in Figure VI-1. The "simple matrix elements" to be associated with these vertices can be worked out by explicit calculation:

— for single pion emission as in Figure VI-1a, a factor $(i2^{-\frac{1}{2}}gM)$, where M is given in Table VI-1.

— for instantaneous proton exchange, as shown in Figure VI-1b, a factor $\delta_{s_4 s_3} g^2/2\eta_0$.

The rest of the rules are just the general rules given in Chapter V-C.

(a)

(b)

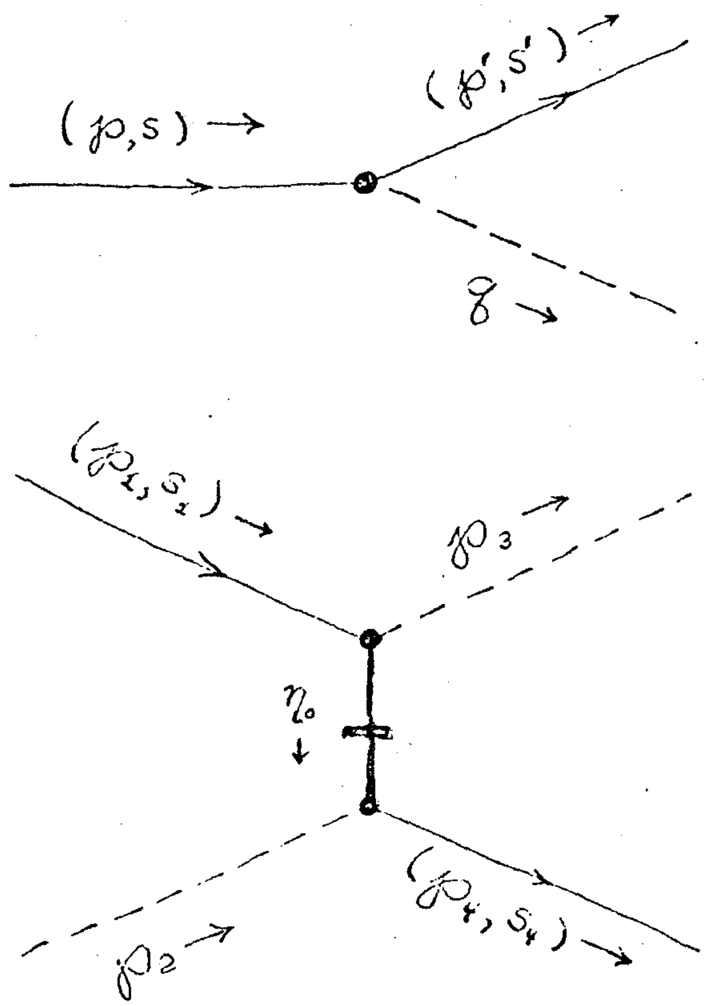

FIGURE VI-1

Pion-Nucleon Vertices

TABLE VI-1

Matrix Elements for Pion Emission
with γ_5 Coupling
$p_\pm = 2^{-\frac{1}{2}}(p^1 \pm ip^2)$

S	S'	M
$\frac{1}{2}$	$\frac{1}{2}$	$2^{-\frac{1}{2}} m\eta_q/\eta\eta'$
$\frac{1}{2}$	$-\frac{1}{2}$	$p_+/\eta - p'_+/\eta'$
$-\frac{1}{2}$	$\frac{1}{2}$	$p_-/\eta - p'_-/\eta'$
$-\frac{1}{2}$	$-\frac{1}{2}$	$-2^{-\frac{1}{2}} m\eta_q/\eta\eta'$

C. Scalar Mesons and Nucleons with $\mathbb{1}$ Coupling

We use a Dirac field $\psi(x)$ and a real scalar field $\phi(x)$ with a Lagrangian

$$\mathcal{L}(x) = \bar{\psi}[\tfrac{i}{2}\overleftrightarrow{\partial}_\mu \gamma^\mu - m]\psi + \tfrac{1}{2}[(\partial_\mu\varphi)(\partial^\mu\varphi) - \kappa^2\varphi^2] - g\varphi\bar{\psi}\psi . \qquad \text{(VI.13)}$$

