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Quantum Physical Origin of Lorentz Transformations

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Abstract. Our aim is to derive the symmetries of the space-time, i.e. the Lorentz transformations, from *discrete* symmetries of the interactions between the most fundamental constituents of matter, in particular quarks and leptons. The role of Pauli's exclusion principle in the derivation of the $SL(2, \mathbb{C})$ symmetry is put forward as the source of the macroscopically observed Lorentz symmetry. Then Pauli's principle is generalized for the case of the Z_3 grading replacing the usual Z_2 grading, leading to ternary commutation relations for quantum operator algebras. In the case of lowest dimension, with two generators only, it is shown how the cubic combinations Z_3 -graded elements behave like Lorentz spinors, and the binary product of elements of this algebra with an element of the conjugate algebra behave like Lorentz vectors.

1. Introduction

Since the advent of quantum physics, great care was taken to demonstrate that all quantum phenomena, once averaged over great number of items, lead to the well known classical limits. The existence of such a limit was considered as one of the cornerstones of proper formulation of quantum mechanics, and Bohr made the “correspondence principle” a central point in his construction of quantum mechanics' basic framework [1].

Despite countless attempts, either using the ideas of “hidden parameters”, or along the lines of the so-called “geometric quantization”, or probabilistic Brownian motion models, a convincing derivation of quantum physics from classical models, no matter how sophisticated, was never found. Therefore it can be stated now without any doubt left, that quantum physics is primordial with respect to other observable phenomena perceived by us on the classical level.

Seen from this angle, the idea to derive the geometric properties of space-time, and perhaps its very existence, from fundamental symmetries and interactions proper to matter's most fundamental building blocks seems quite natural. Many of those properties do not require any mention of space and time on the quantum mechanical level, as was demonstrated by Born and Heisenberg [2] in their version of matrix mechanics, or by von Neumann's formulation of quantum theory in terms of the C^* algebras [3]. The non-commutative geometry is another example of formulation of space-time relationships in purely algebraic terms [4].

In what follows, we shall choose the point of view according to which the space-time relations are a consequence of fundamental *discrete symmetries* which characterize the behavior of matter on the quantum level. In other words, the Lorentz symmetry observed on the macroscopic level, acting on what we perceive as space-time variables, is an averaged version of the symmetry group acting in the Hilbert space of quantum states of fundamental particle systems.



Quantum Mechanics started as a non-relativistic theory, [2] but very soon its relativistic generalization was created. As an immediate result, the wave functions in the Schrödinger picture were required to belong to one of the linear representations of the Lorentz group, which means that they must satisfy the following *covariance principle*:

$$\tilde{\psi}(\tilde{x}) = \tilde{\psi}(A(x)) = S(A)\psi(x). \quad (1)$$

The nature of the representation $S(\Lambda)$ determines the character of the field considered: spinorial, vectorial, tensorial, etc.... As in many other fundamental relations, the seemingly simple equation

$$\tilde{\psi}(\tilde{x}) = \tilde{\psi}(A(x)) = S(A)\psi(x) \quad (2)$$

creates a bridge between two totally different realms: the space-time accessible via classical macroscopic observations, and the Hilbert space of quantum states. It can be interpreted in two opposite ways, depending on which side we consider as the cause, and which one as the consequence. In other words, is the macroscopically observed Lorentz symmetry imposed on the micro-world of quantum physics, or maybe it is already present as symmetry of quantum states, and then implemented and extended to the macroscopic world in classical limit? In such a case, the covariance principle should be written as follows:

$$j^{\mu'}(S(\psi)) = j^{\mu'}(\psi') = \Lambda_{\mu}^{\mu'}(S)j^{\mu}(\psi). \quad (3)$$

In the above formula $j^{\mu} = \bar{\psi}\gamma^{\mu}\psi$ is the Dirac current, ψ is the electron wave function.

In view of the analysis of the causal chain, it seems more appropriate to write the same transformations with Λ depending on S :

$$\psi'(x^{\mu'}) = S\psi(x^{\nu}) = \psi'(\Lambda_{\nu}^{\mu'}(S)x^{\nu}), \quad (4)$$

$$x^{\mu'}(S\psi, \bar{\psi}S) = \Lambda_{\mu}^{\mu'}(S)x^{\mu}(\psi, \bar{\psi}). \quad (5)$$

This form of the same relation suggests that the transition from one quantum state to another, represented by the unitary transformation S is the primary cause implying the transformation of observed quantities such as the electric 4-current, and as a consequence, the apparent transformations of time and space intervals measured with classical physical devices.

Although mathematically the two formulations are equivalent, it seems more plausible that the Lorentz group resulting from the averaging of the action of the $SL(2, \mathbb{C})$ in the Hilbert space of states contains less information than the original double-valued representation which is a consequence of the particle-anti-particle symmetry, than the other way round.

In what follows, we shall draw physical consequences from this approach, concerning the strong interactions in the first place. But before considering these, which describe the forces conveyed by gluons and acting among quarks constituting hadrons, let us first see how the Lorentz group appears through the $SL(2, \mathbb{C})$ group action on fermions, in particular, on the electron states.

2. Pauli's exclusion principle

The Pauli exclusion principle, according to which two electrons cannot be in the same state with identical quantum numbers, is one of the most important foundations of quantum physics [5]. Not only does it explain the structure of atoms and the periodic table of elements, but it also guarantees the stability of matter preventing its collapse as suggested by Ehrenfest, and proved later by Dyson [8-9]. The link between the exclusion principle and particle's spin, known as the "spin-and-statistic theorem", is one of the deepest results in quantum field theory.

