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GLOBAL SIGNATURES OF GAUGE INVARIANCE
VORTICES AND MONOPOLES*

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ABSTRACT

A comprehensive topological classification of vortices and their endpoint Dirac monopoles is formulated in gauge theories with an arbitrary compact Lie group. By way of homotopy theory, a simple analysis is presented for the global groups U(1), O(3), SU(2) then $SU(N)/Z_N$ and SU(N). Finite vortices are achieved through the complementarity of Dirac strings and the nodal lines of the Higgs fields. In general, the varieties of topologically distinct vortices or monopoles are determined solely by the connectivity of the global group, specified by a discrete Abelian fundamental group. The close connection between our work and the Wu-Yang global formulation of gauge fields is pointed out.

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I. PRELUDE

The existence of solitons in certain nonlinear field theories[1] points to a highly attractive mechanism through which strongly interacting objects emerge as collective excitations of weakly interacting fields. A most striking stability feature of solitons is an associated topological charge, a consequence solely of the continuity of the fields. This charge or kink number takes on a discrete set of values which remain invariant under any continuous deformations, particularly in the course of time evolution. Thus obeying homotopic conservation laws, these kinks[2] arise not from any symmetries of the Lagrangian but rather from the global topology of the field manifold, a structure specified by appropriate boundary conditions.

At present, essentially two types of kinks are known in (3+1) dimensional gauge theories. They are the monopoles of 'tHooft[3] and Polyakov[4] and the vortices of Nielsen and Olesen.[5] The remarkable topology of such non-Abelian vortices and its implications for a magnetic confinement scheme for quarks have been studied in specific models by Polyakov[6] in the instance of closed vortices and by Mandelstam[7] and ourselves[8] in the case of finite vortices terminating at Dirac monopoles.[9,10] As they are endowed with a topological "triality," the latter vortices could well provide the requisite "valence bonds" upon which a dual theory of mesons and baryons will be built.[11]

While general topological arguments for the existence of 'tHooft monopoles have been given in gauge theories with any compact group,[12] a corresponding comprehensive analysis for the vortices and their endpoint Dirac monopoles has yet to be performed. It is timely to bridge this gap. Our work takes up the general problem of topological classification of vortices and their Dirac monopoles in non-Abelian gauge theories with any compact Lie group G. Making minimal use of the topology of Lie groups, we show the topologically distinct types of vortices and monopoles allowed in a theory with a global group G to be solely determined by the connectedness of G. By topologically distinct, we mean that vortices of different types cannot be continuously gauge transformed into one another. These "gauge types"[10] are in a simple one to one correspondence to the elements of a finite discrete Abelian group, the fundamental group $\pi_1(G)$ associated with G.

In our analysis, we only require that axially symmetric static solutions have finite energy per unit length, the condition being the Meissner mechanism of flux confinement.[7] Since homotopy groups are not part of every physicist's tool set, we shall clearly illustrate our arguments. We proceed in a pedestrian manner from the simple specific examples of G = U(1), O(3), SU(2), then $SU(N)/Z_N$ and SU(N) to the general instance of any compact Lie group. Due to the global character of our problem, a key distinction must be made between the local versus global invariance groups of any gauge theory. Consider the case of SU(N) versus $SU(N)/Z_N$. While locally isomorphic, they differ in their global structures, one group is simple connected, the other N-fold connected. In this fact lies the essential difference reflected in the allowed topologies of vortices and monopoles in theories with these global groups. Because of continuity the global group G must always be one which acts effectively on the vectors of the physically realized representations. Alternatively, these representations must be faithful under G. Thus in a generalized Nielsen-Olesen model with a local invariance group SU(N), $G = SU(N)/Z_N$ if the Higgs fields $\Phi_i(x)$ belong to an adjoint representation of SU(N) or of any of its tensor products. In this "(N^2-1) fold way" there exists N topologically distinct types of vortices mirroring the N-fold connectedness of G. One of these corresponds to the trivial vacuum sector of no vortex at all. Equivalently, seen from the universal covering group SU(N), these (N-1) nontrivial gauge types of vortices are in a one to one correspondence with the (N-1) nontrivial elements of the center Z_N of SU(N). The details of the necessary symmetry breakdown are irrelevant to their classification provided the vortices exist. Each vortex carries a kink number which through the gauge field potential can be interpreted as a magnetic flux, albeit one defined only by modulo N. The latter qualification makes the definition of flux a gauge invariant one.

On the other hand, if the $\Phi_i(x)$ are chosen in the fundamental representation of SU(N) or any of its faithful representations, then G = SU(N) itself. Due to the simple connectedness of SU(N), there is only one type of vortex, namely the trivial sector of no vortex.

Finite vortices of any shapes with endpoint colorless Dirac monopoles are constructed by way of the complementarity of the Dirac strings and the nodal lines of the Higgs fields. In general, we find there exists a strikingly simple isomorphism between the n gauge types of vortices and their monopoles, their law of flux combination on the one hand and the elements of a discrete, Abelian fundamental group of order n, $\pi_1(G)$, and their law of combination on the other. n is the connectedness of the global group G of the gauge model. The magnetic fluxes carried by the vortices, the strengths of the endpoint Dirac monopoles are defined modulo n, the dimension of $\pi_1(G)$. For specific models, the existence and topological peculiarities of these monopoles were first discussed by Lubkin,[13] then rediscovered recently by Mandelstam,[7] Wu and Yang.[10] Indeed our paper is close in spirit and is seen as complementary to that of Wu and Yang. The connection is one between London's flux quantization and Dirac's monopole quantization, but in a non-Abelian setting.

Our work emerges as an application exercise in the topology of Lie groups. Our presentation will be compact yet self-contained. By restricting ourselves to a few essential topological notions and by starting from standard examples in an Abelian gauge frame, we appeal mainly to the logic of the reader's visual imagination. Our emphasis will be on physical concepts. To the sophisticated a companion paper[14] offers a more rigorous formal analysis by way of the topology of fibre bundles.

Our paper is organized as follows: In Section II we give an intuitive review of some necessary concepts in the topology of groups. In Section III we apply these notions to the classification of infinitely long (or closed) Nielsen-Olesen vortices in specific illustrative examples. In Section IV, with the use of Dirac strings, we construct finite, open vortices and deduce the topology of their endpoint Dirac monopoles. Finally in Section V, we generalize our conclusions to the instance of a gauge theory with an arbitrary compact Lie group. We provide the connection between our work and that of Wu and Yang. We then comment on the relevance of these vortices and monopoles to theories of magnetic confinement in particular.

II. TOPOLOGY OF GROUPS

In the usual perturbative approaches to field theories, focus has been on the local, infinitesimal structures of gauge groups such as their Lie algebras. Not till the recent excitement over extended solutions in field theories does one realize the possibly deep relevance of the global structure of invariance groups.[1, 12] Since almost no previous knowledge is assumed here about the topology of Lie groups, it will be helpful to define some necessary concepts.[15] We shall make use of the global notions of compactness, connectivity, fundamental group and the universal covering group.

A topological space G is <u>compact</u> if any infinite subset of G contains a sequence converging to an element of G. A topological group is compact if it is compact as a topological space. For a Lie group, this notion is intuitive, it is compact if its total group volume is finite. The familiar examples are O(3) which is compact and SL (2C) which is not. From the work of Yang,[16] the compactness of a gauge group is shown as necessary for charge and flux quantization. For that very reason we shall restrict ourselves to compact groups in the following.

In our work, the <u>connectivity</u> of a topological space G is central to the whole classification of vortices and Dirac monopoles, which will be seen as its possible physical manifestations. To define connectivity, consider a point p in G and two closed paths $L_1(t)$, $L_2(t)$ both beginning and ending at p (Fig. 1). Let L_1, L_2 be continuous functions of a parameter t, $0 \le t \le 1$ such that $L_i(0) = L_i(1) = p$, $i=1,2$. Then the curves $L_i(1-t)$, the same curves going in the reverse sense, are the inverses of $L_i(t)$. L_1 and L_2 are called <u>homotopic</u> if there exists a function $L(s,t)$, $0 \le s \le 1$, $0 \le t \le 1$, jointly continuous both in s and t such that $L(0,t) = L_1(t)$ and $L(1,t) = L_2(t)$, i.e., L_1 can be continuously deformed into L_2 with $L(s,t)$

describing the intervening curves as $s=0 \rightarrow 1$. $L(s,t)$ is known as a homotopy. If all closed curves based from any point p in G can be so deformed to zero, the null curve, G is simply connected. If there are n closed curves which cannot be so deformed into one another, G is n-fold connected.

It is easy to verify that homotopy is an equivalence relation denoted by $L_1 \sim L_2$. All curves based at p are distributed into classes C_i, $i=1,2 \ldots n$, the homotopy classes such that all those curves homotopic to each other belong to the same class. A "multiplication" of two curves $L_\alpha L_\beta = L_\gamma$ is defined by identifying the end point of L_α with the starting point of L_β. Then L_γ, a connected path based at p, belongs to one class C_γ uniquely given by C_α and C_β, since

$$L_\alpha \sim L_{\alpha'}, \quad L_\beta \sim L_{\beta'} \rightarrow L_{\alpha\beta} L \sim L_{\alpha'} L_{\beta'} \tag{2.1}$$

and

$$L_\alpha L_\beta \sim L_\alpha \iff L_\beta \sim L_0 \sim 0 \, , \tag{2.2}$$

$$L_\alpha L_\beta^{-1} \sim L_0 \iff L_\alpha \sim L_\beta \, . \tag{2.3}$$

$$(L_\alpha L_\beta) L_\gamma = L_\alpha (L_\beta L_\gamma) \, . \tag{2.4}$$

(L_0 is the null path 0.) All the homotopy classes form thereby a group, the fundamental group of G, denoted by $\pi_1(G)$ also known as the first homotopy group or Poincare group. By the foregoing definition of connectivity, π_1 of a simply connected space, consists only of the unit element I. π_1 of a n-ply connected space has n-elements. Explicit examples will be given in the next section. Three theorems are also worth quoting:[15]

(I) Two locally isomorphic Lie groups are also isomorphic globally if their fundamental groups are isomorphic.

This underscores the importance of the fundamental groups; once the Lie groups are classified with respect to their local properties, $\pi_1(G)$ is the only invariant needed to complete the classification.

(II) The fundamental group of a group space is always Abelian. For an arbitrary topological space, Ω_1, $\pi_1(\Omega)$ is discrete, but in general a non-commutative group.

