ON QUANTIZED SPACE-TIME THEORY

V. G. Kadyshevsky

Joint Institute for Nuclear Research, Dubna

(presented by I. T. Todorov)

1. In the present report we suppose [1, 2, 3] that the hypothetic " fundamental " length l without which it is impossible to describe correctly physical processes occurring at high energies or, respectively, in small space-time scales, is simultaneously a constant defining the intensity of weak interactions(*). This means that the geometric structure of the x-space " in small scale " and the p-space " in large scale " must be closely connected with weak interactions of elementary particles (**) and, possibly, we may now expect the explanation of such an original " geometrical " property of weak processes as the non-conservation of the space parity (***) and non-invariance under " strong " time reversal. This last remark is used by us as an auxiliary argument in the choice of a new geometry for p-space.

We note beforehand that the requirement of invariance under the space inversion P and " strong " time reversal T_s might be eliminated if P and T_s belong to a group of transformations which does not correspond to any physical symmetry. In the usual theory such a group is formed, e.g., of translations in the momentum space. To these transformations there do not correspond any laws of conservation and, in general, they may be neglected, if in the theory there is a vacuum state having a certain smallest (e.g. zero) value of the four-momentum i.e. if in p-space there is a singled cut point. It is natural to expect, even from the arguments of correspondence, that any modification of the existing theory will have no requirement of invariance under translations in the momentum space. Taking into account this circumstance and the above-mentioned arguments we may now assume that the new geometry of p-space must be such that P and T_s are translations in this space.

2. It turns out that already the simplest generalization of the pseudo Euclidean momentum space (space of a constant curvature (****) [5, 6]) possesses the desired properties. Namely, if $\eta_0^2-\eta_1^2-\eta_2^2-\eta_3^2-\varepsilon\eta_4^2 = -\varepsilon$ is a five-dimensional hypersphere (†) simulating such a p-space (diametrically opposite points of the hypersphere are identified,[7] $\varepsilon = \pm 1$) then for $\varepsilon = 1$ among the transformations of translations there is T_s and for $\varepsilon = -1$ the inversion P [3]. Using the arguments of section 1 and supposing in addition that in the theory with a curved p-space the CPT-theorem remains valid we may draw a conclusion about the possibility of elimination of the requirement of T_s and P-invariance in this theory (independing of the choice of ε).

In contrast to [5, 6, 8] we shall call the four-momentum vector p_m $(m = 0, 1, 2, 3)$ the coordinates of the projections of the hypersphere points on the plane $\eta_4 = 0$ i.e. $p_m = \eta_m$. For $\varepsilon = 1$ all the p_m's lie in the region $p^2 = p_0^2 - \mathbf{p}^2 \geqq -1$ (in the following Ω),

---

(*) In [1] l is chosen to be equal to "β—decay length": $\sqrt{G/\hbar c} = 6{\cdot}10^{-17}$ cm (G is the Fermi constant).

(**) Perhaps, in the spirit of the theory of general relativity (see [4]).

(***) Shapiro [4] was the first to point out the possibility of explaining the non-conservation of parity of weak interactions on the basis of a new concept of the space structure in scales of the order of $G/\hbar c$.

(****) By analogy with [5] a corresponding formalism is called the quantized space-time theory.

(†) The hypersphere radius is assumed to be equal to $\hbar/l$ where l is " the fundamental length ". We use a system of units in which $\hbar = c = l = 1$. In this case, obviously, in passing to the usual theory ($l \to 0$) it is sufficient to assume all quantitites having the dimensions of energy-momentum to be much smaller than unity.

for $\varepsilon = -1$ in the region $p^2 \leqq 1$. We choose $\varepsilon = 1$ since in this case there is no restriction on the masses of the physical systems. The boundaries of the region Ω lie in the hypersphere and the points p_m and $-p_m$ for which $p^2 = -1$ are therefore identical. The transformation of the vector p_m under translation by the vector k_m in the curved space is written as follows

$$p_m(+)k_m = \sigma\left[p_m + k_m\left(\sqrt{1+p^2} + \frac{pk}{1+\sqrt{1+k^2}}\right)\right], \qquad \sigma = sgn\left(\sqrt{1+p^2}\sqrt{1+k^2}+pk\right) \tag{2.1}$$