In the case of the electron, it was discovered that this smallest and indivisible carrier of electric charge is also endowed with a magnetic moment and the intrinsic angular momentum, called "the spin", which can take on exclusively two values. This explains the structure of electronic shells in atoms, where for a given main quantum number n there exist $2n^2$ different states. The n^2 states come from the possibility for the electron with energy state given by n to take on different *magnetic number*

states, labeled by the magnetic number m taking values from $-l$ to $+l$, including 0. Here l is the *azimutal* number, varying from $-n$ to n . Then, for a given choice of n , l and m , there are still two possibilities for the electron spin, which can be found in two states, “up” and “down”. The result is well known: the first electronic shell corresponds to the spherically symmetric s -state, with $n=0, l=0, m=0$, so that we have no more than two electrons in the lowest energy shell, with opposite spins. The next shell corresponds to $n=1, l=-1, 0, +1$ and $m=\pm 1, 0$, i.e. three magnetic states. With two possibilities for spin we get the number of states in this subshell equal to 6, and altogether, with the previous shell, 8. The total number of states in all shells will be $2n^2$.

In purely algebraical terms Pauli's exclusion principle amounts to the anti-symmetry of wave functions describing two coexisting particle states. The easiest way to see how the principle works is to apply Dirac's formalism in which wave functions of particles in given state are obtained as products between the “bra” and “ket” vectors [6].

The most important statement concerning the spin of the electron is that two electrons cannot occupy the same state, with all quantum numbers equal including spin. If all quantum numbers except the spin are equal for two electrons, their spins must be opposed. In one word, if the two opposite spin states are created by

$$a_1|0\rangle=|1\rangle, a_2|0\rangle=|2\rangle. \quad (6)$$

Both creation operators are supposed to be nilpotent,

$$a_1 a_1|0\rangle=|1,1\rangle=0, a_2 a_2|0\rangle=|2,2\rangle=0, \quad (7)$$

thus making impossible coexistence of two electrons with the same state of spin.

Let us form an arbitrary linear superposition of two mutually exclusive states.

$$|u\rangle=\lambda|1\rangle+\mu|2\rangle \quad (8)$$

Such a state is created by the corresponding linear combination of creation operators

$$u=\lambda a_1+\mu a_2. \quad (9)$$

This operator, creating a possible state of an electron, must be also nilpotent, so that

$$u^2|0\rangle=u|u\rangle=|u,u\rangle=0. \quad (10)$$

Writing explicitly the square of the linear combination $u=\lambda a_1+\mu a_2$ we get

$$u^2=(\lambda a_1+\mu a_2)^2=\lambda^2 a_1^2+\lambda\mu a_1 a_2+\mu\lambda a_2 a_1+\mu^2 a_2^2.$$

Imposing the nilpotency condition on the operator u makes the above combination vanish. As both operators a_1 and a_2 were supposed to be nilpotent, what remains is

$$\lambda\mu(a_1 a_2+a_2 a_1)=0, \rightarrow a_1 a_2=-a_2 a_1 \quad (11)$$

(the numbers μ and λ commute, so that $\mu\lambda=\lambda\mu$).

Let us introduce a contravariant tensor

$$\varepsilon^\alpha{}^\beta, \alpha, \beta=1, 2. \quad (12)$$

Contracting with two creation operators, we can write:

$$\varepsilon^{\alpha\beta} a_\alpha a_\beta = \frac{1}{2} [\varepsilon^{\alpha\beta} a_\alpha a_\beta + \varepsilon^{\beta\alpha} a_\beta a_\alpha].$$

But $a_\alpha a_\beta = -a_\beta a_\alpha$, therefore

$$\varepsilon^{\alpha\beta} a_\alpha a_\beta = \frac{1}{2} [\varepsilon^{\alpha\beta} - \varepsilon^{\beta\alpha}] a_\alpha a_\beta,$$

from which follows, taking into account that $a_\alpha a_\beta$ are linearly independent,

$$\varepsilon^{\alpha\beta} = -\varepsilon^{\beta\alpha} \rightarrow \varepsilon^{11} = \varepsilon^{22} = 0, \quad \varepsilon^{12} = -\varepsilon^{21}. \quad (13)$$

There exists a natural *dual algebra* of contravariant entities ε^α , $\alpha, \beta=1, 2$ satisfying the anti-commutation relations

$$\zeta^{\alpha} \zeta^{\beta} = -\zeta^{\beta} \zeta^{\alpha}$$

with a corresponding two-form

$$\varepsilon_{\alpha\beta} = -\varepsilon_{\beta\alpha}.$$

The 2-form $\varepsilon_{\alpha\beta}$ can be considered as *inverse* to $\varepsilon^{\alpha\beta}$ because one has:

$$\varepsilon_{\alpha\beta} \varepsilon^{\gamma\beta} = \delta_{\alpha}^{\gamma}. \quad (14)$$

Note the order of indices being contracted: with a different choice, we would get the identity matrix with the minus sign on the right.

