

Gravitational contraction and curvature singularities in $f(R)$ gravity

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Abstract. The discovery of the accelerated expansion of the Universe has had a vast resonance on a number of physical disciplines. In recent years a few viable modified gravity models have been proposed, which naturally lead to a late-time de Sitter stage while reducing to General Relativity in the early Universe. We study two of these models during the contraction of a homogeneous cloud of pressureless dust. We show how the increasing energy/mass density may lead to a curvature singularity and derive the typical timescales for its development.

1. Introduction

The biggest mystery in modern physics is perhaps the nature of Dark Energy, which is responsible for the accelerated expansion of the Universe [1] and accounts for about 75% of its total content. This unknown form of energy seems to have an equation of state close to $w = -1$, which yields “anti-gravity” and makes it phenomenologically analogous to a cosmological Λ -term or to a vacuum energy component (for a review on Dark Energy, see e.g. [2] and references therein).

From a theoretical standpoint, both possibilities are rather natural, since a Λ -term is well within the framework of standard General Relativity (GR) and a zero-point energy is to be expected in the Standard Model as well as in any of its extensions. There remains, however, the terrible discrepancy between the expected value of the vacuum energy density, which can quite easily reach $\rho_{vac} \sim m_{Pl}^4$, and its observed value, $\rho_{vac} \sim 10^{-123} m_{Pl}^4$. This fine-tuning problem is maybe the best motivation to study alternative mechanisms to generate the observed accelerated cosmological expansion.

Modified $f(R)$ gravity theories were first introduced as ultraviolet corrections to GR, originated by loop quantum corrections to matter fields in curved spacetime, as discussed in the seminal paper [3] and in following works (e.g. [4]). These terms are particularly relevant in the early Universe, and may have driven the primordial inflationary stage.

The first infrared $f(R)$ models proposed relied on negative powers of R , which become dominant at small curvatures, in the gravitational action to generate the *late-time* de Sitter stage [5], but suffered from severe instabilities in the presence of matter [6, 7, 8]. In the last few years, the limits for the cosmological viability of $f(R)$ models have been widely studied [9], and recently at least three models evading all such tests have been proposed [10, 11, 12].



Studying these models in an astronomical/astrophysical situation has evidenced the possibility of non-trivial modifications of the standard theory. Investigating both the stability of static, spherically symmetric, dense structures [13, 14] and the contraction of large, rarefied collisionless dust clouds [15, 16], it was found that in these theories it is quite possible to develop a curvature singularity. The additional scalar, massive degree of freedom (dubbed *scalaron*) of $f(R)$ gravity theories moves in a time dependent potential, and the singular point corresponding to $|R| \rightarrow \infty$ can be reached with a finite energy. Due to the dependence of the potential on the external energy/mass density, the situation worsens as the contraction proceeds. The same mechanism is responsible for the past, finite-time singularity found in a cosmological scenario [17]. The structure of the singularity is peculiar, because although $|R| \rightarrow \infty$, the energy density and the metric remain finite; nevertheless, the appearance of such singularity remains hardly desirable in a valid theory. Furthermore, it seems likely that similar features may apply to an extended class of modified gravitational theories, not only $f(R)$ gravity [18].

In this paper, which extends the work of [15, 16], we will focus on the two models of [10, 11], and discuss their behaviour during the contraction of a spherically symmetric, nearly homogeneous cloud of pressureless dust. The possibility of forming a singularity is confirmed, and we propose an improved method to predict the behaviour of curvature while approaching the singularity. These results could be used to constrain models after a careful comparison with observational data, and also to be the starting point for an investigation of further effects of $f(R)$ gravity, e.g. gravitational particle production and the related possibility of detecting cosmic rays [19, 20], which may in principle provide additional signatures of physics beyond GR.

