Conformal Quantum Field Theory in Two and Four Dimensions *

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Abstract

We review basic features and selected topics in conformal field theory, considering memorable results in 2D CFT and a recent attempt to construct a 4D analogue of a chiral algebra generated by local (observable) fields with rational correlation functions from a unified point of view.

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1 Introduction. Conformal symmetry reflects the idea that only ratios of lengths at each point have an invariant meaning.

Angle preserving transformations have been used by the ancients in astronomy. Euler studied them and applied to the "geographic projection of the Earth" (1770-1778, while completely blind); Gauss (1822) solved the general problem of a conformal mapping of a 2–dimensional surface into another. J. Liouville (1850) was the first to describe conformal transformations in more than two dimensions. We begin our exposition by giving a precise formulation (and sketching a proof) of his result for a D-dimensional space of any signature (Liouville has worked out the 3–dimensional euclidean case).

A (locally defined, smooth) coordinate transformation $x \mapsto y(x)$ of a (pseudo-)Riemannian manifold is said to be conformal if the metric is transformed by a similarity:

$$g_{\kappa\lambda}(y)\, \mathrm{d}y^{\kappa}\mathrm{d}y^{\lambda} = g_{\kappa\lambda}(y(x))\, \frac{\partial y^{\kappa}}{\partial x^{\mu}} \frac{\partial y^{\lambda}}{\partial x^{\nu}}\, \mathrm{d}x^{\mu}\mathrm{d}x^{\nu} = \omega^{2}(x)\, g_{\mu\nu}(x)\, \mathrm{d}x^{\mu}\mathrm{d}x^{\nu} \tag{1.1}$$

It is also useful to write down the infinitesimal counterpart of (1.1) setting

$$y^{\mu} = x^{\mu} + \epsilon K^{\mu}(x) , \quad \omega(x) = 1 + \epsilon f(x) , \quad 0 < \epsilon \ll 1 . \tag{1.2}$$

We deduce from (1.1) that the Lie derivative of $g_{\mu\nu}$ is proportional to $g_{\mu\nu}$:

$$\mathcal{L}_{K}\, g_{\mu\nu} := K^{\lambda}\, \partial_{\lambda}\, g_{\mu\nu} + (g_{\mu\nu}\, \partial_{\nu} + g_{\lambda\nu}\, \partial_{\mu})\, K^{\lambda} = 2 f g_{\mu\nu} , \qquad \partial_{\lambda} \equiv \frac{\partial}{\partial x^{\lambda}} . \tag{1.3}$$

We say that two metric tensors $g_{\mu\nu}$ and $\widetilde{g}_{\mu\nu}$ are *conformally equivalent* if they only differ by a positive *conformal factor*

$$\widetilde{g}_{\mu\nu}(x) = \Omega^{2}(x)\, g_{\mu\nu}(x) , \quad \Omega(x) > 0 . \tag{1.4}$$

It is easy to see that conformally equivalent metrics admit the same conformal group: if we distinguish by a tilde the conformal factor and its infinitesimal counterpart for the metric $\{\widetilde{g}_{\mu\nu}\}$ we find

$$\widetilde{\omega}(x) = \frac{\Omega(y(x))}{\Omega(x)}\, \omega(x) , \qquad \widetilde{f}(x) = f(x) + \Omega^{-1}(x)\, K^{\lambda}\, \partial_{\lambda}\, \Omega(x) . \tag{1.5}$$

For a manifold M of a given dimension D the conformal group is maximal if M is *conformally flat*, – i.e. if (for signature $(s,\, D-s)$

$$g_{\mu\nu}(x) = \Omega^{2}(x)\, \eta_{\mu\nu} , \qquad \eta_{\mu\nu} = \epsilon_{\nu}\, \delta_{\mu\nu} , \quad \epsilon_{\nu} = \pm 1 ,$$

$$\sum_{\nu} \epsilon_{\nu} \,=\, 2\,s - D\,. \tag{1.6}$$

Theorem 1.1 (Infinitesimal version of Liouville theorem) *The conformal group for $D = 2$ is infinite dimensional while for $D > 2$ it is isomorphic to the pseudo-orthogonal group $SO\,(s+1,\, D-s+1)$, the* conformal Killing vector $K(x)$ *having the form*

$$K^{\lambda}\,(x) \,=\, a^{\lambda} + \alpha\, x^{\lambda} + l^{\lambda}{}_{\mu}\, x^{\nu} + 2\; (c{\cdot}x)\; x^{\lambda} - x^2\, c^{\lambda}\,, \quad l_{\mu\nu} \,=\, -\, l_{\nu\mu}\,. \tag{1.7}$$

The $\begin{pmatrix} D+2 \\ 2 \end{pmatrix}$ *different parameters,* a^{λ} , α , $l^{\lambda}{}_{\nu}$, c^{λ} , *correspond to translation, dilation, Lorentz rotations, and special conformal transformations.*

To prove the theorem one derives the following infinitesimal form of Eq. (1.1):

$$D\,f \,=\, \partial_{\lambda}\, K^{\lambda}\,, \qquad (D-1)\,\Box\,\partial_{\lambda} K^{\lambda} \,=\, 0 \quad (\,\Box \,=\, \eta^{\mu\nu}\,\partial_{\mu}\,\partial_{\nu}\,)\,,$$

$$(D-2)\;\partial_{\lambda}\,\partial_{\mu}\,\partial{\cdot}K \,=\, 0\,. \tag{1.8}$$

The theorem then follows.

The general orientation preserving conformal transformations in the complex plane and in a pseudo–euclidean 2–dimensional (2D) space have the form

$$z \,\mapsto\, f\,(z)\;, \quad x \pm t \,\mapsto\, f\,(x \pm t) \tag{1.9}$$

for any (single–valued) analytic function f .

We shall be working with the connected group $G = Spin\,(D,\, 2)$ which has a single–valued action on local (Bose and Fermi) fields. It is a 4–fold cover of the conformal group of space time for even D (and its double cover for odd D).

The 2D case is also distinguished by the fact that every (pseudo–)Riemannian metric can be reduced by a reparametrization to a conformally flat form. Indeed, the number $\binom{D+1}{2}$ of independent components of a symmetric tensor $g_{\mu\nu}$ is equal to the number $D+1$ of arbitrary functions at our disposal (the D components of $y(x)$ and the conformal factor) for $D = 2$ only.

The first application of the conformal group to physics came soon after realizing the role of Lorentz invariance for the electromagnetic phenomena ([12] [4]): the Maxwell equations in vacuum are conformally invariant. We shall prove an equivalent statement: for $D = 4$ the Maxwell equations only depend on the conformal class of metrics (not on the choice of an individual pseudo–Riemannian metric in a given conformal class). To this end we shall write Maxwell's equations in terms of the 2–form F (the curvature form of the $\mathrm{U}(1)$ connection A) and its Hodge dual $\star F$ defined by

$$F \, = \, \frac{1}{2} \, F_{\mu\nu} \, \mathrm{d}x^\mu \wedge \mathrm{d}x^\nu \, = \, \mathrm{d}A$$
$$\star F \wedge F \, = \, \mathcal{L} \sqrt{|g|} \, \mathrm{d}x^0 \wedge \mathrm{d}x^1 \wedge \mathrm{d}x^2 \wedge \mathrm{d}x^3 \tag{1.10}$$

where $|g|$ is the absolute value of the determinant of the metric tensor, $\mathcal{L}$ is the Lagrangean density for the (free) electromagnetic field:

$$\mathcal{L} \, = \, - \frac{1}{4} \, g^{\kappa\mu} \, g^{\lambda\nu} \, F_{\kappa\lambda} \, F_{\mu\nu} \quad (\, g^{\lambda\mu} \, g_{\mu\nu} \, = \, \delta^\mu_\nu \,) \, . \tag{1.11}$$

Assuming that F does not change under a conformal rescaling of the metric we find

$$g_{\mu\nu} \, \mapsto \, \Omega^2 \, g_{\mu\nu} \, , \quad g^{\kappa\mu} \, \mapsto \, \Omega^{-2} \, g^{\kappa\mu} \, ,$$
$$F_{\mu\nu} \mapsto F_{\mu\nu} \quad \Rightarrow \quad \sqrt{|g|} \, \mathcal{L} \, \mapsto \, \sqrt{|g|} \, \mathcal{L} \, . \tag{1.12}$$

The free Maxwell equations then assume the conformally invariant form

$$\mathrm{d}F \, = \, 0 \, = \, \mathrm{d} \star F \tag{1.13}$$

The conformally invariant Hodge star acting on 2–forms in 4–dimensional Minkowski space (with $g_{\mu\nu} = \eta_{\mu\nu} = diag\,\{-1,\,+1,\,...,\,+1\}$) satisfies $(\star)^2 = -1$ and hence defines a complex structure in the space of 2–forms on M. A complex Maxwell field splits, according the eigenvalue of $\star$, into the irreducible representations (1, 0) and (0, 1) of $SL\,(2,\,\mathbb{C})$. Indeed, expressing F and $\star F$ in terms of the electric field $\mathbf{E}$ and the magnetic induction $\mathbf{B}$,

$$(F_{\mu\nu}) \;=\; \begin{pmatrix} 0 & -E_1 & -E_2 & -E_3 \\ E_1 & 0 & B_3 & -B_2 \\ E_2 & -B_3 & 0 & B_1 \\ E_3 & B_2 & -B_1 & 0 \end{pmatrix} \,, \quad (\star F_{\mu\nu}) \;=$$

$$\begin{pmatrix} 0 & B_1 & B_2 & B_3 \\ -B_1 & 0 & E_3 & -E_2 \\ -B_2 & -E_3 & 0 & E_1 \\ -B_3 & E_2 & -E_1 & 0 \end{pmatrix} \tag{1.14}$$

we can write

$$F_{A\dot{A}B\dot{B}} \;=\; \eta^{\mu\mu'}\,\eta^{\nu\nu'}\,(\sigma_\mu)_{A\dot{A}}\,(\sigma_\nu)_{B\dot{B}}\,F_{\mu'\nu'} \;=$$

$$=\frac{1}{2}\,\left(\epsilon_{AB}\,\psi_{\dot{A}\dot{B}} + \epsilon_{\dot{A}\dot{B}}\,\varphi_{AB}\right)\,, \quad (\epsilon_{AB}) \;=\; \begin{pmatrix} 0 & 1 \\ -1 & 0 \end{pmatrix} \tag{1.15}$$

where

$$(\psi_{\dot{A}\dot{B}}) \;=\; \begin{pmatrix} E_1 + i\,B_1 + i\,(E_2 + i\,B_2) & E_3 + i\,B_3 \\ E_3 + i\,B_3 & E_1 + i\,B_1 - i\,(E_2 + i\,B_2) \end{pmatrix}\,,$$

$$(\varphi_{AB}) \;=\; \begin{pmatrix} E_1 - i\,B_1 - i\,(E_2 - i\,B_2) & E_3 - i\,B_3 \\ E_3 - i\,B_3 & E_1 - i\,B_1 + i\,(E_2 - i\,B_2) \end{pmatrix}\,.$$

The representations (0, 1) and (1, 0) correspond to self–dual and to anti–self–dual F's, respectively:

$$\star F \;=\; i\,F \quad \Leftrightarrow \quad \varphi_{AB} \;=\; 0\,, \quad \star F \;=\; -i\,F \quad \Leftrightarrow \quad \psi_{\dot{A}\dot{B}} \;=\; 0\,; \tag{1.16}$$

for a real F these two representations are conjugate to each other:

$$F_{\mu\nu} = \overline{F_{\mu\nu}} \quad \Leftrightarrow \quad \epsilon_{\dot{A}\dot{B}} \, \overline{\varphi_{AB}} = \epsilon_{AB} \, \psi_{\dot{A}\dot{B}} \, . \tag{1.17}$$

Maxwell electrodynamics has been instrumental in Einstein's understanding of the geometry of space–time in his special theory of relativity. It is noteworthy that conformal invariance of Maxwell's equations selects $D = 4$ as the dimension of Minkowski space.

The important fact that local conformal transformations preserve causal ordering was noted and exploited much later ([19], [43]). (The necessity of a qualification like "local" will become manifest in Sec. 2 below where we shall see that space- and time-like intervals can be interchanged by proper conformal transformations.)

The presence of natural scales in nature, like masses of elementary particles or discrete wave lengths of atomic spectra, imply that dilation (and *a fortiori* conformal) invariance is not an exact property of matter. It is relevant, however, for the description of critical phenomena in which the correlation length is much larger than the atomic scale [40], and is believed to play a role in understanding the short distance behaviour of quantum field theory (QFT) – in a regime in which masses can be neglected – see, e.g. [34], [35]. This is not the first example when a beautiful symmetry is only exhibited in an idealized situation. Stretching imagination one can draw a parallel with the law of inertia which becomes only manifest if friction is neglected.

After the return of interest for QFT in the late 1960'ies (preceded by a fascination with local current algebras) the study of scale and conformal invariant QFT had several high points.

The first pick of activity was triggered by the discovery of Bjorken scaling in deep inelastic scattering and by two theoretical developments: Polyakov's discovery [40] that 3–point functions are fixed up to normalization by conformal invariance and Wilson's understanding of the role of anomalous dimensions and operator product expansions (OPE) in the Thirring model [49] (and his subsequent development of renormalization group methods). The work in this early period is reviewed in [47].

A break-through in 2D conformal field theory (CFT) was achieved in [5] on the basis of preceding mathematical developments (the computation of Kac's determinant which plays a crucial role in the representation theory of the Virasoro algebra - for a pedagogical review, see [28]). 2D CFT is attracting attention ever since displaying exciting relations with both pure mathematics and various branches of theoretical physics (from down–to–earth applications to surface critical phenomena in condensed matter physics – including the fractional quantum Hall effect, see, e.g. [22] [10] and references to earlier work cited there – to more speculative developments related to the grand string theory project). A textbook survey of the first decade of this work is offered in [13]. Important subsequent work goes under the heading of boundary CFT (see, e.g. [24] [39] and references therein).

