"Bemerkungen zur Diracschen Theorie des Positrons," Zeit. Phys. **90** (1934), 209-231.

Remarks on the Dirac theory of positron

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The purpose of the present paper [1]) is to construct the Dirac theory of the positron [2]) in the formalism of quantum electrodynamics. Thus, we shall demand that the symmetry in nature between positive and negative charge should be expressed in the basic equations from the outset, and that in addition to the well-known difficulties with the divergences that quantum electrodynamics leads to, no new infinities should appear in the formalism, moreover; i.e., that the theory should provide an approximation for the treatment of the circle of problems that have been treated by quantum electrodynamics up to now. By the latter postulate, one distinguishes the present effort from the investigations of Fock [3]), Oppenheimer and Furry [4]), and Peierls [5]), the last of which is similar to it; he is closely linked with the paper of Dirac [6]), moreover. In contrast to the Dirac treatment, one has the work on the meaning of the conservation law for the total system of radiation-matter and the necessity of formulating the basic equations of the theory in a way that grows out of the Hartree approximation.

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[1]) The paper originated in some discussions that I had with Herren Pauli, Dirac, and Weisskopf, in part written and in part oral, and to them I am deeply grateful.

[2]) E. g.: P. A. M. Dirac, *The Principles of Quantum Mechanics*," Oxford (1930), pp. 255.

[3]) V. Fock. C. R. Leningrad (N. S.) no. 6 (1933), 267-271.

[4]) W. H. Furry and J. R. Oppenheimer, Phys. Rev. **45** (1934), 245.

[5]) R. Peierls, *to appear*.

[6]) P. A. M. Dirac, Proc. Camb. Phil. Soc. **30** (1934), 150 (in what follows, this is always referred to by *loc. cit.*).

I. Intuitive theory of matter waves.

1. The inhomogeneous differential equation for the density. Let the most important result of the aforementioned Dirac paper be briefly summarized as follows: A quantum-mechanical system of many electrons that fulfill the Pauli principle and move in a given force field without back-reaction can be characterized by a "density matrix:"

$$(x', t', k' \mid R \mid x'', t'', k'') = \sum_n \psi_n^*(x', t', k')\,\psi_n(x'', t'', k''), \tag{1}$$

when $\psi_n(x', t', k')$ means the normalized eigenfunctions of the states that possess one electron, and x', t', k' (x'', t'', k'', resp.) are position, time, and spin variables. All physically-important properties of quantum-mechanical systems like charge density, current density, etc., can be read off from the density matrix. In general, this is always true in the approximation in which the interaction of the electrons can be ignored; i.e., in which the typical quantum-mechanically intuitive course of events does not enter. The density matrix thus mediates an intuitive, corresponding picture of the actual process that is similar to what the classical-mechanical atomic model does. The demand that the ψ_n in (1), which, according to Dirac, can also be expressed in the form ($t' = t$):

$$R^2 = R, \tag{2}$$

should be normalized can be posed in parallel to the quantum conditions of the previous semi-classical theory.

The temporal change in the density matrix will be determined by the Dirac differential equation:

$$\mathcal{H}R = \left[i\hbar \frac{\partial}{c\,\partial t'} + \frac{e}{c} A_0(x') + \alpha_s \left(i\hbar \frac{\partial}{\partial x_s'} - \frac{e}{c} A_s(x') \right) + \beta mc \right] R = 0. \tag{3}$$

From now on, the following notations shall be applied throughout:

$$\left.\begin{array}{lllll}
\text{Coordinates:} & ct' = x_0' = -x^{0'}, & x_i' = x^{i'}, & x_\lambda' - x_\lambda, \ \dfrac{x_\lambda' + x_\lambda''}{2} = \xi_\lambda, & \\
\text{Potentials:} & A_0 = -A^0, & A_i = A^i, & & \\
\text{Field strengths:} & \dfrac{\partial A^\mu}{\partial \xi_\nu} - \dfrac{\partial A^\nu}{\partial \xi_\mu} = F^{\nu\mu}, & F^{0s} = -F_{0s}, & & \\
 & (F^{01}, F^{02}, F^{03}) = \mathfrak{E}, & (F^{23}, F^{31}, F^{12}) = \mathfrak{H}, & & \\
\text{Spin matrices:} & \alpha^0 = 1, & \alpha_0 = -1, & \alpha^i = \alpha_i. &
\end{array}\right\} \tag{4}$$

Greek indices always run from 0 to 3 and Latin ones from 1 to 3. The raising or lowering of indices shall result from the usual formulas of the theory of relativity. Doubled indices shall always be summed over. Since the α^ν do not transform simply like a vector, the

chosen notation only amounts to a convenient abbreviation for these quantities. Equation (3) now assumes, e.g., the form:

$$\left[\alpha^{\lambda} \left(i\hbar \frac{\partial}{\partial x'_s} - \frac{e}{c} A_s(x') \right) + \beta mc \right] R = 0.$$

If, as the Dirac theory of holes requires, all states of negative energy are occupied, except for finitely many of them, and also only finitely many positive energy states are occupied then the matrix R will be singular on the light-cone that is defined by:

$$x_\rho x^\rho = 0. \tag{5}$$

Following Dirac, one then suitably considers the new matrix [1]):

$$R_S = R - \tfrac{1}{2} R_F\,, \tag{6}$$

in place of the matrix R, in which R_F refers to the value of R for the state of the system in which every electron level is occupied. As one easily confirms, for $t' = t$, R_F goes to the Dirac δ-function of the variables x', k', x'', k''. The matrix R_S already has a symmetry relative to the sign of the charge that will be important in the formalism that follows: Under the addition of $\frac{1}{2} R_F$, it goes to the corresponding matrix R of "hole" theory. Under subtraction of $\frac{1}{2} R_F$, it goes to the negative density matrix of a distribution in which the states of positive energy are occupied and the positive energy states are free. Permuting the points x', t', k' and x'', t'', k'' and switching the sign of R_S are equivalent to a change of sign in the electric charge. The singularity of the matrix R_S on the light-cone was investigated by Dirac; one can represent the matrix in the form:

$$(x', k' \mid R_S \mid x'', k'') = u \frac{\alpha_\rho x_\rho}{(x^\lambda x_\lambda)^2} - \frac{v}{x^\lambda x_\lambda} + w \log \mid x^\lambda x_\lambda \mid, \tag{7}$$

in which:

$$u = -\frac{i}{2\pi^2} e^{-\frac{ei}{c\hbar}\int_{P'}^{P''} A^\lambda dx_\lambda}. \tag{8}$$

(The integral is to be taken along the straight line from P' to P''.)

The quantity w is determined uniquely by a differential equation, but v is determined only up to an additive term of the form $x^\lambda x_\lambda \cdot g$. One ordinarily deduces the charge density, current density, etc., from the density matrix R when one makes the Ansatz, e.g., for the charge density:

$$\rho(x) = e \sum_k (x, k \mid R \mid x, k)\,; \tag{9}$$

[1]) The doubled matrix R_S is the matrix that Dirac denoted by R_1.

corresponding statements are true for the other physical quantities. Now, due to the singularity of the matrix R, this conclusion is obviously incorrect – e.g., when no external field is present – since only the deviation of the density matrix from the matrix of the state in which all of the negative energy levels are filled contributes any charge and current density. From Dirac, one would then have to subtract from the density matrix, another density matrix that is determined uniquely by the external field in order to obtain the "true" density matrix – we call it $(x', k' \mid r \mid x'', k'')$ – that is definitive for the charge and current densities, energy densities, etc., corresponding to equation (9). We set:

$$r = R_S - S, \tag{10}$$

in which S shall be a function of x'_λ, k' and x''_λ, k'' that is uniquely determined by the potentials A^λ.