From (2.1) it follows that

$$1+(p(+)k)^2 = (\sqrt{1+p^2}\sqrt{1+k^2}+pk)^2 \geqq 0$$

i.e. transformations $(+)$ do not take p_m outside Ω. If $p = (m, \mathbf{o})$ and $k = (\mu, \mathbf{o})$ then obviously $\mathbf{p}(+)\mathbf{k} = 0$ and $p_0(+)k_0 = m(+)\mu = m\sqrt{1+\mu^2}+\mu\sqrt{1+m^2}$. The operators of coordinates are defined as infinitesimal translation operators of (2.1) i.e.

$$\phi(p(+)k) = (1 - ix^n k_n)\phi(p)$$

for small k_n's. If $\phi(p)$ is a scalar, then

$$x^n = i\sqrt{1+p^2}\frac{\partial}{\partial p_n} \tag{2.2}$$

the spectrum of $\mathbf{x}$ being discrete, and that of t continuous. From (2.2) follows the existence of the relation $[t, \mathbf{p}] = 0$ which allows to transfer from the $(p_0, \mathbf{p})$-representation to a "mixed" $(t, \mathbf{p})$-representation. This fact seems to be essential for a further development of the theory in the considered scheme (e.g. for formulating the causality principle). It is this fact that underlies the redefinition of the four-momentum.

The volume element of p-space in the coordinates under consideration is

$$d\Omega_p = \sqrt{-g}\, d^4p = \frac{d^4p}{\sqrt{1+p^2}}$$

where g is the determinant of the metric tensor. Obviously, $d\Omega_p = d\Omega_{p(+)k} \neq d\Omega_{k(+)p}$. This property of the "right invariance" of $d\Omega_p$ under translations (2.1) underlies the definition of convolution of functions $f_1(p)$ and $f_2(p)$ in the curved p-space:

$$f_1(p)*f_2(p) = \int_\Omega f_1(q)f_2(-(q(-)p))d\Omega_q \tag{2.3}$$

By replacing the variables $q(-)p = -q'$ which is equivalent to $q = -(q'(-)p)$, we can see that (2.3) is commutative: $f_1*f_2 = f_2*f_1$. However, the usual associative property in (2.3) is lost. Assuming in (2.3) that $f_1(p) = \delta(p)$, $f_2(p) = f(p)$ we have:

$$\delta(p)*f(p) = \sqrt{-g(0)}\,f(-(0(-)p)) = f(p) = \int \delta(q(-)p)f(q)d\Omega_q$$

On the other hand, $\delta(q(-)p) = \delta(p(-)q)$ since if $q(-)p = 0$, then $p(-)q = 0$ and for the remaining p and q we have always $\delta(q(-)p) = \delta(p(-)q) = 0$. Therefore $\delta(p(-)q)$ is analogous to the ordinary fourdimensional function $\delta(p-q)$. Obviously,

$$\delta(p(-)q) = \frac{1}{\sqrt{-g(p)}}\delta(p-q)$$

In considering $F[f(p)]$ in the curved p-space we shall always assume that

$$\delta F[f(p)] = \int \frac{\delta F[f(p)]}{\delta f(q)}\delta f(q)d\Omega_q$$

and hence

$$\frac{\delta f(p)}{\delta f(q)} = \delta(p(-)q) \tag{2.4}$$

3. The author has considered the scalar model of field theory in the curved p-space, to which in the x-representation of the usual theory there corresponds the interaction Lagrangian $\mathscr{L}(x) = g\psi^*(x)\psi(x)\phi(x)$ (ψ and ϕ are the charged and the neutral scalar fields with masses m and μ respectively). The free field formalism was transferred to the new scheme with the following modification: instead of $\delta(p-q)$ entering the relativistically covariant writing of commutation relations, "normal" and "chronological" pairings, we used $\delta(p(-)q)$. To construct the S-matrix it was necessary to generalize the quantity

$$\tilde{\mathscr{L}}(q) = \int e^{-iqx}\mathscr{L}(x)dx = \frac{g}{\sqrt{2\pi}}\psi^*(q)*\psi(q)*\psi(q)$$