Now, if we require that Pauli's principle must apply independently of the choice of a basis in Hilbert space, i.e. that after a linear transformation we get

$$\varepsilon^{\alpha'\beta'} = S_{\alpha}^{\alpha'} S_{\beta}^{\beta'} \varepsilon^{\alpha\beta} = \det S = 1, \quad -\varepsilon^{\alpha'\beta'}, \quad \varepsilon^{1'2'} = 1, \quad (15)$$

then the 2×2 complex matrix $S_{\alpha}^{\alpha'}$ must have the determinant equal to 1, which defines the $SL(2, \mathbb{C})$ group.

3. From Pauli to Dirac

The existence of two internal degrees of freedom had to be taken into account in fundamental equation defining the relationship between basic operators acting on electron states. At the time when Pauli proposed the simplest equation expressing the relation between the energy, momentum and spin:

$$E\psi = mc^2\psi + c\sigma \cdot \mathbf{p}\psi, \quad (16)$$

the relativistic invariance of electromagnetic interactions was firmly established. However, the linear equation (16) is not invariant under the Lorentz group: once squared, it yields the wrong relation between the energy-momentum E , $\mathbf{p}$ and the mass m :

$$E^2 = m^2 c^4 + 2mc^3 \sigma \cdot \mathbf{p} + \mathbf{p}^2.$$

At this point Pauli could have restored the Lorentz invariance by doubling the number of components of the wave function, and by introducing states with negative mass:

$$\begin{aligned} E\psi_+ &= mc^2\psi_+ + c\sigma \cdot \mathbf{p}\psi_+, \\ E\psi_- &= -mc^2\psi_- + c\sigma \cdot \mathbf{p}\psi_-. \end{aligned} \quad (17)$$

It is easy to check that now each of the components satisfies the relativistic equation

$$E^2 = m^2 c^4 + c^2 \mathbf{p}^2 \quad \text{or} \quad \left(\frac{E}{c}\right)^2 - \mathbf{p}^2 = p_{\mu} p^{\mu} = m^2 c^2. \quad (18)$$

The existence of *anti-particles* (in this case the positron), suggests the use of the non-equivalent representation of $SL(2, \mathbb{C})$ group by means of complex conjugate matrices. Along with the time reversal, the Dirac equation can be now constructed. It is invariant under the Lorentz group.

$$\begin{aligned} i\hbar \frac{\partial}{\partial t} \psi_+ - mc^2 \psi_+ &= i\hbar \sigma \cdot \nabla \psi_+, \\ -i\hbar \frac{\partial}{\partial t} \psi_- - mc^2 \psi_- &= -i\hbar \sigma \cdot \nabla \psi_+. \end{aligned} \quad (19)$$

The two coupled Pauli equations, with two masses, positive and negative, can be represented as follows:

$$E[I_2 \otimes I_2]\psi = mc^2[\sigma_3 \otimes I_2]\psi + c[\sigma_1 \otimes \sigma \cdot \mathbf{p}]\psi. \quad (20)$$

Here I_2 stands for the 2×2 identity matrix, and the two Pauli matrices are

$$\sigma_3 = \begin{pmatrix} 1 & 0 \\ 0 & -1 \end{pmatrix}, \quad \sigma_1 = \begin{pmatrix} 0 & 1 \\ 1 & 0 \end{pmatrix}$$

and $\psi = \begin{pmatrix} \psi_+ \\ \psi_- \end{pmatrix}$, so that ψ is a column formed with two two-dimensional Pauli spinors.

After multiplying the entire equation from the left by the matrix $\sigma_3 \otimes I_2$, and moving the momentum term to the left, we get the Dirac equation:

$$E[\sigma_3 \otimes I_2]\psi - c[i\sigma_2 \otimes \sigma \cdot \mathbf{p}]\psi = mc^2 I_2 \otimes I_2 \psi,$$

which can be written as

$$\gamma^\mu p_\mu \psi = mc^2 \psi$$

with

$$\gamma^0 = \sigma_3 \otimes I_2, \quad \gamma^k = -i\sigma_2 \otimes \sigma^k.$$

The price to be paid for the recovery of relativistic invariance was the introduction of negative mass, which Pauli was not ready to accept at that time. A few years later Dirac deduced the same system of equations for the electron introducing the operator acting on a four-component spinors as a “square root” of the d’Alembertian operator. The states with negative mass were interpreted as the “Dirac sea”, and the holes in that sea were interpreted as positrons - electrons' antiparticles.

The transformation properties of γ -matrices can be now written as follows [10]:

$$S\gamma^\mu S^{-1} = A^\mu_{\nu'}(S^{-1})\gamma^{\nu'}, \quad (21)$$

where $A^\mu_{\nu'}$ is the Lorentz transformation acting on space-time 4-vectors, while S is a bi-spinor representation of the Lorentz group, composed of two inequivalent matrices of $SL(2, \mathbb{C})$ group.

The apparent Z_2 symmetry ensuring the relativistic invariance of two coupled Pauli equations echoes a similar situation in the description of classical electromagnetic field. The Maxwell equations can be viewed as a coupled system of linear first-order equations for the components of electric and magnetic fields, $\mathbf{E}$ and $\mathbf{B}$

$$\frac{1}{c} \frac{\partial \mathbf{E}}{\partial t} = \nabla \wedge \mathbf{B}, \quad -\frac{1}{c} \frac{\partial \mathbf{B}}{\partial t} = \nabla \wedge \mathbf{E}.$$

These equations can be decoupled by applying the time derivation twice, which in vacuum, where $\text{div } \mathbf{E} = 0$ and $\text{div } \mathbf{B} = 0$ leads to the d’Alembert equation for both components separately:

$$\frac{1}{c^2} \frac{\partial^2 \mathbf{E}}{\partial t^2} - \nabla^2 \mathbf{E} = 0, \quad \frac{1}{c^2} \frac{\partial^2 \mathbf{B}}{\partial t^2} - \nabla^2 \mathbf{B} = 0.$$

Nevertheless, neither of the components of the Maxwell tensor, be it $\mathbf{E}$ or $\mathbf{B}$, can propagate separately alone; only their combination can travel along the same direction of the common wave vector.

