DOUBLE DISPERSION APPROACH TO NUCLEON-NUCLEON SCATTERING

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1. INTRODUCTION

The first efforts to describe the nucleon-nucleon interaction in terms of a meson field theory were directed toward obtaining a two-nucleon potential to be substituted into a Schrödinger equation. The work of Lévy [1] using a Tamm-Dancoff method started a consistent way of approaching this problem with further developments using the Bethe-Salpeter equation as the basic dynamical equation. In this way both a " potential " and a dynamical equation are obtained from meson field theory. However, the approximations made cannot be easily justified and the difficulties encountered are very large.

It is now recognized that the very concept of a potential cannot be unambiguously defined for strongly interacting particles. The whole approach is therefore clearly insufficient and probably inadequate.

There has been, in the last few years, a change of attitude in the discussion of this problem. The object of the new approach is the determination of the scattering parameters based on some general properties of field theory which are presumably valid independently of a perturbation expansion and which avoid entirely the idea of a potential.

In such a program one is led naturally to the dispersion relation approach. The one-dimensional dispersion relations which use the energy as a variable evidently cannot give a complete description of the scattering because they do not contain any information about the dependence on the momentum transfer.

An investigation of the two-nucleon problem, based on dispersion relations for fixed momentum transfer, was initiated by Goldberger, Nambu and Oehme [2]. They have developed the kinematical formalism which is complicated by the presence of the spins. They also give a detailed discussion of the contribution of the deuteron bound state. However, as we remarked above, their relations are necessarily incomplete from the dynamical point of view.

Two years ago, at the Geneva conference on high energy physics, Mandelstam [3] proposed a generalization of the dispersion relations, embodied in a two-dimensional representation of the scattering amplitude. This representation exhibits the analytic properties of the scattering amplitude as a function of both variables, the energy and the momentum transfer in the whole complex two-dimensional manifold. He then showed that this representation, supplemented by unitarity (in the two particle approximation), provides us with a dynamical system of equations for the scattering, depending on a few parameters that may be identified as coupling constants. The correctness of the Mandelstam representation to all orders in perturbation theory has been brilliantly established by Eden [4].

I want to report here on the result of preliminary work done in Princeton and Berkeley on the nucleon-nucleon problem on the basis of this double dispersion representation by Goldberger, Grisaru, Wong and myself. Work on similar lines has been carried out by the CERN group : Amati, Leader and Vitale.

2. KINEMATICAL SPECIFICATION OF THE PROBLEM

Let us consider the diagram of Fig. 1, which represents nucleon-nucleon scattering. It is clear that if we invert the arrows of particle 2 this diagram will represent nucleon-antinucleon scattering. Therefore the three processes

$$N_1+N_2 \rightarrow N_1'+N_2' \qquad \text{(I)}$$

$$N_1+\overline{N}_2 \rightarrow N_1'+\overline{N}_2' \qquad \text{(II)}$$

$$N_1+\overline{N}_2 \rightarrow \overline{N}_1'+N_2' \qquad \text{(III)}$$

are essentially described by the same Green's function which is represented by the black box taken in different regions of the variables involved, according to the orientation of the external lines. There are two independent scalars in this problem, but in connection with the Mandelstam representation it is convenient to use three variables

$$s = -(P_1+P_2)^2 = -(P_1'+P_2')^2$$

$$\bar{t} = -(P_1-P_2')^2 = -(P_1'-P_2)^2$$

$$t = -(P_1-P_1')^2 = -(P_2'-P_2)^2$$

which are related by: $s+t+\bar{t} = 4m^2$; s is the square of the energy in the center of mass system and $-t$, $-\bar{t}$ are the squares of the momentum transfer for the pairs (1, 1′) and (1, 2′).

Throughout this talk Is hall assume charge independence, thereby neglecting Coulomb effects.

It is well known that for either state of isotopic spin, two-nucleon scattering is described by five independent amplitudes. The same number, of course, is needed to describe nucleon-antinucleon scattering.

