FRAME'S CONJUGATING REPRESENTATION AND GROUP EXTENSIONS

Nigel Backhouse

Department of Applied Mathematics and Theoretical Physics,
University of Liverpool, Liverpool, L69 3BX, England.

Introduction

One of the first things the beginner learns of in representation theory is that every group has two natural representations, namely the trivial representation and the regular representation. There is, however, another natural representation, Frame's Conjugating Representation, which deserves attention for the following reasons: it is easily defined, providing useful examples for students; it presents some unsolved problems and conjectures; it has aroused independent interest in the Pure Mathematics and Mathematical Physics literature in recent years (see ref. 1 for a bibliography).

Let G be a finite group with group algebra $A(G) = \{$complex linear sums of group elements, with product the linear extension of the multiplication in G$\}$. The conjugating representation F is defined on A(G) (or any faithful representation of A(G)) by $g \to F_g$, where $F_g a = gag^{-1}$, for $g \in G$, $a \in A(G)$. By interpreting A(G) as an algebra of group functions, the definition of F can be carried over to the compact group case - see ref. 2 for the proper definition.

F is completely reducible so one can ask for its irreducible constituents. We first observe that F_g is the identity if $g \in Z$, the centre of G, hence F can only contain irreducibles which arise from irreducibles of G/Z. Even so, there are examples which show that F need not contain all irreducibles of G/Z. However, since F is a faithful (and real) representation of G/Z, we know from a theorem of Burnside that $\exists$ an integer N such that $\otimes^n F$ contains all irreducibles of G/Z $\forall\ n \geqslant N$.

If G acts faithfully on something, for example a space of wave functions, then A(G) becomes an algebra of operators which transform tensorially among themselves under F. On reducing F we see that we have a source of irreducible tensor operators for G. F is sometimes called the tensor representation.

Having noted above that F gives rise only to tensor operators which correspond to irreducibles of G/Z, and even then not necessarily all of them directly, we investigate in this paper generalisations of F which can be used to associate tensor operators with all irreducibles of G, hence answering in the affirmative a question of de Vries. Finally, we show that all irreducibles of G/Z appear in F for $G = SU(3)$.

Generalised conjugating representation

Two obvious generalisations of F are obtained by (1) letting G act by conjugation on the group algebra of a group G' containing G; (2) letting G act on its twisted group algebra. Putting these together, suppose $G \leqslant G'$ and let $A(G', \omega)$ be the twisted group algebra of G' corresponding to the factor system ω (recall from ref. 1 that $A(G', \omega)$ is a module over the complex field, with basis the set of objects $\{\nu(g): g \in G'\}$, which has as a multiplication the linear extension of the law $\nu(g_1)\nu(g_2) = \omega(g_1, g_2)\nu(g_1 g_2)$). Now define the representation F^ω of G by $g \to F_g^{\ \omega}$, where $F_g^{\ \omega} a = \nu(g)a\nu(g)^{-1}$ for $g \in G$, $a \in A(G', \omega)$. Using the properties of factor systems and $\nu(g)^{-1} = \nu(g^{-1})/\omega(g, g^{-1})$, we can check that F^ω is indeed a representation of G and that $F_g^{\ \omega}\nu(g') = \nu(gg'g^{-1})\omega(g, g')/\omega(gg'g^{-1}, g)$ for $g \in G$, $g' \in G'$. So $F_g^{\ \omega}$ is the identity operation iff $g \in Z'$, the centre of G', and $\omega(g, g') = \omega(g', g)$ for all $g' \in N'(g)$, the centraliser of g in G' (which for central g is G' itself). But the latter is precisely the condition that g be ω-regular in G' - see ref. 1. If R^ω is the set (not in general a group) of ω-regular elements in G' then it is clear that the kernel of F^ω is $G \cap Z' \cap R^\omega = K^\omega$, which of course must be a subgroup of G. Evidently $A(G', \omega)$ has become a faithful G/K^ω-module, hence is a faithful G-module iff K^ω is trivial.

