DE GRUYTER OPEN Open Phys. 2015; 13:247–262

Research Article Open Access

Mamdouh I. Wanas, Samah N. Osman, and Reham I. El-Kholy

Unification Principle and a Geometric Field Theory

DOI 10.1515/phys-2015-0030
Received December 19, 2014; accepted May 22, 2015

Abstract: In the context of the geometrization philosophy, a covariant field theory is constructed. The theory satisfies the unification principle. The field equations of the theory are constructed depending on a general differential identity in the geometry used. The Lagrangian scalar used in the formalism is neither curvature scalar nor torsion scalar, but an alloy made of both, the W-scalar. The physical contents of the theory are explored depending on different methods. The analysis shows that the theory is capable of dealing with gravity, electromagnetism and material distribution with possible mutual interactions. The theory is shown to cover the domain of general relativity under certain conditions.

Keywords: Anti-Curvature; Unified Field Theory; Poisson Equations; Absolute Parallelism; Schwarzschild Solution

PACS: 02.40.Hw; 04.20.Cv; 11.10.Ef; 12.10.Dm; 04.20.Jb

1 Introduction

It is well known that General Relativity (GR) is the best theory for gravity, so far. However, it suffers from many problems. Some of these problems are old e.g. singularity, horizons, flatness, ...etc. Other problems concern contradiction between observations and GR predictions e.g. rotation curves of stars in spiral galaxies, the accelerating expansion of the Universe, ...etc .

Many authors have tried to modify GR or to write new theories for gravity. This is done in order to account for problems such as those mentioned above. In order to diagnose problems of GR, let us first summarize the main features of this theory in the following points.

1. The theory is constructed in the context of the geometrization philosophy.
2. The theory assumes the validity of two principles: general covariance and equivalence.
3. GR is constructed in context of a 4-dimensional Riemannian geometry. It contains two sets of equations,
   (a) field equations which may be written as:
   $$R_{\mu\nu} - \frac{1}{2} g_{\mu\nu} R = -\kappa \overset{\star}{T}_{\mu\nu}, \tag{1.1}$$
   where $R_{\mu\nu}$, R are Ricci tensor and scalar respectively, $\overset{\star}{T}_{\mu\nu}$ is the material-energy tensor, κ is a conversion constant and $g_{\mu\nu}$ is the metric tensor.
   (b) The second set comprises the geodesic and null-geodesic equations which are used as equations of motion of the theory.
4. Conservation of matter-energy is guaranteed via using Bianchi identity and the field equations (1.1).

The above points are written for the sake of analysis and later comparison with the present work.

The field equations of GR, equation (1.1), can be considered to be composed of two parts: the left hand side (L.H.S) is pure geometric (i.e. constructed from the building blocks[1] of the geometry used, $g_{\mu\nu}$); and the right hand side (R.H.S) is defined from outside the geometry. In deriving the field equations (1.1), an action principle is imposed on a Lagrangian function, the scalar curvature R, producing the L.H.S. of (1.1).

As stated above, there are many attempts to modify GR in order to avoid some of the problems mentioned above. Such modifications can be classified in the following three classes:

1. Modifications of the L.H.S. of (1.1) (geometric class) by adding a cosmological constant term (cf. [6]) to treat the problem of particle horizon or the accelerating expansion of the Universe [40]. This type of modification is carried out in the context of Riemannian geometry. It is successful to some extent since the cosmological constant has its own problems (cf. [26]).

Mamdouh I. Wanas, Samah N. Osman, Reham I. El-Kholy: Department of Astronomy, Faculty of Science, Cairo University, Giza, Egypt, Egyptian Relativity Group (ERG): URL: http://www.erg.eg.net, E-mail: miwanas@sci.cu.edu.eg, samah@sci.cu.edu.eg, relkholy@sci.cu.edu.eg

1 The building blocks (BB) of any geometry are geometric objects using which one can construct the whole geometry. The BB for Riemannian geometry are the components of the metric tensor, $g_{\mu\nu}$.

Another attempt, belonging to this class, is carried out by replacing the scalar curvature R by a general function $f(R)$ (cf. [10]). This type of attempt is also carried out in the context of Riemannian geometry. Despite its partial success, it has the problem that the resulting differential equations are of fourth order (cf. [43]) which is not easy to solve and has unstable solutions [38].

A third type of attempt, belonging to this class, is carried out in the context of a wider geometry with a non-vanishing torsion scalar T (e.g. the AP-geometry). In the AP-geometry, a theory is constructed using a torsion scalar T, defined by:

$$T \stackrel{\text{def}}{=} \gamma^{\alpha\mu\nu}\gamma_{\nu\mu\alpha} - c^{\nu}c_{\nu}, \tag{1.2}$$

where $\gamma^{\alpha}_{.\mu\nu}$ is the contortion tensor and c_ν is the basic vector of the AP-space, is known as the Teleparallel Equivalent of GR (TEGR)(cf. [1, 9]). This is because, one can easily show that, apart from a divergence term, one can write:

$$T \equiv R. \tag{1.3}$$

The type of gravity theories in which $f(T)$ is used, in place of $f(R)$, is known in the literature as $f(T)$-theories (cf. [3, 15, 16, 31, 32]) . An advantage of using $f(T)$ gravity theories is that the resulting differential equations are of second order (cf. [33–36]). But it has been shown that some theoretical problems appear in such type of theories (cf. [24, 37]).

2. Modifications of the R.H.S. of (1.1) (physical class). This class is carried out with or without modifications given in the 1st class, considered above. It is carried out mainly to solve the problem of the SN type Ia observations. It depends mainly on using an equation of state which allows for matter with negative pressure, e.g. phantom energy (cf. [7]), quintessence energy (cf. [8]) and Chaplygin gas (cf. [12, 23]). Although this class has some success, such materials do not exist on the Earth or in the Solar system.
3. Using other scenarios together with orthodox GR. For example the inflation scenario is used to remove particle horizons from FRW-cosmology [17, 25]. Also, MOND is used to remove the problem of rotation curves of spiral galaxies (cf. [5, 42]).

Now, our point of view is that, on one hand, contradictions between the predictions of any gravity theory and experiments or observation may imply new physics. On the other hand, modifications of GR-theory using $f(R)$ or $f(T)$ may imply quantitative improvement of GR and not qualitative improvement. This may not lead to prediction of new physics, if any.

An important note, concerning the above mentioned suggestions, is that such modifications ($f(R)$, $f(T)$) comprise the modification of the geometric part of (1.1), its L.H.S. part, without touching its material distribution part, the R.H.S of (1.1). This may raise the following objections:

1. The material distribution outside the Solar system may be different from that in our laboratories, and consequently the phenomenological $\overset{\star}{T}_{\mu\nu}$ of (1.1) may not be appropriate to describe such material distribution.
2. The link between the geometry used (e.g. L.H.S. of (1.1)) and the material distribution given by $\overset{\star}{T}_{\mu\nu}$ is artificial since the geometry used may not allow for the material distribution given by $\overset{\star}{T}_{\mu\nu}$.
3. A formal difficulty arises as follows. The use of an action principle produces field equations that are equal in number to the unknown functions i.e. the BB of the geometry used. If $\overset{\star}{T}_{\mu\nu}$ is defined from outside the geometry, it will add more unknown functions to the same set of differential equations. In general, this set cannot be solved without adding more conditions, e.g. equations of state, from outside the geometry. This violates the philosophy of geometrization.

We suggest a field theory in which $\overset{\star}{T}_{\mu\nu}$ is defined from the BB of the geometry used. This may remove the above mentioned objections and makes the theory self-consistent. The theory obtained in this case will be a pure geometric theory which would be more consistent with the geometrization philosophy. This may be achieved by a theory which satisfies the Unification principle. This principle may be stated as follows:

Unification Principle

In any geometric field theory, *"all physical quantities and fields are to be induced from one geometric entity, the building blocks of the geometry used"*. This principle has been inspired by Einstein's statement [14]: *"A theory in which the gravitational field and the electromagnetic field do not enter as logically distinct structures, would be much preferable"*.

Satisfying this principle would imply the following:

1. The use of a geometry more wide than the Riemannian one.

2. The Lagrangian density, used to construct the field equations of the theory, is to be composed of the BB of the geometry and their derivatives. In other words, this Lagrangian should not be composed, a priori, from different parts each corresponds to a certain field, as usually done.

In the domain of theoretical physics, quantization and geometrization represent the two main important philosophical ideas of the 20th century. They have been frequently used, successfully, to solve many problems, so far. These two philosophies are not in complete agreement with each other. The main problem between them is that geometrization depends on continuity (of space-time) while quantization depends on discreteness. Many authors thought that in order to quantize geometry one should first quantize space-time [2], i.e. define minimum time and minimum length, which is a very difficult task. Our point of view is different; that is, quantization of gravity starts from trajectories of elementary particles (curves or paths of the geometry used). Fortunately, it has been shown [46, 47] that any non-symmetric geometry, including absolute parallelism, has such curves as will appear in Section 2.

The aim of the present work is to construct a pure geometric field theory, in which fields and material distributions are described using only the BB of the geometry used; in other words, a theory satisfying the Unification Principle. Also we aim to explore whether the suggested theory has a correct GR-limit. The article is arranged as follows. In Section 2, we review briefly the bases of the Absolute Parallelism (AP) geometry that will be used in building the new theory. In Section 3, we apply a variational method to the chosen Lagrangian and get the field equations of the suggested theory. In Section 4, we analyse the field equations into their symmetric and skew parts and compare what we get with other non-linear field theories, e.g. GR. In Section 5, we linearize the field equations and demonstrate that it meets the limits of well-known linear field theories. In Section 6, we show different field types that can be described using the new theory and how to differentiate between them. In Section 7, we apply the new theory to a tetrad vector field having spherical symmetry and show that the results reduce to that of the Schwarzchild exterior solution (under certain conditions). Finally, in Section 8, we point out some important remarks about the suggested theory that are worthy of noting.

2 A Brief Review of AP-Geometry

In this Section we give a brief review of AP-space, (for more details cf. [48], [50], [51] and [52]). The AP-space $(M, \underset{i}{\lambda})$ is an n-dimensional differentiable manifold, each point of which is labelled by n-independent variables $x^\mu (\mu = 0, 1, 2, \dots)$, and at each point we define n-linearly independent contravariant vector fields $\underset{i}{\lambda}^\mu$ ($\mu(= 0, 1, 2, \dots)$ stands for the coordinate components and $i(= 0, 1, 2, \dots)$ stands for the vector numbers). Throughout this paper, we use Latin indices for vector numbers and Greek indices for the coordinate components. As a consequence of this independence the determinant $\|\underset{i}{\lambda}^\mu\| \neq 0$, then we can define the covariant vector fields $\underset{i}{\lambda}_\mu$ which are the normalized cofactors of $\underset{i}{\lambda}^\mu$, in the matrix $(\underset{i}{\lambda}^\mu)$ such that:

$$\underset{i}{\lambda}^\mu \underset{i}{\lambda}_\nu = \delta^\mu_\nu, \tag{2.1}$$

and,

$$\underset{i}{\lambda}^\nu \underset{j}{\lambda}_\nu = \delta_{ij}. \tag{2.2}$$

Using these vectors, one can define the second order tensor:

$$g_{\mu\nu} \overset{\text{def}}{=} \underset{i}{\lambda}_\mu \underset{i}{\lambda}_\nu. \tag{2.3}$$

Since the tensor $g_{\mu\nu}$ is a covariant second order symmetric tensor and the matrix $(g_{\mu\nu})$ is non-degenerate, then $g_{\mu\nu}$ can be used to play the role of the metric of a Riemannian space associated with the AP-space. Also, we can define its contravariant tensor by the relation:

$$g^{\mu\nu} \overset{\text{def}}{=} \underset{i}{\lambda}^\mu \underset{i}{\lambda}^\nu. \tag{2.4}$$

Imposing a metricity condition on $g_{\mu\nu}$, we can construct Christoffel symbols $\{^{\alpha}_{\mu\,\nu}\}$, as usual:

$$\{^{\alpha}_{\mu\,\nu}\} \overset{\text{def}}{=} \frac{1}{2} g^{\alpha\sigma} (g_{\mu\sigma,\nu} + g_{\sigma\nu,\mu} - g_{\mu\nu,\sigma}). \tag{2.5}$$

The AP-space admits a non-symmetric affine connection $\Gamma^\alpha_{\mu\nu}$, which is defined as:

$$\Gamma^\alpha_{.\,\mu\nu} \overset{\text{def}}{=} \underset{i}{\lambda}^\alpha \underset{i}{\lambda}_{\mu,\nu}. \tag{2.6}$$

Using the above connection, one can easily show that:

$$\underset{i}{\lambda}_{\underset{+}{\mu}|\nu} = \underset{i}{\lambda}_{\mu,\nu} - \Gamma^\alpha_{.\,\mu\nu} \underset{i}{\lambda}_\alpha \equiv 0, \tag{2.7}$$

where the stroke and the (+) sign denote tensor differentiation, using (2.6). Some authors call (2.7) the AP-condition (cf. [27]).