2. Curvature dynamics in $f(R)$ gravity

2.1. Action and field equations

We start with the general gravitational action¹

$$A_{grav} = -\frac{m_{Pl}^2}{16\pi} \int d^4x \sqrt{-g} f(R) \equiv -\frac{m_{Pl}^2}{16\pi} \int d^4x \sqrt{-g} [R + F(R)] . \quad (1)$$

Varying (1) with respect to the metric $g_{\mu\nu}$ yields the modified Einstein equations

$$f'(R)R_{\mu\nu} - \frac{1}{2}f(R)g_{\mu\nu} + (g_{\mu\nu}\mathcal{D}^2 - \mathcal{D}_\mu\mathcal{D}_\nu) f'(R) = T_{\mu\nu} , \quad (2)$$

where \mathcal{D} denotes covariant derivative, a prime denotes derivative with respect to R , and

$$T_{\mu\nu} \equiv \frac{8\pi}{m_{Pl}^2} \frac{2}{\sqrt{-g}} \frac{\delta A_m}{\delta g^{\mu\nu}} .$$

Note that the matter action A_m was not explicitly indicated in Eq. 1. The trace of (2) reads

$$3\mathcal{D}^2 F' + RF' - 2F - (R + T) = 0 . \quad (3)$$

Although some results will be rather general, we will specifically consider the two models

$$F_{HS}(R) = -\frac{\lambda R_c}{1 + (R/R_c)^{-2n}} , \quad [10] \quad (4a)$$

$$F_S(R) = \lambda R_c \left[\left(1 + \frac{R^2}{R_c^2} \right)^{-n} - 1 \right] . \quad [11] \quad (4b)$$

¹ We use natural units $c = \hbar = k = 1$, and the Planck mass is defined as $m_{Pl}^2 = G_N^{-1}$. The metric has signature $(+ - - -)$, and we use the conventions $\Gamma_{\mu\nu}^\alpha = \frac{1}{2}g^{\alpha\beta}(\partial_\mu g_{\beta\nu} + \dots)$, $R_{\mu\beta\nu}^\alpha = \partial_\beta \Gamma_{\mu\nu}^\alpha + \dots$, $R_{\mu\nu} = R_{\mu\alpha\nu}^\alpha$, $R = R_\mu^\mu$. With these conventions, $R < 0$ for matter- and vacuum-dominated Universes.

For $\lambda \sim 1$, in both models R_c is of the order of the present cosmological constant; since $f(0) = 0$, the cosmological constant “disappears” in vacuum (using the words of [11]). The late-time constant curvature solutions are roots of the algebraic equation

$$\bar{R} - \bar{R} F'(\bar{R}) + 2F(\bar{R}) = 0, \quad (5)$$

where the matching between theory and observational must be given by

$$\Lambda = \frac{8\pi \Omega_\Lambda \varrho_{vac}}{m_{Pl}^2} = -\frac{\bar{R}}{4}, \quad (6)$$

Ignoring further subtleties, for the sake of definiteness and simplicity, we will assume that λ is of order unity and that $R_c \sim \Lambda_0$; for more information on the cosmological behaviour of the models, we refer the reader to the original papers [10, 11].

2.2. Contracting systems

Let us consider a spherically symmetric cloud of pressureless dust, hence having $T = 8\pi \varrho/m_{Pl}^2$. We make the additional assumption of homogeneity, in the sense that at each instant of time the time derivatives of T and R dominate over the space derivatives; this approach, albeit somewhat rudimentary, is intuitively clear and theoretically justifiable [15].

For definiteness, we assume that such cloud contracts on a typical timescale t_{contr} following:

$$T(t) = T_0 \left(1 + \frac{t}{t_{contr}}\right) = \frac{8\pi \varrho_0}{m_{Pl}^2} \left(1 + \frac{t}{t_{contr}}\right). \quad (7)$$

We must stress that this evolution is not supposed to be physically accurate and is only necessary to perform explicit calculations. Nonetheless, results obtained applying more sophisticated dynamical analyses should be qualitatively in agreement with our results, provided that the contraction law is not terribly different from (7) and in any case until $t \sim t_{contr}$. For a recent dynamical analysis of gravitational contraction in $f(R)$ gravity, see e.g. [21] and references therein.