The Dirac equation for the electron displays a similar Z_2 symmetry, with two coupled equations which can be put in the following form:

$$\begin{aligned} i\hbar \frac{\partial}{\partial t} \psi_+ - mc^2 \psi_+ &= i\hbar \sigma \cdot \nabla \psi_-, \\ -i\hbar \frac{\partial}{\partial t} \psi_- - mc^2 \psi_- &= -i\hbar \sigma \cdot \nabla \psi_+. \end{aligned} \quad (22)$$

where ψ_+ and ψ_- are the positive and negative energy components of the Dirac equation; this is visible even better in the momentum representation:

$$\begin{aligned} [E - mc^2] \psi_+ &= c\sigma \cdot \mathbf{p} \psi_-, \\ [-E - mc^2] \psi_- &= -c\sigma \cdot \mathbf{p} \psi_+. \end{aligned} \quad (23)$$

The same effect (negative energy states) can be obtained by changing the direction of time, and putting the minus sign in front of the time derivative, as suggested by Feynman.

Each of the components satisfies the Klein-Gordon equation, obtained by successive application of the two operators and diagonalization:

$$\left[\frac{1}{c^2} \frac{\partial^2}{\partial t^2} - \nabla^2 - m^2 \right] \psi_{\pm} = 0.$$

As in the electromagnetic case, neither of the components of this complex entity can propagate by itself; only all the components can.

4. Ternary exclusion principle

Our next goal is to find how the Lorentz invariance can be derived from quantum symmetries of strong interactions. According to the present knowledge, the hadrons, behaving like fermions, are in fact composite particles. The true elementary particles carrying both baryonic and fractional electric charges are quarks. The experimental data obtained via deep inelastic scattering reveal that quarks behave like almost point-like objects as compared with proton's or neutron's size: in fact, their dimension is about 1000 times smaller than that of a hadron - about the same ratio as the dimension of atomic nuclei compared with dimension of an atom [11-13].

It seems that at this scale the usual space-time relations are hard to be implemented from outside, especially taking into account that due to the strength of interactions the uncertainty principle should be at work, thus making it even harder to speak of distances and time delays. On the other hand, strict selection rules are apparently at work, because only three-quark states lead to observable fermionic configurations, and quark-antiquark states produce strongly interacting π -mesons. There are two fundamental quark states, called u and d ; the u quark is endowed with fractional electric charge $+2/3$, while the d quark has the charge equal to $-1/3$. The anti-quarks $\bar{u}$ and $\bar{d}$ gave the same charges with opposite signs, $-2/3$ for $\bar{u}$ and $+1/3$ for $\bar{d}$. Thus, a proton is constructed as a product state uud , and neutron corresponds to the product state udd . The three π -mesons are generated by the quark-antiquark pairs according to the scheme

$$\pi^+ \sim u\bar{d}, \quad \pi^- \sim d\bar{u}, \quad \pi^0 \sim \frac{1}{\sqrt{2}}(u\bar{u} - d\bar{d}).$$

Besides, there is another important super-selection rule: three colors are needed to combine three quarks into a hadron, anti-colors for an anti-hadron, and colorless combinations of colors and anti-color are needed to form a meson [14-15].

Although in Quantum Chromodynamics quarks are treated as fermions, there is no direct proof of such assertions, because quarks are never observed as freely propagating particles outside the hadrons where they interact by means of gluon exchange. The fact that *three* quarks combine into stable states, with *two* identical states coexisting, but not three, strongly suggests that a ternary analogue of Pauli's principle may be involved, based on the representations of the cyclic group Z_3 instead of the cyclic group Z_2 . We shall show how certain unusual representations of the $SL(2, \mathbb{C})$ group can be constructed as invariance groups of ternary generalization of Pauli's exclusion principle. This suggests that a convenient generalization of Pauli's exclusion principle would be that no *three* quarks in the same state can be present in a nucleon.

Let us require then the vanishing of wave functions representing the tensor product of *three* (but not necessarily two) identical states. That is, we require that $\hat{O}(x, x, x) = 0$ for any quantum state $|x\rangle$. As in the former case, consider an arbitrary superposition of three different states, $|x\rangle$, $|y\rangle$ and $|z\rangle$,

$$|\omega\rangle = \alpha|x\rangle + \beta|y\rangle + \gamma|z\rangle$$

and apply the same criterion, $\hat{O}(\omega, \omega, \omega) = 0$.

