Symmetries of Differential Equations in Mathematical Physics

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The study of continuous symmetry groups of differential equations dates back almost a century to the oringinal work of Sophus Lie[1]. Since that time most of the work on Lie's theory has turned away from the connection with differential equations probably because it is not general enough; not every differential equation admits a nontrivial Lie group. However, it is precisely the differential equations which are of interest to physicists and applied mathematicians that do admit symmetries[2]. The recent books of Miller[3] and Vilenkin[4] have demonstrated the close connection between Lie theory and the special functions arising from certain differential equations. The symmetry group provides a degree of order and understanding to the conglomerate of special funtions identities. However, one can gain further geometrical insight by starting from certain fundamental differential equations, like the Laplace-Beltrami equation on a Riemannian space of constant curvature. The symmetries of such an equation can be linked to the coordinate curves in which the equations admits a separation of variables. The symmetry group can then be used to derive identities between the various separable solutions. I believe that the major part if not all of special function theory can be understood in this light. This approach has now been developed to the stage where one is gaining information, much of which is new, about not only the more common special functions, but also about Lamé and Ince functions,spheroidal, ellipsoidal, and anharmonic oscillator wave functions and others. I would like to mention at this point that the original idea of relating symmetries of differential equations to separable coordinate systems was formulated about ten years ago by P. Winternitz and his collaborators[5] in the Soviet Union and much of the development since then is due to E. G. Kalnins and W. Miller Jr.[6].

Up until now almost all that I have said pertains to linear differential equations and it is this case which will be discussed in the sequel. However a few words are in order about the more difficult and less extensively developed nonlinear equations. It was noticed some time ago[7] how symmetries could be used to generate similarity solutions of nonlinear differential equations where more general techniques are lacking. These similarity methods are especially applicable to boundary value problems which occur in thermo and hydrodynamics. However, it has only been recently that a systematic approach to these problems has been given[2,8,9].

As far as the separation of variables is concerned, the one nonlinear case which is fairly well known is the spatial separation of the Hamilton-Jacobi equation[10,11]

$$\frac{1}{2m} S_{\underset{\sim}{x}} \cdot S_{\underset{\sim}{x}} + S_t = 0$$

The separation of this equation in mixed space and time coordinates has recently been studied[12] and it appears that it can be related to only a subgroup of the full symmetry group[13]. This subgroup is the subgroup of linear transformations and is related to the time dependent Schrödinger equation, and hence to a quantization prescription[13]. We will have more to say about the separation of variables and symmetries of the Schrödinger equation later. Suffice it now to say that the general connection between Lie theory and separation of variables that will be described breaks down (although not entirely) for nonlinear equations. Indeed the concept of separability is, in general, not well defined.

A. General prescription

Given a general at most second order linear differential equation in n variables x_i (assumed to be real)

$$Qu = 0 \tag{1}$$

with $Q = \alpha_{ij}(\underset{\sim}{x})\partial_{x_i x_j} + \beta_i(\underset{\sim}{x})\partial_{x_i} + \gamma(\underset{\sim}{x})$ where α_{ij}, β_i, γ are locally C^∞ functions. We will only consider second order differential equations although the generalization to higher order can be made.

1) Determine the symmetry group for (1), i.e. the group of local transformations of the form

$$[T(g)u](\underset{\sim}{x}) = \mu(\underset{\sim}{x};g)\, u(\underset{\sim}{x}\cdot g) \tag{2}$$

where μ is a C^∞ function and $x' = (\underset{\sim}{x}\cdot g)$ denotes the group action over R^n, such that $Q'[T(g)u] = 0$, whenever u satisfies (1), where Q' is Q written in the primed variables. It follows from Lie theory that the set of such transformations forms a local Lie group G. For practical purposes it is more convenient to work infinitesimally. Then writing $T(g) \simeq 1 + \epsilon L(\underset{\sim}{x}, \partial_{\underset{\sim}{x}}) + \cdots$, we have the existence of a C^∞ function $\lambda(x)$ such that

$$[Q, L] = \lambda Q \tag{3}$$

Writing $L = a_i(\underset{\sim}{x})\partial x_i + b(\underset{\sim}{x})$ for a_i, $b \in C^\infty$, Eq. (3) determines the functions a_i and b and thus the Lie algebra $\mathfrak{g}$ of G. As mentioned previously there may be no symmetries at all. Of course, we are interested in the case where these is a nontrivial symmetry group.