(we use the convolution symbol) since $\int \mathscr{L}(x)dx = \widetilde{\mathscr{L}}(0)$. It was found that in spite of the non-associativity of (2.3) this generalization is performed in a unique manner, if the requirement of the hermiticity of the Lagrangian is taken into account:

$$\widetilde{\mathscr{L}}(q) = \frac{g}{\sqrt{2\pi}}[\psi^*(q)*\psi(q)]*\phi(q)$$

and hence

$$\widetilde{\mathscr{L}}(0) = \frac{g}{\sqrt{2\pi}}\int \psi^*(p)\psi(-(p(+)k))\phi(k)\mathrm{d}\Omega_p\mathrm{d}\Omega_k$$

Further investigation showed that to construct in the given scheme the scattering matrix having no divergences it is necessary to start from that form of the ordinary S-matrix in which all internal integrations are carried out over the Euclidean space. In such an approach the role of the S-matrix in the developed theory is played by the functional (†) (comp. with [9]).

$$S' = e^{\Delta+\Sigma}\exp\left[i\int \psi^*(p)\psi(-(p(+)k))\phi(k)\mathrm{d}\Omega_p\mathrm{d}\Omega_k\right] \quad (3.1)$$

where

$$\Delta = \frac{1}{4\pi}\int \frac{\mathrm{d}\Omega_k}{\mu^2+k_\nu^2}\,\frac{\delta^2}{\delta\phi(k)\delta\phi(-k)},$$

$$\Sigma = \frac{1}{2\pi}\int \frac{\mathrm{d}\Omega_p}{m^2+p_\nu^2}\,\frac{\delta^2}{\delta\psi(p)\delta\psi^*(-p)},$$

all integrations being carried out over a curved four-dimensional space with a positive definite metric which is analogous to the Euclidean four-dimensional space (††) and the functional derivatives are considered in the sense of (2.4). " True " matrix elements are obtained by taking the variation of S' with respect to ψ, ψ^*, ϕ by putting these arguments equal to zero (comp. with § 47 from [10]) and by analytical continuation of the type $p_4 \to -ip_0$ of the obtained expressions into the physical region of the four components of the external momenta (†††). For example, for a mass diagram of the second order we have (constant factors are omitted)

$$\Sigma_2(p^2) = \underset{p_4\to -ip_0}{\text{anal. continuation}}\int \frac{\mathrm{d}\Omega_k}{[m^2+(p(-)k)_\nu^2][\mu^2+k_\nu^2]}$$

$$= \int_0^1 \frac{\mathrm{d}\alpha}{1+4\alpha(1-\alpha)p^2}\int_{k_\mu^2\leq 1} \frac{\mathrm{d}^4k}{\sqrt{1-k_\mu^2}(L+k_4^2-i\varepsilon)^2} \quad (3.2)$$

where

$$L = \frac{(1-\alpha)\mu^2+\alpha m^2}{\sqrt{1+4\alpha(1-\alpha)p^2}} + \frac{(1+\mathbf{k}^2)-(1-\mathbf{k}^2)\sqrt{1+4\alpha(1-\alpha)p^2}}{2\sqrt{1+4\alpha(1-\alpha)p^2}} > 0,$$

$$\text{if} \quad p^2 < (m(+)\mu)^2$$

A characteristic feature of the integrals obtained in such a way is the absence of the divergences.