We get then, after developing the tensor products,

$$\begin{aligned}
\Phi(\omega, \omega, \omega) &= \alpha^3 \Phi(x, x, x) + \beta^3 \Phi(y, y, y) + \gamma^3 \Phi(z, z, z) + \\
&+ \alpha^2 \beta [\Phi(x, x, y) + \Phi(x, y, z) + \Phi(y, x, x)] + \gamma \alpha^2 [\Phi(x, x, z) + \Phi(x, z, x) + \Phi(z, x, x)] + \\
&+ \alpha \beta^2 [\Phi(y, y, x) + \Phi(y, x, y) + \Phi(x, y, y)] + \beta^2 \gamma [\Phi(y, y, z) + \Phi(y, z, y) + \Phi(z, y, y)] + \\
&+ \beta \gamma^2 [\Phi(y, z, z) + \Phi(z, z, y) + \Phi(z, y, x)] + \gamma^2 \alpha [\Phi(z, z, x) + \Phi(z, x, z) + \Phi(x, z, z)] + \\
&+ \alpha \beta \gamma [\Phi(x, y, z) + \Phi(y, z, x) + \Phi(z, x, y) + \Phi(z, y, x) + \Phi(y, x, z) + \Phi(x, z, y)] = 0
\end{aligned}$$

The terms $\hat{O}(x, x, x)$, $\hat{O}(y, y, y)$ and $\hat{O}(z, z, z)$ do vanish by virtue of the original assumption; in what remains, combinations preceded by various powers of independent numerical coefficients α , β and γ , must vanish separately. This is achieved if the following Z_3 symmetry is imposed on our wave functions:

$$\hat{O}(x, y, z) = j \hat{O}(y, z, x) = j^2 \hat{O}(z, x, y)$$

with $j = \exp\left[\frac{2\pi i}{3}\right]$, $j^3 = 1$, $j + j^2 + 1 = 0$.

Note that the complex conjugates of wave functions $\hat{O}(x, y, z)$ transform under cyclic permutations of their arguments with $j^2 = \bar{j}$ replacing j in the above formula

$$\Psi(x, y, z) = j^2 \Psi(y, z, x) = j \Psi(z, x, y).$$

It is quite easy to imagine simple - and unique - cubic generalization of anti-commutation relations defining the usual Pauli's exclusion principle. The Z_2 group is generated by an idempotent element, represented on the complex plane by multiplication by -1 . The anti-commutation relations are just the faithful representation of the cyclic permutation: $ab = (-1)ba$. Now, there are two different representations of the cyclic group Z_3 in the complex plane, both generated by cubic roots of unity. Let us denote

$$j = \exp\left[\frac{2\pi i}{3}\right], \quad j^2 = \exp\left[\frac{4\pi i}{3}\right], \quad j^3 = 1, \quad j + j^2 + 1 = 0. \quad (24)$$

Let us introduce N generators spanning a linear space over complex numbers, satisfying the following cubic relations [16-17]:

$$\theta^A \theta^B \theta^C = j \theta^B \theta^C \theta^A = j^2 \theta^C \theta^A \theta^B, \quad A, B = 1, 2, \dots, N. \quad (25)$$

We shall also introduce a similar set of *conjugate* generators, $\bar{\theta}^{\dot{A}}$, $\dot{A}, \dot{B}, \dots = 1, 2, \dots, N$, satisfying similar condition with j^2 replacing j :

$$\bar{\theta}^{\dot{A}} \bar{\theta}^{\dot{B}} \bar{\theta}^{\dot{C}} = j^2 \bar{\theta}^{\dot{B}} \bar{\theta}^{\dot{C}} \bar{\theta}^{\dot{A}} = j \bar{\theta}^{\dot{C}} \bar{\theta}^{\dot{A}} \bar{\theta}^{\dot{B}}. \quad (26)$$

Combined with the associativity, these cubic relations impose finite dimension on the algebra generated by the Z_3 -graded generators. As a matter of fact, cubic expressions are the highest order that does not vanish identically. The proof is immediate:

$$\theta^A \theta^B \theta^C \theta^D = j \theta^B \theta^C \theta^A \theta^D = j^2 \theta^B \theta^A \theta^D \theta^C = j^3 \theta^A \theta^D \theta^B \theta^C = j^4 \theta^A \theta^B \theta^C \theta^D \quad (27)$$

and because $j^4 = j \neq 1$, the only solution is

$$\theta^A \theta^B \theta^C \theta^D = 0. \quad (28)$$

Similar conclusion concerns the conjugate generators, for which the highest monomials are also cubic. To complete the algebra, we should impose commutation relations between ordinary and conjugate generators. Our choice is the following relation:

$$\theta^A \bar{\theta}^{\dot{B}} = -j \bar{\theta}^{\dot{B}} \theta^A, \quad \bar{\theta}^{\dot{B}} \theta^A = -j^2 \theta^A \bar{\theta}^{\dot{B}}. \quad (29)$$

Such an algebra is naturally Z_3 -graded: the generators θ^A are of Z_3 -grade 1, their binary products $\theta^A\theta^B$ of Z_3 -grade 2, and cubic products $\theta^A\theta^B\theta^C$ are of Z_3 -grade 3 equivalent to 0. The conjugate operators $\bar{\theta}^{\dot{A}}$ are endowed with grades opposite to that of θ^A 's, namely grade 2 for $\bar{\theta}^{\dot{A}}$, Z_3 -grade 1 (which is 4 modulo 3) for products $\bar{\theta}^{\dot{A}}\bar{\theta}^{\dot{B}}$, and grade 0 for cubic products. (corresponding to 6 modulo 3). Under multiplication the grades add up modulo 3. Such algebras were considered in [18-19]. In principle, one should ask the question what is the effect of non-cyclic (odd) permutations; however, in the case of two generators only, all permutations are equivalent with cyclic ones. From now on, we shall assume that the generalized ternary Pauli's principle is applied to the set of only two quantum operators. Let us symbolize the operator creating an u -quark by θ^1 , and the operator creating a d -quark by θ^2 . Then we have only two linearly independent three-quark states:

$$\theta^1\theta^2\theta^1 = j\theta^2\theta^1\theta^1 = j^2\theta^1\theta^1\theta^2 \quad \text{and} \quad \theta^2\theta^1\theta^2 = j\theta^1\theta^2\theta^2 = j^2\theta^2\theta^2\theta^1.$$

