QCD on-shell recurrence relations and the space-cone gauge

Diana Vaman and York-Peng Yao

Michigan Center for Theoretical Physics
Randall Laboratory of Physics, The University of Michigan
Ann Arbor, MI 48109-1120
email dvaman, yyao@umich.edu

Abstract

We first give a field theoretical derivation of the tree level on-shell gluon (BCFW) recursion relations, by reassembling the tree level gluon Feynman diagrams in a convenient gauge, space-cone. The significance of these recurrence relations is that they allow obtaining multi-gluon tree level amplitudes from the previously computed lower n-point functions. Our proof of the BCFW recursion relations hinges on an algebraic identity in momentum space which is the Fourier transform of Veltman's largest time equation. Then we show that the use of the space-cone gauge is instrumental in selecting the right analytic continuation of the one-loop gluon amplitudes, namely that analytic continuation which leads to an extension of the BCFW recursion relations to the one-loop level.

1 Introduction

QCD calculations are notoriously tedious if one is to follow the usual Feynman-Dyson expansion in some commonly used, such as Feynman, gauge. Over the past few years, great strides have been made to simplify such endeavors. The results for the complete amplitudes at the tree or one- loop level can be quite compact.

Following Witten's proposal for a description of perturbative Yang-Mills gauge theory as a string theory on twistor space [1], and subsequent proposal for an alternative to the usual Feynman diagrams in terms of the so-called maximally helicity violating (MHV) vertices [2], a new set of methods was available for the computation of QCD amplitudes. The latest advance in the form of recursion relations [3, 4], in conjunction with the attendant rules for their construction, is particularly appealing. It is quite obvious from the flavor of such an approach that it bears on the cutting rules in field theory. In fact, some work at the one-loop level under the heading of cut-constructibility clearly points to the same origin [5].

As is well-known, unitarity of the S-matrix and the feasibility of an ordering of a sequence of space time points are intimately related. Indeed, the ordering need not be with respect to time, as is conventionally done. All that is essential in a perturbation series is that one must be able to separate the positive frequency and the negative frequency components in a propagator according to the signature of a certain linear combination of components Δx of the four vector between the two space-time points. For our purpose, a component $\eta \cdot \Delta x$ of the light-cone variables will be a convenient start, where η is a light-like vector. We shall rely on the existence of tubes of analyticity to continue such variables into the space cone, in order to incorporate a gauge condition for QCD. The resulting ordering is the equivalent of the largest time equations. The outcome, for QCD in particular, is that one factorizes a physical amplitude into products of physical amplitudes, with some momenta shifted but still on-shell. This is the content of the BCFW recursion relations [3, 4]:

$$\mathcal{A}(P, \{P_i\}, Q, \{Q_j\}) = \sum_{i,j} \mathcal{A}_L(\hat{P}, \{P_i\}, K) \frac{1}{(P + \sum_i P_i)^2} \mathcal{A}_R(K, \hat{Q}, \{Q_j\}) \,,$$

where $\mathcal{A}_L, \mathcal{A}_R$ are lower n-point functions obtained by isolating two reference gluons with shifted momenta, $\hat{P} = P - z\eta$, $\hat{Q} = Q + z\eta$ with $\eta^2 = \eta \cdot P = \eta \cdot Q = 0$, on the two sides of the cut. The shifting is necessary in order to preserve energy-momentum conservation. We would like to take this opportunity to point out that in so far as factorization is concerned, the masses of the internal propagators have no bearing. However, the demand that the shifted momenta, which will be called reference momenta, should be on-shell will force these external momenta to be light-like.

We now turn to the important step of gauge fixing. In order to facilitate a natural cancellation of terms at every level of a QCD calculation, the gauge that is most convenient for us is the space-cone gauge [6]. A crucial advantage of this gauge is that when we shift the momenta to obtain recursion relations, the dependence on momenta of the vertices will not be affected. Thereupon the factorization of the amplitudes is the same as that in a scalar theory. It is this special attribute which makes the program manageable.