CFT in 4 (and higher) dimension were never quite forgotten (see, e.g. [46] [20] [17]) but it made a real comeback with the discovery of the AdS–CFT correspondence (for an early review – see [1]; later work on the $N = 4$ supersymmetric Yang Mills theory can be traced back from [2] and [6]). The present lectures are chiefly based on [36] and [37] where common features of 2D and 4D CFT are emphasized.

2 Compactified Minkowski space. Conformal orbits of pairs of events.

2.1 The Klein-Dirac quadric and the cone at infinity.

The notion of a conformal space as a projective cone in which the conformal group acts without singularities has been introduced (for euclidean signature) by F. Klein in his famous Erlangen lecture (1872). We proceed to displaying Dirac's construction of compactified Minkowski space $\overline{M}$ [14] which implements Klein's program in the pseudo-euclidean case.

Points in $\overline{M}$ are identified with isotropic rays in $\mathbb{R}^{D,2}$:

$$\overline{M} \, = \, Q \, / \mathbb{R}^* \, ,$$
$$Q \, = \, \left\{ \vec{\xi} \in \mathbb{R}^{D,2} \, ; \; \vec{\xi} \neq 0 \, , \; \vec{\xi}^{\,2} \, := \underline{\xi}^{\,2} + \xi_D^2 - \xi_0^2 - \xi_{-1}^2 \, = \, 0 \right\} . \tag{2.1}$$

Here $\mathbb{R}^*$ is the multiplicative group of non–zero reals, $\underline{\xi}^{\,2} = \sum_{i=1}^{D-1} \xi_i^2$. Let $\vec{e}_a$, $a = -1, 0, 1, ..., D$ be an orthonormal basis in $\mathbb{R}^{D,2}$. The embedding of M in $\overline{M}$ is given by

$$M \, \ni \, x \, \mapsto \, \left\{ \lambda \vec{\xi}_x \right\} \, \in \, \overline{M} \, ,$$
$$\vec{\xi}_x \, = \, x^\mu \, \vec{e}_\mu + \frac{1+x^2}{2} \, \vec{e}_{-1} + \frac{1-x^2}{2} \, \vec{e}_D \tag{2.2}$$

(μ being summed up from 0 to $D-1$ and $x^2 \, = \, x_\mu x^\mu \, = \, \mathbf{x}^2 - x_0^2$). The Minkowski space interval is expressed in terms of the inner product of the representatives of the two points in $\overline{M}$:

$$(x-y)^2 \, = \, -\, 2 \, \vec{\xi}_x \cdot \vec{\xi}_y \, = \, \left(\vec{\xi}_x - \vec{\xi}_y \right)^2 . \tag{2.3}$$

A point in $\overline{M}$ is an image of a point in M iff $\xi^{-1} + \xi^D \neq 0$; then $x^\mu = \frac{\xi^\mu}{\xi^{-1}+\xi^D}$. The points at infinity in $\overline{M}$ form a $D-1$ dimensional cone with tip at p_∞ (cf. [38]):

$$K_\infty \, = \, \left\{ \lambda \, \vec{\xi} \, ; \; \vec{\xi} \cdot \vec{\xi}_\infty \, = \, \xi^{-1} + \xi^D \, = \, 0 \right\} ,$$

$$\vec{\xi}_\infty = (-1,\, 0,\, \mathbf{0},\, 1)\ , \quad p_\infty = \left\{ \lambda\, \vec{\xi}_\infty \right\} . \tag{2.4}$$

An ancient example of a conformal transformation, the conformal inversion $x \mapsto -x/x^2$ (known in the 2D case to Apollonius of Perga, third century BC) maps the origin in M into p_∞ . This is an example of an improper conformal transformation. There is a remarkable proper (and again involutive) conformal transformation, the Weyl inversion,

$$w\, x = \left(\frac{x^0}{x^2}\, ,\, \frac{-\mathbf{x}}{x^2} \right) , \quad w\, \vec{\xi} = \left(-\xi^{-1},\, -\xi^0,\, \xi^1,\, ...,\, \xi^D \right) , \tag{2.5}$$

which also interchanges the light cone at the origin with the light cone at infinity.

The group G acts transitively on $\overline{M}$ since every two isotropic rays can be brought to one another by a pseudo–rotation. The *stability subgroup* G_∞ of p_∞ is the Poincaré group extended by dilations. For example, the images of translations and (uniform) dilations in $SO\,(D,\, 2)$,

$$T_a = \begin{pmatrix} 1+\frac{a^2}{2} & -a^0 & \mathbf{a} & \frac{a^2}{2} \\ a^0 & 1 & \mathbf{0} & a^0 \\ \mathbf{a}^T & \mathbf{0}^T & \mathbb{I} & \mathbf{a}^T \\ -\frac{a^2}{2} & a^0 & \mathbf{a} & 1-\frac{a^2}{2} \end{pmatrix} ,$$

$$\Lambda_\alpha = \begin{pmatrix} \mathrm{ch}\alpha & 0 & \mathbf{0} & -\mathrm{sh}\alpha \\ 0 & 1 & \mathbf{0} & 0 \\ \mathbf{0}^T & \mathbf{0}^T & \mathbb{I} & \mathbf{0}^T \\ -\mathrm{sh}\alpha & 0 & \mathbf{0} & \mathrm{ch}\alpha \end{pmatrix} , \tag{2.6}$$

preserve the ray $\left\{ \lambda \vec{\xi}_\infty \right\}$. The simple linear action of G in $\mathbb{R}^{D,\,2}$ allows to compute the conformal factor in (1.1) for any conformal transformation g . Indeed, if the image gx of $x \in M$ under the action of $g \in G$ also belongs to M then the conformal factor $\omega\,(x,\, g)$ can be defined by

$$g\vec{\xi}_x = \omega\,(x,\, g)\, \vec{\xi}_{gx} \quad \text{for} \quad \vec{\xi}_x \cdot \vec{\xi}_\infty = 1 = \vec{\xi}_{gx} \cdot \vec{\xi}_\infty$$

$$\Rightarrow \quad \omega\,(x,\,g) \;=\; g\,\vec{\xi}_x\cdot\vec{\xi}_\infty \tag{2.7}$$

With this definition we find

$$(gx-gy)^2 \;=\; -2\,\vec{\xi}_{gx}\cdot\vec{\xi}_{gy} \;=\; -2\,\frac{g\vec{\xi}_x\cdot g\vec{\xi}_y}{\omega\,(x,g)\,\omega\,(y,g)} \;=$$

$$= \frac{(x-y)^2}{\omega\,(x,g)\,\omega\,(y,g)}\,. \tag{2.8}$$

in accord with (1.1).

Proposition 2.1 *Any pair* $(p_0,\,p_1)$ *of mutually non-isotropic points of* $\overline{M}$ *can be mapped into any other such pair* $(p_0',\,p_1')$ *by a conformal transformation.*

Proof. Due to the transitivity of the action of G there are elements g_0 and g_0' of G which carry p_0 and p_0' into the point p_∞ : $g_0p_0 = g_0'p_0' = p_\infty$. Then the images g_0p_1 and $g_0'p_1'$ of the two other points will both belong to M (because the original pairs are mutually non isotropic) and can hence be moved to one another by a translation t (in G_∞) which leaves p_∞ invariant: $g_0'p_1' = tg_0p_1$. So the element $g \in G$ which transforms the pair $(p_0,\,p_1)$ into $(p_0',\,p_1')$ is given by $g = (g_0')^{-1}t\,g_0$. $\Box$

2.2 A remarkable complex parametrization of $\overline{M}$.

We shall also need the complexification $\overline{M}_{\mathbb{C}}$ of $\overline{M}$ (and $Q_{\mathbb{C}}$ of Q) since energy positivity (Wightman axiom SC of Sec.3 below) implies that the vector valued distribution $F(x) = \phi(x)|0\rangle$ for an arbitrary local field $\phi(x)$ can be viewed as the boundary value of an analytic function $F(x+iy)$ holomorphic in the (forward) *tube domain* T^+ where

$$T^\pm \;=\; \left\{x \pm i\,y\ ;\ x\,\in\,M\ ,\ y\,\in\,V^+\right\}\,,$$

$$V^\pm \;=\; \left\{y\,\in\,M\ ;\ \pm y^0 > |\mathbf{y}|\right\}\,. \tag{2.9}$$

Clearly, $T^{\pm} \subset \overline{M}_{\mathbb{C}}$ and each of them is a *homogeneous space* of the (real) conformal group G [48], the stabilizer of a point being conjugate to the maximal compact subgroup $Spin\,(D) \times Spin\,(2)$ of G.

Proposition 2.2 *For every point* $p = \left\{\lambda\vec{\xi}\right\} \in T^+$ *and representative* $\vec{\xi} \in Q_{\mathbb{C}}$ *we have*

$$\vec{\xi}\cdot\vec{\xi}^{\,*} < 0 \quad \textit{and} \quad Re\left\{\vec{\xi},\vec{\xi}^{\,*}\right\}^{\perp} :=$$
$$\left\{\vec{\eta} \in \left\{\vec{\xi},\vec{\xi}^{\,*}\right\}^{\perp} \; ; \; \vec{\eta}^{\,*} = \vec{\eta}\right\} = \mathbb{R}^{D,\,0}; \tag{2.10}$$

here $\mathbb{R}^{D,\,0}$ *is a* D*-dimensional (real) euclidean space.*

Proof. Given the transitivity of the action of G on T^+ and the G–invariance of the inner product in $\mathbb{C}^{D+2}$, we can choose, without loss of generality, the vector $\vec{\xi}_+$ associated with ζ_+ according to (2.2) as: $\vec{\xi}_+ := \vec{\xi}_{\zeta_+} = \vec{e}_- + i\,\vec{e}_0$. It follows that $\vec{\xi}_+ \cdot \vec{\xi}_+^{\,*} = (\vec{e}_{-1} + i\,\vec{e}_0) \cdot (\vec{e}_{-1} - i\,\vec{e}_0) = -\,2$. Furthermore, $Re\left\{\vec{\xi},\vec{\xi}^{\,*}\right\}^{\perp}$ is the (real) linear span of $\vec{e}_1, ..., \vec{e}_D$ – i.e., it is a D-dimensional real euclidean space. □

The real basis vectors $\vec{e}_{-1}$, $\vec{e}_0$ can be traded for $\vec{\xi}_{\pm}$:

$$2\,\vec{e}_{-1} = \vec{\xi}_+ - \vec{\xi}_-\,(= \vec{\xi}_+ + \vec{\xi}_+^{\,*}\,)\,,$$
$$2\,\vec{e}_0 = \frac{1}{i}\left(\vec{\xi}_+ + \vec{\xi}_-\right)\,(= i\left(\vec{\xi}_+^{\,*} - \vec{\xi}_+\right)\,) \tag{2.11}$$

allowing to substitute (2.2) for for $x \to \zeta \in T^+$ by

$$\vec{\xi}_\zeta = \frac{1}{2}\left(\frac{1+\zeta^2}{2} - i\,\zeta^0\right)\vec{\xi}_+ - \frac{1}{2}\left(\frac{1+\zeta^2}{2} + i\,\zeta^0\right)\vec{\xi}_- +$$
$$+\zeta^j\,\vec{e}_j + \frac{1-\zeta^2}{2}\,\vec{e}_D = \frac{1+\zeta^2}{2}\,\vec{e}_{-1} + \zeta^\mu \vec{e}_\mu + \frac{1-\zeta^2}{2}\,\vec{e}_D\,. \tag{2.12}$$

Thus ζ is mapped on an euclidean complex D–vector z by a complex conformal transformation (cf. [46]):

$$\underline{z} = \omega^{-1}(\zeta)\,\underline{\zeta}\,, \quad z_D\,(=\,z^D\,)\,=\,\omega^{-1}(\zeta)\,\frac{1-\zeta^2}{2}\,, \quad \omega(\zeta)\,=\,\frac{1+\zeta^2}{2} - i\,\zeta^0$$

$$\Rightarrow \quad z^2 = \sum_{a=1}^{D} z_a^2 \,=\, \frac{\frac{1+\zeta^2}{2} + i\,\zeta^0}{\frac{1+\zeta^2}{2} - i\,\zeta^0} \,=\, \frac{\overline{\omega}(\zeta)}{\omega(\zeta)} \quad \Rightarrow$$

$$\omega^{-1}(\zeta(z)) \,=\, \frac{1+z^2}{2} + z_D \,=:\, \Omega^{-1}(z)$$

$$\Rightarrow \quad \underline{\zeta} = \Omega(z)\,\underline{z}\,, \quad i\,\zeta^0 \,=\, \Omega(z)\,\frac{z^2-1}{2}$$

$$(\Rightarrow \quad \zeta^2 \,=\, \frac{1+z^2-2z_D}{1+z^2+2z_D}\,)\,. \tag{2.13}$$

The fact that the mapping $\zeta \mapsto z$ (2.13) is conformal is reflected in the identity

$$\mathrm{d}z^2\,(=\sum_{a=1}^{D}(\mathrm{d}z^a)^2\,)\,=\,\omega^{-2}(\zeta)\,\mathrm{d}\zeta^2 \quad (\,\mathrm{d}\zeta^2\,=\,\mathrm{d}\underline{\zeta}^2 - \mathrm{d}\zeta_0^2\,)\,. \tag{2.14}$$

The real compactified Minkowski space can be singled out as the set of z invariant under an appropriate involution:

$$z \,\in\, \overline{M} \quad \text{iff} \quad z_a \,=\, z_a^* \,:=\, \frac{\overline{z}_a}{\overline{z}^2} \quad \Rightarrow \quad z^2\,\overline{z}^2 \,=\, 1\,,$$

$$\Pi_{ab} \,:=\, \frac{z_a z_b}{z^2} = \overline{\Pi}_{ab}\,. \tag{2.15}$$

The *real euclidean space* $E\,=\,\mathbb{R}^{D,0}$ corresponding to real $\underline{\zeta}\,=\,\underline{x}$ and pure imaginary ζ^0, gives rise to real $z^a\,(=\,z_a\,)$. Both $\overline{M}$ and $E\backslash\{0\}$ are contained in the submanifold $\mathbb{S}^{D-1}\times\mathbb{C}^*$ of $\overline{M}_{\mathbb{C}}$ (of real dimension $D+1$). This submanifold is characterized by the property that the symmetric tensor $\Pi\,=\,(\Pi_{ab})$ of Eq. (2.15) defines a real (1–dimensional) projection operator:

$$\Pi \,=\, \Pi^* \,=\, \overline{\Pi} = \Pi^2 \quad (\,tr\,\Pi = 1\,)$$

$$\Rightarrow \quad \Pi_{ab} \,=\, u_a\,u_b\,, \quad u \,\in\, \mathbb{S}^{D-1}\,. \tag{2.16}$$

3 Wightman axioms and global conformal invariance.

For the sake of simplicity we shall consider the theory of a single neutral scalar field ϕ. The general case of a system of finite component complex spintensor fields is treated in [36].

