

Short review on noncommutative spacetimes

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Abstract. This review presents noncommutative spacetimes as one of the approaches to Planck scale physics, with the main assumption that at this energy scale spacetime becomes quantized. Spacetime coordinates become noncommutative as observables in Quantum Mechanics. The basic elements of Drinfeld twist deformation theory are reminded. The Hopf algebra language provides natural framework for deformed relativistic symmetries which constitute Quantum Group of symmetry and noncommutative spacetime is in fact Hopf module algebra. The notion of realization for noncommutative coordinates in terms of differential operators is also presented.

1. Introduction

The so called "Planck scale" is the scale at which gravitational effects are equivalently strong as quantum ones and it relates to either a very big energy scale or equivalently to a tiny size scale. There are many theories trying to consistently describe that region. One of the possible assumptions is that below Planck scale spacetime has more general structure, a noncommutative one where (as with quantum mechanics phase-space) uncertainty relations naturally arise. A natural quantization of manifolds can be described in the language of Noncommutative Geometry. Motivation for this fact starts with Gelfand-Naimark theorem [1] which states that there is a one-to-one relation between certain commutative algebras and certain spaces. One can say that the idea of Noncommutative Geometry is to consider noncommutative algebras as noncommutative geometric spaces, i.e. to algebralize geometric notions and then generalize them to noncommutative algebras. Minkowski spacetime can be seen in that picture as well.

Geometrically Minkowski spacetime is a 4-dim affine space over vector space equipped with a nondegenerate, symmetric bi-linear form with Lorentzian signature $\eta_{\mu\nu}$. The position of an event in spacetime is given by point $p = (x^0, x^1, x^2, x^3)$.

However, algebraically Minkowski "spacetime" is an Abelian algebra $\mathcal{A} = \mathfrak{x}^4$ of coordinate functions $x^\mu(p)$ on 4-dim real vector space described by commutation relations: $[x^\mu, x^\nu] = 0$. Algebra $\mathcal{A} = \mathfrak{x}^4 \equiv \text{Poly}(x^\mu) \equiv \mathbb{C}[x^0, \dots, x^3]$ of spacetime coordinates x^μ .

At the Planck scale classical Minkowski spacetime (as commutative algebra) becomes quantized and it is described by noncommutative algebra, i.e. $x_\mu \rightarrow \hat{x}_\mu$. We get in such a way noncommutative model of quantum space-time $[x_\mu x_\nu] = 0 \rightarrow [\hat{x}_\mu \hat{x}_\nu] \neq 0$. For example one can get the so-called Moyal-Weyl (θ) spacetime \mathcal{A}^θ [2],[3] described by:

$$[\hat{x}^\mu, \hat{x}^\nu] = i\hbar\theta^{\mu\nu} \quad (1)$$

with the deformation parameter \hbar which is of length² dimension and it is usually related with Planck length L_P . Another option is to consider the so-called Lie-algebraic one

$$[\hat{x}^\mu, \hat{x}^\nu] = i\hbar\theta_p^{\mu\nu} \hat{x}^\rho \quad (2)$$



deformation parameter h of mass dimension which could correspond, e.g. to Planck mass or Quantum Gravity scale. Its special example is the so-called κ -Minkowski spacetime \mathcal{A}^κ [4],[5]:

$$[\hat{x}^0, \hat{x}^k] = ih\hat{x}^k; \quad [\hat{x}^i, \hat{x}^j] = 0 \quad (3)$$

(where one usually takes $h = \frac{1}{\kappa}$, hence the name of this type of noncommutative spacetime). Those two types (\mathcal{A}^θ and \mathcal{A}^κ) of noncommutative spacetimes have been widely investigated in the literature, since via the deformation parameter, e.g. $\kappa \sim M_P$, they can be naturally connected with the Planck scale regime.