According to (27), no observable states of four or more quarks can be produced. Also the states with three u or three d quarks are prohibited, which agrees with the experiment.

Let us show now how ternary generalization of Pauli's exclusion principle leads to some special representation of the $SL(2, \mathbb{C})$ group. As in the case of the Z_2 symmetry, we shall introduce an invariant *three-form* ρ_{ABC}^α , $\alpha, \beta = 1, 2, \dots, K$, $A, B, \dots = 1, 2, \dots, N$. The upper indices α, β run from 1 to K , where $K = \frac{N^3 - N}{3}$, the number of linearly independent 3-forms satisfying the imposed Z_3 symmetry. The invariant 3-forms are then defined as follows:

$$\begin{aligned} \rho_{ABC}^\alpha \theta^A \theta^B \theta^C &= \frac{1}{3} [\rho_{ABC}^\alpha \theta^A \theta^B \theta^C + \rho_{BCA}^\alpha \theta^B \theta^C \theta^A + \rho_{CAB}^\alpha \theta^C \theta^A \theta^B] = \\ &= \frac{1}{3} [\rho_{ABC}^\alpha \theta^A \theta^B \theta^C + \rho_{BCA}^\alpha (j^2 \theta^A \theta^B \theta^C) + \rho_{CAB}^\alpha j (\theta^A \theta^B \theta^C)], \\ \rho_{ABC}^\alpha \theta^A \theta^B \theta^C &= \frac{1}{3} [\rho_{ABC}^\alpha + \rho_{BCA}^\alpha j^2 + \rho_{CAB}^\alpha j] \theta^A \theta^B \theta^C, \end{aligned} \quad (30)$$

from which we get the following properties of the cubic ρ -matrices:

$$\rho_{ABC}^\alpha = \rho_{BCA}^\alpha j^2 = \rho_{CAB}^\alpha. \quad (31)$$

Even in this minimal and discrete case, there are covariant and contravariant indices: the lower and the upper indices display the inverse transformation property. If a given cyclic permutation is represented by a multiplication by j for the upper indices, the same permutation performed on the lower indices is represented by multiplication by the inverse, i.e. j^2 , so that they compensate each other. The upper index α can take on two values because there are only two independent combinations of three generators, 121 and 212. Hopefully enough, they may coincide with the index of a Pauli spinor.

Similar reasoning leads to the definition of the conjugate forms $\bar{\rho}_{\dot{A}\dot{B}\dot{C}}^{\dot{\alpha}}$ satisfying the relations similar to (31) with j replaced by its conjugate, j^2 :

$$\bar{\rho}_{\dot{A}\dot{B}\dot{C}}^{\dot{\alpha}} = j \bar{\rho}_{\dot{B}\dot{C}\dot{A}}^{\dot{\alpha}} = j^2 \bar{\rho}_{\dot{C}\dot{A}\dot{B}}^{\dot{\alpha}}. \quad (32)$$

In the simplest case of two generators, the j -skew-invariant forms have only two independent components:

$$\begin{aligned} \rho_{121}^1 &= j \rho_{211}^1 = j^2 \rho_{112}^1, \\ \rho_{212}^2 &= j \rho_{122}^2 = j^2 \rho_{221}^2 \end{aligned}$$

and we can set

$$\rho_{121}^1 = 1, \quad \rho_{211}^1 = j^2, \quad \rho_{112}^1 = j,$$

$$\rho_{212}^2 = 1, \quad \rho_{122}^2 = j^2, \quad \rho_{221}^2 = j.$$

5. Lorentz symmetry on quark states

The constitutive cubic relations between the generators of the Z_3 graded algebra can be considered as intrinsic if they are conserved after linear transformations with commuting (pure number) coefficients, i.e. if they are independent of the choice of the basis. Let $U_A^{A'}$ denote a non-singular $N \times N$ matrix, transforming the generators θ^A into another set of generators, $\theta^{B'} = U_B^{B'} \theta^B$. We are looking for the solution of the covariance condition for the ρ -matrices:

$$S_{\beta}^{\alpha'} \rho_{ABC}^{\beta} = U_A^{A'} U_B^{B'} U_C^{C'} \rho_{A'B'C'}^{\alpha'} \quad (33)$$

Now, $\rho_{121}^1 = 1$, and we have two equations corresponding to the choice of values of the index α' equal to 1 or 2. For $\alpha' = 1'$ the ρ -matrix on the right-hand side is $\rho_{A'B'C'}^{1'}$, which has only three components,

$$\rho_{1'2'1'}^{1'} = 1, \quad \rho_{2'1'1'}^{1'} = j^2, \quad \rho_{1'1'2'}^{1'} = j,$$

which leads to the following equation:

$$S_1^{1'} = U_1^{1'} U_2^{2'} U_1^{1'} + j^2 U_1^{2'} U_2^{1'} U_1^{1'} + j U_1^{1'} U_2^{1'} U_1^{2'} = U_1^{1'} (U_2^{2'} U_1^{1'} - U_1^{2'} U_2^{1'}),$$

because $j^2 + j = -1$.

