PROJECTION OPERATOR METHOD AND Q-ANALOG OF ANGULAR MOMENTUM THEORY

Yu.F. Smirnov, V.N. Tolstoy and Yu.I. Kharitonov
Nuclear Physics Institute, Moscow State University
Moscow 119899, USSR

1. Introduction

Quantum algebras were introduced at first in Refs.[1,2].Then this concept was developed in details in Refs.[3,4] and in the papers of other authors (see for example [5-11] and the papers cited there). Because of deep analogy consisting between quantum and usual Lie algebras which is reflected in the fact that the quantum algebra $A_q(l,r)$ of order l and rank r transforms into usual Lie algebra $A(l,r)$ in the limit $q\to 1$ a number of notations and theorems of the theory of Lie algebra representations can be transferred onto quantum algebras. In particular as it was shown in Refs [5-17] the q-analogs of well known quantities of Wigner-Racah algebra (WRA) (3j, 6j, 9j-symbols etc.) can be introduced. The detail investigation of the representation of quantum algebras was begun with the simplest quantum algebra $SU_q(2)$ that is a q-analog of the angular momentum theory (AMT) [18-21].

In this paper we apply to this problem an original approach namely the projection operator method that was developed by us for usual Lie algebras in Refs [22,23] and appears rather effective as well in AMT as for higher Lie algebras. The important advantage of this method is the fact that for the calculation of quantities of WRA any explicit realization of algebra generators is unnecessary. Only commutation rules for generators, their Hermitian properties and the existence of the highest vector are enough for the development of q-algebra unitary representation theory. Below it will be shown that most part of AMT formulae will conserve their shape in the case of $SU_q(2)$ algebra except for exchange of usual numbers (x) by so called q-numbers $[x]$:

$$[x]_q \equiv [x] \equiv (q^x - q^{-x})/(q - q^{-1}) \qquad (1.1)$$

where $q=e^{\hbar}$. Obviously that $[0]=0$, $[1]=1$, $[2]=q+\bar{q}$, $[3]=q^2+1+\bar{q}$, $[-x]=-[x]$, $\lim\limits_{q\to 1}[x]=(x)$, $[x]_q=[x]_{\bar{q}}$, where $\bar{q}\equiv q^{-1}$.Below we shall use the so called q-factorial

$$[n]! \equiv [n][n-1]...[2][1] \qquad (1.2)$$

for nonnegative integer n. As usually we assume $[0]! = 1$ and $[-n]! = \infty$ at $n = 1,2,...$.

The Planck's constant $\hbar$ will be assumed real in our calculation. The special case when q is equal to the root of unit was considered in Refs.[12,13].

2. $SU_q(2)$ algebra and its irreducible representations

The q-analog of $SU(2)$ algebra is defined by three generators J_0, J_+, J_- with following properties

$$[J_0,J_\pm] = \pm J_\pm \;, \tag{2.1}$$

$$[J_+,J_-] = [2J_0] \equiv \left(q^{2J_0} - q^{-2J_0}\right)\Big/(q - \bar{q}) \;, \tag{2.2}$$

$$J_0^+ = J_0 \;, \; J_\pm^+ = J_\pm . \tag{2.3}$$

The irreducible (IR) D^j of highest weight j contains the highest vector $|jj\rangle$ satisfying the equations

$$J_0|jj\rangle = j|jj\rangle \;, \; \langle jj|jj\rangle = 1 \;, \tag{2.4}$$

$$J_\pm|jj\rangle = 0 \;. \tag{2.5}$$

The general basis vector of this IR having the weight m can be constructed using the lowering generator J_-

$$|jm\rangle = \sqrt{\frac{[j+m]!}{[2j]![j-m]!}}\, J_-^{j-m}|jj\rangle \;. \tag{2.6}$$

The normalizing factor calculated by using the relation

$$J_+J_-^n = J_-^nJ_+ + [n]J_-^{n-1}[2J_0-n+1] \;. \tag{2.7}$$

For the finite dimensional IR D^j only $j=0,1/2,1,\ldots$ and $m=j,j-1,\ldots,-j$ are allowed. Thus the structure of $SU_q(2)$ IR's is similar to the IR's of usual $SU(2)$ algebra and the dimension of IR's are the same in both of cases is equal to $(2j+1)$.