The first step in our procedure is to select a set of five invariant amplitudes which are free from kinematical singularities; that is, the only singularities are:

(i) simple poles corresponding to one-particle states, and

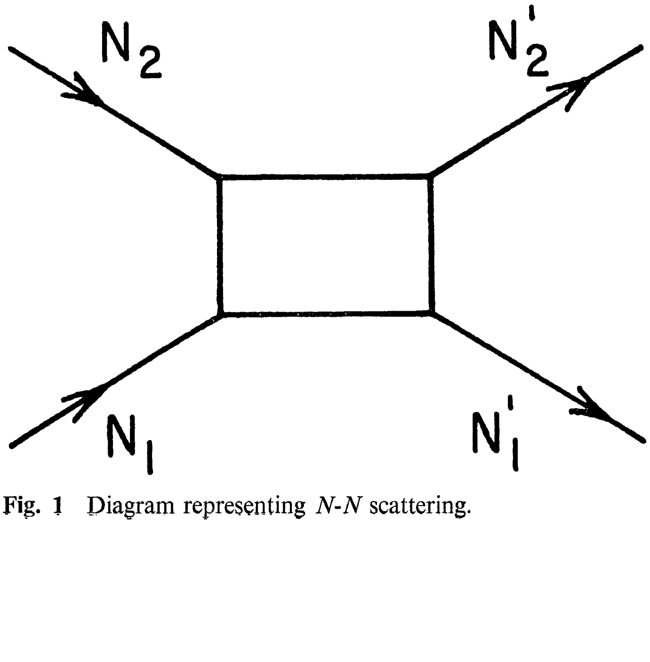

Fig. 1 Diagram representing *N-N* scattering.

(ii) branch points at the thresholds for any allowed channel, in the energy variable for reactions (I), (II), (III). Such amplitudes will then have a Mandelstam representation.

A second criterion for the choice is that they have simple symmetry properties by virtue of the Pauli principle, that is, under the transformation $t \leftrightarrow \bar{t}$.

A third criterion is that the three processes be related by simple crossing. We could not entirely reconcile the last two requirements. We preferred to emphasize the second and obtained a non-trivial matrix transformation for crossing. The set of amplitudes chosen by GNO were obtained by different criteria. No attention was paid to the first two requirements; however, their amplitudes have simple properties under crossing.

One finds that a good set of covariant amplitudes to start with is obtained by writing the Feynman amplitude for process I in the form

$$\bar{u}_2\bar{u}_1\tau u_2 u_1 = \tau_I = [F_1^0(S-\widetilde{S})+F_2^0(T+\widetilde{T})+F_3^0(A-\widetilde{A})+$$
$$+F_4^0(V+\widetilde{V})+F_5^0(P-\widetilde{P})]\Omega_0+$$
$$+[F_1'(S-\widetilde{S})+F_2'(T+\widetilde{T})+F_3'(A-\widetilde{A})+F_4'(V+\widetilde{V})+$$
$$+F_5'(P-\widetilde{P})]\Omega_1$$

where the F's are functions of the scalars $s, t, \bar{t}$. Ω_0 and Ω_1 are the isotopic spin projection operators for singlet and triplet states, and

$$S = \bar{u}(P_2')u(P_2)\bar{u}(P_1')u(P_1)$$

$$V = \bar{u}(P_2')\gamma_\mu u(P_2)\bar{u}(P_1')\gamma_\mu u(P_1)$$

. . . .

$\widetilde{S}$, $\widetilde{V}$, etc. are obtained by interchanging $u(P_2')$ and $u(P_1')$, that is:

$$\begin{pmatrix}\widetilde{S}\\ \widetilde{V}\\ \vdots\end{pmatrix} = (\text{Fierz Matrix})\begin{pmatrix}S\\ V\\ \vdots\end{pmatrix}$$

We write the amplitudes for processes (II) and (III) in exactly the same way in terms of invariant functions $\overline{F}_i$ and $\overline{\overline{F}}_i$. We shall now argue that the amplitudes F_i^I are free from kinematical singularities; that is, they do indeed possess a Mandelstam representation.

Let us consider the invariant functions

$$\tau_i = \mathrm{Tr}\{0_i \Lambda(P_2')\Lambda(P_1')\tau\Lambda(P_1)\Lambda(P_2)\}$$

where O_i are the matrices $1^{(1)}$, $1^{(2)}$, $\gamma_\mu^{(1)}$, $\gamma_\mu^{(2)}$ etc. and $\Lambda(P)$ is the projection operator for positive energy spinor states. According to the Hall-Wightman theorem [5] the τ_i, being Lorentz invariant combinations of the elements of the τ-matrix, are functions of the scalar products of the momenta, analytic in the domain corresponding to the forward tube in momentum space. Since these combinations are formed with constant matrices O_i, it follows by a straightforward generalization of Eden's analysis that the Mandelstam representation is valid for them to all orders of perturbation theory. But

$$\tau_i = D_{ij} F_j$$

and det $D = (st\bar{t})^3$, so that the only possible additional singularities in the F's are poles at $s = 0$, $t = 0$, $\bar{t} = 0$. Later, when we discuss the amplitudes in the center of mass system, we show that the F's are finite in the forward ($t = 0$) and backward ($\bar{t} = 0$) directions.