Also, one can define a third order skew tensor:

$$\Lambda^\alpha_{.\,\mu\nu} \overset{\text{def}}{=} \Gamma^\alpha_{.\,\mu\nu} - \Gamma^\alpha_{.\,\nu\mu}. \tag{2.8}$$

This tensor is the torsion tensor of AP-space. One can also define another third order tensor by:

$$\gamma^{\alpha}_{.\ \mu\nu} \stackrel{\text{def}}{=} \underset{i}{\lambda}^{\alpha} \underset{i}{\lambda}_{\mu;\nu}, \tag{2.9}$$

where the semicolon (;) is used for covariant differentiation using (2.5). This tensor is called the contortion of the space, using which, one can write the relation (cf. [27]):

$$\gamma^{\alpha}_{.\ \mu\nu} = \Gamma^{\alpha}_{.\ \mu\nu} - \left\{ {}^{\ \alpha}_{\mu\ \nu} \right\}. \tag{2.10}$$

The contortion is related to the torsion tensor by the relation [18]:

$$\gamma_{\alpha\mu\nu} = \frac{1}{2}\left(\Lambda_{\alpha\mu\nu} - \Lambda_{\mu\alpha\nu} - \Lambda_{\nu\alpha\mu}\right). \tag{2.11}$$

From (2.10) we can write:

$$\Lambda^{\alpha}_{.\ \mu\nu} = \gamma^{\alpha}_{.\ \mu\nu} - \gamma^{\alpha}_{.\ \nu\mu}. \tag{2.12}$$

It can be shown that $\gamma_{\alpha\mu\nu}$ is skew in its first two indices. A basic vector could be obtained by contraction, using any of the above third order tensors:

$$C_{\mu} \stackrel{\text{def}}{=} \Lambda^{\alpha}_{.\ \mu\alpha} = \gamma^{\alpha}_{.\ \mu\alpha}. \tag{2.13}$$

Using the contortion, we can define the following symmetric third order tensor:

$$\Delta^{\alpha}_{.\ \mu\nu} \stackrel{\text{def}}{=} \gamma^{\alpha}_{.\ \mu\nu} + \gamma^{\alpha}_{.\ \nu\mu}. \tag{2.14}$$

In AP-space we can define at least four linear connections and consequently four types of tensor derivatives as follows:

$$\begin{aligned} A^{\mu}_{\ +|\nu} &\stackrel{\text{def}}{=} A^{\mu}_{\ ,\nu} + \Gamma^{\mu}_{\alpha\nu} A^{\alpha}, \\ A^{\mu}_{\ |\nu} &\stackrel{\text{def}}{=} A^{\mu}_{\ ,\nu} + \Gamma^{\mu}_{(\alpha\nu)} A^{\alpha}, \\ A^{\mu}_{\ -|\nu} &\stackrel{\text{def}}{=} A^{\mu}_{\ ,\nu} + \tilde{\Gamma}^{\mu}_{\alpha\nu} A^{\alpha}, \\ A^{\mu}_{\ ;\nu} &\stackrel{\text{def}}{=} A^{\mu}_{\ ,\nu} + \left\{ {}^{\ \mu}_{\alpha\ \nu} \right\} A^{\alpha}, \end{aligned} \tag{2.15}$$

where A^{μ} is an arbitrary contravariant vector and $\Gamma^{\mu}_{(\alpha\nu)}$, $\tilde{\Gamma}^{\mu}_{\alpha\nu}$ are the symmetric part and the dual of the connection $\Gamma^{\mu}_{.\ \alpha\nu}$ given, respectively, by:

$$\Gamma^{\mu}_{.\ (\nu\alpha)} \stackrel{\text{def}}{=} \frac{1}{2}\left(\Gamma^{\mu}_{.\ \nu\alpha} + \Gamma^{\mu}_{.\ \alpha\nu}\right), \tag{2.16}$$

$$\tilde{\Gamma}^{\mu}_{\alpha\nu} \stackrel{\text{def}}{=} \Gamma^{\mu}_{.\ \nu\alpha}. \tag{2.17}$$

In Riemannian space the only second order tensors, defined using the BB of the space, are Ricci tensor $R_{\mu\nu}$ and the metric tensor $g_{\mu\nu}$. In AP-space there are more second order tensors as shown in Table 1. It can be easily shown that there exists an identity between skew-tensors, given in Table (1), which can be written in the from:

$$\eta_{\mu\nu} + \varepsilon_{\mu\nu} - \chi_{\mu\nu} \equiv 0. \tag{2.18}$$

Table 1: Second Order World Tensors [27]

Skew-symmetric Tensors	Symmetric Tensors
$\xi_{\mu\nu} \stackrel{\text{def}}{=} \gamma_{\mu\nu.\ \underset{+}{\mid\alpha}}^{\ \ \alpha}$	
$\zeta_{\mu\nu} \stackrel{\text{def}}{=} C_{\alpha}\gamma_{\mu\nu.}^{\ \ \alpha}$	
$\eta_{\mu\nu} \stackrel{\text{def}}{=} C_{\alpha}\Lambda^{\alpha}_{.\ \mu\nu}$	$\phi_{\mu\nu} \stackrel{\text{def}}{=} C_{\alpha}\Delta^{\alpha}_{.\ \mu\nu}$
$\chi_{\mu\nu} \stackrel{\text{def}}{=} \Lambda^{\alpha}_{.\ \mu\nu\underset{+}{\mid\alpha}}$	$\psi_{\mu\nu} \stackrel{\text{def}}{=} \Delta^{\alpha}_{.\ \mu\nu\underset{+}{\mid\alpha}}$
$\varepsilon_{\mu\nu} \stackrel{\text{def}}{=} C_{\mu\underset{+}{\mid\nu}} - C_{\nu\underset{+}{\mid\mu}}$	$\theta_{\mu\nu} \stackrel{\text{def}}{=} C_{\mu\underset{+}{\mid\nu}} + C_{\nu\underset{+}{\mid\mu}}$
$\kappa_{\mu\nu} \stackrel{\text{def}}{=} \gamma^{\alpha}_{.\mu\epsilon}\gamma^{\epsilon}_{.\alpha\nu} - \gamma^{\alpha}_{.\nu\epsilon}\gamma^{\epsilon}_{.\alpha\mu}$	$\tilde{\omega}_{\mu\nu} \stackrel{\text{def}}{=} \gamma^{\alpha}_{.\mu\epsilon}\gamma^{\epsilon}_{.\alpha\nu} + \gamma^{\alpha}_{.\nu\epsilon}\gamma^{\epsilon}_{.\alpha\mu}$
	$\omega_{\mu\nu} \stackrel{\text{def}}{=} \gamma^{\epsilon}_{.\mu\alpha}\gamma^{\alpha}_{.\nu\epsilon}$
	$\sigma_{\mu\nu} \stackrel{\text{def}}{=} \gamma^{\epsilon}_{.\alpha\mu}\gamma^{\alpha}_{.\epsilon\nu}$
	$\alpha_{\mu\nu} \stackrel{\text{def}}{=} C_{\mu}C_{\nu}$

where $\Lambda^{\alpha}_{.\mu\nu\underset{+}{\mid\sigma}} \equiv \Lambda^{\alpha}_{\underset{++}{+\mu\nu}\ \mid\sigma}$.

We see from this table that the torsion tensor plays an important role in the structure of AP-space, since all tensors in this table vanish when the torsion tensor vanishes.
It has been found that the Ricci tensor $R_{\mu\nu}$ can be written, using Table (1), in the following equivalent form (cf. [27]):

$$R_{\mu\nu} \stackrel{\text{def}}{=} \frac{1}{2}\left(\psi_{\mu\nu} - \phi_{\mu\nu} - \theta_{\mu\nu}\right) + \omega_{\mu\nu}. \tag{2.19}$$

In the AP-space, as stated above, there are at least four linear connections $\Gamma^{\alpha}_{\ \mu\nu}$, $\Gamma^{\alpha}_{\ (\mu\nu)}$, $\tilde{\Gamma}^{\alpha}_{\ \mu\nu}$ and $\left\{ {}^{\ \alpha}_{\mu\ \nu} \right\}$. For each connection there is a curvature defined by [48]:

$$\begin{aligned} B^{\epsilon}_{.\ \mu\nu\sigma} &\stackrel{\text{def}}{=} \Gamma^{\epsilon}_{.\ \mu\sigma,\nu} - \Gamma^{\epsilon}_{.\ \mu\nu,\sigma} + \Gamma^{\alpha}_{.\ \mu\sigma}\Gamma^{\epsilon}_{.\ \alpha\nu} - \Gamma^{\alpha}_{.\ \mu\nu}\Gamma^{\epsilon}_{.\ \alpha\sigma}, \\ \bar{B}^{\epsilon}_{.\ \mu\nu\sigma} &\stackrel{\text{def}}{=} \Gamma^{\epsilon}_{.(\mu\sigma),\nu} - \Gamma^{\epsilon}_{.(\mu\nu),\sigma} + \Gamma^{\alpha}_{.(\mu\sigma)}\Gamma^{\epsilon}_{.(\alpha\nu)} - \Gamma^{\alpha}_{.(\mu\nu)}\Gamma^{\epsilon}_{.(\alpha\sigma)}, \\ \tilde{B}^{\epsilon}_{.\ \mu\nu\sigma} &\stackrel{\text{def}}{=} \tilde{\Gamma}^{\epsilon}_{.\ \mu\sigma,\nu} - \tilde{\Gamma}^{\epsilon}_{.\ \mu\nu,\sigma} + \tilde{\Gamma}^{\alpha}_{.\ \mu\sigma}\tilde{\Gamma}^{\epsilon}_{.\ \alpha\nu} - \tilde{\Gamma}^{\alpha}_{.\ \mu\nu}\tilde{\Gamma}^{\epsilon}_{.\ \alpha\sigma}, \\ R^{\epsilon}_{.\ \mu\nu\sigma} &\stackrel{\text{def}}{=} \left\{ {}^{\ \epsilon}_{\mu\ \sigma} \right\}_{,\nu} - \left\{ {}^{\ \epsilon}_{\mu\ \nu} \right\}_{,\sigma} + \left\{ {}^{\ \alpha}_{\mu\ \sigma} \right\}\left\{ {}^{\ \epsilon}_{\alpha\ \nu} \right\} - \left\{ {}^{\ \alpha}_{\mu\ \nu} \right\}\left\{ {}^{\ \epsilon}_{\alpha\ \sigma} \right\}, \end{aligned} \tag{2.20}$$

Due to the AP-condition (2.7), it can be shown that one of the curvatures given by (2.20) vanishes identically, i.e.

$$B^{\epsilon}_{.\ \mu\nu\sigma} \equiv 0. \tag{2.21}$$

It is to be considered that, due to (2.21), many authors have considered the AP-space as a flat one. In [44] the AP-space is treated as a non flat one due to the non-vanishing of other curvature tensors as shown above.