For the alternative choice $\alpha' = 2'$ the ρ -matrix on the right-hand side is $\rho_{A'B'C'}^{2'}$, whose three non-vanishing components are

$$\rho_{2'1'2'}^{2'} = 1, \quad \rho_{1'2'2'}^{2'} = j^2, \quad \rho_{2'2'1'}^{2'} = j.$$

The corresponding equation becomes now:

$$S_1^{2'} = U_1^{2'} U_2^{1'} U_1^{2'} + j^2 U_1^{1'} U_2^{2'} U_1^{2'} + j U_1^{2'} U_2^{2'} U_1^{1'} = U_1^{2'} (U_2^{1'} U_1^{2'} - U_1^{1'} U_2^{2'}).$$

The remaining two equations are obtained in a similar manner. We choose now the three lower indices on the left-hand side equal to another independent combination, (212). Then the ρ -matrix on the left hand side must be ρ^2 whose component ρ_{212}^2 is equal to 1. This leads to the following equation when $\alpha' = 1'$:

$$S_2^{1'} = U_2^{1'} U_1^{2'} U_2^{1'} + j^2 U_2^{2'} U_1^{1'} U_2^{1'} + j U_2^{1'} U_1^{1'} U_2^{2'} = U_2^{1'} (U_2^{2'} U_1^{1'} - U_1^{1'} U_2^{2'})$$

and the fourth equation corresponding to $\alpha' = 2'$ is:

$$S_2^{2'} = U_2^{2'} U_1^{1'} U_2^{2'} + j^2 U_2^{1'} U_1^{2'} U_2^{2'} + j U_2^{2'} U_1^{2'} U_2^{1'} = U_2^{2'} (U_1^{1'} U_2^{2'} - U_1^{2'} U_2^{1'}).$$

The determinant of the 2×2 complex matrix $U_B^{A'}$ appears on the right-hand side:

$$S_1^{2'} = -U_1^{2'} [\det(U)]. \quad (34)$$

The remaining two equations are obtained in a similar manner, resulting in the following:

$$S_2^{1'} = -U_2^{1'} [\det(U)], \quad S_2^{2'} = -U_2^{2'} [\det(U)]. \quad (35)$$

The determinant of the 2×2 complex matrix $U_B^{A'}$ appears everywhere on the right-hand side. Taking the determinant of the matrix $S_{\beta}^{\alpha'}$ one gets immediately

$$\det(S) = [\det(U)]^3. \quad (36)$$

However, the U -matrices on the right-hand side are defined only up to the phase, which due to the cubic character of the covariance relations and they can take on three different values: 1, j or j^2 , i.e. the matrices $jU_B^{A'}$ or $j^2U_B^{A'}$ satisfy the same relations as the matrices $U_B^{A'}$ defined above. The determinant of U can take on the values 1, j or j^2 if $\det(S) = 1$. This will be true if we admit that the indices $\alpha, \beta \dots$ relate to the usual Pauli fermions, and that matrices S represent the $SL(2, \mathbb{C})$ group.

A similar covariance requirement can be formulated with respect to the set of 2-forms mapping the quadratic quark-anti-quark combinations into a four-dimensional linear real space. As we already saw, the symmetry (29) imposed on these expressions reduces their number to four. Let us define two quadratic forms, π_{AB}^μ and its conjugate $\bar{\pi}_{\dot{A}\dot{B}}^\mu$

$$\pi_{AB}^\mu \theta^A \bar{\theta}^{\dot{B}} \quad \text{and} \quad \bar{\pi}_{\dot{B}\dot{A}}^\mu \bar{\theta}^{\dot{B}} \theta^A. \quad (37)$$

The Greek indices $\mu, \nu, \dots$ take on four values, and we shall label them 0,1,2,3. The four tensors π_{AB}^μ and their hermitian conjugates $\bar{\pi}_{\dot{B}\dot{A}}^\mu$ define a bi-linear mapping from the product of quark and anti-quark cubic algebras into a linear four-dimensional vector space, whose structure is not yet defined. Let us impose the following invariance condition:

$$\pi_{AB}^\mu \theta^A \bar{\theta}^{\dot{B}} = \bar{\pi}_{\dot{B}\dot{A}}^\mu \bar{\theta}^{\dot{B}} \theta^A. \quad (38)$$

It follows immediately from (29) that

$$\pi_{AB}^\mu = -j^2 \bar{\pi}_{\dot{B}\dot{A}}^\mu. \quad (39)$$

Such matrices are non-hermitian, and they can be realized by the following substitution:

$$\pi_{AB}^\mu = j^2 i \sigma_{AB}^\mu, \quad \bar{\pi}_{\dot{B}\dot{A}}^\mu = -j i \sigma_{\dot{B}\dot{A}}^\mu, \quad (40)$$

where σ_{AB}^μ are the unit 2 matrix for $\mu = 0$, and the three hermitian Pauli matrices for $\mu = 1, 2, 3$. Again, we want to get the same form of these four matrices in another basis. Knowing that the lower indices A and $\dot{B}$ undergo the transformation with matrices $U_B^{A'}$ and $\bar{U}_{\dot{B}}^{\dot{A}'}$, we demand that there exist some 4×4 matrices $A_v^{\mu'}$ representing the transformation of lower indices by the matrices U and $\bar{U}$

$$A_v^{\mu'} \pi_{AB}^\nu = U_A^{A'} U_B^{\dot{B}'} \pi_{A'\dot{B}'}^{\mu'}, \quad (41)$$

This defines the vector (4×4) representation of the Lorentz group.

