The Effect of Retardation on Galactic Rotation Curves

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Abstract.
Galaxies are huge physical systems having dimensions of many tens of thousands of light years. Thus any change at the galactic center will be noticed at the rim only tens of thousands of years later. Those retardation effects seems to be neglected in present day galactic modelling used to calculate rotational velocities of matter in the rims of the galaxy and surrounding gas. The significant differences between the predictions of Newtonian instantaneous action at a distance and observed velocities are usually explained by either assuming dark matter or by modifying the laws of gravity (MOND). In this paper we will show that taking general relativity seriously without neglecting retardation effects one can explain the radial velocities of galactic matter without postulating dark matter.

1. Introduction

The general theory of relativity (GR) is verified by many observations. Nevertheless, some observations seems not to fit GR and observed matter. As soon as 1933 Fritz Zwicky realized that the velocities of the Galaxies within the Comma Cluster are way larger than those predicted by the virial theorem in Newtonian theory [1]. He remarked that the amount of matter needed to account for the velocities could be 400 times that of the visible matter. Which led to postulating an unseen form of matter permeating the cluster. Volders in 1959 remarked that stars in the periphery of the neighbor spiral galaxy M33 do not move as expected [2]. The virial theorem in Newtonian Gravity predicts that $MG/r \sim Mv^2$, that is to say, the rotation curve should increase and at some point bend down and the velocity should drop off as $1/\sqrt{r}$. In the seventies Rubin and Ford [3, 4] showed for a very large sample of spiral galaxies that this behavior is a general feature: velocities at the periphery of the galaxies do not bend down, attain a plateau at some velocity for each galaxy. In figure 1 we see a rotation curves for the M33 galaxy describing this situation. In what follows we will show that such effects can be deduced from GR if retardation effects are not neglected.
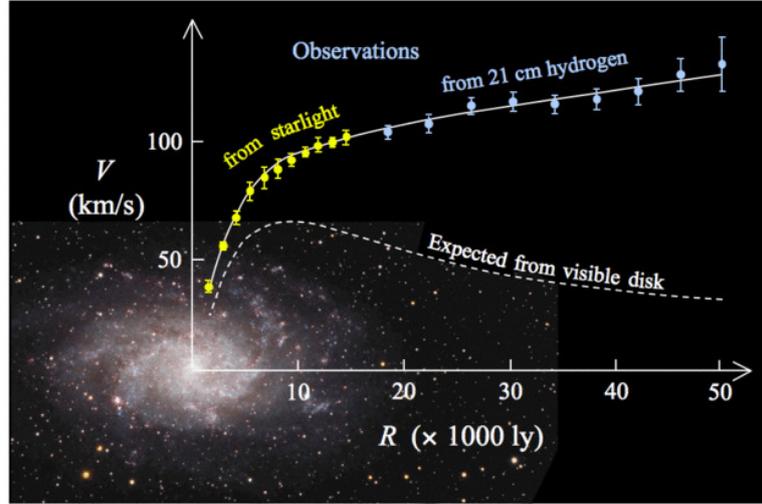
Figure 1. M33 rotation curve [5]

2. General Relativity

The general theory of relativity is based on two fundamental equations, Einstein equations [6, 8, 9, 7]:

$$G_{\mu\nu} = -\frac{8\pi G}{c^4} T_{\mu\nu} \qquad (1)$$

in which $G_{\mu\nu}$ is the Einstein tensor, $T_{\mu\nu}$ is the stress-energy tensor, $G \simeq 6.67\ 10^{-11}\ m^3 kg^{-1} s^{-2}$ is the gravitational constant and $c \simeq 3\ 10^8\ ms^{-1}$ is the velocity of light in vacuum (Greek indices are in the range $0-3$). And the geodesic equation:

$$\frac{d^2 x^\alpha}{ds^2} + \Gamma^\alpha_{\mu\nu} \frac{dx^\mu}{ds} \frac{dx^\nu}{ds} = \frac{du^\alpha}{ds} + \Gamma^\alpha_{\mu\nu} u^\mu u^\nu = 0 \qquad (2)$$

