STEPHANI'S SPACETIMES ARE NO-RADIATIVE

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ABSTRACT

The time dependence of the quadrupole moments of an arbitrary three-dimensional space-like volume of the rotational non symmetric dust spacetimes found by Stephani in 1982 is worked out. Dust is a good model for matter in free fall and the so called quadrupole formulae is in this way evaluated in this regime. According to these formulae the Stephani spacetimes are non radiative.

1. INTRODUCTION

The theory of gravitational waves, as laid by Einstein[1] soon after the publication of his general theory, and developed by himself[2] and others was not trusted, specially by Eddington[3], to be valid for self gravitating systems. Many attempts, starting with the work of Landau and Lifshitz[4], have been made (for one of the most elaborate, see 5) to accomplish a rigorous foundation. Notwithstanding, it is not the general view that success has been achieved. The so called quadrupole formulae are often rendered uncertain for systems like the binary systems (for one of the most radical, see 6). In this paper the time dependence of the quadrupole moments of an arbitrary three-dimensional space-like volume of any of a collection of dust spacetimes found by Stephani[7,8] in 1982 is evaluated. Dust is a good model for matter in free fall and the predictions of my result can be compared with other criteria for radiation. In section 2 the Stephani spacetimes are described; section 3 states the criteria for weak fields; in section 4 I look for appropiate coordinates to evaluate the quadrupole moments and in section 5 the time dependence of the quadrupole moments is found. The last section 6 is the conclusion of this work.

2. STEPHANI'S SPACE-TIMES

The field equation are

$$R_{ik} - \frac{1}{2}g_{ik}R = T_{ik} = \rho u_i u_k, \tag{2.1}$$

where R_{1k} is the Ricci tensor, g_{ik} the metric tensor, $R = R^i_i$ is the Riemann scalar, ρ is the density of the pressureless matter and u_i is the unit four velocity. The metric is of the form

$$ds^2 = dx^2 + dy^2 + N^2 dz^2 - dt^2, \tag{2.2}$$

with N given by:

$$N = ax^2 + by^2 - ct^2 + 2dxy + 2ext + 2fyt + g_1x + g_2y + g_3t + g_4, \tag{2.3}$$

where $a, ...f, g_1, ..., g_4$ being arbitrary functions of z satisfying

$$e^2 = -(a+b)(c+b); f^2 = -(a+b)(a+c); d^2 = (a+c)(b+c), \tag{2.4}$$

with

$$a+b+c \neq 0. \tag{2.5}$$

The mass density is given by

$$\rho(x,y,z,t) = -4\frac{(a+b+c)}{N}. \tag{2.6}$$

The metric has the following properties

i) In the general case the metric admits no killing vectors.

ii) The four velocity with components:

$$2u_x^2 = -\frac{(b+c)}{(a+b+c)},$$
$$2u_y^2 = -\frac{(a+c)}{(a+b+c)},$$
$$2u_t^2 = \frac{(a+b)}{(a+b+c)}, \tag{2.7}$$

has nonzero rotation if $\partial_z u_i \neq 0$. In the special case $\partial_z u_i = 0$, the metric is contained in the class of irrotational dust solutions found by Szekeres[9].

iii) The conditions of positiveness implied by (2.6) and (2.7) cannot be satisfied for arbitrary values of x, y and t, and therefore, the metric is not always positive definite.

We shall assume in this paper that we are working with a portion of the metric in which $\rho > 0$.

3. CONDITIONS FOR A WEAK FIELD

We shall state the flatness condition of a Stephani spacetime and then find weak field solutions as small departures from flatness.

By a detailed calculation of the Riemann tensor R_{ijkl} we have found:

Theorem. $a = b = c = 0$ is a necessary and sufficient condition for a Stephani spacetime to be flat.

We assume that the weakness of the field is governed by the magnitudes of a, b and c, and write:

$$a(z) = \varepsilon\tilde{a}(z); b(z) = \varepsilon\tilde{b}(z); c(z) = \varepsilon\tilde{c}(z), \tag{3.1}$$

where ε is a small parameter. $\tilde{a}(z), \tilde{b}(z), \tilde{c}(z)$ and the remaining arbitrary functions will not be assumed small.

The metric will be expressed as

$$g_{ik} = \eta_{ik} + \gamma_{ik}, \tag{3.2}$$

where η_{ik} is the metric for the flat space-time, that is (2.2) with N given by:

$$N_o = g_1 x + g_2 y + g_3 t + g_4, \tag{3.3}$$

and γ_{ik} is a small deviation from flatness of order ε.