To calculate the irreducible constituents of F^ω we first compute its character, which is the restriction to G of the character χ^ω of the conjugating representation of G' on $A(G', \omega)$. Using the fact that, for fixed $g \in G'$, $g' \to \omega(g, g')/\omega(g', g)$ is a linear character on $N'(g)$, we find that $\chi^\omega(g) = |N'(g)|$ if $g \in N'(g) \cap R^\omega$, but zero otherwise. Then if $\chi^\omega = \sum_\mu c_\mu \chi^{(\mu)}$, we find that $c_\mu = \sum_i{}' \chi_i^{(\mu)}$, where $\chi_i^{(\mu)}$ is the value of the μ^{th} irreducible character of G' on the i^{th} conjugacy class, and where the prime restricts the summation to ω-regular classes only. Another expression is $\chi^\omega = \sum_\lambda \theta^{\omega,\lambda}\theta^{*\omega,\lambda}$, where the $\theta^{\omega,\lambda}$ are the inequivalent irreducible ω-characters of G'. Thus we have two ways of computing the irreducibles of F^ω: either summing entries in the rows of the ordinary character table of G' or using the Clebsch-Gordon series for projective representations of G', and then restricting to G. Which method one chooses of course depends on context and available information, but we must remark that it is only the second method which makes proper sense and is indeed valid in the case of a non-finite compact group.

Examples

E. de Vries has posed the problem: given a finite group G, find some way of associating tensor operators to all irreducibles of G. I hope that the following is the best possible general solution in the context of this paper.

G acts on itself by left translations, thus for $g \in G$, define λ_g by $\lambda_g g' = gg'$. The set $\{\lambda_g : g \in G\}$ forms a group of permutations isomorphic to G itself on the set of elements of G, and hence embeds G in the full permutation group $S_{|G|}$ of order $|G|!$ Now $S_{|G|}$ is centreless for $|G| > 2$, hence $A(S_{|G|})$ provides a faithful G-module. In ref. 1 we show that $A(S_{|G|})$ in fact carries all irreducibles of G. Of course we knew in advance that $\otimes^N A(S_{|G|})$ carries all irreducibles of G for some suitable integer N, but it requires a calculation to prove that N can be taken as unity. I know of no sufficient condition which, in the general case, allows one to take $N = 1$. Kasperkovitz and Dirl, ref. 2, have suggested that a sufficient condition might be the existence of a faithful irreducible representation, but as yet the conjecture is unproved. If the conjecture is true then it implies in particular my own conjecture that Frame's conjugating representation for $G = S_n$ contains all irreducibles of S_n for $n > 2$ - it is strongly verified by looking at character tables for $n = 3, 4, \ldots 10$.

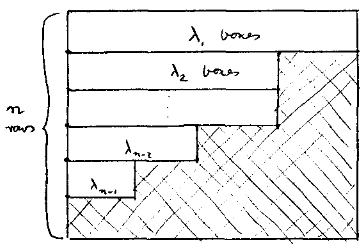
Let me now look at the compact groups SO(3), SU(2), SU(3).

(a) SO(3) is centreless, and has irreducibles D^j, $j = 0, 1, 2\ldots$ We compute the conjugating representation $\bigoplus_{j=0}^{\infty} (D^{2j} \oplus D^{2j-1} \ldots \oplus D^{o})$, hence contains all D^{J}'s infinitely many times.

(b) The projective representations of SO(3) are D^j, $j = {}^1/2, {}^3/2, {}^5/2, \ldots$, hence the twisted conjugating representation of SO(3) is $\bigoplus_{j={}^1/2,{}^3/2} (D^j \otimes D^{j*}) = \bigoplus_{j={}^1/2,\ldots} (D^{2j} \oplus D^{2j-1} \ldots \oplus D^{o})$, so again all irreducibles of SO(3) appear infinitely many times.

(c) SU(2) is the covering group of SO(3), so its representations are the ordinary and projective representations lifted from SO(3). The ordinary conjugating representation (there is no non-trivial conjugating representation) thus contains all D^{J}'s, J an integer, infinitely many times. But these are precisely the ones trivial on the centre of SU(2).