In general such noncommutative algebras can be obtained via deformation procedure from commutative ones. Quantum deformations, which lead to noncommutative algebras (as noncommutative spacetimes), are connected with the Quantum Groups (Hopf algebras) formalism, which constitutes the description of deformed symmetries. Quantum Group as generalized symmetry is described as Hopf algebra $\mathcal{H}(m, \eta, \Delta, \epsilon, S)$. It is composed by an unital associative algebra (\mathcal{H}, m, η) and counital coassociative coalgebra $(\mathcal{H}, \Delta, \epsilon)$. Δ and ϵ are algebra homomorphisms, m and η are coalgebra homomorphisms. The simplest example of Hopf algebra is provided by universal enveloping algebra for given Lie algebra \mathfrak{g} . The universal enveloping algebra $\mathcal{U}_{\mathfrak{g}}$ can be equipped with the primitive coproduct: $\Delta_0(u) = u \otimes 1 + 1 \otimes u$, counit: $\epsilon(u) = 0$, $\epsilon(1) = 1$ and antipode: $S_0(u) = -u$, $S_0(1) = 1$, for $u \in \mathfrak{g}$, and extending them by multiplicativity property to the entire $\mathcal{U}_{\mathfrak{g}}$. Recall that the universal enveloping algebra is a result of the factor construction $\mathcal{U}_{\mathfrak{g}} = T\mathfrak{g}/J_{\mathfrak{g}}$ where $T\mathfrak{g}$ denotes tensor (free) algebra of the vector space \mathfrak{g} quotient out by the ideal $J_{\mathfrak{g}}$ generated by elements $\langle X \otimes Y - Y \otimes X - [X, Y] \rangle$: $X, Y \in \mathfrak{g}$. Noncommutative spacetime \mathcal{A} in this framework is a Hopf module algebra over Hopf algebra \mathcal{H} with the (left) module action $\triangleright : \mathcal{H} \otimes \mathcal{A} \rightarrow \mathcal{A}$; $L \triangleright f$ of $L \in \mathcal{H}$ on $f \in \mathcal{A}$ satisfying: $L \triangleright (f \cdot g) = (L_{(1)} \triangleright f) \cdot (L_{(2)} \triangleright g)$ (for more details see, e.g., [6, 7]). One usually says that such noncommutative spacetime stays in this way invariant under quantum group of transformations in analogy to the classical case.

Hopf module algebras and the Hopf algebras can be deformed e.g. by relevant twisting element which leads to noncommutative spacetime as covariant quantum space over deformed group of symmetry. For Hopf algebra \mathcal{H} , we can consider the twisting two-tensor \mathcal{F} (the so-called Drinfeld twist), as an invertible element in $\mathcal{H} \otimes \mathcal{H}$ such that:

$$\mathcal{F} = f^\alpha \otimes f_\alpha \in \mathcal{H} \otimes \mathcal{H} \quad \text{and} \quad \mathcal{F}^{-1} = \bar{f}^\alpha \otimes \bar{f}_\alpha \in \mathcal{H} \otimes \mathcal{H} \quad (4)$$

As a result of deformation quantized Hopf algebra $\mathcal{H}^{\mathcal{F}}$ has non-deformed algebraic sector (commutators), while coproducts and antipodes are subject of the deformation:

$$\Delta^{\mathcal{F}}(\cdot) = \mathcal{F} \Delta(\cdot) \mathcal{F}^{-1}, \quad S^{\mathcal{F}}(\cdot) = u S(\cdot) u^{-1}, \quad (5)$$

where $u = f^\alpha S(f_\alpha)$. \mathcal{F} satisfies the 2-cocycle and normalization conditions [8, 9]:

$$\mathcal{F}_{12}(\Delta \otimes \text{id})\mathcal{F} = \mathcal{F}_{23}(\text{id} \otimes \Delta)\mathcal{F}, \quad (\epsilon \otimes \text{id})\mathcal{F} = 1 = (\text{id} \otimes \epsilon)\mathcal{F}, \quad (6)$$

which guarantee co-associativity of the deformed coproduct $\Delta^{\mathcal{F}}$ and associativity of the corresponding twisted star-product in the twisted module algebra $\mathcal{A}^{\mathcal{F}}$:

$$f \star g = m \circ \mathcal{F}^{-1} \triangleright (f \otimes g) = (\bar{f}^\alpha \triangleright f) \cdot (\bar{f}_\alpha \triangleright g) \quad (7)$$

for $f, g \in \mathcal{A}$ and the Hopf action \triangleright remains unchanged. $\mathcal{A}^{\mathcal{F}}$ can be represented by deformed \star -commutation relations

$$[x^\mu, x^\nu]_\star \equiv x^\mu \star x^\nu - x^\nu \star x^\mu = ih\theta^{\mu\nu}(x) \equiv ih(\theta^{\mu\nu} + \theta_\lambda^{\mu\nu} x^\lambda + \dots) \quad (8)$$

replacing the nondeformed (commutative) one: $[x^\mu, x^\nu] = 0$ where the coordinate functions (x^μ) play a role of generators for the corresponding algebras: deformed and nondeformed. Instead of $[x^\mu, x^\nu]_\star$ one sometimes writes $[\hat{x}^\mu, \hat{x}^\nu]$, as in e.g. (1) or (3).

