GEOMETRY OF GENERALIZED COHERENT STATES [+]

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ABSTRACT : Various attempts have been made to generalize the concept of coherent states (c.s.). One of them, due to Perelomov, seems to be very promising but not restrictive enough. The Perelomov c.s. are briefly reviewed. One shows how his definition gives rise to Radcliffe's c.s. The relationship between the usual and Radcliffe's c.s. can be investigated either from group contraction point of view (Arecchi et al.) or from a physical point of view (with the aid of the Poincaré sphere of elliptic polarizations of electromagnetic plane waves). The question of finding complete subsets of c.s. is revisited and an attempt is made to restrict the Perelomov definition.

[+] Talk given at the 4th International Colloquium on Group Theoretical Methods in Physics, Nijmegen, 1975.

SEPTEMBER 1975

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I - Introduction

Coherent states (c.s.) introduced by Schrödinger [1] have been shown [2] to play an important role in Quantum Optics [3 - 5]. They have so many nice properties [6 - 7] that many attempts have been made to generalize them. The most attractive attempt is probably that of Perelomov [8] who, emphasizing the role played by the nilpotent Weyl group (also known as the Heisenberg group), defined a way of constructing systems of generalized coherent states (g.c.s.) associated with (almost) any irreducible unitary representation of any Lie group. The property of the ordinary c.s. which has been emphasized by this author in his generalization is the transitive action of the Weyl group on the set of c.s.. The definition of Perelomov will be discussed below.

Another attempt was made a few months earlier by Barut and Girardello [9] where the accent was on the fact that usual c.s. are eigenstates of an unbounded operator, namely the annihilation operator. Their g.c.s. are eigenstates of a nilpotent generator of a given semi-simple (non compact) Lie group. As already stressed by Perelomov [8] their method cannot be extended to all Lie groups, especially to compact ones.

Other systems of g.c.s. have been defined by various authors [10-15] for specific physical problems. The remarkable fact is that all these sets involve Lie groups and appear as special cases of the Perelomov definition. Apart the Weyl and oscillator [16] groups which underly the usual c.s. and the ones of the Landau electron [13], the Lie groups which have been involved are SU(2), SO(4,2) and SU(n,1) : (a) An SU(2) system of g.c.s. has been introduced by Radcliffe [10] in 1971 under the name of spin coherent states ; this system has already been investigated in many works [4, 10, 11, 17-21]. The angular momentum c.s. invented by Atkins and Dobson [11] in relation with the Schwinger [22] - Bargmann [23] approach of the roation group are closely related with the Radcliffe ones ; (b) Gürsey and Orfanidis [12] have used the conformal group to define four vector coherent states associated with four vector position and energy momentum operators ; (c) SU(n,1) sets of g.c.s. have been investigated [14,15] in the special case $n = 3$ for a covariant description of the relativistic harmonic oscillator.

In the present paper, we intend to describe the relationship between the geometric properties of different types of g.c.s. After a brief review of the Perelomov definition of g.c.s. we will show how it allows the introduction of the Radcliffe spin c.s. The connection between harmonic oscillator c.s. (h.o.c.s.) and Radcliffe's ones is investigated. The Perelomov definition is criticized and restricted in order to get a richer structure.

II - Perelomov's definition of a system of g.c.s. [8]

Definition : Let G be a Lie group and $\mathcal{H}$ the Hilbert space of an irreducible unitary continuous representation of G. Let $\hat{\mathcal{H}}$ be the projective space associated with $\mathcal{H}$ ($\hat{\mathcal{H}}$ is the set of rays of $\mathcal{H}$, i.e. the set of one dimensional subspaces of $\mathcal{H}$ referred to as the set of states). Let $\hat{\psi}$ be an arbitrary state, the set of all $g\hat{\psi}$ where $g \in G$ is called by Perelomov a system of generalized c.s.