In place of the differential equation (3), one now has the equation:

$$\mathcal{H}r = -\mathcal{H}S. \tag{11}$$

The right-hand side is a function of the electromagnetic field that must be determined more precisely; the original homogeneous Dirac equation (3) will then be replaced with the inhomogeneous equation (11). Such an equation is the natural expression for the fact that matter can be created and destroyed. The type of creation and annihilation will be established by the form of the quantity HS. If no other external fields are present then S shall be given by the value of R_S for the distribution in which all negative energy states are occupied. We then assume that the matrix r vanishes everywhere in field-free space. The set of all matter that is collectively created when an external field is imposed and the again removed can be ascertained without any closer approximation on S by the presence of external fields. Then, when R_S (and therefore r) is known before the imposition of any sort of field the value of R_S can be ascertained from equation (3) after the field is again removed. However, after the field is removed, S again has its original value, so r can also be calculated. Nonetheless, conversely, the result of the matter created by the imposition and removal of the fields gives the general reference point for the form of the right-hand side of (11) in the presence of fields. For example, a simple perturbative calculation shows that the total set of matter that is created by the imposition and removal of the field is, in general, already infinite when the temporal differential quotient of the electric or magnetic field strength is sometimes discontinuous in the process of imposing and removing it, and first becomes correct when the field strengths or potentials are themselves discontinuous. From this, one concludes that the right-hand side of (11), along with the potentials and field strengths, must also contain the first and second derivatives.

Dirac (*loc. cit.*) carried out the determination of S in the presence of external fields in such a way that he described a certain mathematical process that gave the matrix R_S from the sequence of singular parts; Dirac identified the sum of the singular parts thus obtained by S. However, the mathematical process that was chosen by Dirac did not deliver the aforementioned value of S in the field-free case, but one that differed from it by a matrix that was regular on the light-cone. Whether or not a unique determination of the inhomogeneity in (11) is therefore hardly possible using formal arguments, by

considering the conservation laws for charge, energy, and impulse, one can restrict the possibilities for S in such a way that a definite value can be distinguished as a first hypothesis. We denote the value of S that is valid in the absence of external forces and potentials (cf., above) and the one that is calculated by Dirac, *loc. cit.*, by equations (20) to (22) by S_0 . If no fields are indeed present, but potentials whose rotations vanish enter into equation (3), then S_0 is to be replaced with:

$$e^{-\frac{ei}{\hbar c}\int_{P'}^{P''} A^\lambda dx_\lambda} \cdot S_0 .$$

The quantity S then becomes the most important term that possesses the highest singularity on the light-cone and includes these quantities, where the integral should again be taken along the straight line from P' to P''. We set:

$$S = e^{-\frac{ei}{\hbar c}\int_{P'}^{P''} A^\lambda dx_\lambda} \cdot S_0 + S_1 . \tag{12}$$

If one develops S_1 for small x_λ then, from (7), it must be capable of being represented in the form:

$$S_1 = \frac{a}{x_\lambda x^\lambda} + b \log \left| \frac{x_\lambda x^\lambda}{C} \right| . \tag{13}$$

The tail end of the density matrix is important only for the calculation of the charge, current, and energy densities, so (cf., 2) for the development of the quantity a in x_λ it suffices to know only the terms up to third order in x_λ inclusively and for b, only the terms up to first order in x_λ . Furthermore, on the same grounds, it suffices to calculate only the terms that include the α^λ only linearly. Expressions for a and b that are compatible with equation (7) and Dirac's results on the singularities of the density matrix read (up to higher-order terms):

$$\left.\begin{aligned} a &= u\left\{ \frac{ei}{24\hbar c} x_\rho x^\sigma \alpha^\lambda \left(\frac{\partial F_{\lambda\sigma}}{\partial \xi_\rho} - \delta^\rho_\lambda \frac{\partial F_{\tau\sigma}}{\partial \xi_\tau} \right) - \frac{e^2}{48 c^2 \hbar^2} x_\rho x_\sigma x^\tau \alpha^\rho F^{\mu\sigma} F_{\mu\tau} \right\}, \\ b &= u\left\{ \frac{ei}{24\hbar c} \alpha^\lambda \frac{\partial F_{\tau\lambda}}{\partial \xi_\tau} + \frac{e^2}{48\hbar^2 c^2} x_\lambda \alpha^\mu \left(F_{\tau\mu} F^{\tau\lambda} - \frac{1}{4} \delta^\lambda_\mu F_{\tau\sigma} F^{\tau\sigma} \right) \right\}. \end{aligned} \right\} \tag{14}$$

Here, the field strengths are always to be taken at the position $\frac{x_l' + i x_l''}{2} = \xi_l$. The quantity u is given by equation (8).

If one defines S by means of equations (12) to (14) then the difference $R_S - S$ can become singular on the light cone by way of terms of type $\frac{x_\lambda x_\mu x_\nu x_\pi}{x_\rho x^\rho} \cdot A^{\lambda\mu\nu\pi}$, $x_\lambda x_\mu A^{\lambda\mu}$ log

$| x_\rho x^\rho |$, or $\frac{(\alpha^\lambda \alpha^\mu - \alpha^\mu \alpha^\lambda) A_{\mu\lambda}}{x_\rho x^\rho}$. In these cases, one can, however, deduce the current and charge density, and the energy and impulse density from the density matrix when one goes to the limit $x_\lambda \to 0$, not on the light-cone, but in spacelike or timelike directions. The aforementioned singular terms then contribute nothing. (The terms that are not linear in α_λ already drop out under the passage to the limit.)

The matrices R, R_S , S, and S_0 are all Hermitian; i.e., under an exchange of x', k' with x'', k'' (and thus, under a change of sign for x) they go to the conjugate values.

The calculation of formulas (14) results most simply from the process that was given by Dirac (*loc. cit.*). The mathematical form of the expression (14) shows that the arbitrariness that comes about due to the choice of quantities a and C, if one would like to allow no essential complications in the expression for (14), actually only consists in the fact that an expression of the form $x_\rho x^\rho \alpha^\lambda \frac{\partial F_{\lambda\sigma}}{\partial \xi_\sigma}$ can be added to a, along with another one of the form $x_\rho x^\rho x_\lambda \alpha^\lambda F^{\tau\sigma} F_{\tau\sigma}$, without changing the singularities of the matrix S; C is, moreover, completely arbitrary. For the charge and current densities that follow from the density matrix, the two indeterminacies (in a and C) give rise to an additive charge and current density in the same way. One can therefore establish the first term in a in the manner that was given by (14), and all of the indeterminacy in the charge density is pushed onto C. The second term in a is then, as was shown in 2, determined by the conservation law that was given in equation (14). The arbitrariness in the choice of the constant C is ultimately uninteresting due to the fact that, from Dirac, the equation $\mathcal{H} w = 0$ is true for the matrix w that is defined in (7); i.e., the quantity C drops out of the right-hand side of (11) (up to terms that include α^λ or x_λ quadratically). This is therefore only true when the electromagnetic field, along with all of its derivatives, is continuous and the matrix w can be developed in x_λ and ξ_λ . If one makes these assumptions then one accepts the disadvantage that one cannot simply connect the theory with the special of field-free space (e.g., by perturbative calculations). If one allows discontinuous changes in the higher differential quotients of the fields or other singularities then the equation $\mathcal{H} w = 0$ is no longer true at the singular places in question, and the choice of the quantity C becomes important. In this case, the suitable choice of the quantity C will be found by the following argument: One thinks of a field that is generated by a given external charge density as emerging adiabatically from a "null" field. A matter field will arise from this process that is given by the matrix r. As equations (13) and (14) teach us, according to the choice of C, this matter field will completely or partially compensate for the external charge density or increase it. We will now choose C such that the total charge of the matter field that is given by r vanishes for the process considered. If this were not the case then under the "imposition" the external charge density could not, in fact, be separated from the existing electron density; i.e., one would already have to define the "external" charge density to be the sum of the two densities. We will come back to the mathematical treatment of this question in 3. There, we will also make good the calculation of C, which indeed, from the statements above, has more mathematical than physical meaning. Here, only its value shall be given:

$$C = 4\left(\frac{\hbar}{mc}\right)^2 e^{-2/3-2\gamma}, \tag{15}$$

where γ refers to the Euler constant: $\gamma = 0.577\ldots$

With that, the determination of the inhomogeneity in the differential equation (11) is completed. In regard to the current that follows from the density matrix *r*, our assumptions are equivalent to those of Dirac (*loc. cit.*). On the other hand, as Herr Dirac cordially communicated to us, the choice of matrix *S* that was made here delivers a different energy and impulse density than Dirac's choice does.