The equations of motion are unchanged from Section B except for the substitution $i\gamma_5 \rightarrow \mathbb{1}$. The Hamiltonian is

$$H = \int d\underset{\sim}{x}\, dz \Big\{ \tfrac{1}{2}\varphi(\underset{\sim}{p}^2 + \kappa^2)\varphi + \sqrt{2}\,\psi_+^\dagger[-\underset{\sim}{p}\cdot\underset{\sim}{\gamma} + m + g\varphi]\tfrac{1}{2\eta}[\underset{\sim}{p}\cdot\underset{\sim}{\gamma} + m + g\varphi]\psi_+ \Big\}. \qquad \text{(VI.14)}$$

Again, the old-fashioned perturbation theory rules derived from this Hamiltonian will have the two types of vertices shown in Figure VI-1. The corresponding simple matrix elements are

- for single meson emission as in Figure VI-1a, a factor $(2^{-\frac{1}{2}}gM)$, where M is given in Table VI-2.
- for instantaneous proton exchange as in Figure VI-1b, a factor $(\delta_{s_4 s_3} g^2/2\eta_0)$.

D. Electrodynamics of a Scalar Meson[2]

We use a charged scalar meson field $\phi(x)$ and an electromagnetic potential $A^\mu(x)$. The Lagrangian is

TABLE VI-2

Matrix Elements for Scalar Meson
Emission with 1 Coupling

$$p_{\pm} = 2^{-\frac{1}{2}}(p^1 \pm ip^2)$$

S	S'	M
$\frac{1}{2}$	$\frac{1}{2}$	$2^{\frac{1}{2}} m(\eta+\eta')/\eta\eta'$
$\frac{1}{2}$	$-\frac{1}{2}$	$p_+/\eta - p'_+/\eta'$
$-\frac{1}{2}$	$\frac{1}{2}$	$p_-/\eta - p'_-/\eta'$
$-\frac{1}{2}$	$-\frac{1}{2}$	$2^{\frac{1}{2}} m(\eta+\eta')/\eta\eta'$

$$\mathcal{L}(x) = -\tfrac{1}{4}(\partial^\nu A^\mu - \partial^\mu A^\nu)(\partial_\nu A_\mu - \partial_\mu A_\nu) + \left([i\partial_\mu - eA_\mu]\varphi\right)^\dagger\left([i\partial_\mu - eA_\mu]\varphi\right) - \kappa^2 \varphi^\dagger\varphi \tag{VI.15}$$

This Lagrangian leads to the equations of motion

$$(i\partial_\mu - eA_\mu)(i\partial^\mu - eA^\mu)\varphi = \kappa^2\varphi \tag{VI.16}$$

$$\partial_\nu(\partial^\nu A^\mu - \partial^\mu A^\nu) = 2e\,\varphi^\dagger[\tfrac{i}{2}\overleftrightarrow{\partial}^\mu - eA^\mu]\varphi\,. \tag{VI.17}$$

As in spinor electrodynamics, it is convenient to choose the infinite-momentum gauge, $A^0 = 0$. In this gauge, the $\mu = 0$ component of (VI.17) is

$$-\partial_3(\partial_3 A^3 + \partial_k A^k) = 2e\,\varphi^\dagger\,\tfrac{i}{2}\overleftrightarrow{\partial}_3\,\varphi\,.$$

This equation can be solved at any "time" τ to give A^3 in terms of the independent fields $\underset{\sim}{A}$, ϕ at that τ:

$$A^3 = \frac{1}{\eta^2}\left\{\eta(\underset{\sim}{k}\cdot\underset{\sim}{A}) + e(\varphi^\dagger\overleftrightarrow{\eta}\varphi)\right\}, \tag{VI.18}$$

where $p^k = i\partial^k$, $\eta = i\partial_3$, $\overleftrightarrow{\eta} = i\overrightarrow{\partial}_3 - i\overleftarrow{\partial}_3$, and $1/\eta^2$ is the familiar integral operator with kernel $-\frac{1}{2}\,|\,\mathfrak{z} - \mathfrak{z}'\,|$.