Such systems have the following properties

i) Let H be the stabilizer of $\hat{\psi}$, that is the subgroup of G such that $H\hat{\psi} = \hat{\psi}$. Any element g of G can be written as a product $g = xh$ where $h \in H$ and $x \in G/H$. One readily sees that g.c.s. can be parametrized by the elements of the coset space G/H.

ii) The group G acts transitively on g.c.s. This means that given two g.c.s. $x_1\hat{\psi}$ and $x_2\hat{\psi}$, there exists a group transformation mapping x_1 on x_2 (one also says that the g.c.s. form a homogeneous space of G).

iii) Suppose there exists on G/H an invariant measure dx. If $|x\rangle$ denotes a normalized vector lying on the ray $x\hat{\psi}$, the operator

$$\mathcal{B} = \int |x\rangle\langle x|\, dx \qquad (1)$$

provided it exists, is a multiple of the unit operator : $B = \lambda I$.
It follows that any element $|\psi\rangle$ of $\mathcal{H}$ can be written in the form

$$|\psi\rangle = \frac{1}{\lambda} \int dx \, |x\rangle \langle x|\psi\rangle$$

It follows, from (1) that the system of g.c.s. is complete

iii) From (1) , any wave function on G/H can be written as follows

$$\psi(y) = \langle y|\psi\rangle = \frac{1}{\lambda} \int dx \; \psi(y) \, K(y,x)$$

where $K(y,x) = \langle y|x\rangle$ is a reproducing kernel.

One easily recognizes some important properties of the ordinary c.s. when G is the nilpotent Weyl group. In fact, if we use the Perelomov definition for the Weyl group, we get an infinite number of systems of g.c.s. in which the usual system appears as a very special case. In fact, according to a famous theorem of Von Neumann [24] , the Weyl group only has one kind of continuous irreducible faithful representations. In this representation, any state $\hat{\psi}$ can be shifted in a non trivial way by transformations generated by x_i and $-i\frac{d}{dx_i}$. In other words, any state lies on a two-dimensional system of g.c.s. in the Perelomov sense. The usual c.s. are the ones which lie on the orbit of the ground state of a harmonic oscillator. It follows that the Perelomov definition of g.c.s. does not contain one of the characteristic properties of Schrödinger c.s., namely the closeness of c.s. to the classical states, a property which comes from the minimalization of the Heisenberg uncertainty relations $\Delta x_i \Delta p_i = \frac{\hbar}{2}$
Unfortunately, such a property is not easily generalizable to arbitrary Lie groups.

III - Radcliffe's c.s. from Perelomov definition

According to Perelomov's ideas, given a couple $(G, \mathcal{H})$, we have to decompose $\hat{\mathcal{H}}$ into a union of orbits [25] (homogeneous spaces) of G, each of them corresponding to a set of g.c.s. When G is the ordinary rotation group, such a decomposition has been made in [26] $^{+}$. The results have a simple geometrical description we are going to recall here : first, let us define the concept of constellation.

Constellation of order n : Let $A_1, A_2, \dots, A_k$ be k points of a manifold, with weights $\alpha_1, \alpha_2, \dots \alpha_k$, respectively. The α 's are strictly positive integers and satisfy the relation $\alpha_1 + \alpha_2 + \dots + \alpha_k = n$.

First Example : Any complex polynomial in one variable of degree n is associated with a constellation of order n in the complex plane (its roots) and vice-versa (if the polynomials are defined up to a non-zero factor).

Second Example : Any complex polynomial in one variable of degree $\leq$ n is associated with a constellation of order n on a two dimensional sphere (Proof : if the degree of the polynomial is m, we say that n-m roots are infinite ; then, the extended complex line is mapped on the Riemann sphere through a stereographic projection). The set of constellations of order n on the Riemann sphere will be referred to as the n^{th} sky $S^{(n)}$.

Theorem [26]. The projective space $\hat{\mathcal{H}}$ associated with an (n+1)-dimensional Hilbert space $\mathcal{H}$ can be identified with the n^{th} sky $S^{(n)}$.

As a consequence, finding a finite projective representation of G is equivalent to finding how G acts on the corresponding sky $^{++}$. The answer is quite simple for SU(2) : "JUST ROTATE THE SKY". Therefore, spin j

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$^{+}$ The corresponding decomposition of $\mathcal{H}$ (instead of $\hat{\mathcal{H}}$) into a union of orbits has been made by Mickelsson and Niederle [27].

$^{++}$ About the action of SL(2,C) on the sky $S^{(2j)}$ associated with the representation D_{jo}, see reference [28]. I am grateful to Dr. R. Shaw for having pointed out this reference to me.

states are constellations of order $2j$ and two constellations are on the same orbit if and only if they can be brought into superposition by rotation. Two such constellations will be said to have the same shape. The classification of orbits which are present in a representation of SU(2) has been given in [26] . Let us note that many descriptions are possible + but the following one is quite simple : the state $|jm\rangle$ is represented by a constellation of order $2j$ with one point at the North pole with multiplicity $2m$ and one point at the South pole with multiplicity $2j - 2m$. The operators $J_{\pm}$ act in a very elementary way on such states. It is clear in this scheme that the states $|jm\rangle$ and $|j\ -m\rangle$ lie on the same orbit.