The BCFW recurrence relations were extended to on-shell one-loop level recurrence relations by Bern, Dixon and Kosower [7]. The BCFW proof relied on a certain analytic continuation of the on-shell amplitude in the complex plane, followed by the evaluation of the integral $\oint A(z)dz/z$. The latter step localizes the tree level amplitude onto the residues $ResA(z)$. Since the poles of the tree level amplitudes are simple, and since at the tree level the analytic continuation of BCFW is such that $A(z) \to 0$ as $z \to \infty$, it follows that $ResA(z)$ correspond to setting in sequential order various internal lines on-shell, thus leading to the on-shell recurrence relations. The one-loop story proved to be more complicated. Since the loop amplitudes have cuts, the choice of the integration contour must be such that it is a deformation of the contour at infinity into a contour which hugs closely each cut. Therefore, only the rational parts of the one loop multi-gluon amplitudes will be determined recursively. Another complication comes from the choice of analytic continuation. In certain one-loop amplitudes, the BCFW analytic continuation prescription generates a boundary term, which is a consequence of having $A(z) \to$ finite as $z \to \infty$. However the problem of selecting the right analytic continuation prescription (without knowing the amplitude beforehand) is greatly simplified by the use of the space-cone gauge. We simply require that the analytic continuation, which is essentially a shift of the gluon external momenta into the complex plane, is such that the space-cone vertices are left unchanged. Since the internal propagators necessarily acquire a z-dependence, it is not difficult to see that the boundary term is eliminated.

2 On-shell tree level recursion relations from Feynman diagrams

According to the spinor helicity formalism, null vectors can be decomposed into a product of two commuting spinors (twistors): $P^{\alpha\dot{\beta}} = p^\alpha p^{\dot{\beta}} \equiv |p\rangle\,[p|$. Moreover, we can use the twistors to define a basis on the space of four-vectors

$$P = p^+\,|+\rangle[+| + p^-\,|-\rangle[-| + p\,|-\rangle[+| + \bar{p}\,|+\rangle[-|\,. \tag{1}$$

As shown by Chalmers and Siegel, [6], in the spinor helicity formalism, a powerful simplification is achieved in the Feynman diagramatics by choosing the space-cone gauge: $a = 0$, followed by the elimination of the "auxiliary" component $\bar{a}$ from its equation of motion. The gauge fixed Lagrangian has now only two scalar degrees of freedom

$$\mathcal{L} = Tr\left[\frac{1}{2}a^+\Box a^- - i\left(\frac{\partial^-}{\partial}a^+\right)[a^+,\partial a^-] - i\left(\frac{\partial^+}{\partial}a^-\right)[a^-,\partial a^+] + [a^+,\partial a^-]\frac{1}{\partial^2}[a^-,\partial a^+]\right]. \tag{2}$$

Choosing the space-cone gauge amounts to selecting two of the external momenta to be the reference null vectors for defining a twistor basis: $|+\rangle[+|, |-\rangle[-|$, such that the space-cone gauge fixing is equivalent to $N \cdot A = 0$, where the null vector N is equal to $|+\rangle[-|$.

2.1 The causality ("largest time") equations

As stated in the Introduction, the recursion relations are rooted in the largest time equation. To this end, we briefly revisit here the causality equations as derived by Veltman[8], but appropriately rewriting them in a light-cone frame.

First, we introduce the following set of rules:

-duplicate the Feynman diagram 2^N times, for N vertices, by adding circles around vertices in all possible ways;

-each vertex can be circled or not; a circled vertex will bring a factor of i, and an uncircled vertex will bring a factor of $(-i)$;

-the propagator between two uncircled vertices is $\Delta(x-y)$, while the propagator between two circled vertices is the complex conjugate $\Delta^*(x-y)$;

-the propagator between a circled x_k and an uncircled x_l is $\Delta^+(x_k - x_l)$, while the propagator between an uncircled x_k and a circled x_l is $\Delta^-(x_k - x_l)$.

Clearly, the uncircled Feynman diagram is the usual one, while the fully circled diagram corresponds to its complex conjugate.

The largest time equation states that the sum of all 2^N circled Feynman diagram vanishes:

$$F(x_i) + F^*(x_i) + \mathbf{F}(x_i) = 0\,, \tag{3}$$

where $F(x_i)$ stands for the usual Feynman diagram, $F^*(x_i)$ is its complex conjugate, and $\mathbf{F}(x_i)$ is the sum of $2^N - 2$ diagrams in which at least one vertex is circled and at least one is uncircled.