The argument relies on the fact that certain amplitudes in which the component of the total spin along the direction of motion in the incoming and outgoing beams are different must vanish in the forward (or backward) direction, because otherwise conservation of angular momentum would be violated.

A similar argument applied to nucleon-antinucleon scattering shows that the P's are regular at $s = 0$ as well.

The transformation properties associated with the Pauli principle require that under the interchange of all the coordinates of the final particles the full amplitude must change sign. Now P_0 changes sign, and $s \leftrightarrow \bar{s}$ etc., as well as $t \leftrightarrow \bar{t}$. Therefore the statement of the Pauli principle is:

$$F_i^I(s, \bar{t}, t) = (-1)^{i+I} F_i^I(s, t, \bar{t})$$

Finally, we deduce the crossing symmetry relations for the three processes. Following standard methods one obtains for (I) and (II):

$$F_i(s, \bar{t}, t) = \Gamma_{ij} B \bar{F}_j(\bar{t}, s, t)$$

where

$$B = \tfrac{1}{2}\begin{pmatrix} -1 & 3 \\ 1 & 1 \end{pmatrix}$$

is a matrix in isospin space and Γ is the negative transpose of the Fierz matrix for the amplitudes $(S, -T, A, -V, P)$.

3. PARTIAL WAVE ANALYSIS OF THE SCATTERING MATRIX

One must be careful when writing down the Mandelstam representation for the various covariant amplitudes to ensure that the integrals are convergent. Some subtractions are necessary to fulfill the requirements of unitarity and even more subtractions may be necessary because of the strong nature of the interaction. The outcome of this is that one has to treat the low angular momentum amplitudes separately from the rest of the two-variable problem.

We have then to make a partial wave analysis of the scattering amplitude in the center of mass system. We found that a decomposition in terms of helicity amplitudes [6] is the most convenient for this problem. We then chose the following set of partial amplitudes for each isotopic spin state:

Singlet: $f_0^J =$

$$\langle +\tfrac{1}{2} + \tfrac{1}{2} | T^J(W) | +\tfrac{1}{2} + \tfrac{1}{2} \rangle - \langle +\tfrac{1}{2} + \tfrac{1}{2} | T^J(W) | -\tfrac{1}{2} - \tfrac{1}{2} \rangle$$

Triplet ($J = l$): $f_1^J =$

$$\langle +\tfrac{1}{2} - \tfrac{1}{2} | T^J(W) | +\tfrac{1}{2} - \tfrac{1}{2} \rangle - \langle +\tfrac{1}{2} - \tfrac{1}{2} | T^J(W) | -\tfrac{1}{2} + \tfrac{1}{2} \rangle$$

Triplet ($J = l \pm 1$):

$f_{11}^J =$

$$\langle +\tfrac{1}{2} + \tfrac{1}{2} | T^J(W) | +\tfrac{1}{2} + \tfrac{1}{2} \rangle + \langle +\tfrac{1}{2} + \tfrac{1}{2} | T^J(W) | -\tfrac{1}{2} - \tfrac{1}{2} \rangle$$

$$f_{12}^J = 2\langle +\tfrac{1}{2} + \tfrac{1}{2} | T^J(W) | +\tfrac{1}{2} - \tfrac{1}{2} \rangle$$

$f_{22}^J =$

$$\langle +\tfrac{1}{2} - \tfrac{1}{2} | T^J(W) | +\tfrac{1}{2} - \tfrac{1}{2} \rangle + \langle +\tfrac{1}{2} - \tfrac{1}{2} | T^J(W) | -\tfrac{1}{2} + \tfrac{1}{2} \rangle$$

The Pauli principle requires that transitions occur only in states for which $(-1)^{l+s+I} = -1$. These partial waves may be projected out of five amplitudes $f_1, ..., f_5$ by means of Legendre polynomials. For instance:

$$f_0^J = \frac{p}{2E}\int_{-1}^{+1} f_1(s,z)P_j(z)dz$$

The f's are in turn related to the covariant amplitudes by a matrix $a(p^2, z)$ whose entries are linear functions of p^2 and $z = \cos\theta$. For instance:

$$4\pi f_1 = (E^2+m^2)(F_1+F_5)-4p^2F_3-2p^2zF_2$$

It is from these relationships and the definitions of the f's that one obtains the result mentioned earlier of the regularity of the f's for $t = 0$ and $\bar{t} = 0$.