According to Perelomov's definition, any system of g.c.s. is given by an orbit and thus characterized by a shape of constellation. The system which has been introduced by Radcliffe [10] is the one of "collapsed" constellations that is the one which contains the state $|jj\rangle$. Therefore any Radcliffe c.s. can be labelled by spherical coordinates $\Omega = (\theta, \varphi)$ or by a complex number z . It follows that a spin c.s. can take the value $z = \infty$ in contradistinction with the h.o.c.s. The orbit of Radcliffe c.s. is sometimes referred to as the Bloch sphere [4].

The main properties of spin c.s. have been established in [10, 17, 20] . Let us mention some of them

$$|z\rangle = (1 + |z|^2)^{-j} e^{zJ_+} |j\ -j\rangle \tag{2}$$

$$\langle z'|z\rangle = \left[\frac{(1 + \bar{z}'z)}{(1 + |z'|^2)(1 + |z|^2)} \right]^j \tag{3}$$

In the (θ, φ) notation, one gets $z = \tan\frac{\theta}{2}\, e^{-i\varphi}$; it follows that

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+ Due to the transitive action of $U(n)$ on $S^{(n-1)}$ any state can be represented by a given constellation.

$$\langle \theta'\varphi' | \theta\varphi \rangle = \left(\cos\frac{\Theta}{2}\right)^{2j} \qquad (4)$$

where Θ is the angle between the two corresponding radii on the Bloch sphere S (two orthogonal states are opposite on S) . The completeness relation reads

$$\int d\Omega \, |\Omega\rangle\langle\Omega| = \frac{4\pi}{2j+1} \qquad (5)$$

where $d\,\Omega$ is the usual rotationnally invariant measure.

Remarks : i) The complex parametrization of the Radcliffe c.s. is intimately related to the Riemann sphere used by Vilenkin [29] in his construction of the SU(2) representations.

ii) SU(2) is generally used in physical problems involving two level systems. In the case of the polarization space of the electron, $\hat{\mathcal{H}}$ is a sphere which can be readily embedded in the ordinary space because the rotation group acts in an obvious way on it with an obvious interpretation. In the case of the polarization space of the photon, $\hat{\mathcal{H}}$ is the Poincaré sphere but rotations of this sphere are not related with the physical rotations of the photon states. Isospin and quasi-spin states also correspond to abstract spheres.

iii) According to the work of reference [26] , the sky representation can be used for any finite-dimensional Hilbert space. The Bloch sphere [4] corresponds to symmetrized states of N identical coherent two-level atoms. A generalization of the Bloch sphere for the description of non coherent identical systems appears to be possible with the aid of the constellation concept.

IV - Connection between spin c.s. and h.o.c.s.

Radcliffe [10] has described a relationship between his spin c.s. and the c.s. of the harmonic oscillator in one dimension. It has been shown in [17] that this relationship is better understood with the aid of a group contraction [30 , 31] . Moreover, it follows from the work by Atkins and Dobson [11] , that another relationship can be found between Radcliffe c.s. and the c.s. of the two-dimensional h.o. This is closely related to the Schwinger [22]- Bargmann [23] way of studying the SU(2) group. We intend to show here how this approach can be given a physical interpretation with the Poincaré sphere of elliptic polarizations of an electromagnetic plane wave.

1) The Poincaré sphere and the angular momentum c.s.

Poincaré [32] has shown that every elliptic polarization of an electromagnetic plane wave (propagating in a given direction) is represented by a point on a sphere + . A modern group theoretical approach of the Poincaré sphere geometry would be as follows. Let

$$H = \frac{p_x^2 + p_y^2 + x^2 + y^2}{2} \tag{6}$$

be the (classical) Hamiltonian of the two-dimensional h.o. It can also be written

$$H = \bar{z}_+ z_+ + \bar{z}_- z_- \tag{7}$$

where

$$z_\pm = \frac{1}{2}\left[(x - iy) \pm i(p_x - ip_y)\right] \tag{8}$$

Since it is a two degrees of freedom problem, a complete set of constants of the motion must contain four classical observables. If we discard the phase and energy, the complex number $z = z_+ / z_-$ uniquely define a solution, a polarization. It is clear that z belongs to the extended complex line. With the aid of a stereographic projection, we are led to the Poincaré sphere.