Other causality equations can be obtained by singling out 2 vertices, x_k and x_l:

$$F(x_i) = -\mathbf{F}(k,l,x_i) - \theta((x_l - x_k)^+)\mathbf{F}(k,\mathbf{l},x_i) - \theta((x_k - x_l)^+)\mathbf{F}(\mathbf{k},l,x_i))\,, \tag{4}$$

where $\mathbf{F}(k,l,x_i)$ is the sum of all diagrams with neither k,l circled, but at least one other vertex circled, $\mathbf{F}(k,\mathbf{l},x_i)$ is the sum of all amplitudes with k uncircled , but l circled and finally, $\mathbf{F}(\mathbf{k},l,x_i))$ has k circled and l uncircled.

The η-shift of two of the external momenta, required in the on-shell recurrence relations, is a consequence of the momentum inflow associated with the step-functions which arise in the largest time equation

$$\theta((x_l - x_k)^+) = \int \frac{dz}{2\pi i(z - i\epsilon)} e^{iz\eta\cdot(x_l - x_k)}\,. \tag{5}$$

2.2 Reassembling Feynman diagrams into BCFW recursion relations

A crucial observation is that by performing the BCFW shifts of the external momenta of the two space-cone gauge reference gluons, the momentum dependence of the vertices in any Feynman diagram is left unchanged. Then, after factoring out the vertices, the Feynman diagramatic proof of the tree level on-shell recurrence relations is based on an algrabraic identitiy involving the shifted propagators.

To exploit the full generality of the problem, we derive the identity satisfied by the momentum space scalar propagators working under the assumption that we deal with massive propagators, with arbitrary masses.

We single out two external momenta which do not land on the same vertex as reference vectors and call them P_a and P_b. For a tree graph, there is a unique path through some of the internal lines which connects P_a to P_b. The factorization procedure is to cut these Q_i successively by shifting them by $z\eta$. The on-shell conditions $\hat{Q}_i^2 + m_i^2 = 0,\;\; \hat{Q}_i \equiv q_i - z\eta$ will give us a set of solutions, points in the complex plane, namely $z_i = \frac{Q_i^2+m_i^2}{2\eta\cdot Q_i}$, More precisely stated, the factorization amounts to splicing the graph into a sum of products of two on-shell graphs with shifted momenta $\{P_a - z_i\eta, \ldots, \hat{Q}_i\}$ and $\{-\hat{Q}_i, \cdots, P_b + z_i\eta\}$, where $\cdots$ stand for the other momenta in the left graph segment and similarly for those in the right graph segment, with the propagator $\frac{1}{Q_i^2+m_i^2}$ as the partition. Imposing the condition that the shifted momenta remain on-shell requires that $P_a\cdot\eta = P_b\cdot\eta = 0$, in other words, P_a, P_b must be null.

The identity which we want to establish is

$$\begin{aligned}
\frac{1}{Q_1^2+m_1^2}\frac{1}{Q_2^2+m_2^2}\cdots\frac{1}{Q_{n-1}^2+m_{n-1}^2} \;=\; & \frac{1}{Q_1^2+m_1^2}\frac{1}{(Q_2-z_1\eta)^2+m_2^2}\cdots\frac{1}{(Q_{n-1}-z_1\eta)^2+m_{n-1}^2} \\
& +\frac{1}{(Q_1-z_2\eta)^2+m_1^2}\frac{1}{Q_2^2+m_2^2}\cdots\frac{1}{(Q_{n-1}-z_2\eta)^2+m_{n-1}^2}+\ldots \\
& +\frac{1}{(Q_1-z_{n-1}\eta)^2+m_1^2}\cdots\frac{1}{(Q_{n-2}-z_{n-1}\eta)^2+m_{n-2}^2}\frac{1}{Q_{n-1}^2+m_{n-1}^2},
\end{aligned} \tag{6}$$

and its proof relies on the partial fractioning formula $\oint \frac{dz}{z(z-z_1)(z-z_2)\ldots(z-z_n)} = 0$. Next we notice that eqn. (6) (with massless propagators) is precisely the identity needed to reassemble a generic tree level gluon Feynman diagram into lower on-shell amplitudes.

The connection with the largest time equation comes through the observation that by Fourier transforming (4), we shift the momenta of two gluons landing on the vertices x_k and x_l, as a consequence of the step functions in (4). The graph splicing is induced by the $\Delta^{\pm}$ propagators.