in which $x^\alpha(s)$ are the four dimensional coordinates of the particle in space-time, s is a parameter along the trajectory that for massive particles can be the length of the trajectory, $u^\mu = \frac{dx^\mu}{ds}$ and $\Gamma^\alpha_{\mu\nu}$ is the affine connection (Einstein summation convention is assumed). The stress-energy tensor of matter is usually taken in the form:

$$T_{\mu\nu} = (p + \rho c^2) u_\mu u_\nu - p\ g_{\mu\nu} \qquad (3)$$

In the above p is the pressure, ρ is the density. Lowering and raising indices is done through the metric $g_{\mu\nu}$ and the inverse metric $g^{\mu\nu}$, that is $u_\mu = g_{\mu\nu} u^\nu$. The same metric serves to calculate s:

$$ds^2 = g_{\mu\nu} dx^\mu dx^\nu, \qquad (4)$$

and the affine connection:

$$\Gamma^\alpha_{\mu\nu} = \frac{1}{2} g^{\alpha\beta} \left(g_{\beta\mu,\nu} + g_{\beta\nu,\mu} - g_{\mu\nu,\beta} \right), \qquad g_{\beta\mu,\nu} \equiv \frac{\partial g_{\beta\mu}}{\partial x^\nu} \qquad (5)$$

The affine connection serves to calculate the Riemann and Ricci tensors and the curvature scalar:

$$R^\mu_{\nu\alpha\beta} = \Gamma^\mu_{\nu\alpha,\beta} - \Gamma^\mu_{\nu\beta,\alpha} + \Gamma^\sigma_{\nu\alpha} \Gamma^\mu_{\sigma\beta} - \Gamma^\sigma_{\nu\beta} \Gamma^\mu_{\sigma\alpha}, \quad R_{\alpha\beta} = R^\mu_{\alpha\beta\mu}, \quad R = g^{\alpha\beta} R_{\alpha\beta} \qquad (6)$$

which in turn serves to calculate the Einstein tensor:

$$G_{\alpha\beta} = R_{\alpha\beta} - \frac{1}{2} g_{\alpha\beta} R. \qquad (7)$$

Hence matter distribution determines the metric through equation (1) and metric determines trajectories through equation (2) as is well known.

3. Linear Approximation of GR

Except for the extreme cases of compact objects (black holes and neutron stars) and the very early universe (big bang) one need not consider the full non-linear Einstein equation. In most other cases of astronomical interest (galactic dynamics included) one can linearize those equations around the flat Lorentz metric $\eta_{\mu\nu}$ such that[1]:

$$g_{\mu\nu} = \eta_{\mu\nu} + h_{\mu\nu}, \quad \eta_{\mu\nu} \equiv \text{diag}\,(1, -1, -1, -1), \quad |h_{\mu\nu}| \ll 1 \tag{8}$$

One than defines the quantity:

$$\bar{h}_{\mu\nu} \equiv h_{\mu\nu} - \frac{1}{2}\eta_{\mu\nu}h, \quad h = \eta^{\mu\nu}h_{\mu\nu}, \tag{9}$$

$\bar{h}_{\mu\nu} = h_{\mu\nu}$ for non diagonal terms. For diagonal terms:

$$\bar{h} = -h \Rightarrow h_{\mu\nu} = \bar{h}_{\mu\nu} - \frac{1}{2}\eta_{\mu\nu}\bar{h}. \tag{10}$$

It can be shown ([6] page 75 exercise 37, see also [7, 8, 9]), that one can choose a gauge such that the Einstein equations are:

$$\bar{h}_{\mu\nu,\alpha}{}^{\alpha} = -\frac{16\pi G}{c^4}T_{\mu\nu}, \qquad \bar{h}_{\mu\alpha,}{}^{\alpha} = 0. \tag{11}$$