Now, it is clear from Eq(7) that H is invariant under SU(2) . Therefore SU(2) must act on the Poincaré sphere. The SU(2) generators

+ North (resp.South) hemisphere corresponds to right (resp. left) polarizations ; the latitude angle 2I is such that $\cos I = (A^2-B^2)/(A^2 + B^2)$ where A and B are the half axes lengths of the ellipse ; the longitude angle is 2φ where φ is the angle of the main axis with a given direction in the polarization plane.

are

$$J_3 = \frac{1}{2}(x p_y - y p_x)$$

$$J_1 = \frac{1}{4}(p_x^2 - p_y^2 + x^2 - y^2) \qquad J_2 = \frac{1}{2}(xy + p_x p_y) \tag{9}$$

which are constants of the motion since $[\vec{J},H] = 0$. We readily note that $J^2 = \frac{1}{4}H^2$ and that $\vec{J}$ defines exactly one point on the Poincaré sphere of radius $H/2$ with the very meaning indicated in the last footnote. In other words, the knowledge of $\vec{J}/H$ determines uniquely the shape and the orientation of the ellipse.

The quantum mechanical approach is quite analogous : we define the annihilation operators as in (8)

$$a_{\pm} = \frac{1}{2}\left[(x - iy) \pm i(p_x - ip_y)\right] \tag{10}$$

and the corresponding (Hermitian conjugate) creation operators $a^+_{\pm}$. We get

$$H = a^\dagger_+ a_+ + a^\dagger_- a_- \tag{11}$$

$$[a_\varepsilon, a^\dagger_{\varepsilon'}] = \delta_{\varepsilon\varepsilon'} \tag{12}$$

Then, the $\vec{J}$ operators expressed in terms of a and a^+ are exactly the ones Schwinger [22] introduced in his study of the SU(2) group. If $z_{\pm}$ are the eigenvalues of $a_{\pm}$, we see how we go from h.o.c.s. z_+, z_- to the spin c.s. $|z\rangle$ just by defining equivalence classes

$$|z_+ z_-\rangle = |\lambda z_+, \lambda z_-\rangle \tag{13}$$

each equivalence class defining a spin c.s. In the Poincaré interpretation, two harmonic oscillator motions are equivalent if they correspond to the same polarization (that is if their corresponding ellipses have same eccentricity and orientation)+.

+ Another interesting property is the following one : the operator a^+_+ (resp. a^+_-) can be interpreted as the creator of a point at North (resp. South) pole of the Poincaré sphere. Therefore $J_+ = a^+_+ a_-$ raises a point from South to North pole and $J_- = a^+_- a_-$ does the opposite (see [26]).

Résumé

Two-dimensional h.o.c.s.	$\longleftrightarrow$	spin c.s.
Bargmann-Schwinger study of SU(2)	$\longleftrightarrow$	Vilenkin study of SU(2)
Electromagnetic plane wave	$\longleftrightarrow$	elliptic polarization

2) Group contraction of the rotation group into the oscillator group

A set of Lie algebras $G(\alpha, \beta)$ of dimension 4 which has been extensively investigated by Miller [33] are intimately related with special functions. These Lie algebras have the following commutation brackets

$$\begin{cases} [J_3, J_1] = i\, J_2 \\ [J_3, J_2] = -i\, J_1 \\ [J_1, J_2] = i\alpha J_3 + i\beta E \\ [E, \vec{J}\,] = 0 \end{cases} \tag{14}$$

These real Lie algebras are the ones of $SO(3) \times \mathbb{R}$ for $\alpha > 0$, $SO(2,1) \times \mathbb{R}$ for $\alpha < 0$, $E(2) \times \mathbb{R}$ for $\alpha = \beta = 0$ and $Osc(1)$ for $\alpha = 0$, $\beta \neq 0$. Here $E(2)$ denotes the Euclidean group in the two dimensional space, $Osc(1)$ is the oscillator group [16] and $\mathbb{R}$ is the one dimensional translation group. Miller [33] has shown that these Lie algebras are related through contraction processes [30 , 31] in the following way :

$$\begin{tikzcd} SO(3) \times \mathbb{R} \arrow[dr] \arrow[drr] & & \\ & Osc(1) \arrow[r] & E(2) \times \mathbb{R} \\ SO(2,1) \times \mathbb{R} \arrow[ur] \arrow[urr] & & \end{tikzcd}$$

each arrow denoting a contraction.

