

SLAC – PUB – 3762  
September 1985  
T

AXIAL ANOMALY AND INDEX THEOREM FOR A  
TWO-DIMENSIONAL DISK WITH BOUNDARY\*

ZHONG-QI MA<sup>†</sup>

*Stanford Linear Accelerator Center  
Stanford University, Stanford, California, 94305*

ABSTRACT

As a complement of the paper of Ninomiya and Tan, we obtain a correct index formula for a two-dimensional disk with a boundary which is a special case of the APS theorem.

Submitted to *Physical Review D*

---

\* Work supported by the Department of Energy, contract DE-AC03-76SF00515.

<sup>†</sup> On leave from the Institute of High Energy Physics, Academia Sinica,  
P.O. Box 918, Beijing, People's Republic of China.

Recently, we read a beautiful paper by Ninomiya and Tan (NT)<sup>1</sup> who discussed the index theorem, as a special case of a general index theorem proved by Atiyah, Patodi and Singer (APS),<sup>2</sup> for the manifold with a boundary in terms of “handed” non-local boundary conditions. For a manifold with boundaries NT obtained the general expression of the index theorem, including the boundary contributions:

$$n_+ - n_- = \int_X A(X) dX - \sum_j f(Y_j) \quad , \quad (1)$$

where  $A(X)$  is the anomaly density,  $n_+(n_-)$  is the number of chirality positive (negative) zero-energy solutions to the Dirac eigenvalue equation, and  $f(Y_j)$  is the surface contribution of the  $j$ -th boundary.

In physics, we usually use two kinds of coordinates, the coordinates with length dimension and the angular coordinates. There are some characteristics for the angular coordinates which should be dealt with carefully. Unfortunately, NT made some mistakes in the index theorem for the angular coordinates. In order to make the problem clear, we only discuss the index theorem for a two-dimensional disk with a boundary in this note. We use the same notation as NT.<sup>1</sup>

In a two-dimensional manifold we introduce a circular boundary about an appropriate origin, *i.e.*, restricting  $0 \leq r \leq \rho$  in polar coordinates  $(r, \theta)$ . Adopting a “radial” gauge where the general gauge potential is along  $\hat{\theta}$  direction,

$$\vec{V} = \hat{\theta}V(r, \theta) = (-\sin \theta \hat{x} + \cos \theta \hat{y}) V(r, \theta) \quad . \quad (2)$$

The massless Dirac operator is

$$i\mathcal{D} = -i\sigma_2 (\partial_x - iV_x) + i\sigma_1 (\partial_y - iV_y) = \begin{pmatrix} & L^+ \\ L & \end{pmatrix} \quad (3)$$

$$L = e^{i\theta} [\partial_r + B_r(\theta)] \quad , \quad L^+ = e^{-i\theta} [-\partial_r + B_r(\theta)] \quad , \quad (4)$$

$$B_r(\theta) = \frac{i}{r} \partial_\theta + V(r, \theta) \quad . \quad (5)$$

where  $V(r, \theta)$  is real, and  $B_r(\theta)$  is, for a fixed  $r$ , self-adjoint over the periodic interval.

The eigenfunction of  $i\mathcal{D}$  is denoted by  $\psi_E$ :

$$i\mathcal{D} \psi_E = E \psi_E \quad , \quad \psi_E = \begin{bmatrix} \phi_E(r, \theta) \\ \chi_E(r, \theta) \end{bmatrix} \quad (6)$$

$$L \phi_E = E \chi_E \quad , \quad L^+ \chi_E = E \phi_E \quad (7)$$

As a first step towards deriving an index theorem, proper boundary conditions must be imposed so that  $L$  and  $L^+$  are true adjoint of each other, that is,  $\langle \phi | L^+ | \chi \rangle = \langle \chi | L | \phi \rangle^*$ . However, owing to the factor  $e^{i\theta}$  ( $e^{-i\theta}$ ) in  $L$  ( $L^+$ ), the global boundary condition has a different form from Eq. (3.2) in Ref. 1. From Eq. (4) we have

$$L = \frac{e^{i\theta/2}}{\sqrt{r}} [\partial_r + B_r(\theta)] e^{i\theta/2} \sqrt{r} \quad (4')$$

$$L^+ = \frac{e^{-i\theta/2}}{\sqrt{r}} [-\partial_r + B_r(\theta)] e^{-i\theta/2} \sqrt{r}$$