A new tensor of type (2,2) has been suggested:

$$\underset{i}{\lambda}_{\alpha}\, W^{\alpha\beta}_{\cdot\,\cdot\,\gamma\delta} \stackrel{\text{def}}{=} \underset{i}{\lambda}^{\beta}{}_{\|\gamma\delta} - \underset{i}{\lambda}^{\beta}{}_{\|\delta\gamma}\,, \tag{2.22}$$

where the double stroke represents tensor differentiation using a certain linear connection. If one uses the Levi Civita linear connection in this definition, then $W^{\alpha\beta}_{\cdot\,\cdot\,\gamma\delta}$ is the Riemann-Christoffel curvature tensor. But if we use a different linear connection the resulting object is not the conventional curvature. For this reason the W-tensor has been first called the non-conventional curvature tensor [28, 44]. Afterwards, it is shown that this tensor is neither curvature nor torsion, it is a geometric alloy made of the two tensors and cannot be defined except in the AP-space. So it has been given the name W-tensor [50]. It has been used in constructing the Generalized Field Theory (GFT) using the dual connection ($\tilde{\Gamma}^{\alpha}_{\mu\nu}$). In the case of GFT the W-tensor is defined by [28, 44]:

$$\underset{i}{\lambda}_{\alpha}\, W^{\alpha\mu}_{\cdot\,\cdot\,\nu\sigma} = \underset{i}{\lambda}^{\mu}{}_{\underset{-}{|}\nu\sigma} - \underset{i}{\lambda}^{\mu}{}_{\underset{-}{|}\sigma\nu}. \tag{2.23}$$

It has been found that $W^{\alpha\mu}_{\cdot\,\cdot\,\nu\sigma}$ can be written explicitly in the from:

$$W^{\alpha\mu}_{\cdot\,\cdot\,\nu\sigma} \stackrel{\text{def}}{=} \underset{i}{\lambda}^{\alpha}\left(\underset{i}{\lambda}^{\mu}{}_{\underset{-}{|}\nu\sigma} - \underset{i}{\lambda}^{\mu}{}_{\underset{-}{|}\sigma\nu}\right). \tag{2.24}$$

The path equations corresponding to the four linear connections, mentioned above, have been found to be [46]:

$$\begin{aligned} \frac{dV^{\mu}}{dS^{+}} + \left\{{}^{\mu}_{\alpha\beta}\right\} V^{\alpha} V^{\beta} &= -\Lambda_{(\alpha\beta)\,.}^{\;\;\;\;\;\;\mu}\, V^{\alpha} V^{\beta}, \\ \frac{dW^{\mu}}{dS^{o}} + \left\{{}^{\mu}_{\alpha\beta}\right\} W^{\alpha} W^{\beta} &= -\frac{1}{2}\Lambda_{(\alpha\beta)\,.}^{\;\;\;\;\;\;\mu}\, W^{\alpha} W^{\beta}, \\ \frac{dJ^{\mu}}{dS^{-}} + \left\{{}^{\mu}_{\alpha\beta}\right\} J^{\alpha} J^{\beta} &= 0, \end{aligned} \tag{2.25}$$

where S^{+}, S^{o} and S^{-} are parameters varying along the curves with tangents V^{α}, W^{α} and J^{α}, respectively, and $\Lambda_{(\alpha\beta)\,.}^{\;\;\;\;\;\;\mu}$ is given by:

$$\Lambda_{(\alpha\beta)\,.}^{\;\;\;\;\;\;\mu} \stackrel{\text{def}}{=} \frac{1}{2}\left(\Lambda_{\alpha\beta\,.}^{\;\;\;\;\mu} + \Lambda_{\beta\alpha\,.}^{\;\;\;\;\mu}\right). \tag{2.26}$$

It is to be noted that the coefficient of the torsion term jumps by a step of one half from an equation to the next in the set (2.25). Note also that the set (2.25) contains only three equations, not four as expected. We consider this property as reflecting some type of natural quantization in the geometry used.

3 The Lagrangian Function and Field Equations

When constructing his theory of GR, Einstein has not used an action principle to derive the equations of the theory. He has depended mainly on a differential identity in Riemannian geometry, Bianchi 2[nd] identity, to write these equations. He considered this identity as a geometric representation of conservation. Afterwards, Hilbert has suggested the use of an action principle to reproduce the field equations of GR. In general, the two procedures, identity and least action, seem to be equivalent. However, Einstein has refrained from using the action principle method in deriving the field equations of other field theories, especially when using different geometric structures (cf. [14]).

We prefer to use the identity method since it is more consistent with the geometrization philosophy. In this case one has to look for appropriate differential identities, in the geometry used, or to look for a method for deriving such identities. Fortunately, Dolan and McCrea have suggested such method for Riemannian geometry [13]. This method has been modified [44] to be appropriate for the AP-geometry and has been used to construct a field theory, the GFT [28]. In this modified (Dolan-McCrea) method, the differential identity derived in the AP-space is of the general form:

$$E^{\mu}_{\cdot\,\nu\underset{-}{|}\mu} \equiv 0\,, \tag{3.1}$$

where

$$\lambda\, E^{\mu}_{\nu} \stackrel{\text{def}}{=} \frac{\delta\mathcal{L}}{\delta\underset{i}{\lambda}_{\mu}}\,\underset{i}{\lambda}_{\nu}\,, \tag{3.2}$$

where $\mathcal{L}$ is the Lagrangian density, $\frac{\delta\mathcal{L}}{\delta\underset{i}{\lambda}_{\mu}}$ is the Hamiltonian derivative of $\mathcal{L}$, and λ is the determinant of $\underset{i}{\lambda}_{\mu}$.

The identity (3.1) can be considered as a general geometric representation of conservation of quantities defined by $E^{\mu}_{\cdot\,\nu}$. Consequently, the field equations of the theory can be written in the form:

$$E_{\mu\nu} = 0\,. \tag{3.3}$$

The above procedure is very similar to that used by Einstein in deriving the field equations of GR for the first time.

In the present work we are interested in examining the consequence of using the symmetric linear connection (2.16), in place of the dual one, in the definition of the W-tensor (2.24).

Now we define the W-tensor (2.24) using the symmetric linear connection (2.16) as:

$$\bar{W}^{\alpha\mu}_{\cdot\cdot\,\nu\sigma} \stackrel{\text{def}}{=} \underset{i}{\lambda}^{\alpha}\left(\underset{i}{\lambda}^{\mu}{}_{\overset{o}{|}\nu\sigma} - \underset{i}{\lambda}^{\mu}{}_{\overset{o}{|}\sigma\nu}\right), \tag{3.4}$$

contracting by setting $\alpha = \sigma$ we get:

$$\bar{W}^{\mu}_{\cdot\,\nu} \stackrel{\text{def}}{=} \underset{i}{\lambda}^{\alpha} \left(\underset{i}{\lambda}^{\overset{\mu}{\circ}}{}_{|\nu\alpha} - \underset{i}{\lambda}^{\overset{\mu}{\circ}}{}_{|\alpha\nu} \right) , \tag{3.5}$$

contracting again by setting $\mu = \nu$ we get:

$$\bar{W} = \bar{W}^{\mu}_{\cdot\,\mu} \stackrel{\text{def}}{=} \underset{i}{\lambda}^{\alpha} \left(\underset{i}{\lambda}^{\overset{\mu}{\circ}}{}_{|\mu\alpha} - \underset{i}{\lambda}^{\overset{\mu}{\circ}}{}_{|\alpha\mu} \right) . \tag{3.6}$$

Using one of the definitions of the set (2.15) and substituting from (2.8), (2.10), (2.12) and (2.13), then (3.6) will give, after necessary reductions:

$$\bar{W} = \frac{3}{2}c^{\alpha}_{;\alpha} - \frac{1}{2}c^{\alpha}c_{\alpha} + \frac{1}{4}\gamma^{\alpha\mu\nu}\Lambda_{\mu\alpha\nu} - \frac{1}{2}\gamma^{\alpha\mu\nu}\gamma_{\mu\nu\alpha}. \tag{3.7}$$

So, one can define a Lagrangian density as:

$$\mathcal{L} \stackrel{\text{def}}{=} \lambda\bar{W}, \qquad \left(\lambda \stackrel{\text{def}}{=} \|\underset{i}{\lambda}_{\mu}\| \right), \tag{3.8}$$

i.e. using (3.7):

$$\mathcal{L} = \lambda \left[\frac{3}{2}c^{\alpha}_{;\alpha} - \frac{1}{2}c^{\alpha}c_{\alpha} + \frac{1}{4}\gamma^{\alpha\mu\nu}\Lambda_{\mu\alpha\nu} - \frac{1}{2}\gamma^{\alpha\mu\nu}\gamma_{\mu\nu\alpha} \right]. \tag{3.9}$$

As it is well known, the first term of expression (3.9), $c^{\alpha}_{;\alpha}$, gives no contribution to variation. So, (3.9) can be written as:

$$\mathcal{L} = \lambda \left[-\frac{1}{2}c^{\alpha}c_{\alpha} + \frac{1}{4}\gamma^{\alpha\mu\nu}\Lambda_{\mu\alpha\nu} - \frac{1}{2}\gamma^{\alpha\mu\nu}\gamma_{\mu\nu\alpha} \right]. \tag{3.10}$$

In general, the Lagrangian is assumed to be:

$$L \equiv L \left(\underset{i}{\lambda}_{\mu}, \underset{i}{\lambda}_{\mu,\nu}, \underset{i}{\lambda}_{\mu,\nu\sigma} \right) .$$

In this case, using (3.2), we may define a second order tensor $S^{\beta}_{\cdot\,\sigma}$ as:

$$\lambda\, S^{\beta}_{\cdot\,\sigma} \stackrel{\text{def}}{=} \frac{\delta\mathcal{L}}{\delta\underset{j}{\lambda}_{\beta}} \underset{j}{\lambda}_{\sigma}, \tag{3.11}$$

where,

$$\frac{\delta\mathcal{L}}{\delta\underset{j}{\lambda}_{\beta}} \stackrel{\text{def}}{=} \frac{\partial\mathcal{L}}{\partial\underset{j}{\lambda}_{\beta}} - \frac{\partial}{\partial x^{\nu}}\frac{\partial\mathcal{L}}{\partial\underset{j}{\lambda}_{\beta,\nu}} + \frac{\partial^2}{\partial x^{\mu}\partial x^{\nu}}\frac{\partial\mathcal{L}}{\partial\underset{j}{\lambda}_{\beta,\mu\nu}} , \tag{3.12}$$

is the variational derivative of the function $\mathcal{L}$. It is clear that $S^{\beta}_{\cdot\,\sigma}$ represents the components of a tensor of the character indicated by their indices. The identity (3.1) is a generalization of the Bianchi contracted identity in the AP-space. It is to be considered that (3.1) is a general identity independent of the explicit choice of the Lagrangian. It is just sufficient to assume that the Lagrangian is of the general form $L = L(\lambda_{\alpha}, \lambda_{\alpha,\beta}, \lambda_{\alpha,\beta\gamma})$. If we consider (3.1) as a general geometric representation of some type of conservation in the theory suggested, then we can take:

$$S^{\beta}_{\cdot\,\sigma} \stackrel{\text{def}}{=} \frac{1}{\lambda}\frac{\delta L}{\delta\underset{j}{\lambda}_{\beta}} \underset{j}{\lambda}_{\sigma} = 0, \tag{3.13}$$

as the field equations of this theory, as stated above.