From the general framework [8], a twisted deformation of Lie algebra \mathfrak{g} requires a topological extension of the corresponding enveloping algebra $\mathcal{U}_{\mathfrak{g}}$ into an algebra of formal power series $\mathcal{U}_{\mathfrak{g}}[[h]]$ in the formal parameter h , providing the so-called h -adic topology (see e.g., [6, 7, 9]), it is mainly due to the fact that twisting element has to be invertible. The Hopf module algebra \mathcal{A} has to be extended by h -adic topology to $\mathcal{A}[[h]]$ as well and then deformed into $\mathcal{A}^{\mathcal{F}}[[h]]$. There is a correspondence between twisting element, which can be rewritten as a power series expansion

$$\mathcal{F} = 1 \otimes 1 + \sum_{\alpha=1}^{\infty} h^{\alpha} f^{\alpha} \otimes f_{\alpha} \quad \text{and} \quad \mathcal{F}^{-1} = 1 \otimes 1 + \sum_{\alpha=1}^{\infty} h^{\alpha} \bar{f}^{\alpha} \otimes \bar{f}_{\alpha}, \quad (9)$$

$f^{\alpha}, f_{\alpha}, \bar{f}^{\alpha}, \bar{f}_{\alpha} \in \mathcal{U}_{\mathfrak{g}}$; h is deformation parameter, and classical r -matrix $r \in \mathfrak{g} \wedge \mathfrak{g}$ satisfying classical Yang–Baxter equation and universal (quantum) r -matrix \mathcal{R} :

$$\mathcal{R} = \mathcal{F}^{21} \mathcal{F}^{-1} = 1 + hr \mod (h^2) \quad (10)$$

satisfying quantum Yang–Baxter equation. Classical r -matrices classify non-equivalent deformations. We distinguish between two types of the classical r -matrix:

1) Abelian one, if it has the form

$$r_A = \sum_{i=1}^n a_i \wedge b_i \quad (11)$$

where all elements a_i, b_i ($i = 1, \dots, n$) commute among themselves. The corresponding twist is given as follows

$$F_{r_A} = \exp\left(\frac{1}{2} r_A\right). \quad (12)$$

2) Another situation appears when the classical r -matrix $r = r_{J_n}(h)$ has the form

$$r_{J_n} = h \left(\sum_{\nu=0}^n b_{\nu} \wedge a_{\nu} \right), \quad (13)$$

where the elements a_{ν}, b_{ν} ($\nu = 0, 1, \dots, n$) satisfy the relations

$$\begin{aligned} [a_0, b_0] &= b_0, & [a_0, a_i] &= (1 - t_i) a_i, & [a_0, b_i] &= t_i b_i, \\ [a_i, b_j] &= \delta_{ij} b_0, & [a_i, a_j] &= [b_i, b_j] = 0, & [b_0, a_j] &= [b_0, b_j] = 0, \end{aligned} \quad (14)$$

($i, j = 1, \dots, n$), ($t_i \in \mathbb{C}$). Such an r -matrix is called of Jordanian type. The corresponding twist is given as follows (see e.g. [10])

$$F_{r_{J_n}} = \exp\left(h \sum_{i=1}^n a_i \otimes b_i e^{-t_i \sigma}\right) \exp(a_0 \otimes \sigma), \quad (15)$$

where $\sigma := \ln(1 + hb_0)$.

One can notice that the zero component $r_{J_0}(h) := hb_0 \wedge a_0$ in (13) is the classical Jordanian r -matrix and the corresponding Jordanian twist is given by the formula (15) for $n = 0$, i.e. $F_{r_{J_0}} = \exp(a_0 \otimes \sigma)$ (and this shorter form will be used later on).