Acting by generators J_- and J_+ on vector (2.6) we obtain

$$J_-|jm\rangle = \sqrt{[j+m][j-m+1]}\;|j,m-1\rangle \;, \tag{2.8}$$

$$J_+|jm\rangle = \sqrt{[j-m][j+m+1]}\;|j,m+1\rangle \;. \tag{2.9}$$

Thus the explicit form of D^j IR for $SU_q(2)$ coincides with corresponding formulae for $SU(2)$ except for the substitution of usual number $(j\pm m)$ and $(j\mp m+1)$ by q-numbers in two last rows.

In the theory of $SU_q(2)$ IR the important role plays the Casimir operator

$$C_2 = J_-J_+ + [J_0+1/2]^2 = J_+J_- + [J_0-1/2]^2 , \tag{2.10}$$

which is Hermitian one. The vectors (2.6) are its eigenvectors

$$C_2|jm\rangle = [j+1/2]^2|jm\rangle \ . \tag{2.11}$$

3. Projection operators for $su_q(2)$ algebra

First of all we are interesting in the projection operator (PO) $P^j_{j,j}=P^j$ having the property

$$P^j|j'm=j\rangle = \delta_{j,j'}|jj\rangle, \tag{3.1}$$

i.e. acting on an arbitrary vector $|j)$ of weight $m=j$

$$|m=j) = \sum_{j'\geqslant j} B_{j'}|j'j) \tag{3.2}$$

the operator P^j projects the component $|jj\rangle$ being the highest weight vector of IR D^j

$$P^j|j) = B_j|jj) \ . \tag{3.3}$$

Similarly to Refs.[22-25] we seek this PO as a power series of generators J_+ and J_-

$$P^j = \sum_{r=0}^{\infty} C_r\, J_-^r J_+^r \ . \tag{3.4}$$

The exponents of these generators are the same due to condition

$$[P^j, J_0] = 0 \ . \tag{3.5}$$

Since

$$P^j|jj\rangle = |jj\rangle \ , \tag{3.6}$$

we obtain because of (2.6) that

$$C_0 = 1 \tag{3.7}$$

and

$$J_+\, P^j|j) = J_+ \sum_{r=0}^{\infty} C_r\, J_-^r J_+^r|j) = 0 \ . \tag{3.8}$$

By using of Eq. (2.7) the following recurrent relation for C_r coefficients can be found

$$C_{r-1} + [r][2j+r+1]C_r = 0 \ . \tag{3.9}$$

Solving it we have

$$C_r = (-1)^r \frac{[2j+1]!}{[r]!\,[2j+r+1]!} \ . \tag{3.10}$$

Obviously the PO is Hermitian one

$$(P^j)^+ = P^j. \tag{3.11}$$

By Hermitian conjugation of Eq.(3.8) we obtain an important property of PO

$$P\, J_- = 0 \ . \tag{3.12}$$

The operator

$$P^{j}_{-j,-j} = \sum_{r=0}^{\infty} C_r\, J_+^{\,r} J_-^{\,r} \qquad (3.13)$$

with the same coefficients C_r as in Eq.(3.10) is an extremal projector on the lowest weight of IR D^j.

The most general form of projecting operator

$$P^{j}_{m,m'} = \sqrt{\frac{[j+m]!}{[2j]![j-m]!}}\; J_-^{\,j-m}\, P^{j}\, J_+^{\,j-m'} \sqrt{\frac{[j+m']!}{[2j]![j-m']!}} \qquad (3.14)$$

will need in further calculations. These POs have the properties

$$(P^{j}_{m,m'})^{+} = P^{j}_{m',m}\ ,\quad P^{j}_{m,m'}\,|m'\,) = B_{j,m'}\,|jm\rangle\ . \qquad (3.15)$$

4. "Vector coupling" of q-angular momenta

Now we turn to the of "vector coupling" of angular momenta in the case of $SU_q(2)$ algebra . The generators of summary angular momentum $J(1,2)$ are of the form

$$J^{q}_{0}(1,2) = J_0(1) + J_0(2), \qquad (4.1a)$$

$$J^{q}_{\pm}(1,2) = J_{\pm}(1)\, q^{J_0(2)} + q^{-J_0(1)} J_{\pm}(2)\ . \qquad (4.1b)$$

In standard notation for Hopf algebras the relations (4.1) must be written in the following form

$$J^{q}_{0} = J_0 \otimes I + I \otimes J_0\ ,$$

$$J^{q}_{\pm} = J_{\pm} \otimes q^{J_0} + q^{-J_0} \otimes J_{\pm}\ .$$

However we shall use below notation (4.1) in order to conserve the maximal possible similarity with usual AMT.