2) Determine the second order symmetries of (1), i.e. the set $\mathcal{S}$ of differential operators (modulo Q) $S = c_{ij}(x)\partial_{x_i x_j} + a_i(x)\partial_{x_i} + b(x)$ such that (3) is satisfied upon replacing L by S and where now $\lambda(x)$ is a first order differential operator with C^∞ coefficients. It is emphasized that $\mathcal{S}$ does not necessarily form a Lie algebra; however, it does form a vector space which carries a representation of G. It is important to notice that G acts on $\mathcal{S}$ and splits $\mathcal{S}$ into G-orbits. There are two relevant types[14] of second order symmetries

Type 1: the elements of $\mathcal{S}$ are all second order members of the universal enveloping algebra $\mathcal{U}$ of $\mathcal{G}$.

Type 2: At least one element of $\mathcal{S}$ is not a member of $\mathcal{U}$. In general we are interested in classifying all orbits of (n-1) commuting operators of $\mathcal{S}$.

3) Find the coordinate systems such that (1) separates variables, i.e. introduce new variables v_i with $x_i = X_i(v_j)$ such that $u = R(v_j) \prod_{i=1}^{n} U_i(v_i)$ reduces (1) to a set of ordinary differential equations for the functions $U_i(v_i)$. The function $R(v_i)$ is called a multiplier or modulation factor[15] and is determined from the analysis. When Eq. (1) takes the form

$$[\Delta + E - V(x_i)]u = 0 \tag{4}$$

where $E \neq 0$ and Δ is the Laplace-Beltrami operator for a space of constant Riemannian curvature, the method of Stäckel[10,16,17] can be used to find the separable coordinates. When (1) does not have the form (4) we need recourse to other methods. For example for the wave or Laplace equation (i.e. $E = V = 0$) one can find orthogonal coordinates by the method of obtaining confocal cyclides from hyperspherical coordinates[15,18,6j]. This seems to be related to conformal symmetries. When none of these methods work, one simply uses brute force to find the coordinates. This usually applies to nonorthogonal coordinates[6k]. It should be mentioned that the classification of separable coordinate systems is really a classification of equivalence classes (i.e. orbits) of separable systems defined by some reasonable group of geometric symmetries H which may or may not be the symmetry group G. In general we are interested in both H-and G-inequivalent separable systems.

4) Associate with the (n-1) separation constants λ_i for a given separable system of (1) an H or G orbit of (n-1) commuting members $\{S_i\}$ of $\mathcal{S}$, such that

$$S_i u = \lambda_i u \tag{5}$$

For equations of the form (4), the existence of such a set S_i has been shown[19]. For all other equations treated so far it has always turned out to be the case although no general theorem as yet exists. Let us also mention that if a variable appears in (1) only to first order in its derivative, the corresponding operators will also be of first order and thus a member of $\mathcal{g} \subset \mathcal{S}$.

5) Find a simpler model for the Lie algebra $\mathcal{g}$ and Lie group G in the sense that it acts over a space of lower dimension. If possible construct a Hilbert space and a unitary representation of G for both models and the associated unitary transformation between the two models. Do all calculations in the simple model such as spectral analysis, computation of overlap functions, etc. This provides the derivation of many expansion theorems between the generalized eigenbases for each separable system. If the construction of a Hilbert space is not possible, use Weisner's method[3,20] to obtain generating functions relating various bases.