2. The conservation laws. The charge and current density follows from the density matrix *r* in the usual way, and from an investigation by Tetrode [1]), the energy and impulse tensor of the matter waves can be derived from the following equations:

$$\left.\begin{aligned} s_\lambda(\xi) &= e\sum_{k',k''}\alpha^\lambda_{k'k''}(\xi,k'\,|\,r\,|\,\xi,k''),\\ U^\mu_\nu(\xi) &= \lim_{x\to 0}\left\{ic\hbar\frac{\partial}{\partial x_\mu} - \frac{e}{2}\left[A^\mu\left(\xi+\frac{x}{2}\right)+A^\mu\left(\xi-\frac{x}{2}\right)\right]\right\}\\ &\quad \sum_{k',k''}\alpha^\lambda_{k'k''}\left(\xi+\frac{x}{2},k'\,|\,r\,|\,\xi-\frac{x}{2},k''\right). \end{aligned}\right\} \tag{16}$$

In order to show that the conservations laws for the quantities thus defined have the usual form, we shall first prove the following equation:

$$\sum_{k',k''}\alpha^\lambda_{k',k''}\left[i\hbar\frac{\partial}{\partial\xi^\lambda} - \frac{e}{c}A^\lambda\left(\xi+\frac{x}{2}\right)+\frac{e}{c}A^\lambda\left(\xi-\frac{x}{2}\right)\right]\left(\xi+\frac{x}{2},k'\,|\,r\,|\,\xi-\frac{x}{2},k''\right) = 0, \tag{17}$$

up to terms that are at least quadratic in the x_λ . Equation (17) is equivalent to the assertion that:

$$\sum_{k',k''}\alpha^\lambda_{k',k''}\left[i\hbar\frac{\partial}{\partial\xi^\lambda} - \frac{e}{c}A^\lambda\left(\xi+\frac{x}{2}\right)+\frac{e}{c}A^\lambda\left(\xi-\frac{x}{2}\right)\right]\left(\xi+\frac{x}{2},k'\,|\,S\,|\,\xi-\frac{x}{2},k''\right) = 0, \tag{18}$$

up to quadratic terms in x_λ . The equation $\mathcal{H}R_S = 0$ is then indeed true for the matrix R_S , and certainly so is equation (17). Now, one has:

[1]) H. Tetrode, Zeit. Phys. **49** (1928), 858.

$$\left[i\hbar \frac{\partial}{\partial \xi^{\lambda}} - \frac{e}{c} A^{\lambda}\left(\xi + \frac{x}{2}\right) + \frac{e}{c} A^{\lambda}\left(\xi - \frac{x}{2}\right)\right] e^{-\frac{ei}{\hbar c}\int_{P'}^{P''} A^{\lambda} dx_{\lambda}} = e^{-\frac{ei}{\hbar c}\int_{P'}^{P''} A^{\lambda} dx_{\lambda}} \cdot \frac{e}{c} \int_{P'}^{P''} F^{\lambda\mu} dx_{\mu} \tag{19}$$

and $\frac{\partial}{\partial \xi_{\lambda}} S_0 = 0$. If one then observes that S_0 can be written in the form:

$$\alpha^{\lambda} x_{\lambda} f(x^{\rho} x_{\rho}) + \beta mcg(x^{\rho} x_{\rho})$$

then it follows that equation (18) is true in any event for the first part of S, namely:

$$e^{-\frac{ei}{\hbar c}\int_{P'}^{P''} A^{\lambda} dx_{\lambda}} \cdot S_0 .$$

Thus, we still have to show that equation (18) is true for the S_1 part. Its validity for the $b \log \left| \frac{x_{\rho} x^{\rho}}{C} \right|$ part of S_1 is then self-explanatory, because, from Dirac, the matrix w satisfies $\mathcal{H}w = 0$ (cf., on this, pp. 6). Thus, it remains for us to discuss the $a / x_{\lambda} x^{\lambda}$ part. Calculation shows that, from (14), the terms that arise from the differentiation with respect to ξ_{λ} in the first part of a, due to equation (19), cancel those of the second part precisely. With that, the validity of equation (17) is proved.

The law of conservation of charge follows from equation (17), when one takes it to the limit $x_{\lambda} \to 0$:

$$\frac{\partial}{\partial \xi_{\lambda}} e \sum_{k',k''} \alpha^{\lambda}_{k',k''}(\xi, k' \mid r \mid \xi, k'') = \frac{\partial s_{\lambda}}{\partial \xi_{\lambda}} = 0. \tag{20}$$

From the remarks regarding equation (14), the passage to the limit $x_{\lambda} \to 0$ is to be carried out, not on the light-cone, but on either a spacelike or timelike direction.

For the conservation of energy and impulse, one finds in the same way:

$$\frac{\partial U^{\mu}_{\nu}(\xi)}{\partial \xi_{\nu}} =$$

$$\lim_{x\to 0}\left\{ ic\hbar \frac{\partial}{\partial x_{\mu}} - \frac{e}{2}\left[A^{\mu}\left(\xi + \frac{x}{2}\right) + A^{\mu}\left(\xi - \frac{x}{2}\right)\right]\right\} \sum_{k',k''} \alpha^{\nu}_{k',k''} \frac{\partial}{\partial \xi_{\nu}} \left(\xi + \frac{x}{2}, k' \mid r \mid \xi - \frac{x}{2}, k''\right)$$

$$- \lim_{x\to 0}\left\{ \frac{e}{2} \frac{\partial}{\partial \xi_{\nu}} \left[A^{\mu}\left(\xi + \frac{x}{2}\right) + A^{\mu}\left(\xi - \frac{x}{2}\right)\right]\right\} \sum_{k',k''} \alpha^{\nu}_{k',k''} \frac{\partial}{\partial \xi_{\nu}} \left(\xi + \frac{x}{2}, k' \mid r \mid \xi - \frac{x}{2}, k''\right),$$

and, from (17):

$$\left.\begin{aligned}
&\frac{\partial U_\nu^\mu(\xi)}{\partial \xi_\nu} = \\
&\lim_{x\to 0}\left\{ ic\hbar\frac{\partial}{\partial x_\mu} - \frac{e}{2}\left[A^\mu\left(\xi+\frac{x}{2}\right) + A^\mu\left(\xi-\frac{x}{2}\right)\right]\right\}\left[-\frac{e}{ic\hbar}A^\nu\left(\xi+\frac{x}{2}\right) + \frac{e}{ic\hbar}A^\nu\left(\xi-\frac{x}{2}\right)\right] \\
&\qquad \sum_{k',k''}\alpha^\nu_{k',k''}\frac{\partial}{\partial\xi_\nu}\left(\xi+\frac{x}{2},k'\,|\,r\,|\,\xi-\frac{x}{2},k''\right) \\
&-\lim_{x\to 0}\frac{e}{2}\frac{\partial}{\partial\xi_\nu}\left[A^\mu\left(\xi+\frac{x}{2}\right) + A^\mu\left(\xi-\frac{x}{2}\right)\right]\sum_{k',k''}\alpha^\nu_{k',k''}\left(\xi+\frac{x}{2},k'\,|\,r\,|\,\xi-\frac{x}{2},k''\right) \\
&= -eF^{\nu\mu}(\xi)\sum_{k',k''}\alpha^\nu_{k',k''}(\xi,k'\,|\,r\,|\,\xi,k'') = -F^{\nu\mu}s_\nu .
\end{aligned}\right\} \quad (21)$$