It is easy to prove that the operators (4.1) are satisfying to commutation relations (2.1) and (2.2).

The action of generators (4.1) on basis vectors $|j_1 m_1\rangle|j_2 m_2\rangle$ of the tensor product of IRs is given by formulae

$$J^{q}_{0}(1,2)|j_1 m_1\rangle|j_2 m_2\rangle = (m_1+m_2)|j_1 m_1\rangle|j_2 m_2\rangle\ , \qquad (4.2a)$$

$$J^{q}_{\pm}(1,2)|j_1 m_1\rangle|j_2 m_2\rangle = q^{m_2}\langle j_1, m_1\pm 1|J_{\pm}|j_1 m_1\rangle|j_1 m_1\pm 1\rangle|j_2 m_2\rangle +$$

$$+\ q^{-m_1}\langle j_2, m_2\pm 1|J_{\pm}|j_2 m_2\rangle|j_1 m_1\rangle|j_2 m_2\pm 1\rangle\ . \qquad (4.2b)$$

It should be remarked that the q-analog of the binomial expansion formula is valid

$$\left[J^q_\pm(1,2)\right]^r = \left[J_\pm(1)q^{J_0(2)} + q^{-J_0(1)}J_\pm(2)\right]^r =$$

$$= \sum_s \frac{[r]!}{[s]![r-s]!} J_\pm^s(1)\, J_\pm^{r-s}(2)\, q^{sJ_0(2)-(r-s)J_0(1)} . \qquad (4.3)$$

The generalization of the vector coupling procedure on the case of $SU_q(2)$ consists in the following. It is necessary to construct from the tensor product basis vectors $|j_1m_1\rangle|j_2m_2\rangle$ such linear combinations

$$|j_1j_2;jm\rangle_q = \sum_{m_1,m_2} \langle j_1m_1;j_2m_2|jm\rangle_q \; |j_1m_1\rangle|j_2m_2\rangle \qquad (4.4)$$

which belong to the IR D^j of $SU_q(2)$, i.e. they are eigenvectors of the Casimir operator $C_2^q(1,2)$ with eigenvalues $\Lambda=[j+\frac{1}{2}]^2$:

$$C_2^q(1,2)|j_1j_2;jm\rangle_q = [j+1/2]^2|j_1j_2;jm\rangle_q . \qquad (4.5)$$

The coefficient $\langle j_1m_1;j_2m_2|jm\rangle_q$ in linear combinations (4.4) are called as Clebsch-Gordan coefficients (q-CGC) for $SU_q(2)$ quantum algebra. To find them we shall use the PO approach. Simultaneously the structure of Clebsch-Gordan series for $SU_q(2)$ will be found or more correctly it will be confirmed that the Clebsch-Gordan series for $SU_q(2)$ coincides with the Eq.(4.1).However before to turn to this point it is pertinent to list the orthormality relations for the q-CGCs

$$\sum_{m_1,m_2} \langle j_1m_1;j_2m_2|jm\rangle_q \; \langle j_1m_1;j_2m_2|j'm'\rangle_q = \delta_{j,j'}\delta_{m,m'} , \qquad (4.5a)$$

$$\sum_{j,m} \langle j_1m_1;j_2m_2|jm\rangle_q \; \langle j_1m_1';j_2m_2'|jm\rangle_q = \delta_{m_1,m_1'}\delta_{m_2,m_2'} . \qquad (4.5b)$$

The q-CGCs form an orthogonal matrix and the following equation which is inverse with respect to transformation (4.4) is valid

$$|j_1m_1\rangle|j_2m_2\rangle = \sum_{j,m} \langle j_1m_1;j_2m_2|jm\rangle_q \; |j_1j_2;jm\rangle_q . \qquad (4.6)$$