(III) All connected compact Lie groups can be gotten from connected compact simple groups. Let G_i be a sequence of connected simple compact groups. Define their direct products

$$G = G_1 \otimes G_2 \otimes \ldots G_n$$

and

$$G^{(p)} = G \otimes T^p$$

When T^p is a p-parameter compact Abelian group, i.e., a p-dimensional toroid, then the groups $G^{(p)}$ as well as the factor groups $G^{(p)}/\Delta$ with Δ being a discrete subgroup of the center Z of $G^{(p)}$.

We also need the concept of the universal covering group. It can be proved that for any multiply connected Lie group G, there exists a unique simply connected group $\bar{G}$ such that G is a homomorphic to $\bar{G}$, the universal covering group of G. All Lie groups with the same Lie algebra as $\bar{G}$ and hence locally isomorphic to $\bar{G}$, are of the form

$$G \simeq \frac{\bar{G}}{\Delta} \quad \text{and} \quad \Delta \simeq \pi_1(G) , \tag{2.5}$$

Δ is a subgroup of the center Z of $\bar{G}$. The covering group $\bar{G}$ literally "covers" all other locally isomorphic groups G and determines both their properties and their representations. Finally we gather some definitions concerning the action of a group G on a space M.

- A group G acts transitively on a space M if for every pair of point p and q $\in$ M, there is a group operation g $\in$ G such that $p \xrightarrow{g} q = gp$.
- G acts effectively on M if the identity is the only group operation which leaves every p $\in$ M fixed: $gp = p$ for all $p \in M \Rightarrow g = I$.
- The orbit of the point p $\in$ M under G is the set of all q $\in$ M reachable by the application of some g $\in$ G to p.

Next we apply the above concepts and theorems to the classification problem at hand.

III. TOPOLOGY OF VORTICES

In superfluids and superconductors, the vortex states are well known coherent phenomena.[17,18,19] In He II, vortex lines are spontaneously generated by cooling the liquid in a bucket rotating at an angular velocity $\Omega > \Omega_c$, the critical velocity for vortex creation. Similarly in a superconductor vortices are made by the application of a magnetic field $H > H_c$, the critical field strength. A vortex line consists of a central core domain where there is a depletion of the condensate. The radial extension of the core is the healing or coherence length ξ. Vortices are physical objects exhibiting homotopic conservation laws, the vorticity quantization in Helium II and the flux quantization in type II superconductors respectively. They are thus topologically stable. Dynamical stability is achieved through the circulating supercurrents of particles. In Helium the current pattern of flow outside the vortex dies away with the radial distance r as $1/r$, while in a superconductor, the coupling of the charged current with the magnetic field results in a $\exp(-r/\rho)$ behavior for large r. The latter phenomenon made possible by a minimal gauge coupling is the Meissner effect. ρ is the penetration width, the other length in the vortex structure.

Recent works[7] in classical gauge theories concern vortex-like solutions to relativistic field theories. In the following, the topology of vortices and their Dirac monopoles is studied. It will be shown to mirror perfectly the global topology of the gauge group in a given model. Though our arguments depend only on the continuity of the Higgs fields and the connectivity of the group manifold of the global gauge group G, we will work with an explicit model with a Hermitian Lagrangian density

$$\mathscr{L}(x) = -\frac{1}{4}\vec{F}_{\mu\nu}\cdot\vec{F}_{\mu\nu} - \sum_{i=1}^{k} |D_\mu \Phi_i(x)|^2 - U(\Phi_1, \ldots \Phi_k) \,. \tag{3.1}$$

Equation (3.1) is a generalized Nielsen-Olesen model[5] made up of one Yang-Mills field $\vec{F}_{\mu\nu}$ and k Higgs fields Φ_i. k, the number of Φ_i needed to have a well defined vortex solution, is of no importance to our topological analysis. Via their self-couplings, the Φ_i provide an expedient mean of implementing spontaneous symmetry breakdown. The invariance group of (3.1) is taken to be a compact m parameter simple Lie group G. Extension to a semi-simple group is straightforward. The difference will be the existence of r independent coupling constants, one for each simple group G_i which compose $G = G_1 \otimes G_2 \otimes G_3 \ldots \otimes G_r$. $U(\Phi_i)$ is the usual renormalizable effective G invariant potential. We first define our notations.[20]

Let the group generators T_i of G obey the Lie algebra

$$\left[T_i, T_j\right] = i\, C_{ijk} T_k \,, \qquad i,j,k = 1,2 \ldots m \,, \tag{3.2}$$

where the structure constants C_{ijk} are real numbers. The Higgs fields transform according to

$$\Phi_i(x) \rightarrow \Phi_i'(x) = e^{-i\vec{L}\cdot\vec{\omega}(x)}\, \Phi_i(x) \equiv S(\omega)\, \Phi_i(x) \tag{3.3}$$

where Φ_i is a p-component column vector and the L_i form a p×p matrix representation of the T_i. $\omega^i(x)$ are the parameters of the local transformation.

The covariant derivative is

$$D_\mu \Phi_i = \left(\partial_\mu - ie\, \vec{L}\cdot\vec{A}_\mu\right)\Phi_i \tag{3.4}$$

where the gauge-potentials $\vec{A}_\mu$ assume values in the Lie algebra of G.

$$\vec{A}_\mu' \cdot \vec{L} = S\vec{A}_\mu \cdot \vec{L} S^{-1} - \frac{i}{e}\, \partial_\mu S S^{-1} \tag{3.5}$$

and

$$\vec{F}_{\mu\nu} = \partial_\mu \vec{A}_\nu - \partial_\nu \vec{A}_\mu + e\vec{A}_\mu \times \vec{A}_\nu \ . \tag{3.6}$$

The system (3.1) without the gauge field is seen as the non-Abelian relativistic analog of the Ginzburg-Pitaevski theory of superfluids. Yet the corresponding pure Higgs vortices are shown not to be extrema of the field theory.[4,6] They have infinite energy per unit length since there exists no stabilizing rotation of the ground state which is the infinitely extended vacuum. This divergence can also be seen as an infrared effect of the Goldstone bosons begotten from spontaneous symmetry breakdown. In magnetic confinement schemes,[5] (3.1) is seen as the analog of the Landau-Ginzburg theory of superconductivity. The vacuum is then likened to a type II superconductor perforated by Abrikosov's flux lines.[18] They arise as the gauge field strength could locally restore gauge invariance along the vortex cores, which are nodal lines of the Higgs fields. These are the Nielsen-Olesen vortices which exemplify vacuum excitations. They can be identified in the strong coupling limit with the Nambu strings of dual resonance models.[5,21] The energy per unit length of these vortices is finite thanks to local gauge invariance which allows for the Higgs mechanism, the analog of the Meissner effect.

Since the continuous process of time evolution is a homotopy,[2] it suffices in our considerations to deal with static vortices. We first classify pure vortices without monopoles. So they are either infinitely long or closed. Spontaneous breaking at the tree level is achieved by assuming that the absolute minimum of the effective potential U is reached at some nonzero constant values of the fields $\Phi_i = \Phi_i^o \neq 0$. The Φ_i^o and the $A_\mu^\alpha = 0$ specify the translational invariant classical vacuum. The relevant objects for our purely topological analysis, the static vortex boundary conditions are determined by the following requirement[6,7]

$$\Delta\mathscr{E}_i \propto \int_{|\underset{\sim}{x}|\to\infty} d^2\underset{\sim}{x}\,(D_\mu\Phi_i)^*\,(D_\mu\Phi_i) \to 0\ , \qquad i=1,2,\dots k\ ; \tag{3.7}$$

namely the kinetic energy contribution of each Φ field to the energy density must vanish at asymptotic distances $|\underset{\sim}{x}| = r = \sqrt{x^2+y^2}$ from the vortex core located at $\underset{\sim}{x} = (0,0)$. The vortex is oriented along the z direction. (3.7) is a possible statement of the Higgs mechanism as the asymptotic cancellation between the gauge potentials and the spacetime dependent gradients of the phases of the Higgs fields viz.,

$$D_\mu \Phi_i \underset{|\underset{\sim}{x}|\to\infty}{\longrightarrow} 0 \qquad i=1,2,\ldots k \quad . \tag{3.8}$$

Geometrically, (3.8) means that the Φ_i are covariant constants at infinity.[8] The Higgs fields simply get parallel transported at large distances from the vortex core and reflect the fact that the broken components of the connection $\vec{A}_\mu$ become flat, i.e., the corresponding $F^\alpha_{\mu\nu}$ vanish. For an isolated, infinitely long, static vortex lying along the z direction and having as ϕ the azimuthal angle, the axial symmetry reduces the problem to a two dimensional one.. The finite energy density condition (3.8) becomes

$$\lim_{r\to\infty} \vec{A}_\phi \cdot \vec{L} = -\frac{i}{e}\frac{1}{r}\frac{d}{d\phi}\ln \Phi_i(\infty,\phi) \ . \tag{3.9}$$

Expressed in an integrated form (3.8) gives

$$\Phi_i(y) = S_\ell(y,x)\,\Phi_i(x) \tag{3.10}$$

where

$$S_\ell(y,x) \equiv \mathsf{T} \exp\left(-ie\int_x^y A\!\!\!/_\mu(z)\, dz_\mu\right)$$

is the nonintegrable phase factor. T is the ordering operator for the matrices $A\!\!\!/_\mu(z) = \vec{L}\cdot\vec{A}_\mu$ along the path ℓ at spatial infinity, beginning at the point x and ending at y. The motions of this phase factor $S_\ell(y,x)$ in the group space G, will constitute the very kernel of our analysis.

Having set up the proper framework, we proceed to extract the topological implications of (3.9). We start with the Abelian U(1) model of Nielsen and Olesen.[5] The symmetry is completely broken with one complex Φ field. At $r \to \infty$, Abrikosov's vortex boundary conditions[18] are

$$\underset{\sim}{A}(r) = \frac{i}{e} \underset{\sim}{\nabla} \ln S(\phi) \tag{3.11}$$

$$\Phi(\phi) = S(\phi)\, \Phi^{o}(0) \tag{3.12}$$

where $\Phi^{o}(0)$ is a fixed vector minimizing the potential $U(|\Phi|^2)$. $\underset{\sim}{\nabla}$ denotes the 3-space gradient operator. The above compact forms for $\underset{\sim}{A}$ and Φ follow readily from the polar coordinate representation in (3.9) and the "abnormal" vortex boundary condition $\Phi(\phi) = e^{+i\ell\phi}\, \Phi^{o}(0)$. The phase factor $S(\phi) = e^{+i\ell\phi}$ is a finite group element of U(1). It is single-valued since by continuity of the wave function one must satisfy $\Phi(2\pi) = \Phi(0)$. Yet the multivaluedness of the phase $\ell\phi$ signals the existence of a topological quantum number or kink associated with $\Phi(x)$. By the continuity of $\Phi(x)$, the asymptotic form (3.12) implies a nodal line of Φ, $\Phi(0,0,z) = 0$ along the vortex core with ℓ as the vorticity number.[19] Due to the presence of the gauge field, ℓ has the meaning of units of magnetic flux carried by the vortex. $4\pi g = \oint_c \underset{\sim}{A} \cdot d\underset{\sim}{x}$, c denotes a circle at $r \to \infty$. The London flux quantization is $eg = \ell/2$.