The first four equations relating the 4×4 real matrices $A_v^{\mu'}$ with the 2×2 complex matrices $U_B^{A'}$ and $\bar{U}_{\dot{B}}^{\dot{A}'}$ are as follows:

$$\begin{aligned} A_0^{0'} + A_3^{0'} &= U_1^{1'} \bar{U}_{\dot{1}}^{\dot{1}'} + U_1^{2'} \bar{U}_{\dot{1}}^{\dot{2}'} \\ A_0^{0'} - A_3^{0'} &= U_2^{1'} \bar{U}_{\dot{2}}^{\dot{1}'} + U_2^{2'} \bar{U}_{\dot{2}}^{\dot{2}'} \\ A_0^{0'} - i A_2^{0'} &= U_1^{1'} \bar{U}_{\dot{2}}^{\dot{1}'} + U_1^{2'} \bar{U}_{\dot{2}}^{\dot{2}'} \\ A_0^{0'} + i A_2^{0'} &= U_2^{1'} \bar{U}_{\dot{1}}^{\dot{1}'} + U_2^{2'} \bar{U}_{\dot{1}}^{\dot{2}'} \end{aligned}$$

There are three other sets of four equations similar to the one displayed above, corresponding to three alternative choices of values of the upper primed index $1', 2'$ and $3'$. We do not display them here explicitly; the resulting matrices $A_v^{\mu'}$ span a vector representation of the Lorentz group, although with special complex coefficients.

The metric tensor $g_{\mu\nu}$ can be defined in the following manner.

With the invariant “spinorial metric” in two complex dimensions, ε^{AB} and $\varepsilon^{\dot{A}\dot{B}}$ such that $\varepsilon^{12} = -\varepsilon^{21} = 1$ and $\varepsilon^{\dot{1}\dot{2}} = -\varepsilon^{\dot{2}\dot{1}}$, we can define the contravariant components $\pi^{\nu AB}$. It is easy to show that the Minkowskian space-time metric, invariant under the Lorentz transformations, can be defined as

$$g^{\mu\nu} = \frac{1}{2} \left[\pi_{AB}^\mu \pi^{\nu AB} \right] = \text{diag}(+, -, -, -) \quad (42)$$

According to this picture, the Lorentz symmetry observed on macroscopic scale is the result of the action of the $SL(2, \mathbb{C})$ group on the Hilbert space of quantum states of elementary fermions (leptons), i.e. electrons, μ -mesons and neutrinos, all of which satisfy the Dirac equation and Pauli's exclusion

principle. The essential ingredient is the appearance of two Z_2 symmetries, one due to the dichotomic spin states, another one to the particle-antiparticle symmetry. Due to the last symmetry, Dirac spinors ψ are composed out of two in equivalent Pauli spinors. The Dirac spinors, in turn, produce Lorentz scalars or vectors due to the introduction of charge conjugation, $\bar{\psi} = \psi^\dagger \gamma^0$. Then Lorentz scalars and Lorentz 4-vectors are produced out of binary products $\bar{\psi}\psi = \psi^\dagger \gamma^0 \psi$ and $\bar{\psi}\gamma^k \psi = \psi^\dagger \gamma^0 \gamma^k \psi$.

For hadrons and strongly interacting mesons (here we consider only protons and neutrons, plus the π -mesons) the apparent Lorentz symmetry stems from the Z_3 generalization of Pauli's principle implemented in quarks, so that only two three-quark configurations can be observed, uud and udd , as well as three independent quark-antiquark pairs $u\bar{d}$, $d\bar{u}$ and $\frac{1}{\sqrt{2}}(u\bar{u} - d\bar{d})$. The Z_3 graded

ternary algebra quark generators are preserved by the Z_3 -twisted representation of the $SL(2, \mathbb{C})$ group. Cubic combinations of these operators behave like Pauli spinors, transforming under usual representations of $SL(2, \mathbb{C})$. Their quadratic combinations transform as Lorentz 4-vectors or Lorentz scalars.

The dynamics of quarks is very well described by Quantum Chromodynamics [14-15], where quarks are treated as usual Dirac fermions interacting via vector particles called gluons. All these fields are endowed with an extra property called *color*, and become observable only in "colorless" combinations, with three different colors (red, green and blue) for the three-quark states producing nucleons, and eight color-anticolor combinations for gluons, spanning an octet representation of the $SU(3)$ algebra. On the other hand, the possibility of deriving a new Z_3 -twisted representation of the Lorentz group suggests that a generalization of the Dirac equation incorporating this symmetry coexisting with the $Z_2 \times Z_2$ symmetry in the usual case should be possible. The color dynamics based on this generalization is beyond the scope of the present paper. The generalized Dirac equation mixing three colors, based on the $Z_2 \times Z_3 \times Z_2$ symmetry is constructed and discussed in papers [20-21].

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