4. ANALYTICITY OF PARTIAL WAVE AMPLITUDES

The deduction of partial wave dispersion relations from the Mandelstam representation has been described in many papers for a variety of processes. The procedure is always the same [7] and consists in locating the singularities of the partial amplitudes which occur whenever one of the denominators which appear in the Mandelstam representation vanishes. The variables t and $\bar{t}$ (for our particular problem) are given by:

$$t = -2p^2(1-z)$$

$$\bar{t} = -2p^2(1+z)$$

and z is allowed to vary in the interval $(-1, 1)$ since the partial waves are obtained by integrating over the variable z in this interval. The Born term which comes from the one meson exchange gives rise to a cut which extends from $s = 4m^2-\mu^2$ to $s = -\infty$.

The denominator $s'-s$ gives rise to a physical cut from $4m^2$ to $+\infty$. The denominators $t'-t$ and $\bar{t}'-\bar{t}$ give rise to a cut from $s = 4m^2-4\mu^2$ to $s = -\infty$. The branch point corresponds to the exchange of two mesons.

If we multiply the partial amplitudes by factors (E/p) or (m/p) to avoid kinematical singularities, one can write the following representation for these functions (neglecting subtractions):

$$h_\alpha^J(s) = h_{\alpha B}^J(s)+\frac{1}{\pi}\int_{4m^2}^{\infty}\frac{ds'\,\mathrm{Im}\,h_\alpha^J(s')}{s'-s}+$$

$$+\frac{1}{\pi}\int_{-\infty}^{4(m^2-\mu^2)}\frac{ds'\,\mathrm{Im}\,[h_\alpha^J(s')-h_{\alpha B}^J(s')]}{s'-s}$$

On the positive cut one has not to worry about $\mathrm{Im}\,h_\alpha^J(s')$ since it is given by unitarity; it will be discussed later.

On the negative cut one can relate $\mathrm{Im}\,[h_\alpha^J(s')-$ $-h_{\alpha B}^J(s')]$ to the absorptive amplitude for process (II). The result is very similar to the usual crossing except for a tricky point of taking the real part of the absorptive amplitudes when they become complex. It is exactly when this happens that the Legendre polynomial expansion does not converge.

One obtains:

$$\mathrm{Im}\,[h_\alpha^J(s)-h_{\alpha B}^J(s)] =$$

$$-2\sum_{J'}C^{JJ'}\int_{-1-\frac{2\mu^2}{p^2}}^{1} a(p_1^2z)\Gamma B\,\mathrm{Re}\,\bar{A}(\bar{t},t)P_{J'}(z)dz$$

where $C^{JJ'}$ are numerical coefficients which occur in the projection of the partial waves by means of Legendre polynomials. The factor 2 comes from the contribution of processes (II) and (III), which are exactly equal.

It is a simple matter to determine the boundary of the region where the Legendre expansion converges. It comes from second order perturbation theory. I only mention that the expansion is valid throughout the region where t and $\bar{t}$ are less than $4m^2$. However, this region, which corresponds to nucleon-antinucleon annihilation into mesons is unphysical. For all practical purposes, however, only the two-pion intermediate state can be taken into account. Fortunately it is just for these unphysical regions of the energy that solutions of the nucleon-antinucleon annihilation into two pions are most reliable. The two-pion contribution can be taken into account in second order perturbation theory from which the lower angular momenta, say s and p waves, have been subtracted. We then add explicitly the contributions of these angular momenta, using for instance the results obtained for them by Frazer and Fulco [8], improved by a normalization procedure due to Wong [9].

5. THE INTEGRAL EQUATIONS; METHODS OF SOLUTION

Suppose that one has some means of evaluating $\mathrm{Im}\, h_\alpha^J(s)$ along the negative cut. Then the dispersion relation for this partial wave becomes actually an integral equation since on the positive cut $\mathrm{Im}\, h_\alpha^J(s)$ is given by unitarity. A method to solve this equation was developed by Chew and Mandelstam. Let us consider the singlet amplitudes and write:

$$h(\nu) = \frac{N(\nu)}{D(\nu)}$$

where $N(\nu)$ is analytic except for the negative cut and $D(\nu)$ is analytic except for the positive cut. One must also specify the behavior of these two functions at infinity. This question is rather hard to settle and leads as is well known to the Castillejo-Dalitz-Dyson ambiguities [10]. Since $h(\nu)$ goes to a constant at infinity, one must allow for at least one subtraction in the representations for N and D. If one makes more than one subtraction it is found that the new arbitrary parameters thus introduced can not be determined, even in principle. Since we believe that the physical solution has no undetermined parameters besides the coupling constant and masses, we disregard those other possibilities.