5. Q-analogs of Clebsch-Gordan coefficients

Using PO we can write the vector (4.4) in a form

$$|j_1j_2;jm\rangle_q = (Q_q')^{-1}P^{j,q}_{m,m'}(1,2)|j_1m_1'(1)\rangle|j_2m_2'(2)\rangle, \qquad (5.1)$$

where $m'=m_1'+m_2'$. Thus the q-CGC can be calculated as the matrix element of PO

$$\langle j_1m_1,j_2m_2|jm\rangle_q =$$

$$= (Q_q')^{-1}\langle j_1m_1(1)|\langle j_2m_2(2)|P^{j,q}_{m,m'}(1,2)|j_1m_1'(1)\rangle|j_2m_2'(2)\rangle \qquad (5.2)$$

where Q_q' is a normalizing factor. As for values of m_1' and m_2' in Eqs. (5.1) and (5.2) they can be chosen in arbitrary manner but for simplification of calculations it is convenient to take $m_1'=j_1$ and $m_2'=j-j_1$.

Then the Eq.(5.2) can be rewritten in the form

$$\langle j_1 m_1; j_2 m_2 | jm\rangle_q =$$

$$= (Q'_q)^{-1} \langle j_1 m_1(1)| \langle j_2 m_2(2)| P^{j;q}_{m,j}(1,2) | j_1 j_1(1)\rangle | j_2, j-j_1(2)\rangle, \qquad (5.3)$$

where

$$Q_q^{\,2} = \langle j_1 j_1(1)| \langle j_2, j-j_1(2)| P^{j;q}_{j,j}(1,2) | j_1 j_1(1)\rangle | j_2, j-j_1(2)\rangle. \qquad (5.4)$$

Since $|j_1 j_1(1)\rangle$ is a highest weight vector the generators $J_+(1)$ in PO $P^{j,q}_{m,j}(1,2)$ can be omitted and for the normalizing factor Q_q we obtain

$$Q_q^{\,2} = \langle j_1 j_1; j_2, j-j_1 | jj\rangle^2 = \langle j_2, j-j_1(2)| P^{j;q}_{(j_1)}(2) | j_2, j-j_1(2)\rangle, \qquad (5.5)$$

where

$$P^{j;q}_{(j_1)}(2) = \sum_{r\geqslant 0} \frac{(-1)^r [2j+1]!}{[r]![2j+r+1]!}\, q^{-2rj_1} J_-^{\,r}(2) J_+^{\,r}(2). \qquad (5.6)$$

Let's adopt an usual phase convention for Q_q being positive (arithmetic) square root of $Q_q^{\,2}$. As a results all q-CGCs will be real.

It is clear from Eq.(5.3) that only values of summary angular momentum j satisfying the conditions $-j_2 \leqslant j-j_1 \leqslant j_2$ are possible.It means the following restriction

$$j_1 - j_2 \leqslant j \leqslant j_1 + j_2 \,.$$

Since angular momenta j_1 and j_2 are "equal in rights" the restriction

$$j_2 - j_1 \leqslant j \leqslant j_1 + j_2$$

is valid too. Combining two these conditions we obtain for $SU_q(2)$ the same "rule of vector coupling" of angular momenta as for usual $SU(2)$ algebra: $|j_1 - j_2| \leqslant j \leqslant j_1 + j_2$.

Finally the general expression for q-CGCs can be written in form

$$\langle j_1 m_1; j_2 m_2 | jm\rangle_q =$$

$$= \frac{\langle j_1 m_1(1)| \langle j_2 m_2(2)| P^{j;q}_{m,j}(1,2) | j_1 j_1(1)\rangle | j_2, j-j_1(2)\rangle}{\langle j_1 j_1(1)| \langle j_2, j-j_1(2)| P^{j;q}(1,2) | j_1 j_1(1)\rangle | j_2, j-j_1(2)\rangle^{1/2}}. \qquad (5.7)$$

To calculate the numerator of this expression it is necessary to express PO $P^{j,q}_{m,j}(1,2)$ in terms of generators $J_\pm(1,2)$, then to do the binomial expansion of their powers in terms of $J_\pm(1)$ and $J_\pm(2)$ using the Eq.(4.3). The last step is a calculation of matrix elements of $J_\pm^{\,s}(i)$ using formulas (2.8), (2.9).Here we give an explicit expression only for denominator:

$$\langle j_1 j_1(1)|\langle j_2, j-j_1(2)|P^{j;q}_{m,j}(1,2)|j_1 j_1(1)\rangle|j_2, j-j_1(2)\rangle =$$

$$= \frac{[2j+1]![j_1+j_2-j]!}{[j-j_1+j_2]!} \sum \frac{(-1)^r [j-j_1+j_2+r]!\, q^{-2rj_1}}{[r]![2j+r+1]![j_1+j_2-j-r]!} \qquad (5.8)$$