If one adds the energy-impulse tensor of the Maxwell field:

$$V_\nu^\mu = \frac{1}{4\pi}\left(-F^{\tau\mu}F_{\tau\nu} + \frac{1}{4}\delta_\nu^\mu F^{\tau\sigma}F_{\tau\sigma}\right) \quad (22)$$

to U_ν^μ and sets down the Maxwell equations in the form:

$$\frac{\partial F_{\tau\nu}}{\partial\xi_\nu} = -4\pi s_\tau \quad (23)$$

then the tensor:

$$T_\nu^\mu = U_\nu^\mu + V_\nu^\mu \quad (24)$$

obeys the relation:

$$\frac{\partial T_\nu^\mu}{\partial\xi_\nu} = 0. \quad (25)$$

From Tetrode (*loc. cit.*), the difference $U_{\mu\nu} - U_{\nu\mu}$ is a tensor whose divergence vanishes, moreover. One can also symmetrize the energy-impulse tensor of the matter field without disturbing the validity of (25).

One can also briefly summarize the results up to now in the following way: If one restricts oneself to an intuitive analogue theory of matter fields then the well-known difficulty with the appearance of negative energy levels in the Dirac theory can be avoided in such a way that one replaces the homogeneous Dirac differential equation (3) with an inhomogeneous equation, where the inhomogeneity is indicative of "pair creation." The usual conservation laws are valid for the matter field that satisfies this equation, as well as the Maxwell field, and at the same time the energies of the matter and the radiation field are always individually positive.

One can recognize the invariance of the theory under a change of sign of the elementary charge most simply in the following way: One replaces + *e* with – *e* in equations (11) and (16), as well as (x', k' | r | x'', k'') with – (x'', k'' | $\bar{r}$ | x', k'). The original equations (11) and (16) are then valid once more for the matrix *r*.

3. Applications. Two simple examples shall illustrate the application of the methods that were depicted in 1 and 2. We first assume that a scalar potential A_0, which is regarded as a small perturbation, is slowly introduced and then kept constant, and then ask what sort of matter is created by it from originally empty space; thus, the charge density that gives rise to the potential $A0$ shall be referred to as the "external charge density" [1]).

We next solve the Dirac differential equation for an electron whose state is represented by a plane wave before the imposition of the field. Its eigenfunction is called ψ_n and before the imposition of the field one has:

$$\psi_n(x') = u_n(x')\ e^{\frac{i}{\hbar} p_n^0 x_0'} . \tag{26}$$

We set:

$$\psi_n(x') = \sum_m c_{nm}(x_0')\, u_m(x')\ e^{\frac{i}{\hbar} p_m^0 x_0'} , \tag{27}$$

and from:

$$\left\{ \alpha^l \left[i\hbar \frac{\partial}{\partial x_\lambda'} - \frac{e}{c} A^\lambda(x') \right] + \beta mc \right\} \psi = 0$$

it follows in the usual way that:

$$\frac{d}{dx_0'} c_{nm} = \frac{i}{\hbar} H_{nm} e^{\frac{i}{\hbar}\left(p_n^0 - p_m^0\right) x_0'} , \tag{28}$$

where:

$$H_{nm} = \int u_m^*(x''') \frac{e}{c} \alpha^\lambda A^\lambda(x''') u_n(x''')\, dx''' . \tag{29}$$

Here, $\int dx'''$ means integration over the position variables and summation over the spin indices.

From (28), one deduces, when the H_{nm} are constant in time, that:

$$c_{nm} = H_{nm} \frac{e^{\frac{i}{\hbar}\left(p_n^0 - p_m^0\right) x_0} - \varepsilon_{nm}}{p_n^0 - p_m^0} + \delta_{nm} . \tag{30}$$

The constants ε_{nm} thus depend upon the type of temporal increase in H_{nm} ; we would like to assume that the increase happens so slowly and uniformly that the ε_{nm} vanish in a sufficient approximation. One then has:

$$c_{nm} = H_{nm} \frac{e^{\frac{i}{\hbar}\left(p_n^0 - p_m^0\right) x_0}}{p_n^0 - p_m^0} + \delta_{nm} .$$

and

[1]) This problem has essentially already been treated by Dirac in his report to the Solvay Congress in 1933.

$$\psi_n(x') = \left[\sum_m u_m(x') \frac{H_{nm}}{p_n^0 - p_m^0} + u_n(x') \right] e^{\frac{i}{\hbar} p_n^0 x_0'} . \tag{31}$$

Terms of order higher than the first in H_{nm} will always be neglected in what follows. From its definition, the matrix R_S satisfies:

$$(x', k' \mid R_S \mid x'', k'') = \frac{1}{2} \left[\sum_{n, p_n^0 > 0} \psi_n^*(x', k') \psi_n(x'', k'') - \sum_{n, p_n^0 < 0} \psi_n^*(x', k') \psi_n(x'', k'') \right]. \tag{32}$$

We can now divide the sum over all states into an integral over the impulse and a sum over four possible states for each impulse. The operator:

$$\frac{\alpha^l p^l + \beta mc}{| p^0 |},$$

in which the number l is to be summed from 1 to 3 (as always for Latin indices), has the property that it yields + 1 when it is applied to any state of positive energy and – 1 for a state of negative energy. With the help of this operator, the summations over the spin states may then be easily performed, and all that remains is the integral over the impulse. Thus, in the following, one will always set $t' = t''$; i.e., $x^{0\prime} = x^{0\prime\prime}$:

$(x', k' \mid R_S \mid x'', k'') =$

$$- \frac{1}{2} \int \frac{d\mathfrak{p}}{h^3} \frac{\alpha^l p^l + \beta mc}{| p^0 |} e^{\frac{i}{\hbar} p^\rho (x_\rho'' - x_\rho')} - \frac{1}{8} \int dx''' \int \frac{d\mathfrak{p}'}{h^3} \int \frac{d\mathfrak{p}''}{h^3} e^{\frac{i}{\hbar} [x_l'''(p'^\lambda - p^{l\prime\prime}) + p^{l\prime\prime} x_\rho'' - p'^\lambda x_\rho']}$$

$$\left\{ \left(1 + \frac{\alpha^l p''^l + \beta mc}{| p^{0\prime\prime} |} \right) \frac{\frac{e}{c} A^0(x''')}{| p^{0\prime} | + | p^{0\prime\prime} |} \left(1 - \frac{\alpha^l p'^l + \beta mc}{| p^{0\prime} |} \right) \right.$$

$$\left. \left(1 - \frac{\alpha^l p''^l + \beta mc}{| p^{0\prime\prime} |} \right) \frac{\frac{e}{c} A^0(x''')}{| p^{0\prime} | + | p^{0\prime\prime} |} \left(1 + \frac{\alpha^l p'^l + \beta mc}{| p^{0\prime} |} \right) \right\}$$

$$+ \text{conj.} \tag{33}$$

The first term in (33) represents the matrix S_0 and will be subtracted from R_S , from the definition of r. The next two terms go to:

$$-\frac{1}{4}\int dx'''\int\frac{d\mathfrak{p}'}{h^3}\int\frac{d\mathfrak{p}''}{h^3}e^{\frac{i}{\hbar}\left[x_l'''(p'^l-p''^l)+p''^l x_l''-p'^l x_l'\right]}\cdot\frac{e}{c}\frac{A^0(x''')}{|p^{0'}|+|p^{0''}|}\frac{|p^{0'}p^{0''}|-p_l'p_l''-m^2c^2}{|p^{0'}p^{0''}|}. \qquad (34)$$