We also consider the limit $|R/R_c| \gg 1$, which is usually appropriate since the typical densities in pre-stellar and pre-galactic clouds are much larger than the present critical energy density $\varrho_{vac} \approx 10^{-29} \text{ g cm}^{-3}$, in which $|F/R| \ll 1$ and $|F'| \ll 1$. Moreover, for typical values of energy density we are in low-gravity regime, that is (denoting with M the total mass of the object and with r its radius) $GM/r \ll 1$, so the spacetime is nearly Minkowski. This allows us to rewrite (3) as

$$\frac{\partial^2 F'}{\partial t^2} + \frac{1}{3}(R + T) \simeq 0. \quad (8)$$

2.3. Trace equation and scalaron potential

Defining

$$\xi \equiv -3F', \quad (9)$$

we write (8) as an oscillator equation for the field ξ :

$$\ddot{\xi} + R + T = 0 \quad \Leftrightarrow \quad \ddot{\xi} + \frac{\partial U}{\partial \xi} = 0, \quad (10)$$

with U being the time-dependent potential

$$U(\xi, t) = T(t) \xi + \int^\xi R(\xi') d\xi'. \quad (11)$$

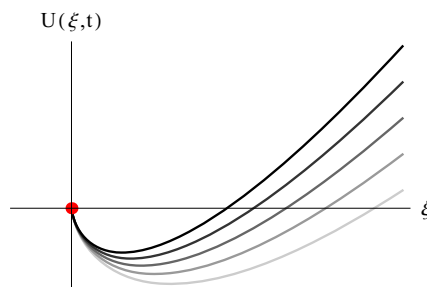


Figure 1. Qualitative shape of potentials for models (4), assuming $R/R_c \gg 1$. For both models $\xi = \xi_{sing} = 0$ (red dot) corresponds to the singular point $|R| \rightarrow \infty$. The typical time evolution is also shown, from light grey (initial times) to black (later times). Note that the bottom of the potential approaches ξ_{sing} .

The field ξ , sometimes dubbed *scalaron*, is the additional scalar, massive degree of freedom originated by the introduction of $F(R)$ in the action, which for both models considered (in the limit $|R/R_c| \gg 1$) is roughly equal to

$$\xi(R) \simeq 6n\lambda \left(\frac{R_c}{R} \right)^{2n+1}. \quad (12)$$

Evidently, the limit $|R| \rightarrow \infty$ corresponds to the finite value $\xi \rightarrow \xi_{sing} = 0$.

In GR this field vanishes identically, and R and T are related simply algebraically; in $f(R)$ gravity, instead, the relation between geometry and matter is differential, and in fact (10) describes oscillations around the GR solution

$$R + T = 0, \quad (13)$$

with a frequency given approximately by the square root of the second derivative of the potential with respect to the field, that is

$$\omega^2 \simeq \frac{\partial^2 U}{\partial \xi^2} = \frac{\partial R}{\partial \xi} \simeq \frac{1}{\partial \xi / \partial R} \Big|_{R+T=0} \simeq -\frac{R_c}{6n\lambda(2n+1)} \left(-\frac{T}{R_c} \right)^{2n+2}. \quad (14)$$

In this equation, we have assumed that Note that since $R_c < 0$, the square of the frequency is always positive; this prevents the instabilities of [7].

As we have mentioned, the potential U contains an explicit time-dependence, contained in the term proportional to $T(t)$, and in fact it will change with time or to be more precise with varying energy/mass density; beside the fact that its minimum moves following $R + T = 0$, the value of U in its minimum and the overall shape change, as well. Using the explicit expression (12), we find up to an additive constant, which can be set equal to zero without loss of generality:

$$U(\xi, t) \simeq T(t) \xi + 3\lambda(2n+1)R_c \left(\frac{\xi}{6n\lambda} \right)^{\frac{2n}{2n+1}}. \quad (15)$$

The qualitative shape of this potential is depicted in Fig. 1. Remarkably, we also notice that $U(\xi_{sing} = 0)$ does not vary with time.

We can define the ‘‘average’’ value of the field

$$\xi_a(t) \equiv \xi(R + T = 0), \quad (16)$$

in order to quantify the displacement of solutions from the GR expectation. Note that ξ_a is not necessarily a solution of (10) with $R+T=0$, so it is not meant to be an actual physical solution, it simply corresponds to the position of the bottom of the potential. Since T is increasing in contracting systems, Eq. (12) indicates that ξ_a decreases with time, approaching the singular point $\xi_{sing}=0$. This will have important consequences, as we will see.