We say that the scalar field is conformally covariant of dimension d if for $\{x\,;\; gx \in M\}$

$$g \;:\; \phi\,(x) \;\mapsto\; [\omega\,(x,\, g)]^d \; \phi\,(gx) \;. \tag{3.1}$$

The requirement that ϕ is single valued implies that d should be an integer in this case. The n–point Wightman function (in fact, tempered distribution)

$$\mathcal{W}\,(x_1,\, ...,\, x_n) \;=\; \langle 0 |\; \phi\,(x_1)\, \dots\, \phi\,(x_n)\, |0\rangle \tag{3.2}$$

is said to satisfy the condition of *global conformal invariance* (GCI) if

$$[\omega\,(x_1,\, g)\, \dots\, \omega\,(x_n,\, g)]^d \; \mathcal{W}\,(gx_1,\, ...,\, gx_n) \;=\; \mathcal{W}\,(x_1,\, ...,\, x_n) \tag{3.3}$$

for $gx_1,\, ...,\, gx_n \in M$. This implies, in particular (unrestricted) Poincaré and dilation invariance.

We shall consider a quantum field theory satisfying GCI as well as the standard Wightman axioms which we proceed to recall (see [44] or [8]).

Spectral condition (SC) The Fourier transform of a translation invariant Wightman function

$$\mathcal{W}\,(x_1,\, ...,\, x_n) \;=\; W\,(x_{12},\, ...,\, x_{n-1\,n})\;, \quad x_{ik} \;=\; x_i - x_k \tag{3.4}$$

defined by

$$\tilde{W}\,(q_1, ..., q_{n-1}) \;=$$

$$= \int_{M^{\times(n-1)}} \cdots \int W(y_1, ..., y_{n-1})\ e^{-i(q_1 y_1 + ... + q_{n-1} y_{n-1})}\ \mathrm{d}^D y_1 \ldots \mathrm{d}^D y_{n-1} \qquad (3.5)$$

has support in the product of closed future light–cones:

$$supp\ \tilde{W} \subseteq \left(\overline{V}^{\,+}\right)^{\times(n-1)} \quad , \quad \overline{V}^{\,\pm} = \left\{ q \in M\,;\ \pm q^0 \geq |\underline{q}| \right\}. \qquad (3.6)$$

This is the relativistic (Lorentz invariant) form of *energy positivity.*

A sharpening of SC includes the *uniqueness of the vacuum.* It implies the vanishing of truncated n–point functions for $\rho_{ij} \mapsto \infty$ and the vanishing of all 1–point functions.

Local commutativity (LC):

$$\mathcal{W}(x_1, ..., x_i, x_{i+1}, ..., x_n) = \mathcal{W}(x_1, ..., x_{i+1}, x_i, ..., x_n)\ . \qquad (3.7)$$

Wightman positivity (WP), a consequence of Hilbert space positivity. Consider the Borchers algebra of finite sequences of (complex valued, smooth) test functions

$$f = (f_0,\ f_1(x_1)\,,\ f_2(x_1, x_2)\,,\ ...,\ f_n(x_1, ..., x_n)\,,\ ...)\ ,$$
$$f_n = 0 \quad \text{for} \quad n \gg 1\,, \qquad (3.8)$$

equipped with a (noncommutative) multiplication

$$f \times g = (f_0\, g_0,\ f_0\, g_1(x_1) + f_1(x_1)\ g_0,\ f_0\, g_2(x_1, x_2) +$$
$$f_1(x_1)\ g_1(x_2) + f_2(x_1, x_2)\ g_0,\ ...) \qquad (3.9)$$

and with an involution $f \mapsto f^*$ given by (term by term) complex conjugation. Define the *Wightman functional* $\mathcal{W}(f)$ by the series (finite sum)

$$\mathcal{W}(f) = f_0 + \int \mathcal{W}(x_1, x_2)\ f_2(x_1, x_2)\ \mathrm{d}^D x_1\, \mathrm{d}^D x_2 + \ldots\ . \qquad (3.10)$$

Then Wightman positivity assumes the form

$$\mathcal{W}(f^* \times f) \geq 0 \quad \text{for any} \quad f\,. \qquad (3.11)$$

The following simple result is an elementary application of the above postulates (see Sec.5.D of [16]).

Proposition 3.1 *The* 2*–point function of* ϕ *as well as its* 3*–point functions with arbitrary tensor fields are determined from the above postulates up to normalization constants. If, for instance,* T_{2l} *is a symmetric traceless tensor of (even) dimension* $2\nu_l$ *,*

$$T_{2l}(x,\,\zeta) \;=\; T_{\mu_1,\ldots,\mu_{2l}}(x)\,\zeta^{\mu_1}\,\ldots\,\zeta^{\mu_{2l}}\,, \quad \Box_\zeta\, T_{2l}(x,\,\zeta) \;=\; 0\,, \tag{3.12}$$

then the 3*–point function of two* ϕ*'s and* T_{2l} *is given by*

$$\begin{gathered} \langle 0|\,\phi\,(x_1)\,\phi\,(x_2)\,T_{2l}\,(x_3,\,\zeta)\,|0\rangle \;= \\ A_l\;(12)^d\,\left(X^2\right)^{\nu_l}\left(\zeta^2\right)^l C_{2l}^{\frac{D}{2}-1}\left(\widehat{\zeta}\cdot\widehat{X}\right)\,, \quad \widehat{v} \;=\; \frac{v}{\sqrt{v^2}}\,. \end{gathered} \tag{3.13}$$

Here (12) *is the free massless propagator (corresponding to* $d=1$*)*

$$(12) \;=\; \frac{1}{4\,\pi^2\rho_{12}}\,, \qquad \rho_{12} \;=\; x_{12}^{\,2} + i\,0\,x_{12}^0\,, \tag{3.14}$$

the D*–vector* X *depends on the three points,*

$$X\,(=\,X_{12}^3\,) \;=\; \frac{x_{13}}{\rho_{13}} - \frac{x_{23}}{\rho_{23}}\,, \quad X^2 \;=\; \frac{\rho_{12}}{\rho_{13}\rho_{23}}\,, \tag{3.15}$$

C_n^λ *is the Gegenbauer polynomial which satisfies the differential equation*

$$\left\{\left(1-z^2\right)\frac{\mathrm{d}^2}{\mathrm{d}z^2} - (2\,\lambda+1)\,z\frac{\mathrm{d}}{\mathrm{d}z} + n\,(n+2\,\lambda)\right\}C_n^\lambda\,(z) \;=\; 0\,, \tag{3.16}$$

and the normalization and orthogonality conditions

$$\begin{gathered} C_n^\lambda\,(1) \;=\; \begin{pmatrix} 2\,\lambda+n-1 \\ n \end{pmatrix}, \\ \frac{1}{\pi}\int_0^\pi C_n^\lambda\,(\cos\theta)\;C_m^\lambda\,(\cos\theta)\;\sin^{2\lambda}\theta\,\mathrm{d}\theta \;=\; \frac{\Gamma\,(2\,\lambda+n)\,\delta_{mn}}{2^{2\,\lambda-1}\,n!\;(n+\lambda)\;\Gamma^2\,(\lambda)}\,. \end{gathered} \tag{3.17}$$

The polynomial in ζ *is the (unique) harmonic extension of its light–cone value*

$$\left(\zeta^2\,X^2\right)^{\frac{n}{2}}\,C_n^{\lambda}\left(\widehat{\zeta}\cdot\widehat{X}\right)\;=\;\begin{pmatrix} n+\lambda-1 \\ n \end{pmatrix}\left(2\,\zeta\cdot X\,\right)^n$$

$$\textit{for}\quad \zeta^{\,2}\;=\;0\,. \tag{3.18}$$

WP implies

$$d\;\geq\;\lambda\;:=\;\frac{D}{2}-1\,(\,\equiv\;\lambda\,(D)\,)\,,\quad \nu_l\;\geq\;\lambda+l$$

$$\textit{for}\quad l\;>\;0\,,\quad \nu_0\;\geq\;0\,. \tag{3.19}$$

Sketch of proof. We shall only indicate how WP gives lower bounds to the dimensions (referring for details to Sec.5 of [16]). The $i0x^0$ term in the definition of rho in (3.14) defines the corresponding distribution in a way to ensure that its Fourier transform vanishes outside the forward cone (i.e. satisfies SC). The 2–point function of a symmetric traceless tensor $T_n\,(x)$ of dimension d_n has the form

$$\begin{aligned}
&\langle 0|\;T_n\,(x_1,\,\zeta_1)\;T_n\,(x_2,\,\zeta_2)\;|0\rangle\;= \\
&=N_n\,\frac{(d_n+n-1)\,\Gamma\,(d_n-1)}{(2\,\pi)^{\frac{D}{2}}}\;\left(\frac{2}{\rho_{12}}\right)^{d_n}\,(\zeta_1\cdot r\,(x_{12})\cdot\zeta_2)^n\;= \\
&=\;\frac{2\,\pi\,N_n}{\Gamma\,(d_n-\lambda)}\;\int\;\frac{\mathrm{d}^Dp}{(2\,\pi)^D}\;\theta\,(p^0)\;\left(-\,\frac{p^{\,2}}{2}\right)_+^{d_n-\lambda-n-1}\;\times \\
&\times\sum_{s\,=\,0}^{n}\,(-1)^s\;\frac{(d_n-2\,\lambda-n)_{n-s}}{(d_n+s-1)_{n-s}}\;\left(\frac{1}{2}\;p^{\,2}\right)^n\;\Pi^{ns}\,(p;\,\zeta_1,\,\zeta_2)
\end{aligned} \tag{3.20}$$

for ${\zeta_1}^2 = 0 = {\zeta_2}^2$; here $(a)_l = \frac{\Gamma(a+l)}{\Gamma(a)}$, $(x)_+ = \max\,(0,\, x)$, $\zeta_1 \cdot r\,(x_{12}) \cdot \zeta_2 = \zeta_1 \cdot \zeta_2 - 2\,\frac{(\zeta_1 \cdot x_{12})(\zeta_2 \cdot x_{12})}{\rho_{12}}$, $\Pi^{ns}\,(p)$ are projection operators in M^n:

$$\Pi^{nl}\,(p)\;\Pi^{ns}\,(p) \;=\; \delta_{ls}\,\Pi^{ns}\,(p)\;; \qquad \sum_{s\,=\,0}^{n} \Pi^{ns}\,(p) \;=\; \mathbb{I}$$

$$\text{i.e.} \quad \sum_{s\,=\,0}^{n} \Pi^{ns}\,(p;\,\zeta_1,\,\zeta_2) \;=\; (\zeta_1 \cdot \zeta_2)^n \;, \tag{3.21}$$

satisfying

$$p_{\mu_s}\;\ldots\;p_{\mu_n}\;[\Pi^{ns}\,(p)]^{\mu_1\,\ldots\,\mu_n}{}_{\nu_1\,\ldots\,\nu_n} \;=\; 0 \;=$$
$$= [\Pi^{ns}\,(p)]^{\mu_1\,\ldots\,\mu_n}{}_{\nu_1\,\ldots\,\nu_n}\;p^{\nu_s}\;\ldots\;p^{\nu_n}\;. \tag{3.22}$$

WP amounts to positivity of the Fourier transform for complex conjugate ζ . It is verified for

$$d_n \;\geq\; \lambda + n\,H \quad \text{for} \quad n \;>\; 0\,, \quad d_0 \;\geq\; \lambda\,, \tag{3.23}$$

as displayed by the expression for the contracted projection operator:

$$(-1)^s \left(\frac{1}{2}\,p^2\right)^n \Pi^{ns}\,\left(p;\,\zeta,\,\overline{\zeta}\right) \;=$$

$$\frac{D + 2\,s - 3}{(n-s)!}\;\;\frac{n!^2\;\left[(p \cdot \zeta)\left(p \cdot \overline{\zeta}\right)\right]^n}{(D-3)_{n+s+1}}\;\;C_s^{\lambda - \frac{1}{2}}\left(1 - \frac{p^2\;\left(\zeta \cdot \overline{\zeta}\right)}{(p \cdot \zeta)\left(p \cdot \overline{\zeta}\right)}\right)\;. \tag{3.24}$$

□

For a systematic study of the unitary positive energy representations of G which provide necessary conditions for Wightman positivity see [41] [32].