Arecchi et al. [17] have stated that the contraction from $SO(3) \times \mathbb{R}$ to $Osc(1)$ transform the spin c.s. into the h.o.c.s. This statement is true but the proof presented by these authors is incomplete. Our purpose is to give an exact derivation of the contraction by using unitary representations of the real groups under consideration, i.e. $SO(3) \times \mathbb{R}$ and $Osc(1)$.

We start from the Lie algebra of $SO(3) \times \mathbb{R}$ with generators $\vec{J}$ and E satisfying (14) with $\alpha = 1$ and $\beta = 0$. We perform the following change of basis

$$\begin{cases} H = J_3 + \left(c + \frac{1}{2} - \frac{1}{2c}\right) E \\ A_{1,2} = \frac{1}{\sqrt{2c}} \; J_{1,2} \\ F = E \end{cases} \tag{15}$$

The commutation rules read :

$$\begin{cases} [H \, , A_\pm] = \pm A_\pm \\ [A_+ , A_-] = \frac{1}{c} H - \left(1 + \frac{1}{2c} - \frac{1}{2c^2}\right) F \\ [F \, , H] = 0 \end{cases} \tag{16}$$

For $c = \frac{1}{2}$, the change of basis is the identity one. When we make c going to infinity, we get the Osc(1) Lie algebra as a contracted Lie algebra. Obviously, it would be possible to perform this contraction by use of a simpler parametrization than (15) . However the one we chose is convenient for the study of c.s.

So far, we have investigated the contraction on the Lie algebra. Let us now see what we get for the representation. We start from the Vilenkin description [29] by polynomial of degree $\leq 2j$

$$\begin{cases} J_3 = z \frac{\partial}{\partial z} - j \\ J_+ = - z^2 \frac{\partial}{\partial z} + 2jz \\ J_- = \frac{\partial}{\partial z} \\ E = 1 \end{cases} \tag{17}$$

Let us renormalize by setting

$$z = \zeta / \sqrt{2j} \tag{18}$$

This corresponds to a redefinition of Radcliffe's states (Eq.2)

$$|\zeta\rangle = \left(1+\frac{|\zeta|^2}{2j}\right)^{-j} e^{\zeta/\sqrt{2j}\, J_+} |j,-j\rangle \tag{19}$$

With this change of normalization, Eqs.(15) and (17) give us

$$\begin{cases} H = \zeta\frac{\partial}{\partial\zeta} - j + c + \frac{1}{2} - \frac{1}{2c} \\ A_+ = -\frac{1}{\sqrt{jc}}\zeta^2\frac{\partial}{\partial\zeta} + \sqrt{\frac{j}{c}}\,\zeta \\ A_- = \sqrt{\frac{j}{c}}\frac{\partial}{\partial\zeta} \\ F = 1 \end{cases} \tag{20}$$

If we now make $c = j$ and <u>then</u> make j going to infinity, we readily obtain the usual h.o.c.s. of the group Osc(1) , namely

$$H = \zeta\frac{\partial}{\partial\zeta} + \frac{1}{2} \qquad A_+ = \zeta \qquad A_- = \frac{\partial}{\partial\zeta} \qquad F = 1 \tag{21}$$

Let us now give a more rigorous description of what we have just arrived at. Let $\mathcal{B}$ be a Bargmann space [7] and let P_j be the projector on the subspace of polynomials of degree less than or equal to $2j$. Each set of the following operators

$$\begin{cases} H^{(j)} = \left(\zeta\frac{\partial}{\partial\zeta} + \frac{1}{2} - \frac{1}{2j}\right)P_j \\ A_+^{(j)} = \left(-\frac{1}{2j}\zeta^2\frac{\partial}{\partial\zeta} + \zeta\right)P_j \\ A_-^{(j)} = \frac{\partial}{\partial\zeta}P_j \\ F^{(j)} = P_j \end{cases} \tag{22}$$

defines an irreducible representation of the Lie algebra generated by $H^{(j)}$, $A^{(j)}$ and $F^{(j)}$. When j goes to infinity, we get the Bargmann representation of the group Osc(1).