Equation (11) can always be integrated to take the form [10][2]:

$$\bar{h}_{\mu\nu}(\vec{x}, t) = -\frac{4G}{c^4}\int \frac{T_{\mu\nu}(\vec{x}', t - \frac{R}{c})}{R}d^3x',$$

$$t \equiv \frac{x^0}{c}, \quad \vec{x} \equiv x^a \quad a, b \in [1, 2, 3], \quad \vec{R} \equiv \vec{x} - \vec{x}', \quad R = |\vec{R}|. \tag{12}$$

The factor before the integral is small: $\frac{4G}{c^4} \simeq 3.3\ 10^{-44}$ hence in the above calculation one can take $T_{\mu\nu}$ which is zero order in $h_{\alpha\beta}$. Let us now calculate the affine connection in the linear approximation:

$$\Gamma^{\alpha}_{\mu\nu} = \frac{1}{2}\eta^{\alpha\beta}\left(h_{\beta\mu,\nu} + h_{\beta\nu,\mu} - h_{\mu\nu,\beta}\right). \tag{13}$$

The affine connection has only first order terms, hence for a first order approximation of $\Gamma^{\alpha}_{\mu\nu}u^{\mu}u^{\nu}$ appearing in the geodesic, $u^{\mu}u^{\nu}$ is zeroth order. In the zeroth order:

$$u^0 = \frac{1}{\sqrt{1 - \frac{v^2}{c^2}}}, \qquad u^a = \vec{u} = \frac{\frac{\vec{v}}{c}}{\sqrt{1 - \frac{v^2}{c^2}}}, \qquad \vec{v} \equiv \frac{d\vec{x}}{dt}, \quad v = |\vec{v}|. \tag{14}$$

For non relativistic velocities:

$$u^0 \simeq 1, \qquad \vec{u} \simeq \frac{\vec{v}}{c}, \qquad u^a \ll u^0 \qquad \text{for} \quad v \ll c. \tag{15}$$

Inserting equation (13) and equation (15) in the geodesic equation we arrive at the approximate form:

$$\frac{dv^a}{dt} \simeq -c^2\Gamma^a_{00} = -c^2\left(h^a_{0,0} - \frac{1}{2}h_{00,}{}^a\right) \tag{16}$$

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[1] Private communication with the late Professor Donald Lynden-Bell
[2] For reasons why the symmetry between space and time is broken see [11, 12]

Let us now look at $T_{\mu\nu} = (p + \rho c^2)u_\mu u_\nu - p\, g_{\mu\nu}$. In the current case $\rho c^2 \gg p$, combining this with equation (15) we arrive at $T_{00} = \rho c^2$ while all other components of the tensor $T_{\mu\nu}$ are significantly smaller. This implies that $\bar{h}_{00}$ is significantly larger than other components of the tensor $\bar{h}_{\mu\nu}$. Of course one should be careful and not deduce from the different magnitudes of quantities that such a difference exist between their derivatives. In fact by the gauge condition in equation (11):

$$\bar{h}_{\alpha 0,}{}^0 = -\bar{h}_{\alpha a,}{}^a \qquad \Rightarrow \bar{h}_{00,}{}^0 = -\bar{h}_{0a,}{}^a, \quad \bar{h}_{b0,}{}^0 = -\bar{h}_{ba,}{}^a. \tag{17}$$

Hence the zeroth derivative of $\bar{h}_{00}$ (contains a $\frac{1}{c}$ factor) is the same order as the spatial derivative of $\bar{h}_{0a}$ and like wise the zeroth derivative of $\bar{h}_{0a}$ (which appears implicitly in equation (16)) is the same order of the spatial derivative of $\bar{h}_{ab}$. However, it is safe to compare spatial derivatives of $\bar{h}_{00}$ and $\bar{h}_{ab}$ and conclude that the former is significantly larger than the later. Using equation (10) and taking the above consideration into account we write equation (16) as:

$$\frac{dv^a}{dt} \simeq \frac{c^2}{4}\bar{h}_{00,}{}^a \Rightarrow \frac{d\vec{v}}{dt} = -\vec{\nabla}\phi = \vec{F}, \qquad \phi \equiv \frac{c^2}{4}\bar{h}_{00} \tag{18}$$