4 Local commutativity and GCI imply Huygens principle. Huygens principle and spectral conditions imply rationality.

The following strong locality property is an immediate consequence of Proposition 2.1.

Theorem 4.1 *GCI and LC imply the* Huygens principle (*a strong form of LC*):

$$[\,\phi\,(x)\ ,\ \phi\,(y)\,]\ =\ 0\quad for$$
$$(x-y)^2\neq 0\quad (\Leftrightarrow\ \ (3.7)\ \ valid\ \ for\ \ x_{i\,i+1}^{\,2}\neq 0\,)\,. \tag{4.1}$$

In fact, the *proof* of the statement for the Wightman function requires a refinement of Proposition 2.1 proven in [36].

Lemma 4.2 *For each set of points* $(x_1, ..., x_m, y_1, y_2)$ *in* M *such that* $y_{12}^{\,2} \neq 0$ *and a pair of mutually non-isotropic* y_1', y_2' *there exists a* $g \in C$ *such that* $gx_i \in M$ *for* $1 \leq i \leq m$ *and* $y_1' = gy_1$, $y_2' = gy_2$.

Lemma 4.2 can be *proven* by induction in m. We will briefly sketch the argument.

For $m = 0$ it reduces to Proposition 2.1. Assume that it is established for some $m \geq 0$. We shall prove that it is then also valid for arbitrary $m+1$ points $x_1, ..., x_{m+1}$ and mutually non-isotropic pairs (y_1, y_2), (y_1', y_2') in M. According to the assumption there exists a $g' \in C$ such that $g'x_i \in M$ for $1 \leq i \leq m$ and $y_1' = g'y_1$, $y_2' = g'y_2$. If $g'x_{m+1} \in M$ we are in business. If $p\, := g'x_{m+1} \in K_\infty$ then there exists an element h, arbitrarily close to the group unit in the stabilizer $C_{y_1', y_2'}$ $(\subset C\,)$ of the pair y_1', y_2' such that $hp \notin K_\infty$. To complete the proof of Lemma 4.2 it remains to choose h so that $hg'x_i \in M$ for $i\,=\,1\,,\,...\,,\,m$. This is possible since M is an open set in $\overline{M}$ and C acts continuously on $\overline{M}$. Hence, $g = hg'$ satisfies the conclusion of Lemma 4.2. $\square$

The main result of this section can be formulated as follows.

Theorem 4.3 *The truncated n–point function of $\phi(x)$ is a rational function of ρ_{ij} which vanishes for odd nd and has the form*

$$\mathcal{W}^t(x_1, .., x_n) = \frac{P(\rho_{kl})}{\prod\limits_{1 \le i < j \le n} \rho_{ij}^{\mu}} \quad for \quad n \ge 4\,, \quad n\,d \quad even\,, \tag{4.2}$$

where μ is a positive integer, P is a homogeneous polynomial of its arguments of degree $n(n-1)\mu - \frac{1}{2}\,nd$. Furthermore, P is symmetric under permutation of x_j and every index j of ρ appears in each term of P, $\mu\,(n-1) - d$ times. The 2– and 3–point functions of ϕ (which coincide with the corresponding truncated functions) are given by Proposition 3.1; *in particular,*

$$\mathcal{W}(x_1, x_2, x_3) = N_{3d}\, [(12)\,(23)\,(13)]^{\frac{d}{2}}\ ,$$

$$N_{3d} = 0 \quad for \quad d \quad odd\,. \tag{4.3}$$

For space-time dimension $D = 4$ the power of the denominator in (4.2) *can be chosen as $\mu = d - 1$.*

Sketch of proof. Since the n-point Wightman function is a Schwartz distribution its singularities have a finite order. WP implies that their order does not exceed the order of the pole of the the 2-point function (see Sec.4 of [36]). It then follows from the Huygens principle (Theorem 4.1) that, for some (sufficiently large) positive integer μ, the (translation invariant) function

$$F(x_{12}, ..., x_{n-1\,n}) = \left(\prod_{1 \le i < j \le n} \rho_{ij}^{\mu} \right) \mathcal{W}^t(x_1, ..., x_n) \tag{4.4}$$

is fully symmetric with respect to permutations of the x's implying, in particular,

$$F(y_1, ..., y_{n-1}) = F(-y_{n-1}, ..., -y_1)\ . \tag{4.5}$$

Applying SC to both sides we conclude that the Fourier transform $\widetilde{F}$ of F in each argument has support in the intersection of the forward and the backward cone. Hence, the support of $\widetilde{F}$ is at the origin in momentum space. In other words, F is a polynomial in its arguments. The symmetry of the numerator is a consequence of LC. Poincaré invariance implies that $F\left(x_{12}, \ldots, x_{n-1\,n}\right) = P\left(\rho_{kl}\right)$ for some polynomial in the ρ's. The last property of P follows from GCI. The fact that the truncated functions (for $n > 2$ and $D = 4$) have strictly lower singularities than the 2–point function follows from WP (see Corollary 4.4 of [36]). Thus we can choose then $\mu = d - 1$. $\square$

Remark 4.1 For $D = 2$ the power μ of the denominator in (4.2) cannot, in general, be chosen smaller than d. For instance the so called "energy field" in the critical Ising model, $\varepsilon\left(x\right) = \psi\left(x^0 + x^1\right)\overline{\psi}\left(x^0 - x^1\right)$ where ψ and $\overline{\psi}$ are free (1–component) Majorana–Weyl spinors of dimension $d_\psi = \frac{1}{2}$, has dimension $d = 1$ and a nontrivial truncated 4–point function which falls off at infinity while $d - 1 = 0$.

As noted in the beginning this basic result remains true for an arbitrary system of GCI fields. The only difference is in the power of ρ_{ij} in the definition of F (4.4). For instance in the theory of a rank n symmetric tensor field of dimension d this power becomes $d + n$.

Corollary 4.4 *The Huygens principle for any pair of GCI Bose fields can be written in the form*

$$\rho_{12}^{N}\left[\phi\left(x_1\right), \phi\left(x_2\right)\right] = 0 \quad \textit{for} \quad N \gg 1, \tag{4.6}$$

i.e. for N sufficiently large (for the commutator of a rank n symmetric tensor field of dimension d with itself it is enough to take $N = d + n$).

Remark 4.2 The canonical scalar and spinor fields (of dimension $\frac{D-2}{2}$ and $\frac{D-1}{2}$, respectively) do not have rational correlation functions and hence cannot satisfy GCI for odd D . It follows that the same is true for the stress–energy tensor T whose 3–point function is necessarily irrational (for instance, the general conformally invariant 3-point function of T satisfying the Ward–Takahashi identity for $D=3$ is a superposition of the 3–point functions of the canonical (composite, traceless) tensors of a scalar and a spinor field of dimensions $\frac{1}{2}$ and 1 , respectively, whose x–space propagators involve square roots of ρ_{ij}). It follows that the Huygens' principle cannot be realized for the full algebra of observable local fields in odd space–time dimensions much in line with what we know from classical field theory.

For any four points $x_1, ..., x_4$, there are exactly two independent conformally invariant cross ratios:

$$\eta_1 \, = \, \frac{\rho_{12}\,\rho_{34}}{\rho_{13}\,\rho_{24}}\,, \quad \eta_2 \, = \, \frac{\rho_{14}\,\rho_{23}}{\rho_{13}\,\rho_{24}}\,. \tag{4.7}$$

Corollary 4.5 *For $D>2$ the most general GCI truncated 4–point function of a scalar field of dimension d can be written in the form*

$$\mathcal{W}^{\,t}\,(x_1,\,...,\,x_4) \,=\, \mathcal{D}_d\,(\rho_{ij})\;\mathcal{P}_d\,(\eta_1,\,\eta_2)\;,$$

$$\mathcal{D}_d\,(\rho_{ij}) \,=\, \frac{(\rho_{13}\,\rho_{24})^{d-2}}{(\rho_{12}\,\rho_{23}\,\rho_{34}\,\rho_{14})^{d-1}}\,, \quad \mathcal{P}_d\,(\eta_1,\,\eta_2) \,=\, \sum_{\substack{i,\,j\,\geq\,0\\ i+j\,\leq\,2\,d-3}} c_{ij}\,\eta_1^i\,\eta_2^j\,. \tag{4.8}$$

Locality implies invariance of the polynomial $\mathcal{P}$ under the 6–element permutation group $\mathcal{S}_3$ generated by

$$\begin{aligned} s_{12}\,:\,\mathcal{P}_d\,(\eta_1,\,\eta_2) \;&\mapsto\; \eta_2^{2d-3}\,\mathcal{P}_d\left(\frac{\eta_1}{\eta_2},\,\frac{1}{\eta_2}\right) \;=\; \mathcal{P}_d\,(\eta_1,\,\eta_2)\;, \\ s_{23}\,:\,\mathcal{P}_d\,(\eta_1,\,\eta_2) \;&\mapsto\; \eta_1^{2d-3}\,\mathcal{P}_d\left(\frac{1}{\eta_1},\,\frac{\eta_2}{\eta_1}\right) \;=\; \mathcal{P}_d\,(\eta_1,\,\eta_2) \end{aligned} \tag{4.9}$$

This also yields the symmetry property $s_{12} = s_{12}s_{23}s_{12} = s_{23}s_{12}s_{23}$: $\mathcal{P}_d(\eta_1, \eta_2) \mapsto \mathcal{P}_d(\eta_2, \eta_1) = \mathcal{P}_d(\eta_1, \eta_2)$. *This leaves us with the following* $\left[\!\left[\frac{d^2}{3}\right]\!\right]$ [1] *independent coefficients*

$$c_{ij} \quad \mathit{for} \quad i \leq j \leq \frac{2d-3-i}{2}$$

$$(c_{ij} = c_{ji} = c_{i,2d-3-i-j} = c_{2d-3-i-j,i} = c_{j,2d-3-i-j} = c_{2d-3-i-j,j}). \tag{4.10}$$

We shall study in detail the case $d = 2$ for $D = 4$, that is the minimal d for which a non–vanishing truncated 4–point function exists. Using the shorthand notation $\langle 0|\,\phi(x_1)\ldots\phi(x_n)\,|0\rangle =: \langle 1\ldots n\rangle$ we shall write in this case

$$\begin{aligned} \langle 12\rangle &= \frac{c}{2}\,(12)^2\,, \quad \langle 123\rangle = c\,(12)\,(23)\,(13)\,, \\ \langle 1234\rangle &= \langle 12\rangle\langle 34\rangle + \langle 14\rangle\langle 23\rangle + \langle 13\rangle\langle 24\rangle + \mathcal{W}_4^t\,(d=2) \end{aligned} \tag{4.11}$$

with (ij) given by (3.14) and

$$\begin{gathered} \mathcal{W}_4^t = c\,(12)\,(23)\,(34)\,(14)\,(1+\eta_1+\eta_2) = \\ c\,\Big\{(12)\,(23)\,(23)\,(14) + (12)\,(13)\,(24)\,(34) + (13)\,(14)\,(23)\,(24)\Big\}\,. \end{gathered} \tag{4.12}$$

5 Stress-energy tensor; the Virasoro algebra.

In a CFT the conformal symmetry generators are expressed as integrals of the conserved stress-energy tensor:

$$\begin{aligned} &P_\mu = \int \Theta_{\mu 0}(x)\,\mathrm{d}\mathbf{x}\,, \;\; L_{\mu\nu} = \int (x_\mu\,\Theta_{\nu 0}(x) - x_\nu\,\Theta_{\mu 0}(x))\,\mathrm{d}\mathbf{x}\,, \\ &D = \int x^\lambda\,\Theta_{\lambda 0}(x)\,\mathrm{d}\mathbf{x}\,, \;\; K_\mu = \int \left(2x_\mu\,x^\lambda - x^2\,\delta_\mu^\lambda\right)\Theta_{\lambda 0}(x)\,\mathrm{d}\mathbf{x}\,, \\ &(\mathrm{d}\mathbf{x} = \mathrm{d}x^1\ldots\mathrm{d}x^{D-1})\,. \end{aligned} \tag{5.1}$$

[1] $[\![a]\!]$ stands for the integer part of a

The infinitesimal translations P_μ and Lorentz transformations $L_{\mu\nu}$ are conserved as a consequence of stress-energy conservation. The generators of dilation (D) and of special conformal transformations (K_μ) are conserved if in addition $\Theta_{\mu\nu}$ is traceless:

$$\frac{\mathrm{d}D}{\mathrm{d}x^0} = \int \Big(\Theta^\nu{}_\nu(x) - x^\lambda \, \partial_\nu \, \Theta_\lambda{}^\nu(x) \Big) \, \mathrm{d}\mathbf{x} = 0 \quad \text{if}$$

$$\partial_\nu \, \Theta^\nu{}_\lambda(x) = 0 \quad \text{and} \quad \Theta^\nu{}_\nu(x) = 0 \, . \tag{5.2}$$

In the theory of a free massless scalar field this requires adding a second derivative term to the canonical stress-energy tensor:

$$\Theta_\nu^\mu(x) = \frac{D}{2\,(D-1)} \, {:}\partial^\mu\varphi\,\partial_\nu\varphi{:} - \frac{\delta_\nu^\mu}{2\,(D-1)} \, {:}(\partial\varphi)^2{:}$$

$$+ \frac{D-2}{8\,(D-1)^2} \, \left(\delta_\nu^\mu \, \Box - D\,\partial^\mu\partial_\nu\right) \, {:}\varphi^2(x){:} \; . \tag{5.3}$$

The last term only vanishes for $D = 2$. We proceed to considering this exceptional case in more detail.