2. Quantum symmetries

The Poincaré symmetry is the full symmetry of special relativity and it includes translations, rotations and boosts. Algebraically it is described by the Poincaré Lie algebra (Lorentz generators $M_{\mu\nu}$ and momenta P_μ), usually denoted as $\text{iso}(1, 3)$, and defined by the following commutation relations

$$[M_{\mu\nu}, M_{\lambda\rho}] = i(M_{\mu\lambda}\eta_{\nu\rho} - M_{\mu\rho}\eta_{\nu\lambda} - M_{\nu\lambda}\eta_{\mu\rho} + M_{\nu\rho}\eta_{\mu\lambda}), \quad (16)$$

$$[M_{\mu\nu}, P_\lambda] = i(P_\nu\eta_{\mu\lambda} - P_\mu\eta_{\nu\lambda}), \quad (17)$$

$$[P_\mu, P_\nu] = 0, \quad (18)$$

where $\eta_{\mu\nu}$ is the metric tensor with Lorentzian signature. As it is known (and shown above) any Lie algebra provides an example of undeformed Hopf algebra. First one has to extend it to universal enveloping algebra and then equip it in comultiplication, counit and antipode maps. Therefore, the universal enveloping algebra of the Poincaré Lie algebra $\mathcal{U}_{\text{iso}(1,3)}$ together with

$$\Delta_0 M_{\mu\nu} = M_{\mu\nu} \otimes 1 + 1 \otimes M_{\mu\nu} \quad \text{and} \quad \Delta_0 P_\mu = P_\mu \otimes 1 + 1 \otimes P_\mu \quad (19)$$

$$S(M_{\mu\nu}) = -M_{\mu\nu}; \quad S(P_\mu) = -P_\mu; \quad \epsilon(M_{\mu\nu}) = \epsilon(P_\mu) = 0 \quad (20)$$

defined on the generators and then extended to the whole $\mathcal{U}_{\text{iso}(1,3)}$ constitutes the undeformed Poincaré Hopf algebra. Such Quantum Group can be further deformed.

First deformations of Poincaré symmetry appeared in the early 90's. The simplest is the so-called θ -deformation corresponding to the Moyal-Weyl noncommutative spacetime (1) and it can be obtained by twist deformation:

$$\mathcal{F} = \exp\left(-\frac{i}{2}h\theta^{\mu\nu}P_\mu \otimes P_\nu\right); \quad \mathcal{F}^{-1} = \exp\left(\frac{i}{2}h\theta^{\mu\nu}P_\mu \otimes P_\nu\right) \quad (21)$$

where $\theta^{\mu\nu}$ is a constant antisymmetric matrix. We say that this twist has support in Poincaré algebra, i.e. $\mathcal{F} \in \mathcal{U}_{\text{iso}(1,3)} \otimes \mathcal{U}_{\text{iso}(1,3)}$. After twisting the Hopf algebra structure becomes twisted Poincaré algebra:

$$\Delta_\theta(P_\mu) = \Delta_0(P_\mu); \quad \Delta_\theta(M_{\mu\nu}) = \Delta_0(M_{\mu\nu}) - h(P\theta)_\mu \wedge P_\nu + h(P\theta)_\nu \wedge P_\mu \quad (22)$$

where $(P\theta)_\mu = P_\rho\theta^{\rho\lambda}\eta_{\lambda\mu}$. One can easily notice that for the deformation parameter $h \rightarrow 0$ the coalgebra becomes undeformed (19) as classical limit should provide. It has been shown (e.g. see [3]) that the θ -spacetime \mathcal{A}^θ (1) stays covariant under θ -deformed version of the Poincaré quantum group. The module algebra \mathcal{A} is also deformed accordingly into $\mathcal{A}^\mathcal{F} = \mathcal{A}^\theta$ (the module action stays the same). In \mathcal{A}^θ we have new (\star) multiplication (7) which leads to the algebra described by (1).

The other type deformation (not coming from twist) is provided by the so-called κ -deformation of Poincaré algebra [11, 12, 13] which originally was obtained by contraction procedure from q -deformed $SO_q(3, 2)$. In the deformed case the κ -Poincaré $\mathcal{U}_{\text{iso}(1,3)}^\kappa$ algebra consists of (16) - (18) as in undeformed case but coalgebra structure is no longer primitive and has the form:

$$\Delta_\kappa(M_i) = \Delta_0(M_i) = M_i \otimes 1 + 1 \otimes M_i, \quad \Delta_\kappa(N_i) = N_i \otimes 1 + \Pi_0^{-1} \otimes N_i - \frac{1}{\kappa}\epsilon_{ijm}P_j\Pi_0^{-1} \otimes M_m \quad (23)$$

$$\Delta_\kappa(P_i) = P_i \otimes \Pi_0 + 1 \otimes P_i, \quad \Delta_\kappa(P_0) = P_0 \otimes \Pi_0 + \Pi_0^{-1} \otimes P_0 + \frac{1}{\kappa}P_m\Pi_0^{-1} \otimes P^m, \quad (24)$$

$$S_\kappa(M_i) = -M_i, \quad S_\kappa(N_i) = -\Pi_0 N_i - \frac{1}{\kappa}\epsilon_{ijm}P_j M_m, \quad S_\kappa(P_i) = -P_i \Pi_0^{-1} \quad (25)$$

