A STUDY OF HEAVY LIGHT FLAVOUR MESONS IN A QCD INSPIRED QUARK MODEL USING APPROXIMATION METHODS

A THESIS

**SUBMITTED TO GAUHATI UNIVERSITY
FOR THE
DEGREE OF DOCTOR OF PHILOSOPHY IN
PHYSICS IN THE FACULTY OF SCIENCE**

SUBMITTED BY

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YEAR: SEPTEMBER,2011

Dedicated to the memory of Professor Nathan Isgur (1947-2001) whose work inspired us to carry out this research.

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To whom it may concern

This is to certify that Mr Bhaskar Jyoti Hazarika has worked under my guidance for the thesis entitled "A STUDY OF HEAVY LIGHT FLAVOUR MESONS IN A QCD INSPIRED QUARK MODEL USING APPROXIMATION METHODS" which is being submitted to the Gauhati University for the degree of Doctor of Philosophy.

The thesis is his own work.He has fulfilled all the requirements under the PhD regulations of Gauhati University and to the best of my knowledge ,the thesis as whole or a part thereof has not been submitted to any other university for any degree or diploma.

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Declaration

I hereby declare that the thesis entitled "A STUDY OF HEAVY LIGHT FLAVOUR MESONS IN A QCD INSPIRED QUARK MODEL USING APPROXIMATION METHODS" which is being submitted to the Gauhati University for the degree of Doctor of Philosophy is my own work done under the guidance of Prof. D K Choudhury.

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This is to the best of my knowledge and belief.

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Acknowledgement

Surely it's the invaluable help of the research supervisor whose physical and mental presence all the time makes it possible to carry out a research work.I am greatly indebted to my supervisor Prof. D K Choudhury for the way he guided me patiently,encouraged from all angles and supported during problematic period. He never pressurized me and it was his greatness what meant a lot for me.Then, I must speak about my family.At any time, they are with me and this support keeps me to stand boldly.Amazingly,my father who has not obtained a high degree of formal education keeps me asking about the progress I have made in research.Certainly,his seriousness greatly influenced me Thank you my dear father

Let me express my sincere gratitude to the respective heads,faculty members,staff of the department of Physics, Gauhati University for the facility and support they provided me.Special thanks to Dr M P Bora ,who innovated the computer laboratory so nicely to give it a new dimension;Borua *da* (Debabrata Barua) for his honorary still so dedicated service and Naba *da* (Naba Kr Sarma , now working in IDOL, Gauhati University) for the support he provided.My well wishers,friends all were and are always with me in this journey . I am greatly thankful to Dr N S Bordoloi whose work indeed set my target.Ranjit *da*, Archana, Sanjeeb,Simanta,Zeen, Rupjyoti,Saiful,Samiran,Nabaratna,Abinash,Kingkor and others helped me in different situations and I am quite thankful to them.

Now a days, it is well known that to be handy in computer is a must for the researchers particularly for those involved in theoretical investigation.Regarding this,I sincerely thank Chabin Thakuria whom I consider as my computer *Guru*.How can I forget those inspirations of my colleagues Dr Projit Das,Dr Dipali Das, Mrs Bandana Das ? I am indebted to all of them.I also thank Dr B Bora ,Principal of Pandu college for his encouragement.

There is a long list of such names (as it happens always)who directly or indirectly helped me a lot.Although it is not possible to pick up those names here,but I will remain ever grateful to them.

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Chapter 1

Introduction

With the gained interest of hadrons containing at least one heavy quark ,the construction of phenomenological models in the nonperturbative regime of QCD is very essential.In most cases, the potential models are very much successful in the prediction of hadron spectrum and decay modes.Under such circumstances,to pursue a QCD inspired potential model and then to study the different static and dynamic properties of hadrons really makes sense.The effectiveness and reliability of such a model can be tested through the calculation of these properties and their comparison with other models and data.

The present work is an attempt to work out a potential model for the relatively simpler mesonic system containing at least one heavy quark,where the basic input equation is the Schrödinger equation .We solve it for heavy-light flavour pseudoscalar mesons like D, D_s, B, B_s etc.The solution i.e. wavefunction is obtained by using diffrent approximation methods like Dalgarno method [1] and Variationally Improved Perturbation Theory (VIPT)[2, 3, 4] .We note that with linear plus Coulombic potential ,we have alternate options of choosing parent-perturbation (i.e. parent-child) combination and then use it in the said approximation methods in the process of obtaining the appropriate wavefunction.Getting an accurate wavefunction is very much essential for any successful potential model to obtain corresponding static and dynamic properties like Isgur-Wise(I-W) function [5, 6, 7, 8],elastic form factors[9, 10, 11], charge radii etc.The I-W function being a universal function parameterizes all the nonperturbative effects of semileptonic decays while the elastic form factor or charge radius measures the charge distribution of the constituents of a hadron and thus they determine the fate of a specific QCD model.

We incorporate relativistic modifications from outside due to the light quarks involved and use fixed values of running coupling constant α_s either form $\overline{MS}$ [12] or from V scheme [13, 14, 15].Both finite and infinite mass limits are taken into account which give a broader aspect of the model.

The obtained results are compared with available data as well as those of other models to confirm the reliability and effectiveness of the model.The comparison convinces us that our work is in reasonably agreement with those models and data.

1.1 QCD:perturbative QCD (pQCD) and nonperturbative QCD (npQCD)

The violation of Pauli priniciple for states like Δ^{++}, Δ^{-} etc and nonexistence of single quark or states like qq,$\overline{qq}$ etc led to the color hypothesis.

According to it, each quark flavour carries three strong color charges,red (r),yellow (y) and blue (b) . As far as quark content is concerned ,only color singlet (colorless) states exist as free particles.This leads to color confinement and explains why no free quark or states like qq,$\overline{qq}$ etc exist.The color of a quark forms the basis of $SU(3)$ color symmetry group. The quarks interact through a non-Abelian gauge field known as gluons which are self interacting .The colors and gluons are experimentally proven facts and the gauge invariant field theory resulted out of their strong interaction is known as QCD.

The important properties of QCD are : (i) the gluons being mediators of strong interaction ,carry color charge and exchange of gluons give rise to attractive forces between color singlet states which provides binding between quarks in a hadron ,(ii) asymptotic freedom which implies that the effective coupling constant decreses logarthimically at short distances or high momentum transfer which is the basis for pQCD that accounts for different phenomena at high energy or momentum transfer,(iii) confinement of quarks which implies that at large distances or low energy ,the binding energy between color charges increases linearly so that no free quark exists or quarks are never found isolated.This property is not properly investigated yet on theoretical basis but supported by lattice theory and quarkonium spectroscopy.The quark confinement leads to the npQCD and takes account of the low energy regime.

The pQCD allows one to calculate the short distance behaviour of quarks and gluons in terms of perturbative expansion of the strong coupling constant α_s.Although pQCD is very useful in the high energy regime through the use of concepts like running coupling constant and renomalization group equation ,but not applicable in the low energy region to take account of certain properties like confinement,chiral symmetry breaking,dynamical mass generation etc.So, a reliable approach of npQCD is very much essential as far as the static and dynamic properties of hadrons are concerned.

In the nonperturbative description, we basically believe in the lattice QCD [16, 17, 18, 19] which is however handicapped at distances less than what is called lattice spacing.Further, the method being based on expensive computation technique is left with inadequate physical insight to understand important properties like confinement etc.The way then lies in the construction of some phenomenological quark models based on confinement or long range forces to predict hadronic properties like mass, decay widths etc. To that end we are left with some models developed earlier like constituent quark model [20, 21], light cone QCD[22] and effective field theories like HQET [23, 24],ChPT [25, 26, 27] besides QCD sum rules [28].

According to CQM,the hadrons are considered as bound state of three valence quarks (baryons) or a quark and an antiquark (mesons).The valence quarks have different masses and internal structure from QCD quarks although the quantum numbers are same for both of them.The NonRelativistic Constituent Quark Model(NRQM) [29, 30]which is a type of CQM,can organize the calculation of α_s in all orders and at the same time,elaborate the relativistic corrections to the conventional formula very successfully.Further,within the framework of NRQM,hadronic spectra was explained successfully by different authors [31, 32, 33, 34, 35, 36, 37].

1.2 Potential model

As pointed above,it is clear that formulation of a phenomenological model is very useful to make a proper analysis in the nonperturbative regime.Various such models are proposed in different context.Out of these,the concept of potential between a heavy quark and another heavy or light quark has been subject of theoretical investigation since long [38, 39, 40, 41, 42, 43].The potential has played a key role in understanding properties like quark confinement and can describe non relativistically bound systems such as heavy quarkonia very successfully. Indeed the potential model is tremendously successful in providing both qualitative and quantitative description of the hadron spectrum and the deacy modes.The potential models seem to reproduce the experimental values much better as they contain more input parameters than lattice QCD or perturbative QCD models.However, it is always preferable to use phenomenological form of flavour independent potential from qualitative arguments [39, 40] and to find out the limitations of the model than to explain experimental data.

To work with a suitable potential model,the choice of the correct potential is the most important thing.As we know that in QCD,the exchange of glucns gives force of attraction between color singlet states and thus provides binding between quarks in a hadron.This potential known as One Gluon Exchange (OGE) potential is attractive because of the color charges present and can at best provide binding at short distance.But,it is not sufficient to explain the confinement of quarks and thus a long distance part of the potential is also required.The long distance potential in QCD is assumed to increase with the distance due to the self interaction of color carrying gluons so that the quarks can be confined in a hadron.Phenomenologically,such a potential is of the form

$$V(r) = V_g(r) + V_L(r) \tag{1.1}$$

where

$$V_g(r) = -\frac{k_s \alpha_s}{r} + V_s \tag{1.2}$$

and $k_s = \frac{4}{3}(\frac{2}{3})$ for $q\overline{q}(qqq)$ system.

Here V_g is the OGE potential potential having spin dependent components V_s while V_L is the corresponding spin and flavour independent confining piece.

With these considerations,Rujula,Georgi and Glashow [29] developed a Hamilotonian resulted out of Fermi-Breit interaction given as:

$$H = H_0 + H_C + H_{FB} + H_L \tag{1.3}$$

where the subscripts o, C, FB, L on the various terms of the Hamiltonian H refer to the zero interaction(free) ,Coulomb,Fermi-Breit (spin-spin,spin-orbit and tensor term of spin) and long range (confining) respectively.

The exact form of H_L is still not calculable.Various workers have used different forms for it with success.The lattice QCD [44] and string theory [45]support a linear potential $H_L \sim r$,some authors prefer a harmonic oscillator potential $H_L \sim r^2$ [46],while others favour the logarithimic dependence $H_L \sim ln\ r$ [42] . These speculations have made the number of parameters fixed from the comparison with experimental data comparable to the number of experimental data.So, while introducing a form of the potential on qualitative ground,one must develop simpler models than to explain experimental data and look for the limitations of the model.

The above considerations lead one to consider the linear plus Coulombic potential [43, 44] as the suitable one to start with and it has already been used in the explanation of hadron spectroscopy with success.With such a potential,the Coulombic term can be calculated with two loop correction of Wilson loop formulae [47] and this can be incorporated in an effective coupling constant in a scheme known as V scheme [13, 14, 15].

For heavy-light flavour mesons,the relativistic effects need consideration due to light quarks involved . Considerable interest has been shown in this regard and there are certain models in which relativistic corrections are treated as perturbation [48] as far as nonrelativistic models are concerned.

The nonrelativistic models are usually based on Schrödinger equation. The advantage of this equation is that it can handle many particle system effectively at least in principle.However,it will be seen later that for a Hamiltonian in equation (1.11),the exact solution of Schrödinger equation is not possible and certain refinements are required for that.Further,relativistic effect is to be put from outside.

As far as relativistic effect is concerned , Dirac equation is a suitable one but it is effective for a single particle only, not for many particles and one needs a wide framework of Quantum Field Theory (QFT).

1.3 Semileptonic decays and heavy quark symmetry

The semileptonic decays offer a perfect testing ground for both pQCD and npQCD effects such as decay constants,form factors as well as the best possible predictions of the CKM matrix elements.So, a potential model can be used to make such predictions for the semileptonic decays for mesons containing at least one heavy quark.

The analysis of semileptonic decays have been greatly simplified for hadrons containing at least one heavy quark due to what is called Heavy Quark Symmetry (HQS) [5, 6, 10, 24, 49, 50].Being a very useful tool to obtain model independent information of weak decays, the HQS arises because the masses (m_Q) of heavy quarks (c, b, t) are much larger than the QCD scale Λ_{QCD} .In other sense,it is the infinite mass limit $m_Q \rightarrow \infty$ that leads to HQS as noted below:

I. The heavy quark and the hadron that contain it have the same velocity.In the hadron's rest frame,the heavy quark is also at rest.The light degree of freedom are blind to the flavour(mass) and thus we obtain a heavy flavour symmetry introduced by Shuryak [51].

II. In this limit,the spin of the heavy quark decouples from the gluon field or light degree of freedom because the hyperfine ,magnetic interaction scale as m_Q^{-1} .The members of a hyperfine multiplet become degenerate in mass.Consequently,there is a new spin symmetry because of which the light degree of freedom in the (heavy-light) mesons are in the same state even if the spin orientation of heavy degree of freedom changes.

The flavour-spin symmetry $SU(2N_h)$ for N_h number of heavy flavours as discussed above is the HQS.Due to HQS , replacement of a heavy quark by another of different mass and spin but same velocity will not effect the light degree of freedom.The underlying theory of QCD in this limit is the HQET [49, 52] which allows a systematic,order by order evaluation of correction to infinite mass limit in inverse powers of the heavy quark masses.The HQS or HQET offer useful contribution to understand the dynamics of systems containing at least one heavy quark.

1.3.1 The Isgur-Wise function

As stated above,the HQS has greatly simplified the analysis of heavy flavour hadrons. Indeed, for the semileptonic decays of B mesons,the HQS implies that all the independent form factors that describe these decays are expressible in terms of a single universal function of velocity transfer commonly known as Isgur-Wise(I-W) function [5, 6, 7, 8].The I-W function parameterizes all the nonperturbative QCD effects of semileptonic decays and thus is a standard factor in determining the reliability of a specific QCD model.

The I-W function measures the overlap of the wavefunctions of the light degrees of freedom of initial and final mesons moving with velocities v and v' respectively. The knowledge of I-W function $\xi(y)$ (where $y = v.v'$) is essential to make a direct connection between heavy hadron and the corresponding quark amplitude and gain insight into m_Q^{-1} corrections of HQET.The condition $\xi(y = 1) = 1$ is the normalization condition of I-W function at the zero recoil point ($v = v'$) well predicted by HQET.

The HQET-the well defined theory of QCD arising out of HQS as said above, make predictions related to decays of heavy hadrons in terms of this single function enabling the description over simplified and less model dependent.Consequently,the experimental predictions become convincing ones.However,the I-W function is not calculable in pQCD and needs nonperturbative means like lattice QCD or QCD sum rules.The perturbative calculations leave new uncalculable functions and thus reduces the predictive power of HQET.Further,HQET can't predict the shape of the I-W function.So,the phenomenological models always remain important.However,the dynamical quark models can be matched with HQET from the normalization condition of I-W function at zero recoil point apart from considering different aspects of the I-W function.

We note that the I-W function (also the elastic form factor,charge radius etc)is directly related to the hadronic wavefunction and so getting an accurate wavefunction is a real test for any specific potential model.One can use different approximation methods like the Dalgarno method [1],VIPT [2, 3, 4] in getting it by solving the Schrödinger equation for the quark-antiquark system.

1.4 The approximation methods

It is clear that while investigating different hadronic properties ,one have to use different approximation methods to obtain the wavefunction. Every method has its own merits and demerits which basically depends on the type of application.

The Dalgarno method [1] is a type of stationary state perturbation theory in which one first uses a trial form of the first order correction to wavefunction in terms of a series cooperated to the perturbed Hamiltonian operating on the unperturbed wavefunction and then adopt the series solution method of Forbenius.

The VIPT [2, 3, 4] is a combination of variational method and perturbation theory.At first,the variational method is used in terms of known trial function (in contrary to the usual perturbation theory) which then allows to exercise the perturbation theory in terms of convergent expansion parameter obtained from the variational method used earlier.

With the linear cum Coulombic potential we have two options of choosing the parent or perturbation which allows a broader prespective of the model.

1.5 A brief description of the thesis

This work is evolved from a QCD inspired potential model with the linear cum Coulombic potential and aims at the calculation of hadronic properties like slope and curvature of I-W function ,elastic form factors,charge radii for the relatively simpler mesonic system containing at least one heavy quark.For that we have solved the Schrödinger equation using the Dalgarno method,VIPT in two different scenarios-linear part and Coulombic part as perturbation .So,our main attention lies in getting an appropriate wavefunction to be used in the calculation of the above quantities. The reliability and effectiveness of the model is tested by the comparison of these calculated values with those of other models and data [53-68].

1.5.1 The model:

The model [30] we have pursued is based on the work of Rujula,Georgi and Glashaow [29] .Being very successful in explaining various aspects of hadronic physics ,their work used a Hamiltonian known as Fermi-Breit Hamiltonian having terms which are more singular than r^{-2} and was not exactly solvable.Their work was improved by Godfrey and Isgur [69] by postulating relativistic potential and smearing function.The smearing of potentials removed all the singularities but it required additional parameters.

In our model,we have adopted the same Hamiltonian in the absence of spin for the ground state only and use Schrödinger equation to solve it.The relativistic effect is incorporated in a parameter free way using the standard Dirac modification [70, 71].

The Fermi-Breit Hamiltonian developed by Rujula et al [29] for two quarks of masses m_i, m_j and three momenta p_i, p_j can also be written as [69] :

$$H(r) = \frac{p_i^2}{2m_i} + \frac{p_j^2}{2m_j} + H^{conf}(r) + H^{hyp}(r) + H^{s.o}(r) \tag{1.4}$$

Here,

$$H^{conf}(r) = \left[-\frac{\alpha_s}{r} + \frac{3br}{4} + \frac{3c}{4} \right] \mathbf{F}_i.\mathbf{F}_j \tag{1.5}$$

$$H^{hyp}(r) = -\frac{\alpha_s}{m_i m_j} \left[-\frac{8\pi \mathbf{S}_i.\mathbf{S}_j \delta^3(r)}{3} + \frac{1}{r^3} \left(\frac{3(\mathbf{S}_i.\mathbf{r})(\mathbf{S}_j.\mathbf{r})}{r^2} - \mathbf{S}_i.\mathbf{S}_j \right) \right] (\mathbf{F}_i.\mathbf{F}_j) \tag{1.6}$$

$$H^{s.o.}(r) = H^{s.o\,(c.m.)} + H^{s.o\,(t.p.)} \tag{1.7}$$

$$H^{s.o.(t.p.)}(r) = -\frac{1}{2r} \frac{\partial H^{conf}}{\partial r} \left(\frac{\mathbf{S}_i}{m_i^2} + \frac{\mathbf{S}_j}{m_j^2} \right) \mathbf{L} \tag{1.8}$$

$$H^{s.o.(c.m.)}(r) = \frac{\alpha_s}{r^3} \left(\frac{1}{m_i} + \frac{1}{m_j} \right) \left(\frac{\mathbf{S}_i}{m_i} + \frac{\mathbf{S}_j}{m_j} \right) (\mathbf{F}_i.\mathbf{F}_j)\mathbf{L} \tag{1.9}$$

Here α_s is the running coupling constant , b is the confinememt parameter and c is another parameter whose significance will be cleared later and $\mathbf{S}_i$ and $\mathbf{S}_j$ are the spins of the i^{th} and j^{th} quark respectively separated by a distance r.Also, for the mesons

$$< \mathbf{F}_i.\mathbf{F}_j >= -\frac{4}{3} \tag{1.10}$$

For the ground state ($l = 0$) the spin independent Hamiltonian becomes:

$$H = \frac{p_i^2}{2m_i} + \frac{p_j^2}{2m_j} - \frac{4\alpha_s}{3r} + br + c \tag{1.11}$$

With this Hamiltonian we have solved the Schrödinger equation namely

$$H|\psi >= (H_0 + H')|\psi >= E|\psi > \tag{1.12}$$

by using the perturbative methods like Dalgarno method and VIPT with the different options of parent-child combination.The unperturbed Hamiltontian is H_0 and H' is the perturbative part to be chosen from the equation (1.11) above.

Chapter 2: It has reported the use of Dalgarno method with Coulombic parent in getting the wavefunction .This work has extensively observed the effectiveness of a scale parameter 'c' within the potential leading to upper bounds on the slope and curvature of I-W function.

Chapter 3: This chapter has introduced the VIPT with Coulombic parent in obtaining the wavefunction and then used it in the calculation of I-W function.The improvents are well recorded over the earlier method done in chapter 2.

Chapter 4: In this chapter,we have chosen the linear part as parent in the Dalgarno method in getting the wavefunction as an alternate approach.This new wavefunction is then used in the calculation of I-W function as done in the earlier chapters and important conclusions are drawn.

Chapter 5: This chapter deals with the last option i.e. linear parent with VIPT in the process of searching an accurate wavefunction which is then used in the calculation of slope and curvature of I-W function in a similar manner.

Chapter 6: It has extended the use of VIPT in the calculation of elastic form factors and charge radii for the same heavy-light mesons to make the model more reliable and effective one.

Chapter 7: It includes the summary,conclusion and future outlook.

Chapter 2

Bounds on the slope and curvature of Isgur-Wise(I-W) function in a QCD inspired quark model

2.1 Introduction

As stated in chapter 1, this work basically solves the Schrödinger equation using approximation methods like Dalgarno method ,Variationally Improved Perturbation Theory (VIPT) and obtains the wave function under two different scenarios-linear part and Coulombic part as perturbation of the linear cum Coulombic potential provided by the approximation methods.The model pursued by us is based on the work of Rujula,Georgi and Glashow as explained in chapter 1,and we are finally left with a spin independent Hamiltonian for ground state to work with as given by equation(1.11) there.

We solve the nonrelativistic Schrödinger equation with this Hamiltonian and then incorporate relativistic effect usmg standard Dirac modification.The wavefunctions thus obtained are used in the calculation of slope and curvature of Isgur-Wise (I-W) function,elastic form factors ,charge radii as mentioned earlier.

In this chapter,we start our search for the wavefunction using the Dalgarno method with Coulombic part as the parent of the total potential $-\frac{4\alpha_s}{3r}+br+c$.This has been already considered in the work of [30, 72, 73] with $c = 0$, and then $c \neq 0$ in [74].The parameter 'c' usually appear in a composite form 'cA_0' as a coefficient in the solution of Schrödinger equation with Dalgarno method where 'A_0' is the undetermined factor appearing in the series solution of the same (*cf.*Appendix A).However,the case $c \neq 0$ in [74] was handicapped by its scaling at natural scale $\sim 1GeV$ with presumably taken $A_0 = 1$ and the adhoc adjustment of the strong coupling constant.

In this work,we consider the $c \neq 0$ case with a different strategy.We use the wavefunction at the origin involving the unknown coefficient cA_0 and fix it from the experimental values of masses and decay constants directly.The reality constraint on cA_0 will be seen to yield lower bounds on the strong coupling constant α_s, which would lead to the upper bounds on the slope and curvature of the I-W function.

The chapter is organized as follows : section 2 contains the formalism,section 3 the results and lastly section 4 encloses the conclusion and remarks.

2.2 Formalism

2.2.1 The Wavefunction

We rewrite the spin independent Fermi-Breit Hamiltonian given by eq.(1.11) for ground state ($l = 0$),with the two body problem reduced to a single one of reduced mass μ :

$$\begin{aligned} H &= H_o + H', \\ &= -\frac{\nabla^2}{2\mu} - \frac{4\alpha_s}{3r} + br + c. \end{aligned} \tag{2.1}$$

We set $b = 0.183 GeV^2$ and look for an effective range of running coupling constant α_s which can lead to better results for slope and curvature of I-W function.

With $H_o = -\frac{\nabla^2}{2\mu} - \frac{4\alpha_s}{3r}$ as the parent Hamiltonian and $H' = br + c$ as the perturbed Hamiltonian , we obtain a ground state wavefunction upto the first order correction using the Dalgarno method [1] of stationary state perturbation theory as :

$$\psi_{conf}(r) = N\left(cA_0 + \frac{1}{\sqrt{\pi a_0^3}} - \frac{\mu b a_0 r^2}{\sqrt{\pi a_0^3}}\right) e^{-\frac{r}{a_0}}. \tag{2.2}$$

where A_0 is the unknown coefficient appearing in the series solution of the Dalgarno method as stated above(details are given in Appendix A).

Including the relativistic effect [70, 71], the wavefunction is :

$$\psi_{conf+rel}(r) = N'\left(cA_0 + \frac{1}{\sqrt{\pi a_0^3}} - \frac{\mu b a_0 r^2}{\sqrt{\pi a_0^3}}\right)\left(\frac{r}{a_0}\right)^{-\epsilon} e^{-\frac{r}{a_0}}, \tag{2.3}$$

Here a_0 is given by:

$$a_0 = \frac{3}{4\mu\alpha_s}, \tag{2.4}$$

and

$$\epsilon = 1 - \sqrt{1 - \frac{4\alpha_s}{3}}, \tag{2.5}$$

N and N' are the normalization constants given by :

$$N^2 = \frac{1}{1 + \frac{45\mu^2 b^2 a_0^6}{8} - 3\mu b a_0^3 + \pi a_0^3 c^2 A_0^2 + \frac{2cA_0\pi a_0^3}{\sqrt{\pi a_0^3}} - \frac{3\pi a_0^6 c A_0 \mu b}{\sqrt{\pi a_0^3}}}, \tag{2.6}$$

and

$$N'^2 = \frac{2^{7-2\epsilon}}{\Gamma(3-2\epsilon) X_1}. \tag{2.7}$$

where X_1 is given in Appendix B.

We note that the equations (2.2),(2.3),(2.6)and (2.7) are obtained from eq.(4),(6),(5) and (7) of ref.[74] exhibiting explicit dependence of cA_0 in them.

2.2.2 Fixing of the coefficient cA_0

The wavefunction at the origin (WFO), is related to the decay constant f_p and the mass of the pseudoscalar meson M_p through the relation [30, 75]:

$$|\psi(0)|^2 = \frac{f_p^2 M_p}{12}. \tag{2.8}$$

Again from equation (2.2), we have :

$$|\psi(0)|^2 = N^2[c^2A_0^2 + \frac{1}{\pi a_0^3} + \frac{2cA_0}{\sqrt{\pi a_0^3}}]. \tag{2.9}$$

Using equation(2.6) and (2.9), we arrive at the quadratic equation for cA_0:

$$A'(cA_0)^2 + B'(cA_0) + C' = 0, \tag{2.10}$$

where

$$A' = \pi a_0^3 |\psi(0)|^2 - 1, \tag{2.11}$$

$$B' = 2\sqrt{\pi a_0^3}|\psi(0)|^2 - 3\mu b a_0^3 \sqrt{\pi a_0^3}|\psi(0)|^2. \tag{2.12}$$

and

$$C' = |\psi(0)|^2 \left[1 + \frac{45\mu^2 b^2 a_0^6}{8} - 3\mu b a_0^3\right] - \frac{1}{\pi a_0^3}. \tag{2.13}$$

Using the experimental values of f_p and M_p [76] , we determine $|\psi(0)|^2$ using equation(2.8) which in turn will yield two solutions for cA_0 in equation (2.10):

$$cA_0 = \frac{-B' \pm \sqrt{B'^2 - 4A'C'}}{2A'}, \tag{2.14}$$

Thus,cA_0 depends on μ,M_P,f_P and α_s.The solution corresponding to the +ve(-ve) sign of equation(2.14)will be termed as +ve(-ve) solution hereafter.It will be shown numerically that for a given μ, M_P,and f_P ,α_s reaches the minimum value when the following condition is satisfied :

$$B'^2 - 4A'C' = 0. \tag{2.15}$$

The formalism involving eq.(2.5)-(2.15) is strictly valid only without relativistic effect as the wavefunction at the origin with such effect [eq.(2.3)] is not well defined due to its singularity at the origin. For a subsequent analysis ,we assume that cA_0 does not deviate significantly from its non-relativistic value so that it can be used to calculate the slope and curvature of the I-W function even without relativistic effect.

2.2.3 Charge radius (slope)and convexity parameter (curvature) of I-W function

The Isgur-Wise function is written as [6, 7, 77] :

$$\xi\left(v_\mu.v'_\mu\right) = \xi(y) = 1 - \rho^2(y-1) + C(y-1)^2 + ... \tag{2.16}$$

where

$$y = v_\mu.v'_\mu, \tag{2.17}$$

and v_μ and v'_μ being the four velocity of the heavy meson before and after the decay.The quantity ρ^2 is the slope of I-W function at $y = 1$ and known as charge radius :

$$\rho^2 = \frac{\partial\xi}{\partial y}|_{y=1}, \tag{2.18}$$

The second order derivative is the curvature of the I-W function known as convexity parameter :

$$C = \frac{1}{2}\left[\frac{\partial^2\xi}{\partial^2 y}|_{y=1}\right]. \tag{2.19}$$

For the heavy-light flavor mesons the I-W function can also be written as [7, 72] :

$$\xi(y) = \int_0^{+\infty} 4\pi r^2 |\psi(r)|^2 \cos prdr, \tag{2.20}$$

where

$$p^2 = 2\mu^2 (y-1). \tag{2.21}$$

Equation (2.20) holds good for both relativistic and nonrelativistic case.The wavefunction $\psi(r)$ takes different form for both the cases. Without relativistic effect, it is given by equation(2.2) and with relativistic effect it is given by (2.3).

With the wavefunction(2.2)in equation(2.20) i.e. including confinement only the charge radius ρ^2_{conf} and convexity parameter C_{conf} are respectively given by:

$$\rho^2_{conf} = \frac{\mu^2[24\pi c^2 A_0^2 a_0^5 + 24a_0^2 + 630\mu^2 b^2 a_0^8 + 48cA_0\sqrt{\pi a_0^7} - 180cA_0\mu b\sqrt{\pi a_0^{13}} - 180\mu b a_0^5]}{8\pi c^2 A_0^2 a_0^3 + 8 + 45\mu^2 b^2 a_0^6 + 16cA_0\sqrt{\pi a_0^3} - 24\mu bcA_0\sqrt{\pi a_0^3} - 24\mu b a_0^3}, \tag{2.22}$$

and

$$C_{conf} = \frac{\mu^4[60\pi c^2 A_0^2 a_0^7 + 60a_0^4 + 4725\mu^2 b^2 a_0^{10} + 120cA_0\sqrt{\pi a_0^{10}} - 840cA_0\mu b\sqrt{\pi a_0^{17}} - 840\mu b a_0^7]}{16\pi c^2 A_0^2 a_0^3 + 16 + 90\mu^2 b^2 a_0^6 + 32cA_0\sqrt{\pi a_0^3} - 48\mu bcA_0\sqrt{\pi a_0^3} - 48\mu b a_0^3}. \tag{2.23}$$

With the wavefunction (2.3)in equation (2.20) i.e. including both relativistic and confinement effect the charge radius $\rho^2_{conf+rel}$ and convexity parameter $C_{conf+rel}$ are given by :

$$\rho^2_{conf+rel} = \frac{\mu^2 a_0^2 (4-2\epsilon)(3-2\epsilon)[X_1]}{4[X_2]}, \tag{2.24}$$

and

$$C_{conf+rel} = \frac{\mu^4 a_0^4 (6-2\epsilon)(5-2\epsilon)(4-2\epsilon)(3-2\epsilon)[X_3]}{96[X_2]}. \tag{2.25}$$

where X_1,X_2 and X_3 are given in Appendix B.

We note that equations (2.24) and (2.25) are equivalent to equations (18) and (19)of ref.[74] exhibiting explicit cA_0 dependence.

2.3 Results

2.3.1 Values of cA_0 and lower bounds on α_s

As noted earlier , cA_0 depends on μ,M_P,f_P and α_s.In fig.(2.1-2.5) we plot cA_0 vs α_s for D,D_s,B,B_s and B_c mesons .It shows that α_s tends to reach the minimum value when two solutions of eq.(2.10) almost merge satisfying the condition(2.15).This feature is true for any set of the parameters μ, f_p and M_p. In table 2.1 ,we give the lower bounds on α_s for these mesons .

The dependence of cA_0 on α_s and μ can be noted as follows :

With constant μ , cA_0 decreases with α_s values rising and vice-versa.On the other hand, with constant α_s , cA_0 increases(decreases) with increase (decrease) in μ.

2.3.2 Bounds on slope and curvature of the I-W function

Using the lower bounds on α_s for each heavy-light and heavy-heavy mesons,we obtain upper bounds on the slope and curvature of the I-W function using equations (2.22),(2.23),(2.24) and (2.25). They are listed in table 2.2.We note that with increasing α_s values, the slope and curvature decrease and henceforth the lower bound on α_s corresponds to the upper bound on ρ^2 and C.

In table 2.3,we record the predictions of the slope and curvature of the I-W function in various models while in table 2.4,we reproduce the corresponding predictions of the model [74] with $c = 1GeV$ and $A_0 = 1$ in V-scheme [13, 14, 15] for various mesons.Two set of values for B, B_s and B_c mesons are shown in the table where case- a represents the actual values for ρ^2 and C in that work with $\alpha_s = 0.261$;while case-b represents those for adhoc adjustable value of $\alpha_s = 0.60$ in order to show the usefulness of large α_s as mentioned in ref.[74] .The α_s values were already large for D and D_s mesons,so no adhoc adjustment was necessary that might lead to two set of values .

Table 2.1: Lower Bounds on α_s.

Mesons	Quark content	μ(GeV) ref.[76]	M_p(GeV) ref.[76]	f_p(GeV) ref.[76]	cA_0	Lower bound on α_s
D	$c\bar{u}/c\bar{d}$	0.276	1.869	0.192	0.9665	~ 0.601
B	$\bar{b}u/\bar{b}d$	0.315	5.279	0.210	0.7653	~ 0.652
D_s	$c\bar{s}$	0.368	1.968	0.157	0.9543	~ 0.49
B_s	$\bar{b}s$	0.44	5.279	0.171	0.999	~ 0.493
B_c	$\bar{b}c$	1.18	5.37	0.36	1.167	~ 0.302

Table 2.2: Upper Bounds on slope and curvature.

Meson (Quark Content)	Slope ρ^2		Curvature C	
	Without relativistic effect	With relativistic effect	Without relativistic effect	With relativistic effect
$D(c\bar{u}/c\bar{d})$	6.78	1.675	13.19	5.138
$B(\bar{b}u/\bar{b}d)$	5.78	1.016	9.58	1.29
$D_s(c\bar{s})$	9.115	3.067	26.48	14.32
$B_s(\bar{b}s)$	11.92	2.652	34.49	6.902
$B_c(\bar{b}c)$	28.46	10.39	219.46	45.23

Table 2.3: Predictions of the slope and curvature of the I-W function in various models.

Model	Value of ρ^2	Value of curvature C
Le Youanc *et al* [53]	≥ 0.75	..
Le Youanc *et al* [54]	≥ 0.75	≥ 0.47
Rosner [55]	1.66	2.76
Mannel [56]	0.98	0.98
Ebert et al [57]	1.04	1.36
Pole Ansatz [58]	1.42	2.71
MIT Bag Model [59]	2.35	3.95
Simple Quark Model [60]	1	1.11
Skryme Model [61]	1.3	0.85
QCD Sum Rule [62]	0.65	0.47
Relativistic Three Quark Model [63]	1.35	1.75
Neubert [64]	0.82±0.09	..
Infinite Momentum Frame Quark Model [65]	3.04	6.81
UKQCD Coll.[66]	0.83^{+15+24}_{-11-22}	..
CLEO Coll. [67]	$0.76 \pm 0.16 \pm 0.08$	..
BELLE Coll. [68]	0.69 ± 0.14	..

Table 2.4: Predictions of the slope and curvature of the I-W function in the QCD inspired quark model according to ref.[74] with $c = 1$ and $A_0 = 1$ taking relativistic and confinement effect in V-scheme.This table is reproduced from the last rows of tables 1,2,3 of ref.[74] .

Meson	α_s	slope (ρ^2)	curvature(C)
D	0.625	1.136	5.377
D_s	0.625	1.083	3.583
B	$a)$0.261	$a)$128.128	$a)$5212
	$b)$0.60	$b)$1.329	$b)$7.2
B_s	$a)$0.261	$a)$112.759	$a)$4841
	$b)$0.60	$b)$1.257	$b)$4.379
B_c	$a)$0.261	$a)$44.479	$a)$2318
	$b)$0.60	$b)$1.523	$b)$0.432

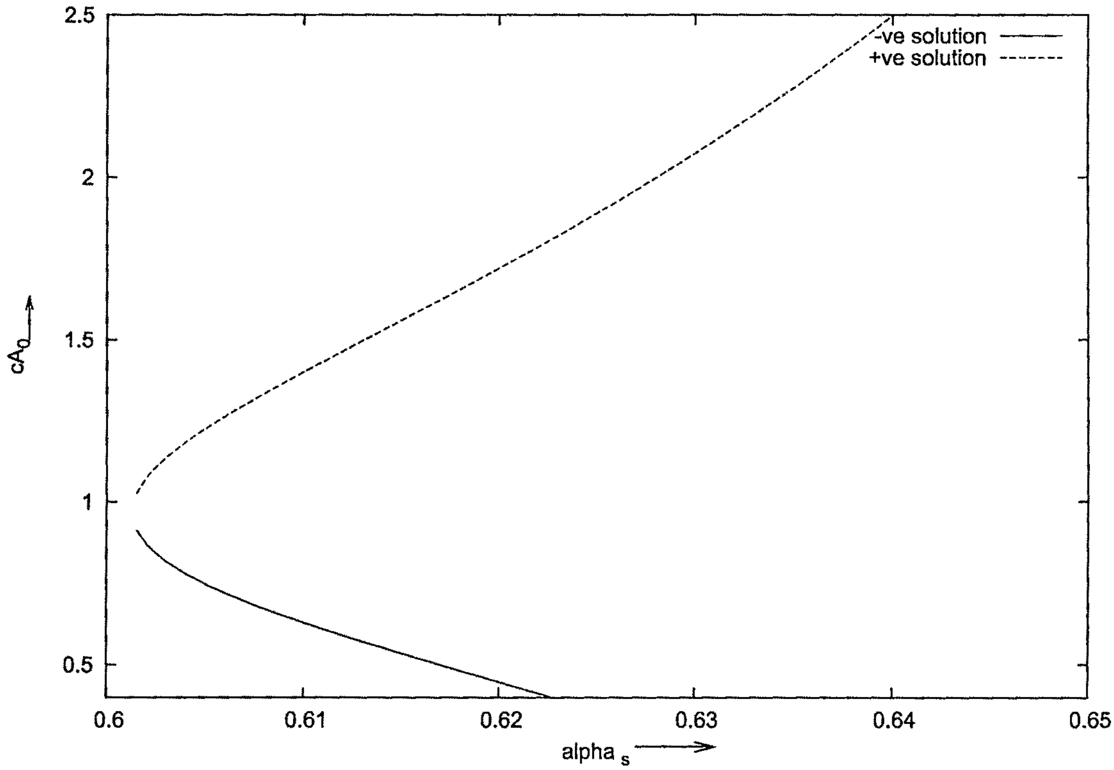

Figure 2.1: Variation of cA_0 vs α_s for D Meson.The +ve (-ve) solution of eq.2.14 corresponds to the dashed (solid) line and the two lines nearly coincide at $\alpha_s \sim 0.601$, the lower bound on α_s corresponding to the eq.2.15 for D Meson.