The 2D field φ does not exist as an operator–valued distribution acting in a Hilbert space: its 2–point function is logarithmic and hence violates WP (in the words of Coleman [11] "there are no Goldstone bosons in two dimensions"). Its gradient, however gives rise to a well defined (free) conformal vector field, a conserved current:

$$\sqrt{\pi} j_\mu(x) = i\,\partial_\mu\varphi(x) \; , \quad \partial_\mu j^\mu(x) = 0 = \partial_\mu j_\nu(x) - \partial_\nu j_\mu(x) \; . \tag{5.4}$$

Equations (5.4) are solved by the *chiral components* of the current which depend on a single light-cone variable:

$$(\partial_0 - \partial_1)\, j_L = 0 \, , \quad j_L\left(x^0 + x^1\right) = \frac{1}{2} \, \left(j^0(x) - j^1(x)\right) \; ,$$

$$(\partial_0 + \partial_1)\, j_R = 0 \, , \quad j_R\left(x^0 - x^1\right) = \frac{1}{2} \, \left(j^0(x) + j^1(x)\right) \; . \tag{5.5}$$

Expressing $\Theta_{\mu\nu}$ in terms of j_μ we obtain from (5.3) the so-called *Sugawara formula* (which can actually be traced back to work of the 1930ies – see

[30]). The traceless stress tensor in 2D also has two independent chiral components:

$$\Theta_L\left(x^0+x^1\right) \;=\; \frac{1}{2}\;\left(\Theta_0^0\,(x)+\Theta_0^1\,(x)\right)\;\left(=\;\pi\,j_L^2\,\right),$$
$$\Theta_R\left(x^0-x^1\right) \;=\; \frac{1}{2}\;\left(\Theta_0^0\,(x)-\Theta_0^1\,(x)\right)\;\left(=\;\pi\,j_R^2\,\right), \qquad (5.6)$$

We shall demonstrate that a chiral conformal field $\Theta\,(t)$ of dimension 2 gives rise to an infinite dimensional Lie algebra with commutation relations (CR) derived from Wightman axioms. The possibility to separate the left and the right movers' algebra is a simple consequence of L (see [23]).

Proposition 5.1 *Locality implies that the left and the right movers' fields mutually commute,*
$\left[j_L\left(x^0+x^1\right),j_R\left(y^0-y^1\right)\right] \;=\; 0 \;=\; \left[j_L\left(x^0+x^1\right),j_R\left(y^0-y^1\right)\right]$, *for all* $x,\,y\in M$.

Next we shall use the stereographic projection to introduce what is called in [23] the *analytic compact picture* fields:

$$t \;=\; 2\,i\,\frac{1-z}{1+z}\quad\left(\,z\;=\;\frac{1+\frac{i}{2}\,t}{1-\frac{i}{2}\,t}\sim 1+i\,t-\frac{t^2}{2}\;+\ldots\right),\quad z\;=\;e^{i\,\tau}$$
$$\Leftrightarrow\quad t\;=\;2\,\mathrm{tg}\left(\frac{\tau}{2}\right)\quad\left(\,t\,\in\,\mathbb{R}\,\Leftrightarrow\,\tau\,\in\,\mathbb{R}\,\right), \qquad (5.7)$$
$$J\,(z)\;=\;2\,\pi\,\left(i\,\frac{\mathrm{d}t}{\mathrm{d}z}\right)j\,(t\,(z))\;=\;\frac{8\,\pi}{(1+z)^2}\;j\left(2\,i\,\frac{1-z}{1+z}\right)\,,$$
$$T\,(z)\;=\;2\,\pi\,\left(i\,\frac{\mathrm{d}t}{\mathrm{d}z}\right)^2\,\Theta_L\,(t\,(z))\;=\;2\,\pi\,\frac{16}{(1+z)^4}\;\Theta_L\left(2\,i\,\frac{1-z}{1+z}\right)\,. \qquad (5.8)$$

Since (5.7) is an embedding of a light ray into the unit circle the physical values of z are the fixed points of the involution $z^*\;=\;\frac{1}{\overline{z}}$ where $\overline{z}$ is the complex conjugate of z. The observable fields are single–valued functions of z on the circle admitting a discrete Fourier (Laurent) expansion:

$$J\,(z)\;=\;\sum_{n\,\in\,\mathbb{Z}}J_n\,z^{-n-1}\,,\quad T\,(z)\;=\;\sum_{n\,\in\,\mathbb{Z}}L_n\,z^{-n-2}\,, \qquad (5.9)$$

With these conventions the reality condition for J and T assumes the form:

$$J_n^* \,=\, J_{-n}\,, \quad L_n^* \,=\, L_{-n}\,. \tag{5.10}$$

Proposition 5.2 *G invariance, for $G \,=\, Spin\,(2,2) \cong\, SU\,(1,1){\times}SU\,(1,1)\,,$ implies Möbius invariance of the chiral fields and of the vacuum state, the infinitesimal Möbius transformations being generated by the modes $L_0\,,\, L_{\pm 1}$ of T:*

$$\begin{aligned}
&[\,L_n\,,\,J\,(z)] \,=\, \frac{\mathrm{d}}{\mathrm{d}z}\,\left(z^{n+1}\,J\,(z)\right)\,, \\
&[\,L_n\,,\,T\,(z)] \,=\, z^n\,\left(z\,\frac{\mathrm{d}}{\mathrm{d}z}\,+2\,(n+1)\right)\,T\,(z)\,, \\
&L_n\,|0\rangle \,=\, 0 \quad \mathit{for} \quad n \,=\, 0,\,\pm 1\,.
\end{aligned} \tag{5.11}$$

The SC yield positivity of the chiral conformal energy L_0 *which implies*

$$L_n\,|0\rangle \,=\, 0 \,=\, J_n\,|0\rangle \quad \mathit{for} \quad n \,\geq\, 0\,. \tag{5.12}$$

The *proof* of this statement is contained in Sec.2.4 of [23].

The following result goes back to an early stage of 2D CFT ([42] [31]); its main part (concerned with the Virasoro algebra) is known as the Lüscher–Mack theorem (see [45] [33]).

Theorem 5.3 *Wightman axioms including G–invariance imply the CR*

$$\begin{aligned}
&[\,J\,(z_1)\,,\,J\,(z_2)\,] \,=\, \partial_2\,\delta\,(z_{12}) \quad (\,z_{12} \,=\, z_1 - z_2\,,\ \partial_2 \,\equiv\, \frac{\partial}{\partial z_2}\,) \\
&\qquad\qquad \Leftrightarrow\ [\,J_n\,,\,J_m] \,=\, n\,\delta_{n,\,-m}
\end{aligned} \tag{5.13}$$

$$\begin{aligned}
&[\,T\,(z_1)\,,\,T\,(z_2)\,] \,=\, \frac{c}{12}\,\partial_2^3\,\delta\,(z_{12}) + (T\,(z_1) + T\,(z_2))\,\partial_2\,\delta\,(z_{12})\,, \\
&\qquad\qquad [\,c\,,\,T\,(z)\,] \,=\, 0\,,
\end{aligned} \tag{5.14}$$

where the z–picture δ–function is defined by its formal power series expansion (*cf.* [27]):

$$\delta\left(z_{12}\right) = \frac{1}{z_1} \sum_{n\,\in\,\mathbb{Z}} \left(\frac{z_2}{z_1}\right)^n = \lim_{|z_1| \searrow |z_2|} \frac{1}{z_{12}} + \lim_{|z_2| \searrow |z_1|} \frac{1}{z_{21}}\,. \tag{5.15}$$

The local CR are equivalent to the Virasoro CR for the modes of T:

$$[\,L_n\,,\,L_m] = n\,L_{n+m} + c\,\frac{n^3-n}{12}\,\delta_{n,\,-m}\,, \quad [\,c\,,\,L_n] = 0\,. \tag{5.16}$$

The quantum Sugawara formula (*implied by* (5.6)) *reads*

$$T\,(z) = \frac{1}{2}\,:J^2\,(z):=$$

$$= J^{(+)}\,(z)\;J^{(-)}\,(z) + \frac{1}{2}\,\left(J^{(+)}\,(z)\;J^{(+)}\,(z) + J^{(-)}\,(z)\;J^{(-)}\,(z)\right)\,, \tag{5.17}$$

where $J^{(-)}\,(z)$ ($J^{(+)}\,(z)$) is the sums of (*non*)*negative powers of z:*

$$J^{(-)}\,(z) = \sum_{n\,=\,0}^{\infty} J_n\,z^{-n-1}\,,$$

$$J^{(+)}\,(z) = \sum_{n\,=\,1}^{\infty} J_{-n}\,z^{n-1} = J\,(z) - J^{(-)}\,(z)\,. \tag{5.18}$$

It fixes the Virasoro central charge to $c = 1$ and further implies the mixed CR

$$[\,T\,(z_1)\,,\,J\,(z_2)\,] = \frac{\partial}{\partial z_2}\,\left(\delta\,(z_{12})\,J\,(z_2)\right)\,. \tag{5.19}$$

Sketch of proof. The Huygens principle for an arbitrary chiral (hermitean) Bose field $O_n\,(z)$ of dimension $n \in \mathbb{N}$ (cf. (4.6)),

$$(z-w)^{2n}\,[\,O_n\,(z)\,,\,O_n\,(w)\,] = 0 \tag{5.20}$$

implies that the commutator can be expanded in a finite sum of derivatives of $\delta\,(z-w)$:

$$[\, O_n\,(z)\ ,\, O_n\,(w)] \;=\; \sum_{l\,=\,0}^{2n-1} A_{2n-l-1}\,(w)\;\partial_w^l\,\delta\,(z-w)$$

$$(\,\partial_w \;\equiv\; \frac{\partial}{\partial w}\,)\,. \tag{5.21}$$

Here $A_k\,(w)$ is a (hermitean) local field of dimension k. The antisymmetry of the commutator together with the identity

$$A_k\,(z)\;\partial_w^l\,\delta\,(z-w) \;=\; \sum_{s\,=\,0}^{l} \begin{pmatrix} l \\ s \end{pmatrix} A_k^{(s)}\,(w)\;\partial_w^{l-s}\,\delta\,(z-w)\;. \tag{5.22}$$

imply the relations

$$(-1)^k\;A_k\,(w) \;=\; \sum_{s\,=\,0}^{k} (-1)^s \begin{pmatrix} 2\,n-k+s-1 \\ s \end{pmatrix} A_{k-s}^{(s)}\,(w)\;. \tag{5.23}$$

The uniqueness of the vacuum requires that A_0 is a real number. Eq. (5.23) then allows to express all A_k in terms of the first non-vanishing A_{2l} for $l > 0$. If, in particular, A_2 is not identically zero then

$$A_1 \;=\; 0\,, \quad 2\,A_3 \;=\; (2\,n-3)\;A_2' \quad \text{etc.} \tag{5.24}$$

Applied to $J\,(z)$ $(\,n\;=\;1\,)$ this gives (5.13) (normalizing J in such a way that $A_0 = 1\,$). In the case of $T(z)$ $(n = 2)$ the normalization is fixed by the infinitesimal Möbius transformations (5.11). Indeed using (5.9), (5.11) and (5.15) we find

$$L_{-1} \;=\; \oint T\,(z)\;\frac{\mathrm{d}z}{2\,\pi\,i}\,, \quad [\,L_{-1}\,,\,T\,(z)] \;=\; T'\,(z)$$

$$(\,\text{as} \oint\limits_{|z_1|\,=\,r_1\,>\,|z_2|} f\,(z_1)\;\delta\,(z_{12})\;\frac{\mathrm{d}z_1}{2\,\pi\,i} \;=\; f\,(z_2)\,)\,. \tag{5.25}$$

This allows to identify A_3 and (in view of (5.24)) A_2: $A_3\,(z) \;=\; T'\,(z) \;=\; \frac{1}{2}\;A_2'\,(z)\,,\; A_2\,(z) \;=\; 2T\,(z)\,;$ setting $A_0\,(z) \;=\; \frac{c}{12}$ we end up with

$$[\,T\,(z_1)\ ,\,T\,(z_2)\,] \;=\; \frac{c}{12}\;\partial_2^3\,\delta\,(z_{12}) + 2\,T\,(z_2)\;\partial_2\,\delta\,(z_{12}) + T'\,(z_2)\;\delta\,(z_{12}) \tag{5.26}$$

which is equivalent to (5.14). $\Box$

Remark 5.1 The CR for $T(z)$ can be established under the (apparently) weaker assumption of scale invariance but still requiring tracelessness of the stress-energy tensor ([31], [23] Sec.3.). This is not surprising since, as we have already noted, the infinitesimal special conformal transformations are conserved whenever there is a conserved traceless stress–energy tensor.

6 Example of a rational CFT: Weyl fermions.

The chiral fields like $T(z)$ and $J(z)$ satisfy GCI and hence have rational correlation functions. A chiral CFT is generated by a "complete set" of positive energy representations of a local *chiral algebra* $\mathfrak{A}$ which are closed under multiplication of the corresponding interpolating fields (thus giving rise to well defined *fusion rules*). If the number of such representations, called *superselection sectors*, is finite then the chiral CFT is called *rational* (RCFT).

The energy and charge distribution in an irreducible $\mathfrak{A}$ module $\mathcal{H}_\kappa$ of a chiral algebra is captured by the chiral partition function or *character* of the representation

$$\chi_1^{(\kappa)}(\tau, \zeta) = tr_{H_\kappa}\left(q^{\widetilde{L}_0} z^{J_0}\right), \quad q = e^{2\pi i \tau},$$
$$Im\,\tau > 0, \quad z = e^{2\pi i \zeta}, \quad \widetilde{L}_0 = L_0 - \frac{c}{24}. \tag{6.1}$$

The characters in an RCFT have the remarkable property to span a finite dimensional representation of the *modular group* $SL(2, \mathbb{Z})$, of fractional linear transformations of the parameter τ (with an accompanying change of ζ and an appropriate multiplicative factor). We shall exhibit these properties in the simplest example of an RCFT: the theory of a (free) Weyl field.