Following the usual procedure one converts the singular equation into a Fredholm equation for D:

$$D(-\nu) = 1 + h(0)j(\nu) + \nu \int_{\frac{\mu^2}{4}}^{\infty} \alpha(-\nu')D(-\nu')K(\nu,\nu')\frac{d\nu'}{\nu'}$$

where

$$j(\nu) = \frac{1}{\pi}\left(\frac{\nu}{\nu-m^2}\right)^{\frac{1}{2}} \ln \frac{1+\sqrt{\nu/(\nu-m^2)}}{\sqrt{\nu/(\nu-m^2)}-1}$$

and

$$K(\nu,\nu') = \frac{j(\nu')-j(\nu)}{\nu'-\nu}$$

Here $\nu = -p^2$.

The s-wave scattering length which is related to $h(0)$ can in principle be determined because at $E^2 = 0$ the following identity must be satisfied:

$$f_1 - f_3 - zf_4 = 0$$

which can be put in terms of partial waves. In practice, however, this is hard to obtain.

We have tried to find a solution which is satisfactory in the low energy physical region. We have completely neglected singularities around $E^2 = 0$. Therefore our solution is not expected to be valid near that point. This practical difficulty in determining the scattering length reflects the fact that short range forces probably contribute appreciably to the scattering lengths and we are unable to take them into account.

This method can be extended to the triplet amplitudes. Following Bjorken, one defines a matrix:

$$h(\nu) = \begin{pmatrix} h_{11} & h_{12} \\ h_{12} & h_{22} \end{pmatrix}$$

and writes $h(\nu) = N(\nu)D(\nu)^{-1}$ and everything that was written before applies here, but now N and D are matrices and we must only be careful to preserve the correct order of products.

In the states of angular momentum $J = 1$ and isospin $I = 0$, the triplet amplitudes have a pole corresponding to the deuteron bound state. This pole might have been introduced explicitly as a singularity of N but we prefer to think that it will appear as a singularity of D^{-1}, that is, as a zero of det D after the D equations are solved. Exploiting the symmetric character of the kernel $K(\nu, \nu')$, we have obtained a variational solution of the integral equation which applies to the single channel as well as to the many channel case. For s-waves we have a variational principle for the derivative of the amplitude. For partial waves other than s-waves we have a stronger variational principle for the amplitudes themselves due to the fact that we know they vanish at the origin.

6. RANGE OF VALIDITY OF THIS PROGRAM

Summing up the results of this investigation, one can say the following: one cannot expect to obtain in practice the s-wave scattering lengths and therefore the binding energy of the deuteron, which is essentially given by the triplet scattering length, but we hope to be able to calculate with reasonable accuracy the deuteron residue, hence to determine the d- to s-state ratio and the effective ranges. The energy dependence of the phase shifts can be studied. One can estimate the range of energies within which the one-pion and two-pion exchanges are dominant. This will happen for energies such that the momentum transfer does not greatly exceed 3μ, the threshold energy for the channel $N+\bar{N}—3\pi$. Allowing for a maximum momentum transfer of 4μ we would get a range of validity for the low angular momenta up to 170 MeV in the lab system. For high angular momentum one expects that these approximations will still be valid for much higher energies.

7. NUCLEON-ANTINUCLEON SCATTERING

Finally, I want to make some short comments on the $N\bar{N}$ problem.

First, some questions of principle. Here one probably has to include the contribution of the deuteron bound state from the beginning since it gives rise to a cut in the partial wave amplitudes. On the other hand, one might expect, by analogy with the appearance of the deuteron in the NN problem, that the one pion state should appear as a pole in the $J=0$, $I=1$ amplitude and not be included at the outset. Such a possibility has meaning only if there is some other source of NN and $N\bar{N}$ forces, for example a four-fermion interaction, or some other boson field. The way in which the one-pion exchange cut appears has been discussed by Blankenbecler in an earlier talk at this Conference [11].

In practice, the nucleon-antinucleon problem offers enormous difficulties. The right hand cut extends into an unphysical region down to $s=4\mu^2$. To obtain a solution valid in the physical region $s>4m^2$ one should take into account the contributions of many-pion intermediate states. The prospects for a solution of the nucleon-antinucleon problem along these lines are therefore not very promising.

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