Gravitational collapse in Painlevé-Gullstrand coordinates

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Abstract

We construct an exact solution for the spherical gravitational collapse in a single coordinate patch. To describe the dynamics of collapse, we use a generalized form of the Painlevé-Gullstrand coordinates in the Schwarzschild spacetime. The time coordinate of the form is the proper time of a free-falling observer so that we can describe the collapsing star not only outside but also inside the event horizon in a single coordinate patch. We show the both solutions corresponding to the gravitational collapse from infinity and from a finite radius.

1 Introduction

Numerous studies have been done on the properties of the black hole and the formation by gravitational collapse. The standard method of describing a spherical contraction of a uniformly distributed dust star [1, 2] is making a physically reasonable junction of the two different spacetimes corresponding to the interior and exterior regions of the collapsing body. The interior and exterior solutions are given by the FLRW metric and the Schwarzschild metric, respectively, and described in different coordinate systems. Although it is nothing wrong to construct solutions in such a manner, one cannot describe the dynamics of the collapsing star in terms of the coordinates of the observer outside the event horizon. The main purpose of this paper is to describe the both regions inside and outside the horizon by a single coordinate system in a physical way.

The Painlevé-Gullstrand coordinates of the Schwarzschild solution [3, 4] is, in fact, the key to a simple physical picture of black hole and gravitational collapse. Unlike the Schwarzschild form, the Painlevé-Gullstrand metric tensor has an off-diagonal element so that it is regular at the Schwarzschild radius and has a singularity only at the origin of the spherical coordinates. In other words, the surfaces of constanttime traverse the event horizon to reach the singularity. Therefore, the Painlevé-Gullstrand coordinates are convenient for exploring the geometry of collapsing star and black hole both inside and outside the horizon altogether by a single coordinate patch. This feature results from the fact that this coordinate system adopts a time coordinate as measured by an observer who is at rest at infinity and freely falls straightforward to the origin. We note that the physics of the collapsing matter is best described by its proper time, i.e., the Painlevé-Gullstrand time coordinate. In the present work, we generalize the Painlevé-Gullstrand metric to incorporate gravitational collapse.

2 Painlevé-Gullstrand coordinates

In this section, we derive generalized Painlevé-Gullstrand coordinates (t_p, r, θ, ϕ) . We say here "generalized" in the sense that we introduce free-falling observers who start not only from infinity but also from other general points. The following mathematical derivation of the generalized Painlevé-Gullstrand coordinates is close to the derivation in [5], but the physical situation and the time coordinate are different.

The time coordinate t_p of this family is the proper time τ of an observer who freely falls radially from rest. Let us start with the Schwarzschild metric given in the standard form

$$ds^{2} = -f(r)dt_{s}^{2} + f^{-1}(r)dr^{2} + r^{2}\left(d\theta^{2} + \sin^{2}\theta d\phi^{2}\right), \qquad (1)$$

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where f(r) = 1 - 2M/r. In the Schwarzschild spacetime, the normalized four-velocity u_s^{μ} of an observer at the spacetime coordinates $x_s^{\mu}(\tau)$ is defined by $u_s^{\mu} = dx_s^{\mu}/d\tau \equiv \dot{x}_s^{\mu}$ with τ being the proper time, and can be explicitly written as $u_s^{\mu} = (\dot{t}_s, \dot{r}, \dot{\theta}, \dot{\phi}) = (\varepsilon/f, -\sqrt{\varepsilon^2 - f}, 0, 0)$, where ε is a constant of motion. Since we choose the Painlevé-Gullstrand time coordinate t_p as the proper time of the free-falling observer, the geodesic is orthogonal to the surfaces $t_p = \text{constant}$ and the geodesic tangent vector $u_{s\mu}$ is equal to the gradient of t_p : $u_{s\mu} = -\partial t_p(x_s^{\mu})/\partial x_s^{\mu}$, that is to say,

$$dt_{\rm p} = \varepsilon dt_{\rm s} + \frac{\sqrt{\varepsilon^2 - f}}{f} dr.$$
⁽²⁾

Consequently, the Painlevé-Gullstrand metric takes the form

$$ds^{2} = -dt_{p}^{2} + \frac{1}{\varepsilon^{2}} \left(dr + v(r) dt_{p} \right)^{2} + r^{2} d\Omega^{2},$$
(3)

where $v(r) = \sqrt{\varepsilon^2 - f(r)}$ is radially free-falling velocity. The observer in geodesic motion have the normalized four-velocity

$$u_{\rm p}{}^{\mu} = \left(\dot{t}_{\rm p}, \dot{r}, \dot{\theta}, \dot{\phi}\right) = (1, -v, 0, 0).$$
(4)

Note that the metric form (3) is different from that given by Martel and Poisson [5], for our time coordinate is ε times larger than theirs. This is because our metric is characterized by the free fall from various points at rest, while theirs by the free fall from infinity at various initial velocities.