2.4. Energy conservation

From (10) we can also derive the exact “energy” conservation equation

$$\frac{1}{2}\dot{\xi}^2 + U(\xi, t) - \int^t dt' \frac{\partial T}{\partial t'} \xi(t') = \text{const.} \quad (17)$$

The last non-local, integral term on the l.h.s. is due to the explicit time-dependence of the external energy density; in the simple case (7), $\partial T/\partial t = T_0/t_{contr}$ and this term is proportional to the integral of ξ over time. In contracting systems, its overall contribution is to increase the “canonical” total energy, that is the sum of kinetic and potential energies².

We study the problem in two different regimes: the “adiabatic” regime and the “slow-roll” regime. Intuitively, the separation depends on how much the potential $U(\xi, t)$ varies during one oscillation of the scalaron, which occurs on a timescale

$$t_{osc} \simeq \frac{2\pi}{\omega}, \quad (18)$$

where ω is given by (14). On the other hand, the typical timescale for substantial variations in the potential is t_{contr} . Therefore, we expect the boundary between the two regimes to be somewhere near the equality

$$\frac{t_{contr}}{t_{osc}} = \frac{\omega t_{contr}}{2\pi} = 1. \quad (19)$$

3. Adiabatic regime

Let us start assuming that t_{contr} is very large compared to ω^{-1} , which means that U does not change noticeably over many scalaron oscillations. This also translates into the fact that the variation of the integral in (17) during a few oscillations is very small. Therefore, the oscillations of ξ are approximately adiabatic, in the sense that at each oscillation ξ moves between two values ξ_{min} and ξ_{max} having, ignoring the time dependence of U ,

$$U(\xi_{min}) \simeq U(\xi_{max}). \quad (20)$$

In this situation, we expand ξ around ξ_a as

$$\xi(t) = \xi_a(t) + \xi_1(t) \quad (21a)$$

$$\equiv \xi_a(t) + \alpha(t) \sin \Phi(t), \quad (21b)$$

where

$$\Phi(t) \simeq \int^t dt' \omega. \quad (22)$$

The function α , which is the amplitude of the scalaron oscillation around its average value, is also assumed to be relatively slowly-varying, that is

$$\frac{\dot{\alpha}}{\alpha} \ll \omega. \quad (23)$$

² Note that this is true for $\xi > 0$, whereas for $\xi < 0$ it would give the opposite behaviour. However, it has been shown that the condition $F' < 0$, corresponding to $\xi > 0$, is crucial for the correct behaviour of modified gravity models at (relatively) low curvatures [17]. Indeed, this is clearly the case in the considered models.

The singularity point $\xi = \xi_{sing}$ is reached if at any moment

$$|\xi_a(t) - \xi_{sing}| \leq |\alpha|, \quad (24)$$

which in our case is equivalent, taking for simplicity and without loss of generality $\alpha > 0$, to

$$\xi_a(t) \leq \alpha(t). \quad (25)$$

3.1. Linear approximation

At first order in ξ_1 , Eq. (10) reads

$$\ddot{\xi}_1 + \omega^2 \xi_1 \simeq -\ddot{\xi}_a. \quad (26)$$

The term on r.h.s. only depends on the external energy density, and acts as a source term for the oscillations of the ‘‘perturbation’’ ξ_1 . In first approximation, however, it can be neglected since it is approximately equal to ξ_0/t_{contr}^2 , and we have assumed that $\omega t_{contr} \gg 1$. For analogous reasons, we can neglect $\ddot{\alpha}$. Using expansion (21b) then yields

$$\frac{\dot{\omega}}{\omega} \simeq -2\frac{\dot{\alpha}}{\alpha} \Rightarrow \alpha(t) \simeq \alpha_0 \sqrt{\frac{\omega_0}{\omega(t)}}. \quad (27)$$

This can be considered a rather general result, and the specific $F(R)$ model will determine the particular evolution of the oscillations. The value α_0 in Eq. (27) is related to the initial conditions we impose, that is to the initial displacement from the GR behaviour. We will set the initial values of R and \dot{R} , and from those derive the initial values of ξ and $\dot{\xi}$. Thus, α_0 can be calculated differentiating Eq. (21b), which gives

$$\dot{\xi}_0(R_0, \dot{R}_0) \simeq \dot{\xi}_{a,0} + \alpha_0 \omega_0 \Rightarrow \alpha_0 \simeq \frac{\dot{\xi}_0 - \dot{\xi}_{a,0}}{\omega_0}. \quad (28)$$

This leads to the explicit solution

$$\alpha(t) \simeq \left| \dot{\xi}_0 - \dot{\xi}_{a,0} \right| [\omega_0 \omega(t)]^{-1/2}. \quad (29)$$

The modulus in the r.h.s. entails no loss of generality and ensures that $\alpha > 0$.