For the evaluation of this expression, one now sets, as appropriate:

$$\mathfrak{p}'=\mathfrak{t}+\frac{\mathfrak{g}}{2},\qquad \mathfrak{p}''=\mathfrak{t}-\frac{\mathfrak{g}}{2},\qquad \mathfrak{r}'-\mathfrak{r}''=\mathfrak{r},\qquad \frac{\mathfrak{r}'+\mathfrak{r}''}{2}=\mathfrak{R}. \qquad (35)$$

It then reads:

$$-\frac{1}{4}\int dx'''\int\frac{d\mathfrak{g}}{h^3}A^0(\mathfrak{r}''')e^{\frac{i}{\hbar}(\mathfrak{r}''-\mathfrak{R})\mathfrak{g}}\int\frac{d\mathfrak{t}}{h^3}e^{\frac{i}{\hbar}\mathfrak{t}\mathfrak{r}}\frac{|p^{0'}p^{0''}|-p'p''-m^2c^2}{(|p^{0'}|+|p^{0''}|)|p^{0'}p^{0''}|}. \qquad (36)$$

The fraction under the integral signs is best developed in $\mathfrak{g}$ for $g \ll mc$, and takes on the value:

$$\frac{1}{2k_0^2}\left[\frac{g^2}{2}-\frac{(\mathfrak{tg})^2}{2k_0^2}-\frac{3g^4}{16k_0^2}+\frac{5(\mathfrak{tg})^2g^2}{2k_0^4}-\frac{7(\mathfrak{tg})^2}{16k_0^6}+\cdots\right], \qquad (37)$$

in which we have set $k_0^2 = k^2 + m^2c^2$. A lengthy calculation leads to the following result for (36) (for small values of $|\mathfrak{r}| = r$):

$$-\frac{\pi}{4}\int dx'''\int\frac{d\mathfrak{g}}{h^3}A^0(\mathfrak{r}''')e^{\frac{i}{\hbar}(\mathfrak{r}''-\mathfrak{R})\mathfrak{g}}\cdot\frac{1}{h^3}\left[g^2\left(\frac{2}{9}-\frac{2}{3}\gamma-\frac{2}{3}\log\frac{mcr}{\hbar}\right)+\frac{1}{3}\frac{(\mathfrak{gr})^2}{r^2}-\frac{g^4}{15m^2c^2}\right]$$

$$=\frac{1}{16\pi h}\left[\frac{2}{3}\left(\frac{1}{3}-\gamma-\log\frac{mcr}{\hbar}\right)(\mathrm{grad}_{\mathfrak{R}})^2+\frac{1}{3r^2}(\mathfrak{r}\,\mathrm{grad}_{\mathfrak{R}})^2\right.$$

$$\left.+\frac{1}{15}\left(\frac{\hbar}{mc}\right)^2(\mathrm{grad}_{\mathfrak{R}})^2(\mathrm{grad}_{\mathfrak{R}})^2\right]\frac{e}{c}A^0(\mathfrak{R}). \qquad (38)$$

The first two terms – when one doubles them, since the complex conjugate must be added to (36) – represent the part:

$$\frac{a}{x_\lambda x^\lambda}+b\log\left|\frac{x_\lambda x^\lambda}{C}\right|$$

of equation (13), and are thus dropped when one goes from R_S to the matrix r. Formally, (38) also subsequently gives the basis for the fact that the constant C in equation (15) was set equal to $4\left(\frac{\hbar}{mc}\right)^2 e^{-2/3-2\gamma}$. We thus arrive at the fact that it is unnecessary to correct

the perturbation calculations of the total charge that generates the field at each new step. Finally, for $|\mathfrak{r}| = 0$, the density matrix $(x', k' \mid r \mid x'', k'')$ becomes:

$$(\xi', k' \mid r \mid \xi'', k'') = \frac{1}{120\pi h}\frac{e}{c}\left(\frac{\hbar}{mc}\right)^2 \Delta\Delta^0(\xi), \tag{39}$$

and the charge density itself [$\Delta A^0(\xi) = -4\pi\rho_0$, where ρ_0 refers to the external charge density]:

$$\rho = -\frac{1}{15\pi}\frac{e^2}{\hbar c}\left(\frac{\hbar}{mc}\right)^2 \Delta\rho_0, \tag{40}$$

which was already computed by Dirac [1]). This additional density, whose total charge vanishes, also has no physical significance. It is then inseparable from the "external" density and will therefore be calculated automatically along with the "external" density.

The "polarization of the vacuum" first becomes a physical problem for the temporal variation of external densities; one imagines, e.g., a charge distribution that moves back and forth periodically. In such a case, one can distribute the external charge density in its temporal mean value and a second density that oscillates periodically around the null value. The spatial integral of the second part vanishes when the external charge density moves back and forth in a finite spatial domain. The considerations up to now are valid for the part, so for them, the "polarization" plays no physical role. The total charge of a particle can thus never change by means of the polarization of the vacuum. In order to ignore what happens for the second part, we consider, in place of the temporally constant scalar potential A^0 in equations (26) to (29), a potential that varies periodically, and set:

$$A^0(x') = B^0(\mathfrak{r}')\; e^{\frac{i}{\hbar} f x_0'} + \text{conj.} \tag{41}$$

The only change that must be made to the expressions (34) to (36) consists in the fact that the fraction:

$$\frac{1}{(|p^{0'}| + |p^{0''}|)\,|p^{0'}p^{0''}|}$$

is replaced with:

$$\frac{|p^{0'}| + |p^{0''}|}{[(|p^{0'}| + |p^{0''}|) - f^2]\,|p^{0'}p^{0''}|}.$$

The new formulas thus go over to the old ones quite simply – we take $f \ll mc$ – in such a way that the expression under the integral sign in (36) is multiplied by $1 + f^2 / 4k_0^2$. In addition, terms in α^l generally appear in the density matrix; we would like to restrict

[1]) P. A. M. Dirac., Report to the Solvay Congress 1933. Dirac's value differs from the one above by a factor of 2, which, as Dirac graciously informed me, is due to an oversight in his equations.

ourselves to the calculation for the charge density, for which the terms in α^l play no role. If one considers only the terms that are proportional to g^2 in (37) then the expression:

$$\frac{1}{2k_0^2}\left[\frac{g^2}{2}-\frac{(\mathfrak{tg})^2}{2k_0^2}\right]\frac{f^2}{4k_0^2} \tag{42}$$

now gets added to (37). The part of the density in question then becomes:

$$-\frac{\pi}{4\cdot 15}\int dx'''\int\frac{d\mathfrak{g}}{h^3}\frac{e}{c}\left[B^0(\mathfrak{r}''')e^{\frac{i}{\hbar}f x_0'''}+\text{conj.}\right]e^{\frac{i}{\hbar}(\mathfrak{r}'''-\mathfrak{R})\mathfrak{g}}\frac{f^2g^2}{h^3(mc)^2} \tag{43}$$

and one thus has the extra density:

$$\rho=-\frac{1}{15\pi}\frac{e^2}{\hbar c}\frac{f^2}{(mc)^2}\cdot\rho_0\,. \tag{44}$$

Here, the periodically oscillating density is denoted by ρ_0, which gives rise to the field $B^0(x)e^{\frac{i}{\hbar}f x_0'}$, and whose spatial integral vanishes. Equation (44) teaches us that the dipole moment that is coupled to an oscillating charge will be reduces by the polarization of the vacuum, and indeed even more so as the frequency of oscillation increases. As Dirac has already suggested, this situation necessitates a change in the scattering formula of Klein and Nishina, which generally amounts to perhaps one-tenth of a percent in the realm of the Compton wavelength.