The important point to note is that the metric (3) indicates an analogue of the conservation of energy in the Newtonian mechanics,

$$E = \frac{1}{2} \left(\frac{dr}{dt_{\rm p}}\right)^2 + \Phi(r),\tag{5}$$

where $E = (\varepsilon^2 - 1)/2$ is a conserved energy and $\Phi(r) = -M/r$ is a gravitational potential energy of a central force field. In particular, if the particle freely falls from rest at infinity, the conserved energy E is zero (i.e., $\varepsilon = 1$) and the metric (3) reduces to the standard form given by Painlevé and Gullstrand:

$$ds^{2} = -dt_{p}^{2} + \left(dr + \sqrt{\frac{2M}{r}}dt_{p}\right)^{2} + r^{2}d\Omega^{2}.$$
 (6)

For a free fall from infinity, the radial velocity v is the Newtonian escape velocity $\sqrt{2M/r}$. It is obvious from the metric (3) and (6) that the Painlevé-Gullstrand coordinates are regular at the horizon r = 2M. This enables us to deal with the geometry of black hole both inside and outside the horizon.

In the subsequent sections, we will consider the solution of the Einstein equation with matter. In general, the energy E and the mass M are functions of t_p and r, not constant values. With the physical picture in mind and motivated by (3), we make the ansatz for the metric in the generalized Painlevé-Gullstrand form

$$ds^{2} = -dt_{p}^{2} + \frac{1}{1 + 2E(t_{p}, r)} \Big(dr + v(t_{p}, r) dt_{p} \Big)^{2} + r^{2} d\Omega^{2},$$
(7)

where

$$v(t_{\rm p}, r) = \sqrt{2E(t_{\rm p}, r) + \frac{2m(t_{\rm p}, r)}{r}}.$$
 (8)

3 Spherical gravitational collapse—from infinity

We solve the Einstein equation in the spherical gravitational collapse from infinity. According to Birkhoff's theorem, the Schwarzschild solution is the only solution of the vacuum Einstein equation for a spherically symmetric spacetime. In particular, even if matter distribution is not static but moving in a spherically symmetric way, the exterior vacuum region is given by the Schwarzschild metric. As shown in the previous

section, we can express the Schwarzschild metric in the Painlevé-Gullstrand form (6). Meanwhile, for the matter solution in the case of the gravitational collapse, the metric is assumed to be of the form

$$ds^{2} = -dt_{p}^{2} + \left(dr + \sqrt{\frac{2m(t_{p}, r)}{r}}dt_{p}\right)^{2} + r^{2}d\Omega^{2},$$
(9)

that is, the generalized form (7) with E = 0. For simplicity, we consider the Einstein equation with uniformly distributed dust matter, $G_{\mu\nu} = 8\pi T_{\mu\nu} = 8\pi \rho(t_{\rm p})u_{\rm p\mu}u_{\rm p\nu}$, and then obtain the solution of the mass function

$$m(t_{\rm p}, r) = \frac{4\pi}{3} r^3 \rho(t_{\rm p}), \quad \rho(t_{\rm p}) = \frac{1}{6\pi t_{\rm p}^2}.$$
 (10)

Let $R(t_p)$ be the surface radius of the star at time t_p and M be the mass inside the surface. It is natural to impose a boundary condition at the surface $r = R(t_p)$, that is, the mass,

$$m(t_{\rm p}, r)|_{r=R(t_{\rm p})} = \frac{4\pi}{3} R^3(t_{\rm p})\rho(t_{\rm p}) \equiv M,$$
 (11)

of the star is constant, and then the radius of the boundary is given by

$$R(t_{\rm p}) = \left(\frac{9M}{2}(-t_{\rm p})^2\right)^{1/3}.$$
(12)

This means that the surface of the star is at rest at infinity and its radius monotonically decreases to zero as $t_{\rm p} \rightarrow 0$. In addition, the motion of the surface is geodesic. From simple calculations, the exterior metric (6) and the interior metric (9) turn out to be smoothly matched at the boundary surface $r = R(t_{\rm p})$. Therefore, we can describe the geometry of all the spacetime by a single coordinate system $(t_{\rm p}, r, \theta, \phi)$.