It is clear from Eq. (29) that if $\dot{\xi}_0 = \dot{\xi}_{a,0}$ the amplitude of oscillations should vanish at all times. This can be seen to be completely wrong, for instance numerically. The disagreement with our estimate is due to the fact that we have neglected $\ddot{\xi}_a$, and in general terms proportional to $1/t_{contr}^2$, in Eq. (26). When α is initially very small, however, those terms should be kept and the approximations used are no longer valid. Therefore, Eq. (29) is reliable when $|\dot{\xi}_0 - \dot{\xi}_{a,0}|$ is large enough, say of the order of $\dot{\xi}_{a,0}$ or slightly less. In figure 2 we show a comparison of numerical solution of (10) with our analytical result.

We stress that the linear approach must not be extended to the analysis of $R(t)$, in fact we are studying the very process of $|R| \rightarrow \infty$. Moreover, as R evolves, $|\xi_1/\xi_a|$ increases so the linear approximation will eventually cease to hold. This happens when ξ approaches $\xi_{sing} = 0$, so it is indeed near the singularity that this behaviour will be less accurate. Still, as one can easily check numerically, Eq. (29) works reasonably well at all stages of the evolution of R .

3.2. Generation of the singularity

We solve the system giving the initial conditions

$$\begin{cases} R_0 = -T_0 \\ \dot{R}_0 = f \dot{T}_0 = -f T_0/t_{contr}, \end{cases} \quad (30)$$

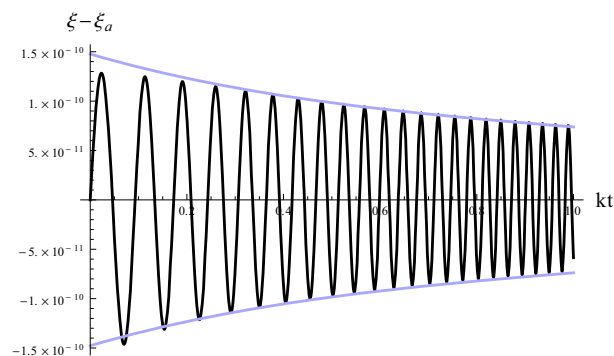


Figure 2. Comparison of the evolution of ξ with the predicted result, see equations (21) and (29). The values of parameters used are: $n = 1$, $\lambda = 1$, $R_{29} = 1$, $\varrho_{29} = 10^3$ and $t_{10} = 5 \cdot 10^{-4}$, $f = 0.5$.

where f is a free parameter quantifying the initial displacement from the GR behaviour $R + T = 0$; in particular, $f = 1$ corresponds to the situation in which R behaves initially exactly as if there were no $F(R)$ at all. Because of the considerations made at the end of the previous paragraph and noting that with these initial conditions

$$\dot{\xi}_0 = f \dot{\xi}_{a,0}, \quad (31)$$

our results will be particularly reliable for values of f not too close to unity.

Substituting ξ and ω from (12) and (14), respectively, equation (25) becomes

$$6n\lambda \left(-\frac{R_c}{T} \right)^{2n+1} \leq \frac{[6n\lambda(2n+1)]^{3/2} |1-f| (-R_c)^{3n+3/2}}{T_0^{(5n+3)/2} T^{(n+1)/2} t_{contr}}. \quad (32)$$

For convenience, we define the following dimensionless quantities:

$$\begin{aligned} R_{29} &\equiv \frac{m_{Pl}^2}{8\pi} \frac{(-R_c)}{10^{-29} \text{ g cm}^{-3}}, \\ \varrho_{29} &\equiv \frac{\varrho_0}{10^{-29} \text{ g cm}^{-3}}, \\ t_{10} &\equiv \frac{t_{contr}}{10^{10} \text{ years}}. \end{aligned} \quad (33)$$