A complex field $\psi\,(z)$ and its conjugate, which carry unit charge,

$$[\,J_0\,,\,\psi^*\,(z)\,]\;=\;\psi^*\,(z)\;,\quad [\,J_0\,,\,\psi\,(z)\,]\;=\;-\,\psi\,(z)\;, \tag{6.2}$$

and obey a homogeneous Möbius transformation law, have the canonical dimension $\frac{1}{2}$ of a Weyl fermion (because of the Sugawara formula (5.17). They satisfy the local anticommutation relations

$$[\,\psi\,(z_1)\;,\,\psi\,(z_2)\,]_+\;=\;0\;=\;[\,\psi^*\,(z_1)\;,\,\psi^*\,(z_2)\,]_+\;,$$
$$[\,\psi\,(z_1)\;,\,\psi^*\,(z_2)\,]_+\;=\;\delta\,(z_{12}) \tag{6.3}$$

which (together with SC) determine their Wightman functions. The chiral algebra $\mathfrak{A}$ of the theory is the maximal Bose field subalgebra of the Clifford algebra $\mathfrak{F}$ of $\psi^{(*)}$. It is generated by the charge 2 bilocal fields $\frac{1}{z_{12}}\,\psi\,(z_1)\,\psi\,(z_2)$ and $\frac{1}{z_{12}}\,\psi^*\,(z_1)\,\psi^*\,(z_2)$ that have finite limits for $z_{12}\to 0$ and provides an example of an 1–dimensional even (charge-)lattice current algebra $\mathfrak{A}=\mathfrak{A}\,(4\,\mathbb{Z})$, the square of the minimal charge vector being 4. There are 4 irreducible positive energy representations of $\mathfrak{A}$ labeled by the elements of the dual lattice *mod* $4\,\mathbb{Z}$. The four $\mathfrak{A}$ modules are combined into two superselection sectors $\mathcal{H}_\kappa$ of the field algebra $\mathfrak{F}$: the Neveu–Schwarz (NS) sector $\mathcal{H}_0$ containing the vacuum $|0\rangle$ and the Ramond (R) sector $\mathcal{H}_{\frac{1}{2}}$ with a pair of oppositely charged minimal energy states carrying charge of absolute value $\kappa=\frac{1}{2}$. The mode expansion of $\psi^{(*)}\,(z)$ depends on the sector; if

$$\psi\,(z)\;=\;\sum_\rho \psi_\rho\,z^{-\rho-\frac{1}{2}}\;,\quad \psi^*\,(z)\;=\;\sum_\rho \psi^+_\rho\,z^{-\rho-\frac{1}{2}}\;,$$
$$(\psi_\rho)^*\;=\;\psi^+_{-\rho} \tag{6.4}$$

then $\rho\in\mathbb{Z}+\frac{1}{2}$ for the NS sector and $\rho\in\mathbb{Z}$ for the R sector (in a compact formula, $\rho=n+\frac{1}{2}-\kappa$), so that $\left(\psi\left(e^{2\,\pi\,i}z\right)-(-1)^{2\,\kappa}\,\psi\,(z)\right)\mathcal{H}_\kappa=0$. There are two equivalent expressions for the modular energy $\widetilde{L}_0$: one, in terms of the current modes, another, in terms of the $\psi^{(*)}$ modes, both given by ζ–function regularized sums of symmetric (rather than normal) products. We have

$$\widetilde{L}_0 \;=\; \frac{1}{2} \sum_{n \in \mathbb{Z}} J_{-n}\, J_n \;=\; L_0 + \frac{1}{2} \sum_{n=1}^{\infty} n \;=$$
$$=\; L_0 + \frac{1}{2}\, \zeta\,(-1) \;=\; L_0 - \frac{1}{24} \tag{6.5}$$

where

$$L_0 \;=\; \frac{1}{2}\, J_0^2 + \sum_{n=1}^{\infty} J_{-n}\, J_n\,, \qquad \left(e^{2\,\pi\, i\, L_0} - e^{i\,\pi\,\kappa^2}\right) \mathcal{H}_\kappa \;=\; 0\,;$$
$$\zeta\,(-n) \;=\; (-1)^n\, \frac{B_{n+1}}{n+1} \tag{6.6}$$

B_m being the Bernoulli numbers (see, e.g. [18], Sec.1.5), $B_1 = -\frac{1}{2}$, $B_2 = \frac{1}{6}$, $B_{2n+1} = 0$ for $n = 1,\, 2,\, \ldots\,$, $B_4 = -\frac{1}{30}$, etc.; similarly,

$$\widetilde{L}_0 \;=\; \frac{1}{2} \sum_{\rho} \rho\, \left(\psi^+_{-\rho}\, \psi_\rho + \psi_{-\rho}\, \psi^+_\rho\right) \;=$$
$$=\; \sum_{\rho > 0} \rho\, \left(\psi^+_{-\rho}\, \psi_\rho + \psi_{-\rho}\, \psi^+_\rho\right) - \sum_{\rho > 0} \rho\,; \tag{6.7}$$

here the c–number term assumes different values in the two sectors:

$$- \sum_{\rho > 0} \rho \;=\; \frac{1}{2}\, \kappa^2 - \frac{1}{24}$$

$$\left(\text{since} \quad \frac{1}{2} \sum (2n+1) = -\frac{1}{2}\, \left(\zeta\,(-1) - 2\,\zeta\,(-1)\right) \;=\; \frac{1}{2}\,\zeta\,(-1)\,\right). \tag{6.8}$$

In order to compute the characters (6.1) we introduce the *mean value* of an operator A in $\mathcal{H}_\kappa$:

$$\langle A \rangle^{\kappa}_{\beta,\, \zeta} \;=\; \frac{1}{\chi_1^{(\kappa)}\,(\tau,\, \zeta)}\; tr_{\mathcal{H}_\kappa} \left(A\, q^{\widetilde{L}_0}\, z^{J_0}\right)$$
$$\text{for} \quad 2\pi i \tau \;=:\; -\,\beta \quad \left(\, q \;=\; e^{-\,\beta}\,\right) \tag{6.9}$$

(β having the physical interpretation of inverse absolute temperature). It satisfies the Kubo-Martin-Schwinger (KMS) boundary condition [25]

$$\langle\, A\, q^{\widetilde{L}_0}\, z^{J_0}\, B\, q^{-\,\widetilde{L}_0} z^{-\,J_0}\, \rangle^{\kappa}_{\beta,\,\zeta} \;=\; \langle\, B\, A\, \rangle^{\kappa}_{\beta,\,\zeta} \tag{6.10}$$

which will allow us to compute the mean value of the energy

$$\langle\, \widetilde{L}_0\, \rangle^{\kappa}_{\beta,\,\zeta} \;=\; -\,\frac{\partial}{\partial\beta}\, \log \left(\chi^{(\kappa)}_1 \,(\tau,\, \zeta) \right) \tag{6.11}$$

for both expressions (6.5) and (6.7) in our model. (For a systematic application of the KMS condition for computing chiral partition functions in $c = 1$ RCFTs see [9] and [23].)

The known CR (5.13) of the current modes and the KMS condition (6.10) allow to compute the mean value of each term in the expansion (6.6):

$$\begin{aligned}
& q^{-n}\, \langle\, J_{-n}\, J_n\, \rangle^{\kappa}_{\beta,\,\zeta} \;=\; \langle\, J_{-n}\, q^{\widetilde{L}_0}\, J_n\, q^{-\,\widetilde{L}_0}\, \rangle^{\kappa}_{\beta,\,\zeta} \;= \\
& = \langle\, J_n\, J_{-n}\, \rangle^{\kappa}_{\beta,\,\zeta} \;=\; n + \langle\, J_{-n}\, J_n\, \rangle^{\kappa}_{\beta,\,\zeta} \\
& \Longrightarrow \qquad \langle\, J_{-n}\, J_n\, \rangle^{\kappa}_{\beta,\,\zeta} \;=\; \frac{n\, q^n}{1 - q^n}\,.
\end{aligned} \tag{6.12}$$

Only the mean value of the first term in the expansion (6.6) depends on z and κ:

$$\langle\, J_0^2\, \rangle^{\kappa}_{\beta,\,\zeta} \;=\; \sum_{n\,\in\,\mathbb{Z}}\, (n + \kappa)^2\; q^{\frac{1}{2}\,(n+\kappa)^2}\; z^{n+\kappa}\,. \tag{6.13}$$

Integrating and exponentiating the result we end up with the following expression for the characters

$$\chi^{(\kappa)}_1\,(\tau,\,\zeta) \;=\; \frac{1}{\eta\,(\tau)}\, \sum_{n\,\in\,\mathbb{Z}}\, q^{\frac{1}{2}\,(n+\kappa)^2}\; z^{n+\kappa}\,, \quad \eta\,(\tau) \;=\; q^{\frac{1}{24}}\, \prod_{n\,=\,1}^{\infty}\, (1 - q^n) \tag{6.14}$$

where $\eta\,(\tau)$ is the Dedekind η–function.

Similarly, the expansion (6.7) yields an infinite product formula for the characters

$$\chi_1^{(\kappa)}(\tau,\,\zeta)\;=\;q^{\frac{\kappa^2}{2}-\frac{1}{24}}z^{\kappa}\prod_{n=1}^{\infty}\left(1+z\,q^{n+\kappa-\frac{1}{2}}\right)\left(1+z^{-1}\,q^{n-\kappa-\frac{1}{2}}\right)\,. \tag{6.15}$$

One thus obtains an independent proof of the identity of the expressions (6.14) and (6.15) which is a consequence of the classical Jacobi triple product formula (see e.g. [26]).

The vacuum (NS) representation of the field algebra $\mathfrak{F}$ is the only one in which $\psi^{(*)}(z)$ are local Fermi fields (they satisfy more general *anyonic* exchange relations with the field of dimension $\frac{1}{8}$ that intertwines the NS with the R sector). The R sector does provide a single valued representation of the bosonic chiral algebra $\mathfrak{A}$ which can be defined as the subalgebra of gauge invariant obseravables in $\mathfrak{F}$ with respect to the finite gauge group $\mathbb{Z}_2$ of inner automorphisms whose nontrivial element is the fermion parity f:

$$f\;:\;B\;\mapsto\;e^{i\,\pi\,J_0}\,B\,e^{-\,i\,\pi\,J_0}$$

$$(\,e^{i\,\pi\,J_0}\,\psi^{(*)}\,e^{-\,i\,\pi\,J_0}\;=\;-\,\psi^{(*)}\,,e^{i\,\pi\,J_0}\,A\,e^{-\,i\,\pi\,J_0}\;=\;A\quad\text{for}\quad A\,\in\,\mathfrak{A}\,)\,. \tag{6.16}$$

In other words, we are dealing here with a simple $\mathbb{Z}_2$ orbifold theory (see, e.g. [15] [29] [7]). The characters (6.1) have now to be supplemented by the chiral partition functions including the fermion parity:

$$\chi_f^{(\kappa)}(\tau,\,\zeta)\;=\;tr_{\mathcal{H}_\kappa}\left(f\,q^{\widetilde{L}_0}\,z^{J_0}\right)\;=\;(-i)^{2\,\kappa}\;\chi_1^{(\kappa)}\left(\tau,\,\zeta+\frac{1}{2}\right)\,. \tag{6.17}$$

We thus have

$$\chi_f^{(\kappa)}(\tau,\,\zeta)\;=\;q^{\frac{\kappa^2}{2}-\frac{1}{24}}\,z^{\kappa}\prod_{n=1}^{\infty}\left(1-z\,q^{n+\kappa-\frac{1}{2}}\right)\left(1-z^{-1}\,q^{n-\kappa-\frac{1}{2}}\right)\;=$$

$$=\;K_{2\,\kappa}(\tau,\,z;\,4)-K_{2\,\kappa+2}(\tau,\,z;\,4) \tag{6.18}$$

where

$$K_l\,(\tau,\,z;\,m) \qquad = \frac{1}{\eta\,(\tau)}\sum_{n\,\in\,\mathbb{Z}} q^{\frac{m}{2}\,\left(n+\frac{l}{m}\right)^2}\,z^{\frac{1}{2}\,(m\,n+l)} = $$
$$= K_{l+m}\,(\tau,\,z;\,m)\ . \tag{6.19}$$

The modular group $SL\,(2,\,\mathbb{Z})$ has two generators S and T satisfying one relation:

$$S = \begin{pmatrix} 0 & -1 \\ 1 & 0 \end{pmatrix}\ ,\quad T = \begin{pmatrix} 1 & 1 \\ 0 & 1 \end{pmatrix}\ ,$$
$$S^2 = (S\,T)^3 \quad (\,S\,\tau = \frac{-1}{\tau}\,, T\,\tau = \tau + 1\,)\,. \tag{6.20}$$

For even m the action of S and T on the characters (6.19) is given by

$$e^{-\,i\,\pi\,\frac{m}{4}\,\frac{\zeta^2}{\tau}}\,K_l\left(-\,\frac{1}{\tau},\,\frac{\zeta}{\tau};\,m\right) = \frac{1}{\sqrt{m}}\sum_{l'\,=\,0}^{m-1} e^{-\,2\,i\,\pi\,\frac{l\,l'}{m}}\,K_{l'}\,(\tau,\,\zeta;\,m)\ ,$$
$$K_l\,(\tau+1,\,\zeta;\,m) = e^{i\,\pi\left(\frac{l^2}{m}\,-\,\frac{1}{12}\right)}\,K_l\,(\tau,\,\zeta;\,m)\ . \tag{6.21}$$

We note that abelian current algebras, the simplest example of which is the above considered $U\,(1)$ current algebra, are associated with RCFT models applied to the study of fractional quantum Hall states (see, e.g. the work of Fröhlich et al. and of Cappelli et al. which can be traced back from [22] and [10]).