Thus ϕ is a gravitational potential of the motion which can be calculated using equation (12):

$$\phi = \frac{c^2}{4}\bar{h}_{00} = -\frac{G}{c^2}\int \frac{T_{00}(\vec{x}', t - \frac{R}{c})}{R}d^3x' = -G\int \frac{\rho(\vec{x}', t - \frac{R}{c})}{R}d^3x' \tag{19}$$

and $\vec{F}$ is the force per unit mass. If ρ is static we are in the realm of the Newtonian instantaneous action at a distance theory. However, it is unlikely that ρ is static as a galaxy will attract mass from the intergalactic medium.

4. Beyond the Newtonian Approximation

The retardation time $\frac{R}{c}$ which may be a few tens of thousands of years is short with respect to the time that the galactic density changes significantly. This means that we can write a Taylor series for the density [15]:

$$\rho(\vec{x}', t - \frac{R}{c}) = \sum_{n=0}^{\infty} \frac{1}{n!}\rho^{(n)}(\vec{x}', t)(-\frac{R}{c})^n, \qquad \rho^{(n)} \equiv \frac{\partial^n \rho}{\partial t^n}. \tag{20}$$

Inserting equation (20) into equation (19) and keeping the first three terms we will obtain:

$$\phi = -G\int \frac{\rho(\vec{x}', t)}{R}d^3x' + \frac{G}{c}\int \rho^{(1)}(\vec{x}', t)d^3x' - \frac{G}{2c^2}\int R\rho^{(2)}(\vec{x}', t)d^3x' \tag{21}$$

The first term will provide the Newtonian potential, the second term does not contribute, the third term will result in the lower order correction to the Newtonian theory:

$$\phi_r = -\frac{G}{2c^2}\int R\rho^{(2)}(\vec{x}', t)d^3x' \tag{22}$$

The total force per unit mass:

$$\begin{aligned} \vec{F} &= \vec{F}_N + \vec{F}_r \\ \vec{F}_N &= -\vec{\nabla}\phi_N = -G\int \frac{\rho(\vec{x}', t)}{R^2}\hat{R}d^3x', \qquad \hat{R} \equiv \frac{\vec{R}}{R} \\ \vec{F}_r &\equiv -\vec{\nabla}\phi_r = \frac{G}{2c^2}\int \rho^{(2)}(\vec{x}', t)\hat{R}d^3x' \end{aligned} \tag{23}$$

While the Newtonian force $\vec{F}_N$ is always attractive the retardation force $\vec{F}_r$ can be either attractive or repulsive. Also notice that while the Newtonian force decreases as $\frac{1}{R^2}$, the retardation force is independent of distance as long as the Taylor approximation of equation (20) is valid. For short distances the Newtonian force is dominant but as the distances increase the retardation force becomes dominant. Newtonian force can be neglected for distances significantly larger than the retardation distance:

$$R \gg R_r \equiv c\Delta t \qquad (24)$$

Δt is the typical duration in which the density ρ changes. Of course for $R \ll R_r$ the retardation effect can be neglected and only Newtonian forces should be considered. For large distances $r = |\vec{x}| \to \infty$ such that $\hat{R} \simeq \frac{\vec{x}}{|\vec{x}|} \equiv \hat{r}$ we obtain:

$$\vec{F}_r = \frac{G}{2c^2}\hat{r} \int \rho^{(2)}(\vec{x}',t)d^3x' = \frac{G}{2c^2}\hat{r}\ddot{M}, \qquad \ddot{M} \equiv \frac{d^2M}{dt^2}. \qquad (25)$$

Now as the galaxy attracts intergalactic gas its mass increases thus $\dot{M} > 0$, however, as the intergalactic gas is depleted the rate at which the mass increases must decrease hence $\ddot{M} < 0$. Thus in the galactic case:

$$\vec{F}_r = -\frac{G}{2c^2}|\ddot{M}|\hat{r} \qquad (26)$$

and the retardation force is attractive.