The topological meaning of the mapping (3.12) can be visualized in the following way. As the field Φ is carried once around a large circle $\phi = 0 \to 2\pi$ centered at the vortex in physical space, the phase factor $S(\phi)$ describes a closed circuit $S(0) = S(2\pi) = 1$ in the U(1) group space. Since U(1) is isomorphic to S0(2), it has the topology of a circle. The closed path winds around ℓ complete turns in S0(2). Let p be any point in S0(2), then a closed curve based at p going j times around the circle cannot be continuously deformed into one going k times around (k≠j), the difference being a path winding (j-k) times around S0(2) and so not homotopic to zero. Consequently, the homotopy classes of all closed paths are infinite in number. They form the fundamental group of the circle $\pi_1(SO_2) \simeq \pi_1(U(1))$, the infinite cyclic group made up of the identity and the integral powers of the class of path winding once around S0(2). As it is isomorphic to Z_∞, the additive group of integers, we write this symbolically as $\pi_1(U(1)) \simeq Z_\infty$. By (3. 12), there is a one to one correspondence between $S(\phi)$ and $\Phi(\phi)$. Hence the vortices are also classified by the fundamental group $\pi_1(U(1)) \simeq Z_\infty$. Each vortex is labelled by a homotopic invariant an integer $\ell \in Z_\infty$. No continuous U(1) gauge transformation can bring a vortex with ℓ units of flux into one with m ($\ell \neq m$) units of flux. There exists an infinite variety of topologically distinct vortices mirroring perfectly the ∞-connectedness of the global group U(1).

Next, we turn to the non-Abelian context, a model (3. 1) with a local SU(2) group.[5] To obtain a vortex solution, Nielsen and Olesen needed two Higgs fields Φ_1 and Φ_2 in the adjoint representation of SU(2). The symmetry is completely broken. As will become clear a judicious choice of representation for the Φ_i is of essential importance. This is so because for a representation to give a complete description of the group, it must be a faithful representation. Equivalently the group G acts effectively on the Φ_i. This point has been particularly stressed by Wu and Yang.[10] Applied to the Nielsen-Olesen choice of representation, the above observation implies that the global group of their model is in fact the adjoint group $SU(2)/Z_2 = O(3)$.

Let Φ_1^o, Φ_2^o be the two fixed isovectors which minimize $U(\Phi_1, \Phi_2)$ and α their relative fixed angle in the O(3) field space. The vortex boundary conditions[5] take their simplest forms in the "Abelian gauge". At $|\underset{\sim}{x}| \to \infty$

$$\underset{\sim}{A}^1 = \underset{\sim}{A}^2 = 0 \qquad (3.13a)$$

$$\underset{\sim}{A}^3 t_3 \equiv \underset{\sim}{A} t_3 = \frac{i}{e} \underset{\sim}{\nabla} \ln S(\phi) \quad , \tag{3.13b}$$

$$\Phi_i(\phi) = S(\phi)\,\Phi_i^o(0) \;, \qquad i=1,2 \;. \tag{3.14}$$

where

$$\Phi_1^o = \Phi_1(0) \;; \qquad \Phi_2^o = \Phi_2(\alpha) \;.$$

In this gauge, the vortex is oriented along the x_3 axis in isospace and the vectors Φ_1 and Φ_2 lies in the (x_1-x_2) plane with a relative angle α and wind around x_3 as $\phi = 0 \rightarrow 2\pi$ (Fig. 2).

Due to continuity of the wave functions, $\Phi_i(\phi)$ are single-valued and we obtain

$$S(\phi) = \exp\left[i\ell\phi t_3\right] \in O(3) = SU(2)/Z_2 \tag{3.15}$$

if the representation is the adjoint one, while

$$S(\phi) = \exp\left[2i\ell\phi\, \tau_3\right] \in SU(2) \tag{3.16}$$

if the representation is the fundamental one. The operators t_3 in (3.15) and τ_3 in (3.16) are a 3×3 matrix and a 2×2 matrix, respectively. They are representations of the generator X_3 of the Lie algebra of SU(2) such that $\left[X_i, X_j\right] = i\epsilon_{ijk} X_k$. First let us consider the case where the global group is $O(3) = SU(2)/Z_2$.

As in the U(1) case, the presence of kinks in Φ_i is signaled by the multiple valued phase $i\ell\phi$ of $S(\phi)$. The "Abelian gauge" might lead one to expect an infinity of distinct vortices.[5] That this is a gauge illusion follows from an analysis of the global meaning of (3.14).

As illustrated in Fig. 3a, the O(3) group space is most suitably parametrized as a sphere of radius π. All rotations $D(\phi, \hat{n}) = \exp(i\phi\, \vec{x}\cdot\hat{n})$ are points inside or on this sphere. Any radial vector $\vec{r} = \phi/\pi\;\; \hat{n}$ has the direction of the rotation $\hat{n}$; the rotation angle is ϕ. The unit element I sits at the origin (r=0). Since a rotation by $\phi=\pi$ and one by $\phi=-\pi$ are the same, diametrically opposite points on the surface of the sphere should be identified. Due to these schizophrenic surface points, all closed curves in O(3) are partitioned into two homotopy classes making up the fundamental group $\pi_1(O(3)) \simeq Z_2$, the group of integers modulo 2. This simply reflects the two-fold connectivity of $O(3) = SU(2)/Z_2$. These curves are the ones crossing the surface of the sphere an even and an odd number of times respectively. The first types of curves are homotopic to zero, the second types are homotopic to an O_2 axis of rotation (Figs. 3b, c).

The consequence for the vortices of this two-fold connectedness of O(3) is immediate. As the $\Phi_i(\phi)$ are transported in physical space around a large circle centered at the vortex, (3.15) implies that the factor $S(\phi)$, $\phi=0 \rightarrow 2\pi$, traces out a closed curve in the O(3) group space, the above described sphere. To be exact, $S(\phi)$ being a one parameter subgroup of O(3) traces out a "geodesic" DD' (see Fig. 3b). Since $S(\phi)$ starts and ends at the unit element $S(0) = S(2\pi) = I$ as $S(\phi) \in O(3)$, it winds n complete turns, n complete diameters in O(3). However, unlike the closed paths in the U(1) circle, the n-even curves are homotopic to zero while the n-odd curves are homotopic to a single rotation.

From the standpoint of the $\Phi_i(\phi)$, there correspond two topologically distinct classes of boundary conditions on the vortices. To use the Wu-Yang terminology,[10] there exist two gauge types of vortices. The first labelled by even ℓ can be continuously gauged into the pure vacuum Φ_i^o and $\vec{A}_\mu=0$ by a transformation $U(\phi, t) \in O(3)$, $0 \leq t \leq 1$, t being another continuous group parameter. $US(\phi)U^{-1}=I$, $U(\phi, t)$ is an example of a homotopy. For an explicit example, the reader is referred to the letter of Mandelstam.[7] The second class of vortices are labelled by odd ℓ. They are homotopic to a vortex having one Dirac unit of flux g=1/2e. We have taken $4\pi g = \oint_c \underset{\sim}{A}^3 \cdot d\underset{\sim}{x} \equiv \oint_c \underset{\sim}{A} \cdot d\underset{\sim}{x}$ as a definition of magnetic flux, the quantization condition is Dirac's $eg=\ell/2$. As it stands, this condition is not yet a manifestly gauge invariant statement. This defect can be readily amended. Since the vortices are in a one to one correspondence with the elements (I, -I) of Z_2, the group of integer modulo 2, a gauge invariant definition is

$$eg = \frac{\ell}{2} \text{ mod. } 2 \quad . \tag{3.17}$$

Namely the magnetic flux is defined only by modulo 2.

Alternatively, the same conclusion is reached by taking $S(\phi)$ in (3.15) as an element of SU(2), the universal covering group of O(3). O(3) is to SU(2) what a cut plane is to a two sheeted Riemann surface. Therefore the SU(2) group space can be represented by two unit spheres with as their centers, I and -I respectively. Here diametrically opposite points located on different spheres must be identified. From Fig. 4 it follows that all closed paths in SU(2) are homotopic to zero; SU(2) is simply connected.

As $\phi=0 \rightarrow 2\pi$, the $S(\phi) \in SU(2)$ with ℓ even traces out a closed curve, a "geodesic" in SU(2) since $S(0) = S(2\pi) = I$. This winding of ℓ = even turns is homotopic to zero. The $S(\phi)$ with ℓ odd trace out open paths beginning at $S(0)=I$ but ending at another focal point $S(2\pi) = -I$ hence the curves cannot be deformed into a null path. Of course I and -I are but the elements of Z_2, which reflects the 2 to 1 homomorphism between SU(2) and O(3). Hence from the viewpoint of the universal covering group, the vortices are classified as they should be by the very same fundamental group Z_2.

If we now choose the Φ_i in the fundamental representation or any other faithful representation of SU(2), the global group is identified as SU(2) itself. In the case of one isodoublet Φ breaking the symmetry completely (3.1) gives the model of Dashen, Hasslacher and Neveu (DHN).[23] While they seek a spherically symmetric solution, which turns out to be unstable,[24] we are after a vortex solution of the type (3.13) - (3.14). The vortex boundary conditions at $|\vec{x}| \to \infty$ are (3.13) and (3.14) but with (3.16) and one Φ field.

The seeming existence of an infinite variety of vortices is again a mirage due to the Abelian gauge. Indeed as Φ is carried around a great circle in physical space, $S(\phi)$ for all integer ℓ describes a closed circuit in the SU(2) group space and such a path is always homotopic to zero (Fig. 4). This means that there exists suitable SU(2) continuous gauge transformations which can bring all the Φ (ϕ) and $\vec{A}$ in (3.13) and (3.14) with (3.16) into the vacuum fields Φ^o, $\vec{A}=0$. So there exists only one gauge type of vortices in SU(2), namely the trivial type of no vortex at all. This fact is solely tied to the simply connectedness of SU(2). Thus there are no Nielsen-Olesen vortices in the DHN model.