The counit ϵ is undeformed, i.e., $\epsilon(A) = 0$ for $A \in (M_i, N_i, P_i)$. We have used the notation:

$$M_{0i} = iN_i; \quad M_{ij} = \epsilon_{ijk}M_k; \quad \Pi_0 = \left(\frac{P_0}{\kappa} + \sqrt{1 - \frac{P_\mu P^\mu}{\kappa^2}}\right).$$

Such quantum group constitutes deformed symmetry for κ -Minkowski spacetime defined by the commutation relations (3). The above Hopf algebra is given in the so-called classical basis [14], but there exist other bases in which κ -Poincaré algebra looks different and even the commutation relations are modified (are no longer of (16) - (18) form). For the so-called bicrossproduct basis see e.g. [13] or for the so-called standard basis see e.g. [11]. Depending on the purpose of application one can use different forms of this quantum group, for example in κ -deformed quantum field theories (see e.g. [15]), doubly special relativity (see e.g. [16]) or in the most recent relative-locality effect (see e.g. [17]) the bicrossproduct basis form was used the most extensively. κ -Minkowski spacetime defined by the relations (3) constitutes a Hopf module algebra over κ -Poincaré algebra defined by (16) - (18), (23) - (25).

The κ -deformation of Poincaré algebra is characterised by the inhomogeneous classical YB equation, which implies that one should not expect to get κ -Minkowski space from a Poincaré twist. However, twists belonging to extensions of the Poincaré algebra are not excluded, e.g. one can consider the twists with the support in \mathcal{U}_{igl} [18, 19] or in the Weyl-Poincaré algebra [20, 21], the minimal one-generator extension of the Poincaré algebra which includes dilatation generator. For those twists we also obtain κ -Minkowski spacetime (3) however no longer as κ -Poincaré module algebras, i.e. with different Hopf algebra as quantum symmetry.

3. Realizations and representations

Another commonly used feature in the noncommutative spacetimes approach is the realization for the noncommutative coordinates. One can express noncommutative coordinates as operators acting on algebra of functions in analogy to Quantum Mechanics when observables of position and momenta become operators on a Hilbert space. The Heisenberg algebra (i.e. quantum mechanical phase space) H can be defined as a free algebra of n coordinate generators and n generators of momenta, satisfying the following relations:

$$[x_\mu, x_\nu] = 0; \quad [P_\mu, x_\nu] = -i\eta_{\mu\nu} \cdot 1; \quad [P_\mu, P_\nu] = 0; \quad (26)$$

where $\eta_{\mu\nu} = (-, +, +, +)$ is diagonal metric tensor with Lorentzian signature.

Now for example let's consider $\mathfrak{g} = \text{igl}(n, \mathbb{R}) = \mathfrak{gl}(n, \mathbb{R}) \oplus t^n$ of the inhomogeneous general linear group as a semidirect product of $\mathfrak{gl}(n, \mathbb{R})$ with translations t^n . We choose a basis $L_\nu^\mu, P_\mu \in \text{igl}(n, \mathbb{R})$ with the following standard set of commutation relations:

$$[L_\nu^\mu, L_\lambda^\rho] = -i\delta_\nu^\rho L_\lambda^\mu + i\delta_\lambda^\mu L_\nu^\rho, \quad [L_\nu^\mu, P_\lambda] = i\delta_\lambda^\mu P_\nu \quad (27)$$

$\mu, \nu, \dots = 0, \dots, n-1$ $\text{igl}(n, \mathbb{R})$ as Lie algebra acting on \mathcal{A} (algebra of (complex-valued) functions on the spacetime manifold $\mathbb{M} = \mathbb{R}^n$) via first-order differential operators (derivations \equiv vectors fields). It has the following realization: $L_\mu^\nu = x^\nu P_\mu$ ¹. The action of differential operators on functions induced by derivations (vector fields) remains the same in deformed and nondeformed cases.