This sum may be calculated using one of Vandermonde formulae [27]. As a result we find

$$Q_q^{\,2} = q^{(j_1+j_2-j)(j-j_1+j_2+1)} \frac{[2j_1]![2j+1]!}{[j+j_1-j_2]![j_1+j_2+j+1]!}.$$

It should be noted that the matrix element (5.8) can be calculated also in recurrent manner [28]. Thus we obtain the following explicit analytical formula for q-CGCs

$$\langle j_1 m_1; j_2 m_2 | jm\rangle_q = \delta_{j,m_1+m_2}\, q^{-\frac{1}{2}(j_1+j_2-j)(j+j_1+j_2+1)+j_1 m_2 - j_2 m_1}$$

$$\times \sqrt{\frac{[2j+1][j+m]![j_2-m_2]![j_1+j_2+j+1]![j_1+j_2-j]![j_1-j_2+j]!}{[j-m]![j_1-m_1]![j_1+m_1]![j_2+m_2]![j-j_1+j_2]!}}$$

$$\times \sum_z \frac{(-1)^{j_1+j_2-j-z}[2j_2-z]![j_1+j_2-m-z]!\, q^{z(j_1+m_1)}}{[z]![j_1+j_2-j-z]![j_2-m_2-z]![j_1+j_2+j+1-z]!}. \qquad (5.9)$$

In a "classical" limit $q=1$ it coincides with the general formula for CGCs obtained in Ref.[18]. It is once more version of q-CGCs formulae alternative to ones derived in Refs.[9-11,14-17].

Simple analytical formulae can be found for important particular cases [28]

$$\langle 00; jm | j'm'\rangle_q = \langle jm; 00 | j'm'\rangle_q = \delta_{j,j'}\, \delta_{m,m'}, \qquad (5.10)$$

$$\langle jm, j'm' | 00\rangle_q = \delta_{j',j}\, \delta_{m,-m'} \frac{(-1)^{j-m}}{\sqrt{[2j+1]}}\, q^m, \qquad (5.11)$$

$$\langle j_1 m_1; j_2 m_2 | jj\rangle_q =$$

$$= \delta_{j,m_1+m_2} (-1)^{j_1-m_1}\, q^{\frac{1}{2}(j_1+j_2-j)(j-j_1+j_2+1)-(j+1)(j_1-m_1)} \times$$

$$\times \sqrt{\frac{[2j+1]![j_1+m_1]![j_2+m_2]![j_1+j_2-j]!}{[j_1-m_1]![j_2-m_2]![j_1-j_2+j]![j-j_1+j_2]![j_1+j_2+j+1]!}}, \qquad (5.12)$$

$$\langle j_1 j_1 ; j_2 m_2 | jm \rangle_q = \delta_{m, j_1 - m_2} \, q^{\frac{1}{2}(j_1+j_2-j)(j-j_1+j_2+1)-j_1(j-m)} \times$$

$$\times \sqrt{\frac{[2j+1][j+m]![2j_1]![j_2-m_2]![j-j_1+j_2]!}{[j-m]![j_2+m_2]![j_1-j_2+j]![j_1+j_2-j]![j_1+j_2+j+1]!}} \, . \qquad (5.13)$$

6. 3j-symbols and their symmetry properties

In Ref.[11] the q-analog of 3j-symbols was defined as a follows

$$\begin{pmatrix} j_1 & j_2 & j_3 \\ m_1 & m_2 & m_3 \end{pmatrix}_q = \frac{(-1)^{j_1-j_2-m_3}}{\sqrt{[2j_3+1]}} \, q^{-\frac{1}{3}(m_1-m_2)} \langle j_1 m_1 ; j_2 m_2 | jm \rangle_q \, . \qquad (6.1)$$

In order to include the Regge symmetry properties of 3j-symbols it is convenient to introduce the Regge symbol

$$\begin{pmatrix} j_1 & j_2 & j_3 \\ m_1 & m_2 & m_3 \end{pmatrix}_q \equiv \begin{bmatrix} j_1+m_1 & j_2+m_2 & j_3+m_3 \\ j_1-m_1 & j_2-m_2 & j_3-m_3 \\ j_3-j_1+j_2 & j_3+j_1-j_2 & -j_3+j_1+j_2 \end{bmatrix}_q \qquad (6.2)$$

that is invariant with respect transposition and even permutations of rows and columns. At odd permutations of rows and columns the phase factor $(-1)^{j_1+j_2+j_3}$ appears and the substitution $q \to \bar{q}$ takes place.