Steps 1)-4) provide the basic procedure for obtaining and relating the symmetries of a differential equation to the separable coordinate systems for that equation. Although step 5) is somewhat extra its importance cannot be overestimated. It is precisely this step which enables one to derive the kind of information about the solutions of a given differential equation which physicists and applied mathematicians are interested in. It provides deep insight into integral relations and expansion formalae between the various special functions which occur as eigenbases for unitary representations of Lie groups. Some of these eigenbases can be related to subgroup reductions $G_n \subset \cdots \subset G_1 \subset G$ where $G_0, \cdots, G_n$ denote continuous subgroups of G. The remaining eigenbases have been called nonsubgroup reductions although it appears possible to relate these to subgroup reduction where the subgroups may now be discrete[21].

In order to illustrate the above procedure in more concrete terms we discuss some examples.

B. <u>Helmholtz Eq. in Euclidean 2-space</u>

We consider the equation

$$u_{x_1 x_1} + u_{x_2 x_2} + \lambda u = 0 \qquad (6)$$

1) It is straight forward to calculate the symmetry group for (6) and the results give the well known group E(2) with its Lie algebra e(2). A basis for e(2) is given by

$$M = x_1 \partial_{x_2} - x_2 \partial_{x_1} \qquad P_1 = \partial_{x_1} \qquad P_2 = \partial_{x_2} \qquad (7)$$

whose integrated group action is well known and will not be given.

2) The second order symmetries $\mathcal{S}/\mathcal{G}$ are of type 1 which are spanned by the second order members of the enveloping algebra M^2, P_1^2, P_2, P_1P_2, $MP_1 + P_1M$, $MP_2 + P_2M$. The action of E(2) splits $\mathcal{S}/\mathcal{G}$ into the orbits $M^2 + \ell P_1^2$, $MP_1 + P_1M$, M^2, P_1^2 .

3) The separation of variables for (6) is well known[17,22] and there are four E(2)-inequivalent systems given by i) Cartesian, ii) polar, iii) parabolic, iv) elliptic.

4) The correspondence between orbit representatives and separable systems is one-one[5c]. Respectively we have

i) $x_1, x_2 \longleftrightarrow P_1^2$

ii) $x_1 = r\cos\theta$, $x_2 = r\sin\theta \longleftrightarrow M^2$

iii) $x_1 = \frac{1}{2}(\xi_1^2 - \xi_2^2)$, $x_2 = \xi_1\xi_2 \longleftrightarrow MP_1 + P_1M$

iv) $x_1 = \ell\,\mathrm{ch}\rho\,\cos\sigma$, $x_2 = \ell\,\mathrm{sh}\rho\,\sin\sigma \longleftrightarrow M^2 + \ell P_1^2$

5) A simpler model for e(2) is

$$\mathcal{M} = \partial_\theta \qquad \mathcal{P}_1 = i\lambda\cos\theta \qquad \mathcal{P}_2 = i\lambda\sin\theta \tag{8}$$

We can set up Hilbert spaces $L^2(R_2)$ such that (7) are skew-adjoint operators and $L^2(S^1)$ where S^1 is the unit circle such that (8) are skew-adjoint operators. A unitary transform between $L^2(S^1)$ and $L^2(R_2)$ is

$$F(\underset{\sim}{x}) = \int_0^{2\pi} d\theta\, e^{i\underset{\sim}{k}\cdot\underset{\sim}{x}\lambda} f(\theta) \tag{9}$$

where $\underset{\sim}{k} = (\cos\theta , \sin\theta)$ and $f \in L^2(S^1)$. We then find the following basis functions for the four systems in the two models.

Type		$L^2(S^1)$	$L^2(R_2)$
i)	Cartesian	delta	Product of exponentials
ii)	Polar	exponential	Product of Bessel and exponential
iii)	Parabolic	Powers of trig. functions	Products of Parabolic Cylinder
iv)	Elliptic	Mathieu	Product of Mathieu

Writing (9) explicitly for each of the above basis functions gives us integral identities. Computing overlap functions in the $L^2(S^1)$ model allows us to derive expansion formulas for the $L^2(R_2)$ model.

C. Free particle Schrödinger equation in two space and one time dimensions[6g,h]

We consider the equation

$$u_{x_1x_1} + u_{x_2x_2} + iu_t = 0 \tag{10}$$

It should also be mentioned here that most of the following remains unchanged if instead of the Schrödinger equation we remove the i and consider the heat equation[8].

