CLIFFORD ALGEBRAS, SPINORS AND FINITE GEOMETRIES

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ABSTRACT

The pleasant incidence properties of the finite projective geometry PG(m,2) are invoked in order to handle nicely certain commutativity/anti-commutativity aspects of the real Clifford algebras Cl(0,d), $d = |PG(m,2)| = 2^{m+1} - 1$, $m = 2, 3, \ldots$

1. Introduction

As in [1],[2] we deal with an irreducible representation of Cl(0,d), $d = 2^{m+1} - 1$, ($m \geq 2$), in which the operators $\Gamma_1, \Gamma_2, \ldots, \Gamma_d$ satisfy

$$(\Gamma_p)^2 = -I, \qquad \Gamma_p\Gamma_q = -\Gamma_q\Gamma_p, \; p \neq q, \tag{1.1}$$

and

$$\Gamma_1\Gamma_2 \ldots \Gamma_d = + I. \tag{1.2}$$

Let $S = \{1,2, \ldots ,d\}$. Then for each $\alpha \in P(S)$(= power set of S) we can define an associated element $\Gamma(\alpha)$ in our representation. For example if $\alpha = \{2,3,7,8\}$ then $\Gamma(\alpha) = \Gamma_2\Gamma_3\Gamma_7\Gamma_8$. Of course $\Gamma(\{p\}) = \Gamma_p$. It follows from (1.1),(1.2) that

$$\Gamma(\alpha)\Gamma(\beta) = \pm\Gamma(\alpha\Delta\beta), \; \Gamma(S) = +I \tag{1.3}$$

where $\alpha\Delta\beta$ denotes the symmetric difference of the subsets α, β of S. Using Δ as addition, observe that P(S) is a vector space, of dimension d, over the field $\mathbb{F}_2 = \{0,1\}$ of order 2. In particular $\alpha\Delta\alpha = \emptyset$ (= the zero vector of P(S)). What is more, noting that $\alpha\cap(\beta\Delta\gamma) = (\alpha\cap\beta)\Delta(\alpha\cap\gamma)$, we see that $(P(S),\Delta,\cap)$ is a $\mathbb{F}_2$-algebra having S as 1 (since $\alpha\cap S = \alpha$).

Let us now interpret S as the set of points of a finite projective geometry PG(m,2), of (projective) dimension m over $\mathbb{F}_2$. Because of the peculiar nature of $\mathbb{F}_2$, in which $\lambda \neq 0$ implies $\lambda = 1$, we may view S as consisting of the nonzero vectors of a vector space V of dimension m+1 over $\mathbb{F}_2$, three distinct points p,q,r of S being collinear if and only if $p+q+r = 0$. Now $|V| = 2^{m+1}$, and so $|S| = 2^{m+1} - 1 = d$, as announced above. Let S_r denote, for $r = 0,1, \ldots,m$, the set of all the r-flats of PG(m,2), and let C_r denote that vector subspace of P(S) which is spanned by the complements of the r-flats:

$$C_r = \langle \alpha^c : \alpha \in S_r \rangle. \tag{1.4}$$

Observe that C_o coincides with the subspace E(S) of P(S) consisting of all the even subsets of S:

$$C_o = E(S) = \{\alpha \in P(S) : |\alpha| \in 2\mathbb{Z}\}. \tag{1.5}$$

Since $\alpha^c = \alpha\Delta S$, we have $P(S) = C_o \oplus \langle S \rangle$. In particular $\dim C_o = d - 1$. (<u>Caution</u>: in [1],[2] we viewed C_o slightly differently, as the quotient $P(S)/\langle S \rangle$, and used multiplicative notation, with C_o isomorphic to the elementary abelian group $(\mathbb{Z}_2)^{d-1}$.)

2. Some abelian results

Denote by $F(S)$ the $\mathbb{F}_2$-vector space consisting of all the forms on S (i.e. all the functions $S \to \mathbb{F}_2$). For $r>0$, let $F_r = F_r(S)$ denote the vector subspace of $F(S)$ consisting of all the forms of degree r. Thus F_r is spanned by forms of the kind $f_1 f_2 \ldots f_r$ where each f_i is a linear form, arising (by restriction to S) from an element of the dual space $\hat{V} = L(V, \mathbb{F}_2)$. For $f \in F(S)$ we define $\beta(f) \in P(S)$ by

$$\beta(f) = \{p \in S : f(p) = 1\} . \qquad (2.1)$$

Lemma A The mapping $\beta : f \mapsto \beta(f)$ yields an isomorphism

$$(F(S), +, \cdot, 0, 1) \to (P(S), \Delta, \cap, \emptyset, S) \qquad (2.2)$$

of $\mathbb{F}_2$-algebras. (In the algebra $F(S)$ the multiplication is pointwise multiplication of forms, and 0, 1 denote the forms taking the constant values 0, 1 respectively.) Upon restriction the algebra isomorphism β yields the isomorphisms of $\mathbb{F}_2$-vector spaces

$$F_r \cong C_{m-r} , \; r = 0,1, \ldots, m, \qquad (2.3)$$

(where, for convenience, we define F_o to be $\{0\}$).