The field algebra $\mathfrak{F}$ of a complex Weyl "spinor" is the antisymmetric tensor square of the algebra of a Majorana Weyl field. It follows that the partition functions of the corresponding "twisted sectors" are squares of Ising model partition functions[2], a fact which is displayed by the infinite product expressions (6.15) and (6.18). The modular invariant partition function for the critical Ising model can be written, accordingly, in the form

$$Z_{Ising} = \frac{1}{2}\left\{\left|\chi_1^{(0)}\,(\tau,\,0)\right| + \left|\chi_1^{(0)}\left(\tau,\,\frac{1}{2}\right)\right| + \left|\chi_1^{\left(\frac{1}{2}\right)}\,(\tau,\,0)\right|\right\} \tag{6.22}$$

[2]For a textbook treatment of the critical Ising model - see [13].

where the absolute values of the characters

$$\left| \chi_1^{(\kappa)} (\tau, z) \right| = \left\{ \chi_1^{(\kappa)} (\tau, \zeta) \, \overline{\chi_1^{(\kappa)} (\tau, \zeta)} \right\}^{\frac{1}{2}} ,$$

$$\overline{\chi_1^{(\kappa)} (\tau, \zeta)} = \chi_1^{(\kappa)} \left(-\overline{\tau}, -\overline{\zeta} \right) \tag{6.23}$$

are expressed as series of (rational) powers of q and $\overline{q}$ (bounded below by $-\frac{1}{48}$) with positive integer coefficients.

2D CFT has become an important domain of pure mathematics. The role of the modular group, noted here, gives only a glimpse of this development. Happily, there are two impressive recent books which expand the modern mathematical point of view on CFT and survey its role in problems of pure mathematics: [3] and [21].

7 Operator product expansions (OPE). Contribution of the conserved tensors.

Given a neutral scalar field ϕ of (integer) dimension d in ($D = 4$) Minkowski space we will look for an expansion of the product of two ϕ's in terms of bilocal fields:

$$\phi (x_1) \, \phi (x_2) = \sum_{\nu = 0}^{d} (12)^{d-\nu} \, V_\nu (x_1, x_2) \, , \quad V_0 = C_\phi \, (> 0) \tag{7.1}$$

where (12) is the free massless field's 2–point function (3.14). One can try to determine V_ν as the part of the OPE involving *twist* (i.e. dimension minus rank) $2\,\nu$ conformal symmetric tensor fields (among others). One can assume without restricting the generality that different V_ν's have disjoint OPEs so that they are orthogonal:

$$\langle 0 | \, V_\lambda (x_1, x_2) \, V_\nu (x_3, x_4) \, | 0 \rangle = 0 \quad \text{for} \quad \lambda \neq \nu \, . \tag{7.2}$$

Bilocality means that two V's commute whenever their arguments are space–like separated. The results of Sec.4 then imply that two V's commute unless a pair of their arguments is light–like.

The field V_1 is of special interest: its OPE involves an infinite set of (even rank) *conserved* symmetric traceless tensors starting with the stress–energy tensor. This OPE has a particularly simple form for $d \leq 2$ – i.e. for free fields, $d = 1$, and for the $d = 2$ case. Then V_1 is expanded in just such tensors so that we can write

$$V_1 (x_1, x_2) \, = \, 2 \sum_{l=0}^{\infty} C_{1l} \int_0^1 \mathrm{d}\alpha \, K_l \, (\alpha, \, \rho_{12} \, \Box_2) \; T_{2l} \, (x_2 + \alpha \, x_{12}; \, x_{12}) \tag{7.3}$$

where

$$K_l \, (\alpha, \, z) \; = \; \frac{(4l+1)!}{(2l!)^2} \, \alpha^{2l} \, (1-\alpha)^{2l} \sum_{n=0}^{\infty} \frac{\left[\alpha \, (\alpha - 1) \, \frac{z}{4}\right]^n}{n! \, (4l+1)_n} \, ,$$

$$\int_0^1 K_l \, (\alpha, \, 0) \; \mathrm{d}\alpha \; = \; 1 \, , \tag{7.4}$$

$\Box_2$ is the d'Alembert operator acting on x_2 for fixed x_{12}. The operator acting on T_{2l} is chosen to transform the 2–point function of T_{2l} into a 3–point one:

$$\int_0^1 \mathrm{d}\alpha \, K_l \, (\alpha, \, \rho_{12} \, \Box_2) \; \frac{(x_{12} \cdot r \, (y \, (\alpha)) \cdot \zeta)^{2l}}{\left(\rho_{y(\alpha)}\right)^{2\,l+2}} \; = \; \frac{(X \cdot \zeta)^{2l}}{\rho_{13} \, \rho_{23}} \, ,$$

$$y \, (\alpha) \; = \; x_{23} + \alpha \, x_{12} \, , \quad \rho_y \; = \; y^2 + i \, 0 \, y^0 \tag{7.5}$$

where X and $r\,(y)$ are defined in (3.15) and after (3.20), respectively. The conservation of the tensor T_{2l} assumes a simple analytic form:

$$\frac{\partial^2}{\partial x \, \partial \zeta} \, T_{2l} \, (x, \, \zeta) \; = \; 0 \, , \quad l \; = \; 0, \, 1, \, \ldots \, . \tag{7.6}$$

(For $l = 0$ Eq. (7.6) just says that T_0 is independent of ζ .) For $d = 2$ the expansion (7.3) necessarily includes the original field $\phi(x)$ itself:

$$\frac{1}{2} V_1(x, x) = \phi(x) \; (\equiv T_0(x)) \quad \text{for} \quad d\,(= d(\phi)) = 2 \quad (C_{10} = 1) \,. \tag{7.7}$$

For higher d we shall be interested in the case when no $d = 2$ field contributes to the OPE so that $C_{10} = 0$. The contribution of T_{2l} to the 4–point function

$$\langle 0|\; V_1(x_1, x_2)\; V_1(x_3, x_4)|0\rangle \tag{7.8}$$

is universal (i.e. independent of d) and can be expressed in terms of hypergeometric functions (see Eq. (3.10) of [17] or (A.6) of [37]). For a light–like x_{34} , i.e. for $\rho_{34} = 0\,(= \eta_1)$ this expression becomes particularly simple:

$$\begin{aligned} &\langle 0|\; V_1(x_1, x_2) \int_0^1 d\alpha\, \frac{\alpha^{2l}\,(1-\alpha)^{2l}}{B\,(2l+1,\, 2l+1)}\, T_{2l}\,(x_4 + \alpha\, x_{34})\,|0\rangle = \\ &\quad = A_l\,(13)\,(24)\; \epsilon^{2l}\, F\,(2l+1,\, 2l+1;\, 4l+2;\, \epsilon)\;, \\ &\epsilon = 1 - \eta_2\,(= 2\left(\frac{x_{24}}{\rho_{24}} - \frac{x_{13}}{\rho_{13}}\right) x_{34} \quad \text{for} \quad \rho_{34} = 0\,) \end{aligned} \tag{7.9}$$

where A_l is defined by the normalization of the 3–point function (3.13). The constants C_l and $A_l = N_l\, C_l$ depend on the normalization of the conserved tensor field T_{2l} (N_l determining its 2-point function). The product $A_l\, C_l = N_l\, C_l^2$ is invariant under rescaling of T_{2l} and can be computed from the 4-point function (1.7). For $d = 2$ this yields the relation

$$c\left(1 + \frac{1}{1-\epsilon}\right) = 2 \sum_{l=0}^{\infty} A_l\, C_l\, \epsilon^{2l}\, F\,(2l+1,\, 2l+1;\, 4l+2;\, \epsilon) \tag{7.10}$$

which gives

$$A_l\, C_l\,(= N_l\, C_l^2\,) = \begin{pmatrix} 4l \\ 2l \end{pmatrix}^{-1} c\,. \tag{7.11}$$

(The result differs from the free field case, $d = 1$, just by the factor c .) The Ward–Takahashi identity for the time ordered 3–point function of the stress-energy tensor (see [34]) allows to compute its normalized 3–point function,

$$\langle 0|\ \phi(x_1)\ \phi(x_2)\ T_2(x_3,\zeta)\,|0\rangle \ =\ \frac{2c}{3}\,(12)\,(23)\,(13)\left(X^2\zeta^2 - 4\,(X\cdot\zeta)^2\right), \tag{7.12}$$

and hence to find A_1 (N_1) and C_1 separately:

$$A_1\,(=\,N_1\,C_1\,)\ =\ -\,\frac{2}{3}\,,\quad C_1\ =\ -\,\frac{1}{4}\,,\quad N_1\ =\ \frac{8}{3}\,c\,. \tag{7.13}$$

It should be noted that the hypergeometric function in (7.9) is not rational (it involves a logarithm of $1-\epsilon$ - see [37], Appendix A) while the infinite sum (7.10) is a rational function of ϵ.

8 An infinite dimensional Lie algebra associated with a $d = 2$ field.

We shall now concentrate on the case $d = 2$ and will use the complex compact picture parametrization of Sec.2B writing (7.1) for this special case in the form

$$\phi(z_1)\ \phi(z_2)\ =\ \langle 12\rangle + (12)\,V(z_1,\,z_2) + :\!\phi(z_1)\ \phi(z_2)\!: \tag{8.1}$$

where

$$(12)\ =\ z_{12}^{-2}\ =\ \frac{1}{z_1^2}\sum_{n\,=\,0}^{\infty}\left(\frac{z_2^2}{z_1^2}\right)^{\frac{n}{2}} C_n^1\,(\widehat{z}_1\cdot\widehat{z}_2)\qquad (\,\widehat{z}_i\ =\ \frac{z_i}{\sqrt{z_i^2}}\,)\,,$$

$$\langle 12\rangle\ =\ \frac{c}{2}\,(12)^2\,. \tag{8.2}$$

(Eqs. (8.1) (8.2) are obtained from (7.1) by setting $V_0 = \frac{c}{2}$, $V_1 \equiv V$, V_2 $(z_1,\,z_2) =\, :\phi(z_1)\,\phi(z_2):.$) The expansion of (12) in terms of Gegenbauer polynomials (see (3.16)–(3.18)) is designed to indicate that the z-picture rule for defining (12) as a distribution consists in viewing it as a limit of an analytic function defined in the domain $|\,z_1\,|\ >\ |\,z_2\,|$.

Theorem 8.1

(1) *The bilocal field* $V\,(z_1,\,z_2)$ *with correlation function*

$$\langle 0|\, V\,(z_1,\,z_2)\; V\,(z_3,\,z_4)\,|0\rangle \;=\; c\,\Big((13)\,(24) + (14)\,(23)\Big) \tag{8.3}$$

(*derived from* (4.11) *and* (4.12)) *is harmonic in each argument and admits an expansion in homogeneous harmonic polynomials*

$$V\,(z,\,w) \;=\; \sum_{n,\,m\,\in\,\mathbb{Z}} V_{nm}\,(z,\,w)\ ,$$

$$\Delta_z\, V_{nm}\,(z,\,w) \;=\; 0 \;=\; \Delta_w\, V_{nm}\,(z,\,w)\ , \tag{8.4}$$

$$\left(z\cdot\frac{\partial}{\partial z} + n + 1\right) V_{nm}\,(z,\,w) \;=\; 0 \;=\; \left(w\cdot\frac{\partial}{\partial w} + m + 1\right) V_{nm}\,(z,\,w)\ . \tag{8.5}$$

For positive n (or m) V_{nm} *is, in fact, a polynomial in* $\frac{z}{z^2}$ (*resp.* $\frac{w}{w^2}$). *Eqs.* (8.4) *and* (8.5) *then imply the vanishing of* V_{0m} *and* V_{n0} *in the vacuum representation.*

(2) *The modes* V_{nm} *obey the CR*

$$\begin{aligned}
&[\,V_{n_1 n_2}\,(z_1,\,z_2)\,,V_{n_3 n_4}\,(z_3,\,z_4)\,] \;=\\
&=\; c\,\prod_{j\,=\,1}^{4}\,\left(z_j^2\right)^{-\,\frac{n_j+1}{2}}\;\Big\{C^1_{|n_1|-1}\,(\widehat{z}_1\cdot\widehat{z}_3)\;C^1_{|n_2|-1}\,(\widehat{z}_2\cdot\widehat{z}_4)\;\delta_{n_1,-n_3}\,\delta_{n_2,-n_4} \,+\\
&+\; C^1_{|n_1|-1}\,(\widehat{z}_1\cdot\widehat{z}_4)\;C^1_{|n_2|-1}\,(\widehat{z}_2\cdot\widehat{z}_3)\;\delta_{n_1,-n_4}\,\delta_{n_2,-n_3}\Big\}\,\epsilon\,(n_1)\;\epsilon\,(n_2)\;+\\
&+\;\left(z_1^2\right)^{-\,\frac{n_1+1}{2}}\,\left(z_3^2\right)^{-\,\frac{n_3+1}{2}}\,C^1_{|n_1|-1}\,(\widehat{z}_1\cdot\widehat{z}_3)\,\epsilon\,(n_1)\;\delta_{n_1,-n_3}\,V_{n_2 n_4}\,(z_2,\,z_4)\;+\\
&+\;\left(z_2^2\right)^{-\,\frac{n_2+1}{2}}\,\left(z_3^2\right)^{-\,\frac{n_3+1}{2}}\,C^1_{|n_2|-1}\,(\widehat{z}_2\cdot\widehat{z}_3)\,\epsilon\,(n_2)\;\delta_{n_2,-n_3}\,V_{n_1 n_4}\,(z_1,\,z_4)\;+\\
&+\;\left(z_1^2\right)^{-\,\frac{n_1+1}{2}}\,\left(z_4^2\right)^{-\,\frac{n_4+1}{2}}\,C^1_{|n_1|-1}\,(\widehat{z}_1\cdot\widehat{z}_4)\,\epsilon\,(n_1)\;\delta_{n_1,-n_4}\,V_{n_2 n_3}\,(z_2,\,z_3)\;+\\
&+\;\left(z_2^2\right)^{-\,\frac{n_2+1}{2}}\,\left(z_4^2\right)^{-\,\frac{n_4+1}{2}}\,C^1_{|n_2|-1}\,(\widehat{z}_2\cdot\widehat{z}_4)\,\epsilon\,(n_2)\;\delta_{n_2,-n_4}\,V_{n_1 n_3}\,(z_1,\,z_3)
\end{aligned} \tag{8.6}$$