LIST OF REFERENCES

1. В. Г. Кадышевский, ДАН СССР, *136*, 70 (1961).
2. В. Г. Кадышевский. Тезисы и программа I Советской гравитационной конференции. Москва, 1961 г.
3. В. Г. Кадышевский, ЖЭТФ, *41*, 1885 (1961).
4. И. С. Шапиро, УФН, *61*, 313 (1957).
5. H. Snyder. Phys. Rev. *71*, 38 (1947).
6. Ю. А. Гольфанд, ЖЭТФ, *37*, 504 (1959).
7. Ф. Коейн. Неэвклидова геометрия. Гостехиздат, 1936.
8. Ю. А. Гольфанд, ЖЭТФ (в печати).
9. S. Hori. Prog. Theor. Phys. *7*, 578 (1952).
10. Н. Н. Боголюбов и Д. В. Ширков. Введение в теорию квантованных полей. Гостехиздат, 1957 г.

(†) A similar functional has been considered by Golfand [8].

(††) The region Ω in this space is of the form $p_\mu^2 \leq 1$, the volume element $\mathrm{d}\Omega_p = \frac{\mathrm{d}^4p}{\sqrt{1-p_\mu^2}}$ etc.

(†††) To describe correctly the singularities arising in the physical region it is necessary to add - iε. to the mass.

DISCUSSION

THIRRING: I think always in these quantized space time theories certain conceptual difficulties arise, namely the co-ordinates x are no longer commuting quantities; they are operators. On what Hilbert space do they act, and in particular what is the physical significance of the eigenstates of the operators x? Do they simply mean that we measured that there was a point x?

TODOROV: Until now in Kadyshevsky's work the x space is almost completely excluded from consideration. The author tries to work only in momentum space. It is wellknown that in present day quantum field theory the co-ordinates x and the momenta p play quite different roles, although in quantum mechanics we say that x- and p-spaces are equivalent. We are usually working in p-space, and for example, the Lorentz invariance in p-space is much better experimentally checked than Lorentz invariance in x-space. We do not even know what is the physical meaning of a point in x-space. We know for example that if we try to localize a particle at a point, we have to give it a big momentum and we will find, instead of one particle, a multiplicity of pairs.

GENERALISED COMMUTATION RELATIONS AND STATISTICS

S. Kamefuchi

Imperial College, London.

and

Y. Takahashi

Dublin Institute for Advanced Studies, Dublin.

(presented by S. Kamefuchi)

Before going into my talk, I must mention that exactly the same conclusion has been reached before independently, in a way somewhat different from ours, by V. Glaser and M. Fierz. This report is, therefore, to be taken as a contribution from these authors and ourselves.

We would like to present here a general method of field quantisation and discuss its application to elementary particle physics. The question we ask ourselves is the following: when we require a field operator $\psi(x)$ to satisfy the relation

$$\frac{\partial\psi(x)}{\partial x_\mu} = i[\psi(x), P_\mu] , \qquad (1)$$

(where P_μ is the energy-momentum four vector) what are the most general commutation relations for the operator ψ?

It is convenient to discuss the problem in momentum space by introducing Fourier coefficients of ψ's a_k and a_k^+. To determine the commutation relations between the a_k and a_k^+ in a self-consistent fashion we consider the following infinitesimal transformation

$$\begin{aligned} a_k \to a_k' &= a_k - i\sum_m a_m \xi_{km} - i\sum_m a_m^+ \eta_{km} , \\ a_k^+ \to a_k'^+ &= a_k^+ + i\sum_m a_m^+ \xi_{km}^* + i\sum_m a_m \zeta_{mk} \; (\eta_{km}^* = \zeta_{mk}) , \end{aligned} \qquad (2)$$

and distinguish the following two cases (R) and (S). Case (R): The transformations (2) belong to an infinite-dimensional *rotation* group, provided

$$\begin{aligned} \xi_{km}^* &= \xi_{mk} , \\ \eta_{km} + \eta_{mk} &= 0 . \end{aligned} \qquad (3)$$