The two components in  $\psi_E$  have different  $\theta$  dependences:

$$\psi_E = \frac{1}{\sqrt{r}} \begin{bmatrix} e^{-i\theta/2} & \phi_{1E}(r, \theta) \\ e^{i\theta/2} & \chi_{1E}(r, \theta) \end{bmatrix} \quad . \quad (6')$$

Therefore, we obtain the eigenvalue equations

$$L_1 = \partial_r + B_r(\theta) \quad , \quad L_1^+ = -\partial_r + B_r(\theta) \quad (8)$$

$$L_1 \phi_{1E} = E \chi_{1E} \quad , \quad L_1^+ \chi_{1E} = E \phi_{1E} \quad (7')$$

and the global boundary condition

$$\int_0^{2\pi} r d\theta e^{i\theta} \chi_{E'}^*(r, \theta) \phi_E(r, \theta) = \int_0^{2\pi} d\theta \chi_{1E'}^*(r, \theta) \phi_{1E}(r, \theta) = 0 \quad , \quad r = 0 \text{ or } \rho . \quad (9)$$

Define the continuous effective potential

$$\bar{V}(r) = \frac{1}{2\pi} \int_0^{2\pi} V(r, \theta) d\theta \quad . \quad (10)$$

The anomaly is

$$A = \frac{1}{2\pi} \iint F r dr d\theta = \frac{1}{2\pi} \iint \left[ \frac{1}{r} \frac{\partial}{\partial r} r V(r, \theta) \right] r dr d\theta = \rho \bar{V}(\rho) - r \bar{V}(r) \Big|_{r=0} . \quad (11)$$

Usually we demand  $\bar{V}(0) = 0$  in order to make gauge potential  $\vec{V}$  single-valued at the origin. Only for the manifold with a hole around the origin, the nonvanishing  $\bar{V}(\delta)$  is allowed, where  $\delta$  is the radius of the hole, and the anomaly is

$$A = \rho \bar{V}(\rho) - \delta \bar{V}(\delta) \quad . \quad (11')$$



With these conditions,  $L_1$  and  $L_1^\dagger$  are adjoints of each other and the APS theorem (1) can in principle be applied. The boundary contributions are

$$f(\rho) = \frac{1}{2} [\eta(\rho) - h_\rho] , \quad f(\delta) = \frac{1}{2} [\eta(\delta) + h_\delta] , \quad (17)$$

$$\eta(\rho) = - \lim_{s \rightarrow \infty} \sum_{\lambda_\rho \neq 0} [\text{sign } \lambda_\rho] |\lambda_\rho|^{-s} , \quad \eta(\delta) = \lim_{s \rightarrow \infty} \sum_{\lambda_\delta \neq 0} [\text{sign } \lambda_\delta] |\lambda_\delta|^{-s} , \quad (18)$$

$$h_r = \dim \ker B_r .$$

Noticing that  $J$  is half of the odd integer, we obtain

$$f(\rho) = \left\langle \rho \bar{V}(\rho) + \frac{1}{2} \right\rangle - \frac{1}{2} , \quad f(\delta) = \frac{1}{2} - \left\langle \delta \bar{V}(\delta) + \frac{1}{2} \right\rangle , \quad (19)$$

where

$$\langle A \rangle = A - [A] , \quad (20)$$

and  $[A]$  denotes the largest integer less than or equal to  $A$ .

Substituting Eqs. (11') and (19) into Eq. (1) we find the index of the massless Dirac operator  $i\mathcal{D}$

$$n_+ - n_- = A - f(\rho) - f(\delta) \quad (1')$$

$$\begin{aligned} &= \rho \bar{V}(\rho) - \delta \bar{V}(\delta) - \left\langle \rho \bar{V}(\rho) + \frac{1}{2} \right\rangle + \left\langle \delta \bar{V}(\delta) + \frac{1}{2} \right\rangle \\ &= \left[ \rho \bar{V}(\rho) + \frac{1}{2} \right] - \left[ \delta \bar{V}(\delta) + \frac{1}{2} \right] . \end{aligned} \quad (21)$$