Now from the Lagrangian function (3.10), we have:

$$L = -\frac{1}{2}c^{\alpha}c_{\alpha} + \frac{1}{4}\gamma^{\alpha\mu\nu}\Lambda_{\mu\alpha\nu} - \frac{1}{2}\gamma^{\alpha\mu\nu}\gamma_{\mu\nu\alpha} . \tag{3.14}$$

Using (3.10), we get:

$$\begin{aligned} \frac{\partial\mathcal{L}}{\partial\underset{j}{\lambda}_{\beta}} &= \frac{\partial(\lambda R)}{\partial\underset{j}{\lambda}_{\beta}} + \lambda\underset{j}{\lambda}^{\beta}\left(\frac{1}{2}c^{\mu}c_{\mu} + \frac{1}{4}\Lambda^{\alpha\mu\nu}(\Lambda_{\nu\alpha\mu} - \Lambda_{\alpha\mu\nu}) \right) \\ &+ \frac{1}{4}\lambda\underset{j}{\lambda}^{\epsilon}\left(4c^{\mu}\Lambda^{\beta}_{\cdot\,\epsilon\mu} - 4c^{\beta}c_{\epsilon} + 2\Lambda_{\alpha\epsilon\gamma}\left(2\Lambda^{\alpha\beta\gamma} - \Lambda^{\gamma\alpha\beta} + \Lambda^{\beta\alpha\gamma} \right) \right) , \end{aligned} \tag{3.15}$$

where R is the Ricci scalar constructed from Christoffel symbols, and:

$$\begin{aligned} \frac{\partial}{\partial x^{\gamma}}\frac{\partial\mathcal{L}}{\partial\underset{j}{\lambda}_{\beta,\gamma}} &= \frac{\partial}{\partial x^{\gamma}}\frac{\partial(\lambda R)}{\partial\underset{j}{\lambda}_{\beta,\gamma}} \\ &+ \lambda\Gamma^{\epsilon}_{\cdot\,\epsilon\gamma}\left(\underset{j}{\lambda}^{\gamma}c^{\beta} - \underset{j}{\lambda}^{\beta}c^{\gamma} + \underset{j}{\lambda}_{\alpha}\left(\frac{1}{2}\Lambda^{\gamma\alpha\beta} - \Lambda^{\alpha\beta\gamma} - \frac{1}{2}\Lambda^{\beta\alpha\gamma} \right) \right) \\ &+ \lambda\left(\underset{j}{\lambda}^{\epsilon}\left(c^{\beta}_{\cdot\,,\epsilon} - c^{\beta}\Gamma^{\gamma}_{\cdot\,\epsilon\gamma} + c^{\gamma}\Gamma^{\beta}_{\cdot\,\epsilon\gamma} \right) - \underset{j}{\lambda}^{\beta}c^{\gamma}_{\cdot\,,\gamma} \right) \\ &+ \lambda\underset{j}{\lambda}_{\epsilon}\Gamma^{\epsilon}_{\cdot\,\alpha\gamma}\left(\frac{1}{2}\Lambda^{\gamma\alpha\beta} - \Lambda^{\alpha\beta\gamma} - \frac{1}{2}\Lambda^{\beta\alpha\gamma} \right) \\ &+ \lambda\underset{j}{\lambda}_{\alpha}\left(\frac{1}{2}\Lambda^{\gamma\alpha\beta}_{\dots,\gamma} - \Lambda^{\alpha\beta\gamma}_{\dots,\gamma} - \frac{1}{2}\Lambda^{\beta\alpha\gamma}_{\dots,\gamma} \right) . \end{aligned} \tag{3.16}$$

It is clear that the last term of (3.12) vanishes identically, since the Lagrangian chosen is a function of $\underset{i}{\lambda}_{\beta}$ and its first derivatives only.

Substituting from (3.15) and (3.16) into (3.12), we get after some manipulation:

$$\begin{aligned} S^{\beta}_{\cdot\,\sigma} =& G^{\beta}_{\cdot\,\sigma} + \delta^{\beta}_{\cdot\,\sigma}\left(\frac{1}{4}\gamma^{\alpha\mu\nu}(\gamma_{\mu\nu\alpha} - \gamma_{\alpha\mu\nu}) - \frac{1}{2}c^{\mu}c_{\mu} + c^{\overset{\gamma}{+}}_{\cdot\,|\gamma} \right) \\ &+ \frac{1}{2}\gamma_{\alpha\mu\sigma}\gamma^{\beta\alpha\mu} + \frac{1}{2}\gamma_{\alpha\mu\sigma}\gamma^{\alpha\mu\beta} + \frac{1}{2}\gamma_{\sigma\alpha\mu}\gamma^{\alpha\mu\beta} + \frac{3}{2}\gamma_{\sigma\mu\alpha}\gamma^{\beta\alpha\mu} \\ &- \frac{1}{2}c^{\mu}\Lambda^{\beta}_{\cdot\,\sigma\mu} - \Lambda^{\cdot\,\beta\gamma}_{\sigma}c_{\gamma} + \frac{1}{2}\Lambda^{\gamma\cdot\,\beta}_{\cdot\,\sigma}c_{\gamma} - c^{\beta}_{\cdot\,|\underset{+}{\sigma}} + \Lambda^{\cdot\,\beta\mu}_{\sigma\,\cdot\cdot\,|\underset{+}{\mu}} \\ &+ \frac{1}{2}\Lambda^{\beta\cdot\,\mu}_{\cdot\,\sigma\cdot\,|\underset{+}{\mu}} - \frac{1}{2}\Lambda^{\mu\cdot\,\beta}_{\cdot\,\sigma\cdot\,|\underset{+}{\mu}}, \end{aligned} \tag{3.17}$$

where

$$G^{\beta}_{\cdot\,\sigma} \stackrel{\text{def}}{=} R^{\beta}_{\cdot\,\sigma} - \frac{1}{2}\delta^{\beta}_{\sigma}R. \tag{3.18}$$

The above expression (3.17) can be written in terms of the second order tensors defined in Table 1 as:

$$\begin{aligned} S^{\beta}_{\cdot\,\sigma} =& G^{\beta}_{\cdot\,\sigma} + \delta^{\beta}_{\sigma}\bar{A} - \frac{1}{2}\left[\theta^{\beta}_{\cdot\,\sigma} + \sigma^{\beta}_{\cdot\,\sigma} - \tilde{\omega}^{\beta}_{\cdot\,\sigma} - 3\left(\omega^{\beta}_{\cdot\,\sigma} \right.\right. \\ &\left.\left. + \frac{1}{2}\psi^{\beta}_{\cdot\,\sigma} - \frac{1}{2}\phi^{\beta}_{\cdot\,\sigma} \right) \right] + \frac{1}{2}\left(\zeta^{\beta}_{\cdot\,\sigma} - \xi^{\beta}_{\cdot\,\sigma} - \frac{1}{2}\varepsilon^{\beta}_{\cdot\,\sigma} \right) , \end{aligned} \tag{3.19}$$

where

$$\bar{A} \stackrel{\text{def}}{=} \frac{1}{2}\left(\theta - \alpha + \frac{1}{2}\sigma - \frac{1}{2}\omega \right) . \tag{3.20}$$

By using equation (3.13) and noting, from (2.3) and (2.7), that:

$$g_{\mu\nu|\underset{+}{\sigma}} = 0, \tag{3.21}$$

we may write the following set of field equations, for the suggested theory, in the form:

$$\begin{aligned} S_{\mu\nu} &= G_{\mu\nu} + g_{\mu\nu}\bar{A} - \frac{1}{2}\Big[\theta_{\mu\nu} + \sigma_{\mu\nu} - \tilde{\omega}_{\mu\nu} \\ &\quad -3\left(\omega_{\mu\nu} + \frac{1}{2}\psi_{\mu\nu} - \frac{1}{2}\phi_{\mu\nu}\right)\Big] + \frac{1}{2}\left(\zeta_{\mu\nu} - \xi_{\mu\nu} - \frac{1}{2}\varepsilon_{\mu\nu}\right) \\ &= 0\,. \end{aligned} \tag{3.22}$$

4 Comparison with Non-linear Field Theories

In order to gain some physical information about the geometric constituents of the theory, we are going to compare the field equations (3.22) with the corresponding field equations of non-linear field theories. To do so, we first split the suggested field equations (3.22) into their symmetric and skew parts.

The Symmetric Part of $S_{\mu\nu}$

The symmetric part of $S_{\mu\nu}$ is defined, as usual, by:

$$S_{(\mu\nu)} \overset{\text{def}}{=} \frac{1}{2}\left(S_{\mu\nu} + S_{\nu\mu}\right). \tag{4.1}$$

Evaluating this definition, using (3.22), one gets after some manipulations:

$$\begin{aligned} S_{(\mu\nu)} &= G_{\mu\nu} - \frac{1}{2}g_{\mu\nu}\left(\alpha - \theta + \frac{1}{2}\omega - \frac{1}{2}\sigma\right) \\ &- \frac{1}{2}\left[\theta_{\mu\nu} + \sigma_{\mu\nu} - \tilde{\omega}_{\mu\nu} - 3\left(\omega_{\mu\nu} + \frac{1}{2}\psi_{\mu\nu} - \frac{1}{2}\phi_{\mu\nu}\right)\right]. \end{aligned} \tag{4.2}$$

From which we write the symmetric field equations ($S_{(\mu\nu)} = 0$) as:

$$\begin{aligned} G_{\mu\nu} &= \frac{1}{2}g_{\mu\nu}\left(\alpha - \theta + \frac{1}{2}\omega - \frac{1}{2}\sigma\right) \\ &+ \frac{1}{2}\left[\theta_{\mu\nu} + \sigma_{\mu\nu} - \tilde{\omega}_{\mu\nu} - 3\left(\omega_{\mu\nu} + \frac{1}{2}\psi_{\mu\nu} - \frac{1}{2}\phi_{\mu\nu}\right)\right]. \end{aligned} \tag{4.3}$$

or, in a more compact form, it can be written as:

$$R_{\mu\nu} - \frac{1}{2}g_{\mu\nu}R = T_{\mu\nu}, \tag{4.4}$$

where

$$T_{\mu\nu} \overset{\text{def}}{=} \frac{1}{2}\left(g_{\mu\nu}\psi + \theta_{\mu\nu} - \frac{3}{2}\psi_{\mu\nu}\right) + B_{\mu\nu}, \tag{4.5}$$

and

$$\begin{aligned} B_{\mu\nu} &\overset{\text{def}}{=} \frac{1}{2}\Big[g_{\mu\nu}\left(\alpha + \frac{1}{2}\omega - \frac{1}{2}\sigma\right) + \sigma_{\mu\nu} - \tilde{\omega}_{\mu\nu} \\ &\quad -3\left(\omega_{\mu\nu} - \frac{1}{2}\phi_{\mu\nu}\right)\Big]. \end{aligned} \tag{4.6}$$

The significance of the tensor $B_{\mu\nu}$ will be shown later in §6.

Equation (4.4) is the symmetric field equations of the suggested theory. They have, apparently, the same form of Einstein field equations of GR. Since the L.H.S of (4.4) satisfies the contracted Bianchi identity, then we can write:

$$T^{\mu\nu}{}_{;\nu} = 0, \tag{4.7}$$

which gives the conservation of the physical quantities represented by the tensor $T_{\mu\nu}$. Consequently, $T_{\mu\nu}$ can be used to represent the distribution of matter and energy. We call this tensor the *"geometric material-energy tensor"*. Also, the comparison between (4.4) and the field equations of GR shows that the tensor $g_{\mu\nu}$, defined by (2.3), plays the role of the gravitational potential. These results will gain more justification in §5.