We calculate the canonical momentum operators P_α ($\alpha = 1, 2, 3$) and find that they are simple in the infinite-momentum gauge:

$$P_\alpha = \int d\underset{\sim}{x} \int dz \left\{ (\partial_3 \varphi^\dagger)(\partial_\alpha \varphi) + (\partial_\alpha \varphi^\dagger)(\partial_3 \varphi) - (\partial_3 A^k)(\partial_\alpha A_k) \right\} .$$

These momentum operators will apparently generate space displacements of the fields if we choose the commutation relations

$$\begin{aligned} \left[A^k(x), A^l(0) \right]_{\tau=0} &= -\delta_{kl} \frac{i}{4} \epsilon(z) \delta^2(\underset{\sim}{x}) \\ \left[\varphi(x), \varphi^\dagger(0) \right]_{\tau=0} &= -\frac{i}{4} \epsilon(z) \delta^2(\underset{\sim}{x}) \\ \left[\varphi(x), \varphi(0) \right]_{\tau=0} &= \left[\varphi(x), A^k(0) \right]_{\tau=0} = 0 . \end{aligned} \tag{VI.19}$$

The corresponding Fourier expansions of the fields at $\tau = 0$ are

$$\begin{aligned} \underset{\sim}{A}(x) &= (2\pi)^{-3} \int d\underset{\sim}{p} \int_0^\infty \frac{d\eta}{2\eta} \sum_{\lambda=\pm 1} \left\{ \underset{\sim}{e}(\lambda)\, e^{-ip\cdot x}\, a(p,\lambda) + \underset{\sim}{e}(\lambda)^* e^{+ip\cdot x} a^\dagger(p,\lambda) \right\} \\ \varphi(x) &= (2\pi)^{-3} \int d\underset{\sim}{p} \int_0^\infty \frac{d\eta}{2\eta} \left\{ e^{-ip\cdot x}\, b(p) + e^{+ip\cdot x}\, d^\dagger(p) \right\} , \end{aligned} \tag{VI.20}$$

where a, b, d destroy photons, mesons, anti-mesons respectively with the usual normalization $[b(\hat{p}), b^\dagger(p)] = (2\pi) 2\eta\, \delta(\eta - \hat{\eta}) \delta^2(\underset{\sim}{p} - \hat{\underset{\sim}{p}})$ etc., and the polarization vectors $\underset{\sim}{e}(\lambda)$ are, as before, $\underset{\sim}{e}(\pm 1) = \mp 2^{-\frac{1}{2}} (1, \pm i)$.

When the canonical Hamiltonian is written in terms of the independent fields and several integrations by parts are performed, we get

$$
\begin{aligned}
H = \int d\underset{\sim}{x}\, dz \Big\{ & \frac{1}{2} \sum_{k=1}^{2} A^k \underset{\sim}{p}^2 A^k + \kappa^2 \varphi^\dagger \varphi \\
& + \frac{1}{2} e^2 (\varphi^\dagger \overset{\leftrightarrow}{\eta} \varphi) \frac{1}{\eta^2} (\varphi^\dagger \overset{\leftrightarrow}{\eta} \varphi) \\
& + e (\varphi^\dagger \overset{\leftrightarrow}{\eta} \varphi) \frac{1}{\eta} (\underset{\sim}{p} \cdot \underset{\sim}{A}) \\
& + \varphi^\dagger (\underset{\sim}{p} - e\underset{\sim}{A}) \cdot (\underset{\sim}{p} - e\underset{\sim}{A}) \varphi \Big\}
\end{aligned}
\tag{VI.21}
$$

The matrix elements of H can be obtained by using the field expansions (VI.20). Apparently, there are three kinds of vertices, as shown in Figure VI-2. The corresponding simple matrix elements are:

— for single photon emission, as shown in Figure VI-2a, a factor

$$2e \frac{\eta\eta'}{\eta_g} \left(p_+/\eta - p_+'/\eta' \right) \qquad \text{for } \lambda = +1$$

$$2e \frac{\eta\eta'}{\eta_g} \left(-p_-/\eta + p_-'/\eta' \right) \qquad \text{for } \lambda = -1$$

— for photon absorption and emission, as shown in Figure VI-2b, a factor

$2e^2 \delta_{\lambda_2 \lambda_4}$.

— for a "Coulomb" interactions as shown in Figure VI-2c, a factor

$e^2 (\eta_1 + \eta_3) \frac{1}{\eta_0^2} (\eta_2 + \eta_4)$.