Therefore by defining the map $L_\mu^\nu \rightarrow x^\nu P_\mu$ which is a Lie algebra isomorphism we obtain "Heisenberg algebra realization" of $\mathcal{U}_{\text{igl}(n)}$. Moreover the Heisenberg realization described above induces Heisenberg representation via $P_\mu = -i\partial_\mu = i\frac{\partial}{\partial x^\mu}$; $L_\mu^\nu = -ix^\nu \partial_\mu$ acting in the vector space $C^\infty(\mathbb{R}^n)$ ².

In noncommutative spacetimes approach the noncommutative coordinates can be introduced as operators acting on algebra of functions as well. It comes from the fact that the twisted star product

¹ We will use the same letter to denote an abstract element in $\text{igl}(n)$ and its (first-order) differential operator realization in \mathcal{XIM} (i.e. assuming the embedding of the $\mathcal{U}_{\text{igl}(n)} \hookrightarrow \mathcal{U}_{\mathcal{XIM}}$).

² The best known representations are given on the space of (smooth) functions on \mathbb{R}^n in terms of multiplication and differentiation operators. For this reason one can identify Heisenberg algebra with an algebra of linear differential operators on \mathbb{R}^n with polynomial coefficients.

enables us to 'realize' the algebra $\mathcal{A}^{\mathcal{F}}$ in terms of (formal) differential operators on a manifold M , i.e. in algebra of vector fields $\mathcal{X}M$. Operator realizations are naturally defined by $\hat{x} \in \mathcal{U}_{\mathcal{X}M}$:

$$\hat{x}^{\mu} = x^{\mu} + \sum_{m=1}^{\infty} h^m (\bar{f}^{\alpha} \triangleright x^{\mu}) \cdot \bar{f}_{\alpha} \quad (28)$$

which can be written also as $\hat{x}^{\mu}(g) = x^{\mu} \star g$, $g \in \mathcal{A}$.

Summarising, one can say that we deal with the realization of given algebra in space of linear operators over \mathfrak{h} , i.e. $\mathcal{L}(\mathfrak{h})$ (where \mathfrak{h} is the Hilbert space) when we determine homomorphism from one algebra to another. Hence the name, e.g. "Heisenberg algebra realization". When linear space is determined we speak about representations (for more detailed analysis and examples, see e.g. [22]).

For example in θ -deformation (corresponding to θ -twist (21)) the (Heisenberg) realization for coordinates is the following:

$$\hat{x}^{\mu} = x^{\mu} - \frac{1}{2} h \theta^{\mu\nu} P_{\nu} \quad (29)$$

One can easily check that such coordinates x^{μ} satisfy commutation relations (1) by using only (26).

The noncommutative κ -Minkowski space (3) also can be realized in terms of generalized differential operators. In fact there exist a huge amount of such Heisenberg realizations of the κ -Minkowski algebra. Particularly important is the so-called non-covariant family of realizations [19, 23, 24]:

$$\hat{x}^i = x^i \phi(A), \quad \hat{x}^0 = x^0 \psi(A) + i h x^k \partial_k \gamma(A) \quad (30)$$

where $A = i h \partial_0$. Functions ϕ, ψ , and γ are taken to be real analytic obeying initial conditions $\phi(0) = 1$ and $\psi(0) = 1$, and $\gamma(0)$ has to be finite in order to ensure a proper classical limit at $h \mapsto 0$. The κ -Minkowski commutation relations [cf. (3)] are equivalent to the property that functions ϕ, ψ , and γ do satisfy the equation [23]:

$$\gamma = 1 + (\ln \phi)' \psi \quad (31)$$

where $\phi' \equiv \frac{d\phi}{dA}$. There are few examples of twists providing κ -Minkowski spacetime (3) with support in $\mathcal{F} \in \mathcal{U}_{\text{igl}} \otimes \mathcal{U}_{\text{igl}}$ for which the realization of noncommutative coordinates fall into the non-covariant class (see e.g. [18, 19, 20]).