The Skew Part of $S_{\mu\nu}$

The skew part, of the tensor $S_{\mu\nu}$ given by (3.22), can be written in the form:

$$S_{[\mu\nu]} \overset{\text{def}}{=} \frac{1}{2}\left(S_{\mu\nu} - S_{\nu\mu}\right), \tag{4.8}$$

which can be explicitly written as:

$$S_{[\mu\nu]} = \frac{1}{2}\left(\zeta_{\mu\nu} - \xi_{\mu\nu} - \frac{1}{2}\varepsilon_{\mu\nu}\right). \tag{4.9}$$

But, we have from Table 1:

$$\varepsilon_{\mu\nu} \overset{\text{def}}{=} c_{\mu|\underset{+}{\nu}} - c_{\nu|\underset{+}{\mu}} = c_{\mu,\nu} - c_{\nu,\mu} - \eta_{\mu\nu}. \tag{4.10}$$

Then substituting from (4.10) into (4.9), then the skew part of the field equations (3.22) can be written as:

$$c_{\mu,\nu} - c_{\nu,\mu} = 2\zeta_{\mu\nu} - 2\xi_{\mu\nu} + \eta_{\mu\nu}. \tag{4.11}$$

Alternatively, this equation can be written in a more compact form:

$$F_{\mu\nu} = c_{\mu,\nu} - c_{\nu,\mu}\,, \tag{4.12}$$

where

$$F_{\mu\nu} \overset{\text{def}}{=} -2\xi_{\mu\nu} + Z_{\mu\nu}\,, \tag{4.13}$$

and

$$Z_{\mu\nu} \overset{\text{def}}{=} 2\zeta_{\mu\nu} + \eta_{\mu\nu}. \tag{4.14}$$

Consequently, $F_{\mu\nu}$ will satisfy the relation:

$$F_{\mu\nu;\sigma} + F_{\sigma\mu;\nu} + F_{\nu\sigma;\mu} = F_{\mu\nu,\sigma} + F_{\sigma\mu,\nu} + F_{\nu\sigma,\mu} \equiv 0\,. \tag{4.15}$$

We can also define the vector density:

$$\mathcal{J}^{\mu} \stackrel{\text{def}}{=} \mathcal{F}^{\mu\nu}{}_{,\nu}\,, \tag{4.16}$$

where

$$\mathcal{F}^{\mu\nu} \stackrel{\text{def}}{=} \lambda\, F^{\mu\nu}, \tag{4.17}$$

then:

$$\mathcal{J}^{\mu}{}_{,\mu} = \mathcal{F}^{\mu\nu}{}_{,\nu\mu} \equiv 0\,. \tag{4.18}$$

since $\mathcal{F}^{\mu\nu}$ is skew tensor density. Also we can define the scalar $\mathcal{J}$ by:

$$\mathcal{J}^{2} \stackrel{\text{def}}{=} \mathcal{J}^{\mu}\mathcal{J}_{\mu} \tag{4.19}$$

A primary comparison between the skew field equations (4.12), of the suggested theory, and the corresponding field equations of Einstein-Maxwell theory [39] shows that:

- The skew tensor (4.13) can be used to represent electromagnetic filed strength.
- The basic vector c_μ (defined by (2.13)) appearing in (4.12) represents a generalized electromagnetic potential.
- Equations (4.12) and consequently (4.15) represent a generalization of Maxwell field equations.
- The vector density $\mathcal{J}^{\mu}$ represents the current density.
- The scalar $\mathcal{J}$ represents the charge density.
- The identity (4.18) represents the conservation of electric charge.

The above physical attributions, to geometric objects, will be more supported in the next Section.

5 Comparison with Linear Field Theories

From the above Section, it appears that the field theory suggested in the present work may be used to describe a unified gravitational and electromagnetic fields since both fields are generated from one entity, the BB of the AP-geometry. Also, the theory represents a material distribution given by $T_{\mu\nu}$. Recalling that the theory suggested is highly non linear in the unified potential (the tetrad vector), one can conclude that the separation of gravity and electromagnetism, appeared in Section 4 is apparent (not complete). In order to obtain a real separation between the two fields, one has to go to low energy limits. This goal may be achieved by examining the case of weak and static fields and a slowly moving test particle. In other words, one has to linearize the theory in order to support the physical meaning attributed, in Section 4, to the geometric objects and also to gain more physics. This linearization will enable us to compare the suggested theory with corresponding linear field theories in which physics is more clear.

The linearization scheme may be as follows:
On one hand the tetrad vector field $\underset{i}{\lambda}_{\mu}$ represents the building blocks of the AP-geometry. On the other hand, in view of the present theory, it represents the unified potential generating both gravitational and electromagnetic potentials, $g_{\mu\nu}$ and c_μ, respectively. Also a material distribution is induced by this tetrad. So we are going to assume that:

$$\underset{i}{\lambda}_{\mu} = \delta_{i\mu} + \epsilon\, \underset{i}{h}_{\mu}. \tag{5.1}$$

where $\delta_{i\mu}$ is Kronecker delta, $\underset{i}{h}_{\mu}$ are functions of the coordinates representing deviations of the tetrad vector field from its flat space values and ϵ is a dimensionless parameter. Weak field can be achieved by neglecting terms of the second order in ϵ in the expansion formulae as follows. We expand any geometric object S in the form:

$$S = \overset{(0)}{S} + \epsilon \overset{(1)}{S} + \epsilon^{2}\overset{(2)}{S} + \ldots + \epsilon^{r}\overset{(r)}{S} + \ldots. \tag{5.2}$$

where $\overset{(r)}{S}$ is the coefficient of ϵ^{r} in the above expansion formula. For example, the metric tensor $g_{\mu\nu}$, using (2.3) and (5.1), can be written as:

$$g_{\mu\nu} = \delta_{\mu\nu} + \epsilon\, y_{\mu\nu} + \epsilon^{2}\, \underset{i}{h}_{\mu}\underset{i}{h}_{\nu} \tag{5.3}$$

where

$$y_{\mu\nu} \stackrel{\text{def}}{=} \underset{\mu}{h}_{\nu} + \underset{\nu}{h}_{\mu}\,. \tag{5.4}$$

Thus,

$$\overset{(0)}{g}_{\mu\nu} = \delta_{\mu\nu}\,, \qquad \overset{(1)}{g}_{\mu\nu} = y_{\mu\nu}\,, \qquad \overset{(2)}{g}_{\mu\nu} = \underset{i}{h}_{\mu}\underset{i}{h}_{\nu}\,. \tag{5.5}$$

So, we can express each of the tensors in the field equations (3.22) in ascending powers of the parameter ϵ. Table 2 gives the results of expansion for different AP-geometric objects [44], using their definitions given in Table 1 and the expansion formula (5.1). Confining ourselves to linear terms only $\big(O(\epsilon)\big)$, we get the following results for the symmetric and skew parts of the field equations.

The Symmetric Part

The linearized form of the symmetric equations (4.4) is given by:

$$\overset{(1)}{R}_{\mu\nu} - \frac{1}{2}\delta_{\mu\nu}\overset{(1)}{R} = \overset{(1)}{T}_{\mu\nu}\,, \tag{5.6}$$

which is equivalent to:

$$\overset{(1)}{R}_{\mu\nu} = \overset{(1)}{T}_{\mu\nu} - \frac{1}{2}\delta_{\mu\nu}\overset{(1)}{T}\,. \tag{5.7}$$

Table 2: Expansion of AP Geometric Objects [44]

Geometric Object	Order of ϵ			
	0 Order	*1st Order*	*2nd Order*	*3rd & Higher Orders*
$\underset{i}{\lambda}_{\mu}$	✓	✓	✘	✘
$g_{\mu\nu}$	✓	✓	✓	✘
$\underset{i}{\lambda}^{\mu}$	✓	✓	✓	✓
$g^{\mu\nu}$	✓	✓	✓	✓
$\Gamma^{\alpha}_{\mu\nu}$	✘	✓	✓	✓
$\left\{ {}^{\alpha}_{\mu\,\nu} \right\}$	✘	✓	✓	✓
$\gamma^{\alpha}_{.\,\mu\nu}$	✘	✓	✓	✓
$\Lambda^{\alpha}_{.\,\mu\nu}$	✘	✓	✓	✓
$\Delta^{\alpha}_{.\,\mu\nu}$	✘	✓	✓	✓
C_{μ}	✘	✓	✓	✓
$\xi_{\mu\nu}$	✘	✓	✓	✓
$\chi_{\mu\nu}$	✘	✓	✓	✓
$\varepsilon_{\mu\nu}$	✘	✓	✓	✓
$\zeta_{\mu\nu}$	✘	✘	✓	✓
$\eta_{\mu\nu}$	✘	✘	✓	✓
$\kappa_{\mu\nu}$	✘	✘	✓	✓
$\theta_{\mu\nu}$	✘	✓	✓	✓
$\psi_{\mu\nu}$	✘	✓	✓	✓
$\phi_{\mu\nu}$	✘	✘	✓	✓
$\tilde{\omega}_{\mu\nu}$	✘	✘	✓	✓
$\omega_{\mu\nu}$	✘	✘	✓	✓
$\sigma_{\mu\nu}$	✘	✘	✓	✓
$\alpha_{\mu\nu}$	✘	✘	✓	✓
$R_{\mu\nu}$	✘	✓	✓	✓
$F_{\mu\nu}$	✘	✓	✓	✓

✓ represents the existence of the term, and ✘ represents its absence.

Using (2.5), the conventional definition of Ricci tensor, and the contraction of $R^{\epsilon}_{.\,\mu\nu\sigma}$ given in the set (2.20), we get:

$$\overset{(1)}{R}_{\mu\nu} = \frac{1}{2}\left(y_{\mu\nu,\alpha\alpha} + y_{\alpha\alpha,\mu\nu} - y_{\mu\alpha,\nu\alpha} - y_{\nu\alpha,\mu\alpha}\right). \tag{5.8}$$

From (4.5) and Table 2, we have:

$$\overset{(1)}{T}_{\mu\nu} = \tfrac{1}{2}\left(\delta_{\mu\nu}\overset{(1)}{\psi} + \overset{(1)}{\theta}_{\mu\nu} - \tfrac{3}{2}\overset{(1)}{\psi}_{\mu\nu}\right), \tag{5.9}$$

$$\overset{(1)}{T} = \tfrac{3}{4}\overset{(1)}{\psi}, \tag{5.10}$$

since $\psi = -\theta$.
Substituting from (5.8), (5.9) and (5.10) into the linear equations (5.7), we get:

$$\begin{aligned}&\frac{1}{2}\left(y_{\mu\nu,\alpha\alpha} + y_{\alpha\alpha,\mu\nu} - y_{\mu\alpha,\nu\alpha} - y_{\nu\alpha,\mu\alpha}\right)\\ &= \frac{1}{2}\left(\delta_{\mu\nu}\overset{(1)}{\psi} + \overset{(1)}{\theta}_{\mu\nu} - \frac{3}{2}\overset{(1)}{\psi}_{\mu\nu}\right) - \frac{3}{8}\delta_{\mu\nu}\overset{(1)}{\psi},\end{aligned} \tag{5.11}$$

or

$$\begin{aligned}&y_{\mu\nu,\alpha\alpha} + \left(\frac{1}{2}y_{\alpha\alpha,\mu} - y_{\mu\alpha,\alpha}\right)_{,\nu} + \left(\frac{1}{2}y_{\alpha\alpha,\nu} - y_{\nu\alpha,\alpha}\right)_{,\mu}\\ &= \overset{(1)}{\theta}_{\mu\nu} - \frac{3}{2}\overset{(1)}{\psi}_{\mu\nu} + \frac{1}{4}\delta_{\mu\nu}\overset{(1)}{\psi}.\end{aligned} \tag{5.12}$$