5. Dark Matter

In what circumstances can one confuse retardation with the effect of a non existent "dark matter"? Let us ignore retardation effects and suppose that radial velocities are a result of some mysterious dark matter. In this case we can write for a spherically symmetric mass distribution [13]:

$$-\frac{v_c^2}{r}\hat{r} = \vec{F}_d = -\frac{GM_d(r)}{r^2}\hat{r} \qquad (27)$$

v_c is the speed of a test particle of constant radius r and $M_d(r)$ is the amount of dark matter inside the radius r. Comparing equation (27) and equation (26) we see that the "dark matter" mass can be calculated as follows:

$$M_d(r) = \frac{r^2|\ddot{M}|}{2c^2} \qquad (28)$$

Now since:

$$M_d(r) = 4\pi \int_0^r r'^2\rho_d(r')dr', \qquad \frac{dM_d(r)}{dr} = 4\pi r^2\rho_d(r) \qquad (29)$$

it follows:

$$\rho_d(r) = \frac{|\ddot{M}|}{4\pi c^2 r} \qquad (30)$$

and the asymptotic rotation velocity is thus:

$$v_c = \sqrt{\frac{G}{2c^2}|\ddot{M}|r} \qquad (31)$$

This is consistent with observational data of [5] who concluded that the "dark matter" density decreases as $r^{-1.3}$ for M33. We notice that the asymptotic rotation velocity can only be achieved for distances which are much larger than the galaxy itself and the exceed considerably the distances for which Newtonian contributions are significant and at the same time do not violate the second order Taylor expansion approximation which we consider.

6. MOND

Another approach to explaining galactic rotation curves is the claim that either the laws of dynamics (Newton's second law) or the laws of Gravitation (GR) should be modified. This approach championed by Milgorm is denoted "MOND" (Modified Newtonian dynamics) [14]. In one version of this approach Newton's law of gravity is modified:

$$\vec{F}_M = -\frac{GM}{\mu(\frac{a}{a_0})r^2}\hat{r} \tag{32}$$

In the above μ is the interpolation function that should be 1 for $a_0 \ll a$. Let us assume:

$$\mu(\frac{a}{a_0}) = \frac{1}{1 + (\frac{a_0}{a})^2} \tag{33}$$

If $a_0 \gg a$, $\mu \simeq (\frac{a}{a_0})^2$. A test particle revolving in a constant radius will have centrifugal acceleration $a = \frac{v^2}{r}$ and thus:

$$\vec{F}_M = -\frac{GMa_0^2}{v^4}\hat{r} \tag{34}$$

For v constant at a far away distance this expression is similar to the retardation force and thus:

$$|\ddot{M}| = \frac{2Ma_0^2c^2}{v^4}. \tag{35}$$

Milgrom found $a_0 = 1.2 \ 10^{-10} ms^{-2}$ to be most fitting to the data. The mass of the M33 galaxy is $9.95 \ 10^{40} kg$ and the velocity far away[3] from the galaxy is $179,000 \ ms^{-1}$. We thus obtain $\ddot{M} \simeq 2.51 \ 10^{17} kgs^{-2}$ and a ratio $\frac{|\ddot{M}|}{M} \simeq 2.52 \ 10^{-24} \ s^{-2}$. This amounts to a typical accumulation acceleration time scale of $20,000$ years and retardation distance of $20,000$ light years which seems reasonable according to figure 1. We notice that such an accumulation acceleration cannot be maintained throughout the life of the galaxy and is connected with "recent" (in cosmological time scale) depletion of gas around the galaxy.

7. Conclusion

We show that "dark matter" and "MOND" effects are explained in the framework of standard GR as effects due to retardation without assuming any exotic matter or modifications of the theory of gravity.

What will happen if the mass out side the galaxy is not yet depleted or totally depleted? In this case $\ddot{M} = 0$ and retardation force should vanish. This was indeed reported recently [16] for the young galaxy NGC1052-DF2.

Retardation effects in electromagnetic theory were discussed in [17, 18, 19].

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[3] This is only a crude estimation, of course the retardation force does not allow strictly constant circular velocity profiles

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