The metric becomes, to first order:

$$ds^2 = dx^2 + dy^2 + N_o^2 dz^2 - dt^2 + \varepsilon \Phi dz^2, \tag{3.4}$$

with

$$\Phi = 2(g_1 x + g_2 y + g_3 t + g_4)(\tilde{a}x^2 + \tilde{b}y^2 - \tilde{c}t^2 + 2\tilde{d}xy + 2\tilde{e}xt + 2\tilde{f}yt), \tag{3.5}$$

whereas the density becomes

$$\rho = -4\varepsilon \frac{(\tilde{a} + \tilde{b} + \tilde{c})}{N_o}. \tag{3.6}$$

4. EQUATIONS OF TRANSFORMATION TO MINKOWSKI COORDINATES

Consider a flat space-time. Let $X^k : X^1 = X, X^2 = Y, X^3 = Z, X^4 = T$ denote a set of Minkowski coordinates, and $x^a : x^1 = x, x^2 = y, x^3 = z, x^4 = t$ a set of Stephani coordinates as in (2.2). In the following, a primed symbol for a quantity means that it is to be calculated in x^k : unprimed quantities are to be calculated in x^a.

If we put

$$\frac{\partial X^k}{\partial x^a} = P^k_a, \tag{4.1}$$

the equations of transformation for the Christoffel symbols are

$$\frac{\partial P^k_a}{\partial x^b} = \left\{ {c \atop a\ b} \right\} P^k_c. \tag{4.2}$$

Hence the problem consists in the determination of the 20 functions X^k, p^k_a satisfying the linear system of differential equations (4.1) and (4.2), and also the 10 finite equations:

$$\eta_{ab} - \eta'_{ik} P^i_a P^k_b = 0, \tag{4.3}$$

which are the equations of transformation of the metric tensor η_{ab}. The integrability conditions of (4.1) are satisfied identically because of (4.2) and the conditions of integrability of (4.2) are

$$R_{abcd} - R'_{ijkl}P^i_a P^j_b P^k_c P^l_d = 0, \tag{4.4}$$

where R_{abcd} is the Riemann tensor, which are also satisfied.

By integration we can find:

$$X^\alpha(x,y,z,T) = \tilde{X}^\alpha(x,y,z)T + \tilde{\tilde{X}}^\alpha(x,y,z),$$
$$T = T; \alpha = 1,2,3, \tag{4.5}$$

where $\tilde{X}^\alpha$ and $\tilde{\tilde{X}}^\alpha$ are functions of their arguments.

5. THE QUADRUPOLE MOMENTS

The quadrupole moments of a region V at constant time T are

$$D^{\alpha\beta} = \int\limits_{V,T=const.} \rho(X,Y,Z,T)[3X^\alpha X^\beta - \delta^{\alpha\beta}X^\mu X^\mu]dXdYdZ$$

$$= \int \rho(X(x,y,z,T),Y(x,y,z,T),Z(x,y,z,T),T)$$
$$\times[3X^\alpha(x,y,z,T)X^\beta(x,y,z,T) - \delta^{\alpha\beta}X^\mu X^\mu(x,y,z,T)]$$
$$\times{}^3J(x,y,z,T)dxdydz, \tag{5.1}$$

with

$${}^3J = \left|\frac{\partial(X,Y,Z)}{\partial(x,y,z)}\right|_{T=const.} \tag{5.2}$$

The Jacobian can be evaluated as:

$${}^3J = e\tilde{T}N_0, \quad e = \pm 1, \tag{5.3}$$

where $\tilde{T}$ is a function of x, y, z.

Finally, we see from (4.5) that the term is square brackets in the quadrupole formula (5.1) is of the form

$$3X^\alpha X^\beta - \delta^{\alpha\beta}X^\mu X^\mu = F^{\alpha\beta}(x,y,z)T^2 + G^{\alpha\beta}(x,y,z)T + H^{\alpha\beta}(x,y,z). \tag{5.4}$$

Inserting (3.6), (5.3) and (5.4) in (5.1) and omitting the parameter ε we obtain

$$D^{\alpha\beta} = -4e\int_V [(\tilde{a}+\tilde{b}+\tilde{c})\tilde{T}](x,y,z)$$

$$\times [F^{\alpha\beta}(x,y,z)T^2 + G^{\alpha\beta}(x,y,z)T + H^{\alpha\beta}(x,y,z)]dxdydz. \quad (5.5)$$

Since the description of V depends only on x, y, z and not on T it is readily apparent that

$$\dddot{D}^{\alpha\beta} = 0. \quad (5.6)$$

We conclude that according to the Einstein's quadrupole formula:

$$\frac{dE}{dt} = -\frac{1}{5}\dddot{D}^{\alpha\beta}\dddot{D}^{\alpha\beta}, \quad (5.7)$$

which gives the rate of energy loss carried away by gravitational waves, the matter in Stephani's dust space-times does not radiate. The result is independent of the selection of the moving mass.

6. CONCLUSION

This work shows that for the dust Stephani space-times the linear approximation predicts that a body of dust moving in a Stephani règime does not radiate gravitational waves. The result of the linear approximation is independent of the shape of the moving body. We expect to work out other criteria of radiation in the future.

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