Substituting these expressions in (32) assuming (7), we find the value of the energy/mass density T_{sing} at which the curvature singularity forms:

$$\begin{aligned} \frac{T_{sing}}{T_0} &= \left(\frac{T_0^{n+1} t_{contr}}{\sqrt{6n} |1-f| \lambda (2n+1)^3 R_c^{n+1/2}} \right)^{2/(3n+1)} \\ &\simeq \left(0.53 \frac{\varrho_{29}^{n+1} t_{10}}{\sqrt{n\lambda} |1-f| (2n+1)^3 R_{29}^{n+1/2}} \right)^{2/(3n+1)}. \end{aligned} \quad (34)$$

Typically we would have $\lambda, R_{29} \sim \mathcal{O}(1)$, so this result tells us that if the initial energy density is not too large and/or the contraction is fast enough, the overall contraction required to reach the singularity is quite realistic, with T_{sing}/T_0 possibly even only slightly larger than unity. Moreover, the larger n , the easier it is to develop the singularity with relatively small T_{sing} .

Once again, we should stress that (7) is not expected to be completely accurate, but our estimate (34) should be more or less correct even with different, more sophisticated contraction laws.

4. Slow-roll regime

Let us now relax the assumption of adiabaticity made above, which can be summarised in the statement

$$\frac{\dot{T}}{T}, \frac{\dot{\alpha}}{\alpha} \sim t_{contr}^{-1} \ll \omega. \quad (35)$$

We will instead focus on the opposite regime, that is (see Eq.19)

$$\frac{\omega t_{contr}}{2\pi} \lesssim 1, \quad (36)$$

showing that a singularity can form in this case, as well. We assume that ξ initially sits on the bottom of the potential (as in the initial conditions of Eq. 30), in $\xi = \xi_0$. Let us also denote ξ_{eq} the point corresponding to the equality

$$U(\xi_{eq}) = U(\xi_{sing}). \quad (37)$$

Clearly, ξ_{eq} can change with time, as the reader can see looking at figure 1. There, ξ_{eq} is the point other than ξ_{sing} at which the potential crosses the x axis; we can clearly see that ξ_{eq} is changing, and specifically decreasing, with time.

If there exists a time τ at which

$$\xi_{eq}(\tau) = \xi_0, \quad (38)$$

with

$$\frac{\omega \tau}{2\pi} \lesssim 1, \quad (39)$$

then $\xi(\tau)$ is, roughly speaking, still equal to $\xi_0 = \xi_{eq}(\tau)$. In the meantime the bottom of the potential has moved, so that now ξ finds itself on a slope dragging the field down towards $\xi = \xi_a$, from a height equal to the height of the singular point ξ_{sing} . In the next oscillation, therefore, ξ will most likely reach ξ_{sing} and hence the singularity.

The mechanism is the following: ξ starts at ξ_0 , and the potential changes, due to the increasing energy density, more rapidly than ξ can oscillate, so that ξ_{eq} reaches ξ_0 before ξ has had the time to move away significantly from its initial position. Eventually, on a timescale roughly equal to ω^{-1} , ξ will start moving and reach ξ_{sing} on the other side of its oscillation; this will also be the typical timescale for the formation of the singularity. Evidently, the smaller the value of $\omega \tau$, the more accurate these assumptions will be.

For the models in exam, we have $U = 0$ in $\xi = \xi_{sing}$ and

$$\xi_{eq} \simeq 6n\lambda \left(-\frac{2n+1}{2n} \frac{R_c}{T} \right)^{2n+1}, \quad (40)$$

while initially the field is in

$$\xi_0 \simeq 6n\lambda \left(-\frac{R_c}{T_0} \right)^{2n+1}. \quad (41)$$

The request that $\xi_{eq} = \xi_0$ then becomes

$$\frac{T}{T_0} = 1 + \frac{1}{2n}, \quad (42)$$

which is always quite close to unity, and even more so for increasing n . This equality occurs at a time (see Eq. 7)

$$\tau = \frac{t_{contr}}{2n}. \quad (43)$$

Note that with this result equation (36) automatically implies (39). We must substitute this value in (39) to find the minimum contraction time, or on the other hand the maximum contraction rate, allowed; otherwise, a singularity would most probably develop within the first oscillation. This results in

$$t_{contr, min} \approx 2.4 \times 10^{11} \text{ years} \frac{n \sqrt{(2n+1)\lambda n} R_{29}^{n+1/2}}{\left(\frac{2n+1}{2n} \varrho_{29}\right)^{n+1}}. \quad (44)$$