At this juncture, conclusions may already be ventured regarding an arbitrary compact Lie group. We shall only do so in Section IV after the construction of finite vortices. Motivated by their potential physical relevance we proceed to higher dimensional unitary groups, to the analysis of model (3.1) with a local SU(N) group. For our purpose, it suffices to assume a completely broken SU(N). The general case of a residual unbroken symmetry is discussed elsewhere.[7,8]

If the Higgs fields Φ_i are chosen in the fundamental representation of SU(N) (or any of its faithful representations thereof), the global group is SU(N). As in the SU(2) case, the vortex boundary conditions in the Abelian gauge are

$$\underset{\sim}{A}^i = 0 \qquad i=1,2,\ldots\ N^2-1$$

$$\underset{\sim}{A}^{N^2-1}\lambda \equiv \underset{\sim}{A}\lambda = \frac{i}{e}\, \underset{\sim}{\nabla}\, \ell n\, S(\phi)\ ; \tag{3.18}$$

$$\Phi_j(\phi) = S(\phi)\, \Phi_j^o(\alpha_c)\ , \qquad j=1,2,\ldots\ k\ . \tag{3.19}$$

The phase factor $S(\phi)$ is given by

$$S(\phi) = \exp(i\ell\phi N\lambda) \in SU(N) \tag{3.20}$$

with

$$\lambda \equiv \frac{1}{N} \mathrm{Diag}\,(1, 1, \dots\ 1-N) \quad , \tag{3.21}$$

the N×N matrix representative of the last element of the (N^2-1) generators of the Lie algebra of SU(N), $\left[\lambda_i, \lambda_j\right] = f_{ijk}\lambda_k$.[7,8]

The vortex is pointed along the (N^2-1)th direction in group space, the important reason for this specific choice of orientation in unitary space will be given subsequently. The k needed Φ_i fields are selected such that Φ_i^o are vectors minimizing the effective potential $U(\Phi_2)$ with relative fixed angles α_i and they define some two dimensional hyperplane orthogonal to the (N^2-1)th axis. Effectively they form a single vector rotating around that axis with an angle $N\ell\phi$.

We observe that the phase factor $S(\phi)$ generates a one parameter subgroup of SU(N). While no visual intuition is now available, that SU(N) is simply connected and $S(0) = S(2\pi) = I$ imply that all the closed curves traced out by $S(\phi)$ as $\phi = 0 \rightarrow 2\pi$ for <u>any</u> integer ℓ can be continuously deformed or gauged into the unit element I by suitable SU(N) transformations. Namely, the $\vec{A}$ and $\Phi_i(\phi)$ in (3.18 - 3.19) can be continuously brought into the vacuum fields $\vec{A} = 0$, Φ_i^o. So there are <u>no</u> vortex solutions if G=SU(N).

On the other hand if the Higgs fields are chosen in the adjoint representation (or any other faithful representations of $SU(N)/Z_N$), the global group is then the factor group $G=SU(N)/Z_N$. Z_N denotes the <u>center</u> of SU(N), the aggregate of all those SU(N) elements which commute with all elements of SU(N). In the fundamental representation the discrete Abelian invariant subgroup Z_N has the form

$$Z_N = \left(I_N, \omega I_N, \omega^2 I_N, \dots\ \omega^{N-1} I_N\right) \quad . \tag{3.22}$$

$\omega = \exp(2\pi i/N)$ is the Nth primitive root of unity and I_N the N×N unit matrix. Clearly the center is, so to say, the Abelian part of the group SU(N). We note further that the residues in (3.22), 0, 1, 2, ... (N-1) with respect to the modules N then make up a cyclic group of order N. The law of composition is addition followed by reduction to the least negative residue relative to N. So Z_N is isomorphic to the additive group of integers modulo N, it is the fundamental group mirroring the n-fold connectedness of G, $\pi_1(G) \simeq Z_N$. In the Abelian gauge, the vortex boundary conditions are given by (3.18) and (3.19) with the difference that the phase factor $S(\phi) \equiv \exp(i\ell\phi\hat{\lambda}) \in SU(N)/Z_N$. $\hat{\lambda}$ is the $(N^2-1)\times(N^2-1)$ matrix representative of the last of the (N^2-1) generators of the Lie algebra of SU(N). Continuity of the wavefunctions is obeyed, $\Phi_i(0) = \Phi_i(2\pi)$.

Topologically, as the $\Phi_i(\phi)$ winds around in asymptotic physical space, $S(\phi)$ generates a closed curve beginning and ending at $S(0) = S(2\pi) = I$, a path made up of ℓ complete turns in $G = SU(N)/Z_N$. We know from Section II that $\pi_1(SU(N)/Z_N) = Z_N$, namely, there exists N homotopy classes of closed curves in a N-fold connected manifold such as the $SU(N)/Z_N$ group space. In a way entirely analogous to our discussion of the $SU(2)/Z_2$ case, the vortices are classified by the center Z_N of SU(N). To the (N-1) nontrivial elements of Z_N correspond (N-1) nontrivial gauge types of vortices. Any two vortices belonging to the same gauge type can be continuously gauged into one another; they are hence physically indistinguishable. The quantization condition can be stated in a gauge invariant way as

$$eg = \frac{\ell}{2} \bmod N \; ; \tag{3.23}$$

The flux is defined mod N. Alternatively, as in the O(3) case, we can work in the universal covering group and take $S(\phi) = \exp(i\ell\phi\lambda) \in SU(N)$, $\lambda \equiv \frac{1}{N}\,\mathrm{Diag}\,(1, 1, \ldots\ (-N))$. Then the (3.18) remains the same while (3.19) takes

takes the matrix form

$$\boldsymbol{\Phi}_i(\phi) = S(\phi)\, \boldsymbol{\Phi}_i(\alpha_i)\, S^{-1}(\phi) \quad . \tag{3.24}$$

$\boldsymbol{\Phi}_i \equiv \Phi_i L$ is now a matrix, the L_i form a N×N matrix representation of the SU(N) generators.

In consequence of S(0)=I and $\boldsymbol{\Phi}_i(2\pi) = \boldsymbol{\Phi}_i(0)$, we deduce

$$\left[S(2\pi),\ \boldsymbol{\Phi}_i^o\right] = 0 \quad . \tag{3.25}$$

Provided the gauge symmetry is entirely broken, any combination of the generators of SU(N) do not commute with all the $\boldsymbol{\Phi}_i^o$, i=1, 2, ... k, it follows that $S(2\pi) \in Z_N$, the group center.

As the Higgs fields wind around the vortex in real space, $S(\phi)$ generates a path in the SU(N) group space, beginning at the unit element but ending at one of the N elements of Z_N. When $S(2\pi)$=I, the path is closed and hence contractible to zero, SU(N) being simply connected. These $S(\phi)$ are labelled by ℓ=Nm, m is an integer, since $\exp(i2\pi mN\lambda) = I$.[8] The corresponding boundary conditions (3.24) - (3.25) can be continuously gauged into the vacuum Φ_i^o, $\vec{A}_\mu$=0 and stand for the zero vortex sector of the model. Otherwise the paths are open curves linking the point I with any one of $\omega^\ell I$, ℓ=1, 2, ... N-1. Any two such mappings $S(\phi)$ and $S'(\phi)$ are not homotopic to zero and are mutually homotopic if and only if $S(2\pi) = S'(2\pi)$. Hence there exists (N-1) such <u>relative</u> homotopy classes of open paths, relative because their end points are kept fixed during the deformation. With the trivial class of closed paths $S(0)=S(2\pi)$, they form the relative fundamental group $\pi_1^{Rel}(SU(N)) = Z_N$ which in turn classifies the corresponding boundary conditions (3.18) - (3.24) and the vortices. The corresponding magnetic flux is defined modulo N. We recover the quantization law (3.23).

The two equivalent viewpoints only reflect the N to 1 homomorphism $SU(N) \Rightarrow SU(N)/Z_N$. In this uncovering transformation all the elements of the center are mapped into the unit matrix. As an illustration we consider the topology of SU(3). Here the center Z_3 is given in the fundamental representation $D^{(3)}(1,0)$ by the three matrices $\{I, \omega I, \omega^2 I\}$ with $\omega = \exp(2\pi i/3)$. So Z_3 is say generated by ωI. The generator is $\exp(2\pi i/3\,(\lambda_1 - \lambda_2))$ for any irreducible representation $D^N(\lambda_1, \lambda_2)$ which are split naturally into three classes isomorphic to Z_3. They are the triality zero class where $\lambda_1 \equiv \lambda_2 (\mathrm{mod}\ 3)$ and the triality ± 1 classes where $\lambda_1 \equiv \lambda_2 \pm 1\ (\mathrm{mod}\ 3)$. Hence only the triality zero representations of SU(3) are also representations of the factor group $SU(3)/Z_3$. We then see how the global structure of the group is tied in with that of the representations.

We close this section by clarifying the specific choice of the (N^2-1) axis of SU(N), e.g., the λ^8 axis for SU(3) to orient the vortex. Unlike the R^3 space of the SU(2) adjoint representation, the R^{N^2-1} space of the SU(N) adjoint representation is <u>not</u> isotropic under the action of the group SU(N). This is due to the existence of families of <u>orbits</u> of special directions. Take the example of SU(3), in the octet representation, a vector originally pointing in the λ^8 direction can be obtained by infinitesimal rotations of the group components along the $\lambda^4, \lambda^5, \lambda^6, \lambda^7$ axes but not along the $\lambda^1, \lambda^2, \lambda^3$ axes. Since we seek vortices connected to the topology of the group SU(N) (mod Z_N) and not those related to any of its possible subgroups, the desired vortices must be oriented along the (N^2-1)th axis in group space. In any other gauge, this direction lies along the orbit $S(x)\,\lambda\, S(x)^{-1}$ of λ, $S(x) \in SU(N)$. Indeed of all the matrix representatives of the generators of SU(N), only λ is such that its Abelian subgroup generated by $S(\phi) = \exp(i\ell\phi\lambda)$ has as its own discrete invariant subgroup, the center Z_N of

SU(N). It can be easily verified that $\exp(2\pi\ell\lambda) = \omega^{\ell} I$. In other words, only the curves spanned by $S(x)\ S(\phi)\ S(x)^{-1}$ probe the whole global structure of $SU(N)/Z_N$ or SU(N).[8]

IV. GENERAL VORTICES AND DIRAC MONOPOLES

We now construct finite, open vortices and classify their endpoint monopoles. An infinitely long vortex can be viewed as a magnetic dipole chain bridging a monopole-antimonopole pair located at antipodal infinities. From this standpoint, the topology of these endpoint monopoles is readily inferred. By the continuity of the flux and gauge invariance, the strength g for the monopole and -g for its antimonopole must be the same as the flux through $g = \frac{n}{2e}$ mod N if the global group is $G = \frac{SU(N)}{Z_N}$. This deduction fulfills the expected equivalence between London's flux quantization[26] and Dirac monopole quantization,[9] albeit in a non-Abelian gauge theory.