Now, on account of the peculiar nature of $\mathbb{F}_2$, we have $f^2 = f$, $f^3 g^4 h^2 = fgh$, etc., and consequently we have the nesting

$$F(S) \supset F_m \supset F_{m-1} \supset \cdots \supset F_o = \{0\} , \qquad (2.4)$$

and also the equalities

$$F_r = F(S) , \text{ for } r > m . \qquad (2.5)$$

From (2.3), (2.4) we obtain immediately the next lemma. Alternatively the inclusions $C_r \supseteq C_{r+1}$ follow from theorem 2.3 of [1], and the fact that, for $r = 0,1, \ldots, m-1$, the inclusion is proper follows, for example, from (2.7) below.

Lemma B $\quad E(S) = C_o \supset C_1 \supset \ldots \supset C_m = \{\emptyset\} . \qquad (2.6)$

Theorem C For $r = 1,2, \ldots, m$ there exists a unique linear isomorphism

$$\varphi_r : \wedge^r \hat{V} \to C_{m-r}/C_{m-r+1} \qquad (2.7)$$

such that, for arbitrary $f_1, \ldots, f_r \in \hat{V}$,

$$\varphi_r(f_1 \wedge \ldots \wedge f_r) = \beta(f_1) \cap \ldots \cap \beta(f_r) \bmod C_{m-r+1} . \qquad (2.8)$$

By inverting the isomorphism φ_{m+1-r} and using the properties of the (unique, over $\mathbb{F}_2$) Poincaré isomorphism $\wedge^{m+1-r} \hat{V} \to \wedge^r V$ we obtain also the next theorem.

Theorem D For $r = 1,2, \ldots, m$ there exists a unique linear surjection

$$\psi_r : C_{r-1} \to \wedge^r V$$

such that

$$\psi_r(\mathrm{join}(v_1, \ldots, v_r)^c) = v_1 \wedge \ldots \wedge v_r$$

holds whenever $v_1, \ldots, v_r$ are independent points of S.
(Thus the usual Plücker map, from S_{r-1} on to those rays of $\wedge^r V$ which are spanned by decomposable r-vectors, "extends" to a linear map from the whole of C_{r-1} on to the whole of $\wedge^r V$.) Moreover $\ker \psi_r = C_r$, and we have the linear isomorphism

$$C_{r-1}/C_r \cong \wedge^r V , \; r = 1, \ldots, m . \qquad (2.9)$$

Convenient bases for the subspaces C_r can be displayed in terms of the faces of a chosen simplex of reference for PG(m,2). Let $\mathcal{F}_s$ denote the set of s-faces of the simplex and let $\mathcal{E}_s = \{\alpha^c : \alpha \in \mathcal{F}_s\}$. By appealing to standard bases in the exterior algebra $\wedge V$, or otherwise, one obtains the next lemma.

<u>LEMMA E</u> $\mathcal{E}_{m-1} \cup \mathcal{E}_{m-2} \cup \ldots \cup \mathcal{E}_r$ is a basis for C_r . Consequently

$$\dim C_r = \binom{m+1}{1} + \binom{m+1}{2} + \ldots + \binom{m+1}{m-r} . \qquad (2.10)$$

Finally, recalling that $\wedge^r V$ is known to be irreducible under the natural action of GL(V) $\cong GL(m+1; \mathbb{F}_2)$, the isomorphism (2.9) yields the following result.

<u>THEOREM F</u> Under the natural action of GL(V), the subspace chain (2.6) is a composition series.

<u>REMARK</u> If $m = 3$ then, by (2.10), $\dim C_1 = 10$. As described in [1], the $2^{10} = 1024$ figures of C_1 fall into seven $GL(4;\mathbb{F}_2)$-orbits. At the time of writing the paper [1] the author was not aware of the isomorphisms (2.3), and so did not see the tie-up with the classification of quadrics in PG(3,2), as given in Tables 15.4 and 15.9 of [3]. Similarly, in the case m=4, the classification in [2] of the $2^{15} = 32{,}768$ figures of C_2 into eight $GL(5;\mathbb{F}_2)$-orbits ties in, via the isomorphism $F_2 \cong C_2$, with the classification of quadrics in PG(4,2). For example, each 15_3 figure, see [2], in C_2 is a non-degenerate quadric whose equation can be taken to be $x_1x_2 + x_3x_4 = (x_5)^2$, and one finds that there are 13,888 such quadrics in PG(4,2), in agreement with equation (4.10) in [2].