The simplest twist is of Abelian type $(\mathcal{F}_s^A)^{-1} = \exp[-i h (s P_0 \otimes D - (1-s) D \otimes P_0)]$ (with s being a numerical parameter labelling different twisting tensors) which provides the following realizations for noncommutative coordinates:

$$\hat{x}_s^i = x^i e^{(1-s)hP_0}, \quad \hat{x}_s^0 = x^0 - h s x^k P_k \quad (32)$$

where $D = x^k P_k$ in Heisenberg realization. All twists \mathcal{F}_s^A (for any s) correspond to the same classical r -matrix: $r_A = D \wedge P_0$ and they have the same universal quantum r -matrix which is of exponential form: $\mathcal{R} = (\mathcal{F}_s^A)^{21} (\mathcal{F}_s^A)^{-1} = e^{i D \wedge P_0}$. The deformed symmetry algebra corresponding to this twist can be found in [18, 20, 22]. Some interesting applications for symmetrized version of this twist (for the choice of parameter $s = \frac{1}{2}$) were considered e.g. in [25].

Another example of κ -space realizations falling into non-covariant class (30) come from Jordanian twist: $(\mathcal{F}_r^J)^{-1} = \exp(-J_r \otimes \sigma_r)$, and have the form:

$$\hat{x}_r^i = x^i (1 - r h P_0)^{-\frac{1}{r}} \quad \text{and} \quad \hat{x}_r^0 = x^0 (1 - r h P_0). \quad (33)$$

where $J_r = i(\frac{1}{r} x^k P_k - x^0 P_0)$ (in Heisenberg realization) with a numerical factor $r \neq 0$ labelling different twists and $\sigma_r = \ln(1 - h r P_0)$ (cf. the remark in point 2) in Introduction). Direct calculations show that,

regardless of the value of r , twisted commutation relations (8) take the form of that for κ -Minkowski spacetime (3). The corresponding classical r -matrices are the following:

$$r_J = J_r \wedge P_0 = \frac{1}{r} D \wedge P_0 - L_0^0 \wedge P_0. \quad (34)$$

Again we refer the reader for the full form of deformed Hopf algebra to [20, 22] as quantum symmetry in this case. It is worth to mention that for the choice of $r = -1$ we can reduce the symmetry group from the whole \mathcal{U}_{igl} to the the minimal one-generator extension of the Poincare algebra which includes dilatation generator, i.e. Poincare-Weyl algebra [20, 22].

Interestingly, for the minimal extension, i.e. Poincare-Weyl algebra there exists also symmetric version of above Jordanian twist (see e.g. [26, 27, 28]) which can be written in compact way as [28]:

$$\mathcal{F}^{-1} = \sum_{m=0}^{\infty} \frac{1}{m!} \left(-\frac{h}{2}\right)^m \sum_{k=0}^m (-1)^k \binom{m}{k} P_0^{m-k} J^{<k>} \otimes P_0^k J^{<m-k>}, \quad (35)$$

where $J^{<0>} = 1, J^{<1>} = J, J^{<2>} = J(J+1), \dots$, and $J = x^\mu P_\mu$ (in realization) and still $[J, P_0] = P_0$. It provides slightly modified realization for noncommutative coordinates with respect to non-covariant family (30):

$$\hat{x}^i = x^i \phi(A); \quad \hat{x}^0 = x^0 \psi(A) + i h x^k \partial_k \gamma(A) + \xi(A) \quad (36)$$

where $\phi = (1 - \frac{1}{2}A)$, $\psi = (1 - \frac{1}{4}A^2)$, $\gamma = \frac{1}{2}(1 - \frac{1}{2}A)$ and $\xi = \frac{i h}{4}A$. In general $\xi(A)$ could be any function satisfying $\xi(0) = 0$ (to provide the classical limit) and still the equation (31) will be satisfied. Which means that this twist via start product (8) also provides κ -Minkowski algebra \mathcal{A}^κ with the commutation relations (3).

4. Conclusions

The structure of spacetime, at the scale where quantum gravity effects take place, is one of the most important questions in fundamental physics. Below the quantum gravity scale the symmetry of spacetime should also be deformed. Noncommutative spacetimes as one of the approaches to the description of Planck scale physics allow to describe the geometry from the quantum mechanical point of view. At the Planck scale the idea of size or distance in classical terms is not valid any more, because one has to take into account quantum uncertainty which naturally arise via noncommutative coordinates. Moreover their realizations as differential operators play the role of observables.

The aim of the review was to introduce in a compact way the variety of noncommutative spacetimes and their deformed relativistic symmetries which have been widely used by many authors in the view of applications in physical theories like quantum field theory or gravity. The deformation of spacetime requires a generalisation of its symmetry group and one deals with the θ - or κ -Poincare Quantum Group as well as with twist-deformations of inhomogeneous linear group.

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