For the disk without a hole around the origin

$$\left. \begin{aligned} f(\rho) &= \left\langle \rho \bar{V}(\rho) + \frac{1}{2} \right\rangle - \frac{1}{2} \\ f(0) &= 0 \end{aligned} \right\} , \quad (19')$$

$$\begin{aligned} n_+ - n_- &= \rho \bar{V}(\rho) - \left\langle \rho \bar{V}(\rho) + \frac{1}{2} \right\rangle + \frac{1}{2} \\ &= \left[ \rho \bar{V}(\rho) + \frac{1}{2} \right] \end{aligned} \quad (21')$$

As a check, we discuss a symmetric gauge with  $V(r, \theta) = V(r)$  in a disk,  $0 \leq r \leq \rho$ , without a hole around the origin. For this case we have

$$\psi_E(r, \theta) = \frac{1}{r} \begin{bmatrix} e^{-i\theta/2} & f_{\lambda E}(r) \\ e^{i\theta/2} & g_{\lambda E}(r) \end{bmatrix} e_\lambda(\theta) \quad (22)$$

$$e_\lambda(\theta) = \exp\{iJ\theta\}, \quad \lambda_r = -\frac{J}{r} + V(r),$$

$$L_1 = \partial_r - \frac{J}{r} + V(r), \quad L_1^+ = -\partial_r - \frac{J}{r} + V(r). \quad (23)$$

The chirality positive and negative zero-energy solutions to the Dirac eigenvalue equation are the following:

$$f_{\lambda 0}(r) = \exp \left\{ \int_1^r \left[ \frac{J}{r'} - V(r') \right] dr' \right\}$$

(23a)

$$\sim \begin{cases} cr^J, & r \sim 0 \\ \exp \left\{ J \ln \rho - \int_1^\rho V(r') dr' \right\}, & r = \rho \end{cases},$$

$$g_{\lambda 0}(r) = \exp \left\{ - \int_1^r \left[ \frac{J}{r'} - V(r') \right] dr' \right\}$$

(23b)

$$\sim \begin{cases} cr^{-J}, & r \sim 0 \\ \exp \left\{ -J \ln \rho + \int_1^\rho V(r') dr' \right\}, & r = \rho \end{cases}.$$

$$\frac{f'_{\lambda 0}(\rho)}{f_{\lambda 0}(\rho)} = \frac{1}{\rho} [J - \rho V(\rho)] = -\lambda_\rho$$

(24)

$$\frac{g'_{\lambda 0}(\rho)}{g_{\lambda 0}(\rho)} = \frac{1}{\rho} [-J + \rho V(\rho)] = \lambda_\rho$$

where we assume that  $\rho > 1$  without loss of generality. Imposing the global boundary conditions (16) at  $r = 0$  and  $r = \rho$ , we get ( $J$  is half of the odd integer!)

$$\begin{aligned} \frac{1}{2} \leq J \leq \rho V(\rho) & \quad \text{for} \quad f_{\lambda 0} \\ -\frac{1}{2} \geq J > \rho V(\rho) & \quad \text{for} \quad g_{\lambda 0} \end{aligned}$$

(25)

and

$$\begin{aligned} n_+ &= \left[ \rho V(\rho) + \frac{1}{2} \right] \theta[V(\rho)] \\ n_- &= - \left[ \rho V(\rho) + \frac{1}{2} \right] \theta[-V(\rho)] \end{aligned} \quad (26)$$

so Eq. (21') follows.

If, for example,

$$V(r) \sim \begin{cases} 0, & r \sim 0 \\ \frac{F}{r}, & r \lesssim \rho \end{cases}, \quad (27)$$

we have

$$n_+ - n_- = \left[ F + \frac{1}{2} \right]. \quad (28)$$

Boyanovsky and Blankenbecler<sup>4</sup> also obtained this result for  $F + \frac{1}{2} \neq \text{integer}$ .