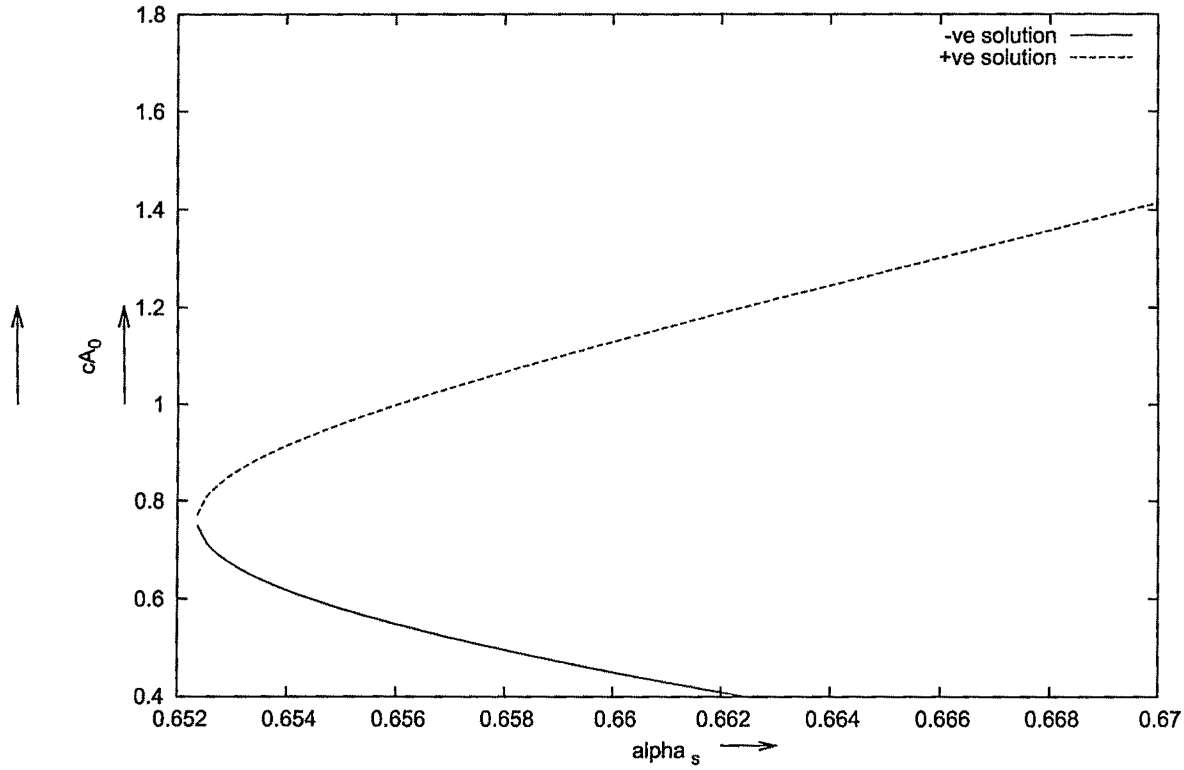

Figure 2.2: Variation of cA_0 vs α_s for B Meson.The +ve (-ve) solution of eq.2.14 corresponds to the dashed (solid) line and the two lines nearly coincide at $\alpha_s \sim$ 0.652 , the lower bound on α_s corresponding to the eq.2.15 for B Meson.

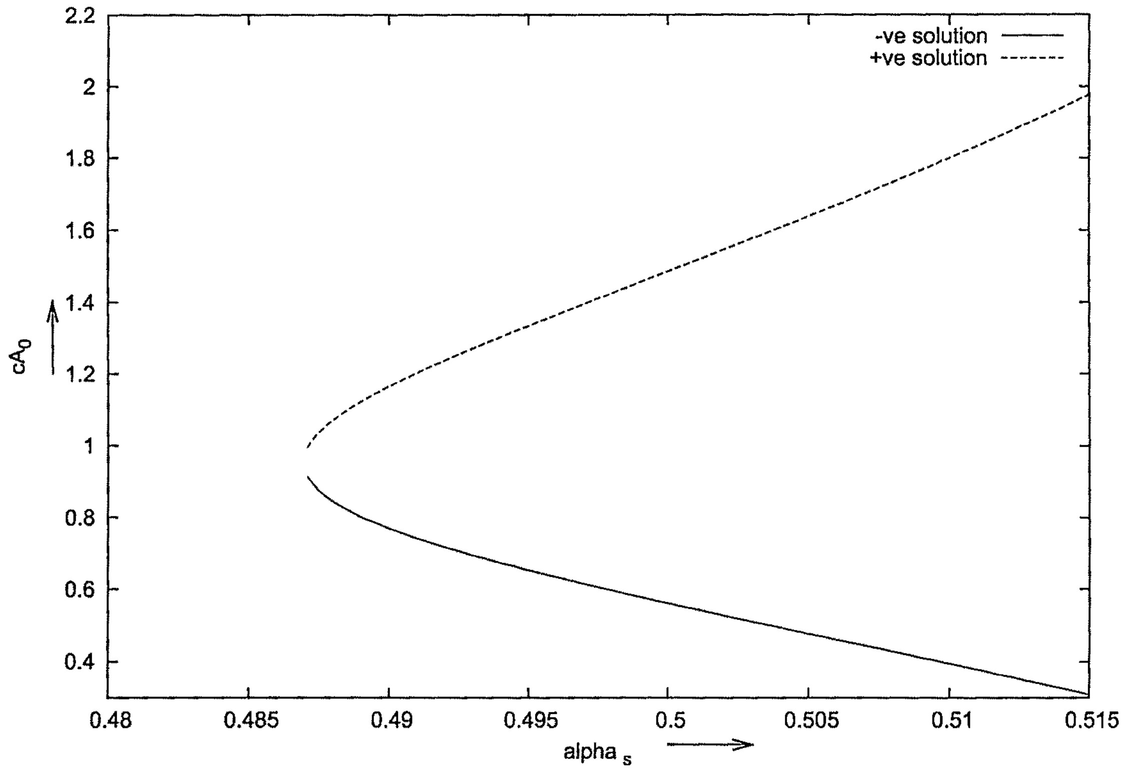

Figure 2.3: Variation of cA_0 vs α_s for D_s Meson.The +ve (-ve) solution of eq.2.14 corresponds to the dashed (solid) line and the two lines nearly coincide at $\alpha_s \sim 0.49$, the lower bound on α_s corresponding to the eq.2.15 for D_s Meson.

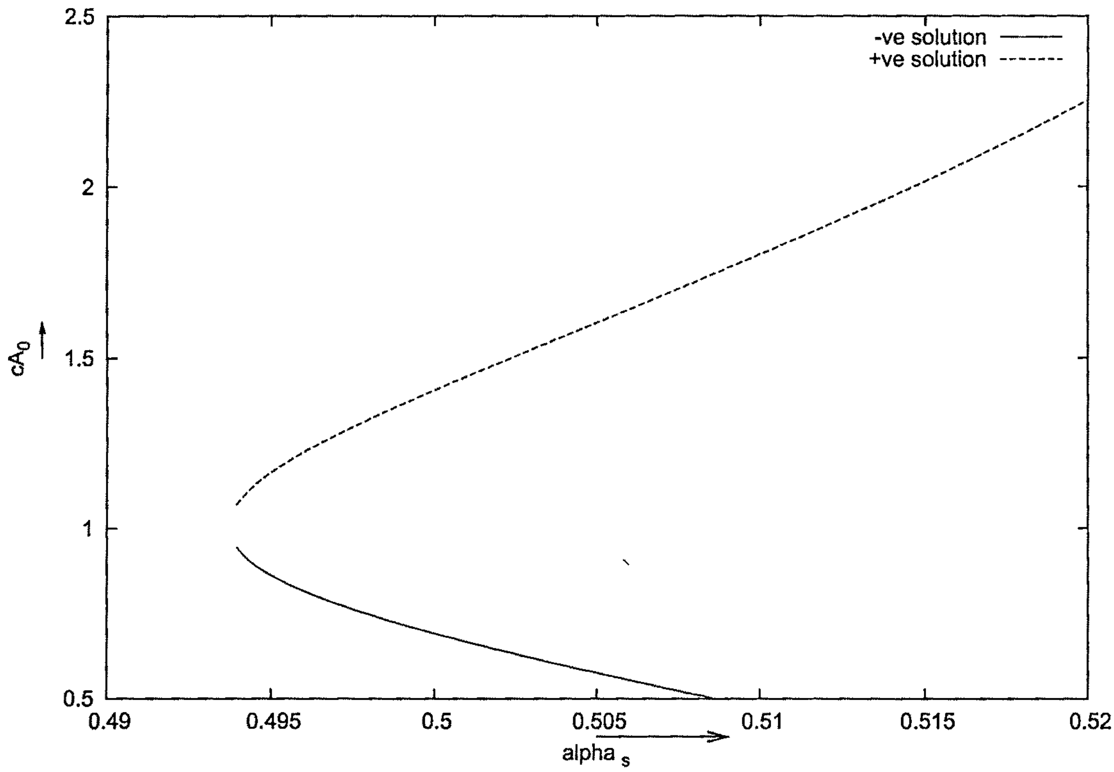

Figure 2.4: Variation of cA_0 vs α_s for B_s Meson.The +ve (-ve) solution of eq.2.14 corresponds to the dashed (solid) line and the two lines nearly coincide at $\alpha_s \sim 0.493$, the lower bound on α_s corresponding to the eq.2.15 for B_s Meson.

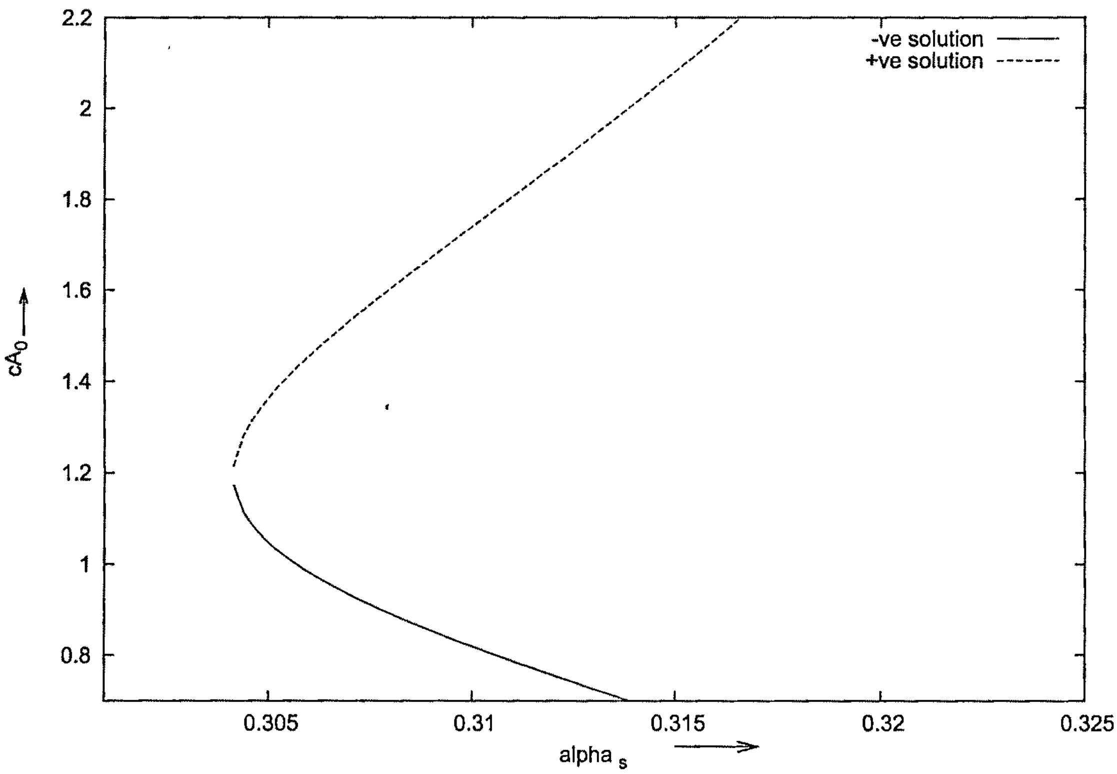

Figure 2.5: Variation of cA_0 vs α_s for B_c Meson.The +ve (-ve) solution of eq.2.14 corresponds to the dashed (solid) line and the two lines nearly coincide at $\alpha_s \sim$ 0.302 , the lower bound on α_s corresponding to the eq.2.15 for B_c Meson.

2.4 Conclusion and Remarks

In this chapter,we have shown that the reality bound on cA_0 puts lower limit on α_s and correspondingly upper limit on ρ^2 and C.

Furthermore,with cA_0 , the upper bounds on ρ^2 and C decrease which is evident from the above list of bounds(table-2.2). The estimated upper bounds on ρ^2 and C for all the mesons are found to be consistent with other models and data (table-2.3) without making any adhoc enhancement of the strong coupling constant as had been done in ref.[74](table-2.4) .From the phenomenological point of view we note that in the nonrelativistic limit ,the universal form factor and Isgur-Wise function for semileptonic decay $B \rightarrow D^* l\nu$ are identical when subleading terms in velocity and terms of order $O\left(\frac{E_b}{m_Q}\right)$ are neglected with E_b as the binding energy and m_Q as the mass of heavy quark [50].However even if we make calculation for the universal form factor for finite mass, we obtain to first order in $(y-1)$ as 0.8 -2.57$(y-1)$ which seems to be satisfactory [50, 52].

It is worthwhile to note that in the limit $cA_0 \rightarrow 0$, there will be no bounds on α_s as well as on ρ^2 and C ; rather fixed values of α_s have to be used to get definite set of ρ^2 and C.So,in that case, the analysis will turn to that of ref. [73,74] where large confinement i.e.$b = 0.183 GeV^2$ [78] could not be incorporated e.g. tables -(1,3) of ref.[73] and tables -(2,3) of ref.[74].

We conclude this chapter with a comment on the physical significance of the factor 'c' that has become so crucial for our analysis of bounds on slope and curvature.

It is common wisdom that a constant potential like 'c' just scales the energies and doesnot affect the wavefunction nor does it change physics.This can be seen from the hydrogen atom problem with the potential $V(r) = \frac{-A}{r} + c$. However , if one uses 'c' as the perturbation instead of as parent in the Dalgarno method of perturbation theory [1],the normalized wave function for the H-atom becomes:

$$\psi(r) = \left[\frac{1}{1+\pi a_0^3 c^2 A_0^2 + \frac{2cA_0 \pi a_0^3}{\sqrt{\pi a_0^3}}}\right]\left(cA_0 + \frac{1}{\sqrt{\pi a_0^3}}\right)e^{-\frac{r}{a_0}}$$

which is to be compared with the normalized wave function with 'c' as parent:

$$\psi(r) = \left(\frac{1}{\sqrt{\pi a_0^3}}\right) e^{-\frac{r}{a_0}}$$

Thus, the perturbative child 'c' rather than the parent 'c' plays the crucial role in the present analysis.

Chapter 3

Slope and curvature of Isgur-Wise function using Variationally Improved Perturbation Theory (VIPT) with Coulombic parent in a QCD inspired potential model

3.1 Introduction

We have seen in chapter 2 ,how the reality constraint on the coefficient 'cA_0' led to bounds on the slope and curvature of I-W function instead of fixed values for them [79].In this chapter,we use recently introduced Variationally Improved Perturbation Theory (VIPT)[3-5] as an alternate approach in finding the wavefunction and then use it in the calculation of slope and curvature I-W function.

The VIPT is being a recent entry into the literature which shows a great expectation regarding the use of approximation methods.The work by Aitchison and Dudek [4] inspired us to apply the method to the QCD inspired model which had limitations some of which may be due to the use of conventional perturbation technique.We know that the results of perturbation theory are expressed in terms of finite power series (in an expansion parameter which is taken to be very small)that seem to converge to the exact values when summed to higher order .After a certain order,however the results become increasingly worse since the series is usually divergent (being asymptotic series).At this juncture, the variational method which estimates variationally optimized parameter (through energy minimization)helps in converting the divergent perturbation expansion into a convergent one which can be evaluated for large expansion parameter .We note that using only the variational method [1, 80] is quite cumbersome as it is difficult to choose an appropriate trial wavefunction in terms of unknown parameter(s)which is later optimized to estimate the parameter(s).But in VIPT, we use a known wavefunction as a trial one (e.g.the 1s state H-atom wavefunction)and then optimize it to get the new parameter(s)(e.g.$\overline{\alpha}'_{10}$ in our case [eq.3.10 of this work])which make the perturbation series convergent.Further,we know that the perturbation theory is efficient to systems which have good unperturbed Hamiltonian ,while variational method is robust even in cases where it is hard to determine a good unperturbed Hamiltonian .On the other hand,VIPT becomes independent of the fact whether we have a good unperturbed Hamiltonian or not.

Question arises regarding the use of the Coulombic piece as the parent and linear part as the perturbed one of the total Cornell potential - that upto what distance this consideration is valid ? Indeed,it was shown in ref.[4] that if $< r > < r_0$ then the Coulomb base will perform better . Here $< r >$ is the expectation value of the distance r which reasonably gives the size of a state(in this case meson) and r_0 is a point at which linear cum Coulomb potential becomes zero(fig.1 of Aitchison and Dudek ,ref. [4]).Further,for low lying mesons i.e. $n = 1, l = 0$,(*cf.Equation 8* of ref.[4])the expectation value $< r >$ is inversely proportional to the parameter $\alpha = \frac{4\alpha_s}{3}$ for a given reduced mass μ .Using VIPT we get variably optimized $\overline{\alpha}'_{10}$ (*cf.Equation 3.10* of this work) as the new parameter which assumes substantially larger value than α.As a result,it effectively makes the "linear term" weaker so that

Coulombic piece becomes the parent.This ensures us that the distance between the quarks is short enough to treat the binding effect mainly in terms of the Coulombic potential.Thus VIPT is a convenient and strong tool in treating the Coulombic potential as parent and linear as perturbation of the total Cornell potential.

It would be evident from equation(3.10) that $\overline{\alpha}'_{10}$ increases with the increase in α_s and greater values of $\overline{\alpha}'_{10}$ strongly support the binding effect mainly in terms of Coulombic potential. For the B-sector meson, the α_s values are small. It raises the question of applicability of the Coulombic part as parent. However the corresponding $\overline{\alpha}'_{10}$ values are sufficiently large enough to conform to the expectation $< r > < r_0$ but probably not as large enough to make the results of slope and curvature of the Isgur-Wise function compatible one for these mesons.

The aim of the chapter is to apply the VIPT method to the QCD inspired quark model [29, 30] referred earlier and to calculate the I-W function ,its slope and curvature .Using the same Hamiltonian and treating linear confinement as perturbation we arrive at the hadronic wavefunction which enables to calculate the I-W function. Relativistic modification of the wavefunction [30, 70, 71] as well as the two loop effect of strong coupling constant using V-scheme [13, 14, 15] is also taken into account.

Section 2 has reported the formalism,section 3 the results and finally section 4 the discussion and conclusion.

3.2 Formalism

3.2.1 Variationally Improved Perturbation Theory-VIPT

The recently introduced VIPT method [4, 5, 6], combines two procedures, namely stationary state perturbation theory and the variational method.We have total Hamiltonian as:

$$H = H_0 + H' \tag{3.1}$$

where H_0 being the parent Hamiltonian containing a physical parameter$P$ (say) and H' is the perturbed Hamiltonian.The corresponding wavefunctions also contain P. In VIPT, we make :

$$P = P + P' - P' \tag{3.2}$$

where P' is the variational parameter such that :

$$\begin{aligned} H &= H_{oP'} + H_o - H_{oP'} + H' \\ &= H_{oP'} + H'_{P'} \end{aligned} \tag{3.3}$$

The parent Hamiltonian is now $H_{oP'}$ instead of H_o which depends on the variational parameter P' and $H'_{P'}$ is the new perturbed Hamiltonian instead of H' which also depends on P'.Correspondingly the wavefunctions will also change with P being replaced by P'. Now,one can treat these wavefunctions as trial wavefunctions with P' as the variational parameter and would find the value of P' which gives minimum value of energy corrected upto the first order. This will yield variationally improved unperturbed wavefunction upon which the usual perturbation theory will be applied. The wavefunction corrected upto the first order of j^{th} state is given by [4] :

$$\psi_j = \psi_j^{(0)} + \sum_{k \neq j} \frac{\int \psi_k^{(0)*} H'_{P'j} \psi_j^{(0)} dv}{E_j^{(0)} - E_k^{(0)}} \tag{3.4}$$

The energy corrected upto first order for the same state is :

$$\begin{aligned} E_j &= \int \psi_j^{(0)*} H \psi_j^{(0)} dv \\ &= \int \psi_j^{(0)*} (H_{oP'} + H'_{P'}) \psi_j^{(0)} dv \end{aligned} \tag{3.5}$$

where ψ_k, E_k are the wavefunction and energy eigen values of the k^{th} states which are orthonormal to j^{th} state .The superscript(0) means zeroeth order correction of the corresponding quantities .

With the Cornell potential [44], we can have two possibilities of choosing parent (and hence perturbed) Hamiltonian as said earlier.In one,Coulombic piece is the parent and in the other linear one as the second possibility.

The summation in equation (3.4),can include any number of k^{th} states .In this work,we consider upto three terms in the summation.

3.2.2 Coulomb plus linear potential and wavefunctions using VIPT

(i)With one term in the summation

As explained earlier , we use variational parameter α' instead of the physical parameter $\alpha = \frac{4\alpha_s}{3}$ (as we are taking the Coulombic potential as the parent one). The Hamiltonian takes the form [equation (3.3)]:

$$\begin{aligned} H &= H_o + H' \quad (3.6) \\ &= -\frac{\nabla^2}{2\mu} - \frac{4\alpha_s}{3r} + br + c \\ &= -\frac{\nabla^2}{2\mu} - \frac{\alpha}{r} + br + c \\ &= -\frac{\nabla^2}{2\mu} - \frac{\alpha'}{r} + \frac{(\alpha' - \alpha)}{r} + br + c \\ &= H_{o\alpha'} + H'_{\alpha'} \quad (3.7) \end{aligned}$$

where $\alpha = \alpha - \alpha' + \alpha'$.

Now ,$H_{o\alpha'} = -\frac{\nabla^2}{2\mu} - \frac{\alpha'}{r}$ is the parent Hamiltonian with α' and $H'_{\alpha'} = \frac{(\alpha'-\alpha)}{r} + br + c$ is the perturbed Hamiltonian with the same variational parameter α' .We notice that the physical parameter α is replaced by the variational parameter α'.

We consider j^{th} as $1s$ state ($n = 1$,$l = 0$) and in the summation of equation(3.4), we consider only one number of k^{th} state which is the $2s$ state ($n = 2$,$l = 0$).

The trial $1s$ state can be written (analogous to H- atom case)with variational parameter α' as (this being the unperturbed wavefunction):

$$\psi_{10}^{(0)} = \frac{(\mu\alpha'_{10})^{\frac{3}{2}}}{\sqrt{\pi}} e^{-\mu\alpha'_{10}r} \tag{3.8}$$

where subscript 10 in α' indicates the quantum number (n,l) of the j^{th} state.

We now find the value of α'_{10} which leads to minimum E_j given by (3.5) in the following way :

In the variational method, we are interested only in the 'r'-dependence of the Hamiltonian ,so 'c' in $H'_{\alpha'}$ has no role to play in the calculation [1].

Using equation (3.5),(3.7),(3.8) :

$$E_{10}\left(\alpha'_{10}\right) = \frac{\mu\alpha'^2_{10}}{2} - \mu\alpha\alpha'_{10} + \frac{3b}{3\mu\alpha'_{10}} \tag{3.9}$$

Minimization of equation (3.9) gives :

$$\alpha'^3_{10} - \alpha\alpha'^2_{10} - \frac{3b}{2\mu^2} = 0 \tag{3.10}$$

The solution of (3.10) is the required value of α'_{10} which gives minimum $E_{10}(\alpha'_{10})$ and we denote it by $\overline{\alpha}'_{10}$.

Thus ,unperturbed wavefunction in VIPT is :

$$\psi_{10}^{(0)}\left(\overline{\alpha}'_{10}\right) = \frac{(\mu\overline{\alpha}'_{10})^{\frac{3}{2}}}{\sqrt{\pi}} e^{-\mu\overline{\alpha}'_{10}r} \tag{3.11}$$

Here $\overline{\alpha}'_{10}$ will be different for different mesons as solution of equation (3.10)depends on μ and α with $b = 0.183GeV^2$. We list the values of $\overline{\alpha}'_{10}$ in table 3.1 using known values of α_s under $\overline{MS}$ [12] and those in table 3.2 with α_s in V-scheme [13, 14, 15].

Table 3.1: Values of $\overline{\alpha}'_{10}$ for different mesons with α_s values under $\overline{MS}$ scheme .

Mesons	μ	α_s	$\alpha = \frac{4\alpha_s}{3}$	$\overline{\alpha}'_{10}$
D	0.2761	0.39	0.52	1.7271
D_s	0.3648	0.39	0.52	1.4642
B	0.31	0.22	0.2933	1.5104
B_s	0.44	0.22	0.2933	1.23
B_c	1.18	0.22	0.2933	0.6979

Table 3.2: Values of $\overline{\alpha}'_{10}$ for different mesons with α_s values under V-scheme .

Mesons	μ	α_s	$\alpha = \frac{4\alpha_s}{3}$	$\overline{\alpha}'_{10}$
D	0.2761	0.693	0.924	1.9105
D_s	0.3648	0.693	0.924	1.6593
B	0.31	0.261	0.348	1.531
B_s	0.44	0.261	0.348	1.2521
B_c	1.18	0.261	0.348	0.724

Now we consider the single k^{th} state in the summation of equation (3.4)which is the $2s$ state given by :

$$\begin{aligned} \psi_k^{(0)}\left(\overline{\alpha}'_{10}\right) &= \psi_{20}^{(0)}\left(\overline{\alpha}'_{10}\right) \\ &= \frac{\left(\mu\overline{\alpha}'_{10}\right)^{\frac{3}{2}}}{\sqrt{8\pi}} e^{-\frac{\mu\overline{\alpha}'_{10}r}{2}} \left(1 - \frac{\mu\overline{\alpha}'_{10}r}{2}\right) \end{aligned} \tag{3.12}$$

Therefore equation (3.4)gives wavefunction corrected upto first order as :

$$\psi_{10}\left(\overline{\alpha}'_{10}\right) = \psi_{10}^{(0)}\left(\overline{\alpha}'_{10}\right) + \frac{\int \psi_{20}^{(0)*}\left(\overline{\alpha}'_{10}\right) H'_{\overline{\alpha}'_{10}} \psi_{10}^{(0)}\left(\overline{\alpha}'_{10}\right) dv}{E_{10}^{(0)}\left(\overline{\alpha}'_{10}\right) - E_{20}^{(0)}\left(\overline{\alpha}'_{10}\right)} \psi_{20}^{(0)}\left(\overline{\alpha}'_{10}\right) \tag{3.13}$$

The energy eigenvalues are given by :

$$E_{n0}^{(0)}\left(\overline{\alpha}'_{10}\right) = -\frac{\mu\overline{\alpha}'^2_{10}}{2n^2} \tag{3.14}$$

The summation in equation (3.13)is dropped as we are considering single k^{th} state.Also, we have $n = 1$,and $n = 2$, due to the single state consideration in equation (3.4). Carrying out the integration in (3.13), we find that the wavefunction corrected upto the first order as :

$$\psi_{10}\left(\overline{\alpha}'_{10}\right)=\psi^{(0)}_{10}\left(\overline{\alpha}'_{10}\right)-\frac{4\sqrt{\mu}}{3\sqrt{\pi}\left(\overline{\alpha}'_{10}\right)^{\frac{1}{2}}}\left(\frac{4\mu\overline{\alpha}'_{10}\left(\alpha-\overline{\alpha}'_{10}\right)}{27}-\frac{32b}{81\mu\overline{\alpha}'_{10}}\right)\left(1-\frac{\mu\overline{\alpha}'_{10}r}{2}\right)e^{\frac{\mu\overline{\alpha}'_{10}r}{2}} \tag{3.15}$$

The relativistic version of (3.15)is [70, 71]:

$$\psi_{10,rel}\left(\overline{\alpha}'_{10}\right)=\psi_{10}\left(\overline{\alpha}'_{10}\right)\left[\left(r\mu\overline{\alpha}'_{10}\right)^{-\epsilon}\right] \tag{3.16}$$

with ϵ given by eq(2.5).

The expressions for I-W function, charge radius and convexity parameter with confinement only(which corresponds to wavefunction given by eq.3.15) are:

$$\xi_{S,conf}\left(y\right)=1-\rho^2_{S,conf}\left(y-1\right)+C_{S,conf}\left(y-1\right)^2+... \tag{3.17}$$

where the charge radius is :

$$\rho^2_{S,conf}=\frac{4\pi N_1^2}{\mu^3\overline{\alpha}'^5_{10}}\left[\frac{3c_1'^2}{4}+84A^2+\frac{1024c_1'A}{243}\right]. \tag{3.18}$$

and the convexity parameter is :

$$C_{S,conf}=\frac{4\pi N_1^2}{6\mu^3\overline{\alpha}'^7_{10}}\left[\frac{45c_1'^2}{8}+5760A^2+\frac{20\times 2^{12}c_1'A}{3^6}\right] \tag{3.19}$$

Here,

$$c_1'=\frac{\mu\overline{\alpha}'_{10}}{\pi^{\frac{1}{3}}} \tag{3.20}$$

and

$$A=\frac{4\sqrt{\mu}}{3\sqrt{\pi}\left(\overline{\alpha}'_{10}\right)^{\frac{1}{2}}}\left[\frac{4\mu\overline{\alpha}'_{10}\left(\alpha-\overline{\alpha}'_{10}\right)}{27}-\frac{32b}{81\mu\overline{\alpha}'_{10}}\right] \tag{3.21}$$

The subscript 'S' corresponds to the single term in the summation of equation(3.4). The normalization constant N_1 is given by:

$$4\pi N_1^2=\frac{1}{\left[\frac{c_1'^2}{4\mu^3\overline{\alpha}'^3_{10}}+\frac{2A^2}{\mu^3\overline{\alpha}'^3_{10}}\right]} \tag{3.22}$$

The respective relativistic versions are :

$$\xi_{S,rel+conf}(y) = 1 - \rho^2_{s,rel+conf}(y-1) + C_{s,rel+conf}(y-1)^2 + ... \tag{3.23}$$

with

$$\rho^2_{S,rel+con} = \frac{4\pi N_1'^2 \Gamma(3-2\epsilon)(4-2\epsilon)(3-2\epsilon)}{\mu^3 \overline{\alpha}'^5_{10}} \left[\frac{c_1'^2}{32} + X_1 + X_2\right] \tag{3.24}$$

and

$$C_{S,rel+conf} = \frac{4\pi N_1'^2 \Gamma(3-2\epsilon)(6-2\epsilon)(5-2\epsilon)(4-2\epsilon)(3-2\epsilon)}{6\mu^3 \overline{\alpha}'^7_{10}} \left[\frac{c_1'^2}{128} + X_3 + X_4\right] \tag{3.25}$$

Here the normalization constant N_1' is given by :

$$4\pi N_1'^2 = \frac{\mu^3 \overline{\alpha}'^3_{10}}{\Gamma(3-2\epsilon)\left[\frac{c_1'^2}{8} + X_5 + X_6\right]} \tag{3.26}$$

All the functions $X_i(\epsilon); i = 1, 2, ..6$ are defined in the Appendix C.

(ii)Two terms in the summation

In this step, we consider the $3s$ state ($n = 3, l = 0$) in addition to $2s$ state (as done in the single term case).The $3s$ state with the variational parameter $\overline{\alpha}'_{10}$ is written as :

$$\psi^{(0)}_{30}(\overline{\alpha}'_{10}) = \frac{(\mu\overline{\alpha}'_{10})^{\frac{3}{2}}}{\sqrt{27\pi}}\left(1 - \frac{2\mu\overline{\alpha}'_{10}r}{3} + \frac{2\mu^2\overline{\alpha}'^2_{10}r^2}{27}\right)e^{-\frac{\mu\overline{\alpha}'_{10}r}{3}} \tag{3.27}$$

With the inclusion of this state , the summation and integration in (3.4) gives the wavefunction corrected upto the first order as :

$$\psi_{10}(\overline{\alpha}'_{10}) = \psi^{(0)}_{10}(\overline{\alpha}'_{10}) - A\left(1 - \frac{\mu\overline{\alpha}'_{10}r}{2}\right)e^{\frac{\mu\overline{\alpha}'_{10}r}{2}} + B\left(1 - \frac{-2\mu\overline{\alpha}'_{10}r}{3} + \frac{2\mu^2\overline{\alpha}'^2_{10}r^2}{27}\right)e^{-\frac{\mu\overline{\alpha}'_{10}r}{3}} \tag{3.28}$$

where

$$B = \frac{\sqrt{\mu}}{\sqrt{\pi}\left(\overline{\alpha}'_{10}\right)^{\frac{1}{2}}}\left[\frac{3\mu\overline{\alpha}'_{10}\left(\alpha - \overline{\alpha}'_{10}\right)}{64} - \frac{27b}{256\mu\overline{\alpha}'_{10}}\right] \tag{3.29}$$

The relativistic version is obtained by multiplying (3.28) by $(r\mu\overline{\alpha}'_{10})^{-\epsilon}$. The I-W function ,charge radius and convexity parameter for the wavefunction (3.28) which is to be normalized are given by (i.e. with confinement only):

$$\xi_{D,conf}(y) = 1 - \rho^2_{D,conf}(y-1) + C_{D,conf}(y-1)^2 + ... \tag{3.30}$$

where the charge radius is :

$$\rho^2_{D,conf} = \frac{4\pi N_2^2}{\mu^3\overline{\alpha}'^5_{10}}\left[\frac{3c_1'^2}{4} + 84A^2 + \frac{1024c_1'A}{243} - \frac{3^4 \times 211 \times B^2}{4} + \frac{3^6 \times 39 \times c_1'B}{2^8} + \frac{6^6 \times 69 \times 16 \times AB}{3 \times 5^7}\right] \tag{3.31}$$

and the convexity parameter is :

$$C_{D,conf} = \frac{4\pi N_2^2}{6\mu^3\overline{\alpha}'^7_{10}}\left[\frac{45c_1'^2}{8} + 5760A^2 + \frac{20 \times 2^{12}c_1'A}{3^6} + 414163 \times B^2 + \frac{3^9 \times 185 \times c_1'B}{4^5} + \frac{6^9 \times 24608 \times AB}{3 \times 5^9}\right] \tag{3.32}$$

with normalizaion constant N_2 given by :

$$4\pi N_2^2 = \frac{1}{\left[\frac{c_1'^2}{4\mu^3\overline{\alpha}'^3_{10}} + \frac{2A^2}{\mu^3\overline{\alpha}'^3_{10}} + \frac{27B^2}{4\mu^3\overline{\alpha}'^3_{10}} + \frac{27c_1'B}{4\mu^3\overline{\alpha}'^3_{10}} - \frac{6^3 \times 492 \times AB}{5^5\mu^3\overline{\alpha}'^3_{10}}\right]} \tag{3.33}$$

The subscript 'D' corresponds to two terms in the summation. The respective relativistic versions of (3.30),(3.31) and (3.32) are :

$$\xi_{D,rel+conf}(y) = 1 - \rho^2_{D,rel+conf}(y-1) + C_{D,rel+conf}(y-1)^2 - ... \tag{3.34}$$

where

$$\rho^2_{D,rel+conf} = \frac{4\pi N_2'^2(4-2\epsilon)(3-2\epsilon)\Gamma(3-2\epsilon)}{\mu^3\overline{\alpha}'^5_{10}}\left[\frac{c_1'^2}{32} + X_1 + X_2 + \sum_{i=7}^{11} X_i\right] \tag{3.35}$$

and

$$C_{D,rel+conf} = \frac{4\pi N_2'^2 (6-2\epsilon)(5-2\epsilon)(4-2\epsilon)(3-2\epsilon)\Gamma(3-2\epsilon)}{6\mu^3\overline{\alpha}_{10}'^{\,7}} \times [\frac{c_1'^2}{128} + X_3 + X_4 + \sum_{i=12}^{16} X_i] \tag{3.36}$$

The normalization constant N_2' is given as :

$$4\pi N_2'^2 = \frac{\mu^3\overline{\alpha}_{10}'^{\,3}}{\Gamma(3-2\epsilon)[\frac{c_1'^2}{8} + X_5 + X_6 + \sum_{i=17}^{21} X_i]} \tag{3.37}$$

and $X_i(\epsilon); i = 7, 8, ..., 21$ are defined in the Appendix C.

(iii)With three terms in the summation

In addition to the $2s$ and $3s$ states we now add the $4s$ state :

$$\psi_{40}^{(0)}(\overline{\alpha}_{10}') = \frac{\left(\mu\overline{\alpha}_{10}'\right)^{\frac{3}{2}}}{\sqrt{2\pi}} \left(\frac{1}{4} - \frac{3\mu\overline{\alpha}_{10}'r}{16} + \frac{\mu^2\overline{\alpha}_{10}'^2r^2}{32} - \frac{\mu^3\overline{\alpha}_{10}'^3r^3}{8\times 96}\right) e^{-\frac{\mu\overline{\alpha}_{10}'r}{4}} \tag{3.38}$$

With the inclusion of this state , the first order wavefunction now becomes :

$$\psi_{10}(\overline{\alpha}_{10}') = \psi_{10}^{(0)}(\overline{\alpha}_{10}') - A\left(1 - \frac{\mu\overline{\alpha}_{10}'r}{2}\right) e^{-\frac{\mu\overline{\alpha}_{10}'r}{2}} + B\left(1 - \frac{2\mu\overline{\alpha}_{10}'r}{3} + \frac{2\mu^2\overline{\alpha}_{10}'^2r^2}{27}\right) e^{-\frac{\mu\overline{\alpha}_{10}'r}{3}} + D'\left(\frac{1}{4} - \frac{3\mu\overline{\alpha}_{10}'r}{16} + \frac{\mu^2\overline{\alpha}_{10}'^2r^2}{32} - \frac{\mu^3\overline{\alpha}_{10}'^3r^3}{8\times 96}\right) e^{-\frac{\mu\overline{\alpha}_{10}'r}{4}} \tag{3.39}$$

where

$$D' = \frac{(\mu\overline{\alpha}_{10}')^{\frac{3}{2}}}{\sqrt{\pi}} [\frac{36(\alpha - \overline{\alpha}_{10}')}{15625\overline{\alpha}_{10}'} - \frac{384b}{78125\mu^2\overline{\alpha}_{10}'^3}] \tag{3.40}$$

As usual , the relativistic version of this wavefunctin is obtained by multiplying above expression by $(r\mu\overline{\alpha}_{10}')^{-\epsilon}$.

Thus,with confinement only ,the I-W function is :

$$\xi_{T,conf}(y) = 1 - \rho_{T,conf}^2(y-1) + C_{T,conf}(y-1)^2 + ... \tag{3.41}$$

where charge radius is :

$$\rho^2_{T,conf} = \frac{4\pi N_3^2}{\mu^3\overline{\alpha}'^5_{10}}[\frac{\rho^2_{D,conf}\mu^3\overline{\alpha}'^5_{10}}{4\pi N_2^2} + 10368 \times D'^2 - 2.51 \times D'c'_1$$
$$-109.88 \times D'A - 2558.46 \times D'B] \qquad (3.42)$$

and convexity parameter is :

$$C_{T,conf} = \frac{4\pi N_3^2}{6\mu^3\overline{\alpha}'^7_{10}}[\frac{C_{D,conf}6\mu^3\overline{\alpha}'^7_{10}}{4\pi N_2^2} + 9123840 \times D'^2$$
$$-19.32 \times D'c'_1 - 3196.4 \times D'A - 183755.94 \times D'B] \qquad (3.43)$$

with

$$4\pi N_3^2 = \frac{1}{[\frac{c'^2_1}{4\mu^3\overline{\alpha}'^3_{10}} + \frac{2A^2}{\mu^3\overline{\alpha}'^3_{10}} + \frac{27B^2}{4\mu^3\overline{\alpha}'^3_{10}} + \frac{27c'_1B}{4\mu^3\overline{\alpha}'^3_{10}} - \frac{6^3\times 492\times AB}{5^5\mu^3\overline{\alpha}'^3_{10}} + \frac{16D'^2}{\mu^3\overline{\alpha}'^3_{10}}]} \qquad (3.44)$$

Here,the subscript 'T' refers to three terms in the summation.

The corresponding relativistic expresions are :

$$\xi_{T,rel+conf}(y) = 1 - \rho^2_{T,rel+conf}(y-1) + C_{T,rel+conf}(y-1)^2 + ... \qquad (3.45)$$

where

$$\rho^2_{T,rel+conf} = \frac{4\pi N'^2_3(4-2\epsilon)(3-2\epsilon)\Gamma(3-2\epsilon)}{\mu^3\overline{\alpha}'^5_{10}}[\frac{c'^2_1}{32} + \sum_{i=22}^{29} X_i] \qquad (3.46)$$

and

$$C_{T,rel+conf} = \frac{4\pi N'^2_3(6-2\epsilon)(5-2\epsilon)(4-2\epsilon)(3-2\epsilon)\Gamma(3-2\epsilon)}{6\mu^3\overline{\alpha}'^7_{10}}$$
$$\times[\frac{c'^2_1}{128} + \sum_{i=30}^{37} X_i] \qquad (3.47)$$

The normalization constant is given by :

$$4\pi N_3'^2 = \frac{\mu^3 \overline{\alpha}_{10}'^3}{\Gamma(3-2\epsilon)\left[\frac{c_1'^2}{8} + \sum_{i=38}^{45} X_i\right]} \tag{3.48}$$

and the functions $X_i(\epsilon); i = 22, 23, .., 45$ are defined in the Appendix C.

3.3 Calculation and Results

We have listed the values of charge radius and convexity parameter of the calculated I-W function for various heavy-light flavor mesons in the present method considering single state , two states and three states in the summation occurred in VIPT with confinement and relativistic effect.

In making the tables we have used two sets of α_s values : one under $\overline{MS}$- scheme [12] and the second under V-scheme [13, 14, 15] at 'c' and 'b' -quark mass scale so that we get two sets of readings for the same quantities .Table 3.3 represents the numerical values of the parameters c_1' ,A,B,D' given by equations (3.20),(3.21),(3.29)and(3.40) respectively with α_s under $\overline{MS}$- scheme;while table 3.4 represents those values with α_s values under V-scheme.Similarly,tables (3.5-3.7) give charge radius and convexity parameter for different combination of states with α_s values under $\overline{MS}$-scheme;whereas tables (3.8-3.10) give the same quantities with α_s values under V-scheme. The values of $\overline{\alpha}_{10}'$ are taken from the tables 3.1 and 3.2.