Sketch of proof. (1) Eq. (8.3) and the Wightman axioms imply harmonicity of V (cf. Proposition 2.1 of [37]). The properties of the modes of V follow from the argument that establishes Proposition 2.2 of [37]. The *vacuum* *module* $\mathcal{H}_V$ of the mode algebra is an inner product space with a unique vacuum state given by the 1-dimensional projection operator $|0\rangle\,\langle 0|$ such that

$$\begin{aligned} V_{nm}\,(z,\,w)\,|0\rangle \;&=\; 0 \quad \text{if} \quad n \geq 0 \quad \text{or} \quad m \geq 0\,, \\ \langle 0|\,V_{nm} \;&=\; 0 \quad \text{unless} \quad n > 0 \quad \text{and} \quad m > 0 \end{aligned} \tag{8.7}$$

and $|0\rangle$ is a cyclic vector for the modes V_{nm}. The positivity of the inner product (i.e., Wightman positivity) then implies

$$V_{0m}\,\mathcal{H}_V \;=\; 0 \;=\; V_{n0}\,\mathcal{H}_V\,. \tag{8.8}$$

(2) The proof of (8.6) requires using the fact that the truncated 5– and 6–point functions of ϕ are also given by (crossing symmetric) sums of 1–loop graphs (see Proposition 2.3 of [37]). $\Box$

The hermiticity condition for ϕ implies the presence of a conjugation in the mode algebra such that if e_1 and e_2 are two real unit vectors then

$$V_{nm}\,(e_1,\,e_2)^* \;=\; V_{-m\,-n}\,(e_1,\,e_2) \qquad (\,\text{for} \quad e_{1,\,2} \,\in\, \mathbb{S}^3\,)\,. \tag{8.9}$$

The resulting real (with respect to the involution so defined) Lie algebra $\mathfrak{L}_V$ is a central extension of the infinite dimensional real symplectic algebra $sp\,(\infty,\,\mathbb{R})$. To each unit vector $e \in \mathbb{S}^3$ corresponds a subalgebra $\mathfrak{L}_V^e$ of $\mathfrak{L}_V$ of the same type that is much simpler to realize. It is generated by

$$v_{nm} \;:=\; V_{nm}\,(e,\,e) \;\in\; \mathfrak{L}_V^e \subset \mathfrak{L}_V\;, \quad n,\,m \,\in\, \mathbb{Z}\;, \quad e^2 \;=\; 1 \tag{8.10}$$

satisfying

$$\begin{aligned} [\,v_{n_1m_1}, v_{n_2m_2}\,] \;=\;\; & c\,n_1\,m_1\,(\delta_{n_1,\,-n_2}\,\delta_{m_1,\,-m_2} + \delta_{n_1,\,-m_2}\,\delta_{m_1,\,-n_2}) \;+ \\ & + n_1\,(\delta_{n_1,\,-n_2}\,v_{m_1m_2} + \delta_{n_1,\,-m_2}\,v_{m_1n_2}) \;+ \\ & + m_1\,(\delta_{m_1,\,-n_2}\,v_{n_1m_2} + \delta_{m_1,\,-m_2}\,v_{n_1n_2})\;\;. \end{aligned} \tag{8.11}$$

It is easy to verify that for integer $c = N$ this algebra is generated by $O\,(N)$ invariant normal products

$$v_{lm}^{(c)} = :\vec{J}_l \cdot \vec{J}_m: \equiv \sum_{i=1}^{c} :J_l^i\, J_m^i: \tag{8.12}$$

of N commuting $U\,(1)$ currents $\vec{J}_n = \left\{ J_n^i,\ i = 1,\ ...,\ N \right\}$ where the current modes satisfy the Heisenberg type CR

$$\left[\, J_m^i\,,\, J_n^j \,\right] \,=\, m\, \delta_{m,\,-n}\, \delta_{ij}\ , \quad m,\, n\, \in\, \mathbb{Z}\ , \quad i,\, j\, =\, 1,\, ...,\, N\ . \tag{8.13}$$

Remark 8.1 The finite dimensional subalgebra of $\mathfrak{L}_V^e$ generated by v_{nm} for n, m of the same sign ($nm > 0$) and $|n| \leq K$, $|m| \leq K$ equipped with the above involution is just $sp\,(2\,K,\,\mathbb{R})$. Indeed the commutator

$$[\, v_{kl}\,,\, v_{-n\,-m}\,] \,=\, k\,(s_{lm}\,\delta_{kn} + s_{ln}\,\delta_{km}) + l\,(s_{kn}\,\delta_{lm} + s_{km}\,\delta_{ln}) \tag{8.14}$$

where k, l, m, n are all positive is expressed in terms of the symmetric products

$$s_{ln} \,=\, \frac{1}{2}\, \left(\vec{J}_l \cdot \vec{J}_{-n} + \vec{J}_{-n} \cdot \vec{J}_l \right) \quad (\, l\, >\, 0\,, \quad n\, >\, 0\,)\,. \tag{8.15}$$

One has to pass to normal products and hence to a central extension in the infinite dimensional case only. Indeed then the series defining the modes of the original field ϕ,

$$2\, \phi_n\,(z) \,=\, \sum_{\nu\,\in\,\mathbb{Z}} V_{\nu,\,n-\nu}\,(z,\, z) \quad (\, V_{mn}\,(z,\, z) \,=\, V_{nm}\,(z,\, z)\,) \tag{8.16}$$

would not have made sense (for $n = 0$) had we used the symmetric products instead of the normal ones.

We shall now demonstrate that unitarity of the vacuum representation of $\mathfrak{L}_V^e$ actually implies the realization (8.12). Moreover, the result extends to the full algebra $\mathfrak{L}_V$.

Theorem 8.2 *The inner product in the vacuum space $\mathcal{H}_V$ is positive semidefinite iff $c \in \mathbb{Z}_+ = \{0, 1, 2, ...\}$.*

Proof. For each positive integer n there is a vector $|\Delta_n\rangle$ whose norm square is a multiple of $c\,(c-1)\,...\,(c-n+1)$:

$$\langle\Delta_n| \;=\; \frac{1}{n!}\;\langle 0| \begin{vmatrix} v_{11} & v_{12} & \dots & v_{1n} \\ v_{21} & v_{22} & \dots & v_{2n} \\ \dots & \dots & \dots & \dots \\ v_{n1} & v_{n2} & \dots & v_{nn} \end{vmatrix} \;,$$

$$\langle\Delta_n|\,\Delta_n\rangle \;\equiv\; \|\,|\,\Delta_n\rangle\|^2 \;=\; (n+1)!\;c\,(c-1)\;\dots\;(c-n+1)\;\;. \qquad (8.17)$$

Indeed, it follows from (8.11) that the norm square of $|\Delta_n\rangle$ is a polynomial of degree n in c. We shall demonstrate that it vanishes for $c = 0,\,1,\,...,\,n{-}1$. To this end we insert for integer $c = N$ the normal products (8.12) for v_{kl} in the definition of $\langle\Delta_n|$ (8.17). We observe that the result is expressed by the Gram determinant of n vectors in an N dimensional space which vanishes for $N < n$. The coefficient $(n+1)!$ to the leading (nth) power of c is computed as a sum of norm squares of terms entering the expansion of the determinant.

It follows that if c is not a natural number then the representation of $\mathfrak{L}_V$ cannot be unitary. To see that it is unitary for positive integer $c = N$ it is sufficient to note that in this case V can be presented in the form

$$V\,(z_1,\,z_2) \;=\; \sum_{i\,=\,1}^{c}\;:\varphi_i\,(z_1)\,\varphi_i\,(z_2): \qquad (8.18)$$

with φ_i mutually commuting free zero mass fields and to recall that a system of free fields satisfies all Wightman axioms, including positivity. $\square$

It follows as a corollary that if a GCI field of dimension 2 satisfies all Wightman axioms then it belongs to the Borchers class of free fields.

9 Outlook and discussion

The present notes being somewhat inhomogeneous, we shall look back, in conclusion, at what we had and will try to distribute some accents.

These lectures were not aimed at introducing the student to current fads in CFT. They were intended, instead, to

1. provide a general view on the subject from a historical perspective (Sec. 1) including a survey of compactified Minkowski space (Sec. 2), the playground for global conformal transformations;
2. remind the (Wightman) axiomatic approach to QFT (Sec. 3) that provides the natural framework for a theory with an enhanced symmetry, such as GCI;
3. survey some memorable results of 2D CFT (Secs. 5, 6);
4. describe a current attempt [37] to construct 4D CFT models with rational correlation functions (Secs. 7, 8).

We tried to present these differently looking topics from a unified point of view. Thus Sec. 4, which has a central part in our approach, serves both as a unorthodox introduction to 2D CFT and as a starting point in our search of soluble 4D CFT models in the last two sections.

The existence of a 4D CFT model with rational correlation functions outside the class of normal products of free massless fields still remains an open problem. We should like to end up our discussion with the remark that the (positive or negative) answer to the existence question appears to be within reach. Indeed, if such a model does exist its algebra of (gauge invariant) local fields should include apart from the stress energy tensor T also a Lagrangean density: a GCI scalar field $\mathcal{L}(x)$ of dimension 4. Using Corollary 4.5 we can write its truncated 4-point function in the form:

$$
\begin{aligned}
\mathcal{W}_4^t(4) \;&=\; \frac{[(12)\,(34)\,(34)\,(14)]^2}{\eta_1\,\eta_2} \times \\
&\times \Big\{ c_0\,\big(1+\eta_1^5+\eta_2^5\big) + c_1\,\big(\eta_1+\eta_2+\eta_1^4+\eta_2^4+\eta_1\,\eta_2\,\big(\eta_1^3+\eta_2^3\big)\big) + \\
&+ c_2\,\big(\eta_1^2+\eta_2^2+\eta_1^3+\eta_2^3+\eta_1^2\,\eta_2^2\,(\eta_1+\eta_2)\big) + \\
&+ b_1\,\eta_1\,\eta_2\,\big(1+\eta_1^2+\eta_2^2\big) + b_2\,\eta_1\,\eta_2\,(\eta_1\,\eta_2+\eta_1+\eta_2)\Big\} \\
&(\,c_i \,\equiv\, c_{0i} \quad \text{for} \quad i \,=\, 0,\,1,\,2\,; \quad b_i \,\equiv\, c_{1i} \quad \text{for} \quad i \,=\, 1,\,2\,)\;. \qquad (9.1)
\end{aligned}
$$

Demanding that no field of dimension 2 appears in the OPE of $\mathcal{L}\,(x_1)\,\mathcal{L}\,(x_2)$ (and that this OPE contains T) we find the constraint

$$c_2 \;=\; -c_0 \;-\; c_1\,(\,\neq\; 2\,c_0\,)\,. \qquad (9.2)$$

The problem is whether Wightman axioms (including positivity) would allow a more general solution within the resulting 4–parameter family of truncated 4-point functions (9.1), (9.2) than the 1-parameter subset

$$c_0 \;=\; c_2 \;=\; b_1 \;=\; -c_1/2\,, \quad b_2 \;=\; 0\,. \qquad (9.3)$$

If the parameters are restricted by (9.3) then one can prove by the method of Sec. 8 that the remaining parameter ($c_0 = c_2 = \ldots$) should be a positive integer multiple of a fixed (positive) number, recovering a sum of normal products of free Maxwell fields:

$$\mathcal{L}_F\,(x) \;=\; -\,\frac{1}{4}\,\sum_{a\,=\,1}^{N_F}\; :F_{\mu\nu}^{\;a}\,(x)\;F_a^{\mu\nu}\,(x): \quad (\,N_F\,\in\,\mathbb{N}\,)\,. \qquad (9.4)$$

The truncated n-point function of $\mathcal{L}\,(x)$ for even n can again be given as a sum of $\frac{(n-1)!}{2}$ 1–loop graphs, the propagator associated with the line joining the vertices 1 and 2 being

$$
\begin{aligned}
&\mathcal{D}_{\lambda_1\mu_1\lambda_2\mu_2}(x_{12}) = \\
&= \frac{1}{4}\, \Big\{ \partial_{\lambda_1} \left(\partial_{\lambda_2}\, \eta_{\mu_1\mu_2} - \partial_{\mu_2}\, \eta_{\mu_1\lambda_2} \right) \,-\, \partial_{\mu_1} \left(\partial_{\lambda_2}\, \eta_{\lambda_1\mu_2} - \partial_{\mu_2}\, \eta_{\lambda_1\lambda_2} \right) \Big\}\, \frac{1}{4\,\pi^2 \rho_{12}} \\
&= \frac{r_{\lambda_1\lambda_2}(x_{12})\; r_{\mu_1\mu_2}(x_{12}) \,-\, r_{\lambda_1\mu_2}(x_{12})\; r_{\mu_1\lambda_2}(x_{12})}{4\,\pi^2\, \rho_{12}^2}\ . \qquad (9.5)
\end{aligned}
$$

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