7. Tensor operators, Wigner-Eckart theorem

As a q-analog of rank k tensor operator we shall consider a set of $2k+1$ operators $T^k_{æ}(q)$ ($æ=k,k-1,\dots,-k+1,-k$) satisfying the following commutation relations with generators of $SU_q(2)$ algebra

$$[J_0, T^k_{æ}(q)] = æ \, T^k_{æ}(q) \, , \qquad (7.1)$$

$$J_{\pm} T^k_{æ}(q) = q^{æ} T^k_{æ}(q) J_{\pm} + \sqrt{[k \mp æ][k \pm æ + 1]} \, T^k_{æ \pm 1}(q) q^{-J_0} . \qquad (7.2)$$

Acting by generators $J_{0,\pm}$ on vectors $\Phi^{j,k}_{m,æ}(q) \equiv T^k_{æ}(q)|jm\rangle$ and taking account (7.1) and (7.2) we obtain

$$J_0 \left[T^k_{æ}(q)|jm\rangle \right] = (m+æ) T^k_{æ}(q)|jm\rangle \, , \qquad (7.3)$$

$$J_{\pm} \left[T^k_{æ}(q)|jm\rangle \right] = q^{æ} \sqrt{[j \mp m][j \pm m+1]} \, T^k_{æ}(q)|j, m\pm 1\rangle +$$

$$+ \, q^{-m} \sqrt{[k \mp æ][k \pm æ + 1]} \, T^k_{æ \pm 1}(q)|jm\rangle \, . \qquad (7.4)$$

From the comparison of these expressions with (4.2) it is clear that vectors $\Phi^{j,k}_{m,\varkappa}(q)$ are transforming as basis vectors of the tensor product $D^j \otimes D^k$ of IRs of $SU_q(2)$ algebra. Therefore it is possible to expand these vectors on components $\Phi^{j,k:j''}_{m''}(q)$ belong to the IR $D^{j''}$ of $SU_q(2)$

$$\Phi^{j,k}_{m,\varkappa}(q) = \sum_{j''m''} \langle jm;k\varkappa|j''m''\rangle_q \; \Phi^{j,k:j''}_{m''}(q) \; . \tag{7.5}$$

Multiplying both sides of this equation by vector $\langle j'm'|$ and taking account the orthogonality property

$$\left\langle j'm' \left| \Phi^{j,k:j''}_{m''}(q) \right\rangle \right. = \delta_{j',j''} \; \delta_{m',m''} \; f^{j,k}_{j'}(q) \; , \tag{7.6}$$

where $f^{j,k}_{j'}(q)$ is independent on m', m, $\varkappa$ we obtain the q-analog of well known Wigner-Eckart theorem:

$$\langle j'm'|T^k_\varkappa(q)|jm\rangle = \langle jm;k\varkappa|j'm'\rangle_q \; (j'\|T^k(q)\|j) \tag{7.7}$$

or in more standard form

$$\langle j'm'|T^k_\varkappa(q)|jm\rangle = \frac{\langle jm;k\varkappa|j'm'\rangle_q}{\sqrt{[2j+1]}}(-1)^{2k}\langle j'\|T^k(q)\|j\rangle . \tag{7.8}$$

As an example the tensor operator the first rank $J^1_\varkappa(q)$ ($\varkappa=0,\pm1$) is constructed by us from generators $J_{0,\pm}$ in explicit form:

$$J^1_{\pm1}(q) = \frac{\mp1}{\sqrt{[2]}} \, q^{-J_0} \, J_\pm \; , \tag{7.9a}$$

$$J^1_0(q) = \frac{1}{\sqrt{[2]}} \left[q^{-1}[2J_0] + (q-q^{-1})J_+J_-\right] =$$

$$= \frac{1}{\sqrt{[2]}} \left[q^{-1}[2J_0] + (q-q^{-1})(C_2-[J_0-1/2]^2)\right] \; . \tag{7.9b}$$