3 On-shell loop recurrence for all same helicity gluons

We begin by making the observation that in the space-cone gauge a one loop same helicity gluon amplitude is built only out of 3-point vertices. To be specific, let us consider 1-loop amplitudes with positive helicity external gluons. Then all the vertices will be trivalent: $(++-)$.

First, we notice that we cannot specify the space-cone vector $\eta = |+\rangle[-|$ completely in terms of external gluon momenta. The reason is that the external (on-shell) gluons have all the same helicity. We do have an alternative, though: we can select three external gluons and shift their momenta according to

$$\begin{aligned}
P_1 = |1\rangle[1| \to \hat{P}_1 = |\hat{1}\rangle[1| = (|1\rangle + z[23]|+\rangle)[1| \\
P_2 = |2\rangle[2| \to \hat{P}_2 = |\hat{2}\rangle[2| = (|2\rangle + z[31]|+\rangle)[2| \\
P_3 = |3\rangle[3| \to \hat{P}_3 = |\hat{3}\rangle[3| = (|3\rangle + z[12]|+\rangle)[3|
\end{aligned} \tag{7}$$

The sum of the three momenta does not change under the holomorphic shift, beacuse of the Schouten identity: $[23][1|+[31][2|+[12][3|=0$. Incidentally, we notice that this is the same holomorphic shift which Risager [13] used to prove the CSW rules starting from the BCFW recursion relations. It is also clear, by inspection, that the shifted momenta remain on shell. Lastly, since the vertices for amplitudes with all external gluons of the same helicity are all of the type $(++-)$, then these shifts will not change the vertices, nor the external line factors.

Armed with this observation we can proceed to derive the on-shell recurrence relation following BCFW and compute $\oint \frac{dz}{z} A(z)$, where $A(z)$ is the amplitude evaluated with the shifted momenta. We know that the all plus one-loop amplitude is a rational function, since it vanishes at the tree level. We have also seen that the vertices do not change with the shift (7), but the internal propagators do, with at least one of them being affected. Then we infer that $A(z) \sim 1/z^n$, with $n \geq 1$ as $z \to \infty$. Therefore, $\oint \frac{dz}{z} A(z) = 0$ when the contour is taken at infinity. In other words, there is no boundary term to contend with. This means that we can factorize the all plus amplitude into lower n-point functions associated with the residues of $A(z)$:

$$\begin{aligned} A_n^{(1)}(\mathbf{P_1}, \mathbf{P_2}, \mathbf{P_3}, P_4, \ldots P_n) &= A_3^{(0)}(\widehat{\mathbf{P}}_\mathbf{1}, \widehat{\mathbf{P}}_\mathbf{2}, K) \frac{1}{2\mathbf{P_1} \cdot \mathbf{P_2}} A_{n-1}^{(1)}(K, \widehat{\mathbf{P}}_\mathbf{3}, P_4 \ldots P_n) \\ &+ A_3^{(0)}(\widehat{\mathbf{P}}_\mathbf{2}, \widehat{\mathbf{P}}_\mathbf{3}, K) \frac{1}{2\mathbf{P_2} \cdot \mathbf{P_3}} A_{n-1}^{(1)}(K, P_4, \ldots P_n \widehat{\mathbf{P}}_\mathbf{1}) \\ &+ A_3^{(0)}(\widehat{\mathbf{P}}_\mathbf{3}, P_4, K) \frac{1}{2\mathbf{P_3} \cdot P_4} A_{n-1}^{(1)}(K, P_5 \ldots P_n, \widehat{\mathbf{P}}_\mathbf{1}, \widehat{\mathbf{P}}_\mathbf{2}) \\ &+ A_3^{(0)}(P_n, \widehat{\mathbf{P}}_\mathbf{1}, K) \frac{1}{2P_n \cdot \mathbf{P_1}} A_{n-1}^{(1)}(K, \widehat{\mathbf{P}}_\mathbf{2}, \widehat{\mathbf{P}}_\mathbf{3}, P_4 \ldots P_{n-1}) \end{aligned} \tag{8}$$

where the superscripts $0, 1$ indicate whether the one-shell amplitude is tree or one-loop level, the boldface letters denote the triplet of external gluon momenta, and the hats denote the shifts made such that the line cut is put on-shell. In each of the four terms K is placed on-shell by the appropriate z shift.

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