For the constant  $H$ -field example,

$$V(r) = \frac{1}{2} H r, \quad (29)$$

we have

$$\phi_0(r, \theta) = r^{J-(1/2)} \exp \left\{ i \left( J - \frac{1}{2} \right) \theta - \frac{1}{4} H r^2 \right\} \quad (30a)$$

$$\chi_0(r, \theta) = r^{-J-(1/2)} \exp \left\{ i \left( J + \frac{1}{2} \right) \theta + \frac{1}{4} H r^2 \right\} \quad (30b)$$

Due to the left-handed condition at  $r = 0$ , only  $J \geq \frac{1}{2}$ , *i.e.*,  $J - \frac{1}{2} \geq 0$ , is accepted for Eq. (30a), and  $J \leq -\frac{1}{2}$ , *i.e.*,  $J + \frac{1}{2} \leq 0$ , for Eq. (30b). Equation (16b) at

$r = \rho$  leads to the result that  $\frac{1}{2} \leq J \leq \frac{1}{2} H \rho^2$  for  $\phi_0$  and  $-\frac{1}{2} \geq J > \frac{1}{2} H \rho^2$  for  $\chi_0$ . Together we obtain a compact expression for the index

$$n_+ - n_- = \left[ \frac{1}{2} H \rho^2 + \frac{1}{2} \right]. \quad (31)$$

For the r.h.s. of the APS theorem (1'), the first term contributes  $\frac{1}{2} H \rho^2$ , and the surface term, from Eq. (19'), is

$$f(\rho) = \left\langle \frac{1}{2} H \rho^2 + \frac{1}{2} \right\rangle - \frac{1}{2} \quad (33)$$

$$f(0) = 0$$

which is precisely what is needed for the APS theorem (1'). This result holds whether or not there is a small hole around the origin, because the boundary conditions (9) and (16a) at  $r = 0$  are satisfied even though  $J = \pm 1/2$ . If there is a small hole with a radius  $\delta$  around the origin and  $V(\delta) \neq 0$ , we have

$$f(\rho) = \left\langle \frac{1}{2} H \rho^2 + \frac{1}{2} \right\rangle - \frac{1}{2} \quad (34)$$

$$f(\delta) = \frac{1}{2} - \left\langle \frac{1}{2} H \delta^2 + \frac{1}{2} \right\rangle,$$

and  $\frac{1}{2} H \delta^2 < J \leq \frac{1}{2} H \rho^2$  for  $\phi_0$  and  $-\frac{1}{2} H \delta^2 \geq J > \frac{1}{2} H \rho^2$  for  $\chi_0$ ,

$$\begin{aligned} n_+ - n_- &= \left[ \frac{1}{2} H \rho^2 + \frac{1}{2} \right] - \left[ \frac{1}{2} H \delta^2 + \frac{1}{2} \right] \\ &= \frac{1}{2} H (\rho^2 - \delta^2) - f(\rho) - f(\delta). \end{aligned} \quad (35)$$

In summary, Ninomiya and Tan wrote a beautiful paper; unfortunately, with a defect. This note is only a complement for their paper.

## ACKNOWLEDGEMENTS

The author would like to thank Professor R. Blankenbecler and Dr. D. Boyanovsky for discussion. It is a pleasure to thank Professor S. D. Drell and the Theory Group at SLAC for their warm hospitality. This paper is supported by the Department of Energy, under contract DE-AC03-76SF00515.

## References

1. M. Ninomiya and Chung-I Tan, Nucl. Phys. **B257**, 199 (1985).
2. M. F. Atiyah, V. K. Patodi and I. M. Singer, Math. Proc. Camb. Phil. Soc. **77**, 43 (1975); **78**, 405 (1975); **79**, 71 (1976).
3. If the global boundary condition at  $r = \rho$  of NT is imposed [see Eqs. (6.5)–(6.8) in Ref. 1 and  $f_\lambda(\rho) \neq 0$  when  $\lambda \geq 0$ ,  $g_\lambda(\rho) \neq 0$  when  $\lambda < 0$ ], the nonvanishing  $f_{\lambda_1}(\rho) e_{\lambda_1}(\theta)$  with the lowest  $\lambda = \lambda_1$  and the nonvanishing  $g_{\lambda_2}(\rho) e_{\lambda_2}(\theta)$  with the largest  $\lambda = \lambda_2$  will not satisfy Eq. (9), because  $e^{i\theta} e_{\lambda_1} = e_{\lambda_2}$ .
4. D. Boyanovsky and R. Blankenbecler, Phys. Rev. **D31**, 3234 (1985). They imbedded the radial half-line problem onto the full line and recovered the problem of interest by a simple and tame limiting procedure, instead of the global boundary condition (16). See Eqs. (2.54) and (2.76) in their paper.