Correspondingly,the graphs which show the variation of I-W function $\xi(y)$ versus velocity transfer ratio 'y' consist of total two figures out of which the first one (i.e.fig.3.1) correspond to $\overline{MS}$ scheme and the last one (i.e. fig.3.2) to V-scheme.

Table 3.3: Various parameters with α_s values under $\overline{MS}$ scheme.

Mesons	α_s	c_1'	A	$B \times 10^{-2}$	$D' \times 10^{-4}$
D	0.39	0.33	-0.0712	0.304	5.055
D_s	0.39	0.37	-0.08	0.34	5.60
B	0.22	0.32	-0.082	0.35	5.77
B_s	0.22	0.37	-0.094	0.401	6.66
B_c	0.22	0.5625	-0.134	0.57	9.51

Table 3.4: Various parameters with α_s values in V-scheme.

Mesons	α_s	c_1'	A	$B \times 10^{-2}$	$D' \times 10^{-4}$
D	0.693	0.36	-0.0613	0.3166	4.345
D_s	0.693	0.42	-0.066	0.34	4.65
B	0.261	0.33	-0.08	0.41	5.67
B_s	0.22	0.376	-0.09	0.47	6.48
B_c	0.22	0.58	-0.0127	0.66	9.01

Table 3.5: Slope (charge radius) and curvature (convexity parameter) with single term in eq.(3.4) under $\overline{MS}$ scheme.

Mesons	$\rho^2_{s,conf}$	$C_{s,conf}$	$\rho^2_{s,rel+conf}$	$C_{s,rel+conf}$
D	3.73	13.92	2.197	5.61
D_s	5.06	26.18	2.53	10.54
B	5.83	29.08	4.132	18.72
B_s	9.49	71.48	6.30	34.44
B_c	25.54	592.1	18.1	379.7

Table 3.6: Slope (charge radius) and curvature (convexity parameter) with two terms in eq.(3.4) under $\overline{MS}$ scheme.

Mesons	$\rho^2_{D,conf}$	$C_{D,conf}$	$\rho^2_{D,rel+conf}$	$C_{D,rel+conf}$
D	2.84	9.37	1.83	5.184
D_s	3.9	17.72	2.50	9.776
B	4.14	18.55	3.72	14.92
B_s	6.56	44.47	5.66	34.31
B_c	18.64	385.32	16.55	305.23

Table 3.7: Slope(charge radius) and curvature(convexity parameter) with three terms in eq.(3.4)under $\overline{MS}$ scheme.

Mesons	$\rho^2_{T,conf}$	$C_{T,conf}$	$\rho^2_{T,rel+conf}$	$C_{T,rel+conf}$
D	2.83	9.15	1.80	5.04
D_s	3.88	17.28	2.46	9.45
B	4.13	18.1	3.68	14.53
B_s	6.45	43.37	5.59	33.4
B_c	18.55	375.91	16.35	298.31

Table 3.8: Slope(charge radius) and curvature(convexity parameter) with single term in eq.(3.4)under V-scheme.

Mesons	$\rho^2_{s,conf}$	$C_{s,conf}$	$\rho^2_{s,rel+conf}$	$C_{s,rel+conf}$
D	2.19	6.22	0.433	0.525
D_s	2.62	9.55	0.56	0.85
B	5.43	26.26	3.57	15.27
B_s	8.12	58.65	5.33	34.11
B_s	21.4	447.03	13.86	258.35

Table 3.9: Slope(charge radius) and curvature(convexity parameter) with two terms in eq.(3.4) under V-scheme.

Mesons	$\rho^2_{D,conf}$	$C_{D,conf}$	$\rho^2_{D,rel+conf}$	$C_{D,rel+conf}$
D	1.82	4.57	0.432	0.524
D_s	2.28	7.31	0.55	0.84
B	3.60	16.2	3.16	12.32
B_s	5.42	36.4	4.72	27.52
B_s	15.05	294.87	42.78	243.8

Table 3.10: Slope(charge radius) and curvature(convexity parameter) with three terms in eq.(3.4) under V-scheme.

Mesons	$\rho^2_{T,conf}$	$C_{T,conf}$	$\rho^2_{T,rel+conf}$	$C_{T,rel+conf}$
D	1.79	4.36	0.430	0.516
D_s	2.25	6.98	0.545	0.815
B	3.55	15.43	3.12	11.77
B_s	5.3	34.67	4.66	26.29
B_s	14.82	278.3	12.61	204.34

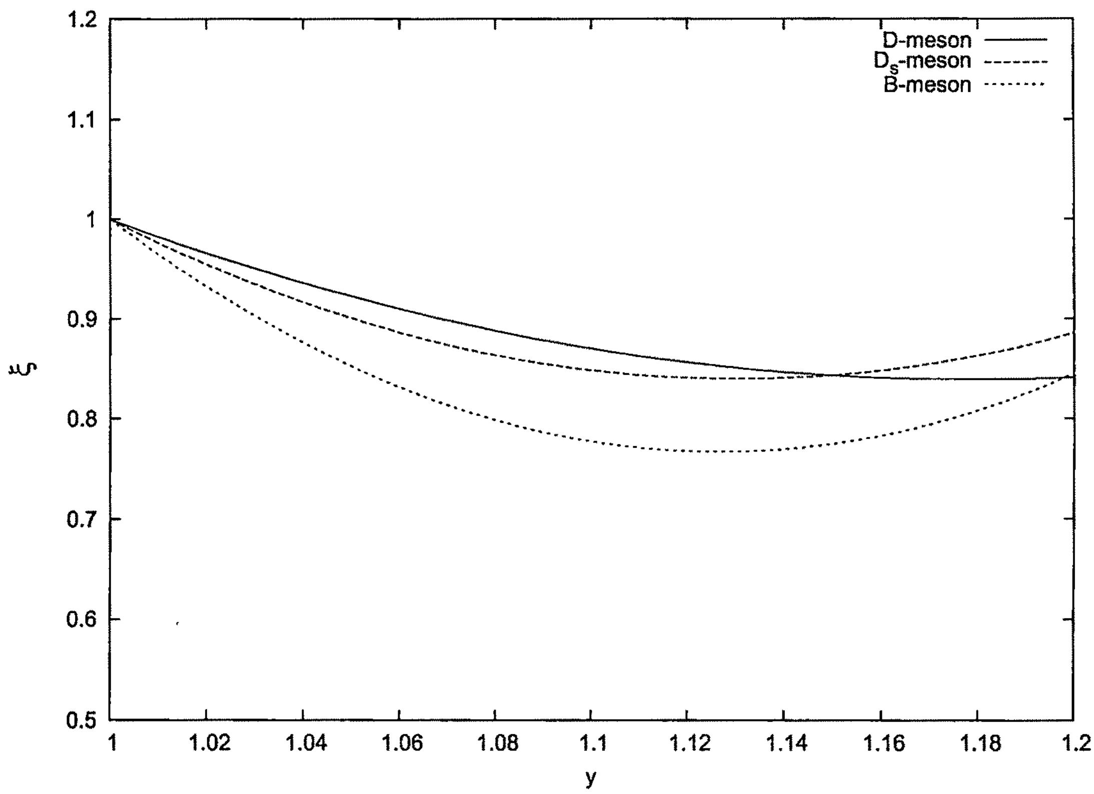

Figure 3.1: Variation of Isgur-Wise Function $\xi(y)$ vs velocity transfer ratio 'y' as given by eq.3.45 for $\overline{MS}$ scheme,(*cf.*Table-3.7).

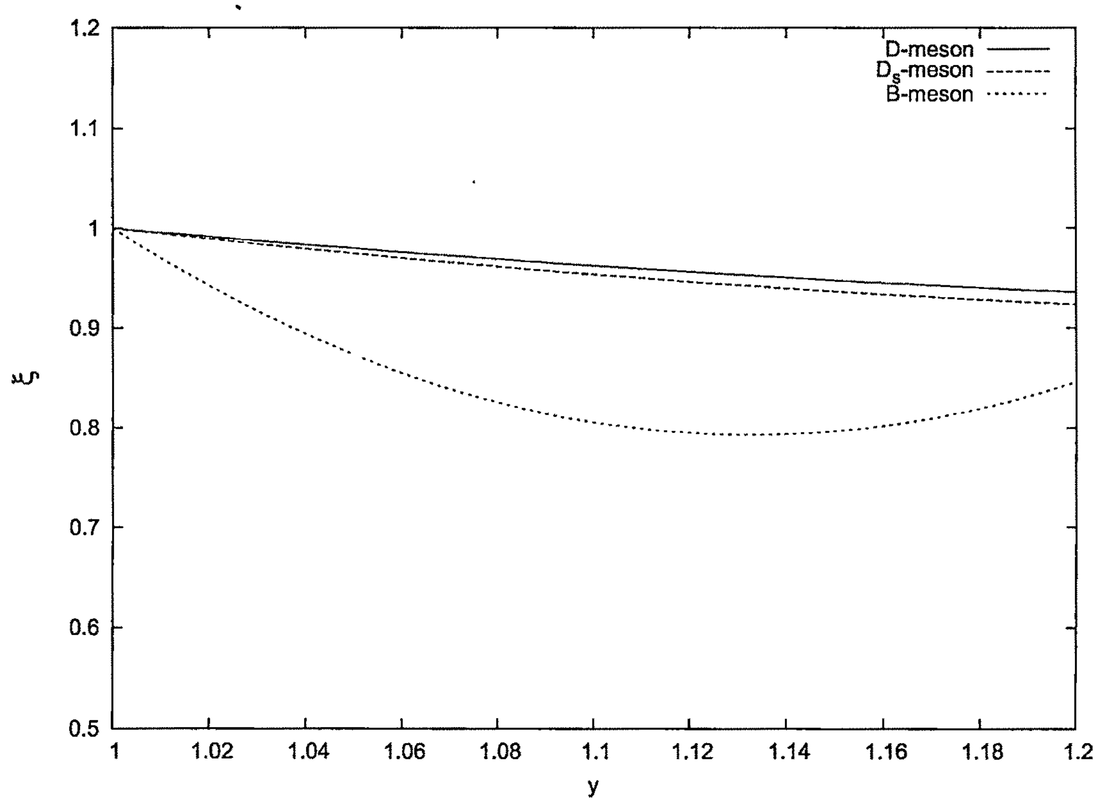

Figure 3.2: Variation of Isgur-Wise Function $\xi(y)$ vs 'y' as given by eq. (3.45) for V scheme,(*cf.*Table-3.10).

3.4 Discussion and Conclusion

In this chapter ,we have calculated the slope and curvature of the I-W function using VIPT method in the QCD inspired quark model [30,72-74].In this approach, we notice that with the inclusion of more states in the summation of equation(3.4), the results come closer to the predictions of the other models [53-68].Further, An analysis of the tables (3.5-3.10) indicates that relativistic effects invariably reduces the values of ρ^2 and C so as to bring them close to the predictions of other models.We have seen from the results that the slope and curvature agree quite well with the values and bounds of other models in table 2.3 for D and D_s mesons but not as expected for B, B_s, B_c mesons .This is due to the low value of α_s for the B sector mesons. Such feature was earlier noticed in ref.[74] too,suggesting the necesssity of higher order effects beyond O(α_s^3) in V-scheme.

We also note that the equations (3.18),(3.19),(3.24),(3.25),(3.31),(3.32),(3.35), (3.36),(3.42),(3.43),(3.46) and (3.47) along with (3.22),(3.26),(3.33),(3.37),(3.44) and (3.48) of the text contain several large numerical factors appearing to be divergent compared to the leading order term which is in contrary to the expectation of a perturbation theory.However, a careful study reveals that actually it is not so.

As an illustration, the correct leading order term in equation (3.18) with $b = 0$, $\overline{\alpha}'_{10} = \alpha$ becomes $\rho^2_{S,conf,LO} = \frac{3}{\alpha^2} = \frac{27}{16\alpha_s^2}$; which for $\alpha_s = 0.693$ is ~ 3.51 not far away from the results of table 3.8 .Similar analysis can be done for the other equations as well .

With the Coulombic parent, this approach shows unsatisfactory results for the B-sector mesons. Definitely, it will be interesting to explore if the linear potential as parent can improve the results of the present analysis as far as B-sector mesons are concerned.

Chapter 4

Isgur-Wise function in a QCD potential model with linear parent in Dalgarno method

4.1 Introduction

We have seen from the above analysis in chapters 2 and 3 that while calculating the slope and curvature of I-W function ,the Coulombic parent leads to unsatisfactory results for heavier mesons like B_s, B_c whether one uses Dalgarno method or VIPT [79, 81].It definitely paves the way for considering the linear potential as parent in the solution of Schrödinger equation using the approximation methods.

In this chapter,we calculate the slope and curvature of I-W function using the linear potential as parent with the help of Dalgarno method.The linear parent gives rise to the Airy functions as the unperturbed wave functions.The corrected wave function upto first order for the Coulombic part as perturbation can be calculated in the same way as for Coulombic parent in chapter 2.

This chapter includes the formalism as section 2,the results as section 3 and lastly the discussion and conclusion is the section 4.

4.2 Formalism

4.2.1 The wavefunction

In this case, for the Hamiltonian (eq.1.11) we have considered ,the perturbed Hamiltonian is :

$$H' = -\frac{4\alpha_s}{3r} + c \tag{4.1}$$

and the unperturbed Hamiltonian is:

$$H_o = -\frac{\nabla^2}{2\mu} + br \tag{4.2}$$

The constant 'c' at its natural scale is taken to be 1 GeV [74].

The unperturbed wave function coresponding to H_0 are the Airy functions which after normalization can be written as :

$$\psi_n^{(0)}(r) = \frac{N}{2\sqrt{\pi}r} Ai((2\mu b)^{\frac{1}{3}} + \rho_{0n}) \tag{4.3}$$

where ρ_{0n} s are the zeros of the Airy function $Ai(\rho_{0n}) = 0$, $n = 1, 2, 3..$ represent the principal quantum no.(of course for the ground state n=1) and N is the normalization constant.

The ρ_{0n} s are given as [4, 82]:

$$\rho_{0n} = -[\frac{3\pi(4n-1)}{8}]^{\frac{2}{3}} \tag{4.4}$$

The first order correction to wave function $\psi_n^{(1)}$ and energy $W_n^{(1)}$ are related as:

$$H_0\psi_n^{(1)} + H'\psi_n^0 = W_n^0\psi^1 + W_n^1\psi^0 \tag{4.5}$$

where W_n^0 is the unperturbed energy given as [4]

$$W_n^0 = E_n = -\left(\frac{b^2}{2\mu}\right)^{\frac{1}{3}} \rho_{0n} \tag{4.6}$$

and

$$W_n^{(1)} = \int_0^{+\infty} r^2 H' \left|\psi^{(0)}(r)\right|^2 dr \tag{4.7}$$

Since we consider the ground state ($n = 1$) , so we drop the 'n' from W_n^0,$W_n^{(1)}$, $\psi_n^{(0)}$ and ψ_n^1. The first order correction is :

$$\psi^1(r) = -\frac{4\alpha_s}{3}\left(\frac{a_1}{r} + a_2 + a_3 r\right) \tag{4.8}$$

As Airy function $Ai(r)$ involve infinite series in 'r', so in calculating the coefficients a_1,a_2 and a_3 we have considered upto order r^3 and they are given by :

$$a_1 = \frac{0.8808\,(b\mu)^{\frac{1}{3}}}{(E-c)} - \frac{a_2}{\mu\,(E-c)} + \frac{4W^1 \times 0.21005}{3\alpha_s\,(E-c)} \tag{4.9}$$

$$a_2 = \frac{ba_0}{(E-c)} + \frac{4 \times W^1 \times 0.8808 \times (b\mu)^{\frac{1}{3}}}{3\alpha_s\,(E-c)} - \frac{0.6535 \times (b\mu)^{\frac{2}{3}}}{(E-c)} \tag{4.10}$$

$$a_3 = \frac{4\mu W^1 \times 0.1183}{3\alpha_s} \tag{4.11}$$

The total wave function corrected upto first order with normalization is:

$$\psi_{coul}(r) = \psi^{(0)}(r) + \psi^{(1)}(r) \tag{4.12}$$

$$= \frac{N_1}{2\sqrt{\pi}}\left[\frac{Ai((2\mu b)^{\frac{1}{3}} + \rho_{01})}{r} - \frac{4\alpha_s}{3}\left(\frac{a_1}{r} + a_2 + a_3 r\right)\right] \tag{4.13}$$

where N_1 is the normalization constant for the total wave function $\psi_{coul}(r)$ with subscript 'coul' means Coulombic potential as perturbation.

The relativistic version of eq (4.13)is obtained by multiplying it with $\left(\frac{r}{a_0}\right)^{-\epsilon}$ i.e.

$$\psi_{rel}(r) = \psi_{coul}(r)\left(\frac{r}{a_0}\right)^{-\epsilon} \tag{4.14}$$

where a_0 is given by eq.(2.4) and ϵ is the relativistic factor given by (2.5).

4.3 Calculation and Results

We have calculated the values of charge radius and convexity parameter of the Isgur-Wise function given by eq.(2.20) for two set of coupling constants both in $\overline{MS}$[12] and V-scheme [13-15] to facilitate the comparison of our result with the previous work[73,74].

For these calculations,we have used the expressions for E, W^1 ,a_1,a_2,a_3 given by equations (4.6),(4.7),(4.9),(4.10),(4.11) respectively.These are shown in the table 4.1 and table 4.2.The result of ρ^2 and C in the present work is shown in table 4.3. We also compare the present result with that of previous work with linear as the perturbation [74] in V-scheme which was an improvement over $\overline{MS}$-scheme and is shown in table 4.3.

In evaluating the various integrations, we use numerical method of integration in mathematica software.

Table 4.1: The values of E and W^1 in GeV

Mesons	E	W^1	
		$\overline{MS}$ scheme	V scheme
D	0.3898	0.0467	0.08314
D_s	0.4291	0.5137	0.0915
B	0.4072	0.02742	0.0327
B_s	0.4553	0.0308	0.0366
B_c	0.6327	0.0451	0.051

Table 4.2: List of a_1,a_2 and a_3

Mesons	a_1		$a_2(GeV)$		$a_3(GeV^2)$	
	V scheme	$\overline{MS}$ scheme	V scheme	$\overline{MS}$ scheme	V scheme	$\overline{MS}$ scheme
D	0.2143	0.1943	-0.006138	-0.007877	0.00293	0.002933
D_s	0.238	0.21387	-0.00916	-0.01257	0.0043	0.004304
B	0.2245	0.2029	-0.00749	-0.0099	0.00349	0.00348
B_s	0.254	0.2269	-0.0114	-0.01604	0.005446	0.00547
B_c	0.38	0.3222	-0.0188	-0.023	0.02034	0.2035

Table 4.3: Values of slope(ρ^2) and curvature(C) in our work and its comparision to earlier work in this model for $c = 1GeV$.

Our work			
Scheme	Mesons	ρ^2	C
	D	0.7936	0.0008
$\overline{MS}$-scheme	D_s	1.186	0.002
	B	0.89	0.0004
	B_s	1.41	0.0012
	B_c	5.49	0.0322
	D	0.896	0.00306
V-scheme	D_s	1.352	0.0077
	B	0.912	0.0007
	B_s	1.421	0.00155
	B_c	5.67	0.065
Earlier work			
ref.[74]	D	1.136	5.377
	D_s	1.083	3.583
	B	128.13	5212
	B_s	112.759	4841
	B_c	44.479	2318

4.4 Discussion and Conclusion

Our calculated values of slope of I-W function in this work are found to be in good agreement with the other results (table 2.3). The lattice QCD evaluation of $\rho^2 = 0.83^{+15+24}_{-11-22}$ for B meson[66] and the experimental values of D meson $\rho_D^2 = 0.76 \pm 0.16 \pm 0.08$ [67] and $\rho_D^2 = 0.69 \pm 0.14$ [68] are also in good agreement with our calculated results.However, the values of curvature for each meson are found to be smaller in comparison to other predicted values. The reason may be presumably due to cut off the infinite series of $Ai(z)$ upto $O(r^3)$ as noted earlier.But, still such small values can be considered as a success particularly for the B sector mesons as these values were very large in case of Coulombic potential as parent[72-74].

This study of the Isgur-Wise function with Coulombic part as perturbation shows a different picture as compared to the earlier work [72-74] With linear part as perturbation done earlier, the slope and curvature decrease with the increase of α_s; while in this work , we have observed a reverse effect. Further,this analysis shows a great reduction in the values of ρ^2 and C for all the mesons as compared to the previous work with linear part as perturbation.

Let us conclude the section with a few comments.

The strong coupling constant entering the coulombic potential is a function of the momentum in full QCD.But in potential model,it is nothing but a mere parameter. However,here we have used the strong coupling constant in the $\overline{MS}$ and V-scheme to facilitate only a proper comparison with the previous work with linear part as perturbation[73,74].The comparison between the two schemes for all the work done so far [73,74,81] shows that the V-scheme is the preferable one.

Chapter 5

Isgur-Wise function in a QCD inspired potential model with confinement as parent in the Variationally Improved Perturbation Theory (VIPT)

5.1 Introduction

As noted in chapter 4 ,while using the Dalgarno method with linear potential as parent[83],the results for the slope and curvature of I-W function were quite satisfactory except for the B_c meson .So, to make further study with VIPT again for the linear parent is meaningful which will widen the applicability of VIPT at the same time.

A careful investigation shows that the linear part with significant confinement effect($b = 0.183GeV^2$) usually comes out to be dominant over the Coulombic one for mesons having greater reduced mass μ.Further, as pointed in ref.[4], the linear parent comes out to be quite handy in predicting the mass, energy etc for different states over the Coulombic piece.Under such circumstances,it will be definitely worthwhile to test the model with linear parent expecting success for the B_s, B_c mesons also which have greater reduced mass μ.

We recall that [4] for the linear potential to be dominant we require $< r > > r_0$,where $< r >$ is the expectation value of the distance r which reasonably gives the size of a state(in this case meson) and r_0 is a point at which linear cum Coulomb potential becomes zero(fig.1 of Aitchison and Dudek ,ref.[4]). The condition of applicability of VIPT to linear as parent conforms to low value of α_s and high value of b.This is because r_0 is directly proportional to α_s and inversely to b and we need a small r_0 for the linear potential to dominate.So, with linear parent, one can suitably handle large b and small α_s which is necessary in this QCD inspired potential model for the B-sector mesons (e.g. B, B_s, B_c) usually incorporated with small running coupling constant α_s due to their large mass.The linear parent is thus expected to be effective for heavier mesons.

Our approach is further is boosted by the success of the work [83] where we have used the Dalgarno method with linear parent for D, D_s, B, B_s mesons .

With this idea in mind,this chapter is devoted to the calculation of slope and curvature of I-W function using VIPT for the linear parent.

Section 2 is the formalism,section 3 is the result and section 4 contains the disscussion and conclusion.

5.2 Formalism

5.2.1 First order corrected wavefunction and energy in VIPT

The wavefunction corrected upto the first order of j^{th} state is given by eq.(3.4) and the energy corrected upto first order for the same state is given by eq.(3.5). We note that in this case, $P' = b'$ is the variational parameter related to the physical parameter b as in eq.(3.2) :

$$b = b + b' - b' \tag{5.1}$$

In this work also , we consider terms upto three states in the summation (3.4) as was done in ref.[81].

5.2.2 Wavefunctions using VIPT with linear potential as the parent

(i)With one term in the summation

As explained earlier , we take b' as the variational parameter instead of the physical parameter b in the parent linear potential to write the Hamiltonian as[4, 81]:

$$\begin{aligned} H &= H_o + H' \\ &= -\frac{\nabla^2}{2\mu} + br - \frac{4\alpha_s}{3r} + c \\ &= -\frac{\nabla^2}{2\mu} + br - \frac{\alpha}{r} + c \\ &= -\frac{\nabla^2}{2\mu} + b'r - \frac{\alpha}{r} - b'r + br + c \\ &= H_{ob'} + H'_{b'} \end{aligned} \tag{5.2}$$

Now ,$H_{ob'} = -\frac{\nabla^2}{2\mu} - b'r$ is the parent Hamiltonian with the new parameter b' and $H'_{b'} = \frac{\alpha}{r} - b'r + br + c$ is the perturbed Hamiltonian with the same variational parameter b' instead of the physical parameter b .

We consider j^{th} as $1s$ state ($n = 1$,$l = 0$) and in the summation of equation(3.4), we consider a single k^{th} state which is the $2s$ state ($n = 2$,$l = 0$).

We again note that in the variational method, we are interested only in the 'r' dependence of the Hamiltonian, and so 'c' in $H'_{b'}$ has no role to play in the calculation[1].

The unperturbed wavefunctions with linear parent with appropriate boundary conditions are the Airy functions given by [4]:

$$\psi_{n0}(r) = \frac{N_n}{2\sqrt{\pi}r} Ai\left((2\mu b')^{\frac{1}{3}} r + \rho_{on}\right) \tag{5.3}$$

where ρ_{0n} s are the zeros of the Airy function $Ai(\rho_{on}) = 0$ given by eq(4.4) and N_n is the normalization constant.Eq.(5.3) is identical to eq.(4.3) except to the replacement $b \rightarrow b'$.

As an illustration , we reproduce for s states a few of the zeros of the Airy function in table 5.1.

The corresponding energies are given as :

$$E_n = -\left(\frac{b'^2}{2\mu}\right)^{\frac{1}{3}} \rho_{0n} \tag{5.4}$$

Of course $n = 1, 2, 3, 4, ...$ is the principal quantum number.

Thus the trial 1s state $(n = 1, l = 0)$ wavefunction is (which is also the unperturbed wavefunction) :

$$\begin{aligned} \psi^{(0)} &= \psi_{10}^{(0)} \\ &= \frac{N_1}{2\sqrt{\pi}r} Ai\left((2\mu \bar{b}')^{\frac{1}{3}} r - 2.3194\right) \\ &= \frac{N_1}{2\sqrt{\pi}r} Ai(z_1) \end{aligned} \tag{5.5}$$

where

$$z_1 = \left((2\mu \bar{b}')^{\frac{1}{3}} r - 2.3194\right) \tag{5.6}$$

and the subscript 10 indicates the quantum number (n, l) of the j^{th} state.

We note that b' is to be replaced by $\overline{b}'$ which is obtained by minimizing E_j given by equation(3.5).It is essential since in VIPT we have to use the values of variational parameter leading to minimum energy (for example in ref.[81], α_s was replaced by $\overline{\alpha}'_{10}$). The values of $\overline{b}'$ for different mesons are listed in table 5.2.

Now we consider the single k^{th} state in the summation of equation (3.4)which is the $2s$ state given by :

$$\begin{aligned} \psi_{20}^{(0)} &= \frac{N_2}{2\sqrt{\pi}r} Ai\left((2\mu\overline{b}')^{\frac{1}{3}}r - 4.083\right) \\ &= \frac{N_2}{2\sqrt{\pi}r} Ai\left(z_2\right) \end{aligned} \tag{5.7}$$

where

$$z_2 = \left((2\mu\overline{b}')^{\frac{1}{3}}r - 4.083\right) \tag{5.8}$$

The wavefunction corrected upto first order is :

$$\psi_S = N\left[\psi^{(0)} + \frac{(2\mu)^{\frac{1}{3}}}{(\rho_{02} - \rho_{01})\,\overline{b}'^{\frac{2}{3}}}\left(\left(b - \overline{b}'\right) < r >_{2,1} - \alpha < \frac{1}{r} >_{2,1}\right)\psi_{20}\left(r\right)\right] \tag{5.9}$$

where

$$< r >_{2,1} = N_1 N_2 \int_0^{+\infty} r Ai\left((2\mu\overline{b}')^{\frac{1}{3}}r - 2.3194\right) Ai\left((2\mu\overline{b}')^{\frac{1}{3}}r - 4.083\right) dr \tag{5.10}$$

and N is the normalization constant.

(ii)With two terms in the summation

We next consider the $3s$ state ($n = 3, l = 0$) in addition to $2s$ state (as done in the single term case)given by :

$$\psi_{30}^{(0)} = \frac{N_3}{2\sqrt{\pi}r} Ai\left((2\mu\bar{b}')^{\frac{1}{3}} r - 5.5153\right) = \frac{N_3}{2\sqrt{\pi}r} Ai\left(z_3\right) \quad (5.11)$$

where

$$z_3 = \left((2\mu\bar{b}')^{\frac{1}{3}} r - 5.5153\right) \quad (5.12)$$

With the inclusion of this state , the wavefunction corrected upto the first order is :

$$\psi_D = N'[\psi^{(0)} + \frac{(2\mu)^{\frac{1}{3}}}{\left(\rho_{02} - \rho_{01}\right)\bar{b}'^{\frac{2}{3}}} \left(\left(b - \bar{b}'\right) < r >_{2,1} - \alpha < \frac{1}{r} >_{2,1}\right) \psi_{20}\left(r\right) + \frac{(2\mu)^{\frac{1}{3}}}{\left(\rho_{03} - \rho_{01}\right)\bar{b}'^{\frac{2}{3}}} \left(\left(b - \bar{b}'\right) < r >_{3,1} - \alpha < \frac{1}{r} >_{3,1}\right) \psi_{30}\left(r\right)] \quad (5.13)$$

where

$$< r >_{3,1} = N_1 N_3 \int_0^{+\infty} r Ai\left((2\mu\bar{b}')^{\frac{1}{3}} r - 2.3194\right) Ai\left((2\mu\bar{b}')^{\frac{1}{3}} r - 5.5153\right) dr \quad (5.14)$$

and N' is the normalization constant.

(iii)With three terms in the summation

In addition to the $2s$ and $3s$ states we now add the $4s$ state :

$$\psi_{40}^{(0)} = \frac{N_4}{2\sqrt{\pi}r} Ai\left((2\mu\bar{b}')^{\frac{1}{3}} r - 6.782\right) = \frac{N_4}{2\sqrt{\pi}r} Ai\left(z_4\right) \quad (5.15)$$

where

$$z_4 = \left((2\mu\overline{b}')^{\frac{1}{3}}r - 6.782\right) \tag{5.16}$$

With the inclusion of this state , the first order wavefunction now becomes :

$$\begin{aligned}\psi_T = N''[\psi^{(0)} + &\frac{(2\mu)^{\frac{1}{3}}}{(\rho_{02}-\rho_{01})\overline{b}'^{\frac{2}{3}}}\left(\left(b-\overline{b}'\right) < r >_{2,1} -\alpha < \frac{1}{r} >_{2,1}\right)\psi_{20}(r) + \\ &\frac{(2\mu)^{\frac{1}{3}}}{(\rho_{03}-\rho_{01})\overline{b}'^{\frac{2}{3}}}\left(\left(b-\overline{b}'\right) < r >_{3,1} -\alpha < \frac{1}{r} >_{3,1}\right)\psi_{30}(r) \\ &+\frac{(2\mu)^{\frac{1}{3}}}{(\rho_{04}-\rho_{01})\overline{b}'^{\frac{2}{3}}}\left(\left(b-\overline{b}'\right) < r >_{4,1} -\alpha < \frac{1}{r} >_{4,1}\right)\psi_{40}(r)]\end{aligned} \tag{5.17}$$

where

$$< r >_{4,1} = N_1 N_4 \int_0^{+\infty} r Ai\left((2\mu\overline{b}')^{\frac{1}{3}}r - 2.3194\right) Ai\left((2\mu\overline{b}')^{\frac{1}{3}}r - 6.782\right) dr \tag{5.18}$$

and N'' is the normalization constant.

The relativistic version of these wavefunctions are obtained in an analogous way by multiplying the above expression by $(r\mu\alpha)^{-\epsilon}$ [81]. Thus,relativistic version of all these wavefunctions is:

$$\psi_{i,rel} = \psi_i (r\mu\alpha)^{-\epsilon} \tag{5.19}$$

where $i = S, D, T$ and ϵ is the relativistic factor defined in eq.(2.5).Putting all these wavefunctions i.e. equations (5.9),(5.13),(5.17) and (5.19) in (2.20) we can calculate the Isgur-Wise function for the different cases.

5.3 Calculation and Results

We have listed the values of slope and curvature of the calculated I-W function for various heavy-light flavor mesons in the present method considering single state , two states and three states of eq.(3.4) with and without relativistic effect.

Table 5.1: A few of the zeros of Airy function for s states .

State	ρ_{0n}
$1s(n=1, l=0)$	-2.3194
$2s(n=2, l=0)$	-4.083
$3s(n=3, l=0)$	-5.5183
$4s(n=4, l=0)$	-6.782

Table 5.2: Values of $\overline{b}'$ with $b = 0.183 GeV^2$.

Mesons	Reduced mass μ	$\alpha = \frac{4\alpha_s}{3}$	$\overline{b}'$without relativistic effect	$\overline{b}'$with relativistic effect
D	0.2761	0.924	5.406	16.24
D_s	0.368248	0.924	5.876	19.8
B	0.31464	0.348	4.33	5.587
B_s	0.4401	0.348	4.497	5.954
B_c	1.1803	0.348	5.39	8.103

Table 5.1 gives the zeros of Airy function while table 5.2 gives the values of $\overline{b}'$. In tables (5.3-5.5), we record our predictions of slope and curvature for single term,two terms and three terms of eq.(3.4) respectively . Table 5.6 gives a comparison of VIPT and Dalgarno method for both the options .

The α_s values are taken from the V-scheme [13, 14, 15] and the integrations are done numerically for all these calculations.

Table 5.3: Values of slope (ρ^2) and curvature (C) with single term in equation(3.4).

Meson	ρ_S^2	C_S	$\rho_{S,rel}^2$	$C_{S,rel}$
D	1.36	0.01	0.53	0.0022
D_s	1.867	0.03	0.702	0.0036
B	1.93	0.02	1.41	0.013
B_s	2.923	0.046	2.113	0.0283
B_c	9.442	0.484	6.274	0.2522

Table 5.4: Values of slope (ρ^2) and curvature (C) with two terms in equation(3.4).

Meson	ρ_D^2	C_D	$\rho_{D,rel}^2$	$C_{D,rel}$
D	1.201	0.013	0.57	0.0026
D_s	2.001	0.0242	0.74	0.0041
B	2.004	0.0244	1.44	0.0133
B_s	3.031	0.0565	2.16	0.0297
B_c	10.2	0.61	6.51	0.275

Table 5.5: Values of slope (ρ^2) and curvature (C) with three terms in equation(3.4).

Meson	ρ_T^2	C_T	$\rho_{T,rel}^2$	$C_{T,rel}$
D	1.33	0.016	0.604	0.00326
D_s	2.023	0.0305	0.78	0.0054
B	2.027	0.031	1.54	0.0217
B_s	3.087	0.071	2.29	0.047
B_c	10.25	0.767	6.99	0.441

Table 5.6: Comparison of the values of slope ρ^2 and curvature C in VIPT and Dalgarno method for both the options .For comparison we take the best representative values of ρ^2 and C from the available data for D, D_s, B mesons.

VIPT					
		I. Linear Parent [this work]		II.Coulombic Parent [81]	
Terms considered in eq.(3.4)	meson	ρ_S^2	C_S	ρ_S^2	C_S
single term	D	0.53	0.0022	0.433	0.525
	D_s	0.702	0.0036	0.56	0.85
	B	1.41	0.0126	3.6	15.3
two terms	D	0.57	0.0026	0.432	0.524
	D_s	0.74	0.0041	0.55	0.84
	B	1.44	0.0133	3.16	12.32
three terms	D	0.604	0.0033	0.43	0.516
	D_s	0.78	0.0054	0.545	0.815
	B	1.54	0.0213	3.12	11.8
Dalgarno Method					
		I. Linear Parent [83]		II.Coulombic Parent [74]	
-	meson	ρ_S^2	C_S	ρ_S^2	C_S
-	D	0.896	0.0031	1.136	5.377
	D_s	1.352	0.0077	1.083	3.583
	B	1.41	0.013	128.13	5212

5.4 Discussion and Conclusion

This analysis with linear parent shows a complete different picture in comparison to that with Coulombic parent[81].With more terms in (3.4),the slope and curvature have increased in contrary to Coulombic parent.Also, an analysis of table 5.6 indicates that for a definite term,the slope has assumed larger values than those of ref.[81] while for the curvature,the pattern is reversed i.e.it has assumed smaller values than those of ref.[81].

Regarding the number of terms considered in the summation (3.4),we have seen that the most satisfactory and comparable result is for the single term consideration.This is undoubtedably a great phenomenological advantage as involvement of more terms in equation(3.4) makes the calculation quite cumbersome which happened in ref.[81].However, relativistic correction in this case also decrease the slope and curvature of Isgur-Wise function as observed earlier [81].If we look back at our Dalgarno method approach with linear parent in chapter 4, we have observed larger values of slope and curvature for D, D_s mesons while smaller values for B, B_s, B_c mesons in this work over those in that work .

To conclude , the present approach based on VIPT for the calculation of I-W function within the QCD inspired potential model appears to be preferable over the one of ref.[81] where the linear potential was considered as perturbation.

Chapter 6

Form factors and charge radii in a QCD inspired potential model using the Variationally Improved Perturbation Theory

6.1 Introduction

We recall that the reliability and effectiveness of a QCD inspired model is determined by the standard factors like I-W function,elastic form factors,charge radii etc, the basic ingredient of which is the wavefunction .So far, we have determined the wave functions using approximation methods like Dalgarno method,VIPT for both linear and Coulombic parent and use them in the calculation of I-W function.We,in this chapter , report the calculation of elastic form factors and charge radii with the same wavefunctions obtained from VIPT which were used in calculating the I-W function.

It is well known that the elastic form factor and charge radius are dependent on the momentum transform of the wavefunction.So,getting an appropriate wave-function is very essential for a fruitful analysis.With the success of VIPT in the calculation of Isgur-Wise function as pointed above [81,84],one can expect a similar success here also .It is worthwhile to note that while investigating the form factor ,one must take into account of a proper range of four momentum transfer Q^2.The Q^2 range usually determines the applicability of perturbative QCD(pQCD) or nonperturbative(npQCD).So,an accurate selection of Q^2 range within the nonperturbative approach is necessary which will also fall within the experimental regime.This facilitates a direct comparison between theory and experiment.This has been done both theoretically and experimentally since long [85-90] for the light π, K etc mesons.However,for the mesons which contain at least one heavy quark,very little have been investigated theoretically [9-11].In the absence of any experimental data for them,our results may be helpful in future in the experimental set up regarding the Q^2 range .

As far as our model is concerned, the perturbative or nonperturbative regime of QCD can be interpreted through the relativistic factor '$\epsilon = 1 - \sqrt{1 - \frac{4\alpha_s}{3}}$ ' [30]. The reality constraint on the form factor $F(Q^2)$ leads to the condition $0 < \epsilon < 1$, where the case $\epsilon \rightarrow 0$ ($\epsilon \rightarrow 1$) corresponds to the perturbative (nonperturbative) limit of QCD .The $\epsilon \rightarrow 1$ limit demands large α_s or low Q^2.So, discussing the nonperturbative effects of QCD with large confinement parameter b ,we must consider the low Q^2 limit of α_s in this model .However, we have observed in ref.[30] that large value of $b(= 0.183GeV^2)$ prohibits the use of low Q^2 compelling one to involve with small α_s which corresponds to the perturbative regime and thus can't be accounted in this nonperturbative approach.

We reanalyze all these observations in this approach of VIPT for both the scenarios -linear or confinement part as perturbation and Coulombic part as perturbation.We will explore the possibility of incorporating significant value of α_s even with large confinement ($b = 0.183GeV^2$). This work will also check the status of both confinement and Coulombic part as perurbation and observe the consequences regarding the usable range of Q^2 to work , in the absence of experimental data for the said mesons .The calculations are done with a fixed value of α_s from V-scheme [13-15] with large confinement effect $b = 0.183 GeV^2$ instead of variation in both.Even with this single value of α_s and b one can draw similar conclusion regarding the effective range of Q^2. The calculated form factors are plotted graphically to show their variation with Q^2 for both the scenarios.

Basically, this work explores the possibility of improving the results for form factors and charge radii over those of ref.[11,30] with the help of VIPT.In the process, we also try to find a useful range of Q^2 which may be workable for the experimental investigations in the later course of time.Comparison of both the options is being made to arrive at a conclusion in using VIPT.

The rest of the chapter is organized as follows : section 2 contains the formalism, section 3 the result and calculation while section 4 includes the discussion and conclusion.

6.2 Formalism

6.2.1 VIPT with Coulombic potential as parent

(i)Wave function

The physical parameter in this scenario is α_s and the variational parameter is α'_s (eq.3.2).As stated above,we use the wave functions obtained earlier which are given by equations (3.15),(3.28),(3.39) respectively for single,double,triple term consideration of eq.(3.4). The wavefunctions for single and double term consideration can be obtained by putting $B = D' = 0$ and $D' = 0$ respectively in the equation (3.39).