In the following we shall discuss briefly an alternative and more profitable way of reaching the foregoing conclusion. Thus far, the topological classification only requires the use of static, axially symmetric boundary conditions. The question then arises as to whether simple equivalent boundary conditions may exist, ones which also allow for space-time varying, finite, open vortices with monopoles at their ends. The answer is affirmative. Our method consists in a judicious introduction of Dirac strings along the cores of the vortices. For completeness' sake, we shall limit ourselves to the essential aspects of this method. The reader is referred to two other works[8,21] on Abelian and non-Abelian vortices for a detailed illustration of this method. To be specific we consiter again the system (3.1) with a local group SU(N). As we have emphasized in Section III, the global group of the system must be $SU(N)/Z_N$ for the vortices to exist. The Higgs fields are taken in the adjoint representation. We also choose to work in the covering group SU(N) so that the boundary conditions are of the forms (3.18) and (3.24) with $S(\phi) = \exp(i\ell\phi\lambda) \in SU(N)$. By performing on these static vortex solutions the singular gauge transformation

$$\hat{S}(\phi) = \exp(-in\phi\lambda) \tag{4.1}$$

they are brought into the pure vacuum $\Phi_i' = \Phi_i^0$ and $\vec{\underset{\sim}{A}}' = 0$. However a price must be paid for these simpler forms of the fields at infinity. As the sole signature of the existence and the nature of the vortex solution, the multiple-valuedness of the phase $n\phi$ in $S(\phi)$ is not lost in the process but only transferred to the new potential $\vec{\underset{\sim}{A}}'$ at <u>finite</u> distances from the vortex. Since

$$\underset{\sim}{A}'(x) \underset{r \to 0}{\longrightarrow} \underset{\sim}{A}'(\phi) = \hat{S}(\phi)\underset{\sim}{A}\hat{S}^{-1}(\phi) - \frac{n}{e}\underset{\sim}{\nabla}\phi\lambda , \tag{4.2}$$

the component Tr $[A'\lambda]$ of A' acquires a $1/r$-type singularity of strength $\frac{n}{e}$ at the position of the vortex core. The other components $\underset{\sim}{A}$ are finite but non-analytic as a result of the induced ϕ-dependence.[7] By way of (4.2) the field $F_{\mu\nu}$ acquires a stringlike singularity

$$G^+_{ij} = \frac{n}{e}(\partial_i\partial_j - \partial_j\partial_i)\phi \tag{4.3}$$

in its (N^2-1)th component. G^+_{ij} has the meaning of a fictitious gauge line having only support along the vortex core. It gives the desired boundary conditions along the vortex core equivalent to the forms (3.18) - (3.24) specified at spatial infinity.

The major advantage of this string boundary condition is twofold. On the one hand, (4.3) readily admits a space-time dependent generalization. It is recognized[21] as the static, axially symmetric form of the tensor distribution

$$G^+_{\mu\nu} = g\mathscr{E}_{\mu\nu\alpha\beta}\int d\sigma d\tau\, \delta^4(x-Z)[Z_\alpha, Z_\beta] \tag{4.4}$$

where

$$[Z_\alpha, Z_\beta] \equiv \frac{\partial(Z_\alpha, Z_\beta)}{\partial(\sigma, \tau)} , \quad \mu,\nu,\alpha,\beta = (0,1,2,3) .$$

That $g = \frac{n}{2e}$ (mod N) will be clear subsequently. A celebrated object in monopole theory with a single potential A_μ, $G^+_\mu(x)$ parametrizes a Dirac string as it sweeps out a world sheet (σ, τ) in space-time. Since $\oint_c \underset{\sim}{A}'^{N^2-1} \underset{\sim}{dx} = 0$ (mod N) for an infinite circle c centered at the vortex, such a Dirac string must carry the same but opposite flux as its associated vortex. Any continuous SU(N) Transformation S(x) cannot remove the string but can modify its space-time position as well as its orientation in group space within the orbit of L. More importantly, such S(x) cannot change the homotopy class of $S(\phi)$ in (4.1). It follows that, by construction, there must be (N-1) nontrivial gauge types of Dirac strings in a biunique correspondence to the (N-1) gauge types of Nielsen-Olesen vortices. The return flux flowing through the string is therefore $g = \frac{n}{2e}$ (mod N). Just as in Dirac's U(1) theory, the flux carried by the vortex in this singular gauge is to be computed through $\oint_\epsilon \underset{\sim}{A}'^{N^2-1} \cdot \underset{\sim}{d\vec{x}}$ where ϵ is now an infinitesimal circle around the string. The quantization condition (3.23) can similarly be seen as a consequence of the unphysicalness of the string.[8,9]

Moreover, the identification (4.4) with (4.3) affords an easy way of generating space-time dependent finite closed as well as open vortices. For closed moving vortices of any shapes, we can construct an equivalent system[8] having the same Lagrangian (3.1) but with the crucial difference of a modified expression for its $F_{\mu\nu} = \vec{F}_{\mu\nu} \cdot \vec{L}$

$$F_{\mu\nu} = \partial_\mu A_\nu - \partial_\nu A_\mu - ie[A_\mu, A_\nu] - \lambda G^+_{\mu\nu} \quad , \tag{4.5}$$

$$G^+_{\mu\nu}(x) = \mathscr{E}_{\mu\nu\alpha\beta} \sum_{i=1}^{r} g_i \int d\sigma d\tau \; \delta^4(x - Z^i_\mu)[Z^i_\alpha, Z^i_\beta] \; . \tag{4.6}$$

The subtracted term $\lambda G^+_{\mu\nu}$ is the singular gauge fixing term needed to implement the boundary conditions for the existences of r vortices with the respective fluxes $g_i = \frac{n_i}{2e}$ mod N. The chosen gauge is one with the Dirac strings all pointing along the (N-1)th axis in SU(N) space. Any other gauge choice can be reached by way of a regular gauge transformation S(x) such that $\lambda G^{+}_{\mu\nu}$ in (4.5) becomes $CG^{+}_{\mu\nu}$ where $C = S\lambda S^{-1}$ is in the orbit of λ. So, without loss of generality, we shall work in the Abelian gauge.

To obtain finite closed vortices of any shapes, the prescription is then to solve for the fields equations in (4.1) with $F_{\mu\nu}$ given by (4.5) and the vacuum boundary conditions

$$\Phi_i = \Phi_i^o , \quad \vec{A}_\mu = 0 . \tag{4.7}$$

The solutions obtained have the vortices carrying along their cores their own fictitious Dirac strings, the custodian of their respective topological quantum number, the flux g_i.

To complete our circle of arguments, we now prove that we can indeed gauge away these Dirac strings from the solution of the system (4.1) with (4.5) with as resulting system nothing but (4.1) with the suitable space-time dependent generalizations of (3.18) - (3.24). For simplicity we consider the case of a single vortex. The needed singular gauge transformation is then

$$\hat{S}(x) = e^{i\Lambda(x)\lambda} \tag{4.8}$$

such that

$$(\partial_\mu \partial_\nu - \partial_\nu \partial_\mu)\Lambda(x) = eG^{+}_{\mu\nu}(x) \tag{4.9}$$

The transformed boundary conditions are

$$\Phi'(x) = \hat{S}\ \Phi(x)\,\hat{S}^{-1} \tag{4.10}$$

$$\vec{L}\cdot \vec{A}'_\mu(x) = \hat{S}\,\vec{L}\cdot \vec{A}_\mu \hat{S}^{-1} - \frac{i}{e}\,\partial_\mu \hat{S}\,\hat{S}^{-1} \tag{4.11}$$

For a static, infinitely long vortex along the z-direction, the example of (4.8) is already given in (4.1). Generally, as we are concerned with closed or infinitely long vortex, there are no magnetic pole sources, $\partial_\mu G_{\mu\nu} = 0$ and Λ has the solution

$$\Lambda(x) = \int^{x} dy_\mu \, \partial_\mu \Lambda(y) \tag{4.12}$$

with

$$\partial_\mu \Lambda(y) = e \int d^4y' \Delta(y-y') \partial_\nu G^+_{\mu\nu} \tag{4.13}$$

and $\Delta(y-y')$ is the Green function in $\Box^2 \Delta(y) = \delta^4(y)$.[21] The path dependence of $\Lambda(x)$ only reflects its multiple-valuedness. While infinitely long or closed vortices are describable either by the system (4.1) with (4.10) and (4.11) or by (4.1) with (4.5) and $\vec{\Phi}' = \vec{\Phi}^0$, $\vec{A}_\mu = 0$, this is no longer the case for finite open vortices ending at monopoles. The latter are obtained by simply "cutting" the Dirac strings in (4.4). Since the string carries magnetic flux, at the severed points will appear classical point sources, the Dirac monopoles with strength $g_i = \frac{n_i}{2e}$ mod N. As

$$j^{(m)}_\nu = \partial_\mu G_{\mu\nu}(x) = g \int \frac{dZ}{d\tau} \, \delta^4(x-Z_\mu) \tag{4.14}$$

for a finite, open Dirac string, a trivially conserved magnetic monopole current $j^{(m)}_\nu$ appears in the λth component of $D_\mu F^+_{\mu\nu} = \nabla_\mu F^+_{\mu\nu} - ie\left[A_\mu, F^+_{\mu\nu}\right]$.[8] This additional current reflects the fact that A^μ is no longer integrable and must be singular in the presence of the Dirac monopoles.[9] In contrast to the case of closed Dirac strings, we now show that these monopoles must always be accompanied by their Dirac strings, i.e., that the latter cannot be gauged away by any gauge transformation.