4 Spherical gravitational collapse—from a finite radius

We show the solution of the collapse from a finite radius. The standard model of a collapsing star is collapse from rest at a finite initial radius. It is not trivial to apply the idea of the Painlevé-Gullstrand coordinates to the situation of the collapse starting with a finite radius.

To begin with, we consider the boundary surface that freely falls from rest at a radius R_0 . At the surface, the conservation law (5) gives the energy

$$E = -\frac{M}{R_0} \tag{13}$$

and the infall velocity

$$V(t_{\rm p}) = \sqrt{\frac{2M}{R(t_{\rm p})} - \frac{2M}{R_0}}$$
(14)

where $R(t_p)$ is the surface radius smaller than the initial radius R_0 .

Next, when we assume that the exterior solution ranges from the contracting surface radius $R(t_p)$ to the initial radius R_0 (i.e., $R(t_p) < r < R_0$), the energy remains the same as (13),

$$E_{+} = -\frac{M}{R_0},\tag{15}$$

and therefore the infall velocity becomes

$$v_{+}(r) = \sqrt{\frac{2M}{r} + 2E_{+}} = \sqrt{\frac{2M}{r} - \frac{2M}{R_{0}}}$$
(16)

in the exterior region. The exterior metric is therefore given by

$$ds_{+}^{2} = -dt_{p}^{2} + \frac{1}{1 - \frac{2M}{R_{0}}} \left(dr + \sqrt{\frac{2M}{r} - \frac{2M}{R_{0}}} dt_{p} \right)^{2} + r^{2} d\Omega^{2}.$$
 (17)

Finally, in the interior region $0 < r < R(t_p)$, since the energy is now a function of time and radius coordinates, we make the ansatz

$$E_{-}(t_{\rm p}, r) = -\frac{m(t_{\rm p}, r)}{R_0 \frac{r}{R(t_{\rm p})}}$$
(18)

for the energy and

$$v_{-}(t_{\rm p}, r) = \sqrt{\frac{2m(t_{\rm p}, r)}{r} + 2E_{-}(t_{\rm p}, r)} = \sqrt{\frac{2m(t_{\rm p}, r)}{r} \left(1 - \frac{R(t_{\rm p})}{R_{0}}\right)}$$
(19)

for the velocity in the generalized Painlevé-Gullstrand metric (7).

In the case of the uniformly distributed dust, we can solve the Einstein equation taking into account the boundary condition as before. The mass function reduces to

$$m(t_{\rm p}, r) = \frac{4\pi}{3} r^3 \rho(t_{\rm p}), \quad \rho(t_{\rm p}) = \frac{3M}{4\pi R^3(t_{\rm p})}, \tag{20}$$

where the surface radius and the time coordinate are

$$R(t_{\rm p}) = \frac{R_0}{2} \left(1 + \cos \eta \right), \quad t_{\rm p} = \sqrt{\frac{R_0^3}{8M}} \left(\eta + \sin \eta \right), \tag{21}$$

respectively, and the parameter η takes the value from 0 to π . This interior solution

$$ds_{-}^{2} = -dt_{p}^{2} + \frac{1}{1 - \frac{2M}{R_{0}} \left(\frac{r}{R(t_{p})}\right)^{2}} \left(dr + \sqrt{\frac{2M}{R(t_{p})} - \frac{2M}{R_{0}}} \frac{r}{R(t_{p})} dt_{p}\right)^{2} + r^{2} d\Omega^{2}$$
(22)

is smoothly matched with the exterior metric (17) at the boundary surface $r = R(t_p)$. In fact, when the initial radius is large enough to satisfy $R_0 \gg M$, the energy and the velocity in the both regions become the same as those of the collapse from infinity.

5 Summary

For the description of gravitational collapse of a dust star, we have introduced the generalized Painlevé-Gullstrand coordinates with the time coordinate being the proper time of a free-falling observer. We gave the solutions of the Einstein equation in the cases of the collapse from a finite radius as well as from infinity. The metric describes both the interior and exterior regions of the star, which smoothly match at the surface of the star. More precisely, the metric is of C^1 class, while the metric component is of C^{1-} class. The choice of the Painlevé-Gullstrand time coordinate enables us to write the solutions inside and outside the event horizon in a single coordinate patch.

References

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