However we will consider the relativistic version ($\epsilon \neq 0$) of the above wave function obtained by multiplying the nonrelativistic version by $(r\mu\bar{\alpha}'_{10})^{-\epsilon}$ [for example eq.(3.16) of ref.81].

(ii)The elastic charge form factor and charge radii

The form factor can be expressed as [91]:

$$eF\left(Q^2\right) = \sum \frac{e_i}{Q_i} \int_0^{+\infty} r \left|\psi_{T,Rel}\left(r\right)\right|^2 \sin Q_i r dr \tag{6.1}$$

where

$$Q_i = \frac{\sum_{j\neq i} m_j Q}{\sum m_i} \tag{6.2}$$

and we have used the relativistic wave function for three term consideration $\psi_{T,Rel}$.

Putting the relativistic wave function $\psi_{T,Rel}$(relativistic version of eq.(3.39)) in (6.1) we get the form factor as :

$$eF\left(Q^2\right) = \sum e_i N_3'^2 \Gamma(3-2\epsilon)\left[q_1+q_2+q_3+q_4+q_5+q_6+q_7+q_8+q_9+q_{10}\right] \tag{6.3}$$

where $N_3'^2$ is the same normalization constant as appeared in equation(54)of ref.[81] and the different $q_i(Q_i)$ s$(i = 1, 2, ..., 10)$ are defined in the Appendix D.
The charge radius is derived as [30] :

$$\begin{aligned} < r^2 > &= -\frac{d\left(eF\left(Q^2\right)\right)}{dQ^2}\Big|_{Q^2=0} \quad (6.4)\\ &= N_3'^2\Gamma(3-2\epsilon)[r_1+r_2+r_3+r_4+r_5+r_6+r_7+r_8 \\ &+ r_9+r_{10}] \quad (6.5)\end{aligned}$$

where the different r_i s $(i = 1, 2, ..., 10)$ are defined in the Appendix D.

(iii)Status of linear potential as perturbation

The momentum transform of $\psi_{T,Rel}$ is [92, 93]:

$$\begin{aligned}\psi_{T,Rel}\left(Q^2\right) &= \sum \frac{e_i}{Q_i}\sqrt{\frac{2}{\pi}} \int_0^{+\infty} r\psi_{T,Rel}\left(r\right) sinQ_i r dr \quad (6.6)\\ &= \sum e_i N_3' \sqrt{\frac{2}{\pi}}\Gamma(3-2\epsilon)\left[p_1 - p_2 + p_3 + p_4\right] \quad (6.7)\end{aligned}$$

The p_i s which depend on Q_i^2, ϵ etc are given in the Appendix D.

If linear potential is treated as perturbation then from equation(6.7) above the following inequality must be preserved:

$$p_1 > p_2 - p_3 - p_4 \tag{6.8}$$

This inequality leads to a lower limit of Q^2 namely Q_0^2 [30] above which one have to use the values of Q^2.The Q_0^2 is determined from the condition:

$$p_1 = p_2 - p_3 - p_4 \tag{6.9}$$

Due to the quark mass dependence ,Q_0^2 s have different values and they are shown in table 6.2.In the Dalgarno method approach [30], the lower limits Q_0^2 were large and the formalism failed to account for large confinement effect ($b = 0.183GeV^2$)in the nonperturbative QCD regime where α_s values were taken to be large.Only in the limit $b \to 0$,the Q_0^2 values were lowered and the formalism worked for low Q^2 range [30].In this method of VIPT,the values of Q_0^2 are shown to be quite small even with large confinement effect ($b = 0.183GeV^2$) enabling us to work in the nonperturbative QCD regime with large α_s.

We also note that for single term consideration only p_2 exists on the RHS of the inequality (6.8) and for double term both p_2 and p_3 exist.We have also recorded the values of Q_0^2 for single and double term consideration in table 6.2 .

6.2.2 VIPT with linear potential as parent

(i) Wavefunction

As pointed in ref.[4,83,84],the linear parent gives rise to Airy functions.The physical parameter is b and the optimized variational parameter is $\overline{b'}$ .We use the wavefunctions for single,double,triple term consideration obtained earlier which are given by equations (5.9),(5.13),(5.17) respectively in the same way as was done for the

Coulombic parent above.

We note that for single (double) term consideration of equation(3.4) the third and fourth term (fourth term) is dropped from the equation (5.17) and normalization constants also changes to different one (eq.17 and 21 of ref. [84]).

For this case also we take the relativistic version of these wavefunctions obtained by multiplying the nonrelativistic version by $(r\mu\overline{a}'_{10})^{-\epsilon}$.

Like the expressions we have adopted the same values of $b, \overline{b}', \rho_{0n}$ as given in chapter 5(tables 5.1,5.2).

(ii)The elastic charge form factor and charge radii

For the relativistic version ($\psi_{T,Rel}$) of the wave function ψ_T given by eq.(5.17) considered above, the form factor is found to be :

$$eF\left(Q^2\right) = \sum e_i N''^2 \left[C - C'\frac{Q_i^2}{6}\right] \tag{6.10}$$

The coefficients C, C's are given in table 6.3 and N'' is the normalization constant.They are of course different for single,double or more than two term consideration in eq.(3.4).Numerical integrations are done in getting the above result. The corresponding charge radius is obtained by using eq.(6.4) which are recorded in table 6.3.

(iii)Status of Coulombic potential as perturbation

The momentum transform of $\psi_{T,Rel}$ is:

$$\psi_{T,Rel}\left(Q^2\right) = \sum \frac{e_i}{Q_i}\sqrt{\frac{2}{\pi}}\int_0^{+\infty} r\psi_{T,Rel}(r)\, sinQ_i r dr \tag{6.11}$$

$$= \sum e_i N'_3 \sqrt{\frac{2}{\pi}}\Gamma(3-2\epsilon)[p'_1 + p'_2 + p'_3 + p'_4 - \frac{Q_i^2}{6}(p'_5 + p'_6 + p'_7 + p'_8)] \tag{6.12}$$

The p_i' s $(i = 1, 2, .., 8)$ are given in the Appendix D.

If Coulombic potential is treated as perturbation then from equation(6.12) above the following inequality must be preserved:

$$p_1' + p_2' + p_3' + p_4' > \frac{Q_i^2}{6} (p_5' + p_6' + p_7' + p_8') \tag{6.13}$$

This inequality leads to a upper limit of Q^2 namely Q_0^2 below which one have to use the values of Q^2.The Q_0^2 is determined from the condition:

$$p_1' + p_2' + p_3' + p_4' = \frac{Q_i^2}{6} (p_5' + p_6' + p_7' + p_8') \tag{6.14}$$

The different values of upper limit Q_0^2 s are shown in table 6.4.The corresponding values for single and double term consideration are also shown.

6.3 Calculation and Results

In table 6.1 , we record the charge radii for single,double and triple terms of eq.(3.4) for Coulombic potential as parent; whereas the same is recorded for linear potential as parent in table 6.3.We have also listed the lower and upper limit of Q_0^2 for single,double and triple term consideration in tables 6.2 and 6.4.The infinite mass limit shown by the subscript ∞ is also included for triple (single) term consideration for Coulombic (linear) parent.The table 6.5 shows charge radii of different mesons obtained from other models and data.

The α_s values are taken from the V-scheme [13, 14, 15] and the integrations are done numerically for all these calculations.

Table 6.1: Values of charge radii for different mesons with Coulombic parent for single,double and triple terms in eq.(3.4).The subscripts 'S, D, T' correspond to single,double and triple terms respectively whereas 'F' means finite mass consideration.The infinite mass limit (subscript ∞ is used) is shown for the triple term alone.

Meson	D^0	D^+	D_s^+	B^+	B^0	B_s^0	B_c^+
$< r_{S,F}^2 >^{\frac{1}{2}}$ $in fm$	-0.121	0.115	0.11	0.2545	-0.1822	-0.168	0.108
$< r_{D,F}^2 >^{\frac{1}{2}}$ $in fm$	-0.119	0.112	0.106	0.2512	-0.1788	-0.164	0.105
$< r_{T,F}^2 >^{\frac{1}{2}}$ $in fm$	-0.118	0.109	0.101	0.2464	-0.1736	-0.158	0.1034
$< r_{T,\infty}^2 >^{\frac{1}{2}}$ $in fm$	-0.131	0.12	0.113	0.263	-0.186	-0.1742	0.1325

Table 6.2: Values of lower limit of four momentum transfer Q_0^2 with Coulombic parent taking single,double and triple terms in eq.(3.4).We have to use Q^2 values above these.

Meson	D^+	D^-	D_s^+	B^+	B^0	B_s^0	B_c^+
$Q_{0,S}^2$	0.0004	0.0004	0.001	0.053	0.053	0.075	0.211
$Q_{0,D}^2$	0.00036	0.00036	0.0009	0.052	0.052	0.072	0.209
$Q_{0,T}^2$	0.0003	0.0003	0.0007	0.05	0.05	0.07	0.205

The graphs show the variation $eF(Q^2)$ vs Q^2 for D, D_s and B_c mesons for both the options.

Table 6.3: Values of charge radii for different mesons with linear parent for single,double and triple terms in eq.(3.4) . The subscripts 'S, D, T' correspond to single,double and triple terms respectively whereas 'F' means finite mass consideration .The infinite mass limit (subscript ∞ is used) is shown for the single term alone.

Meson	D^0	D^+	D_s^+	B^+	B^0	B_s^0	B_c^+
C_S	8.24	8.24	5.916	14.22	14.22	10.91	4.50
C'_S	2.266	2.266	1.173	7.71	7.71	4.52	0.78
$< r^2_{S,F} >^{\frac{1}{2}}$ $infm$	-0.197	0.1494	0.104	0.425	-0.2996	-0.2227	0.1125
C_D	13.1	13.1	8.1	26.7	26.7	16.5	5.2
C'_D	2.69	2.69	1.67	9.89	9.89	5.7	1.1
$< r^2_{D,F} >^{\frac{1}{2}}$ $infm$	-0.21	0.161	0.121	0.473	-0.336	-0.2489	0.127
C_T	18.14	18.14	11.18	43.09	43.09	25.56	6.636
C'_T	3.365	3.365	2.199	13.14	13.14	7.655	1.363
$< r^2_{T,F} >^{\frac{1}{2}}$ $infm$	-0.24	0.182	0.143	0.555	-0.391	-0.289	0.148
$< r^2_{S,\infty} >^{\frac{1}{2}}$ $infm$	-0.246	0.174	0.125	0.453	-0.32	-0.246	0.144

Table 6.4: Values of upper limit of four momentum transfer Q_0^2 with linear parent taking single,double and triple terms in eq.(3.4).We have to use Q^2 values lower than these.

Meson	D^+	D^-	D_s^+	B^+	B^0	B_s^0	B_c^+
$Q^2_{0,S}$	2.297	2.297	2.92	1.43	1.43	1.676	3.11
$Q^2_{0,D}$	2.1	2.1	2.67	1.31	1.31	1.56	2.89
$Q^2_{0,T}$	1.88	1.88	2.387	1.177	1.177	1.386	2.55

Table 6.5: Prediction of $< r^2 >^{\frac{1}{2}}$ $infm$ for finite and infinite mass consideration in other models.The subscript 'F' ('∞') means finite (infinite) mass limit.

Meson	D^0	D^+	D_s^+	B^+/B^-	B^0	$B_s^0/\overline{B_s^0}$	B_c^+/B_c^-
$< r_F^2 >^{\frac{1}{2}}$[9]	...	0.506	0.491	0.258(B^-)	...	0.256($\overline{B_s^0}$)	0.236(B_c^-)
$< r_F^2 >^{\frac{1}{2}}$[10]	-0.551	0.43	0.352	0.612(B^+)	-0.432	-0.345(B_s^0)	0.207(B_c^+)
$< r_\infty^2 >^{\frac{1}{2}}$[10]	-0.704	0.498	0.425	0.704(B^+)	-0.498	-0.425(B_s^0)	...(B_c^+)
$< r_F^2 >^{\frac{1}{2}}$[11]	-0.484	0.366	0.355	1.72(B^+)	-1.21	-1.17(B_s^0)	1.43(B_c^+)
$< r_\infty^2 >^{\frac{1}{2}}$[11]	-0.6025	0.427	0.427	1.836(B^+)	-1.29	-1.29(B_s^0)	1.84(B_c^+)

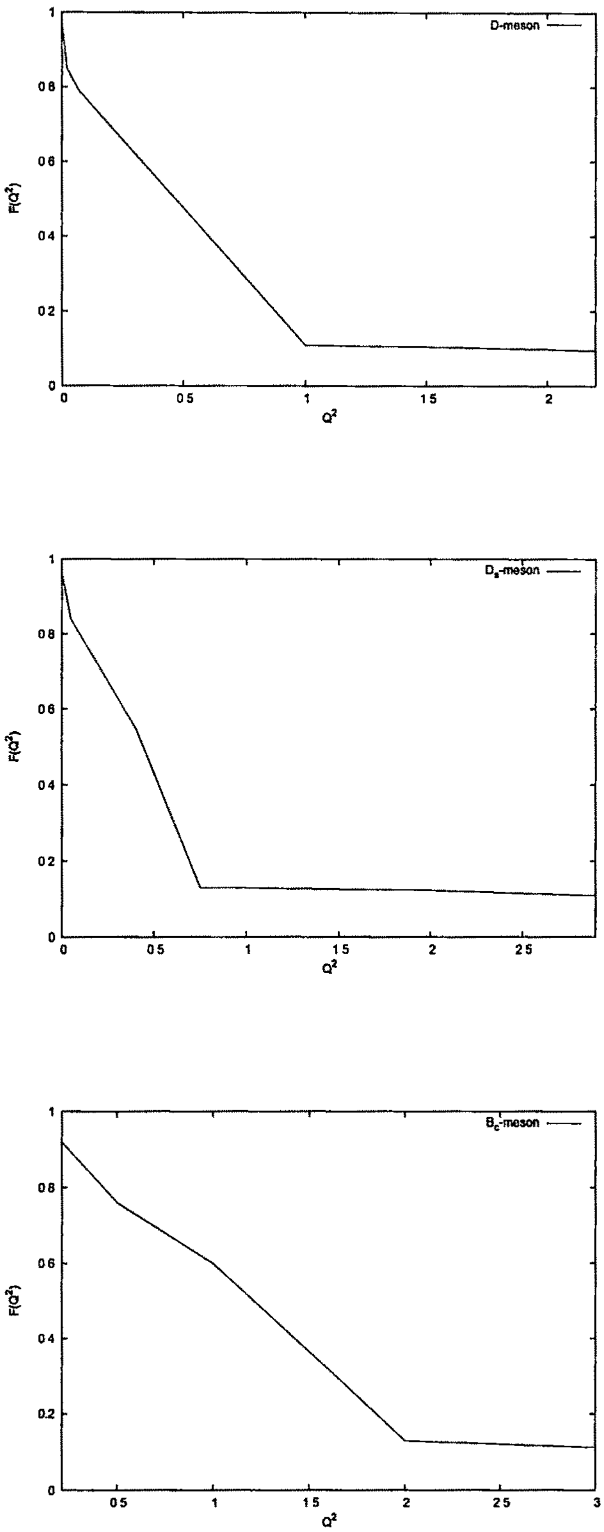

Figure 6.1: Variation of $eF(Q^2)$ vs Q^2 for D ,D_s and B_c-meson with Coulombic parent .

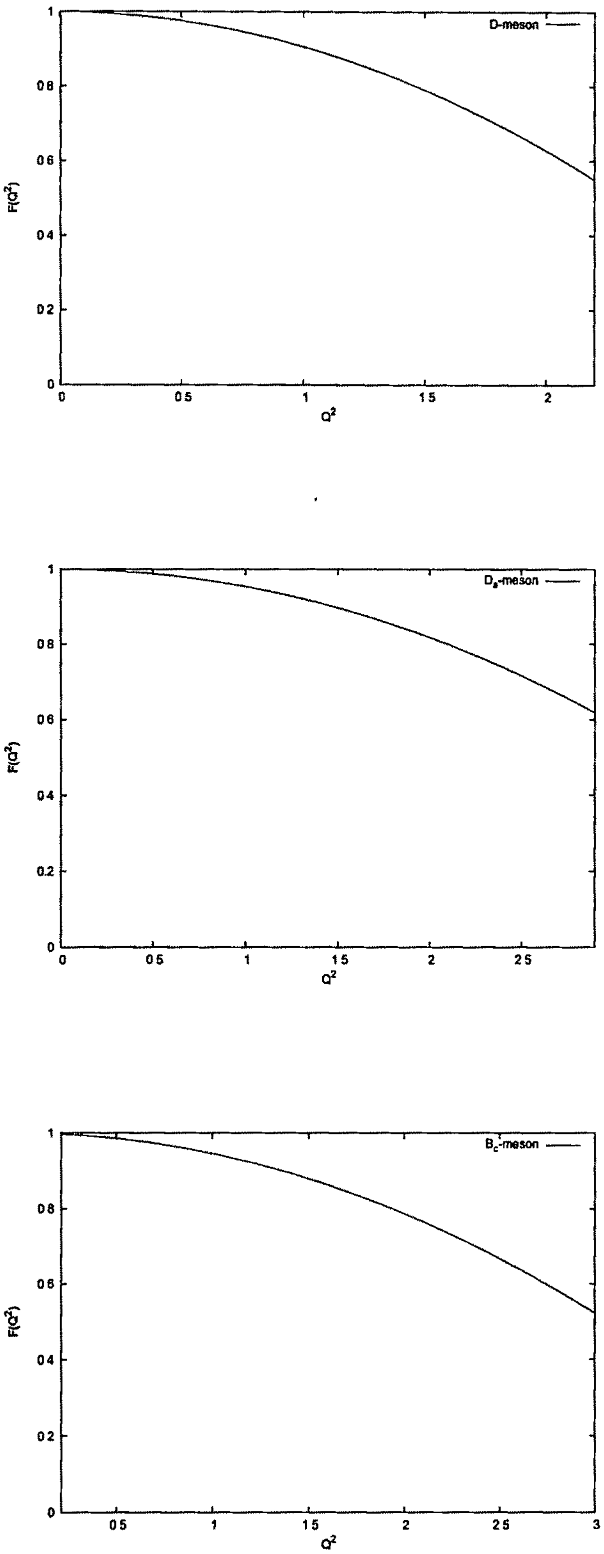

Figure 6.2: Variation of $eF(Q^2)$ vs Q^2 for D ,D_s and B_c-meson with linear parent .

6.4 Discussion and Conclusion

We have analyzed elastic form factors and charge radii in a QCD inspired potential model using VIPT under two scenarios- Coulombic and linear potential as parent . We summarize our achievements below:

I. The form factor $eF(Q^2)$ decreases with the increase of Q^2 (as it should) for both the scenarios .

II. The form factor is either very small (for D-sector mesons)or small (for B-sector mesons)with Coulombic parent as compared to those with linear parent.

The charge radii is also observed to be smaller with Coulombic parent as compared to linear parent.

III. We use a fixed set of values for α_s under V-scheme[13-15] in the calculation , for example it is 0.693 for the D, D_s mesons which is larger than the value 0.261 for the B, B_s, B_c mesons.This consideration directly results in the smaller values of charge radii for D, D_s mesons as compared to the B, B_s, B_c mesons.Larger α_s values are responsible for smaller charge radius.

IV. While checking the status of confinement as perturbation i.e. Coulombic parent (or Coulombic part as perturbation i.e. linear parent) we end up with a lower (or upper) limit on Q^2 .This allows us a useful range of Q^2 to show the variation of form factor which is shown in fig.6.1 and fig.6.2.

V. In the present analysis, even with large b,the lower limit of Q^2(for linear perturbation) are really small as shown in table 6.3 for fixed α_s.We have seen that for $\alpha_s = 0.693$, the lower limit of Q^2 for D, D_s mesons are respectively $0.0003, 0.0007$, whereas with $\alpha_s = 0.261$,the lower limit of Q^2 for B, B_s, B_c mesons are respectively $0.05, 0.07, 0.205$ .These values for B, B_s, B_c mesons will be lowered if we put $\alpha_s > 0.261$. This is clearly advantageous over the Dalgarno method with linear perturbation as done in ref.[30] where the formalism broke down for large b.Thus, this approach allows a large value of $\alpha_s(Q^2)$ in the limit $Q^2 \rightarrow 0$ even with large confinement , an important feature absent in ref.[30].

VI. The Coulombic perturbation leads to an upper limit of Q^2 instead of a lower limit(table 6.4). This allows us to use any value of $\alpha_s(Q^2)$ in the limit $Q^2 \to 0$.

VII. Further, consideration of different terms in eq.(3.4) leads to different charge radii and the limiting values of Q^2 for both the cases. The charge radii and the lower limit of Q^2 decrease with more terms for the linear part as perturbation (i.e. Coulombic parent) whereas the charge radii increase and upper limit of Q^2 decreases for the Coulombic part as perturbation (i.e. linear parent).

VIII. The infinite mass consideration in this work shows that the charge radii are larger than those for finite mass consideration to agree well with other models (table 6.5).

The above list as a whole suggests success of VIPT over the Dalgarno method [11, 30] as far as large confinement and limiting values of Q^2 are concerned. Further,the difference in the values of form factors and charge radii for both the scenarios may be attributed to the use of same α_s (i.e. Q^2) under V-scheme as the Coulombic potential is dominant for large Q^2 (i.e.low r)and the linear potential in the low Q^2 (i.e. large r) regime.We must note that we have used the low Q^2 (like $sinQ_i r \simeq Q_i r - \frac{Q_i^3 r^3}{6}$) assumption[30] in the calculation of form factors and this clearly effects the upper limit of Q^2 corresponding to the validity of Coulombic perturbation (i.e. linear parent). The larger value of α_s for D-sector as compared to B-sector is also another point to take account of this .Although, the linear parent has shown more flexibility and hence is the better option than Coulombic parent in VIPT,but it has used terms up to a particular order in 'r' in the integration involved with Airy function (which is an infinite series).This may lead to the loss of certain information as far as physics is concerned. In the absence of any experimental results for these mesons , it is quite difficult to make a direct conclusion but there is clear indication that one must be careful in choosing the parameter $\alpha_s(Q^2)$ as well as the confinement parameter in the calculation of form factor and charge radius within the QCD framework.

The above discussion led to the conclusion that there is scope to use this approach in the study of meson decays.The lower and upper limit on Q^2 (i.e. range of Q^2) in this analysis may be useful in the experimental set up to investigate cross-section and form factor in future for these mesons.Further, from the model specific values of form factors and charge radii, this method allows to investigate the behaviour of α_s w.r.t Q^2 in the nonperturbative regime of QCD.

Chapter 7

Summary,conclusion and future outlook

This work has dealt with the nonperturbative domain of QCD and discussed the various hadronic properties of heavy-light flavour mesons in terms of a Non Relativistic Quark Model(NRQM)[30] based on potential concept.It is basically evolved from the work of Rujula,Georgi and Glashow [29] who used the linear cum Coulombic pieces as the basis for QCD potential model approach.To test the model we have calculated the slope and curvature of Isgur-Wise(I-W) function [79, 81, 83, 84],elastic form factors ,charge radii [94] etc which are directly dependent on the wave function.We use different approximation methods like Dalgarno method [1] and Variationally Improved Perturbation Theory (VIPT)[2-4] to solve the Schrödinger equation in getting appropriate wavefunctions.We note that the linear cum Coulombic potential allows us to use one part either as parent or perturbation (i.e. child) in these approximation methods and we have used both the options.

Although nonrelativistic in nature,we have incorporated relativistic effect using the standard Dirac modification in a parameter free way. We have tried to incorporate significant confinement effect [78] and fixed numerical values of running coupling constant α_s from either $\overline{MS}$ [12] or V-scheme [13, 14, 15] throughout the whole work.Being successful in predicting the slope and curvature of I-W functions,elastic form factors,charge radii etc,this work suggests novel application of quantum mechanical approximation methods in QCD.

While chapter 1 gives an introduction to the subject matter we want to study, chapter 2 is involved with the use of the Dalgarno method with Coulombic parent.In the process,upper bounds on the slope and curvature of I-W function is obtained.The work [79]has revealed that a scale parameter ('c' in this work) in the potential can modify the wavefunction when treated as perturbation and significantly influence the I-W function .The bounds are well within the predictions of other models and data [53]-[68].Although large confinement ($b = 0.183GeV^2$) was successfully incorporated , this work hints at the necessity of large α_s.

Chapter 3 has summarized the attempt of VIPT with Coulombic parent in the calculation of slope and curvature of I-W function.The results are shown to be good for the D-sector mesons having large α_s ,but not the same for B-sector having small α_s.The analysis shows that more term consideration in the expression for first order corrected wavefunction leads to satisfactory results and thus it is very essential for B-sector mesons.The calculation however becomes cumbersome with more and more terms.

In chapter 4,we study the I-W function using the Dalgarno method with linear parent as an alternate option to improve the results.The results are significantly improved except for the B_c-meson.

In chapter 5,we worked out the last option i.e. linear parent with VIPT in predicting the slope and curvature of I-W function .The predictions are comparable to those of other workers [53-68] even with a single term in the expression for the first order correction to wavefunction unlike the case of Coulombic parent in

VIPT.However,the B_c meson still needs more attention.

The success of VIPT in the calculation of I-W function has tempted us to extend it in the prediction of elastic form factors and charge radii for all the above mesons.In chapter 6, we have done it for both linear and Coulombic parent.In the absence of experimental data,our results are compared with those of other models [9, 10, 11].We have succeded in including large confinement effect and significant value of α_s in this approach which was absent in earlier work [30] with Dalgarno method within this QCD inspired model.Further,it leads to a useful range of four momentum transfer Q^2 in the calculation of form factors and charge radii. Performances of both the options are also compared.

Thus in this thesis,we have used different approximation methods, namely the Dalgarno method and VIPT in solving the nonrelativistic Schrödinger equation and use the solution i.e. wavefunction in the calculation of slope and curvature of I-W function ,elastic form factors,charge radii etc for the heavy-light mesons in a QCD potential model.The relativistic effects are introduced too reasonably.The properties we have calculated are of great importance as they act like standard factors of determining the reliability and effectiveness of a specific model.Although this work has been successful in predicting these standard factors (which thus suggests the effectiveness and reliability of our model),there are certain limitations or aspects which need modifications for greater applicability of the model.

We have listed those aspects as below:

I. At first,the spin effects need serious attention.We have considered a Hamiltonian which is spin independent.So,incorporation of spin effect can be applied which will extend the model for vector mesons and at the same time,provide the mass splitting between pseudoscalar and vector mesons.

II. As we have noticed that whether it is Dalgarno method or VIPT,the linear parent ends up with much more satisfactory results.This approach basically deals with Airy functions [4, 83, 84] which is an infinite series in interquark distance 'r'.In our calculation,we truncate the series upto $O(r^3)$ for the convenience of calculation.As it is not a good idea at all to consider a certain order at will ,so a similar analysis can be carried to investigate the effective order of r leading to most satisfactory results for I-W function or the order (of r)dependence of I-W function.

III. We have restricted our work for the ground state only ($n = 1, l = 0$).So, consideration of excited states will widen the applicability of the present method in new areas of hadron physics.

IV. One of the most common feature throughout the whole work is the use of a fixed value of confinement parameter i.e. $b = 0.183 GeV^2$ .This value taken from the charmonium spectroscopy [78], may be reconsidered as far as B-sector mesons are concerned .This might change the used range of α_s values taken from either $\overline{MS}$-scheme or V-scheme and consequently lead to change in the predictions.

We hope to work for the improvement of the above aspects in future.

Appendix

A Appendix A

In chapter 2,we have calculated the wavefunction corrected upto first order using the Dalgarno method as follows.

We start from the basic equation of perturbation theory :

$$H_0\psi_1^{(1)} + H'\psi_1^{(0)} = W_1^{(0)}\psi_1^{(1)} + W_1^{(1)}\psi_1^{(0)} \tag{A1}$$

This can be put in the form :

$$\left(H_0 - W_1^{(0)}\right)\psi_1^{(1)} = \left(W_1^{(1)} - H'\right)\psi_1^{(0)} \tag{A2}$$

The unperturbed Hamiltonian is :

$$\begin{aligned} H_o &= -\frac{\nabla^2}{2\mu} - \frac{4\alpha_s}{3r} \\ &= -\frac{\nabla^2}{2\mu} - \frac{A}{r} \end{aligned} \tag{A3}$$

where

$$A = \frac{4\alpha_s}{3} \tag{A4}$$

The perturbed Hamiltonian is :

$$H' = br + c \tag{A5}$$

The unperturbed energy is :

$$\begin{aligned} W_1^{(0)} &= -\frac{\mu A^2}{2} \\ &= -\frac{8\mu\alpha_s^2}{9} \end{aligned} \tag{A6}$$

The unperturbed wavefunction is :

$$\psi_1^{(0)} = \frac{1}{\sqrt{\pi a_0^3}} e^{-\frac{r}{a_0}} \tag{A7}$$

The first order corrected energy is:

$$\begin{aligned} W_1^1 &= W \\ &= \int \psi_{100}^{(0)*} H' \psi_{100}^{(0)} dv \end{aligned} \tag{A8}$$

The subscripts reflect the fact that the state we have considered is ground state.

We can put equation(2) in the form :

$$\left(\nabla^2 + \frac{2}{a_0 r} - \frac{1}{a_0^2}\right) \psi_1^{(1)} = \frac{2\mu}{\sqrt{\pi a_0^3}} (br + c - W) e^{-\frac{r}{a_0}} \tag{A9}$$

We take the first order correction to wavefunction as :

$$\psi_1^{(1)} = (br + c) R(r) \tag{A10}$$

Equation(9) becomes :

$$\left(\frac{d^2}{dr^2} + \frac{2d}{rdr} + \frac{2}{a_0 r} - \frac{1}{a_0^2}\right) (br + c) R(r) = D (br + c - W) e^{-\frac{r}{a_0}} \tag{A11}$$

where

$$D = \frac{2\mu}{\sqrt{\pi a_0^3}} \tag{A12}$$

Eq.(11) becomes:

$$(br + c) \frac{d^2 R}{dr^2} + 2b \frac{dR}{dr} + 2b \frac{R}{r} + \frac{2(br + c) dR}{rdr} + \frac{2(br + c) R}{a_0 r} - \frac{(br + c) R}{a_0^2} = D (br + c - W) e^{-\frac{r}{a_0}} \tag{A13}$$

Again let us take

$$R(r) = F(r) e^{-\frac{r}{a_0}} \tag{A14}$$

So, equation(13) becomes:

$$(br+c)\,F''(r)+\left[\frac{2\,(br+c)}{r}+2b-\frac{2\,(br+c)}{a_0}\right]F'(r)+ \\ \left[\frac{2b}{r}-\frac{2b}{a_0}\right]F(r)=D\,(br+c-W) \qquad \text{(A15)}$$

We now take series solution:

$$F(r)=\sum_{n=0}^{\infty}A_n r^n \qquad \text{(A16)}$$

Therefore

$$F'(r)=\sum_{n=0}^{\infty}nA_n r^{n-1} \qquad \text{(A17)}$$

$$F''(r)=\sum_{n=0}^{\infty}n\,(n-1)\,A_n r^{n-2} \qquad \text{(A18)}$$

and so on. Equation(15)is modified to:

$$(br+c)\sum_{n=0}^{\infty}n\,(n-1)\,A_n r^{n-2}+\left[\frac{2c}{r}+4b-\frac{2br}{a_0}-\frac{2c}{a_0}\right]\sum_{n=0}^{\infty}nA_n r^{n-1}+ \\ \left[\frac{2b}{r}-\frac{2b}{a_0}\right]\sum_{n=0}^{\infty}A_n r^n=D\,(br+c-W) \qquad \text{(A19)}$$

After simplification it leads to:

$$\left[c\sum_{n=0}^{\infty}n\,(n-1)\,A_n+2c\sum_{n=0}^{\infty}nA_n\right]r^{n-2}+ \\ \left[b\sum_{n=0}^{\infty}n\,(n-1)\,A_n+4b\sum_{n=0}^{\infty}nA_n-\frac{2c}{a_0}\sum_{n=0}^{\infty}nA_n+2b\sum_{n=0}^{\infty}A_n\right]r^{n-1}- \\ \left[\frac{2b}{a_0}\sum_{n=0}^{\infty}nA_n+\frac{2b}{a_0}\sum_{n=0}^{\infty}A_n\right]r^n=D\,(br+c-W) \qquad \text{(A20)}$$

Let us equate the various coefficients. Equating r^{-1} , $n=1,0$ we get:

$$cA_1+bA_0=0 \qquad \text{(A21)}$$

Equating r^0,$n = 2, 1, 0$ we get :

$$cA_2 + bA_1 = \frac{D}{6}(c - W) \tag{A22}$$

Equating r^1,$n = 3, 2, 1$ we get :

$$cA_3 + bA_2 = \frac{D}{12}\left[b + \frac{2(c-W)}{3a_0}\right] \tag{A23}$$

Equating r^2,$n = 4, 3, 2$ we get :

$$cA_4 + bA_3 = \frac{D}{40a_0}\left[b + \frac{2(c-W)}{3a_0}\right] \tag{A24}$$

Now,

$$F(r) = A_0r^0 + A_1r^1 + A_2r^2 + A_3r^3 + A_4r^4 + ... \tag{A25}$$

Thus,

$$\begin{aligned}\psi_1^{(1)} &= (br + c)\,F(r)\,e^{-\frac{r}{a_0}} \\ &= (br + c)\left(A_0r^0 + A_1r^1 + A_2r^2 + A_3r^3 + A_4r^4 + ...\right)e^{-\frac{r}{a_0}}\end{aligned} \tag{A26}$$

Using the above equations for cA_0, cA_1, cA_2, cA_3 etc we get:

$$\psi_1^{(1)} = \left[cA_0r^0 + (cA_1 + bA_0)\,r^1 + (cA_2 + bA_1)\,r^2 + (cA_3 + bA_2)\,r^3 + (cA_4 + bA_3)\,r^4 + ...\right]e^{-\frac{r}{a_0}}$$

or

$$\psi_1^{(1)} = \left[cA_0r^0 + \frac{D}{6}(c-W)r^2 + \frac{D}{12}\left(b + \frac{2(c-W)}{3a_0}\right)r^3 + \frac{D}{120a_0}\left(b + \frac{2(c-W)}{3a_0}\right)r^4 + ...\right]e^{-\frac{r}{a_0}}$$

Also,

$$\begin{aligned}W &= \int \psi_{100}^{(0)*}\,(br + c)\,\psi_{100}^{(0)}dv \\ &= \frac{3ba_0}{2} + c\end{aligned} \tag{A27}$$

where definition of Gamma functions is used. Thus,

$$c - W = -\frac{3ba_0}{2} \tag{A28}$$

Now,

$$\begin{aligned} b + \frac{2(c-W)}{3a_0} &= b + \frac{2}{3a_0}\left(-\frac{3ba_0}{2}\right) \\ &= 0 \end{aligned} \tag{A29}$$

This leads to :

$$\begin{aligned} \psi_1^{(1)} &= \left[cA_0 - \frac{Dba_0r^2}{4}\right] e^{-\frac{r}{a_0}} \\ &= \left[cA_0 - \frac{\mu ba_0r^2}{2\sqrt{\pi a_0^3}}\right] e^{-\frac{r}{a_0}} \end{aligned} \tag{A30}$$

Thus, the wavefunction corrected upto first order is :

$$\begin{aligned} \psi_1 &= \psi_1^{(0)} + \psi_1^{(1)} \\ &= \left[cA_0 + \sqrt{\pi a_0^3} - \frac{\mu ba_0r^2}{2\sqrt{\pi a_0^3}}\right] e^{-\frac{r}{a_0}} \end{aligned} \tag{A31}$$

With normalization this becomes :

$$\psi_1 = N\left(cA_0 + \sqrt{\pi a_0^3} - \frac{\mu ba_0r^2}{2\sqrt{\pi a_0^3}}\right) e^{-\frac{r}{a_0}} \tag{A32}$$

where N is the normalization constant.

Of course this wavefunction is same as that in equation (2.2) of the thesis (pp-15).