It suffices to consider a static, axially symmetric, finite, open vortex with its Dirac string and monopoles positioned as shown in Fig. 5a. At far enough distances from the vortex, the fields take their vacuum values (4.7). Under the action of any singular gauge transformation $S(\phi, r, z)$ which might get rid of the Dirac string the fields become

$$\Phi'(\phi, r, z) = \hat{S}\Phi^0\hat{S}^{-1}, A'_\mu(\phi, r, z) = \frac{i}{e}\partial_\mu \ln \hat{S}$$

in the asymptotic region. By a complete rotation around the vortex symmetry axis, we end up with the phase factor

$$\hat{S}(2\pi, r, z) = \mathbf{T}\, e^{-ie\oint dx_\mu A'_\mu(\phi, r, z)}$$

If this loop is taken at $z = 0$ and $\ell >> r >> m_V^{-1}$, ℓ being the length and m_V^{-1} the penetration width of the vortex, the field configuration should be the same as one for the infinitely long vortex. Hence

$$\hat{S}(2\pi, r, z) = \omega^m I_N \in Z_N$$

with $\omega^m \neq 1$ since the vortex exists. As we slide this loop continuously by changing z and r, $\hat{S}(2\pi, r, z)$ should also change continuously due to the continuity of the wave function Φ'. As we reach $|z| >> \ell$ and $r \to 0$, the loop shrinks to zero outside the vortex region and $S(2\pi, r, z)$ should become I_N if there were no singularities in A'_μ. Since $\hat{S}$ cannot be discontinuous, A'_μ must have Dirac string singularities ending at the monopoles. We have shown that no gauge transformation can remove the Dirac strings completely: If they do remove the one terminating at the monopoles in Fig. 5a they create instead two separate other strings (Fig. 5b). Equivalently, (3.1) has no solutions (4.5) - (4.7) for finite length, open Dirac strings.

Here readers might recall that a similar argument was made by 't Hooft[3] to show that his monopole exists without a Dirac string. For completeness we make a short comment. A detailed analysis is given in our recent paper.[14] The 't Hooft monopole does not obey the topological triality. Therefore, even if $\omega^m = 1$, there is a monopole solution for $m \neq 0$. Then the above analysis shows that the Dirac string is gauged away if m = 0 (mod N). Namely, a monopole with m-Dirac unit, m = 0 (mod N) can exist without the Dirac string in $SU(N)/Z_N$ theory when the monopole does not obey triality. The 't Hooft model is just an example with N = 2. Thus, our arguments about the necessity of the strings are consistent with the one given in 't Hooft's paper.[3]

In consequence, there cannot be open vortex solutions in the system (3.1) with (3.6). To generate these vortices their monopoles must be introduced explicitly by hand at the start with their associated strings as in (4.5) where $\partial_\mu G_{\mu\nu} \neq 0$. Furthermore, since the non-Abelian gauge fields are themselves electrically charged, the (Abelian) duality symmetry between electric and magnetic fields is lost. This is reflected in a more intricate resulting mathematical structure of the monopole theory.[7,8]

Our Dirac monopoles are to be contrasted with those of t'Hooft and Polyakov which need not be accompanied by such strings.[14] Hence if a gauge theory is to admit both closed and open vortices of the Nielsen-Olesen type and if a single potential A_μ is assumed valid for the whole of space-time, then working in a singular gauge is no longer a choice but a necessity. Having thus completed a rather detailed analysis of vortices and their monopoles in some instructive examples, we will take up next the problem of extension to a general compact Lie group.

V. EXTENSION AND RELEVANCE

Thus far our analysis has been focused on specific examples. However the generality of the method and the simplicity of the results afford a straightforward extension to any compact Lie group. The necessary condition for the existence of Nielsen-Olesen vortices and Dirac monopoles is remarkably simple: the gauge group G must be multiply connected. In other words, its fundamental group must be nontrivial, $\pi_1(G) \neq 0$. In the following we shall expand on this conclusion.

Some time ago, Yang[16] showed the existence of an intimated logical relationship between the quantization of electric charges and of magnetic fluxes in superconductors on the one hand and the global concept of compactness of the gauge group on the other. Here we have shown the existence of yet another logical link, a biunique correspondence between the n allowed gauge types of vortices and monopoles and another key global notion, the n-fold connectivity of G. In this light, vortices and monopoles are seen as possible physical phenomena correlated with the structure "in the large" of the group G; they are truly global signatures of gauge invariance.

In more precise terms, we have shown the fundamental group or first homotopy group $\pi_1(G)$ to be an ideal expression of this global connection. Given a compact n-ply connected Lie group G, $\pi_1(G)$ is a finite, discrete Abelian group of order n. In the same manner as in our illustrative examples of Sections III and IV, the possible vortex solutions $\left\{A_\mu, \Phi_i\right\}$ and/or Dirac monopole solutions $\left\{A_\mu\right\}$ are partitioned into n equivalence classes, in a one to one correspondence to the n elements of $\pi_1(G)$. It is also appropriate to name these homotopy classes "gauge classes" since any two solutions, members of the same class, can be continuously gauged into each other by suitable transformations $S(x) \in G$.

As such they must be only different "gauge images" of one and the same physical object. There are thus n topologically distinct gauge types of vortices and monopoles classifiable according to the n elements of $\pi_1(G)$. In particular, to the identity element of π_1, the class of closed loops in G which are homotopic to zero, corresponds the vacuum sector of no vortex nor monopole. It can be shown[6] that no minima exist for solutions of this gauge type; by suitable continuous gauge transformations they energies can be decreased to zero continuously. Corresponding to the other elements of $\pi_1(G)$ there are the (n-1) nontrivial gauge types of vortices and monopoles.

From Section II, we recall that all connected Lie groups G can be obtained from $\bar{G}$ their universal covering group as factor groups

$$G \simeq \frac{\bar{G}}{\Delta} \; , \qquad \Delta \simeq \pi_1(G) \; . \tag{5.1}$$

The kernel Δ of such a covering map is a subgroup of the center $Z(\bar{G})$ of $\bar{G}$, and with $\bar{G}$ is a connected group. Δ has as its order the connectivity of G. In this paper, we have explicitly verified the conjecture of Lubkin[13] that the global homotopic conservation law of magnetic flux (defined modulo n) is intimately related to the n-ply connectedness of G. As a consequence our general classification problem for vortices and monopoles reduces to one of the topological classification of compact Lie groups, a topic well known to mathematicians.[28] For completeness and possible future use in physics we summarize the relevant results. At this point a further simplifying restriction is made. As all connected compact Lie groups can be obtained from compact simple Lie groups, by way of the general theorem III quoted in Section II, there is no loss of generality in considering only the latter. The different classes of locally isomorphic compact simple Lie groups are denoted symbolically as A_r $(r \geq 1)$, B_r $(r \geq 2)$, C_r $(r \geq 3)$, D_r $(r \geq 4)$ for the classical structures and G_2, F_4, E_6, E_7, E_8 the exceptional structures. The corresponding dimensions of these groups are respectively r (r+2), r (2r+1), r (2r+1), r (2r-1), 14, 52, 78, 133, 248. r denotes the rank.

For A_r, B_r, C_r, their universal coverings are

$$SU(r+1)\ , \qquad Spin\,(2r+1)\ , \qquad Sp(r) \qquad (5.2)$$

whose centers are cyclic of order r+1, 2, 2. We write this as $Z(SU(r+1)) \simeq Z_{r+1}$, $Z(spin(2r+1)) \simeq Z(spin(r)) \simeq Z_2$ respectively. The quotients or factor groups of the full centers are then PU(r+1), $\frac{SU(r+1)}{Z_{r+1}}$, SO(2r+1), P sp(r) of the projective transformations induced in the r-dimensional, real 2r dimensional and quaternionic (r-1) dimensional projective space respectively.

For D_r, the simply connected group is Spin (2r), its center is of order 4, cyclic if r is odd, noncyclic otherwise; the quotient of order 4 is the projective orthogonal group PSO (2r). For r odd, SO(2r) is the only one quotient of order 2; for r even besides SO(2r) there are two "semi-spinor groups" which are mutually homomorphic for all r and isomorphic for r=4 to SO(8).

Finally the simply connected universal coverings of G_2, F_4, E_6, E_7, E_8 have respectively cyclic centers of order 1, 1, 3, 2, 1.[29] This completes the topological classification of compact simple Lie groups and thereby the topological classification of the vortices and monopoles in gauge theories with such n-ply connected groups G. We close this topic by some further illustrations of the above groups as regards their vortex and monopole structures. In Section III, instead of $SU(N)/Z_N$, we could have chosen any $G_r = SU(N)/Z_r$ $Z_r = (I, \omega I, \ldots \omega^{r-1} I)$ with $\omega = e^{i2\pi/r}$ and N/r = integer. Such cyclic Abelian groups of order r are all the discrete invariant subgroups of SU(N) and the corresponding G_r give all the Lie groups locally isomorphic to SU(N). G_r being r-ply connected, there exist r gauge types of vortices and monopoles. With

$\pi_1(SO(n)) \simeq Z_2$ for $n \geq 3$, there are two gauge types for SO(n), one of which is the trivial vacuum sector. Since $\pi_1(Sp(n)) \simeq 0$, there is no nontrivial gauge type for these simplistic groups. The same is true with SU(N), G_2, F_4 and E_8. As for other examples, e.g., for semi-simple compact Lie groups, we leave them to the interested reader.[30]

For the compact simple Lie groups G enumerated above, the centers $Z(\bar{G})$ of the universal covering groups $\bar{G}$ are cyclic Abelian groups. Since $G \simeq \bar{G}/\Delta$ with $\Delta \subset Z(\bar{G})$ and $\Delta \simeq \pi_1(G)$, the corresponding fundamental groups $\pi_1(G)$ are therefore cyclic Abelian of order n, n being the connectivity of G. Now all cyclic groups of the same order n are Abelian and isomorphic, particularly to the group of the residues 0, 1, 2, ... n-1, if the law of composition is addition followed by reduction to the least non-negative residue to m. Equivalently $\pi_1(G)$ is isomorphic to Z_n, the additive group of real integers modulo n. Hence to each gauge type of vortices and monopoles, we can associate a homotopic invariant or kink number, a magnetic flux or charge labelled by an element of Z_n. Namely the magnetic flux or charge is only defined by modulo n. The general law of quantization then reads, $eg = \frac{k}{2}\ell \bmod n$, where e is the group "electric" charge, $\ell = -\infty, \ldots -2, -1, 0, 1, 2 \ldots +\infty$ and k some integer characteristic of the group G in question.