Despite seeming at first sight very large, about twenty times the age of the Universe, this number is usually much shorter. In fact, for λ and R_{29} of order unity and $\varrho_{29} \gtrsim 10^3 - 10^5$, which are typical values for densities in (pre-)galactic clouds, the minimum contraction time already shortens by several orders of magnitude. However, if $t_{contr, min}$ becomes too short, basically every observed contracting system in the Universe would fulfil condition (44); in this case, we must use the analysis of section 3.

Nonetheless, for a reasonable portion of parameter space, in particular the region with small n and not-too-large ϱ_{29} , a contracting structure would develop a singularity within the first scalaron oscillation, thus on a timescale

$$t_{osc} \simeq \frac{2\pi}{\omega(\tau)} \approx 1.2 \times 10^{11} \text{ years} \frac{\sqrt{(2n+1)\lambda n} R_{29}^{n+1/2}}{\left(\frac{2n+1}{2n} \varrho_{29}\right)^{n+1}}. \quad (45)$$

For typical values of physical and model parameters (see above), this timescale could well be shorter than the age of the Universe, hence in principle observable.

5. Conclusions and discussion

The possibility of curvature singularities in DE $f(R)$ gravity models has been confirmed and studied in a rather simple fashion. The trace of the modified Einstein equations has been rewritten, under the simplifying assumptions of homogeneity and low-gravity, as an oscillator equation for the scalaron field ξ , which moves in a potential U depending on the external energy/mass density and thus on time. In the two models considered [10, 11], the potential is finite in the point corresponding to the curvature singularity $|R| \rightarrow \infty$, that is $\xi_{sing} = 0$; the energy conservation equation associated with ξ indicates that in contracting systems the probability of reaching this singularity increases with increasing energy/mass density.

The ratio between the typical contraction time and the inverse frequency of the scalaron determines two distinct regimes. In the adiabatic regime ξ oscillates very fast compared to relevant variations of U , and we can perform a linear analysis to estimate the scalaron amplitude and frequency analytically. The singularity is expected to be reached when the amplitude of the oscillations of ξ exceeds the separation between the ‘‘average’’ value ξ_a and the singular point. Results show that this can happen for reasonable values of physical and model parameters provided that the contraction lasts long enough. In the slow-roll regime, the field ξ is practically ‘‘frozen’’ in its initial value for a time longer than the time required by the potential to ‘‘lift’’ the field up to values which make ξ_{sing} accessible on the next oscillation. Basically, the energy of the field has increased enough for the singularity to be reached, which happens on a timescale comparable to the inverse frequency of the scalaron oscillations.

In principle, the results of this article could provide simple methods to constrain and possibly rule out models [10, 11], and most likely the same technique could be applied to other models already proposed. However, two effects could on one hand hinder the development of singularities, and on the other hand provide additional methods to constrain models: ultraviolet

gravity modifications and gravitational particle production.

Ultraviolet corrections to the gravitational action start dominating at large R , and should set a limit to its growth; in turn, R would never reach the singularity (for details see e.g. [15, 17]).

Gravitational particle production, as is well known, is universal whenever curvature oscillates, and could in principle be a detectable source of high energy cosmic rays [20]. The back-reaction on curvature is a damping of its oscillations, so this damping may also prevent R from reaching infinity. This is particularly important in the adiabatic regime, where there can be very many oscillations before ξ reaches ξ_{sing} and therefore a large amount of energy could be released into SM particles. Fortunately, these cosmic rays would carry model-dependent signatures which could provide us valuable information to improve the constraints on the known models and maybe even suggest new gravitational theories.

Additional investigations involving, for instance, the High Energy Cosmic Rays “ankle” problem and the GZK cutoff, could stimulate further research on this very fascinating subject.

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