B Appendix B

X_1, X_2 and X_3 as occured in chapter 2 are evaluated as :

$$X_1 = 64\pi c^2 A_0^2 a_0^3 + 64 + \mu^2 b^2 a_0^6 (8-2\epsilon)(7-2\epsilon)(6-2\epsilon)(5-2\epsilon)$$
$$+128cA_0\sqrt{\pi a_0^3} - 16cA_0\mu b\sqrt{\pi a_0^9}(6-2\epsilon)(5-2\epsilon)$$
$$-16\mu b a_0^3 (6-2\epsilon)(5-2\epsilon) \qquad \text{(B1)}$$

$$X_2 = 64\pi c^2 A_0^2 a_0^3 + 64 + \mu^2 b^2 a_0^6 (6-2\epsilon)(5-2\epsilon)(4-2\epsilon)(3-2\epsilon)$$
$$+128cA_0\sqrt{\pi a_0^3} - 16cA_0\mu b\sqrt{\pi a_0^9}(4-2\epsilon)(3-2\epsilon)$$
$$-16\mu b a_0^3 (4-2\epsilon)(3-2\epsilon) \qquad \text{(B2)}$$

$$X_3 = 64\pi c^2 A_0^2 a_0^3 + 64 + \mu^2 b^2 a_0^6 (10-2\epsilon)(9-2\epsilon)(8-2\epsilon)(7-2\epsilon)$$
$$+128cA_0\sqrt{\pi a_0^3} - 16cA_0\mu b\sqrt{\pi a_0^9}(8-2\epsilon)(7-2\epsilon)$$
$$-16\mu b a_0^3 (8-2\epsilon)(7-2\epsilon) \qquad \text{(B3)}$$

Not only the above expressions ,but all the integrals in the analysis are evaluated with the help of Gamma function given by :

$$\frac{\Gamma(n+1)}{\alpha^{n+1}} = \int_0^{+\infty} r^n e^{-\alpha r} dr \qquad \text{(B4)}$$

C Appendix C

$X_\imath (\imath = 1, 2, ..., 45)$ as occured in chapter 3 are:

$$X_1 = A^2 \left(1 + \frac{(6-2\epsilon)(5-2\epsilon)}{4} - (5-2\epsilon)\right) \tag{C1}$$

$$X_2 = 64c_1'A\left(\frac{(5-2\epsilon)}{729} - 1729\right) \tag{C2}$$

$$X_3 = A^2 \left(1 + \frac{(8-2\epsilon)(7-2\epsilon)}{4} - (7-2\epsilon)\right) \tag{C3}$$

$$X_4 = 256c_1'A\left(-\frac{1}{2187} + \frac{(7-2\epsilon)}{6561}\right) \tag{C4}$$

$$X_5 = A^2 \left[1 + \frac{(4-2\epsilon)(3-2\epsilon)}{4} - (3-2\epsilon)\right] \tag{C5}$$

$$X_6 = 16c_1'A\left(\frac{(3-2\epsilon)}{81} - \frac{1}{27}\right) \tag{C6}$$

$$X_7 = B^2 \left[\frac{243}{32} + \frac{324(6-2\epsilon)(5-2\epsilon)}{32} - \frac{243(5-2\epsilon)}{16}\right] \tag{C7}$$

$$X_8 = B^2 (7-2\epsilon)(6-2\epsilon)(5-2\epsilon)\left[\frac{9(8-2\epsilon)}{128} - \frac{81}{32}\right] \tag{C8}$$

$$X_9 = \frac{243c_1'B}{4^5}\left[2 - 243(5-2\epsilon) + \frac{3(6-2\epsilon)(5-2\epsilon)}{4}\right] \tag{C9}$$

$$X_{10} = \frac{6^5 AB}{5^5}\left[\frac{14(5-2\epsilon)}{5} - 2\right] \tag{C10}$$

$$X_{11} = \frac{2 \times 6^7 AB(6-2\epsilon)(5-2\epsilon)}{3 \times 5^7}\left[\frac{6(7-2\epsilon)}{45} - \frac{11}{9}\right] \tag{C11}$$

$$X_{12} = B^2 \left[\frac{3^7}{128} + \frac{3^6(8-2\epsilon)(7-2\epsilon)}{32} - \frac{3^7(7-2\epsilon)}{64}\right] \tag{C12}$$

$$X_{13} = \frac{B^2 \times 3^5 (9-2\epsilon)(8-2\epsilon)(7-2\epsilon)}{128}\left[\frac{(10-2\epsilon)}{4} - 3\right] \tag{C13}$$

$$X_{14} = \frac{3^7 \times c_1'B}{4^7}\left[2 - (7-2\epsilon) + \frac{3(8-2\epsilon)(7-2\epsilon)}{4}\right] \tag{C14}$$

$$X_{15} = \frac{6^7 \times AB}{5^7}\left[\frac{14\,(5-2\epsilon)}{5} - 2\right] \tag{C15}$$

$$X_{16} = \frac{2 \times 6^9 AB\,(6-2\epsilon)\,(5-2\epsilon)}{3 \times 5^9}\left[\frac{6\,(7-2\epsilon)}{45} - \frac{11}{9}\right] \tag{C16}$$

$$X_{17} = B^2\left[\frac{27}{8} + \frac{9\,(4-2\epsilon)\,(3-2\epsilon)}{2} - \frac{27\,(3-2\epsilon)}{4}\right] \tag{C17}$$

$$X_{18} = B^2\,(5-2\epsilon)\,(4-2\epsilon)\,(3-2\epsilon)\left[\frac{3\,(6-2\epsilon)}{32} - \frac{9}{8}\right] \tag{C18}$$

$$X_{19} = \frac{3^3 \times c_1' B}{4^3}\left[2 - (3-2\epsilon) + \frac{3\,(4-2\epsilon)\,(3-2\epsilon)}{4}\right] \tag{C19}$$

$$X_{20} = \frac{6^3 \times AB}{5^3}\left[\frac{14\,(3-2\epsilon)}{5} - 2\right] \tag{C20}$$

$$X_{21} = \frac{2 \times 6^5 AB\,(4-2\epsilon)\,(3-2\epsilon)}{3 \times 5^5}\left[\frac{6\,(5-2\epsilon)}{45} - \frac{11}{9}\right] \tag{C21}$$

$$X_{22} = X_1 + X_2 + \sum_{i=7}^{11} X_i \tag{C22}$$

$$X_{23} = D'^2\left[32 + 104\,(6-2\epsilon)\,(5-2\epsilon) - \frac{152\,(7-2\epsilon)\,(6-2\epsilon)\,(5-2\epsilon)}{3} - 96\,(5-2\epsilon)\right] \tag{C23}$$

$$X_{24} = D'^2\,(8-2\epsilon)\,(7-2\epsilon)\,(6-2\epsilon)\,(5-2\epsilon)\,[12 - \frac{4\,(9-2\epsilon)}{3} + \frac{(10-2\epsilon)\,(9-2\epsilon)}{18}] \tag{C24}$$

$$X_{25} = \frac{2 \times 4^5 c_1' D'}{5^5}[1 - \frac{3\,(5-2\epsilon)}{5} + \frac{16\,(6-2\epsilon)\,(5-2\epsilon)}{200} - \frac{(7-2\epsilon)\,(6-2\epsilon)\,(5-2\epsilon)}{375}] \tag{C25}$$

$$X_{26} = \frac{2 \times 4^5 D'A}{3^5}[\frac{5\,(5-2\epsilon)}{3} - \frac{16\,(6-2\epsilon)\,(5-2\epsilon)}{18} - 1] \tag{C26}$$

$$X_{27} = \frac{2 \times 4^5 D'A\,(7-2\epsilon)\,(6-2\epsilon)\,(5-2\epsilon)}{3^9}[13 - \frac{4\,(8-2\epsilon)}{6}] \qquad (C27)$$

$$X_{28} = \frac{2 \times 12^5 D'B}{7^5}[1 - \frac{17\,(5-2\epsilon)}{7} + \frac{151 \times 12^2\,(6-2\epsilon)\,(5-2\epsilon)}{216 \times 7^2}] \qquad (C28)$$

$$X_{29} = \frac{2 \times 12^8 D'B\,(7-2\epsilon)\,(6-2\epsilon)\,(5-2\epsilon)}{7^8}[-\frac{83}{576} + \frac{132\,(8-2\epsilon)}{6048} - \frac{12\,(9-2\epsilon)\,(8-2\epsilon)}{8 \times 27 \times 7^2}] \qquad (C29)$$

$$X_{30} = X_3 + X_4 + \sum_{i=12}^{16} X_i \qquad (C30)$$

$$X_{31} = D^2[128 + 416\,(8-2\epsilon)\,(7-2\epsilon) - \frac{608\,(9-2\epsilon)\,(8-2\epsilon)\,(7-2\epsilon)}{3} - 384\,(7-2\epsilon)] \qquad (C31)$$

$$X_{32} = D'^2\,(10-2\epsilon)\,(9-2\epsilon)\,(8-2\epsilon)\,(7-2\epsilon)\,[48 - \frac{16\,(11-2\epsilon)}{3} + \frac{2\,(12-2\epsilon)\,(11-2\epsilon)}{9}] \qquad (C32)$$

$$X_{33} = \frac{2 \times 4^7 c_1' D}{5^7}[1 - \frac{3\,(7-2\epsilon)}{5} + \frac{16\,(8-2\epsilon)\,(7-2\epsilon)}{200} - \frac{(9-2\epsilon)\,(8-2\epsilon)\,(5-2\epsilon)}{375}] \qquad (C33)$$

$$X_{34} = \frac{2 \times 4^7 D'A}{3^7}[\frac{5\,(7-2\epsilon)}{3} - \frac{16\,(8-2\epsilon)\,(7-2\epsilon)}{18} - 1] \qquad (C34)$$

$$X_{35} = \frac{2 \times 4^7 D'A\,(9-2\epsilon)\,(8-2\epsilon)\,(7-2\epsilon)}{3^{11}}[13 - \frac{4\,(10-2\epsilon)}{6}] \qquad (C35)$$

$$X_{36} = \frac{2 \times 12^7 D'B}{7^7}[1 - \frac{17\,(7-2\epsilon)}{7} + \frac{151 \times 12^2\,(8-2\epsilon)\,(7-2\epsilon)}{216 \times 7^2}] \qquad (C36)$$

$$X_{37} = \frac{2 \times 12^8 D'B\,(9-2\epsilon)\,(8-2\epsilon)\,(7-2\epsilon)}{7^{10}} \times[-\frac{83}{576} + \frac{132\,(10-2\epsilon)}{6048} - \frac{12\,(11-2\epsilon)\,(10-2\epsilon)}{8 \times 27 \times 7^2}] \qquad (C37)$$

$$X_{38} = X_5 + X_6 + \sum_{i=17}^{21} X_i \tag{C38}$$

$$X_{39} = D'^2 [8 + 26\,(4-2\epsilon)\,(3-2\epsilon) - \frac{38\,(5-2\epsilon)\,(4-2\epsilon)\,(3-2\epsilon)}{3} - 24\,(3-2\epsilon)] \tag{C39}$$

$$X_{40} = D'^2\,(6-2\epsilon)\,(5-2\epsilon)\,(4-2\epsilon)\,(3-2\epsilon) \times [3 - \frac{(7-2\epsilon)}{3} + \frac{(8-2\epsilon)\,(7-2\epsilon)}{72}] \tag{C40}$$

$$X_{41} = \frac{2\times 4^3 c_1' D'}{5^3}[1 - \frac{3\,(3-2\epsilon)}{5} + \frac{2\,(4-2\epsilon)\,(3-2\epsilon)}{25} - \frac{(5-2\epsilon)\,(4-2\epsilon)\,(3-2\epsilon)}{375}] \tag{C41}$$

$$X_{42} = \frac{2\times 4^3 D'A}{3^3}[\frac{5\,(3-2\epsilon)}{3} - \frac{16\,(4-2\epsilon)\,(3-2\epsilon)}{18} - 1] \tag{C42}$$

$$X_{43} = \frac{2\times 4^3 D'A\,(5-2\epsilon)\,(4-2\epsilon)\,(3-2\epsilon)}{3^7}[13 - \frac{4\,(6-2\epsilon)}{6}] \tag{C43}$$

$$X_{44} = \frac{2\times 12^3 D'B}{7^3}[1 - \frac{17\,(3-2\epsilon)}{7} + \frac{151\times 12^2\,(4-2\epsilon)\,(3-2\epsilon)}{216\times 7^2}] \tag{C44}$$

$$X_{45} = \frac{2\times 12^6 D'B\,(5-2\epsilon)\,(4-2\epsilon)\,(3-2\epsilon)}{7^6} \times [-\frac{83}{576} + \frac{132\,(6-2\epsilon)}{6048} - \frac{12\,(7-2\epsilon)\,(6-2\epsilon)}{8\times 27\times 7^2}] \tag{C45}$$

D Appendix D

The various expressions for q_i, r_i, p_i, p_i' as occured in chapter 6 are given as :

Expressions for q_i s :

$$q_1 = \frac{c_1'^2}{(4\mu^2\overline{\alpha}'^2 + Q_i^2)^{(1-\epsilon)}} \tag{D1}$$

$$q_2 = A^2\left[\frac{1}{(\mu^2\overline{\alpha}'^2 + Q_i^2)^{(1-\epsilon)}} - \frac{(3-2\epsilon)\mu\overline{\alpha}'}{(\mu^2\overline{\alpha}'^2 + Q_i^2)^{(1.5-\epsilon)}} + \frac{(4-2\epsilon)(3-2\epsilon)\mu^2\overline{\alpha}'^2}{(\mu^2\overline{\alpha}'^2 + Q_i^2)^{(2-\epsilon)}}\right] \tag{D2}$$

$$q_3 = B^2\Big[\frac{1}{(\frac{\mu^2\overline{\alpha}'^2}{9} + Q_i^2)^{(1-\epsilon)}} - \frac{4(3-2\epsilon)\mu\overline{\alpha}'}{3(\frac{\mu^2\overline{\alpha}'^2}{9} + Q_i^2)^{(1.5-\epsilon)}} + \frac{16(4-2\epsilon)(3-2\epsilon)\mu^2\overline{\alpha}'^2}{27(\frac{\mu^2\overline{\alpha}'^2}{9} + Q_i^2)^{(2-\epsilon)}} - \frac{8(5-2\epsilon)(4-2\epsilon)(3-2\epsilon)\mu^3\overline{\alpha}'^3}{81(\frac{\mu^2\overline{\alpha}'^2}{9} + Q_i^2)^{(2.5-\epsilon)}} + \frac{4(6-2\epsilon)(5-2\epsilon)(4-2\epsilon)(3-2\epsilon)\mu^4\overline{\alpha}'^4}{729(\frac{\mu^2\overline{\alpha}'^2}{9} + Q_i^2)^{(3-\epsilon)}}\Big] \tag{D3}$$

$$q_4 = D'^2\Big[\frac{1}{16(\frac{\mu^2\overline{\alpha}'^2}{4} + Q_i^2)^{(1-\epsilon)}} - \frac{3(3-2\epsilon)\mu\overline{\alpha}'}{32(\frac{\mu^2\overline{\alpha}'^2}{4} + Q_i^2)^{(1.5-\epsilon)}} + \frac{17(4-2\epsilon)(3-2\epsilon)\mu^2\overline{\alpha}'^2}{256(\frac{\mu^2\overline{\alpha}'^2}{4} + Q_i^2)^{(2-\epsilon)}} - \frac{19(5-2\epsilon)(4-2\epsilon)(3-2\epsilon)\mu^3\overline{\alpha}'^3}{1536(\frac{\mu^2\overline{\alpha}'^2}{4} + Q_i^2)^{(2.5-\epsilon)}} + \frac{7(6-2\epsilon)(5-2\epsilon)(4-2\epsilon)(3-2\epsilon)\mu^4\overline{\alpha}'^4}{6144(\frac{\mu^2\overline{\alpha}'^2}{9} + Q_i^2)^{(3-\epsilon)}} - \frac{(7-2\epsilon)(6-2\epsilon)(5-2\epsilon)(4-2\epsilon)(3-2\epsilon)\mu^5\overline{\alpha}'^5}{12288(\frac{\mu^2\overline{\alpha}'^2}{4} + Q_i^2)^{(3.5-\epsilon)}}\Big] \tag{D4}$$

$$q_5 = 2c_1'A\left[\frac{1}{(\frac{9\mu^2\overline{\alpha}'^2}{4} + Q_i^2)^{(1-\epsilon)}} - \frac{(3-2\epsilon)\mu\overline{\alpha}'}{2(\frac{9\mu^2\overline{\alpha}'^2}{4} + Q_i^2)^{(1.5-\epsilon)}}\right] \tag{D5}$$

$$q_6 = 2c_1'B\left[\frac{1}{(\frac{16\mu^2\overline{\alpha}'^2}{9} + Q_i^2)^{(1-\epsilon)}} - \frac{2(3-2\epsilon)\mu\overline{\alpha}'}{3(\frac{16\mu^2\overline{\alpha}'^2}{9} + Q_i^2)^{(1.5-\epsilon)}} + \frac{2(4-2\epsilon)(3-2\epsilon)\mu^2\overline{\alpha}'^2}{27(\frac{16\mu^2\overline{\alpha}'^2}{9} + Q_i^2)^{(2-\epsilon)}}\right] \tag{D6}$$

$$q_7 = 2c_1' D'[\frac{1}{4(\frac{25\mu^2\overline{\alpha}'^2}{16} + Q_i^2)^{(1-\epsilon)}} - \frac{3(3-2\epsilon)\mu\overline{\alpha}'}{16(\frac{25\mu^2\overline{\alpha}'^2}{16} + Q_i^2)^{(1.5-\epsilon)}} + \frac{(4-2\epsilon)(3-2\epsilon)\mu^2\overline{\alpha}'^2}{32(\frac{25\mu^2\overline{\alpha}'^2}{16} + Q_i^2)^{(2-\epsilon)}} - \frac{(5-2\epsilon)(4-2\epsilon)(3-2\epsilon)\mu^3\overline{\alpha}'^3}{768(\frac{25\mu^2\overline{\alpha}'^2}{16} + Q_i^2)^{(2.5-\epsilon)}}] \quad \text{(D7)}$$

$$q_8 = -2AB[\frac{1}{(\frac{25\mu^2\overline{\alpha}'^2}{36} + Q_i^2)^{(1-\epsilon)}} - \frac{5(3-2\epsilon)\mu\overline{\alpha}'}{6(\frac{25\mu^2\overline{\alpha}'^2}{36} + Q_i^2)^{(1.5-\epsilon)}} + \frac{20(4-2\epsilon)(3-2\epsilon)\mu^2\overline{\alpha}'^2}{(27\frac{25\mu^2\overline{\alpha}'^2}{36} + Q_i^2)^{(2-\epsilon)}} - \frac{(5-2\epsilon)(4-2\epsilon)(3-2\epsilon)\mu^3\overline{\alpha}'^3}{27(\frac{25\mu^2\overline{\alpha}'^2}{36} + Q_i^2)^{(2.5-\epsilon)}}] \quad \text{(D8)}$$

$$q_9 = -2AD'[\frac{1}{4(\frac{9\mu^2\overline{\alpha}'^2}{16} + Q_i^2)^{(1-\epsilon)}} - \frac{5(3-2\epsilon)\mu\overline{\alpha}'}{16(\frac{9\mu^2\overline{\alpha}'^2}{16} + Q_i^2)^{(1.5-\epsilon)}} + \frac{(4-2\epsilon)(3-2\epsilon)\mu^2\overline{\alpha}'^2}{8(\frac{9\mu^2\overline{\alpha}'^2}{16} + Q_i^2)^{(2-\epsilon)}} - \frac{13(5-2\epsilon)(4-2\epsilon)(3-2\epsilon)\mu^3\overline{\alpha}'^3}{768(\frac{9\mu^2\overline{\alpha}'^2}{16} + Q_i^2)^{(2.5-\epsilon)}} + \frac{(6-2\epsilon)(5-2\epsilon)(4-2\epsilon)(3-2\epsilon)\mu^4\overline{\alpha}'^4}{1536(\frac{9\mu^2\overline{\alpha}'^2}{16} + Q_i^2)^{(3-\epsilon)}}] \quad \text{(D9)}$$

$$q_{10} = 2BD'[\frac{1}{4(\frac{49\mu^2\overline{\alpha}'^2}{144} + Q_i^2)^{(1-\epsilon)}} - \frac{25(3-2\epsilon)\mu\overline{\alpha}'}{48(\frac{49\mu^2\overline{\alpha}'^2}{144} + Q_i^2)^{(1.5-\epsilon)}} + \frac{151(4-2\epsilon)(3-2\epsilon)\mu^2\overline{\alpha}'^2}{864(\frac{25\mu^2\overline{\alpha}'^2}{144} + Q_i^2)^{(2-\epsilon)}} - \frac{27.66(5-2\epsilon)(4-2\epsilon)(3-2\epsilon)\mu^3\overline{\alpha}'^3}{768(\frac{25\mu^2\overline{\alpha}'^2}{144} + Q_i^2)^{(2.5-\epsilon)}} + \frac{3.66(6-2\epsilon)(5-2\epsilon)(4-2\epsilon)(3-2\epsilon)\mu^4\overline{\alpha}'^4}{1152(25\frac{\mu^2\overline{\alpha}'^2}{144} + Q_i^2)^{(3-\epsilon)}} - \frac{(7-2\epsilon)(6-2\epsilon)(5-2\epsilon)(4-2\epsilon)(3-2\epsilon)\mu^4\overline{\alpha}'^4}{10368(25\frac{\mu^2\overline{\alpha}'^2}{144} + Q_i^2)^{(3.5-\epsilon)}}] \quad \text{(D10)}$$

Expressions for r_i s:

$$r_1 = 3c_1'^2(1+\frac{m_i}{m_j})^{-2}(4\mu^2\overline{\alpha}'^2)^{\epsilon-2}(2-2\epsilon) \tag{D11}$$

$$r_2 = 3A^2(1+\frac{m_i}{m_j})^{-2}[(2-2\epsilon)(4\mu^2\overline{\alpha}'^2)^{\epsilon-2} - 3\mu\overline{\alpha}'(3-2\epsilon)^2(\mu^2\overline{\alpha}'^2)^{\epsilon-2.5} + 0.75\mu^2\overline{\alpha}'^2(4-2\epsilon)^2(3-2\epsilon)(\mu^2\overline{\alpha}'^2)^{\epsilon-3}] \tag{D12}$$

$$r_3 = 3B^2(1+\frac{m_i}{m_j})^{-2}[(\frac{4\mu^2\overline{\alpha}'^2}{9})^{\epsilon-2}(2-2\epsilon) - 4\mu\overline{\alpha}'(3-2\epsilon)^2(\frac{4\mu^2\overline{\alpha}'^2}{9})^{\epsilon-2.5} + \frac{16\mu^2\overline{\alpha}'^2}{9}(4-2\epsilon)^2(3-2\epsilon)(\frac{4\mu^2\overline{\alpha}'^2}{9})^{\epsilon-3} - \frac{8\mu^3\overline{\alpha}'^3}{81}(5-2\epsilon)^2(4-2\epsilon)(3-2\epsilon)(\frac{4\mu^2\overline{\alpha}'^2}{9})^{\epsilon-3.5} + \frac{4\mu^4\overline{\alpha}'^4}{729}(6-2\epsilon)^2(5-2\epsilon)(4-2\epsilon)(3-2\epsilon)(\frac{4\mu^2\overline{\alpha}'^2}{9})^{\epsilon-4}] \tag{D13}$$

$$r_4 = 3D'^2(1+\frac{m_i}{m_j})^{-2}[(\frac{\mu^2\overline{\alpha}'^2}{4})^{\epsilon-2}(2-2\epsilon) - \frac{9\mu\overline{\alpha}'}{32}(3-2\epsilon)^2(\frac{\mu^2\overline{\alpha}'^2}{4})^{\epsilon-2.5} + \frac{57\mu^2\overline{\alpha}'^2}{32}(4-2\epsilon)^2(3-2\epsilon)(\frac{\mu^2\overline{\alpha}'^2}{4})^{\epsilon-3} - \frac{19\mu^3\overline{\alpha}'^3}{512}(5-2\epsilon)^2(4-2\epsilon)(3-2\epsilon)(\frac{\mu^2\overline{\alpha}'^2}{4})^{\epsilon-3.5} + \frac{21\mu^4\overline{\alpha}'^4}{6144}(6-2\epsilon)^2(5-2\epsilon)(4-2\epsilon)(3-2\epsilon)(\frac{\mu^2\overline{\alpha}'^2}{4})^{\epsilon-4} - \frac{3\mu^5\overline{\alpha}'^5}{12288}(7-2\epsilon)^2(6-2\epsilon)(5-2\epsilon)(4-2\epsilon)(3-2\epsilon)(\frac{4\mu^2\overline{\alpha}'^2}{9})^{\epsilon-4.5}] \tag{D14}$$

$$r_5 = 2c_1'A(1+\frac{m_i}{m_j})^{-2}[3(\frac{9\mu^2\overline{\alpha}'^2}{4})^{\epsilon-2}(2-2\epsilon) - 1.5\mu\overline{\alpha}'(3-2\epsilon)^2(\frac{9\mu^2\overline{\alpha}'^2}{4})^{\epsilon-2.5}] \tag{D15}$$

$$r_6 = 2c_1'B(1+\frac{m_i}{m_j})^{-2}[3(\frac{16\mu^2\overline{\alpha}'^2}{9})^{\epsilon-2}(2-2\epsilon) - 2\mu\overline{\alpha}'(3-2\epsilon)^2(\frac{16\mu^2\overline{\alpha}'^2}{9})^{\epsilon-2.5} + \frac{2\mu^2\overline{\alpha}'^2}{9}(4-2\epsilon)^2(3-2\epsilon)(\frac{16\mu^2\overline{\alpha}'^2}{9})^{\epsilon-3}] \tag{D16}$$

$$r_7 = 2c_1' D'(1 + \frac{m_i}{m_j})^{-2}[(\frac{25\mu^2\overline{\alpha}'^2}{16})^{\epsilon-2}\frac{3\,(2-2\epsilon)}{4} \quad - \\ \frac{9\mu\overline{\alpha}'(3-2\epsilon)^2}{16}(\frac{25\mu^2\overline{\alpha}'^2}{16})^{\epsilon-2.5} \quad + \\ \frac{3\mu^2\overline{\alpha}'^2}{32}(4-2\epsilon)^2(3-2\epsilon)(\frac{25\mu^2\overline{\alpha}'^2}{16})^{\epsilon-3} \quad - \\ \frac{3\mu^3\overline{\alpha}'^3}{768}(5-2\epsilon)^2(4-2\epsilon)(3-2\epsilon)(\frac{25\mu^2\overline{\alpha}'^2}{16})^{\epsilon-3.5}] \tag{D17}$$

$$r_8 = -2AB(1 + \frac{m_i}{m_j})^{-2}[(\frac{25\mu^2\overline{\alpha}'^2}{36})^{\epsilon-2}3\,(2-2\epsilon) \quad - \\ 2.5\mu\overline{\alpha}'(3-2\epsilon)^2(\frac{25\mu^2\overline{\alpha}'^2}{36})^{\epsilon-2.5} \quad + \\ \frac{20\mu^2\overline{\alpha}'^2}{9}(4-2\epsilon)^2(3-2\epsilon)(\frac{25\mu^2\overline{\alpha}'^2}{36})^{\epsilon-3} \quad - \\ \frac{\mu^3\overline{\alpha}'^3}{9}(5-2\epsilon)^2(4-2\epsilon)(3-2\epsilon)(\frac{25\mu^2\overline{\alpha}'^2}{36})^{\epsilon-3.5}] \tag{D18}$$

$$r_9 = -2AD'(1 + \frac{m_i}{m_j})^{-2}[0.75(\frac{9\mu^2\overline{\alpha}'^2}{16})^{\epsilon-2}\,(2-2\epsilon) \quad - \\ \frac{15\mu\overline{\alpha}'}{16}(3-2\epsilon)^2(\frac{9\mu^2\overline{\alpha}'^2}{16})^{\epsilon-2.5} \quad + \\ \frac{3\mu^2\overline{\alpha}'^2}{8}(4-2\epsilon)^2(3-2\epsilon)(\frac{9\mu^2\overline{\alpha}'^2}{16})^{\epsilon-3} \quad - \\ \frac{13\mu^3\overline{\alpha}'^3}{256}(5-2\epsilon)^2(4-2\epsilon)(3-2\epsilon)(\frac{9\mu^2\overline{\alpha}'^2}{16})^{\epsilon-3.5} \quad + \\ \frac{\mu^4\overline{\alpha}'^4}{512}(6-2\epsilon)^2(5-2\epsilon)(4-2\epsilon)(3-2\epsilon)(\frac{9\mu^2\overline{\alpha}'^2}{16})^{\epsilon-4}] \tag{D19}$$

$$r_{10} = 2BD'(1 + \frac{m_i}{m_j})^{-2}[0.75(\frac{49\mu^2\overline{\alpha}'^2}{144})^{\epsilon-2}\,(2-2\epsilon) \quad - \\ \frac{17\mu\overline{\alpha}'}{16}(3-2\epsilon)^2(\frac{49\mu^2\overline{\alpha}'^2}{144})^{\epsilon-2.5} \quad + \\ \frac{151\mu^2\overline{\alpha}'^2}{288}(4-2\epsilon)^2(3-2\epsilon)(\frac{49\mu^2\overline{\alpha}'^2}{144})^{\epsilon-3} \quad - \\ \frac{83\mu^3\overline{\alpha}'^3}{768}(5-2\epsilon)^2(4-2\epsilon)(3-2\epsilon)(\frac{49\mu^2\overline{\alpha}'^2}{144})^{\epsilon-3.5} \quad + \\ \frac{33\mu^4\overline{\alpha}'^4}{3456}(6-2\epsilon)^2(5-2\epsilon)(4-2\epsilon)(3-2\epsilon)(\frac{9\mu^2\overline{\alpha}'^2}{16})^{\epsilon-4} \quad -$$

$$\frac{\mu^5\overline{\alpha}'^5}{3456}(7-2\epsilon)^2(6-2\epsilon)(5-2\epsilon)(4-2\epsilon)(3-2\epsilon)(\frac{49\mu^2\overline{\alpha}'^2}{144})^{\epsilon-4.5}] \tag{D20}$$

Expressions for p_i s:

$$p_1 = \frac{c_1'}{(\frac{\mu^2\overline{\alpha}'^2}{4}+Q_i^2)^{\frac{(3-\epsilon)}{2}}} \tag{D21}$$

$$p_2 = A[\frac{1}{(\frac{\mu^2\overline{\alpha}'^2}{4}+Q_i^2)^{\frac{(3-\epsilon)}{2}}} - \frac{(3-2\epsilon)\mu\overline{\alpha}'}{2(\frac{\mu^2\overline{\alpha}'^2}{4}+Q_i^2)^{\frac{(4-\epsilon)}{2}}}] \tag{D22}$$

$$p_3 = B[\frac{1}{(\frac{\mu^2\overline{\alpha}'^2}{9}+Q_i^2)^{\frac{(3-\epsilon)}{2}}} - \frac{0.67(3-2\epsilon)\mu\overline{\alpha}'}{(\frac{\mu^2\overline{\alpha}'^2}{9}+Q_i^2)^{\frac{(4-\epsilon)}{2}}} - \frac{2(4-2\epsilon)(3-2\epsilon)\mu^2\overline{\alpha}'^2}{27(\frac{\mu^2\overline{\alpha}'^2}{9}+Q_i^2)^{\frac{(5-\epsilon)}{2}}}] \tag{D23}$$

$$p_4 = D'[\frac{0.25}{(\frac{\mu^2\overline{\alpha}'^2}{16}+Q_i^2)^{\frac{(3-\epsilon)}{2}}} - \frac{3(3-2\epsilon)\mu\overline{\alpha}'}{16(\frac{\mu^2\overline{\alpha}'^2}{16}+Q_i^2)^{\frac{(4-\epsilon)}{2}}} - \frac{(4-2\epsilon)(3-2\epsilon)\mu^2\overline{\alpha}'^2}{32(\frac{\mu^2\overline{\alpha}'^2}{16}+Q_i^2)^{\frac{(5-\epsilon)}{2}}} - \frac{(5-2\epsilon)(4-2\epsilon)(3-2\epsilon)\mu^3\overline{\alpha}'^3}{768(\frac{\mu^2\overline{\alpha}'^2}{16}+Q_i^2)^{\frac{(6-\epsilon)}{2}}}] \tag{D24}$$

Expressions for p_i' s :

$$p_1' = n_1 \times \left(\overline{b}'\mu\right)^{\frac{2}{3}} \tag{D25}$$

$$p_2' = n_2 \times \left(\overline{b}'\mu\right)^{\frac{2}{3}} \tag{D26}$$

$$p_3' = n_3 \times \left(\overline{b}'\mu\right)^{\frac{2}{3}} \tag{D27}$$

$$p_4' = n_4 \times \left(\overline{b}'\mu\right)^{\frac{2}{3}} \tag{D28}$$

$$p_5' = n_5 \times \left(\overline{b}'\mu\right)^{\frac{2}{3}} \tag{D29}$$

$$p_6' = n_6 \times \left(\overline{b}'\mu\right)^{\frac{2}{3}} \tag{D30}$$

$$p_7' = n_7 \times \left(\overline{b}'\mu\right)^{\frac{2}{3}} \tag{D31}$$

$$p'_8 = n_8 \times \left(\overline{b}'\mu\right)^{\frac{2}{3}} \tag{D32}$$

Each of the constants $n_1, n_2, ..., n_8$ are different for different mesons and they have been obtained by numerical integration.

Bibliography

[1] A K Ghatak and S Lokanathan in "Quantum Mechanics" ,McGraw Hill ,**1997,pp-270,pp-291**.

[2] S K You ,K J Jeon ,C K Kim and K Nahm , Eur.J.Phys. **19**,179(1998).

[3] F M Fernandez , Eur. J. Phys. **24**,289(2003).

[4] I J R Aitchison and J J Dudek ,Eur. J. Phys. **23**,605 (2002).

[5] N Isgur and M B Wise,Phys.Lett.B **232**,113(1989).

[6] N Isgur and M B Wise,Phys.Lett.B **237**,527(1990).

[7] F E Close and A Wambach, **RAL-93-022,OUTP 93 06 P**,28.04.1993.

[8] M G Olsson and S Veseli,Phy. Rev. D **51**,2224(1995).

[9] J N Pandya and P C Vinodkumar,Pramana J. Phy. **57**,821(2001).

[10] C W Hwang , Eur.Phy. J. C **23**,585(2002).

[11] N S Bordoloi and D K Choudhury, Ind J. Phy. **82(6)**,779(2008).

[12] Review of Particle Physics ,Particle Data Group, Euro.Phys.J **C-3**,1998).

[13] Y Schroeder , Phys.Lett.B **447**,321 (1999).

[14] Y Schroeder ,Nucl.Phy.Proc.Suppl. **86**, 525,(2000).

[15] M Peter ,Phys. Rev.Lett. **78**,603 (1997);Nucl.Phys.B **501**,471(1997).

[16] K G Wilson, Phys. Rev. D **10**,2445 (1974).

[17] A M Polyakov, Phys. Lett. B **59**,79 (1975).

[18] A M Polyakov, Phys. Lett. B **59**,82 (1975).

[19] F J Wegner,J. Math. Phys. **12**,2259 (1971).

[20] M H Mac Cregor,Nuovo Cimento A, **103**,983(1990).

[21] D Mustaki,arXiv:hep-ph/ **9404206**,1 April 1994.

[22] E V Shuryak,Nucl. Phys. B **198**,83 (1982).

[23] J G Körner *et al* ,Prog. Part. Nucl. Phys. **33**,787 (1994).

[24] M Neubert, Phys. Rep. **245**,259 (1994).

[25] B Stech , M Wirbel,M Bauer, Z Phys. C **29**, 637(1985).

[26] S J Brodsky, G P Lepage, Phys. Rev. D **22**,2157 (1980).

[27] E Jenkins, arXiv:hep-ph/ **9212295**

[28] A Vainshtein,M Shifman, V Zakharov,Nucl. Phys. B **147**,385 (1979).

[29] A D Rujula ,H Georgi and S L Glashow ,Phy.Rev.D **12**,147(1975).

[30] D K Choudhury,P Das,D D Goswami and J K Sharma,Pramana J. Phys. **44**,519(1995).

[31] N Isgur and G Karl ,Phys. Lett. B **72**,109(1977).

[32] N Isgur and G Karl ,Phys. Lett. B **74**,353(1978).

[33] N Isgur and G Karl ,Phys. Rev. D **18**,4178(1978).

[34] N Isgur and G Karl ,Phys. Rev. D **19**,2653(1979).

[35] G Karl,N Isgur and R Koniuk ,Phys. Rev. Lett. **41**,1269(1978).

[36] N Isgur and G Karl ,Phys. Rev. D **20**,1191(1979).

[37] F F Schoberl,W Luchat and D Gromes,Phys. Rep. **200**,127(1999).

[38] Buchmuller and Tye, Phys. Rev. D **24**,132 (1981).

[39] A Martin , Phys. Lett. B **93**,338 (1980).

[40] A Martin , Phys. Lett. B **82**,272 (1979).

[41] J L Richardson, Phys. Lett. B **82**,272 (1979).

[42] C Quigg and J L Rosner, Phys. Lett. B **71**,153 (1977).

[43] C Quigg and J L Rosner, Phys. Rep. **56**,167 (1979); E Eichten,K Gottfrid,T Kinoshita, K D Lane and T M Yan, Phys. Rev. D **21**,203 (1980).

[44] Riazuddin and Fiyyazuddin in "A Modern Introduction to Particle Physics" ,(Allied Publishers Limited), **2000, pp-256.**

[45] Y Nambu,Phy. Rev. D **10**,4262(1974).

[46] S Godfrey and N Isgur,Phy. Rev. D **32**,189(1985).

[47] W Fishler, Nucl. Phys. B **129**,157 (1977).

[48] S Godfrey and N Isgur, Phy. Rev. D **32**,189(1985);S Capstick and S Godfrey,ibid. **41**,2856(1990);S Capstick,ibid. **46**,2864(1992).

[49] H Georgi,Phys. Lett. B **240**,447(1990).

[50] F Jugeau,A Le Yaouanc,L Oliver and J C Raynal, Phy. Rev. D **70**,114020(2004).

[51] E V Shuryak, Phys. Lett. B **93**,134 (1980).

[52] F E Close and A Wambach, **RAL-94-041,OUTP-94 09P**,April 1994.

[53] A Le Yaouanc,L Oliver ,O Pene and J C Raynal ,Phys.Lett. B **365**,319 (1996).

[54] A Le Yaouanc ,L Oliver and J C Raynal ,Phy.Rev. D **69**,094022 (2004).

[55] J L Rosner , Phy.Rev. D **42**,3732(1990).

[56] T Mannel ,W Roberts and Z Ryzak, Phy .Rev. D **44**,R18 (1991);T Mannel,W Roberts and Z Ryzak, Phys.Lett. B **255**;593 (1993).

[57] D Ebert,R N Faustov,V O Galkin, hep-ph/ **0611307v1**.

[58] M Neubert , Phys.Lett. B **264**,455 (1991).

[59] B Holdom ,M Sutherland and J Mureika ,Phy.Rev. D **49**, 2359(1994).

[60] E Jenkins ,A Manohar ,M B Wise ,Nucl.Phys. B **396**,38(1996).

[61] Y B Dai ,C S Huang ,M K Huang and C Liu C,Phys.Lett. B **387**,379 (1996).

[62] M A Ivanov ,V E Lyubouvitskij ,L G Körner and P Kroll ,Phy.Rev. D **56**,348(1997).

[63] B König ,J G Körner ,M Krämer and P Kroll ,Phy.Rev. D **56**,4282(1997).

[64] M Neubert,Int. Jou. Mod. Phy. A **11**,4173(1996).

[65] M Sadzikowski and K Zalewski ,Z.Phys. C **59**,667 (1993).

[66] UKQCD Collaboration,K C Bowler *et al*,Nucl.Phys. B **637**,293(2002).

[67] CLEO Collaboration,J Bartlet *et al*,Phys.Rev.Lett **82**,3746(1999).

[68] BELLE Collaboration,K Abe *et al*,Phys.Lett. B **526**,258(2002).

[69] S Godfrey and N Isgur, Phy. Rev D **32**,189(1985).

[70] J J Sakurai in "Advanced Quantum Mechanics" ,(Massachusetts ,Addison-Willey Publishing Company) ,**1986** ,**pp-128**.

[71] C Itzykson and J Zuber in "Quantum Field Theory",(International Student Edition , McGraw Hill ,Singapore) ,**1986,pp-79**.

[72] D K Choudhury and N S Bordoloi,Int. J. Mod. Phys. A **15**,3667(2000).

[73] D K Choudhury and N S Bordoloi ,Mod. Phys.Lett. A **Vol.17**,No.29,1909(2002).

[74] D K Choudhury and N S Bordoloi , Mod. Phys. Lett. A, **26**,443(2009).

[75] V O Galkin,A Yu Mishurov and R N Faustov,Sov. J. Nucl. Phys. **53**,1026(1991).

[76] D E Groom *et al*(Particle Data Group),Eur. Phy. J. C **15**,1(2000).

[77] M G Olsson and S Veseli,FERMILAB-PUB- **96/418-T**.

[78] E Eichten and K Gottfrid,T Kinoshita, K D Lane and T M Yan, Phys. Rev. D **17**;3090 (1978).

[79] B J Hazarika and D K Choudhury, Bra. Jou. Phys. **41**,159(2011).

[80] Y B Ding ,X Q Li and P N Shen , Phy.Rev. D **60**,074010(1999)

[81] B J Hazarika and D K Choudhury,Pramana J. Phy.**75**,423(2010).

[82] Abramowitz and Stegun in "Handbook of Mathematical Functions,1964" .

[83] B J Hazarika , K K Pathak and D K Choudhury,Mod. Phys. Lett. A **26**,1547 (2011).

[84] B J Hazarika and D K Choudhury,Pramana J. Phy. **78**,555(2012).

[85] S J Brodsky , SLAC-PUB- **5013**(1989).

[86] C J Bebek *et al*, Phy. Rev. D **17**,1693(1978).

[87] C R Ji and F Amiri ,Phy. Rev. D **42**,3764(1990).

[88] T Applequist and E Poggio,Phy. Rev.D **10**,3280(1974).

[89] H Pagels and S Stoker,Phy. Rev. D **20**,2947 (1979).

[90] N Isgur and C H Llewellyn Smith,Phy. Lett. B **217**,535(1989); CERN-TH **5013/88.**

[91] D P Stanley and D Robson , Phy. Rev. D **21**,3180(1980);Phy. Rev. D **26**, 223(1982).

[92] S Flugge in Practical Quantum Mechanics ,Springer-Verlag,New York, Hedelberg, Berlin (1974).

[93] H B Dwight in Tables of integrals and other mathematical data,McMillan Company (1969).

[94] B J Hazarika and D K Choudhury; "Form factors and charge radii in a QCD inspired potential using the Variationally Improved Perturbation Theory",arXiv:hep-ph/ **1112.1477**(submitted for publication).

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ADDENDA

A. LIST OF PUBLICATIONS

[1] **"Slope and curvature of Isgur-Wise function using variationally improved perturbation theory in a quantum chromodynamics inspired potential model"** (with D K Choudhury)--- PRAMANA-journal of physics, Vol.75, No. 3, pp.423-438 (2010).

[2] **"Bounds on the Slope and Curvature of Isgur-Wise Function in a QCD-Inspired Quark Model"** (with D K Choudhury)--- Brazilian Journal of Physics, 41, pp.159-166 (2011).

[3] **"ISGUR-WISE FUNCTION IN A QCD POTENTIAL MODEL WITH COULOMBIC POTENTIAL AS PERTURBATION"** (with K K Pathak and D K Choudhury)--- Modern Physics Letters A (MPLA), Vol.26, No. 21, pp.1547-1554 (2011).

[4] **"ISGUR-WISE FUNCTION IN A QCD INSPIRED POTENTIAL MODEL WITH CONFINEMENT AS PARENT IN THE VIPT"** (with D K Choudhury--- PRAMANA-journal of physics, Vol.78, No.4, pp.555-564 (2012).

B. arXiv PUBLICATIONS and SUBMITTED TO JOURNALS

[1] "FORM FACTORS AND CHARGE RADII IN A QCD INSPIRED POTENTIAL MODEL USING THE VIPT." (with D K Choudhury)---arXiv:**hep-ph/1112.1477.**

[2] "ISGUR-WISE FUNCTION IN A QCD INSPIRED POTENTIAL MODEL WITH WKB APPROXIMATION" (with D K Choudhury)---arXiv:**hep-ph/1112.2800.**

[3] "ISGUR-WISE FUNCTION FOR HEAVY LIGHT MESONS IN D DIMENSIONAL POTENTIAL MODEL" (with S Roy and D K Choudhury)---arXiv:**hep-ph/1205.5330.**

PRAMANA — journal of physics

Vol. 75, No. 3
September 2010
pp. 423–438

Slope and curvature of Isgur–Wise function using variationally improved perturbation theory in a quantum chromodynamics inspired potential model

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MS received 9 January 2010; revised 12 April 2010; accepted 28 April 2010

Abstract. We used variationally improved perturbation theory (VIPT) in calculating the slope and curvature of Isgur–Wise (I–W) function with the Cornell potential $-\frac{4\alpha_s}{3r} + br + c$ instead of the usual stationary state perturbation theory as done earlier. We used $-(4\alpha_s/3r)$, i.e. the Coulombic potential, as the parent and the linear one, i.e. $br + c$ as the perturbed potential in the theory and calculated the slope and curvature of Isgur–Wise function including three states in the summation involved in the first-order correction to wave function in the method.

Keywords. Variationally improved perturbation theory; Isgur–Wise function; charge radii; convexity parameter.