We observe that the quantization of g mirrors the discreteness of $\pi_1(G)$. The Abelian character of $\pi_1(G)$ and hence the additive combination law for the elements of Z_n is reflected in the additive law of combination of the magnetic charges and fluxes defined only by modulo n. Finally for the non-Abelian G, $\pi_1(G)$ is of finite order. It will be seen that this finiteness or modulo n (n finite) property has striking implications for quark confinement schemes.

Recently Wu and Yang[10] considered, among other aspects of pure Yang-Mills fields, the topology of Dirac monopoles where the global groups are U(1), O(3), and SU(2). Our classifications concurs with theirs in these revealing cases. By also dealing with the vortices ending at these Dirac monopoles, our study offers a complementary viewpoint in a two-fold way. First, the connection between the work of Wu and Yang and ours is one between Dirac's monopole quantization[9] and London's flux quantization,[26] albeit for some non-Abelian superconductor. By way of the method of Dirac strings, we give a unified treatment of the topology of vortices and monopoles through finite, open vortices. Secondly, we take up the general classification problem for any compact Lie group using only simple concepts in the topology of Lie groups.

To facilitate a comparison between the two works we have adopted the Wu-Yang terminology of gauge type for homotopy class of solutions and their notation $S(\phi)$ for the phase factor which forms the kernel of both analyses.

In our language, the Wu-Yang global approach to Dirac monopoles in non-Abelian gauge theories can be generalized and summarized as follows. To have Dirac monopoles in gauge theories, the field manifold for the potential A_μ must have a nontrivial topology. There must be more than one overlapping coordinate patches for the A_μ covering all of space time. Defined in their respective patches, the A_μ only have the physical singularities associated with electric and magnetic charges. A necessary phase factor $S(\phi)$ plays the role of a seaming matrix relating two A_μ in the overlap region. The varieties of topologically distinct A_μ are determined uniquely by the connectivity of the global group G. Given a n-ply connected G, there exist only n topologically distinct ways of partitioning the A_μ in the "normal form"[31] of the two overlapping hemispheres centered at a Dirac monopole. The field manifold M for the A_μ is split into n disconnected components M_i in a biunique correspondence with the gauge classes or homotopic closed curves generated by all possible $S(x)\ S(\phi)\ S^{-1}(x)$ where $S(x) \in G$ is continuous. These classes form the fundamental group $\pi_1(G)$ of order n. Thus to each M_i spanned by all A_μ of the same gauge type is associated a homotopic invariant, a magnetic charge g for the monopole such that $e\,g = \frac{k}{2} \ell \bmod n$.

Stated as above, the situation with respect to the vortices is the same topologically. The confinement condition $D_\mu \Phi_i \xrightarrow[r\to\infty]{} 0$ (3.8) which allows for the existence of vortices, also implies that the topologies of the field manifolds for the Higgs fields $\Phi_i(x)$ and the $A_\mu(x)$ are the same. We showed that the possible vortex boundary conditions $\left\{\Phi_i, \vec{A}_\mu\right\}$ are classified according to the homotopy classes of the loops generated by the phase factor $S(x)\ S(\phi)\ S^{-1}(x)$ with $S(x) \in G$ regular as $\phi = 0 \to 2$. These classes form the group $\pi_1(G)$. Hence given a n-ply connected G, n coordinate patches are necessary to cover the entire field manifold of the Φ_i or the $\vec{A}_\mu$. For a more compact treatment of the same problem in terms of fibre bundles, we refer the reader to a companion paper.[14]

A point of importance emphasized by Wu and Yang concerns the empirical determination of the global group G. As in the instance of global symmetries,[15] the starting point should not be the abstract group G itself but rather the physically realized representations, the particle multiplets occurring in nature. The global group is then one where these representations transform faithfully. This is so because only then are all the properties of the group G, its global properties in particular, also possessed by the representations. In Sections II and IV, we have seen this global correlation between the representation content of the Lagrangian of the system and its global invariance group. This connection is a stringent consequence of the continuity of the fields[2] which results in the possible existence of kinks such as our vortices and monopoles. We only mention one amusing consequence of this global correlation.

We recall that the quark confinement hypothesis traces back partly to the triality puzzle, to the impossibility of locally distinguishing between SU(3) and $SU(3)/Z_3$, the Eightfold way. Suppose we have a Lagrangian field theory with quark fields in a faithful representation of SU(3). It follows that the global group is the simply connected SU(3) and there cannot be any Dirac monopoles of flavor. On the other hand, if the system has the global invariance $SU(3)/Z_3$ it has only fields in the triality zero representations of SU(3), the theory then allows for three gauge types of Dirac monopoles of flavor. Hence flavored quarks and flavored Dirac monopoles are mutually exclusive (and both are equally elusive experimentally). Of course, exactly the same arguments can be repeated for flavored vortices. This deduction has all the earmarks of a superselection rule, a possible topological quark confinement mechanism. It needs further scrutiny.

In dealing with Dirac monopoles terminating at Nielsen-Olesen vortices, we made use of Dirac strings which provide a gauge dependent alternative to the more global approach of coordinate patches of Wu and Yang. As in the Abelian case,[9] Dirac monopoles are explicitly introduced via Dirac strings which are fictitious singular gauge lines. While they may be considered unesthetic in a pure gauge theory, these strings were shown to be a great practical usefulness in theories of non-Abelian type II superconductivity. By working in a gauge where the strings are made coincidental with the physical lines of Higgs zeroes, i.e., the vortex cores ending at monopoles, we showed the Dirac strings to turn into physical Nambu strings.[21,8] The latter are just the geometric idealizations of the vortex cores in the strong coupling,[5] London limit of an infinitely small coherence length. For example, in this limit,[8] a gauge field theory with the global group $G = [SU(3)]_{flavor} \otimes [SU(3)/Z_3]_{broken\ color}$, with the quarks seen as colorless Dirac monopoles, turn into a more tractable effective system made up of "triality" Nambu strings terminating at quark-monopoles. The resulting finite, closed and open vortices self and mutually interact via their enveloping flux tubes of massive gluonic clouds. Figures 6a, b, and c illustrate the possible configurations of a mesonic, a diquark-quark and a Y-shaped baryonic strings respectively. More details on this type of semi-classical models of magnetic quark confinement are available in Refs. 7, 8 and 21.

In this work we have formulate in a simple language the necessary condition for the existence of finite, open and closed vortices. These objects exemplify classically indestructible kinks.[2] The classification theorem and the generalized Dirac-London quantization condition are exact statements about these nondissipative, finite energy solutions to gauge field theories. These results can be deduced because only questions of topology are involved. Simple considerations of continuity show an isomorphism between the topology of the vortices and that of the Lie group in question. The problem then reduces to one of homotopic calculations without any recourse to knowledge of dynamics. On the other hand the sufficient condition requires one to solve for the equations of motions, the resulting solutions being in general nonanalytical. Moreover classical solutions are of importance only if the couplings are weak so that a soliton expansion makes sense.[1] As for the possible connection with dual resonance models and through the latter with theories of hadrons, the problem is much more difficult. One must face up to the inescapable challenge of strong coupling quantum field theories in four dimensions. The first concrete steps in this direction has been taken for instance by Drell, Weinstein, and Yankielowicz.[32]

As witnessed by the abundant literature,[33] monopoles have often been appealed to as possible ways out of a host of physical puzzles. Foremost among these have been those of triality and quark confinement. We have only mentioned two applications of relevance to current ideas of hadronic structure. Yet it is apparent that as highly nontrivial solutions to gauge theories, the vortices and their monopoles studied here are of great intrinsic interest as we learn ever more about the structure of non-Abelian extension of electromagnetism.

At the completion of this work, we learned of Coleman's beautiful 1975 Erice Lectures.[34] While focusing primarily on the topology of 'tHooft monopoles, he also discussed very briefly non-Abelian vortices and Dirac monopoles. We thank Poul Olesen for drawing our attention to these lectures notes.

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FIGURE CAPTIONS

1. Homotopy and homotopic curves.
2. Isospace view of the vortex in the Abelian gauge.
3. (a) Parametrization of the O(3) group space.

   (b) Two homotopic curves, one a geodesic, in O(3) which are not contractible to zero.

   (c) Example of two homotopic curves in O(3) which are contractible to zero.
4. SU(2) group space and example of an always contractible closed path and an open path from I to -I.
5. (a)-(b) The unavoidability of strings accompanying Dirac monopoles.
6. (a), (b), (c) Mesonic and baryonic vortices.