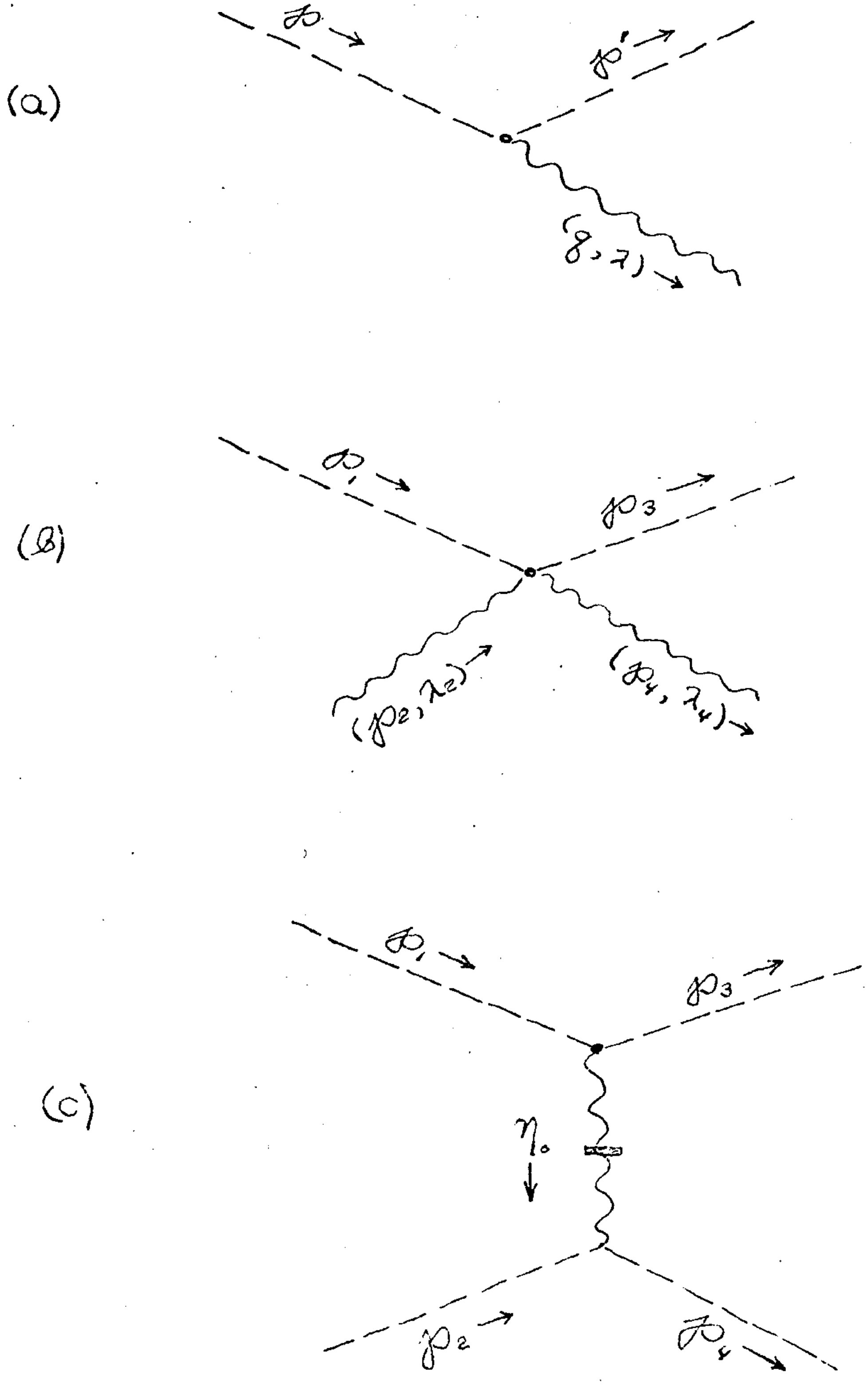

FIGURE VI-2

Vertices for Scalar Electrodynamics

References — Chapter VI

1. S. Weinberg, Phys. Rev. 150, 1313 (1966).

2. Cf. R. A. Neville and R. Rohrlich, Phys. Rev. D3, 1692 (1971).