PACS Nos 12.39.-x; 12.39.Jh; 12.39.Pn

1. Introduction

The Isgur–Wise (I–W) function is a single unknown form factor which includes all the independent form factors occurring in weak decay amplitudes in the heavy quark limit because in the heavy quark limit, two additional symmetries appear in QCD which gives rise to a $SU(2N)$ symmetry called the heavy quark or Isgur–Wise (I–W) symmetry, where N is the number of quarks. The heavy quark symmetry enormously simplifies the analysis of semileptonic decays [1]. The I–W function and the relevant phenomenology are important topics in QCD as they act as a test for the correctness of any specified QCD-inspired model. Also, as the I–W function is related to the wave function directly, a correct estimation of the wave function is an essential tool to understand the decay processes and the relevant mechanism.

In recent years, a QCD-inspired quark model has been pursued by us [2,3] and I–W function has been calculated. In the model, the two-body Schrödinger equation was solved and first-order perturbed wave function for the ground state was obtained using the Dalgarno method [4]. Also in the model, the spin-independent ground state Fermi–Breit Hamiltonian with no contact term was considered [5] and the linear confinement was treated as perturbation keeping the Coulombic term as the parent one.

As an alternative approach one can use the variationally improved perturbation theory (VIPT) method [7] instead of the Dalgarno method in getting the wave function which combines the variational method and the perturbation theory.

The VIPT is a recent entry in the literature [6–8] which shows great promise regarding the use of approximation methods. The work by Aitchison and Dudek [6] inspired us to apply the method to the QCD-inspired model which had some limitations. Some of these limitations may be due to conventional perturbation technique. We know that the results of perturbation theory are expressed in terms of finite power series (in an expansion parameter which is taken to be very small) that seem to converge to the exact values when summed to higher order. After a certain order, however the results become increasingly worse since the series is usually divergent (being asymptotic). At this juncture, the variational method which estimates variationally optimized parameters (through energy minimization) helps in converting the divergent perturbation expansion to a convergent one which can be evaluated for large expansion parameters. We note that the variational method [4,9] is quite cumbersome as it is difficult to choose an appropriate trial wave function in terms of unknown parameter(s) which is later optimized to estimate the parameter(s). But in VIPT, we use a known wave function as the trial one (e.g. the $1s$ state H-atom wave function) and then optimize it to get the new parameter(s) (e.g. $\bar{\alpha}'_{10}$ in our case (eq. (16)) which make the perturbation series convergent. Further, we know that the perturbation theory is suitable to systems which have good unperturbed Hamiltonian, while variational method is robust even in cases where it is hard to determine a good unperturbed Hamiltonian. On the other hand, VIPT can be applied whether we have a good unperturbed Hamiltonian or not.

Question arises regarding the use of the Coulombic piece as the parent and linear part as the perturbed one of the total Cornell potential – that upto what distance this consideration is valid? Indeed, it was shown in ref. [6] that if $\langle r \rangle < r_0$ then the Coulomb base will perform better. Here $\langle r \rangle$ is the expectation value of the distance r which reasonably gives the size of a state (meson in this case) and r_0 is a point at which linear and Coulomb potentials become zero (figure 1 of Aitchison and Dudek [6]). Further, for low-lying mesons, i.e. $n = 1$, $l = 0$ (see eq. (8) of ref. [6]) the expectation value $\langle r \rangle$ is inversely proportional to the parameter $\alpha = 4\alpha_s/3$ (see eq. (12) of this work) for a given reduced mass μ. Using VIPT we get variably optimized $\bar{\alpha}'_{10}$ (see eq. (16) of this work) as the new parameter instead of α which assumes substantially larger value (see table 1 of this work) than α effectively making the 'linear term' weaker so that Coulombic piece becomes the parent. This ensures that the distance between the quarks is short enough to treat the binding effect mainly in terms of the Coulombic potential. Thus VIPT is a convenient and strong tool in treating the Coulombic potential as parent and linear potential as perturbed of the total Cornell potential.

It is evident from eq. (16) that $\bar{\alpha}'_{10}$ increases with the increase in α_s and greater values of $\bar{\alpha}'_{10}$ strongly support the binding effect mainly in terms of Coulombic potential. For the B-meson, the α_s values are small. It raises the question of applicability of the Coulombic part as the parent. However, the corresponding $\bar{\alpha}'_{10}$ values are sufficiently large to conform to the expectation $\langle r \rangle < r_0$ but not large enough to make the results of slope and curvature of the Isgur–Wise function compatible with the constraints as referred by Neubert [10].

This paper aims to apply the VIPT method to the QCD-inspired quark model [2,3] referred earlier and to calculate the I–W function, its slope and curvature. Using the same Hamiltonian and treating linear confinement as perturbation, we arrive at the hadronic wave function which enables us to calculate I–W function. Relativistic modification of the wave function [11,12] as well as the two-loop effect of strong coupling constant using V-scheme [13–16] are also taken into account.

The rest of the paper is organized as follows: Section 2 contains the formalism, §3 the result and calculation and §4 the discussion and conclusion.

2. Formalism

2.1 *Isgur–Wise function: Its slope and curvature*

The Isgur–Wise function is written as [1]

$$\begin{aligned}\xi(v_\mu \cdot v'_\mu) &= \xi(y) \\ &= 1 - \rho^2(y-1) + C(y-1)^2 + \cdots,\end{aligned} \tag{1}$$

where

$$y = v_\mu \cdot v'_\mu \tag{2}$$

with v_μ and v'_μ being the four velocity of the heavy meson before and after the decay. The quantity ρ^2 is the slope of I–W function at $y = 1$ and known as charge radius:

$$\rho^2 = \left.\frac{\partial \xi}{\partial y}\right|_{y=1}. \tag{3}$$

The second-order derivative is the curvature of the I–W function known as convexity parameter:

$$C = \frac{1}{2}\left.\left(\frac{\partial^2 \xi}{\partial y^2}\right)\right|_{y=1}. \tag{4}$$

For the heavy–light flavour mesons, the I–W function can also be written as [3,17]

$$\xi(y) = \int_0^{+\infty} 4\pi r^2 \left|\psi(r)\right|^2 \cos pr \, \mathrm{d}r \tag{5}$$

where

$$p^2 = 2\mu(y-1). \tag{6}$$

Here μ and ψ are respectively the reduced mass and wave function of the hadronic system.

2.2 *Variationally improved perturbation theory*

The VIPT method is not too old [6–8] and it combines two procedures, namely, stationary state perturbation theory and the variational method. We have the total Hamiltonian as

$$H = H_0 + H', \tag{7}$$

where H_0 is the parent Hamiltonian containing a physical parameter P (say) and H' is the perturbed Hamiltonian. The corresponding wave functions also contain P.

In VIPT,

$$P = P + P' - P', \tag{8}$$

where P' is the variational parameter such that

$$\begin{aligned} H &= H_{oP'} + H_o - H_{oP'} + H' \\ &= H_{oP'} + H'_{P'}. \end{aligned} \tag{9}$$

The parent Hamiltonian is now $H_{oP'}$ instead of H_o which depends on the variational parameter P' and $H'_{P'}$ is the new perturbed Hamiltonian instead of H' which also depends on P'. Correspondingly the wave functions will also change when P is replaced by P'. Now, one can treat these wave functions as trial wave functions with P' as the variational parameter and would find the value of P' which gives minimum value of energy corrected upto the first order. This will yeild variationally improved unperturbed wave function upon which the usual perturbation theory will be applied.

The wave function corrected up to the first order of jth state is given by [6]

$$\psi_j = \psi_j^{(0)} + \sum_{k \neq j} \frac{\int \psi_k^{(0)*} H'_{P'j} \psi_j^{(0)} \mathrm{d}v}{E_j^{(0)} - E_k^{(0)}}. \tag{10}$$

The energy corrected up to first order for the same state is

$$\begin{aligned} E_j &= \int \psi_j^{(0)*} H \psi_j^{(0)} \mathrm{d}v \\ &= \int \psi_j^{(0)*} (H_{oP'} + H'_{P'}) \psi_j^{(0)} \mathrm{d}v, \end{aligned} \tag{11}$$

where ψ_k and E_k are the wave function and energy eigenvalues of the kth state which are orthonormal to jth state. The superscript (0) is the zeroeth-order correction of the corresponding quantities.

With Cornell potential [18], we can have two possibilities to choose parent (and hence perturbed) Hamiltonian. In one, Coulombic one is the parent and in the other, linear one is the parent.

The summation in eq. (10) can include any number of kth states. In this work, terms upto three states in the summation are considered.

2.3 *Coulomb cum linear potential and wave functions using VIPT*

2.3.1 *With one term in the summation*

As explained earlier, variational parameter α' is used instead of the physical parameter $\alpha = 4\alpha_s/3$ (here Coulombic potential is the parent one). The Hamiltonian takes the form (eq. (9)):

$$\begin{aligned} H &= H_o + H' \\ &= -\frac{\nabla^2}{2\mu} - \frac{4\alpha_s}{3r} + br + c \\ &= -\frac{\nabla^2}{2\mu} - \frac{\alpha}{r} + br + c \\ &= -\frac{\nabla^2}{2\mu} - \frac{\alpha'}{r} + \frac{(\alpha' - \alpha)}{r} + br + c \\ &= H_{o\alpha'} + H'_{\alpha'}, \end{aligned} \tag{12}$$

where $\alpha = \alpha - \alpha' + \alpha'$. Now, $H_{o\alpha'} = -\frac{\nabla^2}{2\mu} - \frac{\alpha'}{r}$ is the parent Hamiltonian with α' and $H'_{\alpha'} = \frac{(\alpha'-\alpha)}{r} + br + c$ is the perturbed Hamiltonian with the same variational parameter α'. We notice that the physical parameter α is replaced by the variational parameter α'.

We consider j as $1s$ state ($n = 1$, $l = 0$) and in the summation of eq. (10), we consider only one kth state which is the $2s$ state ($n = 2$, $l = 0$).

The trial $1s$ state can be written (analogous to H-atom) with variational parameter α' as (this being the unperturbed wave function)

$$\psi_{10}^{(0)} = \frac{(\mu\alpha'_{10})^{3/2}}{\sqrt{\pi}} \mathrm{e}^{-\mu\alpha'_{10}r}, \tag{13}$$

where subscript 10 in α' indicates the quantum number (n, l) of the jth state.

We now find the value of α'_{10} which leads to minimum E_j given by (11) in the following way:

In the variational method, we are interested only in the 'r'-dependence of the Hamiltonian, and so c in $H'_{\alpha'}$ has no role to play in the calculation [4]. Using eqs (11), (12), (13)

$$E_{10}\left(\alpha'_{10}\right) = \frac{\mu\alpha'^2_{10}}{2} - \mu\alpha\alpha'_{10} + \frac{3b}{3\mu\alpha'_{10}}. \tag{14}$$

Minimization of eq. (14) gives

$$\alpha'^3_{10} - \alpha\alpha'^2_{10} - \frac{3b}{2\mu^2} = 0. \tag{15}$$

The solution of (15) is the required value of α'_{10} which gives minimum $E_{10}(\alpha'_{10})$ and we denote it by $\bar{\alpha}'_{10}$. Thus, unperturbed wave function in VIPT is

Table 1. $\bar{\alpha}'_{10}$ (eq. (15) for different mesons with α_s values under $\overline{\text{MS}}$ scheme.

Mesons	μ	α_s	$\alpha = 4\alpha_s/3$	$\bar{\alpha}'_{10}$
D	0.2761	0.39	0.52	1.7271
D_s	0.3648	0.39	0.52	1.4642
B	0.3100	0.22	0.2933	1.5104

Table 2. $\bar{\alpha}'_{10}$ (eq. (15)) for different mesons with α_s values in V-scheme.

Mesons	μ	α_s	$\alpha = 4\alpha_s/3$	$\bar{\alpha}'_{10}$
D	0.2761	0.693	0.924	1.9105
D_s	0.3648	0.693	0.924	1.6593
B	0.3100	0.261	0.348	1.531

$$\psi_{10}^{(0)}(\bar{\alpha}'_{10}) = \frac{(\mu\bar{\alpha}'_{10})^{3/2}}{\sqrt{\pi}} \mathrm{e}^{-\mu\bar{\alpha}'_{10}r}. \tag{16}$$

Here α'_{10} will be different for different mesons as solution of eq. (15) depends on μ and α with $b = 0.183$. We list the values of $\overline{\alpha}'_{10}$ in table 1 using known values of α_s under $\overline{\text{MS}}$ [3] and those in table 2 with α_s in V-scheme [13–16].

Now we consider the single kth state in the summation of eq. (10) which is the $2s$ state given by

$$\begin{aligned}\psi_k^{(0)}(\bar{\alpha}'_{10}) &= \psi_{20}^{(0)}(\bar{\alpha}'_{10}) \\ &= \frac{(\mu\bar{\alpha}'_{10})^{3/2}}{\sqrt{8\pi}} \mathrm{e}^{-\mu\bar{\alpha}'_{10}r/2}\left(1 - \frac{\mu\bar{\alpha}'_{10}r}{2}\right).\end{aligned} \tag{17}$$

Therefore, eq. (10) gives wave function corrected up to first order:

$$\psi_{10}(\bar{\alpha}'_{10}) = \psi_{10}^{(0)}(\bar{\alpha}'_{10}) + \frac{\int \psi_{20}^{(0)*}(\bar{\alpha}'_{10}) H'_{\bar{\alpha}'_{10}} \psi_{10}^{(0)}(\bar{\alpha}'_{10})\,\mathrm{d}v}{E_{10}^{(0)}(\bar{\alpha}'_{10}) - E_{20}^{(0)}(\bar{\alpha}'_{10})} \psi_{20}^{(0)}(\bar{\alpha}'_{10}). \tag{18}$$

The energy eigenvalues are given by

$$E_{n0}^{(0)}(\bar{\alpha}'_{10}) = -\frac{\mu\bar{\alpha}'^{2}_{10}}{2n^2}. \tag{19}$$

The summation in eq. (19) is dropped as we are considering single kth state. Also, we have $n = 1$ and 2, due to the single-state consideration in eq. (10). Carrying out the integration in (19) we find the wave function corrected up to the first order as

$$\begin{aligned}\psi_{10}(\bar{\alpha}'_{10}) = \psi_{10}^{(0)}(\bar{\alpha}'_{10}) \\ -\frac{4\sqrt{\mu}}{3\sqrt{\pi}(\bar{\alpha}'_{10})^{1/2}}\left(\frac{4\mu\bar{\alpha}'_{10}(\alpha - \bar{\alpha}'_{10})}{27} - \frac{32b}{81\mu\bar{\alpha}'_{10}}\right) \\ \times\left(1 - \frac{\mu\bar{\alpha}'_{10}r}{2}\right)\mathrm{e}^{\mu\bar{\alpha}'_{10}r/2}.\end{aligned} \tag{20}$$

The relativistic version of (20) is [11,12]

$$\psi_{10,\mathrm{rel}}(\bar{\alpha}'_{10}) = \psi_{10}(\bar{\alpha}'_{10})[(r\mu\bar{\alpha}'_{10})^{-\epsilon}] \tag{21}$$

with

$$\epsilon = 1 - \sqrt{1 - \frac{4\alpha_s}{3}}. \tag{22}$$

The expressions for I–W function, charge radius and convexity parameter with confinement only (which corresponds to wave function given by eq. (20)) are

$$\xi_{S,\mathrm{conf}}(y) = 1 - \rho^2_{S,\mathrm{conf}}(y-1) + C_{S,\mathrm{conf}}(y-1)^2 + \cdots, \tag{23}$$

where the charge radius is

$$\rho^2_{S,\mathrm{conf}} = \frac{4\pi N_1^2}{\mu^3\bar{\alpha}'^5_{10}}\left[\frac{3c_1'^2}{4} + 84A^2 + \frac{1024c_1'A}{243}\right], \tag{24}$$

and the convexity parameter is

$$C_{S,\mathrm{conf}} = \frac{4\pi N_1^2}{6\mu^3\bar{\alpha}'^7_{10}}\left[\frac{45c_1'^2}{8} + 5760A^2 + \frac{20\times 2^{12}c_1'A}{3^6}\right]. \tag{25}$$

Here,

$$c_1' = \frac{\mu\bar{\alpha}'_{10}}{\pi^{1/3}} \tag{26}$$

and

$$A = \frac{4\sqrt{\mu}}{3\sqrt{\pi}(\bar{\alpha}'_{10})^{1/2}}\left[\frac{4\mu\bar{\alpha}'_{10}(\alpha - \bar{\alpha}'_{10})}{27} - \frac{32b}{81\mu\bar{\alpha}'_{10}}\right]. \tag{27}$$

The subscript S refers to the single-state consideration in the summation of eq. (10). The normalization constant N_1 is given by

$$4\pi N_1^2 = \frac{1}{[(c_1'^2/4\mu^3\bar{\alpha}'^3_{10}) + (2A^2/\mu^3\bar{\alpha}'^3_{10})]}. \tag{28}$$

The respective relativistic versions are

$$\xi_{S,\mathrm{rel+conf}}(y) = 1 - \rho^2_{S,\mathrm{rel+conf}}(y-1) + C_{S,\mathrm{rel+conf}}(y-1)^2 + \cdots \tag{29}$$

$$\rho^2_{S,\mathrm{rel+conf}} = \frac{4\pi N_1'^2\Gamma(3-2\epsilon)(4-2\epsilon)(3-2\epsilon)}{\mu^3\bar{\alpha}'^5_{10}}\left[\frac{c_1'^2}{32} + X_1 + X_2\right] \tag{30}$$

and

$$C_{S,\mathrm{rel+conf}} = \frac{4\pi N_1'^2 \Gamma(3-2\epsilon)(6-2\epsilon)(5-2\epsilon)(4-2\epsilon)(3-2\epsilon)}{6\mu^3 \bar{\alpha}_{10}'^7} \times \left[\frac{c_1'^2}{128} + X_3 + X_4\right]. \tag{31}$$

Here the normalization constant N_1' is given by

$$4\pi N_1'^2 = \frac{\mu^3 \bar{\alpha}_{10}'^3}{\Gamma(3-2\epsilon)\left[\frac{c_1'^2}{8} + X_5 + X_6\right]}. \tag{32}$$

All the functions $X_i(\epsilon), i = 1, 2, ..., 6$ are defined in the Appendix.

2.3.2 *With two terms in the summation*

In this step, we consider the $3s$ state ($n = 3$, $l = 0$) in addition to $2s$ state (as done in the single-term case). The $3s$ state with the variational parameter $\bar{\alpha}_{10}'$ is written as

$$\psi_{30}^{(0)}(\bar{\alpha}_{10}') = \frac{(\mu\bar{\alpha}_{10}')^{3/2}}{\sqrt{27\pi}} \left(1 - \frac{2\mu\bar{\alpha}_{10}' r}{3} + \frac{2\mu^2 \bar{\alpha}_{10}'^2 r^2}{27}\right) \mathrm{e}^{-(\mu\bar{\alpha}_{10}' r/3)}. \tag{33}$$

By including this state, the summation and integration in (10) gives the wave function corrected upto the first order as

$$\psi_{10}(\bar{\alpha}_{10}') = \psi_{10}^{(0)}(\bar{\alpha}_{10}') - A\left(1 - \frac{\mu\bar{\alpha}_{10}' r}{2}\right) \mathrm{e}^{\mu\bar{\alpha}_{10}' r/2} + B\left(1 - \frac{-2\mu\bar{\alpha}_{10}' r}{3} + \frac{2\mu^2 \bar{\alpha}_{10}'^2 r^2}{27}\right) \mathrm{e}^{-(\mu\bar{\alpha}_{10}' r/3)}, \tag{34}$$

where

$$B = \frac{\sqrt{\mu}}{\sqrt{\pi}(\bar{\alpha}_{10}')^{1/2}} \left[\frac{3\mu\bar{\alpha}_{10}'(\alpha - \bar{\alpha}_{10}')}{64} - \frac{27b}{256\mu\bar{\alpha}_{10}'}\right]. \tag{35}$$

The relativistic version is obtained by multiplying (34) by $(r\mu\bar{\alpha}_{10}')^{-\epsilon}$. The I–W function, charge radius and convexity parameter for the wave function (34) which is to be normalized are given by (i.e. with confinement only)

$$\xi_{D,\mathrm{conf}}(y) = 1 - \rho^2_{D,\mathrm{conf}}(y-1) + C_{D,\mathrm{conf}}(y-1)^2 + ..., \tag{36}$$

where the charge radius is

$$\rho^2_{D,\mathrm{conf}} = \frac{4\pi N_2^2}{\mu^3 \bar{\alpha}_{10}'^5} \left[\frac{3c_1'^2}{4} + 84A^2 + \frac{1024 c_1' A}{243} - \frac{3^4 \times 211 \times B^2}{4} + \frac{3^6 \times 39 \times c_1' B}{2^8} + \frac{6^6 \times 69 \times 16 \times AB}{3 \times 5^7}\right] \tag{37}$$

and the convexity parameter is

$$C_{D,\mathrm{conf}} = \frac{4\pi N_2^2}{6\mu^3\bar{\alpha}_{10}'^7}\left[\frac{45c_1'^2}{8} + 5760A^2 + \frac{20\times 2^{12}c_1'A}{3^6} + 414163\times B^2 + \frac{3^9\times 185\times c_1'B}{4^5} + \frac{6^9\times 24608\times AB}{3\times 5^9}\right] \quad (38)$$

with normalizaion constant N_2 given by

$$4\pi N_2^2 = \frac{1}{\left[\frac{c_1'^2}{4\mu^3\bar{\alpha}_{10}'^3} + \frac{2A^2}{\mu^3\bar{\alpha}_{10}'^3} + \frac{27B^2}{4\mu^3\bar{\alpha}_{10}'^3} + \frac{27c_1'B}{4\mu^3\bar{\alpha}_{10}'^3} - \frac{6^3\times 492\times AB}{5^5\mu^3\bar{\alpha}_{10}'^3}\right]}. \quad (39)$$

The subscript D refers to two terms in the summation. The respective relativistic versions of (36), (37) and (38) are

$$\xi_{D,\mathrm{rel+conf}}(y) = 1 - \rho^2_{D,\mathrm{rel+conf}}(y-1) + C_{D,\mathrm{rel+conf}}(y-1)^2 + \cdots, \quad (40)$$

where

$$\rho^2_{D,\mathrm{rel+conf}} = \frac{4\pi N_2'^2(4-2\epsilon)(3-2\epsilon)\Gamma(3-2\epsilon)}{\mu^3\bar{\alpha}_{10}'^5} \times\left[\frac{c_1'^2}{32} + X_1 + X_2 + \sum_{i=7}^{11} X_i\right] \quad (41)$$

and

$$C_{D,\mathrm{rel+conf}} = \frac{4\pi N_2'^2(6-2\epsilon)(5-2\epsilon)(4-2\epsilon)(3-2\epsilon)\Gamma(3-2\epsilon)}{6\mu^3\bar{\alpha}_{10}'^7} \times\left[\frac{c_1'^2}{128} + X_3 + X_4 + \sum_{i=12}^{16} X_i\right]. \quad (42)$$

The normalization constant N_2' is given as

$$4\pi N_2'^2 = \frac{\mu^3\bar{\alpha}_{10}'^3}{\Gamma(3-2\epsilon)\left[\frac{c_1'^2}{8} + X_5 + X_6 + \sum_{i=17}^{21} X_i\right]} \quad (43)$$

and $X_i(\epsilon)$, $i = 7, 8, \ldots, 21$ are defined in the Appendix.

2.3.3 *With three terms in the summation*

In addition to the 2s and 3s states, we now add the 4s state:

$$\psi_{40}^{(0)}(\bar{\alpha}_{10}') = \frac{(\mu\bar{\alpha}_{10}')^{3/2}}{\sqrt{2\pi}} \times\left(\frac{1}{4} - \frac{3\mu\bar{\alpha}_{10}'r}{16} + \frac{\mu^2\bar{\alpha}_{10}'^2r^2}{32} - \frac{\mu^3\bar{\alpha}_{10}'^3r^3}{8\times 96}\right)\mathrm{e}^{-(\mu\bar{\alpha}_{10}'r/4)}. \quad (44)$$

With the inclusion of this state, the first-order wave function now becomes

$$\psi_{10}(\bar{\alpha}'_{10}) = \psi_{10}^{(0)}(\bar{\alpha}'_{10}) - A\left(1 - \frac{\mu\bar{\alpha}'_{10}r}{2}\right) e^{-(\mu\bar{\alpha}'_{10}r/2)}$$
$$+B\left(1 - \frac{2\mu\bar{\alpha}'_{10}r}{3} + \frac{2\mu^2\bar{\alpha}'^2_{10}r^2}{27}\right) e^{-(\mu\bar{\alpha}'_{10}r/3)}$$
$$+D'\left(\frac{1}{4} - \frac{3\mu\bar{\alpha}'_{10}r}{16} + \frac{\mu^2\bar{\alpha}'^2_{10}r^2}{32} - \frac{\mu^3\bar{\alpha}'^3_{10}r^3}{8\times 96}\right) e^{-(\mu\bar{\alpha}'_{10}r/4)}, \quad (45)$$

where

$$D' = \frac{(\mu\bar{\alpha}'_{10})^{3/2}}{\sqrt{\pi}}\left[\frac{36(\alpha - \bar{\alpha}'_{10})}{15625\bar{\alpha}'_{10}} - \frac{384b}{78125\mu^2\bar{\alpha}'^3_{10}}\right]. \quad (46)$$

As usual, the relativistic version of this wave function is obtained by multiplying the above expression by $(r\mu\bar{\alpha}'_{10})^{-\epsilon}$. Thus, with confinement only the I–W function is

$$\xi_{T,\text{conf}}(y) = 1 - \rho^2_{T,\text{conf}}(y-1) + C_{T,\text{conf}}(y-1)^2 + \cdots, \quad (47)$$

where charge radius is

$$\rho^2_{T,\text{conf}} = \frac{4\pi N_3^2}{\mu^3\bar{\alpha}'^5_{10}}\left[\frac{\rho^2_{D,\text{conf}}\mu^3\bar{\alpha}'^5_{10}}{4\pi N_2^2} + 10368\times D'^2 - 2.51\times D'c'_1\right.$$
$$\left.-109.88\times D'A - 2558.46\times D'B\right] \quad (48)$$

and convexity parameter is

$$C_{T,\text{conf}} = \frac{4\pi N_3^2}{6\mu^3\bar{\alpha}'^7_{10}}\left[\frac{C_{D,\text{conf}}6\mu^3\bar{\alpha}'^7_{10}}{4\pi N_2^2} + 9123840\times D'^2\right.$$
$$\left.-19.32\times D'c'_1 - 3196.4\times D'A - 183755.94\times D'B\right] \quad (49)$$

with

$$4\pi N_3^2 = \frac{1}{\left[\frac{c'^2_1}{4\mu^3\bar{\alpha}'^3_{10}} + \frac{2A^2}{\mu^3\bar{\alpha}'^3_{10}} + \frac{27B^2}{4\mu^3\bar{\alpha}'^3_{10}} + \frac{27c'_1B}{4\mu^3\bar{\alpha}'^3_{10}} - \frac{6^3\times 492\times AB}{5^5\mu^3\bar{\alpha}'^3_{10}} + \frac{16D'^2}{\mu^3\bar{\alpha}'^3_{10}}\right]}. \quad (50)$$

Here, the subscript T refers to three terms in the summation.

The corresponding relativistic expressions are

$$\xi_{T,\text{rel+conf}}(y) = 1 - \rho^2_{T,\text{rel+conf}}(y-1) + C_{T,\text{rel+conf}}(y-1)^2 + \cdots, \quad (51)$$

where

$$\rho^2_{T,\mathrm{rel+conf}} = \frac{4\pi N_3'^2 (4-2\epsilon)(3-2\epsilon)\Gamma(3-2\epsilon)}{\mu^3 \bar{\alpha}_{10}'^5} \left[\frac{c_1'^2}{32} + \sum_{\imath=22}^{29} X_\imath \right] \tag{52}$$

and

$$C_{T,\mathrm{rel+conf}} = \frac{4\pi N_3'^2 (6-2\epsilon)(5-2\epsilon)(4-2\epsilon)(3-2\epsilon)\Gamma(3-2\epsilon)}{6\mu^3 \bar{\alpha}_{10}'^7} \times \left[\frac{c_1'^2}{128} + \sum_{\imath=30}^{37} X_\imath \right]. \tag{53}$$

The normalization constant is given by

$$4\pi N_3'^2 = \frac{\mu^3 \bar{\alpha}_{10}'^3}{\Gamma(3-2\epsilon)[\frac{c_1'^2}{8} + \sum_{\imath=38}^{45} X_\imath]} \tag{54}$$

and the functions $X_\imath(\epsilon)$, $\imath = 21, 22, \ldots, 45$ are defined in the Appendix.

3. Calculation and results

We have listed the values of charge radius and convexity parameter of the calculated I–W function for various heavy–light flavour mesons in the present method considering single state, two states and three states in the summation occurred in VIPT with confinement and relativistic effect.

To set the tables we have used two sets of α_s values: one under $\overline{\mathrm{MS}}$ scheme [3] and the other under V-scheme [13–16] at 'c' and 'b'-quark mass scale so that we get two sets of readings for the same quantities. Table 3 represents the numerical values of the parameters c_1', A, B, D' given by eqs (26), (27), (35) and (46) respectively with α_s under $\overline{\mathrm{MS}}$ scheme while table 4 represents those values with α_s values under V-scheme. Similarly, tables 5–7 give charge radius and convexity parameter for different combinations of states with α_s values under $\overline{\mathrm{MS}}$ scheme whereas tables 8–10 give the same quantities with α_s values under V-scheme. The values of $\bar{\alpha}_{10}'$ are taken from tables 1 and 2.

In table 11, we record the predictions of ρ^2 and C for the present model [19] using Dalgarno method [4] while in table 12, we refer to the predicted values of ρ^2 and C for different models [19–31]. In table 11, only one set of result is shown for the D-, D_s-mesons while two sets are shown for B-meson taken from the tables 1, 2 and 4 of ref. [19] to show the preference of higher α_s values for this meson. Specifically, it is seen that for $\alpha_s = 0.261$, as computed in the V-scheme at b-quark scale, the predictions overshoot the predictions of other models (table 12) by two orders of magnitude. However, for $\alpha_s = 0.60$, the results are comparable. In ref. [19] such an enhanced value of α_s was attributed to the necessity of potentially large flavour-dependent higher-order effects beyond $O(\alpha_s^3)$ in the V-scheme [14–16].

An analysis of tables 5–10 shows that relativistic effects invariably reduce the values of ρ^2 and C so as to bring them close to the predictions of other models. This feature further improves as we take two and three terms in the summation of eq. (10).

Table 3. Various parameters with α_s values under $\overline{\mathrm{MS}}$ scheme.

Mesons	α_s	c_1'	A	$B \times 10^{-2}$	$D' \times 10^{-4}$
D	0.39	0.33	−0.0712	0.304	5.055
D_s	0 39	0.37	−0.0800	0.340	5.600
B	0.22	0.32	−0.0820	0.350	5.770

Table 4. Various parameters with α_s values in the V-scheme.

Mesons	α_s	c_1'	A	$B \times 10^{-2}$	$D' \times 10^{-4}$
D	0.693	0.36	−0.0613	0.3166	4.345
D_s	0.693	0.42	−0.0660	0.3400	4.650
B	0.261	0.33	−0.0800	0.4100	5.670

Table 5. Charge radius and convexity parameter with single term in eq. (10) under $\overline{\mathrm{MS}}$ scheme.

Mesons	$\rho^2_{S,\mathrm{conf}}$	$C_{S,\mathrm{conf}}$	$\rho^2_{S,\mathrm{rel+conf}}$	$C_{S,\mathrm{rel+conf}}$
D	3.73	13.92	2.197	5.61
D_s	5.06	26.13	2.530	10.54
B	5.83	29.03	4.132	18.72

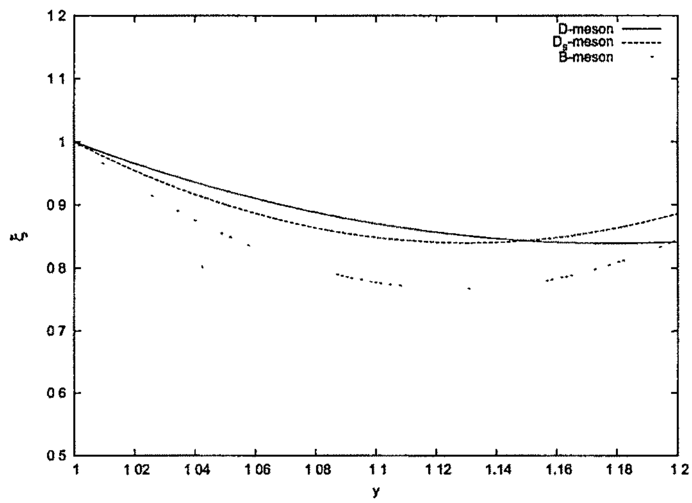

Figure 1. Variation of Isgur–Wise function $\xi(y)$ vs. velocity transfer ratio 'y' with three terms in the summation of eq. (10) (see table 7).

Correspondingly, the graphs which show the variation of I–W function $\xi(y)$ vs. velocity transfer ratio 'y' consist of two figures out of which the first one (i.e. figure 1) correspond to $\overline{\mathrm{MS}}$ scheme and the last one (i.e. figure 2) to V-scheme.

Table 6. Charge radius and convexity parameter with two terms in eq. (10) under $\overline{\text{MS}}$ scheme.

Mesons	$\rho^2_{D,\text{conf}}$	$C_{D,\text{conf}}$	$\rho^2_{D,\text{rel+conf}}$	$C_{D,\text{rel+conf}}$
D	2.84	9.37	1.83	5.184
D_s	3.90	17.72	2.50	9.776
B	4.14	18.55	3.72	14.92

Table 7. Charge radius and convexity parameter with three terms in eq. (10) under $\overline{\text{MS}}$ scheme.

Mesons	$\rho^2_{T,\text{conf}}$	$C_{T,\text{conf}}$	$\rho^2_{T,\text{rel+conf}}$	$C_{T,\text{rel+conf}}$
D	2.83	9.15	1.80	5.04
D_s	3.88	17.28	2.46	9.45
B	4.13	18.10	3.68	14.53

Table 8. Charge radius and convexity parameter with single term in eq. (10) under V-scheme.

Mesons	$\rho^2_{S,\text{conf}}$	$C_{S,\text{conf}}$	$\rho^2_{S,\text{rel+conf}}$	$C_{S,\text{rel+conf}}$
D	2.19	6.22	0.433	0.525
D_s	2.62	9.55	0.560	0.850
B	5.43	26.26	3.570	15.270

Table 9. Charge radius and convexity parameter with two terms in eq. (10) under V-scheme.

Mesons	$\rho^2_{D,\text{conf}}$	$C_{D,\text{conf}}$	$\rho^2_{D,\text{rel+conf}}$	$C_{D,\text{rel+conf}}$
D	1.82	4.57	0.432	0.524
D_s	2.28	7.31	0.550	0.840
B	3.60	16.20	3.160	12.320

Table 10. Charge radius and convexity parameter with three terms in eq. (10) under V-scheme.

Mesons	$\rho^2_{T,\text{conf}}$	$C_{T,\text{conf}}$	$\rho^2_{T,\text{rel+conf}}$	$C_{T,\text{rel+conf}}$
D	1.79	4.36	0.430	0.516
D_s	2.25	6.98	0.545	0.815
B	3.55	15.43	3.120	11.770

Table 11. Predictions of the slope and curvature of the I–W function with $b = 0.183$ GeV^2, $A_0 = 1$ and $c = 1$ GeV in V-scheme for the model of ref. [19] with relativistic and confinement effect.

Mesons	α_s	$\rho^2_{\text{rel+conf}}$	$C_{\text{rel+conf}}$
D	0.625	1.136	5.377
D_s	0.625	1.083	3.583
B	0.261	128.1	5212
	0.600	1.329	7.2

Table 12. Predictions of the slope and curvature of the I–W function in various models.

Model	Value of ρ^2	Value of curvature C
Le Youanc *et al* [20]	≥ 0.75	–
Le Youanc *et al* [21]	≥ 0.75	≥ 0.47
Rosner [28]	1.66	2.76
Mannel [29,30]	0.98	0.98
Pole ansatz [31]	1.42	2.71
MIT bag model [27]	2.35	3.95
Simple quark model [26]	1	1.11
Skryme model [24]	1.3	0.85
QCD sum rule [25]	0.65	0.47
Relativistic three-quark model [23]	1.35	1.75
Infinite momentum frame quark model [22]	3.04	6.81
Neubert [10]	0.82±0.09	–

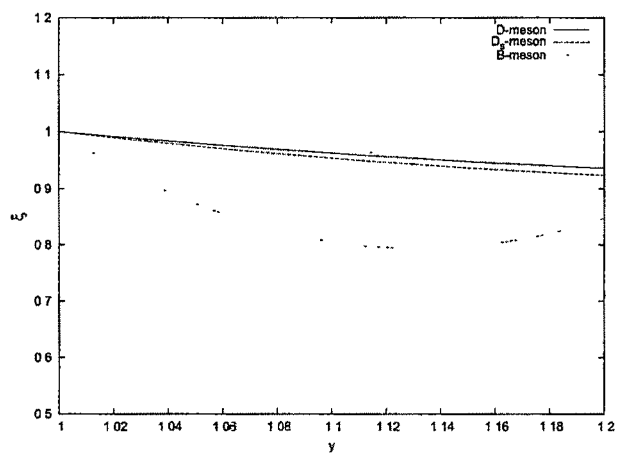

Figure 2. Variation of Isgur–Wise function $\xi(y)$ vs. 'y' with three terms in the summation of eq. (10) (see table 10).

4. Discussion and conclusion

In this paper, we have calculated the slope and curvature of the I–W function using VIPT method in the QCD-inspired quark model [3,13,19]. In this approach, we notice that with the inclusion of more states in the summation of eq. (10), the results come closer to the predictions of the other models [19–31]. We have seen from the results that the slope and curvature agree quite well with the values and bounds of other models in table 12 for D- and D_s-mesons but not as expected for B-meson. This is due to the low value of α_s for B-meson. Such a feature was earlier noticed in ref. [19] too, suggesting the necessity of higher-order effects beyond $O(\alpha_s^3)$ in V-scheme.

We also note that eqs (24), (25), (30), (31), (37), (38), (41), (42), (48), (49), (52) and (53) along with (28), (32), (39), (43), (50) and (54) of the text contain several large numerical factors appearing to be divergent compared to the leading order term which is contrary to the expectation of perturbation theory. However, a careful study reveals that actually it is not so.

As an illustration, the correct leading order term in eq. (24) with $b = 0$, $\bar{\alpha}'_{10} = \alpha$ becomes $\rho^2_{S,\mathrm{conf,LO}} = 3/\alpha^2 = 27/16\alpha_s^2$; which for $\alpha_s = 0.693$ is $\sim$3.51 not far away from the results of table 8. Similar analysis can be done for the other equations as well.

It will also be interesting to explore if the linear potential as parent incorporating more terms in the correction for wave function can improve the results of the present analysis as far as B-meson is concerned. Such an investigation is currently under progress.

Appendix

$$X_1 = A^2\left(1 + \frac{(6-2\epsilon)(5-2\epsilon)}{4} - (5-2\epsilon)\right) \tag{A1}$$

$$X_2 = 64c_1'A\left(\frac{(5-2\epsilon)}{729} - 1243\right) \tag{A2}$$

$$X_3 = A^2\left(1 + \frac{(8-2\epsilon)(7-2\epsilon)}{4} - (7-2\epsilon)\right) \tag{A3}$$

$$X_4 = 256c_1'A\left(-\frac{1}{2187} + \frac{(7-2\epsilon)}{6561}\right) \tag{A4}$$

$$X_5 = A^2\left[1 + \frac{(4-2\epsilon)(3-2\epsilon)}{4} - (3-2\epsilon)\right]. \tag{A5}$$

Rest of the equations can be obtained from the authors on request.

Acknowledgement

The authors thank N S Bordoloi, S Kalita, Z H Devi, C Thakuria, B R Baruah and A Baruah for their technical support. The authors also thank F M Fernandez and I J R Aitchison for inspiring correspondence.