It is clear that these expressions are more complicate then in $SU(2)$ case but in the limit $q=1$ they coincide with standard cyclic components of angular momentum. Calculating necessary CGCs we find the following expression for the reduced matrix elements of the tensor (7.9)

$$\langle j'\|J^1(q)\|j\rangle = \delta_{jj'}\frac{1}{[2]}\sqrt{[2j][2j+1][2j+2]} \; . \tag{7.10}$$

For the unit operator we have

$$(j'\|I\|j) = \delta_{jj'}\sqrt{[2j+1]} \; . \tag{7.11}$$

On the base of the definition (7.1) and Wigner-Eckart theorem the total q-analog of the tensor operator algebra can be formulated.

In conclusion of this section it should be noted that the derivation of Wigner-Eckart theorem in terms of quantum group $SU_q(2)$ was given by A.Klimyk.The definition of tensor operators close to our one was given and actively exploited in the frame of q-boson calculus by L. Biedenharn [6].

8. Recoupling of angular momenta, 6j-symbol

The vector coupling of three angular momenta can be realized in two ways: $(j_1+j_2)+j_3$ and $j_1+(j_2+j_3)$. The transition between two these schemes can be done using Racah coefficients

$$|j_1j_2(j_{12}).j_3:jm\rangle_q=\sum_{j_{23}} U(j_1j_2j_3;j_{12}j_{23})_q|j_1j_2j_3(j_{23}):jm\rangle_q \tag{8.1}$$

It is useful to introduce a q-analog of 6j-symbol instead of Racah coefficients

$$U(abcd;ef)_q = \sqrt{[2e+1][2f+1]}\ (-1)^{a+b+c+d} \begin{Bmatrix} a & b & e \\ d & c & f \end{Bmatrix}_q . \tag{8.2}$$

If to express it in terms of 3j-symbols

$$\begin{Bmatrix} j_1 & j_2 & j_3 \\ l_1 & l_2 & l_3 \end{Bmatrix}_q = \sum_{m\ n} (-q)^{-m_1-m_2-m_3-n_1-n_2-n_3} (-1)^{j_1+j_2+j_3+l_1+l_2+l_3} \times$$

$$\times \begin{bmatrix} j_1 & j_2 & j_3 \\ m_1 & m_2 & m_3 \end{bmatrix}_q \begin{bmatrix} j_1 & l_2 & l_3 \\ -m_1 & n_2 & -n_3 \end{bmatrix}_q \begin{bmatrix} l_1 & j_2 & l_3 \\ -n_1 & -m_2 & n_3 \end{bmatrix}_q \begin{bmatrix} l_1 & l_2 & j_3 \\ n_1 & -n_2 & -m_3 \end{bmatrix}_q \tag{8.3}$$

then the symmetry properties of 6j-symbols can be easy found. Namely the 6j-symbols are invariant with respect to permutations of columns

$$\begin{Bmatrix} j_1 & j_2 & j_3 \\ l_1 & l_2 & l_3 \end{Bmatrix}_q = \begin{Bmatrix} j_2 & j_1 & j_3 \\ l_2 & l_1 & l_3 \end{Bmatrix}_q = \dots \ . \tag{8.4}$$

Also they are invariant with respect to substitution two arbitrary momenta in the first row by corresponding momenta from the second row

$$\begin{Bmatrix} j_1 & j_2 & j_3 \\ l_1 & l_2 & l_3 \end{Bmatrix}_q = \begin{Bmatrix} l_1 & l_2 & j_3 \\ j_1 & j_2 & l_3 \end{Bmatrix}_q = \dots \ . \tag{8.5}$$

Finally Racah coefficients $U(\dots)$ and 6j-symbols are invariant with respect to substitution $q\to\bar{q}$ as it can be seen from the general analytical formula (8.14) for Racah coefficients. The last one may be derived using the projection operators in the manner used in Ref [18] for usual AMT. As a result we have

$$U(J_1J_2JJ_3;J_{12}J_{23})_q = \sqrt{[2J_{12}+1][2J_{23}+1]}\ (-1)^{J-J_{23}-J_{12}+J_2}$$

$$\times \frac{\Delta(J_1J_2J_{12})\ \Delta(J_2J_3J_{23})\ \Delta(J_{12}J_3J)\ \Delta(J_1J_{23}J)}{[J_1-J_2+J_{12}]![-J_1+J_2+J_{12}]![J_2-J_3+J_{23}]![-J_2+J_3+J_{23}]!} \times$$