If one carries out an analogous calculation, in order to compute, say, the matter density that is induced by a light wave then this gives the result that the periodically varying field of a monochromatic plane light wave generates either charge or current density. One can easily see that this result also remains true to an arbitrary approximation: One cannot distinguish any sign for the charge by means of an electromagnetic field in empty space, so the induced charge density must vanish. On the grounds of invariance, the current density also vanishes then. Certainly, the vanishing of the energy density does not follow even from this, and in fact two plane waves that pass through each other can already give rise to the creation of matter. The intuitive theory of matter waves is thus no longer appropriate for the treatment of such problems (pair creation and annihilation), and we thus go on to the quantum theory of waves.

II. Quantum theory of wave fields.

1. Presentation of the basic equations. In the quantum theory of matter waves, the Dirac density matrix corresponds to the product of wave functions with their conjugates; we then set:

$$R = \psi^*(x', k')\ \psi(x'', k''). \tag{45}$$

The commutation relation:

$$\psi^*(x', k')\ \psi(x'', k'') + \psi(x'', k'')\ \psi^*(x', k') = \delta(x'', k'')\ \delta_{k'k''} \tag{46}$$

is true for the wave function (for $x_0' = x_0''$). If one considers the Maxwell field as a given *c*-field then the Dirac matrix is simply the expectation value for the matrix that is defined by (45). Due to the commutation relation (46), one has, in the quantum theory of waves:

$$R_S = \tfrac{1}{2}[\psi^*(x', k')\ \psi(x'', k'') - \psi(x'', k'')\ \psi^*(x', k')]. \tag{47}$$

The equations:

$$\mathcal{H} R_S = 0 \tag{3a}$$

and $R_S = r + S$ remain unchanged and only in the form of the inhomogeneity $\mathcal{H}S$ in:

$$\mathcal{H} r = -\mathcal{H} S \tag{11a}$$

can a change become necessary due to the non-commutation of field strengths with potentials. Now, no non-commuting functions appear in the first term:

$$e^{-\frac{ei}{\hbar c}\int_{P'}^{P''} A^\lambda dx_\lambda} \cdot S_0 .$$

Terms enter into S_1 [cf., (13) and (14)] that are quadratic in the field strengths and play a role when one calculates energy and impulse density from the density matrix. As long as one restricts oneself to the calculation of charge and current density these terms will longer appear. Now, since the Maxwell equations, together with the inhomogeneous equation (11a), determine the physical evolution completely, the reasoning of the formalism that was depicted in I in the context of quantum theory results from a process that was given for ordinary quantum electrodynamics in a note of the author [1]) in connection with previous research of Klein [2]). This process starts with the Maxwell equations and the wave equation, which are treated as *q*-number relations and are integrated according to the usual methods of the intuitive theory. Ordinarily, a perturbation process is applied to the integration of the basic equations, in which one assumes that the interaction between light and matter is small and is developed in powers of the charge. The plane light waves in empty space and the plane electron waves in field-free space then take the form of the unperturbed system. Such a perturbation

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[1]) W. Heisenberg, Ann. d. Phys. **9** (1931), 338.
[2]) O. Klein, Zeit. Phys. **41** (1927), 407.

process is also applicable in the present theory with no further assumptions. It is then necessary only to also develop the matrix S that is appropriate for the inhomogeneity in the wave equation in powers of charge, and to consider the individual terms in the development that are produced by the perturbation process in turn at the successive degrees of approximation. In order to deduce R_S and r – and therefore, the charge and current density, in the zeroth order approximation – one will then only have to subtract the matrix S_0 from R_S . If one represents the wave function in the form:

$$\psi(x, k) = \sum_n a_n u_n(x,k)\,, \tag{48}$$

where the equations:

$$a_n a_m^* + a_m^* a_n = \delta_{nm} \tag{49}$$

are valid, then one has (in the sequel, we shall always set $x_0' = x_0''$):

$$\begin{aligned} R_S &= \tfrac{1}{2}[\,\psi^*(x', k')\ \psi(x'', k'') - \psi(x'', k'')\ \psi^*(x', k')] \\ &= \sum_{n,m} \tfrac{1}{2}(a_n^* a_m - a_m a_n^*)\, u_n^*(x', k')\, u_m(x'', k'')\,. \end{aligned} \tag{50}$$

From this, it follows for r, when one considers the definition of S_0 , that:

$$r = \sum_{n,m} \tfrac{1}{2}\left(a_n^* a_m - a_m a_n^* + \frac{p_n^0}{|\,p_n^0\,|}\cdot\delta_{nm} \right) u_n^*(x', k')\, u_m(x'', k'')\,. \tag{51}$$

From Jordan and Wigner [1]), one represents the operators a_n in the form:

$$a_n^* = N_n\, \Delta_n\, V_n\,, \qquad a_n = V_n\, \Delta_n\, N_n\,, \tag{52}$$

in which Δ_n converts the number N_n into $1 - N_n$, and one sets:

$$V_n = \Pi_{t<n}(1 - 2N_t).$$

For the states of negative energy, one can now introduce [2]):

$$\left.\begin{aligned} a_n^* = a_n' = V_n' \Delta_n' N_n' = V_n \Delta_n N_n', \\ a_n = a_n'^* = N_n' \Delta_n' V_n' = N_n' \Delta_n V_n. \end{aligned}\right\} \tag{53}$$

One will then have $N_n' = 1 - N_n$.

One finally obtains for the matrix r:

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[1]) P. Jordan and E. Wigner, Zeit. Phys. **47** (1928), 631.

[2]) Cf., e.g., W. Heisenberg, Ann. d. Phys. **10** (1931), 888.

$$r = \sum_{p_0, n>0} a_n^* a_n u_n^*(x', k') u_n(x'', k'') - \sum_{p_0, n<0} a_n'^* a_n' u_n^*(x', k') u_n(x'', k'') + \sum_{n \neq m} a_n^* a_m u_n^*(x', k') u_m(x'', k'')$$

$$= \sum_{p_0, n>0} N_n u_n^*(x', k') u_n(x'', k'') - \sum_{p_0, n<0} N_n' u_n^*(x', k') u_n(x'', k'') + \sum_{n \neq m} a_n^* a_m u_n^*(x', k') u_m(x'', k'') . \tag{54}$$

This representation of the density matrix agrees with the representation that was chosen by Pauli and Peierls [1]), Oppenheimer and Furry, Fock (*loc. cit.*). N_n means the number of electrons, N_n', that of the positrons, and the symmetry in the theory on the sign of charge is assumed from the outset. This representation is, however, only correct in the zeroth-order approximation. If one goes on to the first-order approximation then, on the one hand, the coefficients a_n will also contain, as functions of time, terms that are linear in the field strengths [cf., e.g., *loc. cit.*, Ann. d. Phys. **9**, pp. 341, equation (9)], and on the other hand, the terms that are linear in e in the definition of r must be subtracted from the matrix S, and thus the terms:

$$\frac{ei}{\hbar c} \int_{P'}^{P''} A^\lambda dx_\lambda \cdot S_0 + \frac{e}{48\pi^2 \hbar c} \left\{ \frac{x_\rho x^\sigma}{x_\tau x^\tau} \cdot \alpha^\lambda \left(\frac{\partial F_{\lambda\sigma}}{\partial \xi_\rho} - \delta_\lambda^\rho \frac{\partial F_{\mu\sigma}}{\partial \xi_\mu} \right) + \alpha^\lambda \frac{\partial F_{\tau\lambda}}{\partial \xi_\tau} \log \left| \frac{x_\rho x^\rho}{C} \right| \right\} . \tag{55}$$

These terms, together with the terms in the coefficients a_n that are linear in e, then give a contribution to the matrix r that leads to a finite charge and current density (in the first approximation) and which can therefore assist in the calculation of the electromagnetic field in the second approximation, etc.