<u>3. Some Clifford algebra consequences</u>

Loosely speaking, we now deal with m-dimensional projective geometry in which the "points" Γ_p anticommute. The chief link-up of the incidence properties of PG(m,2) with commutativity/anti-commutativity properties of Cl(0,d) is by way of the next lemma. The first part of this lemma follows from (1.1), (1.2) upon using the fact that a projective subspace has an odd number of points.

<u>LEMMA G</u> If $\alpha \in S_r$, $\beta \in S_s$, with $r \geq 0$, $s \geq 0$, then

$$\Gamma(\alpha)\Gamma(\beta) = \begin{cases} \Gamma(\beta)\Gamma(\alpha), \text{ if } \alpha \text{ meets } \beta \\ \Gamma(\beta)\Gamma(\alpha), \text{ if } \alpha \text{ is skew to } \beta \end{cases} \qquad (3.1)$$

Also, for $r \geq 1$, we have $\Gamma(\alpha)^2 = +I$.

For $r = 0,1, \ldots ,m$ we shall be interested in the finite groups

$$G_r = \langle \pm \Gamma(\alpha) : \alpha \in S_r \rangle . \qquad (3.2)$$

For $r \geq 1$, G_r is a proper subgroup of the finite group G_o generated by the Γ_p. This group is of order 2^d, and is isomorphic to the "even Dirac group" consisting of products of an even number of elements drawn from a usual orthonormal set $\{e_1, \ldots ,e_d\}$ of vectors generating Cl(0,d). Clearly

$$G_r/\{\pm I\} \cong C_r . \qquad (3.3)$$

(Incidentally, the fact that the commutator subgroup, Frattini subgroup and centre of G_o are all equal to $\{\pm I\}$ means that G_o is an extra-special 2-group; see for example [4], where further references can be found.) Consequently, from lemma B, we have the subgroup chain

$$G_o \supset G_1 \supset \dots \supset G_m = \{\pm I\} . \qquad (3.4)$$

LEMMA H For $r = 0,1, \dots ,m$, G_r lies inside the centralizer of G_{m-r} within G_o.

This follows from lemma G, since each r-flat meets every (m-r)-flat. However, lemma H can be strengthened as in the next theorem which, as pointed out in section VI of [1], is a fairly easy consequence of theorem F and lemma H. (In [1] our present theorem F appeared only as a conjecture.)

THEOREM I For $r = 0,1, \dots ,m$, G_r is the full centralizer of G_{m-r} within G_o.

COROLLARY J If $m = 2\ell$ is even, then G_ℓ is a maximal abelian normal subgroup of G_o.

ILLUSTRATION In the case $m = 4$, i.e. Cl(0,31), a maximal abelian normal subgroup of G_o is $G_2 \cong \{\pm I\} \times K_2$, where $K_2 \cong C_2$. A possible choice of fifteen independent generators of K_2 is accordingly, by lemma E, the set $\{\Gamma(\alpha) : \alpha \in \mathcal{F}_3 \cup \mathcal{F}_2\}$ associated with the ten 2-faces and five 3-faces of the chosen simplex of reference for PG(4,2). The 2^{15} sets of simultaneous eigenvalues $(\pm 1, \dots , \pm 1)$ of the fifteen mutually commuting involutions $\Gamma(\alpha)$ will label the $2^{15} = 32{,}768$ linearly independent spinor states of our irreducible representation of Cl(0,31).

LEMMA K Let α,β denote arbitrary subsets of S. Then

i) $\Gamma(\alpha)^2 = (-1)^{q(\alpha)}$, where $q(\alpha) = \frac{1}{2}|\alpha|(|\alpha| + 1) + 2\mathbf{Z}$.

ii) $\Gamma(\alpha)\Gamma(\beta) = \epsilon(\alpha,\beta)\Gamma(\beta)\Gamma(\alpha)$, where $\epsilon(\alpha,\beta) = (-1)^{b(\alpha,\beta)}$, with $b(\alpha,\beta) = |\alpha\cap\beta| + |\alpha||\beta| + 2\mathbf{Z} \in \mathbf{Z}/2\mathbf{Z} = \mathbb{F}_2$.

iii) $b(\cdot,\cdot)$ is an alternating bilinear form on P(S).

LEMMA L Let b_o denote the restriction of b to $C_o \times C_o$. (So $b_o(\alpha,\beta) = |\alpha\cap\beta| + 2\mathbf{Z}$.) Then b_o is a non-degenerate scalar product on C_o and, within C_o, C_r is the orthogonal subspace to C_{m-r} :

$$C_r = (C_{m-r})^{\perp}, \; r = 0,1, \dots ,m . \qquad (3.5)$$

The equality (3.5) follows by dimensions (lemma E), after noting that we have the inclusion $C_r \subseteq (C_{m-r})^{\perp}$ (because each r-flat meets every (m-r)-flat).

REMARK Since $b_o(\alpha,\beta) = 0$ if and only if $\Gamma(\alpha)$ commutes with $\Gamma(\beta)$, observe that (3.5) provides us with a second proof of the full centralizer property of theorem I.

References

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