$$\times \frac{[J_{12}+J_3+J+1]![J_1+J_{23}+J+1]!}{[J_1-J_{23}+J]![-J_{12}+J_3+J]!} \times \tag{8.6}$$

$$\times \sum_r \frac{(-1)^r[J_1+J-J_{23}+r]![J_3+J-J_{12}+r]![J_2-J+J_{12}+J_{23}-r]!}{[r]![2J+r+1]![J_1-J+J_{23}-r]![J_3+J_{12}-J-r]![J_2+J-J_{12}-J_{23}+r]!}.$$

Here

$$\Delta(abc) = \sqrt{\frac{[a+b-c]![a-b+c]![-a+b+c]!}{[a+b+c+1]!}}\ .$$

In the particular case of one vanishing angular momentum in the 6j-symbol we obtain

$$\begin{Bmatrix} J_1 & J_2 & J_3 \\ 0 & J_3 & J_2 \end{Bmatrix}_q = \frac{(-1)^{J_1+J_2+J_3}}{\sqrt{[2J_2+1][2J_3+1]}}\ . \tag{8.7}$$

Authors are thankful to A.N. Kirillov, A.U. Klimyk and Ya. Soibelman for illuminating discussions.

Refereces

1. E.K. Sklyanin: Funct. Anal. Appl. **16** 263 (1982).
2. P.P. Kulish, N.Yu. Reshetikhin: Zapiski. Nauch. Semin. LOMI **101** 112 (1980).
3. V.G. Drinfeld: DAN SSSR **32** 254 (1985).
4. M. Jimbo: Lett. Math. Phys. **12** 247 (1986).
5. N.Yu. Reshetikhin: LOMI preprints E-4-87, E-17-87 (1988).
6. L.C. Biedenharn: J. Phys.A **22** L873 (1989).
7. A.J. Macfarlane: J. Phys.A **22** 4581 (1989).
8. M. Rosso: Comm. Math. Phys. **117** 581 (1988).
9. L.L. Vaksman: DAN SSSR **306** 269 (1989).

10. A.N. Kirillov, N.Yu. Reshetikhin: Preprint LOMI E-9-88 (1988).

11. Zhong-Qi-Ma: Preprint ICTP IC/89/162 (1989).

12. A.Ch. Ganchev, V.B. Petkova: Preprint ICTP IC/89/158 (1989).

13. R. Roche, D. Arnaudon: Lett. Math. Phys. **17** 295 (1989).

14. I.I. Kachurik, A.U. Klimyk: Preprint ITP-89-48E (1989).

15. H. Ruegg: Preprint UGVA-DPT 1989/08-625 (1989).

16. H.T. Koelink, T.H. Kornwinder: Preprint of Mathematical Institute, University of Leiden, W-88-12 (1988).

17. M. Nomura: J. Math. Phys. **30** 2397 (1989).

18. D.T. Sviridov, Yu.F. Smirnov: *Theory of optical spectra of transition metal ions*, Moscow, Nauka (1977) (in Russian).

19. D.A. Varshalovich, A.N. Moskalev, V.K. Khersonsky: *Quantum Theory of Angular Momentum*, Leningrad, Nauka (1975) (in Russian).

20. A.P. Yutsis, A.A. Bandzaitis: *Theory of angular momentum in quantum mechanics*, Vilnjus, Mintis (1965) (in Russian).

21. L.C. Biedenharn, J.D. Louck: *Angular Momentum in Quantum Physics*, Addison-Wesley (1981).

22. R.M. Asherova, Yu.F. Smirnov, V.N. Tolstoy: Theor. Math.Fiz. **8**, 255 (1971).

23. R.M. Asherova, Yu.F. Smirnov, V.N. Tolstoy: Matem. Zametki.**36** 15 (1979).

24. J. Shapiro: J. Math. Phys. **6** 1680 (1965).

25. D.P. Zelobenko: DAN SSSR **4** 317 (1984).

26. G. Racah: Phys. Rev. **62** 438 (1942).

27. G. Gasper, M. Rahman: *Basic Hypergeometric series*, Cambridge University Press (1989).

28. Yu.F. Smirnov, V.N. Tolstoy, Yu.I. Kharitonov:Preprint LINP №1607 (1990).