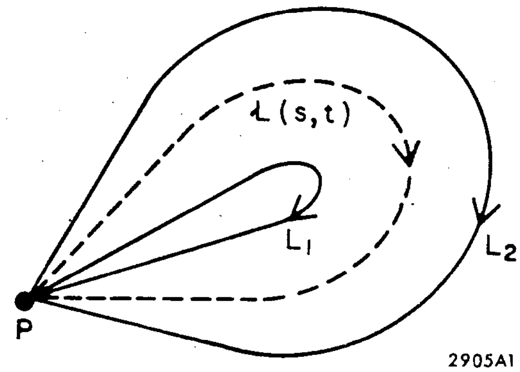

Fig. 1

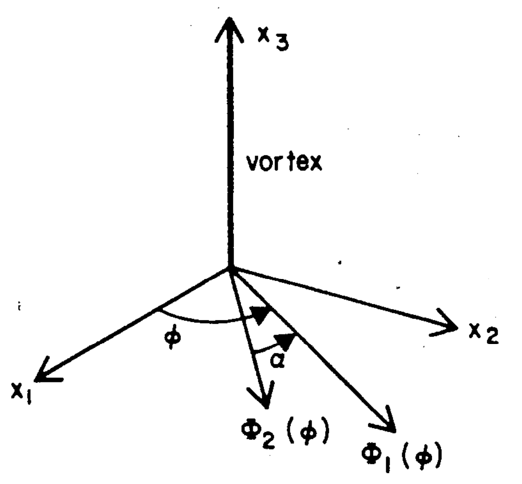

Fig. 2

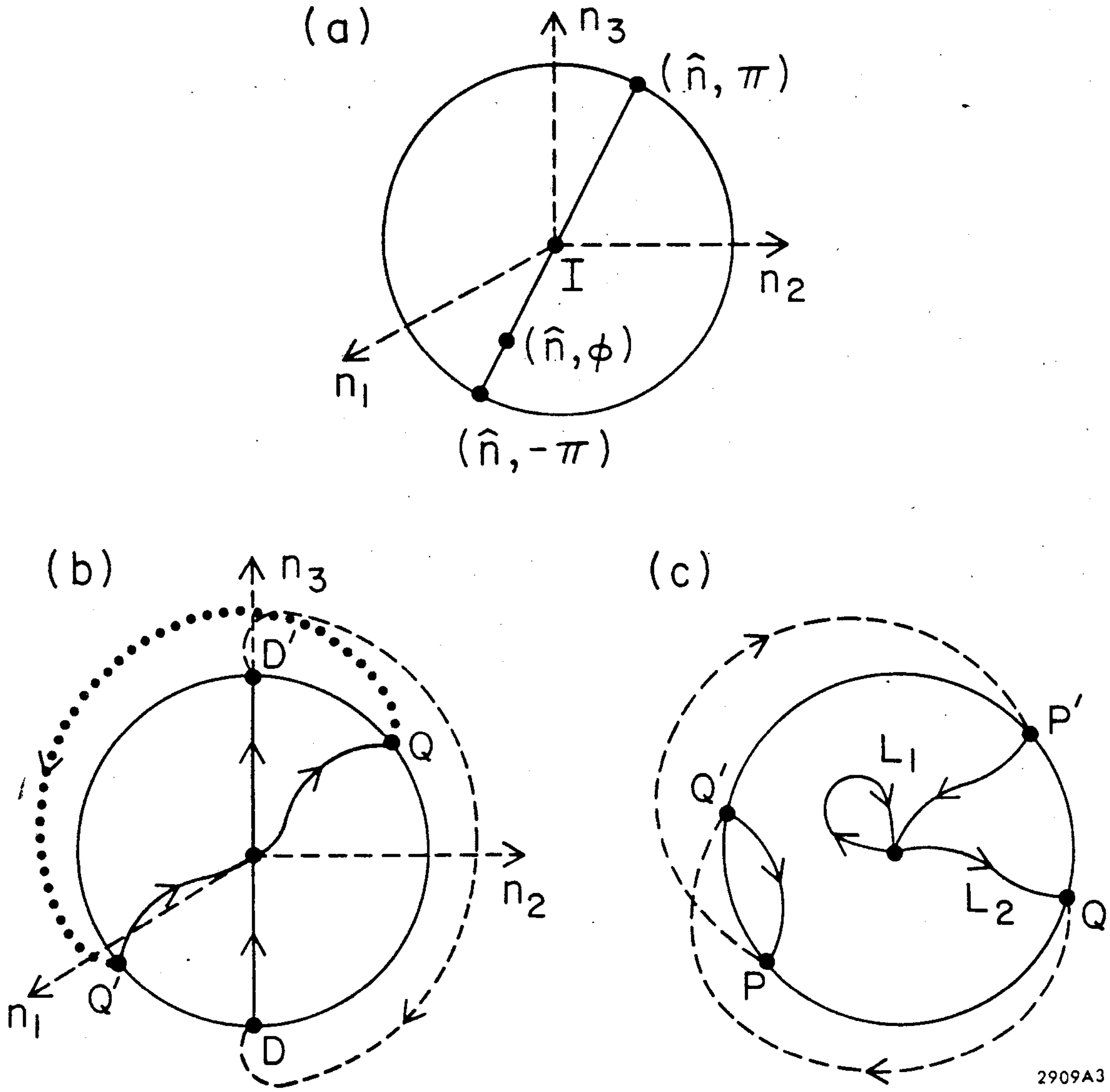

Fig.3

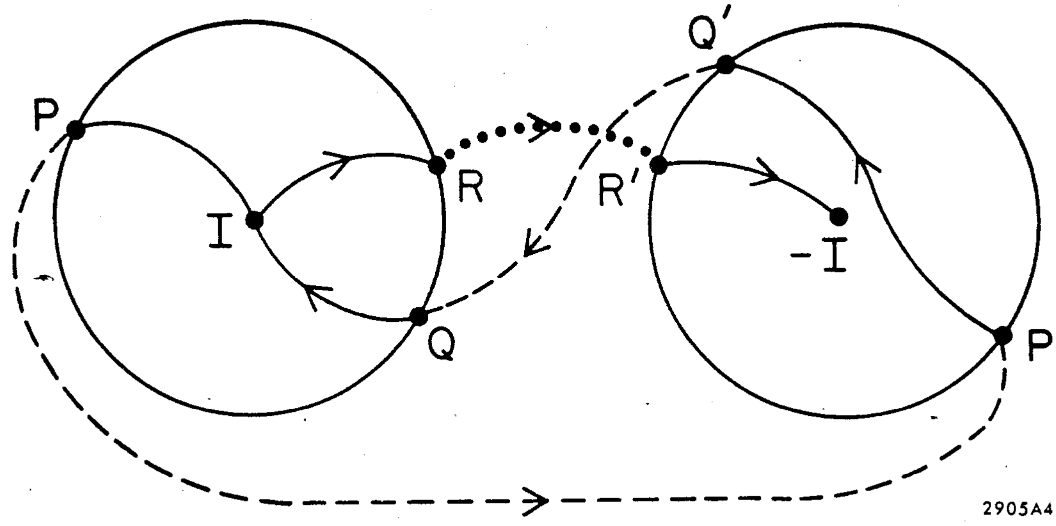
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Fig. 4

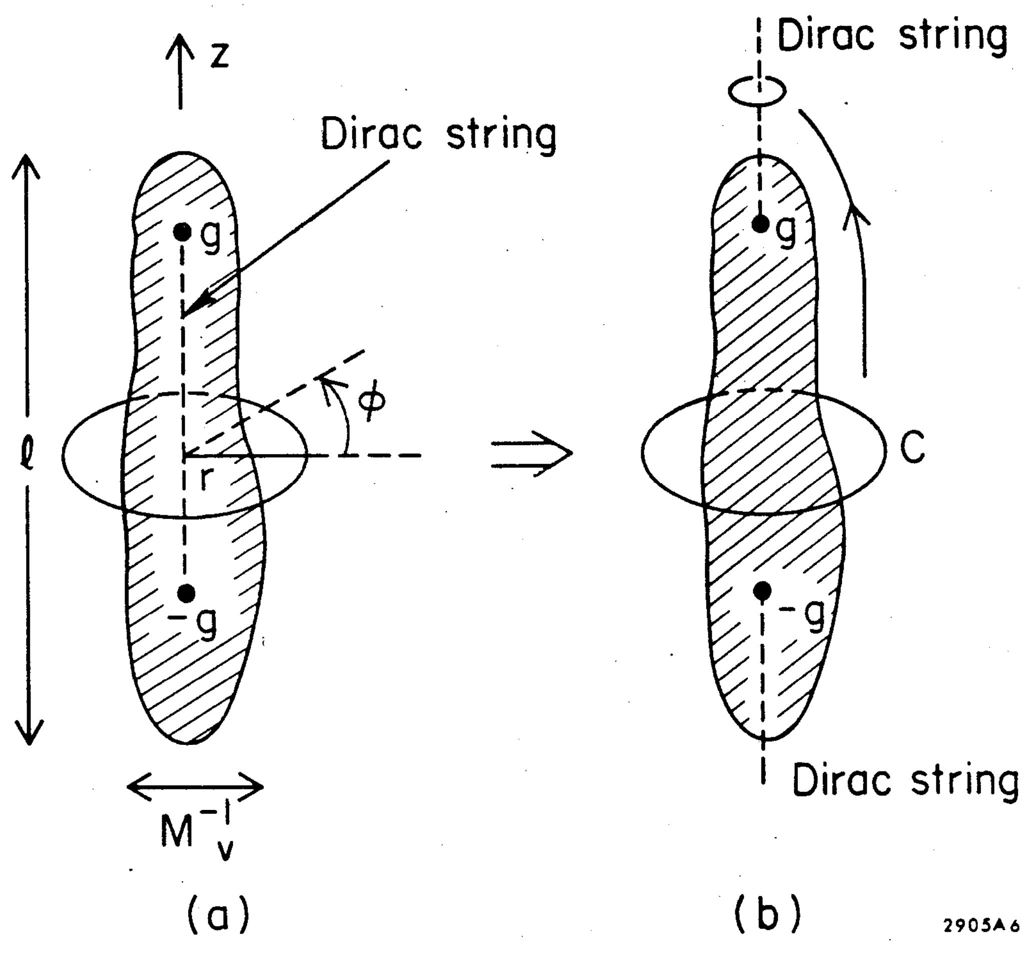
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Fig. 5

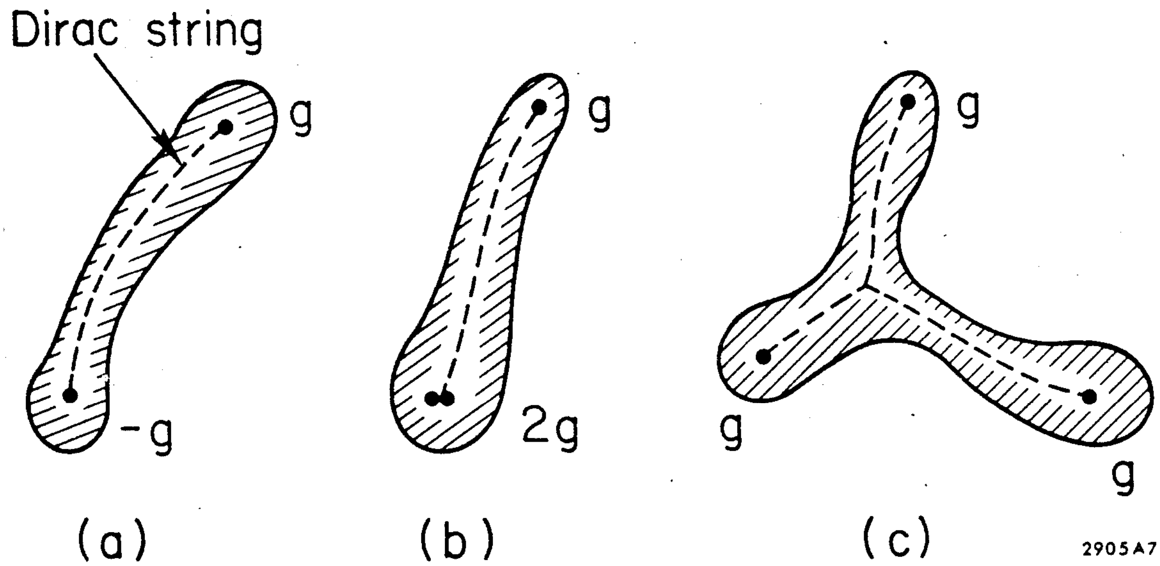

Fig. 6