Remarks : i) The change (18) from z to ς corresponds to giving to the $2j^{th}$ sky a radius $\sqrt{2j}$.

ii) The question arises how to define a Radcliffe c.s. as a function of ς instead of the ket defined in (19) . The answer follows from the identity

$$e^{\bar{\varsigma}'\left[-\frac{1}{2j}\varsigma^2\frac{\partial}{\partial\varsigma}+\varsigma\right]}\,1 = \left(1+\frac{\bar{\varsigma}'\varsigma}{2j}\right)^{2j} \tag{23}$$

In fact, since 1 is the function associated with the state $|\,j\ -j\,\rangle$, one readily sees that the Radcliffe c.s. (in the ς variable) corresponds to the function

$$R^{(j)}_{\bar{\varsigma}'}(\varsigma) = \left(1+\frac{|\varsigma'|^2}{2j}\right)^{-2j}\left(1+\frac{\bar{\varsigma}'\varsigma}{2j}\right)^{2j} \tag{24}$$

which, when j goes to infinity provide us with the usual c.s. functions

$$R^{(\infty)}_{\bar{\varsigma}'}(\varsigma) = e^{-\frac{|\varsigma'|^2}{2}}\,e^{\bar{\varsigma}'\varsigma} \tag{25}$$

iii) A set of fundamental invariants of the algebra (16') is given by F and

$$Q_j = \frac{H^2}{2j} - \left(1+\frac{j-1}{2j^2}\right)HF + \frac{1}{2}\left(A_+A_- + A_-A_+\right) \tag{26}$$

We readily see that Q_∞ is the invariant of the group Osc(1) +

Summary

In F_1 , the Bargmann spaces of entire functions of ς , one can define a sequence of representations D_j of the $SU(2)\times\mathbb{R}$ Lie algebra on an increasing sequence of subspaces P_jF_1 (representations given by Eqs.(15) and (22) with $c = j$). When j tends to infinity, this Lie algebra contracts into that of Osc(1) . The operators (22) which are bounded for

+ The Casimir operator of SU(2) , the eigenvalues of which are $j(j+1)$ is given by $Q_j' = 2j\,Q_j + (j+\frac{1}{2}-\frac{1}{2j})^2\,F^2$.

j finite tend to the unbounded operators (21) in the following sense : if f belongs to H domain, $\lim_{j\to\infty} \| H^{(j)} f - Hf \| = 0$. Moreover the Radcliffe c.s. $R^{(j)}_{\bar{\xi}}, (\xi)$ of Eq.(24) tends to the usual c.s. (25) , i.e.

$$\lim_{j\to\infty} \| R^{(j)}_{\bar{\xi}'} - R^{(\infty)}_{\bar{\xi}'} \| = 0$$

V - Completeness of subsets of g.c.s.

Any system of g.c.s. being complete, it is natural to look for some complete subset. Such a question has already been answered for the usual c.s. by Von Neumann [25] , by Bargmann et al [34] and Perelomov [35]. In this special case an interesting complete set of c.s. which has been investigated, is generated by a discrete subgroup of the Weyl group, namely the group of discrete translations of a lattice in phase space

$$e^{imbx} \, e^{inap} \tag{27}$$

where x and p are position and momentum operators, m and n are integers and a and b are related by the condition $ab = 2\pi$.

It is therefore natural to look, following Perelomov [8] , for complete subsets of g.c.s. which are orbits of some subgroup of the group under consideration. In the case of g.c.s. associated with the Weyl group, a necessary and sufficient condition has been given in [36] for a state to generate a complete set under the lattice group (27).

For the case of spin c.s. , it can be readily seen that any subset of $2j+1$ spin c.s. of spin j is complete, i.e. form a (nonorthogonal) basis of the representation space. The proof is as follows : a spin c.s. $|z_1\rangle$ has components of the form [26] :

$$\begin{vmatrix} 1 \\ \alpha_1 z_1 \\ \alpha_2 z_2^2 \\ \\ \\ \alpha_{2j} z^{2j} \end{vmatrix}$$

Consider $2j+1$ such states : $|z_1\rangle$, $|z_2\rangle$, ... $|z_{2j+1}\rangle$. For these c.s. to be independent, it is necessary that the determinant

$$\begin{vmatrix} 1 & 1 & 1 \ldots\ldots\ldots\ldots & 1 \\ \alpha_1 z_1 & \alpha_1 z_2 & \alpha_1 z_3 & \alpha_1 z_{2j+1} \\ \alpha_2 z_1^2 & \alpha_2 z_2^2 & \alpha_2 z_3^2 & \alpha_2 z_{2j+1} \\ \alpha_{2j} z_1^{2j} & \alpha_{2j} z_2^{2j} & \alpha_{2j} z_3^{2j} & \alpha_{2j} z_{2j+1}^{2j} \end{vmatrix}$$

does not vanish. It is readily seen that this determinant is different from zero if and only if all the z_i's are distinct. This proves the statement.