In the case of classical mechanics, weak and static field, the only non-zero component of the material energy tensor is the 00-component. In order to compare with the classical field equation we write the above field equation (5.11) for $\mu, \nu = 0$, in the form:

$$\nabla^2 y_{00} = \frac{1}{4}\left(\overset{(1)}{\psi}_{11} + \overset{(1)}{\psi}_{22} + \overset{(1)}{\psi}_{33} - 5\overset{(1)}{\psi}_{00}\right). \quad (5.13)$$

The θ_{00} term disappears since, for a static field, we have:

$$\overset{(1)}{\theta}_{00} = \underset{\alpha}{h}_{0,\alpha 0} + \underset{\alpha}{h}_{0,\alpha 0} - 2\underset{\alpha}{h}_{\alpha,00} = 0.$$

It is clear that the symmetric field equations, in its linearized form (5.13), relates the gravitational potential y_{00} (the linear part of $g_{\mu\nu}$ (5.3)) to its source, represented by the R.H.S. of (5.13). To gain more physics, let us examine the R.H.S. of (5.13). It seems that $\psi_{\mu\nu}$ is the tensor responsible for static material distribution. Now we take into account the following points:

1. if we take $\overset{(1)}{\psi}_{00}$ as a geometric representation of the density of a static fluid, then the spatial components $\overset{(1)}{\psi}_{11}$, $\overset{(1)}{\psi}_{22}$, $\overset{(1)}{\psi}_{33}$ can be considered as geometric representations of the hydrostatic pressure of the fluid,
2. classically speaking, and considering slowly moving particles of the fluid, the pressure would be of the 2nd order in the velocity. So the effect of the spatial components $\overset{(1)}{\psi}_{ab}$ (where $a, b = 1, 2, 3$) could be neglected in the linearization regime,
3. it is well-known classically, that pressure has no effect on the gravitational potential. So to compare with classical known results, one has to neglect the effect of $\overset{(1)}{\psi}_{11}$, $\overset{(1)}{\psi}_{22}$, $\overset{(1)}{\psi}_{33}$ in (5.13).

Considering the above points we can write the linearized symmetric field equations (5.13) in the form:

$$\nabla^2 y_{00} = -\frac{5}{4}\overset{(1)}{\psi}_{00}. \quad (5.14)$$

In classical hydrodynamics, the gravitational potential ϕ within a fluid of density ρ is given by Poisson equation (cf. [4]):

$$\nabla^2 \phi = 4\pi\gamma\rho, \quad (5.15)$$

where γ is the universal gravitational constant.
The comparison between the two equations (5.14) and (5.15) justify the assumption that ψ_{00} represents the fluid density and y_{00} is the gravitational potential.
Consequently we can arrive to the following physical results:

1. The symmetric field equation (4.4) is a generalization of Poisson equation.
2. The tensor $\psi_{\mu\nu}$ is a geometric representation of a static material distribution, while the tensor $\theta_{\mu\nu}$ will offer a contribution to this distribution in the non-static case. The tensor $B_{\mu\nu}$ (4.6) will give non-linear effects.
3. The tensor $T_{\mu\nu}$ (4.5), composed of the above mentioned components, can be used to represent the energy and material distribution.
4. In the case of vanishing material distribution, i.e. $\overset{(1)}{\psi}_{00} = 0$, equation (5.14) will reduce to:

$$\nabla^2 y_{00} = 0, \quad (5.16)$$

which is the well known Laplace equation.

In addition to the above justification for attributing material distribution to the tensor $T_{\mu\nu}$, let us add one more. The generalized conservation law of material-energy (4.7) can be written as:

$$T^{\mu\nu}{}_{;\nu} \stackrel{\text{def}}{=} T^{\mu\nu}{}_{,\nu} + T^{\alpha\nu}\left\{{}^{\mu}_{\alpha\,\nu}\right\} + T^{\mu\alpha}\left\{{}^{\nu}_{\nu\,\alpha}\right\} = 0. \quad (5.17)$$

Now in the linearization regime, this law is reduced to:

$$\overset{(1)}{T}{}^{\mu\nu}{}_{,\nu} = 0, \quad (5.18)$$

since other terms are of the 2nd and higher orders. Equation (5.18) is the well known classical conservation law implying that $\overset{(1)}{T}{}^{\mu\nu}$ represents conserved quantities.

The Skew Part

The skew part of the field equations (3.22), in its linearized form, can be written as:

$$\overset{(1)}{F}_{\mu\nu} = \overset{(1)}{C}_{\mu,\nu} - \overset{(1)}{C}_{\nu,\mu}, \quad (5.19)$$

where,

$$\overset{(1)}{F}_{\mu\nu} \stackrel{\text{def}}{=} -2\overset{(1)}{\xi}_{\mu\nu}. \quad (5.20)$$

Note that $Z_{\mu\nu}$, given by (4.14), vanishes in the first approximation scheme (see Table (2)).

It is to be noted that, in the linearized form, raising and lowering of indices are carried out using Kronecker deltas; and the determinant λ will have, to the first order, the value:

$$\lambda = 1 + \epsilon\left(\underset{0}{h}_0 + \underset{1}{h}_1 + \underset{2}{h}_2 + \underset{3}{h}_3\right).$$

So, substituting from (5.19) into (4.16), we can write the current vector J_μ in the following linearized form:

$$\overset{(1)}{J}_{\mu} = \overset{(1)}{C}_{\mu,\nu\nu} - \overset{(1)}{C}_{\nu,\nu\mu}. \quad (5.21)$$

Contracting (5.12), using (5.1) and Table 1, we get:

$$\underset{\nu}{h}_{\alpha,\nu\alpha} = \underset{\nu}{h}_{\nu,\alpha\alpha}. \quad (5.22)$$

Expanding c_μ using (5.1), we get:

$$\overset{(1)}{c}_{\mu,\mu} = \left(\underset{\nu}{h}_{\mu,\nu} - \underset{\nu}{h}_{\nu,\mu}\right)_{,\mu} \,. \tag{5.23}$$

So using (5.22), we get:

$$\overset{(1)}{c}_{\mu,\mu} = 0\,. \tag{5.24}$$

Substituting from (5.24) into (5.21), we get, for a static field:

$$\nabla^2 \overset{(1)}{c}_{\mu} = \overset{(1)}{J}_{\mu}\,. \tag{5.25}$$

In order to compare with the classical static field equations, we write the above equations for $\mu = 0$; we get:

$$\nabla^2 \overset{(1)}{c}_{0} = \overset{(1)}{J}_{0}\,. \tag{5.26}$$

It is clear that the skew-symmetric field equation, in the linearized form (5.26), relates the electric potential $\overset{(1)}{c}_0$ to its source, represented by the R.H.S. of (5.26). To gain more physics, let us examine the R.H.S. of (5.26). It seems that $\overset{(1)}{J}_0$ is the vector component responsible for the electric charge distribution, in the static case.

In classical electrostatics, the electric potential ϕ in an electric charge distribution of an electric charge density q is given by Poisson equation (cf. [22]):

$$\nabla^2 \phi = \frac{4\pi q}{\varepsilon}\,, \tag{5.27}$$

where ε is the dielectric constant.
The two equations (5.26) and (5.27) would be the same if, apart from some conversing constants, $\overset{(1)}{c}_0$ and $\overset{(1)}{J}_0$ are the electric potential and the electric charge density, respectively.

Consequently we can arrive at the following physical results:

1. The skew-symmetric field equations (4.12) are a generalization of Poisson equations (5.27).
2. The vector component J_0 is a geometric representation of a static electric charge distribution while the components J_1, J_2, J_3 have contributions to the non-static case and the tensor $Z_{\mu\nu}$ (4.14) will give non-linear effects in the non-static case.
3. The tensor $F_{\mu\nu}$ (4.13) can be used to represent the strength of the electromagnetic field; and consequently c_μ (2.13) can be considered as the generalized electromagnetic potential.
4. In the case of a vanishing electric charge density, i.e. $J_0 = 0$, equation (5.26) will reduce to:

   $$\nabla^2 \overset{(1)}{c}_0 = 0\,, \tag{5.28}$$

   which is the well known Laplace equation for the electromagnetic potential $\overset{(1)}{c}_0$.

In addition to the above justifications for attributing the electric charge distribution to the tensor $F_{\mu\nu}$, here is one more. The linearized form of the identity (4.18) can be written as:

$$\overset{(1)}{J}{}^{\mu}{}_{,\mu} = \overset{(1)}{F}{}^{\mu\nu}{}_{,\nu\mu} \equiv 0\,, \tag{5.29}$$

which is the well known classical conservation law indicating that $F_{\mu\nu}$ represents conserved quantities.

6 Transition to Physical Application

The two methods given in Sections 4 & 5 can be used for any geometric field theory to explore its physical contents before solving its field equations. In the present Section, we review a third important method, not well known in the literature, for analysing any geometric field theory, before confronting its predictions with observations or experiments. This method was suggested in 1981 [29] and has been used in many applications (cf. [11, 30, 45, 49]). This method is known as "Type Analysis". It measures the capabilities of a certain geometric structure to represent physical systems in the context of the theory concerned.

In order to confront a covariant field theory with observations or experiments, one has to solve its field equations. It is well known that there is no general solution of any geometric covariant field equations (e.g. $R_{\mu\nu} = 0$ of GR) unless one constraints the building blocks of the geometry to certain symmetries (spherical symmetry, axial symmetry, …). This is usually done via Killing equations which contain the building blocks of the geometry together with the generators of certain group of motion or more. Let us call the resulting geometric model a "geometric structure". The main idea of type analysis is very simple. Let us give an example in the case of Riemannian geometry. It is important to carry out the following test: if one uses a geometric structure, a line element, for which all the components of the curvature tensor vanish identically, then one can conclude that this structure cannot be used to study gravitational systems in the context of GR. This test can be applied before solving the field equations of GR. Consequentially, one can classify Riemannian geometric structures, in the context of GR, into two classes. The first is characterized by a vanishing curvature tensor (not appropriate for gravity) and the second is characterized by a non-vanishing curvature (appropriate for gravity applications).

This idea has been extended for theories constructed in wider geometries than the Riemannian one. For example, in the AP-geometry, the identical vanishing of all second-order skew tensors for a certain geometric AP-structure indicates that this structure cannot be used to represent electromagnetism, even if the field theory considered has an electromagnetic sector.

In the context of the present work we give examples of what the term *strength* means. We can see from (4.6), and Table 2, that $B_{\mu\nu}$, when expanded, includes only terms of the second and higher orders in ϵ. So, we can take $B_{\mu\nu}$ as an indicator of the strength of the gravitational field. Also, from (4.14), it is clear that the expansion form of $Z_{\mu\nu}$ involves only terms of the second and higher orders in ϵ. It follows that we can consider $Z_{\mu\nu}$ as an indicator of the strength of the electromagnetic field. The identical vanishing of such tensors, in any AP-structure, indicates that the corresponding fields are weak.

As a result, we can specify some distinct classes of gravitational fields (denoted by the letter G) and electromagnetic fields (denoted by the letter F) according to Table 3.