References

[1] N Isgur and M B Wise, *Phys. Lett.* **B232**, 113 (1989)
[2] D K Choudhury, P Das, D D Goswami and J N Sharma, *Pramana – J. Phys.* **44**, 519 (1995)
[3] D K Choudhury and N S Bordoloi, *Int. J. Mod. Phys.* **A15**, 3667 (2000)
[4] A K Ghatak and S Lokanathan, in: *Quantum mechanics* (McGraw Hill, 1997) pp. 291
[5] A D Rujula, H Georgi and S L Glashow, *Phys. Rev.* **D12**, 147 (1975)
[6] I J R Aitchison and J J Dudek, *Eur. J. Phys.* **23**, 605 (2002)
[7] S K You, K J Jeon, C K Kim and K Nahm, *Eur. J. Phys.* **19**, 179 (1998)
[8] F M Fernandez, *Eur. J. Phys.* **24**, 289 (2003)
[9] Y B Ding, X Q Li and P N Shen, *Phys. Rev.* **D60**, 074010 (1999)
[10] M Neubert, *Int. J. Mod. Phys.* **A11**, 4173 (1996)
[11] J J Sakurai, in: *Advanced quantum mechanics* (Addison-Wesley Publishing Company, Massachusetts, 1986) p. 128
[12] C Itzykson and J Zuber, in: *Quantum field theory* (International Student Edition, McGraw Hill, Singapore, 1986) p. 79
[13] D K Choudhury and N S Bordoloi, *Mod. Phys. Lett.* **A17(29)**, 1909 (2002)
[14] M Peter, *Phys. Rev. Lett.* **78**, 603 (1997); *Nucl. Phys.* **B501**, 471 (1997)
[15] Y Schroeder, *Phys. Lett.* **B447**, 321 (1999)
[16] Y Schroeder, *Nucl. Phy. Proc. Suppl.* **86**, 525 (2000)
[17] F E Close and A Wambach, *Nucl. Phys.* **B412**, 169 (1994)
[18] Riazuddin and Fiyyazuddin, in: *A modern introduction to particle physics* (Allied Publishers Limited, 2000) p. 256
[19] D K Choudhury and Bordoloi, *Mod. Phys. Lett.* **A26**, 443 (2009)
[20] A Le Yaouanc, L Oliver, O Pene and J C Raynal, *Phys. Lett.* **B365**, 319 (1996)
[21] A Le Yaouanc, L Oliver and J C Raynal, *Phys. Rev.* **D69**, 094022 (2004)
[22] B König, J G Körner, M Krämer and P Kroll, *Phys. Rev.* **D56**, 4282 (1997)
[23] M A Ivanov, V E Lyubouvitskij, L G Körner and P Kroll, *Phys. Rev* **D56**, 348 (1997)
[24] E Jenkins, A Manohar and M B Wise, *Nucl. Phys.* **B396**, 38 (1996)
[25] Y B Dai, C S Huang, M K Huang and C Liu, *Phys. Lett.* **B387**, 379 (1996)
[26] B Holdom, M Sutherland and J Mureika, *Phys. Rev.* **D49**, 2359 (1994)
[27] M Sadzikowski and K Zalewski, *Z Phys.* **C59**, 667 (1993)
[28] J L Rosner, *Phys. Rev.* **D42**, 3732 (1990)
[29] T Mannel, W Roberts and Z Ryzak, *Phys. Rev.* **D44**, R18 (1991)
[30] T Mannel, W Roberts and Z Ryzak, *Phys. Lett.* **B255**, 593 (1993)
[31] M Neubert, *Phys. Lett.* **B264**, 455 (1991)

Braz J Phys (2011) 41:159–166
DOI 10.1007/s13538-011-0021-6

Bounds on the Slope and Curvature of Isgur-Wise Function in a QCD-Inspired Quark Model

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Received: 22 February 2011 / Published online: 17 May 2011

Abstract The quantum chromodynamics-inspired potential model pursued by us earlier has been recently modified to incorporate an additional factor '*c*' in the linear cum Coulomb potential. While it felicitates the inclusion of standard confinement parameter $b = 0.183\ \text{GeV}^2$ unlike in previous work, it still falls short of explaining the Isgur-Wise function for the B mesons without ad hoc adjustment of the strong coupling constant. In this work, we determine the factor '*c*' from the experimental values of decay constants and masses and show that the reality constraint on '*c*' yields bounds on the strong coupling constant as well as on slope and curvature of Isgur-Wise function allowing more flexibility to the model.

Keywords Dalgarno method · Isgur-Wise function · Slope · Curvature · Nonrelativistic quark model · Potential models

PACS 12.39.-x · 12.39.Jh · 12.39.Pn

1 Introduction

In recent years, considerable experimental and theoretical efforts have been undertaken to understand the physics of hadrons containing a heavy quark [1]. The Isgur-Wise (I-W) function [2] is an important quantity in this area of hadron physics. It is in this spirit that this function has been studied in various quark models [3–12] besides quantum chromodynamics (QCD) sum rule approach [13], the MIT bag model [14] and the Skryme model [15].

Since one of the basic ingredients of the I-W function is the hadron wavefunction involving heavy quark [3–12], it is therefore meaningful to test any specific QCD-inspired quark model by calculating the I-W function and studying it phenomenologically. Sometimes back, a specific QCD-inspired quark model was proposed by us [16] which had later been used to calculate the I-W function as well [17–19].

One of the drawback of the model is that significant confinement effects could not be accommodated in the model [16–18] due to perturbative constraints coming from using the Dalgarno's method [20]. Only recently [19], the standard confinement effect $b = 0.183\ \text{GeV}^2$ [21] was accommodated in the improved version of QCD-inspired quark model, brought through the introduction of parameter '*c*' in the potential: $V = \frac{-4\alpha_s}{3r} + br + c$ taking $c \sim 1$ GeV as its natural scale and fixing $A_0 = 1$, where A_0 is an undetermined factor appearing in the series solution of the Schrödinger equation ((8) of [19]). In earlier work [16–18], the unknown coefficient cA_0 occurred in the wavefunction was set to zero.

One of the drawback of work [19] was the ad hoc enhancement of strong coupling constant needed to take into account of the slope and curvature of B, B_s and B_c mesons. Also, the scaling of $c \sim 1$ GeV as natural is questionable.

In this work, we take an alternative strategy to remove this ad hoc enhancement as well as the scaling

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of c. We use the wavefunction at the origin (WFO) involving the unknown coefficient cA_0 and fix it from the experimental values of masses and decay constants directly. The reality constraint on cA_0 will be seen to yield lower bounds on the strong coupling constant α_s, which would lead to the upper bounds on the slope and curvature of the I-W function.

The rest of the paper is organised as follows: Section 2 contains the formalism of the improved QCD-inspired quark model, Section 3 encloses the results and in Section 4 we draw conclusion and remarks.

2 Formalism

2.1 The Wavefunction

The spin-independent Fermi–Breit Hamiltonian for ground state ($l = 0$), neglecting the contact term proportional to δ^3, is [16, 17].

$$H = H_o + H',$$
$$= -\frac{\nabla^2}{2\mu} - \frac{4\alpha_s}{3r} + br + c. \quad (1)$$

where α_s is the running coupling constant, b is the confinement parameter and c is another parameter whose significance will be cleared later.

As our objective is to look for the improvement over the earlier work [19], so in this work also we retain the same choice of α_s values taken from the V-scheme [18, 26, 27] and $b = 0.183$ GeV^2 [19, 21] to investigate whether this approach leads to better results or not. With $H_o = -\frac{\nabla^2}{2\mu} - \frac{4\alpha_s}{3r}$ as the parent Hamiltonian and $H' = br + c$ as the perturbed Hamiltonian, we obtain a ground state wavefunction up to the first-order correction using the Dalgarno method [20] of stationary state perturbation theory as:

$$\psi_{\text{conf}}(r) = N\left(cA_0 + \frac{1}{\sqrt{\pi a_0^3}} - \frac{\mu b a_0 r^2}{\sqrt{\pi a_0^3}}\right) e^{-\frac{r}{a_0}}. \quad (2)$$

where A_0 is the unknown coefficient appearing in the series solution of the Dalgarno method.

Including the relativistic effect [22, 23], the wavefunction is:

$$\psi_{\text{conf+rel}}(r)$$
$$= N'\left(cA_0 + \frac{1}{\sqrt{\pi a_0^3}} - \frac{\mu b a_0 r^2}{\sqrt{\pi a_0^3}}\right)\left(\frac{r}{a_0}\right)^{-\epsilon} e^{-\frac{r}{a_0}}. \quad (3)$$

Here a_0 is given by:

$$a_0 = \frac{3}{4\mu\alpha_s}, \quad (4)$$

and

$$\epsilon = 1 - \sqrt{1 - \frac{4\alpha_s}{3}}, \quad (5)$$

N and N' are the normalization constants given by:

$$N^2 = \frac{1}{1 + \frac{45\mu^2 b^2 a_0^6}{8} - 3\mu b a_0^3 + \pi a_0^3 c^2 \cdot A_0^2 + \frac{2cA_0\pi a_0^3}{\sqrt{\pi a_0^3}} - \frac{3\pi a_0^6 c A_0 \mu b}{\sqrt{\pi a_0^3}}} \quad (6)$$

and

$$N'^2 = \frac{2^{7-2\epsilon}}{\Gamma(3 - 2\epsilon) X_1}. \quad (7)$$

where X_1 is given in "Appendix".

We note that the (2), (3), (6) and (7) are obtained from (4), (5), (6) and (7) of [19] exhibiting explicit dependence of cA_0 in them.

2.2 Fixing of the Coefficient cA_0

The WFO is related to the decay constant f_p and the mass of the pseudoscalar meson M_p through the relation [16, 24]:

$$|\psi(0)|^2 = \frac{f_p^2 M_p}{12}. \quad (8)$$

Again from (2), we have:

$$|\psi(0)|^2 = N^2\left[c^2 A_0^2 + \frac{1}{\pi a_0^3} + \frac{2cA_0}{\sqrt{\pi a_0^3}}\right]. \quad (9)$$

Using (6) and (9), we arrive at the quadratic equation for cA_0:

$$A'(cA_0)^2 + B'(cA_0) + C' = 0, \quad (10)$$

where

$$A' = \pi a_0^3 |\psi(0)|^2 - 1, \quad (11)$$

$$B' = 2\sqrt{\pi a_0^3}|\psi(0)|^2 - 3\mu b a_0^3 \sqrt{\pi a_0^3}|\psi(0)|^2, \quad (12)$$

and

$$C' = |\psi(0)|^2\left[1 + \frac{45\mu^2 b^2 a_0^6}{8} - 3\mu b a_0^3\right] - \frac{1}{\pi a_0^3}. \quad (13)$$

Using the experimental values of f_p and M_p [25], we determine $|\psi(0)|^2$ from (8) which in turn will yield two solutions for cA_0 in (10):

$$cA_0 = \frac{-B' \pm \sqrt{B'^2 - 4A'C'}}{2A'}, \tag{14}$$

which will depend on μ, M_p, f_p and α_s. The solution corresponding to the +ve(−ve) sign of (14) will be termed as +ve(−ve) solution hereafter. It will be shown numerically that for a given μ, M_p, and f_p, α_s reaches the minimum value when the following condition is satisfied:

$$B'^2 - 4A'C' = 0. \tag{15}$$

The formalism involving (5)–(15) is strictly valid only without relativistic effect as the wavefunction at the origin with such effect (3) is not well-defined due to its singularity at the origin. For a subsequent analysis, we assume that cA_0 does not deviate significantly from its non-relativistic value so that it can be used to calculate the slope and curvature of the I-W function even without relativistic effect.

2.3 Charge Radius (Slope) and Convexity Parameter (Curvature) of I-W Function

The Isgur-Wise function is written as [2, 17]:

$$\xi(v_\mu . v'_\mu) = \xi(y) = 1 - \rho^2 (y-1) + C(y-1)^2 + ..., \tag{16}$$

where

$$y = v_\mu . v'_\mu, \tag{17}$$

and v_μ and v'_μ being the four velocity of the heavy meson before and after the decay. The quantity ρ^2 is the slope of I-W function at $y = 1$ and known as charge radius:

$$\rho^2 = \frac{\partial \xi}{\partial y}|_{y=1}. \tag{18}$$

The second-order derivative is the curvature of the I-W function known as convexity parameter:

$$C = \frac{1}{2}\left[\frac{\partial^2 \xi}{\partial^2 y}|_{y=1}\right]. \tag{19}$$

For the heavy-light flavor mesons, the I-W function can also be written as [6, 17]:

$$\xi(y) = \int_0^{+\infty} 4\pi r^2 |\psi(r)|^2 \cos pr dr. \tag{20}$$

where

$$p^2 = 2\mu^2 (y-1). \tag{21}$$

Equation (20) holds good for both relativistic and non-relativistic case. The wavefunction $\psi(r)$ takes different form for both the cases. Without relativistic effect, it is given by (2) and with relativistic effect it is given by (3).

With the wavefunction (2) in (20), i.e. including confinement, only the charge radius ρ^2_{conf} and convexity parameter C_{conf} are, respectively, given by:

$$\rho^2_{conf} = \frac{\mu^2\left[24\pi c^2 A_0^2 a_0^5 + 24a_0^2 + 630\mu^2 b^2 a_0^8 + 48cA_0\sqrt{\pi a_0^7} - 180cA_0\mu b\sqrt{\pi a_0^{13}} - 180\mu b a_0^5\right]}{8\pi c^2 A_0^2 a_0^3 + 8 + 45\mu^2 b^2 a_0^6 + 16cA_0\sqrt{\pi a_0^3} - 24\mu b c A_0\sqrt{\pi a_0^3} - 24\mu b a_0^3}, \tag{22}$$

and:

$$C_{conf} = \frac{\mu^4\left[60\pi c^2 A_0^2 a_0^7 + 60a_0^4 + 4725\mu^2 b^2 a_0^{10} + 120cA_0\sqrt{\pi a_0^{10}} - 840cA_0\mu b\sqrt{\pi a_0^{17}} - 840\mu b a_0^7\right]}{16\pi c^2 A_0^2 a_0^3 + 16 + 90\mu^2 b^2 a_0^6 + 32cA_0\sqrt{\pi a_0^3} - 48\mu b c A_0\sqrt{\pi a_0^3} - 48\mu b a_0^3}. \tag{23}$$

With the wavefunction (3) in (20), i.e. including both relativistic and confinement effect, the charge radius $\rho^2_{conf+rel}$ and convexity parameter $C_{conf+rel}$ are given by:

$$\rho^2_{conf+rel} = \frac{\mu^2 a_0^2 (4-2\epsilon)(3-2\epsilon)[X_1]}{4[X_2]}, \tag{24}$$

and

$$C_{conf+rel} = \frac{\mu^4 a_0^4 (6-2\epsilon)(5-2\epsilon)(4-2\epsilon)(3-2\epsilon)[X_3]}{96[X_2]}. \tag{25}$$

where X_1, X_2 and X_3 are given in "Appendix".

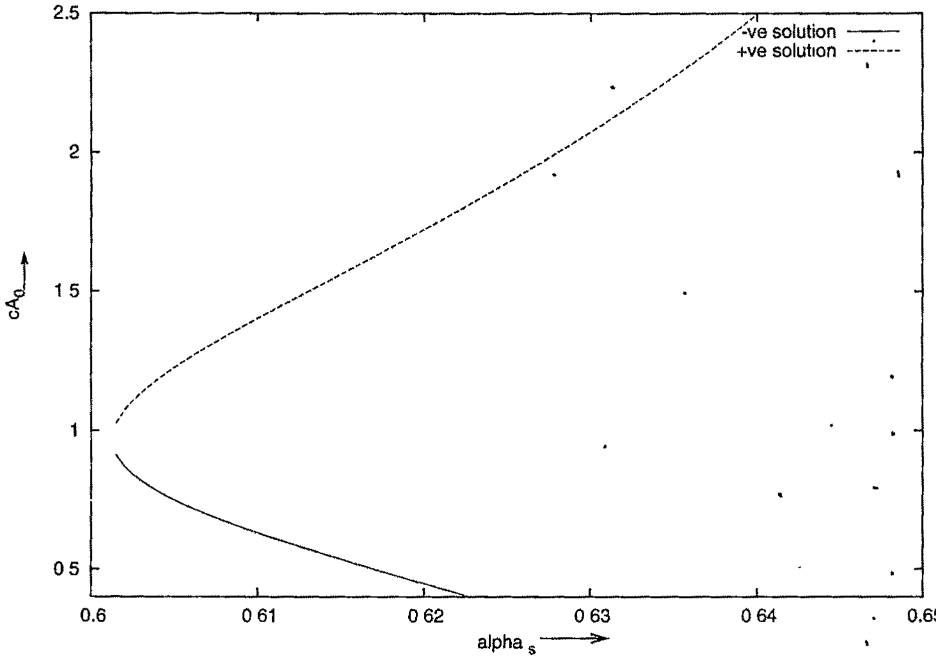

Fig. 1 Variation of cA_0 vs α_s for D meson The +ve(−ve) solution of (14) corresponds to the *dashed* (*solid*) *line* and the *two lines* nearly coincide at α_s ~0 601, the lower bound on α_s corresponding to the solution of (15) for D meson

We note that (24) and (25) are equivalent to (18) and (19) of [19] exhibiting explicit cA_0 dependence.

3 Results

3.1 Values of cA_0 and Lower Bounds on α_s

As noted earlier, cA_0 depends on μ, M_p, f_p and α_s. In Fig. 1, 2, 3, 4 and 5, we plot cA_0 vs α_s for D, D_s, B, B_s and B_c mesons. It shows that α_s tends to reach the minimum value when two solutions of (14) almost merge satisfying the condition (15). This feature is true for any set of the parameters μ, f_p and M_p. In Table 1, we give the lower bounds on α_s for mesons having c and b quarks.

The dependence of cA_0 on α_s and μ can be noted as follows: With constant μ, cA_0 decreases with α_s values rising and vice versa. On the other hand, with constant

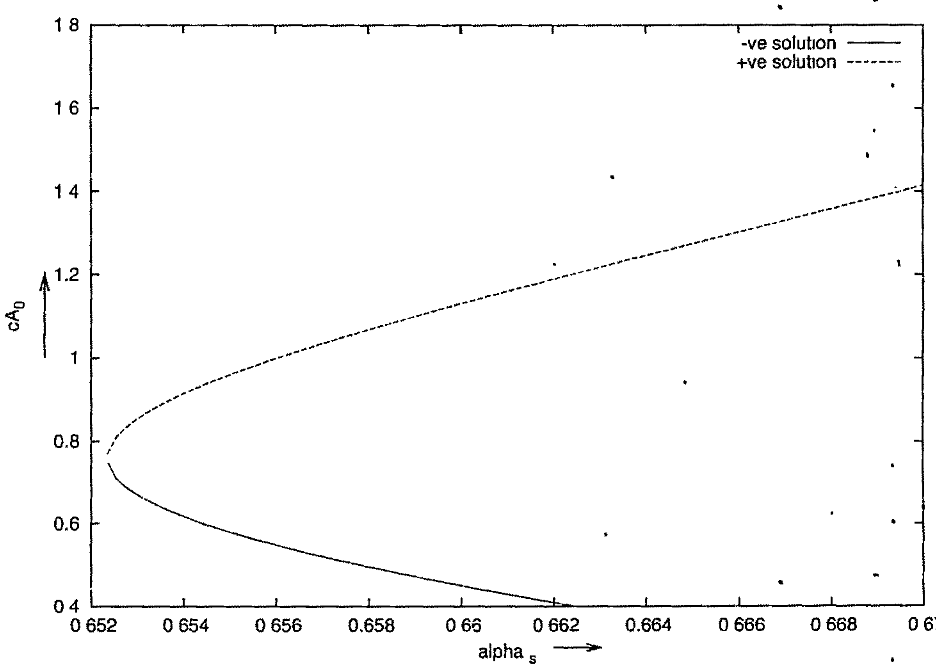

Fig. 2 Variation of cA_0 vs α_s for B meson The +ve(−ve) solution of (14) corresponds to the *dashed* (*solid*) *line* and the *two lines* nearly coincide at α_s ~0.652, the lower bound on α_s corresponding to the solution of (15) for B meson

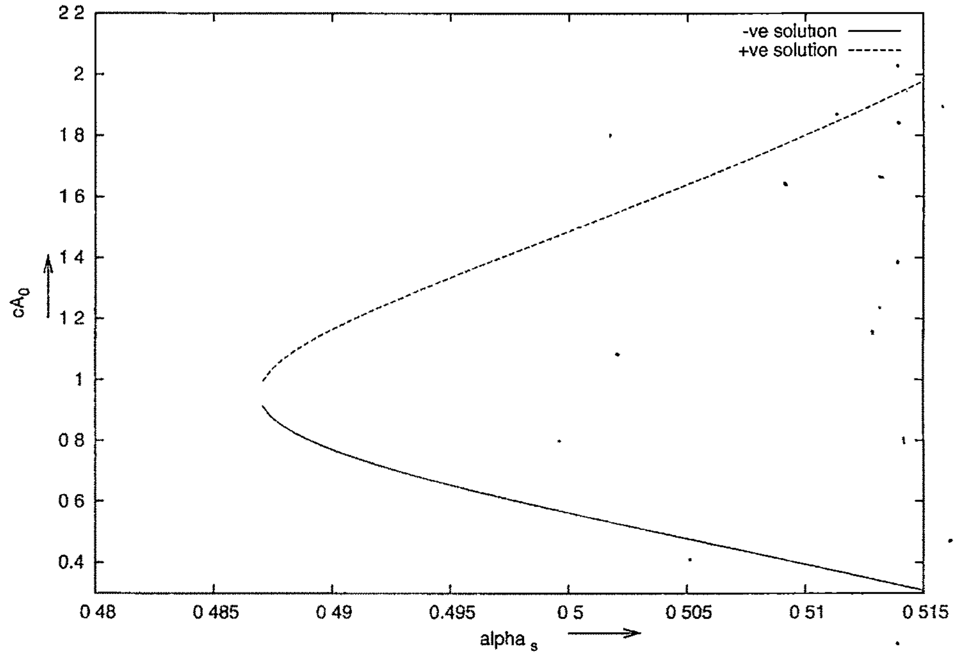

Fig. 3 Variation of cA_0 vs α_s for D_s meson The +ve(−ve) solution of (14) corresponds to the *dashed* (*solid*) *line* and the *two lines* nearly coincide at α_s ~0.49, the lower bound on α_s corresponding to the solution of (15) for D_s meson

α_s, cA_0 increases (decreases) with increase (decrease) in μ.

3.2 Bounds on Slope and Curvature of the I-W Function

Using the lower bounds on α_s for each heavy–light and heavy–heavy mesons, we obtain upper bounds on the slope and curvature of the I-W function using (23), (24), (25) and (26). They are listed in Table 2. We note that with increasing α_s values, the slope and curvature decrease and henceforth the lower bound on α_s corresponds to the upper bound on ρ^2 and C.

In Table 3, we record the predictions of the slope and curvature of the I-W function in various models while in Table 4, we reproduce the corresponding predictions of the model of [19] with $c = 1$ GeV and $A_0 = 1$ in

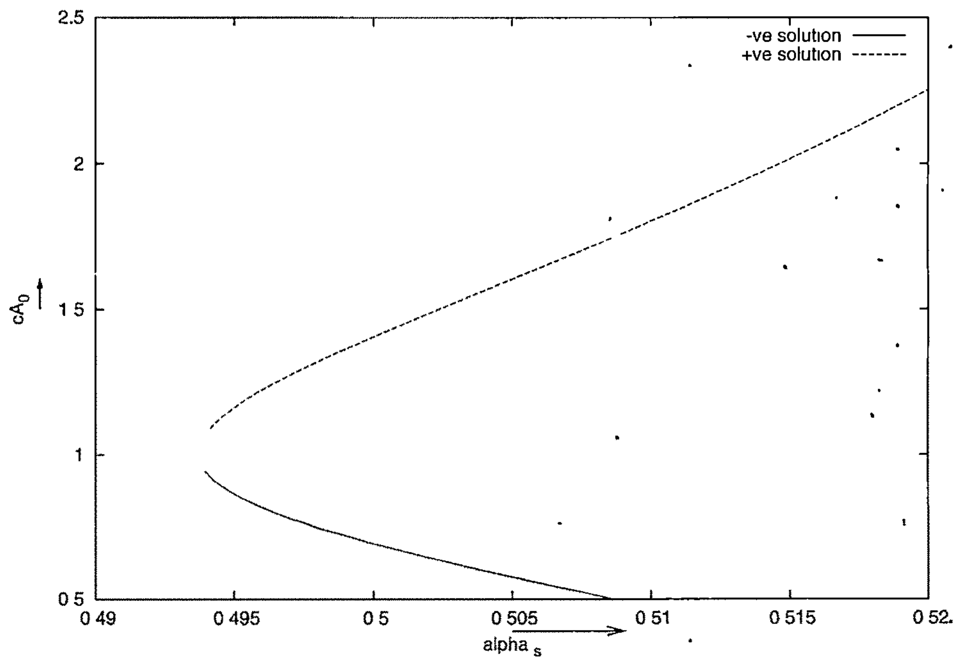

Fig. 4 Variation of cA_0 vs α_s for B_s meson. The +ve(−ve) solution of (14) corresponds to the *dashed* (*solid*) *line* and the *two lines* nearly coincide at α_s ~0.493, the lower bound on α_s corresponding to the solution of (15) for B_s Meson

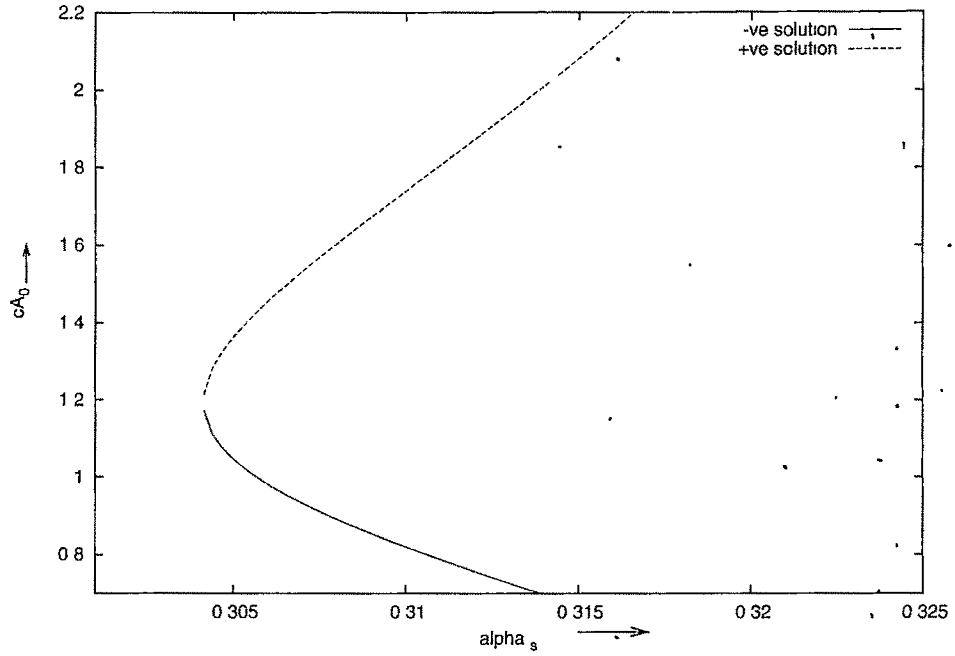

Fig. 5 Variation of cA_0 vs α_s for B_c Meson. The +ve(−ve) solution of (14) corresponds to the *dashed* (*solid*) *line* and the *two lines* nearly coincide at α_s ~0.302, the lower bound on α_s corresponding to the solution of (15) for B_c meson

Table 1 Lower bounds on α_s

Mesons	Quark content	μ (GeV) [25]	M_p (GeV) [25]	f_p (GeV) [25]	cA_0	Lower bound on α_s
D	$c\bar{u}/c\bar{d}$	0.276	1.869	0.192	0.9665	~0.601
B	$\bar{b}u/\bar{b}d$	0.315	5.279	0.210	0.7653	~0.652
D_s	$c\bar{s}$	0.368	1.968	0.157	0.9543	~0.49
B_s	$\bar{b}s$	0.44	5.279	0.171	0.999	~0.493
B_c	$\bar{b}c$	1.18	5.37	0.36	1.167	~0.302

Table 2 Upper bounds on slope and curvature

Meson	Slope ρ^2		Curvature C	
(quark content)	Without relativistic effect	With relativistic effect	Without relativistic effect	With relativistic effect
$D(c\bar{u}/c\bar{d})$	6.78	1.675	13.19	5.138
$B(\bar{b}u/\bar{b}d)$	5.78	1.016	9.58	1.29
$D_s(c\bar{s})$	9.115	3.067	26.48	14.32
$B_s(\bar{b}s)$	11.92	2.652	34.49	6.902
$B_c(\bar{b}c)$	28.46	10.39	219.46	45.23

Table 3 Predictions of the slope and curvature of the I-W function in various models

Model	Value of ρ^2	Value of curvature C
Yaouanc et al. [28]	≥0.75	..
Yaouanc et al. [12]	≥0.75	≥0.47
Rosner et al. [29]	1.66	2.76
Mannel et al. [30, 31]	0.98	0.98
Pole Ansatz [32]	1.42	2.71
MIT bag model [14]	2.35	3.95
Simple quark model [3]	1	1.11
Skryme model [15]	1.3	0.85
QCD sum rule [13]	0.65	0.47
Relativistic three quark model [4]	1.35	1.75
Infinite momentum frame quark model [5]	3.04	6.81

Table 4 Predictions of the slope and curvature of the I-W function in the QCD inspired quark model according to [19] with $c = 1$ and $A_0 = 1$ taking relativistic and confinement effect in V-scheme

Meson	α_s	Slope (ρ^2)	Curvature(C)
D	0.625	1 136	5.377
D_s	0 625	1.083	3 583
B	(a)0.261	(a)128.128	(a)5212
	(b)0.60	(b)1.329	(b)7 2
B_s	(a)0.261	(a)112.759	(a)4841
	(b)0 60	(b)1.257	(b)4 379
B_c	(a)0.261	(a)44 479	(a)2318
	(b)0 60	(b)1.523	(b)0 432

This table is nothing but a replica of the last rows of Tables 1, 2 and 3 of [19]

V-scheme [26, 27] for various mesons. Two set of values for B, B_s and B_c mesons are shown in the table where case (a) represents the actual values for ρ^2 and C in that work with $\alpha_s = 0.261$, while case (b) represents those for an ad hoc adjustable value of $\alpha_s = 0.60$ in order to show the usefulness of large α_s as mentioned in [19]. The α_s values were already large for D and D_s mesons, so no ad hoc adjustment was necessary that might lead to two set of values.

4 Conclusion and Remarks

We have shown that the reality bound on cA_0 puts a lower limit on α_s and a corresponding upper limit on ρ^2 and C. Furthermore, with cA_0, that the upper bounds on ρ^2 and C decrease, which is evident from the above list of bounds (Table 2). The estimated upper bounds on ρ^2 and C for all the mesons are found to be consistent with other models and data (Table 3) without making any ad hoc enhancement of the strong coupling constant as had been done in [19] (Table 4). From the phenomenological point of view, we note that in the nonrelativistic limit, the universal form factor and Isgur-Wise function for semileptonic decay $B \to D^* l\nu$ are identical when subleading terms in velocity and terms of order $O\left(\frac{E_b}{m_Q}\right)$ are neglected with E_b as the binding energy and m_Q as the mass of heavy quark [33]. However, even if we make calculation for the universal form factor for finite mass, we obtain to first order in $(y-1)$ as 0.8–2.57 $(y-1)$ which seems to be satisfactory [33, 34].

It is worth notable that in the limit $cA_0 \to 0$, there will be no bounds on α_s as well as on ρ^2 and C; rather, fixed values of α_s have to be used to get definite set of ρ^2 and C. So, in that case, the analysis will turn to that of [17, 18] where large confinement could not be (i.e. $b = 0.183\ \text{GeV}^2$) incorporated, e.g. Tables 1 and 3 of [17] and Tables 2 and 3 of [18].

We conclude this paper with a comment on the physical significance of the factor 'c' that has become so crucial for our analysis of bounds on slope and curvature. It is common wisdom that a constant potential like 'c' just scales the energies and does not affect the wavefunction nor does it change physics. This can be seen from the hydrogen atom problem with the potential $V(r) == \frac{-A}{r} + c$. However, if one uses 'c' as the perturbation instead of as parent in the Dalgarno method of perturbation theory [20], the normalized wavefunction for the H-atom becomes

$$\psi(r) = \left[\frac{1}{1 + \pi a_0^3 c^2 A_0^2 + \frac{2cA_0\pi a_0^3}{\sqrt{\pi a_0^3}}}\right]\left(cA_0 + \frac{1}{\sqrt{\pi a_0^3}}\right) e^{-\frac{r}{a_0}}.$$

which is to be compared with the normalized wavefunction with 'c' as parent:

$$\psi(r) = \left(\frac{1}{\sqrt{\pi a_0^3}}\right) e^{-\frac{r}{a_0}}.$$

Thus, the perturbative child 'c' rather than the parent 'c' plays the crucial role in the present analysis.

Appendix

X_1, X_2 and X_3 are evaluated as

$$\begin{aligned} X_1 = {} & 64\pi c^2 A_0^2 a_0^3 + 64 + \mu^2 b^2 a_0^6 \\ & \times (8-2\epsilon)(7-2\epsilon)(6-2\epsilon)(5-2\epsilon) \\ & + 128cA_0\sqrt{\pi a_0^3} - 16cA_0\mu b\sqrt{\pi a_0^9}(6-2\epsilon)(5-2\epsilon) \\ & - 16\mu b a_0^3(6-2\epsilon)(5-2\epsilon), \end{aligned} \quad (26)$$

$$\begin{aligned} X_2 = {} & 64\pi c^2 A_0^2 a_0^3 + 64 + \mu^2 b^2 a_0^6 \\ & \times (6-2\epsilon)(5-2\epsilon)(4-2\epsilon)(3-2\epsilon) \\ & + 128cA_0\sqrt{\pi a_0^3} - 16cA_0\mu b\sqrt{\pi a_0^9}(4-2\epsilon)(3-2\epsilon) \\ & - 16\mu b a_0^3(4-2\epsilon)(3-2\epsilon), \end{aligned} \quad (27)$$

and

$$\begin{aligned} X_3 = {} & 64\pi c^2 A_0^2 a_0^3 + 64 + \mu^2 b^2 a_0^6 \\ & \times (10-2\epsilon)(9-2\epsilon)(8-2\epsilon)(7-2\epsilon) \\ & + 128cA_0\sqrt{\pi a_0^3} - 16cA_0\mu b\sqrt{\pi a_0^9}(8-2\epsilon)(7-2\epsilon) \\ & - 16\mu b a_0^3(8-2\epsilon)(7-2\epsilon). \end{aligned} \quad (28)$$

Not only the above expressions but also all the integrals in the analysis are evaluated with the help of Gamma function, given by:

$$\frac{\Gamma(n+1)}{\alpha^{n+1}} = \int_0^{+\infty} r^n e^{-\alpha r} dr. \quad (29)$$

References

1 M. Neubert. Phys. Rep. **245**, 259 (1994)
2 N. Isgur, M B. Wise, Phys. Lett. B **232**, 113 (1989)
3. B Holdom, M. Sutherland, J. Mureika, Phys. Rev. D **49**, 2359 (1994)
4. M A. Ivanov, V.E. Lyubouvitskij, L.G. Korner, P. Kroll, Phys. Rev. D **56**, 348 (1997)
5. B. Konig, J.G. Körner, M Krämer, P. Kroll, Phys. Rev **56**, 4282 (1997)
6 F.E. Close, A. Wambach, Nucl. Phys. B **412**, 169 (1994)
7. H.W. Huang, Phys. Rev. D **56**, 1579 (1979)
8. D. Melikhov, Phys. Rev. D **53**, 2460 (1996)
9. M.R. Ahmady, R.R Mendel, J.D. Talman, Phys. Rev D **52**, 254 (1995)
10. M.G. Olsson, S. Veseli, Fermilab-Pub-96/418-T (1997)
11 M.G Olsson, S. Veseli, Phys. Rev. D **51**, 2224 (1995)
12. A. Le Yaouanc, L. Oliver, J.C Raynal, Phys. Rev. D **69**, 094022 (2004)
13. Y.B. Dai, C.S. Huang, M.Q Huang, C. Liu, Phys Lett B **387**, 379 (1996)
14. M. Sadzikowski, K. Zalewski, Z. Phys. C
15. E. Jenkins, A Manohar, M.B. Wise, Nu (1996)
16. D.K. Choudhury, P. Das, D.D. Gosw Pramana J Phys **44**, 519 (1995)
17. D.K. Choudhury, N.S. Bordoloi, IJN (2000)
18. D.K. Choudhury, N.S. Bordoloi, MPLA 1
19. D.K. Choudhury, N.S. Bordoloi, MPLA 2
20. A. Ghatak, S. Lokanathan, in *Quantum M* Hill, New York, 1997), p. 291
21 E. Eichten, Phys Rev. D **17**, 3090 (1978)
22. J J. Sakurai, in *Advanced Quantum M* Wesley, Boston, 1986), p. 128
23. C. Itzykson, J. Zuber. in *Quantum Fiel* tional Student Edition, McGraw Hill, Sin
24. V.O. Galkin, A. Yu Mishurov, R.N Fa Phys. **53**, 1026 (1991)
25. D E. Groom et al., Particle Data Group. (2000)
26. Y. Schröder, Phys. Lett. B **447**, 321 (1999
27. Y. Schröder, Nucl. Phys. Proc. Suppl. **86**,
28. A. Le Yaouanc, L. Oliver, O. Pene, J C. B **365**, 319 (1996)
29. J L Rosner, Phys. Rev. D **44**, 3732 (1990)
30. T. Mannel, W. Roberts, Z. Ryzak, Phy (1991)
31. T. Mannel. W. Roberts, Z. Ryzak, Phy (1993)
32. M Neubert, Phys Lett B **264**, 455 (1991)
33. F. Jugeau, A. Le Yaouanc, L. Oliver, J C D **70**, 114020 (2004)
34. F.E. Close, A. Wambach, RAL-94-041, C

MODERN PHYSICS LETTERS A

Volume 26, Number 21
July 10, 2011

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Modern Physics Letters A
Vol. 26, No. 21 (2011) 1547–1554

DOI: 10.1142/S0217732311036000

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ISGUR–WISE FUNCTION IN A QCD POTENTIAL MODEL WITH COULOMBIC POTENTIAL AS PERTURBATION

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Received 27 March 2011
Revised 24 April 2011

We study heavy light mesons in a QCD inspired quark model with the Cornell potential $-\frac{4\alpha_S}{3r}+br+c$. Here we consider the linear term br as the parent and $-\frac{4\alpha_S}{3r}+c$, i.e. the Coloumbic part as the perturbation. The linear parent leads to Airy function as the unperturbed wave function. We then use the Dalgarno method of perturbation theory to obtain the total wave function corrected up to first order with Coulombic piece as the perturbation. With these wave functions, we study the Isgur–Wise function and calculate its slope and curvature.

Keywords: Dalgarno method, Isgur–Wise function; slope and curvature.

PACS Nos.. 12.39 -x, 12.39.Jh, 12.39.Pn

1. Introduction

Considerable efforts have been made in understanding the physics of hadrons containing at least one heavy quark since long.[1–9] It is well known that the heavy quark symmetry in the heavy quark limit leads to a single form factor called the Isgur–Wise (I-W) function which can describe the heavy quark bilinear current matrix elements of weak decay. The basic ingradient of the I-W function is the hadronic wave function, the determination of which becomes such a crucial factor. The potential models for this purpose is quite helpful as they contain more input parameters and hence has its firm basis.

Under such circumstances the I-W function has been investigated[3–9] with considerable success of valid degrees in different models. In the potential models, "Cornell potential" is found to be more useful than the others. It leaves two options of choosing the parent (1) the Coulombic part $-\frac{4\alpha_S}{3r}$ and (2) the linear potential part br.

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The slope and curvature of I-W function with the Coulombic potential as the parent has already been reported for different heavy-light flavor mesons,[10–15] which however had certain limitations. In Coulombic potential as parent and linear as perturbation, the value of slope (ρ^2) and curvature (C) was found to be too large in $\overline{\text{MS}}$-scheme. Imposing V-scheme,[14–18] with larger α_s the values were found to be improved[13] but still larger than expectations. As an alternate approach, in the present work we choose linear term "br" as the parent and Coulombic piece as the perturbation in finding the wave function.

As usual, two-body Schrödinger equation is used with the ground state Fermi–Breit Hamiltonian in the absence of contact term and with Coulombic perturbation, the wave function corrected up to first order is obtained by using the Dalgarno method.[10,19] The relativistic effect is incorporated by using standard Dirac modification[20,21] in a parameter free way. These wave functions are used in the calculation of slope and curvature of I-W function.

The rest of the paper is organized as follows: Sec. 2 contains the formalism, Sec. 3 the result and Sec. 4 the conclusion and discussion.

2. Formalism

2.1. *The wave function*

We start with the ground state ($l = 0$) spin independent Fermi–Breit Hamiltonian without the contact term given by[10,11]:

$$H = -\frac{\nabla^2}{2\mu} - \frac{4\alpha_s}{3r} + br + c \tag{1}$$

so that

$$H' = -\frac{4\alpha_s}{3r} + c \tag{2}$$

can be treated as perturbation to the unperturbed Hamiltonian:

$$H_0 = -\frac{\nabla^2}{2\mu} + br\,. \tag{3}$$

In Eq. (1), the strong coupling constant connected to the potential is a function of the momentum as

$$\alpha_s(\mu^2) = \frac{4\pi}{(11 - \frac{2n_f}{3})\ln(\frac{\mu^2}{\Lambda^2})}\,, \tag{4}$$

where n_f is the number of flavor. The constant "c" at its natural scale is taken to be 1 GeV.[13] The two-body nonrelativistic Schrödinger wave equation can be recasted as

$$H|\psi\rangle = (H_0 + H')|\psi\rangle = E|\psi\rangle\,. \tag{5}$$

The unperturbed wave function corresponding to H_0 are the Airy functions which after normalization can be written as:

$$\psi_n^{(0)}(r) = \frac{N}{2\sqrt{\pi} r} Ai((2\mu b)^{\frac{1}{3}} + \rho_{0n}), \quad (6)$$

where ρ_{0n} are the zeros of the Airy function $Ai(\rho_{0n}) = 0$, $n = 1, 2, 3, \ldots$ represent the principal quantum number (of course for the ground state $n = 1$) and N is the normalization constant.