Instead of this process, which is closely connected with the integration methods of the intuitive theory, one can, however, also define a Hamiltonian function in the usual way and then carry out the perturbation theory for the associated Schrödinger equation. To this end, we employ the expression for the total energy that follows from equation (16), so we do not go to the limit $x^\lambda = 0$. The total energy takes the form:

$$E = \int d\xi \left\{ -\left(ei\hbar \frac{\partial}{\partial x_l} - \frac{e}{2} \left[A^l \left(\xi + \frac{x}{2} \right) + A^l \left(\xi - \frac{x}{2} \right) \right] \right) \right.$$

$$\times \sum_{k', k''} \alpha_{k'k''}^l \sum_{n,m} \tfrac{1}{2} (a_n^* a_m - a_m a_n^*) u_n^* \left(\xi + \frac{x}{2}, k' \right) u_m \left(\xi - \frac{x}{2}, k'' \right)$$

$$\left. - \sum_{k', k''} \beta_{k'k''} mc^2 \sum_{n,m} \tfrac{1}{2} (a_n^* a_m - a_m a_n^*) u_n^* \left(\xi + \frac{x}{2}, k' \right) u_m \left(\xi - \frac{x}{2}, k'' \right) \right\} .$$

[1]) I would like to cordially thank Herrn W. Pauli for the written communication of this result.

$$- \sum_{k'}\left(ci\hbar\frac{\partial}{\partial x_0} - \frac{e}{2}\left[A^0\left(\xi+\frac{x}{2}\right) + A^0\left(\xi-\frac{x}{2}\right)\right]\right)\left(\xi-\frac{x}{2},k' \mid S \mid \xi-\frac{x}{2},k'\right) + \frac{1}{8\pi}(\mathfrak{E}^2+\mathfrak{H}^2)\Big\}. \tag{56}$$

If one again develops the Hamiltonian function in powers of the elementary charge and additionally drops the terms $\mathfrak{E}^2 + \mathfrak{H}^2$, as well as the corresponding terms in the expression (14) for the zero-point energy for radiation, then in the limit $x \to 0$ one gets for the zero-order Hamiltonian:

$$H_0 = \sum_{E_n>0} N_n E_n - \sum_{E_n<0} N'_n E_n + \sum_{\mathfrak{g},\mathfrak{e}} M_{\mathfrak{g},\mathfrak{e}} h\nu_{\mathfrak{g},\mathfrak{e}}\,, \tag{57}$$

where one has set $E_n = -\,c\,p_n^0$ and $M_{\mathfrak{g}\mathfrak{e}}$ means the number of light quanta in the state $\mathfrak{g}$ with polarization $\mathfrak{e}$. Likewise, this yields for the perturbation energy of first order in the limit $x \to 0$ (for the sake of simplicity, A^0 is set to zero):

$$H_1 = \int d\xi\, eA^l(\xi) \sum_{k',k''} \alpha^l_{k'k''}\left[\sum_{E_n>0} N_n u_n^*(\xi,k') u_n(\xi,k'') \right.$$
$$\left. - \sum_{E_n<0} N_n u_n^*(\xi,k') u_n(\xi,k'') + \frac{1}{2}\sum_{n\neq m}(a_n^* a_m - a_m a_n^*) u_n^*(\xi,k') u_n(\xi,k'')\right]. \tag{58}$$

The present theory thus agrees with the results of Oppenheimer and Furry in the expressions for H_0 and H_1. We thus obtain terms of higher order that come from the matrix S. The passage to the limit $x \to 0$ cannot be performed immediately in these terms, either. Moreover, in carrying out the perturbation calculations to the second order, the terms in H_2 must be combined with the terms of type $\dfrac{H_1^{nl} H_1^{lr}}{W_n - W_l}$ that originate in H_1 before one can then carry out the passage to the limit $x \to 0$ and yield a definite result for the energy to second order.

In this way, the perturbation process can be, in principle, performed when no infinite self-energy, as in quantum electrodynamics up to now, leads to a divergence in the process [1]). The perturbation energy H_2 has the following form:

$$H_2 = \int d\xi \left[i\frac{e^2}{\hbar c}\left(\int A^\lambda dx_\lambda\right)^2 \frac{\partial}{\partial x_0} + \frac{1}{48\pi^2}\frac{e^2}{\hbar c}\frac{x_\lambda x^\sigma}{x_\rho x^\rho} A^\lambda \left[\frac{\partial F_{0\sigma}}{\partial \xi_0} - \frac{\partial F_{\tau\sigma}}{\partial \xi_\tau}\right]\right.$$
$$\left. - \frac{1}{96\pi^2}\frac{e^2}{\hbar c}\frac{x_\sigma x^\tau}{x_\rho x^\rho} F^{\mu\sigma} F_{\mu\tau} + \frac{1}{48\pi^2}\frac{e^2}{\hbar c}\log\left|\frac{x_\sigma x^\tau}{C}\right| \cdot \left(F_{\tau 0} F^{\tau 0} - \frac{1}{2} F_{\tau\mu} F^{\tau\mu}\right)\right]. \tag{59}$$

[1]) Cf., on this, V. Weisskopf, Zeit. Phys. **89** (1934), 27; furthermore, on the search for ways to avoid the infinite self-energy of the electron, see M. Born, Proc. Roy. Soc. (A) **143** (1934), 410; M. Born and L. Infeld, *ibid.*, **144** (1934), 425.

Due to the integration over ξ, H_2 gives rise to only matrix elements that correspond to the creation or annihilation of light quanta of the same impulse. For the ordinary processes in which light quanta are emitted or absorbed or scattered, these matrix elements then play no role in the first approximation. In the perturbation energy H_3 , which has the form:

$$H_3 = \int d\xi \left[-ic\hbar \frac{1}{24} \left(-\frac{ie}{\hbar c} A^\lambda x_\lambda \right)^4 \frac{\partial S_0}{\partial x_0} \right] = \frac{1}{48\pi^2} \left(\frac{e^2}{\hbar c} \right)^2 \frac{1}{\hbar c} \int d\xi \frac{(A^\lambda x_\lambda)^4}{(x_\rho x^\rho)^2}, \qquad (60)$$

and gives rise to matrix elements that lead to the scattering of light by light (the annihilation and creation of two light quanta with equal impulse sums). Halpern [1]) and Debye [2]) have independently proved the fact that the Dirac theory of the positron has the scattering of light by light a consequence – even when the energy of the light quanta is not sufficient for pair creation. However, the matrix elements in H_4 give no accounting of the magnitude of this scattering, since it must have been previously combined with the contributions that originated in the lower-order approximations in order produce a measure for the probability of a scattering process. Higher perturbation terms than H_4 do not appear; H_5, H_6, etc. all vanish in the limit $x = 0$.