(d) The case of SU(3) requires a little more setting up. I begin by reminding ourselves of some aspects of the representation theory of SU(n). Irreducibles of SU(n) are labelled by Young tableaux $\underline{\lambda} = (\lambda_1, \lambda_2, \ldots \lambda_{n-1})$ containing at most n - 1 rows of square boxes. Actually Young tableaux with at most n rows will do, but it then turns out that $(\lambda_1, \lambda_2, \ldots \lambda_n) \equiv (\lambda_1 - \lambda_n, \lambda_2 - \lambda_n, \ldots, \lambda_{n-1} - \lambda_n)$. Now if $D^{\underline{\lambda}}$ has tableau $\underline{\lambda}$ then the tableaux of $D^{\underline{\lambda}^*}$ is given by rotating through 180° the shaded region in the diagram below.



Also $D^{\underline{\lambda}}$ is trivial on the centre of SU(n) iff $\sum_{i=1}^{n-1} \lambda_i$ is a multiple of n.

Now in order to reduce the product $D^{\underline{\lambda}} \otimes D^{\underline{\mu}}$ to irreducibles, we set up the tableaux $\underline{\lambda}$, $\underline{\mu}$ side by side, writing in a fixed symbol, a_1 say, in the μ_1 boxes in the first row of $\underline{\mu}$, a symbol, a_2 say, in the μ_2 boxes of the second row of $\underline{\mu}$, etc. Then we consider all Young tableaux obtainable from $\underline{\lambda}$ by the adjunction one by one of the labelled boxes of $\underline{\mu}$ consistent with the following restrictions:

(1) at each stage in the process the augmented diagrams must be Young tableaux with at most n rows;

(2) adjoin all boxes from the i^{th} row of $\underline{\mu}$ before adjoining any from the $i + 1^{th}$;

(3) no two boxes with the same label can be in the same column;

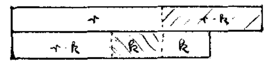
(4) each final tableau must be such that if one records the occurrence of the symbols a_1, a_2, etc., reading the rows as one would read lines of mirror English, then at each stage in the count the number of a_1's $\geq$ number of a_2's $\ldots \geq$ number of a_{n-1}'s.

Finally all tableaux with n rows can be reduced to n - 1 rows. This procedure only tells one whether or not a given irreducible occurs in a Kronecker product, but not its multiplicity. Now let me apply the above to show that the conjugating representation of SU(3) contains all irreducibles of SU(3)/Z. Given $\underline{\lambda} = (\lambda_1, \lambda_2)$ with $\lambda_1 + \lambda_2 = 3r$, r an integer, I will find $\underline{\mu} = (\mu_1, \mu_2)$ such that $D^{\underline{\lambda}}$ occurs in $D^{\underline{\mu}} \otimes D^{\underline{\mu}^*}$. Now always we have $\lambda_1 \geq 3r/2$, but either $\lambda_1 \leq 2r$ or $\lambda_1 > 2r$.

<u>Case $\lambda_1 \leq 2r$</u>

Write $\lambda_1 = 2r - k$, $\lambda_2 = r + k$, where $k \leq r/2$. I claim that I can take $\underline{\mu} = (r, k)$ (then $\underline{\mu}^* = (r, r - k)$). The proof is implicit in the following tableau multiplication

contains

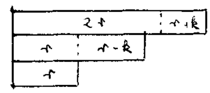


which is precisely $\underline{\lambda}$.

<u>Case $\lambda_1 > 2r$</u>

Write $\lambda_1 = 2r + k$, $\lambda_2 = r - k$, where $0 < k \leqslant r$. I take $\underline{\mu} = (2r, r)$, in which case $\underline{\mu}^* = \underline{\mu}$. Then

contains



which can be reduced to $(2r + k, r - k) = \underline{\lambda}$. This concludes the proof.

<u>References</u>

1. N. B. Backhouse, J. Math. Phys. 16 (1975), 443-7.

2. P. Kasperkovitz and R. Dirl, J. Math. Phys. 15 (1974), 1203-10.