1) The symmetry group for (10) can be calculated[23] and we find the structure $G = [SL(2,R) \otimes O(2)] \circledS W_2$ the semi-direct product of the special linear group $SL(2,R)$ with the 6 dimensional Weyl group W_2. A basis for its Lie algebra is

$$\begin{aligned} K_{-2} &= \partial_t \qquad K_2 = t^2\partial_t + t\,\underline{x}\cdot\partial_{\underline{x}} + \frac{i\underline{x}^2}{4} + \frac{t}{2} \\ D &= 2t\partial_t + \underline{x}\cdot\partial_{\underline{x}} + 1 \qquad M = x_1\partial_{x_2} - x_2\partial_{x_1} \\ P_j &= \partial_{x_j} \qquad B_j = -t\partial_{x_j} + \frac{i}{2}x_j \qquad E = i \end{aligned} \tag{11}$$

Again (11) can be integrated to a local Lie group whose details are omitted.

2) The second order symmetries are of type 1 and turn out to be all symmetric quadratic forms of B_j, P_j, E and M, plus the elements of $\mathcal{G}$. Here we are interested in G-orbits of both $\mathcal{G}$ and the factor space $\mathcal{S}/\mathcal{G}$.

3) We can first separate off the t variable[6f,g,24]. This can be done in 4 different G-inequivalent ways. This reduces the problem to the Helmholtz equation with the addition of one of four different potentials which correspond to the four different t separations, i.e.

$$u_{v_1 v_1} + u_{v_2 v_2} + [E - V(v_1, v_2)]u = 0 \tag{12}$$

where V is one of the following four potentials $V = 0, v_1, \pm(v_1^2 + v_2^2)$; i. e. a free particle, linear potential, attractive or repulsive harmonic oscillator. Then the problem reduces to finding the separable systems of the Helmholtz equation which are compatible with the added potentials. In all there are 15 G-inequivalent systems listed as follows: free particle, systems i)-iv); linear potential, system i) and iii); attractive and repulsive oscillators with systems i), ii) and iv) each; three systems obtained by separating the x_1 variable and solving the corresponding one-space dimensional version[6f] of (10). However, the whole group G is not so easy to visualize geometrically so it is of interest to classify the separable systems up to equivalence of the more geometrically meaningful subgroup $D \circledS G_2$ of the two-dimensional Galilei group extended by dilatations. With this definition of equivalence, there are 26 separable coordinate system. A doubling occurs for all but the attractive harmonic oscillator types.

4) To each separable coordinate system there corresponds a commuting orbit pair (K,S) where $K \in \mathcal{G}/\{E\}$ and $S \in \mathcal{S}/\mathcal{G}$. The correspondence, however, is not one-one. The following orbits in $\mathcal{G}/\{E\}$ correspond to t-separation: K_{-2} or $K_2 \longleftrightarrow$ free particle; $K_{-2} + B_1$ or $K_2 + P_1 \longleftrightarrow$ linear potential; $K_{-2} + K_2$ or $D \longleftrightarrow$ repulsive harmonic oscillator; $K_{-2} - K_2 \longleftrightarrow$ attractive harmonic oscillator. The choice for S then is similar to what we had

previously for the Helmholtz equation with an additional term corresponding to the potential.

5) A simpler two variable model for $\mathcal{G}$ is obtained by putting $t = 0$ in (11) and making the replacement $\partial_t \rightarrow i(\partial_{x_1x_1} + \partial_{x_2x_2}) = i\Delta$. We will denote the generators in this model by script letters corresponding to the generators (11). The group action for this model then gives rise to a group representation by integral transforms[25]. A one-parameter subgroup of this group representation is the transform which connects the two models

$$F(\underset{\sim}{x},t) = e^{t\mathcal{K}_2} f(x) = \frac{1}{4\pi i t} \iint_{-\infty}^{\infty} d^2y \exp[\bar{\lambda}(\underset{\sim}{x}-y)^2/4t] \; f(y) \tag{13}$$