2. Applications. For the most practical applications – e.g., pair creation, annihilation, Compton scattering, etc. – the theory described here does not yield anything new compared to the formulation of the Dirac theory all along. Thus, in all of the cases mentioned, one can break off the perturbation calculation at the second-order approximation and the new terms in H_2, due to their special form, contribute nothing to the transition probabilities that were sought. Things are different for the aforementioned problem of the scattering of light by light and for the coherent scattering of γ-rays from fixed charge centers that was discussed by Delbrück [3]); the calculations in these problems are so complicated that they will not be attempted here.

We would therefore like to restrict the applications to an example in which the term H_2 in equation (59) becomes important; we shall treat the matter density that is linked to a light quantum, and in particular, the self-energy of the light quantum that is given on the basis of this matter density. If one first ignores the term H_2 and calculates with the usual methods heretofore then the process can be represented as follows: Since matrix elements appear in H_1 [equation (58)] that correspond to the conversion of a light quanta into a pair, a light quantum generates a matter field in its neighborhood in a manner that is similar to the way that an electron generates a Maxwell field. The energy of this matter field becomes infinite in complete analogy to the infinite self-energy of electrons. Now, part of the singular terms in the infinite self-energy of the light quantum vanishes when one considers the perturbation term H_2 . They are then arranged such that no infinite self-energy would appear for a classical light wave. Nevertheless, the following calculation

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[1]) O. Halpern, Phys. Rev. **44** (1934), 885.

[2]) I am deeply grateful to Herrn Debye for cordially communicating his reasoning.

[3]) M. Delbrück, Discussion of the experimental results of L. Meitner and her colleagues, Zeit. Phys. **84** (1933), 144.

shows that an infinite part of the self-energy that is required by the application of quantum theory remains. The analogy with the self-energy of the electrons is complete now. In the Maxwell theory, a continuous charge distribution would also lead to a finite self-energy; it is the "quantization" that leads to the infinite self-energy. If one represents the quantization of the electromagnetic field by point-like light quanta then the infinitude of the self-energy also emerges in the intuitive theory of matter waves, since the inhomogeneity in equation (11) includes the field strengths and their first and second derivatives, which become singular in the context of light quanta.

For the calculation of the desired self-energy, one can start from a known formula of perturbation theory for the energy of second order:

$$W_2 = H_0 s_1^2 - s_1 H_0 s_1 + s_1 H_1 - H_1 s_1 + H_2 . \tag{62}$$

In this, H_0, H_1, H_2 mean the various terms in the Hamiltonian function, and s_1 is the first term of the characteristic matrix for the canonical transformation:

$$W = sHs^{-1}, \tag{63}$$

namely:

$$s = 1 + s_1 + \ldots \tag{64}$$

The sense in which the methods described in the previous section are to be used is in the sense that the matrices H in equation (62) are first taken at a finite distance x_λ and then it is only at the conclusion that one first takes the limit as $x_\lambda \to 0$. The matrix s_1 is to be calculated from H_1 in the limit $x_\lambda = 0$ in the usual way:

$$s_1^{lm} = \frac{H_1^{lm}(x=0)}{W_0^l - W_0^m} . \tag{65}$$

The element of the matrix H_1 ($x = 0$), which belongs to the simultaneous creation of an electron of impulse $\mathfrak{p}''$ and a positron of impulse $\mathfrak{p}'$ and the annihilation of a light quantum of impulse $\mathfrak{g}$ (and polarization $\mathfrak{e}$), has the form:

$$\frac{e\hbar}{\sqrt{V}}\sqrt{\frac{e}{g}}(\mathfrak{p}', p_0 < 0 \,|\, \alpha e \,|\, \mathfrak{p}'', p_0'' > 0)\cdot M_{\mathfrak{g},\mathfrak{e}}^{1/2} , \tag{66}$$

where V represents the volume that the periodic boundary conditions is given on and $M_{\mathfrak{g},\mathfrak{e}}$ means the number of light quanta in the state $\mathfrak{g}$, $\mathfrak{e}$. Furthermore, one sets:

$$(\mathfrak{p}',\ p_0' < 0 \,|\, \alpha\, \mathfrak{e} \,|\, \mathfrak{p}'',\ p_0'' > 0) = \int d\mathfrak{r} \sum_{k'k''} u^*_{\mathfrak{p}', p_0'<0}(\alpha^l \mathfrak{e}_l) u_{\mathfrak{p}'', p_0''>0} . \tag{67}$$

If one now introduces the expressions for s_1 that follow from (65) and (66) into equation (62) – in which one must consider not only the matrix elements (66), but the ones that correspond to the process of the simultaneous creation of an electron, positron,

and light quantum – then one obtains contributions from the term $s_1H_1 - H_1s_1$ that remain finite as long as $x_\lambda\, x^\lambda$ does not vanish, and which also yield a finite contribution to W_2 in the limit $x_\lambda \to 0$ when one combines them with the corresponding terms in H_2. This is therefore not true for the part $(H_0\, s_1 - s_1H_0)\, s_1$. If one decomposes H_0 into a part that belongs to the matter waves and one that belongs to the light waves then the first one, in fact, also gives a finite contribution for $x_\lambda\, x^\lambda \neq 0$ that contributes finitely to W_2 when combined with H_2 in the limit $x_\lambda \to 0$. The part that belongs to the electromagnetic field thus does not depend upon x_λ, so it leads to the sum:

$$\sum_{\mathfrak{p}'+\mathfrak{p}''=\mathfrak{g}} |(\mathfrak{p}'|\, s_1\, |\mathfrak{p}'')|^2\; gc\,. \tag{68}$$

This sum diverges; one can immediately refer to the expression (68) as the infinite self-energy of the light quantum. If one carries out the summation in (68) only up to large, but finite, values $|\,\mathfrak{p}'\,| = P$ and considers only the part in (62) that is proportional to $M_{\mathfrak{g},\mathfrak{e}}$ then one obtains an expression of the form:

$$g \cdot c \cdot M_{\mathfrak{g},\mathfrak{e}} \cdot \frac{e^2}{\hbar c} \log \frac{P}{mc}\,. \tag{69}$$

In the quantum theory of wave field, the domain of applicability of the Dirac formulation of the theory of positrons is therefore not essentially larger than the domain of applicability for the elementary formulas of Pauli, Peierls, Fock, Oppenheimer, and Furry. Equations (48) to (61) then show how these formulas can be regarded as the first step in a sequence of approximations that satisfy the requirements of relativistic and gauge invariance. Furthermore, the formalism that is described here also yields finite expectation values for present and energy densities in the first approximation where the elementary formulas would give infinite values. The fact that divergences would appear in the second approximation of the quantum theory of wave fields was to be expected from results of quantum electrodynamics up to now.

The situation that the application of the quantum theory first leads to divergences that do not appear in the intuitive theory of wave fields suggests that this intuitive theory, in fact, already contains the essence of the correct, corresponding description of how things happen, so one cannot carry out the transition to quantum theory in the original way that was sought for in the current theory heretofore. In the Dirac theory of positrons, moreover, a pure separation of the fields that are involved into matter fields and electromagnetic fields is scarcely possible any more. In particular, this comes from the fact that in the quantum theory of waves it is the matrix R_S – not the matrix r – that can be represented simply by the matter wave functions ψ. It is therefore only a unified theory of matter and light fields that gives the Sommerfeld constant $e^2/\hbar c$ a definite value that will make possible a contradiction-free union of the demands of quantum theory with those of a correspondence with intuitive field theory.