The ρ_{0n} are given as[22,23]:

$$\rho_{0n} = -\left[\frac{3\pi(4n-1)}{8}\right]^{\frac{2}{3}}. \quad (7)$$

The first-order correction to wave function $\psi_n^{(1)}$ and energy $W_n^{(1)}$ are respectively given by

$$H_0\psi_n^{(1)} + H'\psi_n^0 = W_n^0\psi^1 + W_n^1\psi^0, \quad (8)$$

where W_n^0 is the unperturbed energy given as[22]

$$W_n^0 = E_n = -\left(\frac{b^2}{2\mu}\right)^{\frac{1}{3}} \rho_{0n} \quad (9)$$

and

$$W_n^{(1)} = \int_0^{+\infty} r^2 H' |\psi^{(0)}(r)|^2 \, dr. \quad (10)$$

Since we consider the ground state ($n = 1$), so we drop the "n" from W_n^0, $W_n^{(1)}$, $\psi_n^{(0)}$ and ψ_n^1. The first-order correction is:

$$\psi^1(r) = -\frac{4\alpha_s}{3}\left(\frac{a_0}{r} + a_1 + a_2 r\right). \quad (11)$$

As Airy function $Ai(r)$ involve infinite series in r, so in calculating the coefficients a_0, a_1 and a_2 we have considered up to order r^3 and are given by:

$$a_0 = \frac{0.8808(b\mu)^{\frac{1}{3}}}{(E-c)} - \frac{a_2}{\mu(E-c)} + \frac{4W^1 \times 0.21005}{3\alpha_s(E-c)}, \quad (12)$$

$$a_1 = \frac{ba_0}{(E-c)} + \frac{4 \times W^1 \times 0.8808 \times (b\mu)^{\frac{1}{3}}}{3\alpha_s(E-c)} - \frac{0.6535 \times (b\mu)^{\frac{2}{3}}}{(E-c)}, \quad (13)$$

$$a_2 = \frac{4\mu W^1 \times 0.1183}{3\alpha_s}. \quad (14)$$

The total wave function corrected up to first order with normalization is

$$\psi_{\text{coul}}(r) = \psi^{(0)}(r) + \psi^{(1)}(r) \quad (15)$$

$$= \frac{N_1}{2\sqrt{\pi}}\left[\frac{Ai((2\mu b)^{\frac{1}{3}} + \rho_{01})}{r} - \frac{4\alpha_s}{3}\left(\frac{a_0}{r} + a_1 + a_2 r\right)\right], \quad (16)$$

where N_1 is the normalization constant for the total wave function $\psi_{\text{coul}}(r)$ with subscript "coul" means Coulombic potential as perturbation.

The relativistic version of Eq. (17) is obtained by multiplying it with $\left(\frac{r}{a_{\text{Bohr}}}\right)^{-\epsilon}$. a_{Bohr} depends on α_s as:

$$a_{\text{Bohr}} = \frac{3}{4\mu\alpha_s} \tag{17}$$

and

$$\epsilon = 1 - \sqrt{1 - \left(\frac{4\alpha_s}{3}\right)^2}\,. \tag{18}$$

Thus, relativistic wave function is:

$$\psi_{\text{rel}}(r) = \psi_{\text{coul}}(r)\left(\frac{r}{a_{\text{Bohr}}}\right)^{-\epsilon}. \tag{19}$$

2.2. *Isgur–Wise function*

The I-W function is written as[1,2]:

$$\xi(y) = 1 - \rho^2(y-1) + C(y-1)^2 + \cdots\,, \tag{20}$$

where

$$y = v_\mu \cdot v'_\mu \tag{21}$$

and v_μ and v'_μ being the four-velocity of the heavy meson before and after the decay. The quantity ρ^2 is the slope of I-W function at $y = 1$ and known as charge radius:

$$\rho^2 = \left.\frac{\partial \xi}{\partial y}\right|_{y=1}. \tag{22}$$

The second-order derivative is the curvature of the I-W function known as convexity parameter:

$$C = \frac{1}{2}\left.\left(\frac{\partial^2 \xi}{\partial y^2}\right)\right|_{y=1}. \tag{23}$$

For the heavy–light flavor mesons, the I-W function can also be written as[6,11]:

$$\xi(y) = \int_0^{+\infty} 4\pi r^2 |\psi(r)|^2 \cos pr\, dr\,, \tag{24}$$

where

$$p^2 = 2\mu^2(y-1) \tag{25}$$

the wave function Eq. (19) with relativistic effect is used in the calculation of $\xi(y)$ given by Eq. (24).

3. Calculation and Results

We have calculated the values of charge radius and convexity parameter of the I-W function given by Eq. (20) for two set of coupling constants both in $\overline{\text{MS}}$ and V-scheme.[14–17]

Regarding the use of the above-mentioned schemes[11,12] we note that with $n_f = 4$ and $n_f = 5$ and fixing $\Lambda_{\text{QCD}} = 0.216$ GeV,[24] the corresponding value of $\alpha_{\overline{\text{MS}}}$ at the scale of 1.5 GeV and 8 GeV are respectively 0.39 and 0.22.[24] The respective change of $\alpha_{\overline{\text{MS}}}$ to $\alpha_v(\frac{1}{r^2})$ in the V-scheme[14–17] for three different choices of scale $\bar{\mu}$ are calculated[12] and shown in Table 1. Although there is no fundamental reason for the choice, we have chosen the two renormalization schemes ($\overline{\text{MS}}$ and V-schemes) to facilitate the comparison of our result with the previous work.[12,13] Also we use the same model parameter $b = 0.183$ GeV^2 from charmonium spectroscopy.[25,26]

For these calculations, we have used the expressions for E, W^1, a_0. a_1 given by Eqs. (10), (12)–(14) respectively. These are shown in Tables 2 and 3. The result of ρ^2 and c in the present work is shown in Table 4. We also compare the present

Table 1. The value of $\alpha_v\left(\frac{1}{r^2}\right)$ for different choices of $\bar{\mu}$.

Choices	$\bar{\mu} = \frac{1}{r}$	$\bar{\mu} = \frac{e^{-\gamma_E}}{r}$,	$\bar{\mu} = \frac{e^{-\gamma_E - \frac{a_1}{2\beta_0}}}{r}$
$\alpha_{\overline{\text{MS}}}(m_b) = 0\ 22$, $n_f = 5$	0.259	0.261	0.258
$\alpha_{\overline{\text{MS}}}(m_c) = 0.39$, $n_f = 4$	0.693	0.651	0.604

Table 2. The values of W^1 and E in GeV.

		W^1	
Mesons	E	$\overline{\text{MS}}$-scheme	V-scheme
D	0 3898	0.0467	0 08314
D_s	0.4291	0.5137	0.0915
B	0.4072	0.02742	0.0327
B_s	0.4553	0 0308	0.0366

Table 3 List of a_0, a_1 and a_2.

	a_0		a_1 (GeV)		a_2 (GeV^2)	
Mesons	V-scheme	$\overline{\text{MS}}$-scheme	V-scheme	$\overline{\text{MS}}$-scheme	V-scheme	$\overline{\text{MS}}$-scheme
D	0.2143	0.1943	−0 006138	−0.007877	0.00293	0.002933
D_s	0.238	0.21387	−0.00916	−0 01257	0.0043	0.0043036
B	0.2245	0.2029	−0.00749	−0.0099	0.00349	0.00348
B_s	0.254	0.2269	−0.0114	−0.01604	0.005446	0.00547

Table 4 Values of ρ^2 and C in our work and its comparison to other work.

Scheme	Mesons	ρ^2	C
Our work			
$\overline{\text{MS}}$-scheme	D	0.7936	0 0008
	D_s	1.186	0.002
	B	0.89	0 0004
	B_s	1.41	0.0012
V-scheme	D	0.896	0 00306
	D_s	1.352	0.0077
	B	0.912	0.0007
	B_s	1.421	0.00155
Other work			
Previous work[12,13]	D	1.136	5.377
	D_s	1.083	3.583
	B	128.28	5212
	B_s	112.759	4841
Le Youanc *et al.*[27]		$\geq 0\ 75$	...
Le Youanc *et al.*[28]		≥ 0.75	≥ 0.47
Rosner[29]		1.66	2.76
Mannel[30,31]		0.98	0.98
Pole Ansatz[32]		1.42	2.71
Ebert *et al.*[36]		1.04	1.36
Simple Quark Model[3]		1.00	1.11
Skryme Model[35]		1.3	0.85
QCD Sum Rule[34]		0.65	0.47
Relativistic Three Quark Model[4]		1.35	1 75
Neubert[33]		$0\ 82 \pm 0.09$	...

result with that of previous work with linear as the perturbation[13] in V-scheme which was an improvement over $\overline{\text{MS}}$-scheme and is shown in Table 4.

In Table 4, we give a list of predictions of ρ^2 and C in different theoretical models.

In evaluating the various integrations, we use numerical method of integration in Mathematica software.

4. Discussion and Conclusion

Our calculated values of slope of I-W function in this work are found to be in good agreement with the other theoretical results (Table 4). The lattice QCD evaluation of $\rho^2 = 0.83^{+15+24}_{-11-22}$ for B-meson[37] and the experimental values of D-meson $\rho_D^2 = 0.76 \pm 0.16 \pm 0.08$ and[38] $\rho_D^2 = 0.69 \pm 0.14$ are[39] also in good agreement with our calculated results. However, the values of C for each meson are found to be smaller in comparison to other theoretical values. The reason may be presumably due to the cutoff of the infinite series of $Ai(z)$ up to $O(r^3)$ as noted earlier and still such small values can be considered as a success particularly for the B-sector mesons as these values were very large in case of Coulombic potential as parent.[11–14]

This study of the I-W function with Coulombic part as perturbation shows a different picture as compared to the earlier work.[11–13] With linear part as perturbation, the slope and curvature decrease with the increase of α_s; while in this work, we have observed a reverse effect. Further, this analysis shows a great reduction in the values of ρ^2 and C for all the mesons as compared to the previous work with linear part as perturbation.

Let us conclude the section with a few comments.

The strong coupling constant entering the Coulombic potential is a function of the momentum in full QCD. But in potential model, it is nothing but a mere parameter. Here we have used the strong coupling constant in the $\overline{\text{MS}}$ and V-scheme to facilitate a proper comparison with the previous work with linear part as perturbation.[12,13]

However, instead of using a particular renormalization scheme we could as well have considered the strong coupling constant merely as a free parameter in the potential model to be fitted from data. Such a possibility is currently under study.

References

1. N. Isgur and M. B. Wise, *Phys. Lett. B* **232**, 113 (1989).
2. N. Isgur and M. B. Wise, *Phys. Lett. B* **237**, 527 (1990).
3. B. Holdom, M. Sutherland and J. Mureika, *Phys. Rev. D* **49**, 2359 (1994).
4. M. A. Ivanov, V. E. Lyubouvitskij, L. G. Kórner and P. Kroll, *Phys. Rev. D* **56**, 348 (1997).
5. B. Kónig, J. G. Körner, M. Krämer and P. Kroll, *Phys. Rev. D* **56**, 4282 (1997).
6. F. E. Close and A. Wambach, *Nucl. Phys. B* **412**, 169 (1994).
7. H. W. Huang, *Phys. Rev. D* **56**, 1579 (1997).
8. D. Melikhov, *Phys. Rev. D* **53**, 2460 (1996).
9. M. R. Ahmady, R. R. Mandel and J. D. Talman, *Phys. Rev. D* **52**, 254 (1995).
10. D. K. Choudhury, P. Das, D. D. Goswami and J. N. Sharma, *Pramana J. Phys.* **44**, 519 (1995).
11. D. K. Choudhury and N. S. Bordoloi, *Int. J. Mod. Phys. A* **15**, 3667 (2000).
12. D. K. Choudhury and N. S. Bordoloi, *Mod. Phys. Lett. A* **17**, 1909 (2002).
13. D. K. Choudhury and N. S. Bordoloi, *Mod. Phys. Lett. A* **26**, 443 (2009).
14. D. K. Choudhury and B. J. Hazarika, *Pramana J. Phys.* **75**, 423 (2010).
15. M. Peter, *Phys. Rev. Lett.* **78**, 603 (1997).
16. M. Peter, *Nucl. Phys. B* **501**, 471 (1997).
17. Y. Schroeder, *Phys. Lett. B* **447**, 321 (1999).
18. Y. Schroeder, *Nucl. Phy. Proc. Suppl.* **86**, 525 (2000).
19. A. K. Ghatak and S. Lokanathan, in *Quantum Mechanics* (McGraw-Hill, 1997), p 291.

20. J. J. Sakurai, in *Advanced Quantum Mechcnics* (Addison-Wiley, 1986), p. 128.
21. C. Itzykson and J. Zuber, in *Quantum Field Theory* (McGraw-Hill, 1986), p. 79.
22. I. R. Aitchison and J. J. Dudek, *Eur. J. Phys.* **23**, 605 (2002).
23. Abramowitz and Stegun, in *Handbook of Mathematical Functions* (Dover, 1964).
24. Review of Particle Physics, Particle Data Group, *Eur. Phys. J. C* **3**, 19 (1998).
25. E. Eichten *et al.*, *Phys. Rev. Lett.* **34**, 369 (1975).
26. E. Eichten *et al.*, *Phys. Rev. D* **17**, 3090 (1978).
27. A. Le Yaouanc, L. Oliver, O. Pene and J. C. Raynal, *Phys. Lett. B* **365**, 319 (1996).
28. A. Le Yaouanc, L. Oliver and J. C. Raynal, *Phys. Rev. D* **69**, 094022 (2004).
29. J. L. Rosner, *Phys. Rev. D* **42**, 3732 (1990).
30. T. Mannel, W. Roberts and Z. Ryzak, *Phys. Rev. D* **44**, R18 (1991).
31. T. Mannel, W. Roberts and Z. Ryzak, *Phys. Lett. B* **255**, 593 (1993).
32. M. Neubert, *Phys. Lett. B* **264**, 455 (1991).
33. M. Neubert, *Int. J. Mod. Phys. A* **11**, 4173 (1996).
34. Y. B. Dai, C. S. Huang, M. K. Huang and C. Liu, *Phys. Lett. B* **387**, 379 (1996).
35. E. Jenkins, A. Manohar and M. B. Wise, *Nucl. Phys. B* **396**, 38 (1996).
36. D. Ebert, R. N Faustov and V. O. Galkin, arXiv:hep-ph/0611307v1
37. UKQCD Collab. (K. C. Bowler *et al.*), *Nucl. Phys. B* **637**, 293 (2002).
38. CLEO Collab. (J. Bartlet *et al.*), *Phys. Rev. Lett.* **82**, 3746 (1999).
39. BELLE Collab. (K. Abe *et al.*), *Phys. Lett. B* **526**, 258 (2002).

PRAMANA
— journal of
physics

Vol. 78, No. 4
April 2012
pp. 555–564

Isgur–Wise function in a quantum chromodynamics-inspired potential model with confinement as parent in the variationally improved perturbation theory

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MS received 14 June 2011; revised 1 November 2011; accepted 8 November 2011

Abstract. We have recently reported the calculation of slope and curvature of Isgur–Wise function based on variationally improved perturbation theory (VIPT) in a quantum chromodynamics (QCD)-inspired potential model. In that work, Coulombic potential was taken as the parent while the linear one as the perturbation. In this work, we choose the linear one as the parent with Coulombic one as the perturbation and see the consequences.

Keywords. Variationally improved perturbation theory; Isgur–Wise function; charge radii; convexity parameter.

PACS Nos 12.39.–x; 12.39.Jh; 12.39.Pn

1. Introduction

Being a universal form factor, the Isgur–Wise (I–W) function has been instrumental in the analysis of semileptonic decays [1] and so far various QCD-inspired models have been developed for its proper understanding. In this spirit, the I–W function had been investigated for the last few years in a QCD-inspired model [2,3] where two-body Schrödinger equation was solved for the spin-independent Fermi–Breit Hamiltonian consisting of the linear cum Coulombic potential with the contact term being neglected [3,4]. The Dalgarno method was the method used to obtain the wave function which could predict the I–W function [15] with either Coulombic piece as the parent [3,5–7] or the linear one as the parent [8]. While refs [5–7] demanded either small confinement (i.e. b) or large coupling constant (α_s), ref. [8] was quite successful in predicting satisfactory results for the slope and curvature with the same range of values for the parameters b and α_s.

As an alternative to Dalgarno method, one can use the recently introduced [9–11] variationally improved perturbation theory (VIPT) to solve the Schrödinger equation to obtain the wave function. The disadvantage of conventional perturbation theory is that it needs a very small expansion parameter which leads to diverging results after a certain order.

Similarly, the variational method needs an appropriate trial wave function in terms of unknown parameter(s) which is quite tedious and this makes it an inconvenient method. However, in VIPT one uses the variational method in terms of a known trial function and through optimization process, new parameters are obtained which are then applied to the perturbation theory to make the perturbation expansion a convergent one [12]. Thus, the VIPT removes the specific problems of variational method and perturbation theory by combining both of them properly and thus hope to handle large perturbation.

With linear cum Coulombic potential [13], we have two options to use in VIPT: (i) Coulombic potential as the parent and linear one as the perturbation and (ii) linear one as the parent and Coulombic potential as perturbation in the potential model we have adopted. We have already reported such an attempt [12] in the calculation of slope and curvature of I–W function with Coulombic parent. It had successfully analysed the said for D, D_s, B mesons taking into account the three terms in the summation of equation expressing the first-order corrected wave function. Although the results were shown to be improved with more terms in that equation, it was quite cumbersome. Further, larger α_s values were felt necessary for B-meson for which the result was not so satisfactory when compared to D, D_s mesons.

A careful investigation shows that the linear part with significant confinement effect ($b = 0.183\ \text{GeV}^2$) is usually dominant over the Coulombic one for mesons having greater reduced mass μ. Further, as pointed out in ref. [10], the linear parent is quite handy in predicting the mass, energy etc. for different states compared to the Coulombic one. So, it is definitely worthwhile to test the model with linear parent-including also the B_s, B_c mesons which have greater reduced mass μ.

We recall that [10] for the linear potential to be dominant we require $\langle r \rangle > r_0$, where $\langle r \rangle$ is the expectation value of the distance r which reasonably gives the size of a state (in this case meson) and r_0 is a point at which linear cum Coulomb potential becomes zero (figure 1 of Aitchison and Dudek [10]). The condition of applicability of VIPT to linear potential as parent conforms to low value of α_s and high value of b because r_0 is directly proportional to α_s and inversely proportional to b and we need a small r_0 for the linear potential to dominate. So, with a linear parent, one can suitably handle large b and small α_s which is necessary in this QCD-inspired potential model for the B-sector mesons (e.g. B, B_s, B_c) usually incorporated with small running coupling constant α_s due to their large mass. The linear parent is thus expected to be effective for heavier mesons.

Our approach is further boosted by the success of the work [8] where we have used the Dalgarno method with linear parent for D, D_s, B, B_s, B_c mesons.

The rest of the paper is organized as follows: Section 2 contains the formalism, §3 the result and calculation while §4 includes the discussion and conclusion.

2. Formalism

2.1 *Isgur–Wise function; its slope and curvature*

The Isgur–Wise function is written as [1]

$$\begin{aligned}\xi(v_\mu \cdot v'_\mu) &= \xi(y) \\ &= 1 - \rho^2 (y-1) + C (y-1)^2 + \cdots ,\end{aligned} \tag{1}$$

where

$$y = v_\mu \cdot v'_\mu \tag{2}$$

and v_μ and v'_μ are the four velocity of the heavy meson before and after the decay. The quantity ρ^2 is the slope of the I–W function at $y = 1$ and known as charge radius:

$$\rho^2 = \left.\frac{\partial \xi}{\partial y}\right|_{y=1} . \tag{3}$$

The second-order derivative is the curvature of the I–W function known as convexity parameter:

$$C = \frac{1}{2}\left.\left(\frac{\partial^2 \xi}{\partial y^2}\right)\right|_{y=1} . \tag{4}$$

For the heavy–light flavour mesons the I–W function can also be written as [3,14]

$$\xi(y) = \int_0^{+\infty} 4\pi r^2 \, |\psi(r)|^2 \cos pr \, \mathrm{d}r, \tag{5}$$

where

$$p^2 = 2\mu^2 (y - 1) . \tag{6}$$

Now the wave function ψ of the hadronic system is determined by taking the linear potential as the parent.

2.2 *First-order corrected wave function and energy in VIPT*

The wave function corrected upto the first order of jth state is given by (eq. (10) of ref. [12])

$$\psi_j = \psi_j^{(0)} + \sum_{k \neq j} \frac{\int \psi_k^{(0)*} H'_{P',j} \psi_j^{(0)} \mathrm{d}v}{E_j^{(0)} - E_k^{(0)}} \psi_k^{(0)} . \tag{7}$$

The energy corrected upto the first order for the same state is

$$\begin{aligned} E_j &= \int \psi_j^{(0)*} H \psi_j^{(0)} \mathrm{d}v \\ &= \int \psi_j^{(0)*} (H_{0P'} + H'_{P'}) \psi_j^{(0)} \mathrm{d}v, \end{aligned} \tag{8}$$

where ψ_k, E_k are the wave function and energy eigenvalues of the kth states which are orthonormal to the jth state. The superscript (0) means zeroth-order correction of the corresponding quantities. Also, we note that P' is the variational parameter and $H_{0P'}$, $H'_{P'}$ are as defined in eq. (9) of ref. [12].

The summation in eq. (7) can include any number of kth states. In this work, we consider terms upto three states in the summation as was done in ref. [12].

2.3 *Wave functions using VIPT with linear potential as the parent*

2.3.1 *With one term in the summation.* As explained earlier, we take b' as the variational parameter instead of the physical parameter b in the parent linear potential to write the Hamiltonian as [3,12]

$$\begin{aligned} H &= H_0 + H' \\ &= -\frac{\nabla^2}{2\mu} + br - \frac{4\alpha_s}{3r} + c \\ &= -\frac{\nabla^2}{2\mu} + br - \frac{\alpha}{r} + c \\ &= -\frac{\nabla^2}{2\mu} + b'r - \frac{\alpha}{r} - b'r + br + c \\ &= H_{0b'} + H'_{b'}, \end{aligned} \tag{9}$$

where $\alpha = 4\alpha_s/3$. Now, $H_{0b'} = -(\nabla^2/2\mu) - b'r$ is the parent Hamiltonian with the new parameter b' and $H'_{b'} = (\alpha/r) - b'r + br + c$ is the perturbed Hamiltonian with the same variational parameter b' instead of the physical parameter b.

We consider jth state as the $1s$ state ($n = 1, l = 0$) and in the summation of eq. (7), we consider a single kth state which is the $2s$ state ($n = 2, l = 0$).

We note that in the variational method, we are interested only in the 'r' dependence of the Hamiltonian, and so 'c' in $H'_{b'}$ has no role to play in the calculation [15].

The unperturbed wave functions with linear parent with appropriate boundary conditions are the Airy functions given by [10]

$$\psi_{n0}(r) = \frac{N_n}{2\sqrt{\pi}r}\mathrm{Ai}\left((2\mu b')^{1/3}r + \rho_{0n}\right), \tag{10}$$

where ρ_{0n}s are the zeroes of the Airy function $\mathrm{Ai}(\rho_{0n}) = 0$ given by [10,16]:

$$\rho_{0n} = -\left[\frac{3\pi(4n-1)}{8}\right]^{2/3} \tag{11}$$

and N_n is the normalization constant.

As an illustration, we reproduce for s states a few of the zeroes of the Airy function in table 1. The corresponding energies are given as

$$E_n = -\left(\frac{b'^2}{2\mu}\right)^{1/3}\rho_{0n}. \tag{12}$$

Table 1. A few of the zeroes of Airy function for s states.

State	ρ_{0n}
$1s$ ($n = 1, l = 0$)	−2.3194
$2s$ ($n = 2, l = 0$)	−4.083
$3s$ ($n = 3, l = 0$)	−5.5183
$4s$ ($n = 4, l = 0$)	−6.782

Of course $n = 1, 2, 3, 4, \ldots$ is the principal quantum number.

Thus the trial $1s$ state ($n = 1, l = 0$) wave function is (which is also the unperturbed wave function):

$$\begin{aligned}\psi^{(0)} &= \psi_{10}^{(0)} \\ &= \frac{N_1}{2\sqrt{\pi}r}\mathrm{Ai}\left((2\mu\bar{b}')^{1/3}r - 2.3194\right) \\ &= \frac{N_1}{2\sqrt{\pi}r}\mathrm{Ai}\left(z_1\right),\end{aligned} \tag{13}$$

where

$$z_1 = ((2\mu\bar{b}')^{1/3}r - 2.3194) \tag{14}$$

and the subscript 10 indicates the quantum number (n, l) of the jth state.

We note that b' is replaced by $\bar{b}'$ which is obtained by minimizing E_j given by eq. (8). It is essential since in VIPT we have to use the values of variational parameter leading to minimum energy (for example in ref. [12], α_s was replaced by $\bar{\alpha}'_{10}$). The values of $\bar{b}'$ for different mesons are listed in table 2.

Now we consider the single kth state in the summation of eq. (7) which is the $2s$ state given by

$$\psi_{20}^{(0)} = \frac{N_2}{2\sqrt{\pi}r}\mathrm{Ai}\left((2\mu\bar{b}')^{1/3}r - 4.083\right) = \frac{N_2}{2\sqrt{\pi}r}\mathrm{Ai}\left(z_2\right), \tag{15}$$

where

$$z_2 = \left((2\mu\bar{b}')^{1/3}r - 4.083\right). \tag{16}$$

The wave function corrected upto first order is

$$\psi_S = N\left[\psi^{(0)} + \frac{(2\mu)^{1/3}}{(\rho_{02} - \rho_{01})\,\bar{b}'^{2/3}}\left((b - \bar{b}')\langle r\rangle_{2,1} - \alpha\left\langle\frac{1}{r}\right\rangle_{2,1}\right)\psi_{20}\left(r\right)\right], \tag{17}$$

where

$$\langle r\rangle_{2,1} = N_1 N_2 \int_0^{+\infty} r\mathrm{Ai}((2\mu\bar{b}')^{1/3}r - 2.3194)\mathrm{Ai}\left((2\mu\bar{b}')^{1/3}r - 4.083\right)\mathrm{d}r \tag{18}$$

and N is the normalization constant.

2.3.2 *With two terms in the summation.* We next consider the $3s$ state ($n = 3, l = 0$) in addition to $2s$ state (as done in the single term case) given by

$$\psi_{30}^{(0)} = \frac{N_3}{2\sqrt{\pi}r}\mathrm{Ai}\left((2\mu\bar{b}')^{1/3}r - 5.5153\right) = \frac{N_3}{2\sqrt{\pi}r}\mathrm{Ai}\left(z_3\right), \tag{19}$$

Table 2. Values of $\bar{b}'$ with $b = 0.183$ GeV^2.

Mesons	Reduced mass μ	$\alpha = 4\alpha_s/3$	$\bar{b}'$ without relativistic effect	$\bar{b}'$ with relativistic effect
D	0.2761	0.924	5.306	16.24
D_s	0.368248	0.924	5.876	19.8
B	0.31464	0.348	4.33	5.587
B_s	0.4401	0.348	4.497	5.954
B_c	1.1803	0.348	5.39	8.103

where

$$z_3 = \left((2\mu\bar{b}')^{1/3}r - 5.5153\right). \tag{20}$$

With the inclusion of this state, the wave function corrected upto the first order is

$$\psi_D = N'\left[\psi^{(0)} + \frac{(2\mu)^{1/3}}{(\rho_{02}-\rho_{01})\,\bar{b}'^{2/3}}\left((b-\bar{b}')\langle r\rangle_{2,1} - \alpha\left\langle\frac{1}{r}\right\rangle_{2,1}\right)\psi_{20}(r) + \frac{(2\mu)^{1/3}}{(\rho_{03}-\rho_{01})\,\bar{b}'^{2/3}}\left((b-\bar{b}')\langle r\rangle_{3,1} - \alpha\left\langle\frac{1}{r}\right\rangle_{3,1}\right)\psi_{30}(r)\right], \tag{21}$$

where

$$\langle r\rangle_{3,1} = N_1N_3\int_0^{+\infty} r\,\mathrm{Ai}\left((2\mu\bar{b}')^{1/3}r - 2.3194\right)\mathrm{Ai}\left((2\mu\bar{b}')^{1/3}r - 5.5153\right)\mathrm{d}r \tag{22}$$

and N' is the normalization constant.

2.3.3 *With three terms in the summation.* In addition to the 2s and 3s states we now add the 4s state:

$$\psi_{40}^{(0)} = \frac{N_4}{2\sqrt{\pi}r}\mathrm{Ai}\left((2\mu\bar{b}')^{1/3}r - 6.782\right) = \frac{N_4}{2\sqrt{\pi}r}\mathrm{Ai}(z_4), \tag{23}$$

where

$$z_4 = \left((2\mu\bar{b}')^{1/3}r - 6.782\right). \tag{24}$$

With the inclusion of this state, the first-order wave function now becomes

$$\psi_T = N''\left[\psi^{(0)} + \frac{(2\mu)^{1/3}}{(\rho_{02}-\rho_{01})\,\bar{b}'^{2/3}}\left((b-\bar{b}')\langle r\rangle_{2,1} - \alpha\left\langle\frac{1}{r}\right\rangle_{2,1}\right)\psi_{20}(r) + \frac{(2\mu)^{1/3}}{(\rho_{03}-\rho_{01})\,\bar{b}'^{2/3}}\left((b-\bar{b}')\langle r\rangle_{3,1} - \alpha\left\langle\frac{1}{r}\right\rangle_{3,1}\right)\psi_{30}(r) + \frac{(2\mu)^{1/3}}{(\rho_{04}-\rho_{01})\,\bar{b}'^{2/3}}\left((b-\bar{b}')\langle r\rangle_{4,1} - \alpha\left\langle\frac{1}{r}\right\rangle_{4,1}\right)\psi_{40}(r)\right], \tag{25}$$

where

$$\langle r\rangle_{4,1} = N_1 N_4 \int_0^{+\infty} r\mathrm{Ai}((2\mu\bar{b}')^{1/3}r - 2.3194)\mathrm{Ai}((2\mu\bar{b}')^{1/3}r - 6.782)\mathrm{d}r \quad (26)$$

and N'' is the normalization constant.

The relativistic version of these wave functions is obtained by multiplying the above expression by $(r\mu\alpha)^{-\epsilon}$ [17,18]. The relativistic modification is felt necessary as the light quark moves faster relative to the static heavy quark. Thus, relativistic version of all these wave functions is

$$\psi_{i,\mathrm{rel}} = \psi_i (r\mu\alpha)^{-\epsilon}, \quad (27)$$

where $i = S, D, T$ and

$$\epsilon = 1 - \sqrt{1 - \frac{4\alpha_s}{3}}. \quad (28)$$

Putting all these wave functions, i.e. eqs (17), (21), (25) and (27) in (5) we can calculate the Isgur–Wise function for different cases.

3. Calculation and results

We have listed the values of charge radius and convexity parameter of the calculated I–W function for various heavy–light flavour mesons in the present method considering single state, two states, and three states of eq. (7) with and without relativistic effect.

Table 1 gives the zeroes of Airy function while table 2 gives the values of $\bar{b}'$. In tables 3–5, we record our predictions of slope and curvature for single term, two terms and three terms of eq. (7) respectively. Table 6 gives a summary of these in other models while in table 7, we give a comparison of VIPT and Dalgarno methods for both the options.

The α_s values are taken from the V-scheme [6,19–21] and the integrations are done numerically for all these calculations.

Table 3. Values of slope ρ^2 and curvature C with single term in eq. (7).

Meson	ρ_S^2	C_S	$\rho_{S,\mathrm{rel}}^2$	$C_{S,\mathrm{rel}}$
D	1.36	0.01	0.53	0.0022
D_s	1.867	0.03	0.702	0.0036
B	1.93	0.02	1.41	0.013
B_s	2.923	0.046	2.113	0.0283
B_c	9.442	0.484	6.274	0.2522

Table 4. Values of slope ρ^2 and curvature C with two terms in eq. (7).

Meson	ρ_D^2	C_D	$\rho_{D,\mathrm{rel}}^2$	$C_{D,\mathrm{rel}}$
D	1.201	0.013	0.57	0.0026
D_s	2.001	0.0242	0.74	0.0041
B	2.004	0.0244	1.44	0.0133
B_s	3.031	0.0565	2.16	0.0297
B_c	10.2	0.61	6.51	0.275

Table 5. Values of slope ρ^2 and curvature C with three terms in eq. (7).

Meson	ρ_T^2	C_T	$\rho_{T,\mathrm{rel}}^2$	$C_{T,\mathrm{rel}}$
D	1.33	0.016	0.604	0.00326
D_s	2.023	0.0305	0.78	0.0054
B	2.027	0.031	1.54	0.0217
B_s	3.087	0.071	2.29	0.047
B_c	10.25	0.767	6.99	0.441

Table 6. Predictions of the slope and curvature of the I–W function in various models.

Model	Value of ρ^2	Value of curvature C
Le Yaouanc *et al* [22]	≥ 0.75	–
Le Yaouanc *et al* [23]	≥ 0.75	≥ 0.47
Rosner [29]	1.66	2.76
Mannel [30,31]	0.98	0.98
Pole ansatz [32]	1.42	2.71
MIT bag model [28]	2.35	3.95
Ebert *et al* [34]	1.04	1.36
Simple quark model [27]	1	1.11
Skryme model [25]	1.3	0.85
QCD sum rule [26]	0.65	0.47
Relativistic three-quark model [24]	1.35	1.75
Neubert [33]	0.82±0.09	–
Infinite momentum frame quark model [35]	3.04	6.81
UKQCD Collaboration [36]	0.83^{+15+24}_{-11-22}	–
CLEO Collaboration [37]	$0.76 \pm 0.16 \pm 0.08$	–

Table 7. Comparison of the values of ρ^2 and C in VIPT and Dalgarno methods for both the options. For comparison we take the best representative values of ρ^2 and C from the available data for D, D_s, B mesons.

VIPT					
		I. Linear parent (this work)		II. Coulombic parent [12]	
Terms considered in eq (7)	Meson	ρ_S^2	C_S	ρ_S^2	C_S
Single term	D	0.53	0.0022	0.433	0.525
	D_s	0.702	0.0036	0.56	0.85
	B	1.41	0.0126	3.6	15.3
Two terms	D	0.57	0.0026	0.432	0.524
	D_s	0.74	0.0041	0.55	0.84
	B	1.44	0.0133	3.16	12.32
Three terms	D	0.604	0.0033	0.43	0.516
	D_s	0.78	0.0054	0.545	0.815
	B	1.54	0.0213	3.12	11.8
Dalgarno Method					
		I. Linear parent [8]		II. Coulombic parent [6]	
–	Meson	ρ_S^2	C_S	ρ_S^2	C_S
–	D	0.896	0.0031	1.136	5.377
	D_s	1.352	0.0077	0.912	0.0007
	B	1.41	0.013	128.13	5212

4. Discussion and conclusion

This analysis with linear parent shows a completely different picture in comparison to that with Coulombic parent [12]. With more terms in (7), the slope and curvature have increased contrary to Coulombic parent. Also, an analysis of table 6 indicates that for a definite term, the slope has assumed larger values than those of ref. [12] while for the curvature, the pattern is reversed, i.e. it has assumed smaller values than those of ref. [12].

Regarding the number of terms considered in the summation (7), we have seen that the result is the most satisfactory and comparable for the single term consideration. This is undoubtably a great phenomenological advantage as involvement of more terms in eq. (7) makes the calculation quite cumbersome which happened in ref. [12]. However, relativistic correction in this case also decreases the slope and curvature of Isgur–Wise function as observed earlier [12]. If we look back at our Dalgarno method with linear parent [8], we have observed larger values of slope and curvature for D, D_s mesons and smaller values for B, B_s, B_c mesons in this work compared to that in [8].

To conclude, the present approach based on VIPT for calculating of I–W function within the QCD-inspired potential model appears to be preferable over the one in ref. [12] where the linear potential was considered as perturbation.

References

[1] N Isgur and M B Wise, *Phys. Lett.* **B232**, 113 (1989); *Phys. Lett.* **B237**, 527 (1990)
[2] D K Choudhury, P Das, D D Goswami and J N Sharma, *Pramana – J. Phys.* **44**, 519 (1995)
[3] D K Choudhury and N S Bordoloi, *Int. J. Mod. Phys.* **A15**, 3667 (2000)
[4] A D Rujula, H Georgi and S L Glashow, *Phys. Rev.* **D12**, 147 (1975)
[5] D K Choudhury and N S Bordoloi, *Mod. Phys. Lett.* **A17(29)**, 1909 (2002)
[6] D K Choudhury and N S Bordoloi, *Mod. Phys. Lett.* **A26**, 443 (2009)
[7] B J Hazarika and D K Choudhury, arXiv:hep-ph/1102.4970, Accepted in *Bra. J. Phys.* (2011)
[8] B J Hazarika, K K Pathak and D K Choudhury, arXiv:hep-ph/1012.4377
[9] S K You, K J Jeon, C K Kim and K Nahm, *Eur. J. Phys.* **19**, 179 (1998)
[10] I J R Aitchison and J J Dudek, *Eur. J. Phys.* **23**, 605 (2002)
[11] F M Fernandez, *Eur. J. Phys.* **24**, 289 (2003)
[12] B J Hazarika and D K Choudhury, *Pramana – J. Phys.* **75**, 423 (2010)
[13] Riazuddin and Fiyyazuddin, in: *A modern introduction to particle physics* (Allied Publishers Limited, 2000) p. 256
[14] F E Close and A Wambach, *Nucl. Phys.* **B412**, 169 (1994)
[15] A K Ghatak and S Lokanathan, in: *Quantum mechanics* (McGraw Hill, 1997) p. 291
[16] *Handbook of mathematical functions with formulas, graphs and mathematical tables* edited by M Abramowitz and I A Stegun (National Bureau of Standards Applied Mathematics Series, 55, Department of Commerce, USA, 1974)
[17] J J Sakurai, in: *Advanced quantum mechanics* (Addison-Willey Publishing Company, Massachusetts, 1986) p. 128
[18] C Itzykson and J Zuber, in: *Quantum field theory* (International Student Edition, McGraw Hill, Singapore, 1986) p. 79
[19] M Peter, *Phys. Rev. Lett.* **78**, 603 (1997); *Nucl. Phys.* **B501**, 471 (1997)
[20] Y Schroeder, *Phys. Lett.* **B447**, 321 (1999)
[21] Y Schroeder, *Nucl. Phys. Proc. Suppl.* **86**, 525 (2000)
[22] A Le Yaouanc, L Oliver, O Pene and J C Raynal, *Phys. Lett.* **B365**, 319 (1996)
[23] A Le Yaouanc, L Oliver and J C Raynal, *Phys. Rev.* **D69**, 094022 (2004)
[24] M A Ivanov, V E Lyubouvitskij, L G Körner and P Kroll, *Phys. Rev.* **D56**, 348 (1997)
[25] E Jenkins, A Manohar and M B Wise, *Nucl. Phys.* **B396**, 38 (1996)
[26] Y B Dai, C S Huang, M K Huang and C Liu, *Phys. Lett.* **B387**, 379 (1996)
[27] B Holdom, M Sutherland and J Mureika, *Phys. Rev.* **D49**, 2359 (1994)
[28] M Sadzikowski and K Zalewski, *Z. Phys.* **C59**, 667 (1993)
[29] J L Rosner, *Phys. Rev.* **D42**, 3732 (1990)
[30] T Mannel, W Roberts and Z Ryzak, *Phys. Rev.* **D44**, R18 (1991)
[31] T Mannel, W Roberts and Z Ryzak, *Phys. Lett.* **B255**, 593 (1993)
[32] M Neubert, *Phys. Lett.* **B264**, 455 (1991)
[33] M Neubert, *Int. J. Mod. Phys.* **A11**, 4173 (1996)
[34] D Ebert, R N Faustov and V O Galkin, arXiv:hep-ph/0611307
[35] B König, J G Körner, M Krämer and P Kroll, *Phys. Rev.* **D56**, 4282 (1997)
[36] UKQCD Collaboration: K C Bowler *et al*, *Nucl. Phys.* **B637**, 293 (2002)
[37] CLEO Collaboration: J Bartlet *et al*, *Phys. Rev. Lett.* **82**, 3746 (1999)