Table 3: Type Analysis

Indicator	Field Represented	Type
$F_{\mu\nu} = 0$	No electromagnetic field	F0
$F_{\mu\nu} \neq 0$ $Z_{\mu\nu} = 0$	Electromagnetic field, not strong	FI
$F_{\mu\nu} \neq 0$ $Z_{\mu\nu} \neq 0$	Strong electromagnetic field	FII
$R^{\alpha}_{.\mu\nu\sigma} = 0$	No gravitational field	G0
$R^{\alpha}_{.\mu\nu\sigma} \neq 0$ $T_{\mu\nu} = 0$	Gravitational field in empty space, not strong	GI
$R^{\alpha}_{.\mu\nu\sigma} \neq 0$ $T_{\mu\nu} \neq 0$ $B_{\mu\nu} = 0$	Gravitational field within a material distribution, not strong	GII
$R^{\alpha}_{.\mu\nu\sigma} \neq 0$ $T_{\mu\nu} \neq 0$ $B_{\mu\nu} \neq 0$	Strong gravitational field within a material distribution	GIII

The type of a unified field is specified by its electromagnetic class (denoted by F) and its gravitational class (denoted by G). For example, a pure weak gravitational field within a material distribution is denoted by F0GII .

It is to be considered that the procedure of type analysis is generally covariant since it depends on the vanishing of certain tensors. This procedure in the AP-geometry depends on two elements: the geometric field theory and Table 2 which is independent of the field theory used. It can be established for any field theory constructed in the AP-geometry. The application of the scheme of type analysis will be more clear in Section 7, which gives a clear example illustrating how to use this scheme in physical applications.

7 A Solution with Spherical Symmetry

In this section, we apply the present theory to the tetrad vector field having spherical symmetry. This tetrad, which has been derived by Robertson [41], can be written in the coordinates ($x^0 \equiv t$, $x^1 \equiv r$, $x^2 \equiv \theta$, $x^3 \equiv \phi$) as:

$$\underset{i}{\lambda}{}^{\mu} = \begin{array}{c} \mu = 0,1,2,3 \\ \longrightarrow \end{array} \quad i = 0,1,2,3 \downarrow \begin{pmatrix} A & D\,r & 0 & 0 \\ 0 & B\sin\theta\cos\phi & \frac{B}{r}\cos\theta\cos\phi & -\frac{B}{r}\frac{\sin\phi}{\sin\theta} \\ 0 & B\sin\theta\sin\phi & \frac{B}{r}\cos\theta\sin\phi & \frac{B}{r}\frac{\cos\phi}{\sin\theta} \\ 0 & B\cos\theta & -\frac{B}{r}\sin\theta & 0 \end{pmatrix}, \tag{7.1}$$

where A, B, and D are unknown functions of r only. From (7.1) we can get the metric tensor of the associated Riemannian space, using (2.3):

$$g_{\mu\nu} = \begin{pmatrix} \frac{1}{A^2B^2}(D^2r^2 + B^2) & -\frac{Dr}{AB^2} & 0 & 0 \\ -\frac{Dr}{AB^2} & \frac{1}{B^2} & 0 & 0 \\ 0 & 0 & \frac{r^2}{B^2} & 0 \\ 0 & 0 & 0 & \frac{r^2}{B^2}\sin^2\theta \end{pmatrix}. \tag{7.2}$$

Evaluating the required second-order tensors (relevant to the field equations (3.22)) using Table 1 and substituting into (4.5), (4.6), (4.13), and (4.14), we find that the tetrad (7.1) corresponds to type FIIGIII, following Table 3. This means that the tetrad vector field (7.1) is capable of representing strong unified field within a material distribution, in the context of the present theory.

Now substituting into the field equations (3.22), we get the following set of differential equations:

$$
\begin{aligned}
S^{0}_{.\,0} :& [D^2 r^2 (-2\frac{A''}{A} + 14\frac{B''}{B} + 2\frac{D''}{D} + 3\frac{A'^2}{A^2} - 35\frac{B'^2}{B^2} + \frac{D'^2}{D^2} \\
&+ 6\frac{A'B'}{AB} - 4\frac{A'D'}{AD} + 8\frac{B'D'}{BD} - 8\frac{A'}{Ar} + 56\frac{B'}{Br} - 6\frac{D'}{Dr} - \frac{21}{r^2}) \\
&+ B^2(-2\frac{A''}{A} + 16\frac{B''}{B} + 3\frac{A'^2}{A^2} - 26\frac{B'^2}{B^2} + 2\frac{A'B'}{AB} - 4\frac{A'}{Ar} + 32\frac{B'}{Br})] \\
&= 0, \\
S^{0}_{.\,1} :& (\frac{A''}{A} - 7\frac{B''}{B} - \frac{D''}{D} - \frac{A'^2}{A^2} + 7\frac{B'^2}{B^2} - 10\frac{A'B'}{AB} \\
&+ \frac{A'D'}{AD} + 3\frac{B'D'}{BD} + 11\frac{A'}{Ar} - 7\frac{B'}{Br} - 4\frac{D'}{Dr}) = 0, \\
S^{1}_{.\,1} :& [D^2 r^2 (\frac{A'^2}{A^2} - 21\frac{B'^2}{B^2} + \frac{D'^2}{D^2} - 14\frac{A'B'}{AB} - 2\frac{A'D'}{AD} \\
&+ 14\frac{B'D'}{BD} + 14\frac{A'}{Ar} + 42\frac{B'}{Br} - 14\frac{D'}{Dr} - \frac{21}{r^2}) \\
&+ B^2(\frac{A'^2}{A^2} - 6\frac{B'^2}{B^2} - 16\frac{A'B'}{AB} + 16\frac{A'}{Ar} + 12\frac{B'}{Br})] = 0 \\
S^{2}_{.\,2} = S^{3}_{.\,3} :& [D^2 r^2 (8\frac{A''}{A} + 14\frac{B''}{B} - 8\frac{D''}{D} - 17\frac{A'^2}{A^2} - 35\frac{B'^2}{B^2} - 9\frac{D'^2}{D^2} \\
&- 24\frac{A'B'}{AB} + 26\frac{A'D'}{AD} + 38\frac{B'D'}{BD} + 32\frac{A'}{Ar} + 56\frac{B'}{Br} - 46\frac{D'}{Dr} - \frac{21}{r^2}) \\
&+ B^2(8\frac{A''}{A} + 6\frac{B''}{B} - 17\frac{A'^2}{A^2} - 6\frac{B'^2}{B^2} + 2\frac{A'B'}{AB} + 8\frac{A'}{Ar} + 6\frac{B'}{Br})] = 0,
\end{aligned}
\tag{7.3}
$$

where the prime represents first derivative w.r.t. r and the double prime represents the second derivative.

It can be shown that, on choosing $D = 0$, the type of the model (7.1) will change to F0GIII, since this choice will switch off all second-order skew tensors relevant to the present theory. In this case the above set of differential equations (7.3) reduces to:

$$
\begin{aligned}
S^{0}_{.\,0} :& -2\frac{A''}{A} + 16\frac{B''}{B} + 3\frac{A'^2}{A^2} - 26\frac{B'^2}{B^2} + 2\frac{A'B'}{AB} - 4\frac{A'}{Ar} + 32\frac{B'}{Br} = 0, \\
S^{1}_{.\,1} :& \frac{A'^2}{A^2} - 6\frac{B'^2}{B^2} - 16\frac{A'B'}{AB} + 16\frac{A'}{Ar} + 12\frac{B'}{Br} = 0, \\
S^{2}_{.\,2} = S^{3}_{.\,3} :& 8\frac{A''}{A} + 6\frac{B''}{B} - 17\frac{A'^2}{A^2} - 6\frac{B'^2}{B^2} + 2\frac{A'B'}{AB} + 8\frac{A'}{Ar} + 6\frac{B'}{Br} = 0.
\end{aligned}
\tag{7.4}
$$

The above equations can be written in the compact form:

$$
\begin{aligned}
&L(r) + 8\frac{B''}{B} - 12\frac{B'^2}{B^2} + 16\frac{B'}{Br} = 0, \\
&M(r) - 4\frac{B'^2}{B^2} - 8\frac{A'B'}{AB} + 8\frac{A'}{Ar} + 8\frac{B'}{Br} = 0, \\
&N(r) + 4\frac{A''}{A} + 4\frac{B''}{B} - 8\frac{A'^2}{A^2} - 4\frac{B'^2}{B^2} + 4\frac{A'}{Ar} + 4\frac{B'}{Br} = 0,
\end{aligned}
\tag{7.5}
$$

where,

$$
\begin{aligned}
L(r) &\stackrel{\text{def}}{=} -2\tfrac{A''}{A} + 8\tfrac{B''}{B} + 3\tfrac{A'^2}{A^2} - 14\tfrac{B'^2}{B^2} + 2\tfrac{A'B'}{AB} - 4\tfrac{A'}{Ar} + 16\tfrac{B'}{Br}, \\
M(r) &\stackrel{\text{def}}{=} \tfrac{A'^2}{A^2} - 2\tfrac{B'^2}{B^2} - 8\tfrac{A'B'}{AB} + 8\tfrac{A'}{Ar} + 4\tfrac{B'}{Br}, \\
N(r) &\stackrel{\text{def}}{=} 4\tfrac{A''}{A} + 2\tfrac{B''}{B} - 9\tfrac{A'^2}{A^2} - 2\tfrac{B'^2}{B^2} + 2\tfrac{A'B'}{AB} + 4\tfrac{A'}{Ar} + 2\tfrac{B'}{Br}.
\end{aligned}
\tag{7.6}
$$

This set (7.5), corresponding to the type F0GIII, represents a pure strong gravitational field within a material distribution. This situation has no successful correspondence in the domain of the classical non-linear field theories. So, let us reduce this set to match the type F0GI, which can be compared with GR. We found that if we take:

$$L(r) = 0 \quad , \quad M(r) = 0 \quad , \quad N(r) = 0\,, \tag{7.7}$$

all the components of $T_{\mu\nu}$ vanish and the field equations (7.5) will correspond to F0GI and are reduced to:

$$
\begin{aligned}
2\tfrac{B''}{B} - 3\tfrac{B'^2}{B^2} + 4\tfrac{B'}{Br} &= 0, \\
-\tfrac{B'^2}{B^2} - 2\tfrac{A'B'}{AB} + 2\tfrac{A'}{Ar} + 2\tfrac{B'}{Br} &= 0, \\
\tfrac{A''}{A} + \tfrac{B''}{B} - 2\tfrac{A'^2}{A^2} - \tfrac{B'^2}{B^2} + \tfrac{A'}{Ar} + \tfrac{B'}{Br} &= 0.
\end{aligned}
\tag{7.8}
$$

This set represents pure gravitational field with spherical symmetry outside material distribution. The resulting solution can be compared with the corresponding GR solution.

Integration of the 1st equation of the set (7.8) gives:

$$\frac{B'}{B^{3/2}} = \frac{\tilde{c}_1}{r^2}$$

where $\tilde{c}_1$ is a constant of integration. Integrating again, we get:

$$B = \left(\frac{c_1}{r} - c_2\right)^{-2}. \tag{7.9}$$

Substituting into the 2nd equation of the set (7.8), we get:

$$\frac{A'}{A} = -2\frac{c_1 c_2}{(c_1^2 - c_2^2 r^2)},$$

which can be integrated to get:

$$A = c_3 \left(\frac{c_2 - \frac{c_1}{r}}{c_2 + \frac{c_1}{r}}\right), \tag{7.10}$$

where c_1, c_2 and c_3 are arbitrary constants of integration. The solution given by (7.9) and (7.10) satisfies the 3rd equation of the set (7.8) without any further condition.