This transform is a unitary transform from $L^2(R_2)$ to $L^2(R_2)$ solutions of (10). We provide an explicit example by applying (13) to the case of the linear potential in parabolic coordinates. In this case the separable coordinates are $x_1 = \tau(\xi_1^2 - \xi_2^2)/2 + \frac{1}{\tau}, x_2 = \xi_1\xi_2\tau, t = \tau$ with the commuting orbit pair ($K_2 + P_1$, $B_2M + MB_2 + P_2^2$) and the corresponding script generators for the simple model. Without going into further details[24] we simple remark that the script generators give rise to an exponential times an anharmonic oscillator wave function, i. e. an $L^2(R_1)$ solution of

$$h'' - (\mu + \lambda x^2 + \tfrac{1}{4}x^4) = 0$$

denoted by $h_n(x;\lambda,\frac{1}{4})$. Then in the three variable model we have the separable solutions $h_n(\xi_1;\lambda,\frac{1}{2})\,h_n(i\xi_2;\lambda,-\frac{1}{2})$ times a modular factor (exponential here). The transform (13) then gives rise to an integral identity

$$h_n(\xi_1;\lambda,\tfrac{1}{2})\,h_n(i\xi_2;\lambda,-\tfrac{1}{2}) = \tau_n(\lambda)\int_{-\infty}^{\infty} dy\, e^{i\xi_1\xi_2 y} Ai\left[\tau^{-1/3}\left(\frac{\xi_1^2-\xi_2^2}{2} - 2\lambda + \frac{y^2}{2}\right)\right] h_n(y;\lambda,\tfrac{1}{4})$$

where $Ai(z)$ is an Airy function. It is remarkable that Lie theory can provide information about problems as asymmetrical as the cuartic anharmonic oscillator. Many other identities can be found by similar techniques. An addition theorem[26] can be derived for the functions h_n. This method has also been applied to the wave functions for the Stark effect[26] of the H-atom, the sextic anharmonic oscillator[27] as well as many other functions.

C. Conclusion

As a conclusion we present a table listing the differential equations which have been studied from the point of view of symmetry groups and separation of variables along with the relevant references to the literature. It should also be mentioned that there is a forthcoming book on the subject of symmetries and separation of variables by W. Miller, Jr.

Equation	Group	Separable System	Reference
$\Delta u + \lambda u = 0$			
i) 2-dimensions			
Euclidean			
$\lambda \neq 0$	E(2)	4, in text	5c,6a
$\lambda = 0$	infinite	infinite, conform.	
Pseudo-Euclidean			
$\lambda \neq 0$	E(1,1)	11	6c,d
$\lambda = 0$	infinite	infinite, conform.	
Sphere	O(3)	2, sph., ellpt.	5d,6e,22,28a
2-sheet Hyperboloid	O(2,1)	9	5a,b,d,6e,22,28b
ii) 3-dimensions			
Euclidean			
$\lambda \neq 0$	E(3)	11	6b,17,22,29,30
$\lambda = 0$	O(4,1)	17	18,30
Pseudo-Euclidean			
$\lambda \neq 0$	E(2,1)	53 or more	6c,j,22
$\lambda = 0$	O(3,2)	over 90	6i-k
Sphere	O(4)	6	17,21,22
2-sheet Hyperboloid	O(3,1)	34	19,22
$\Delta u + iu_t = 0$			
i) 1-dimension	$SL(2,R) \otimes W_1$	7	6f
ii) 2-dimensions			
Euclidean	$[SL(2,R) \otimes O(2)] \otimes W_2$	26	6g,h
Pseudo-Euclidean	$[SL(2,R) \otimes O(1,1)] \otimes W_2$	58	31
$\Delta u + iu_t - \frac{\alpha}{x_1^2} - \frac{\beta}{x_2^2}$			
Euclidean $\alpha, \beta \neq 0$	SL(2,R)	15	26
$\alpha = 0$	$SL(2,R) \otimes W_1$	25	26
$\Delta u + \frac{\alpha}{x_1^2} u = 0$			
Euclidean	SL(2,R)	9	6ℓ,30
Pseudo-Euclidean	SL(2,R)	9	6ℓ

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