Now we write the metric of the associated Riemannian space. Using (7.9) and (7.10), noting that $D = 0$, we can write the metric using (7.2) in the form:

$$ds^2 = \left(c_3\left(\frac{c_2 - \frac{c_1}{r}}{c_2 + \frac{c_1}{r}}\right)\right)^{-2} dt^2 + \left(\frac{c_1}{r} - c_2\right)^4 \left[dr^2 + r^2\left(d\theta^2 + \sin^2\theta^2 d^2\phi\right)\right] \quad (7.11)$$

Choosing the arbitrary constants to be $c_1 = \frac{m}{2}$, $c_2 = -1$, and $c_3 = i$, the above metric becomes:

$$ds^2 = \left(1 + \frac{m}{2r}\right)^4 \left[dr^2 + r^2\left(d\theta^2 + \sin^2\theta d^2\phi\right)\right] - \left(\frac{1 - \frac{m}{2r}}{1 + \frac{m}{2r}}\right)^2 dt^2\,, \quad (7.12)$$

or

$$d\tau^2 = \left(\frac{1 - \frac{m}{2r}}{1 + \frac{m}{2r}}\right)^2 dt^2 - \left(1 + \frac{m}{2r}\right)^4 \left[dr^2 + r^2\left(d\theta^2 + \sin^2\theta d^2\phi\right)\right] \quad (7.13)$$

where τ is the proper time. This represents the metric of the well known Schwarzchild exterior solution in the GR case. The solution given in the present section will be discussed in Section 8.

8 Concluding Remarks

1. In the present work we have constructed a pure geometric unified field theory in the sense that all physical quantities and fields are defined from the building blocks of the AP-geometry, the tetrad vector field components. The theory is constructed depending on two principles
   (a) The general covariance principle,
   (b) The unification principle as mentioned in the introduction.
2. The suggested theory has two main sectors: field and motion. In the geometrization philosophy, motion of test particles is described by curves of the geometry used. According to the curves of the AP-geometry, given by (2.25), trajectories of elementary particles are already quantized. Consequently, the field described by the theory is implicitly quantized. The explicit quantization of the field is not yet explored. As far as we see, it needs much effort.
3. The theory has, in general, sixteen field equations (3.22) to be solved for the sixteen field variables, the tetrad vectors, $\underset{i}{\lambda}{}_{\mu}$.
4. The Lagrangian density (3.8) used to construct the field equations of the theory depends mainly on the W-scalar. The philosophy of using the W-tensor is inspired by Einstein's statement and the unification principle given in the introduction. In general, and as stated above, the W-tensor is neither curvature nor torsion. It is a geometric alloy made of curvature and torsion. Now, if curvature represents gravity, as agreed upon by most of the authors, and torsion represents other physical interactions, as many authors pointed out, then it is better, to construct a general field theory to use a geometric object as the W-tensor. The advantage of using this tensor is that both curvature and torsion are not artificially combined in it. This is in agreement with the geometrization philosophy and the unification principle.
5. Another point in favour of using the W-tensor in the formalism is that: if we start composing the Lagrangian from two added terms one depends on the curvature and the other depends on the torsion, then mutual interactions between the corresponding physics of these two objects are not guaranteed. It is preferable to use the geometric alloy, called the W-tensor, in order to explore such interactions, if any, without adding it by hand.
6. It is well known that pure geometric objects have no physical meaning unless we have a theory. Any geometric field theory should be analysed in order to explore its physical contents, before applying the theory to real physical problems. In the present work, the field theory suggested is theoretically analysed by two methods given in Sections 4 & 5. The third method, given in Section 6, can be considered as located in the midway between theory and application.
7. The application of the theory in a spherically symmetric case (F0GI), given in Section 7, produces a unique solution (7.9), (7.10). This solution gives rise to the well known Schwarzchild exterior solution of GR (7.13). The treatment given in Section 7 shows that:
   (a) The suggested theory covers the successful domain of GR in a similar case.
   (b) The important role of the scheme of type analysis in solving the field equations of the theory.
8. Table 4 gives a brief comparison between the present theory and other field theories: general relativity (GR), the teleparallel equivalent of GR (TEGR), Einstein-Cartan theory (EC), Metric-Affine Gauge

Table 4: Comparison with Non-linear Field Theories

Theory	Geometry	No. of field variables	Momentum-energy tensor	Minimal-coupling	Poisson Equation
GR [14]	Riemannian	10	Phenomenological	Yes	No
TEGR (cf.[1])	AP-space	16	Phenomenological	Yes	No
EC [19, 21]	Riemann-Cartan	16+24=40	Phenomenological	Yes	No
MAG [20, 21]	Metric-Affine	10+16+64=90	Phenomenological	Yes	No
GFT [28]	AP-space	16	Geometric	No	No
Present Theory	AP-space	16	Geometric	No	Yes

theory (MAG) and the generalized field theory (GFT). The criteria used for this comparison are: the geometry used in construction (2^{nd} column), the number of field variables or the building blocks of the geometry (3^{rd} column), the type of energy-momentum tensor used (4^{th} column), the use of minimal coupling (5^{th}) and the appearance of geometric Poisson equations (last column).

9. It is clear that the present theory is not, in general, a gauge field theory. However its skew part (4.12) is invariant under gauge transformation. It is well known, in the domain of electrodynamics, that in order to solve Maxwell's equations, it is necessary to impose some conditions to fix the gauge. The most famous of these conditions is the Lorentz condition. In the present work Lorentz condition (5.24) is obtained from the theory and not imposed from outside. The condition is obtained as a consequence of the symmetric part of the field equation of the theory (4.4) in its linearized form. This simply means that according to the present theory, gauge in electromagnetism is fixed by the gravitational field. This gives the effect of gravity on electromagnetism. In other words it gives a type of interaction between the two fields even in low energy. This supports our choice of the Lagrangian of the theory.
10. A new feature of the present theory is the appearance of Poisson's equations, for material distribution (5.14) and for charge distribution (5.26), from pure geometric considerations. This supports our choice of the tensor $T^{\mu\nu}$ (4.5) to represent the material distribution and the tensor $F_{\mu\nu}$ (4.13) to represent the electromagnetic field. Also it supports our choice of the tensors $g_{\mu\nu}$ and c_μ to represent the potentials of gravity and electromagnetism, respectively. The appearance of the geometric Poisson's equations is a consequence of two features:
    (a) choosing geometric objects to represent the properties of matter.
    (b) the non-minimal coupling guaranteed by the theory.

    From Table 4, it is clear that although the GFT satisfies the two features given above, yet no Poisson's equation appeared. This is because in the low energy limit GFT has no geometric representatives for the material distribution.

References

[1] R. Aldrovandi, J. G. Pereira, K. H. Vu, Braz. J. Phys. 34, 4A (2004)
[2] A. Ashtekar, M. Reuter, C. Rovelli, arXiv:1408.4336v1 [gr-qc]
[3] G. R. Bengochea, R. Ferraro, Phys. Rev. D 79, 124019 (2009)
[4] R. J. Blakely, Potential Theory in Gravity and Magnetic Applications (Cambridge University Press, England, 1996)
[5] R. Bottema, A&A 393, 453 (2002)
[6] S. M. Carroll, Living Rev. Relativ. 4, 1 (2001)
[7] J. Cepa, Astron. Astrophys. 422, 831 (2004)
[8] L. P. Chimento, A. S. Jakubi, D. Pavon, arXiv:0010079v1 [astro-ph]
[9] V. C. De Andrade, L. C. T. Guillen, J. G. Pereira, In: V. G. Gurzadyan, R. T. Jantzen, R. Ruffini (Ed.), the Ninth Marcel Grossmann Meeting, July 2000 (Rome, Italy) (2002) B 1022
[10] A. De Felice, S. Tsujikawa, Living Rev. Relativ. 13, 3 (2010)
[11] R. D. De Souza, R. Opher, JCAP 22, 1002 (2010)
[12] A. Dev, J. S. Alcaniz, D. Jain, Phys. Rev. D 67, 023515 (2003)
[13] P. Dolan, W. H. McCrea, Fundamental Identity in Electrodynamics and General Relativity (Personal communication, 1963)

[14] A. Einstein, The Meaning of Relativity, 5th ed. (Princeton, 1955)
[15] R. Ferraro, F. Fiorini, Phys. Rev. D 75, 084031 (2007)
[16] R. Ferraro, F. Fiorini, Phys. Rev. D 78, 124019 (2008)
[17] A. H. Guth, Phys. Rev. D 23, 347 (1981)
[18] K. Hayashi, T. Shirafuji, Phys. Rev. D 19, 3524 (1979)
[19] F. W. Hehl, P. von der Heyde, G. D. Kerlick, Rev. Mod. Phys. 48, 393 (1976)
[20] F. W. Hehl et al., Phys. Rep. 258, 1 (1995)
[21] F. W. Hehl, arXiv:1204.3672v2 [gr-qc]
[22] J. D. Jackson, Classical Electrodynamics, 3rd ed. (John Wiley & Sons Ltd., New York, 1999)
[23] E. O. Kahya, M. Khurshudyan, B. Pourhassan, R. Myrzakulov, A. Pasqua, arXiv:1402.2592v4 [gr-qc]
[24] M. Li et al., JHEP 1107, 108 (2011)
[25] A. D. Linde, Phys. Lett. B 108, 389 (1982)
[26] J. Martin, arXiv:1205.3365v1 [astro-ph]
[27] F. I. Mikhail, Ain Shams Sci. Bul. 6, 87 (1962)
[28] F. I. Mikhail, M. I. Wanas, Proc. Roy. Soc. Lond. A 356, 471 (1977)
[29] F. I. Mikhail, M. I. Wanas, Int. J. Theor. Phys. 20, 671 (1981)
[30] F. I. Mikhail, M. I. Wanas, A. M. Eid, Astrophys.Space Sci. 228, 221 (1995)
[31] R. Myrzakulov, arXiv:1212.2155v1 [gr-qc]
[32] R. Myrzakulov, D. Sáez-Gómez, P. Tsyba, arXiv:1311.5261v1 [gr-qc]
[33] G.G.L. Nashed, Chin. Phys. Lett. 29, No.5, 050402 (2012)
[34] G.G.L. Nashed, Phys. Rev. D 88, 104034 (2013)
[35] G.G.L. Nashed, Astrophys. Space Sci. 348, 591 (2013)
[36] G.G.L. Nashed, W. El Hanafy, Eur. Phys. J. C 74, No.10, 3099 (2014)
[37] Y. C. Ong et al., Phys.Rev. D 88, 24019 (2013)
[38] M. Ostrogradski, Mem. Ac. St. Petersbourg VI 4, 385 (1850)
[39] H. Reissner, Ann. Phys. (Berlin) 59, 106 (1916)
[40] A. G. Riess et al., Astron. J. 116, 1009 (1998)
[41] H. P. Robertson, Ann. Math. Princeton (2) 33, 496 (1932)
[42] R. H. Sanders, E. Noordermeer, Mon. Not. Roy. Astron. Soc. 379, 402 (2007)
[43] T. P. Sotiriou, V. Faraoni, Rev. Mod. Phys. 82, 451 (2010)
[44] M. I. Wanas, Ph.D. thesis, Cairo university (Giza, Egypt, 1975)
[45] M. I. Wanas, Int. J. Theor. Phys. 24, 639 (1985)
[46] M. I. Wanas, M. Melek, M.E. Kahil, Astrophys.Space Sci. 228, 273 (1995)
[47] M. I. Wanas, M. I. Kahil, Gen. Rel. Grav. 31, 1921 (1999)
[48] M. I. Wanas, Stud. Cercet. Stiin. Ser. Mat. 10, 297 (2001)
[49] M. I. Wanas, Int. J. Geom. Meth. Mod. Phys. 6, 373 (2007)
[50] N. L. Youssef, A. M. Sid-Ahmed, Rep. Math. Phys. 60, 39 (2007)
[51] N. L. Youssef, A. M. Sid-Ahmed, Int. J. Geom. Meth. Mod. Phys. 5,7, 1109 (2008)
[52] N. L. Youssef, W. A. Elsayed, Rep. Math. Phys. 72, 1 (2013)