VI - Conclusion

The Perelomov definition of g.c.s. is only based on the transitivity property and no physical justification has been given for that. Moreover, according to this definition, any state is coherent. This is not very satisfactory and it is desirable not only to justify the need of an orbit but also to restrict the definition by using some physical argument. The most physical argument which is used for usual c.s. is probably the closeness of h.o.c.s. to classical states, a property which is expressed by the minimalization of the Heisenberg inequality. Unfortunately, we do not know how to generalize the Heisenberg uncertainty principle to all Lie groups but it is clear that if we were able to express the closeness of g.c.s. to classical states, the g.c.s. would be parametrized by coordinates in the phase space of the system. A phase space is a particular case of a symplectic manifold and it has already been shown [37-41] how non trivial symplectic manifolds are naturally involved in the description of classical relativistic or non relativistic elementary systems. In this case, the symplectic manifolds are canonically related to the Poincaré and Galilei group by the Kostant-Souriau theorem [42] . Let us underline that the transitivity property only expressed the elementary character of the classical system under consideration.

It follows from our discussion that, whatever is the way of introducing the concept of closeness to classical states, the orbit of g.c.s. must be a symplectic one. Let us examine how strong is the restriction for an orbit to be a symplectic one in the case of the SU(2) group. According to the Kostant-Souriau theorem [42] , the only symplectic homogeneous spaces of a Lie group G are the orbits of G on the dual vector space of the Lie algebra. It is quite simple to see that the only symplectic homogeneous space of SU(2) is the sphere S_2 (as a coset space it is SO(3)/SO(2)). According to [26] the only states which have SO(2) as stability subgroup are the states of type $|jm\rangle$ with $m \neq 0$. On the $2j^{th}$ sky there are $j + \frac{1}{2}$ or j such orbits according to j is half integral or integral. It is quite remarkable that the restriction of symplecticness only select a finite number of orbits among an infinity.+ Obviously, the Radcliffe choice is the most natural one.

The restricted definition we proposed is unfortunately not so successful in the case of the Heisenberg-Weyl or oscillator group, because we are still left with an infinite number of symplectic orbits. However, in the case of the Galilei group, it can be shown, for a spinless particle, that one of possible systems of g.c.s. would be of the form

$$e^{i\frac{\vec{p}^2}{2m}\tau} \, e^{i\vec{p}\hat{a}} \, f((\vec{p}-\vec{k})^2)$$

+ Except in the case $j = \frac{1}{2}$ for which the projective space is a single orbit.

where τ , $\vec{a}$, $\vec{k}$ are parameters. We immediately note that the ordinary c.s. belong to this kind. Similar g.c.s. could be defined for relativistic particles with the aid of the Poincaré group[+]. One of the most promising sets of g.c.s. seems to be the twistor space [43] .

[+] All symplectic manifolds invariant under the Poincaré group have been classified in [41].

- REFERENCES -

[1] E. SCHRÖDINGER
Naturwissenchaften 14 , 664 (1926)

[2] R.J. GLAUBER
Phys. Rev. 131 , 2766 (1963)

[3] J.R.KLAUDER and E.C.G.SUDARSHAN
Fundamental of Quantum Optics (Benjamin 1968)

[4] M. NUSSENZWEIG
Introduction to Quantum Optics (Gordon and Breach, 1973)

[5] F. ROCCA
in the present proceedings

[6] P. CARRUTHERS and M.M. NIETO
Rev. Mod. Phys. 40 , 411 (1968)

[7] V. BARGMANN
Comm. Pure Appl. Math. 14, 187 (1961)

[8] A.M. PERELOMOV
Commun.Math.Phys. 26, 222 (1972)

[9] A.O.BARUT and L. GIRARDELLO
Commun.Math.Phys. 21, 41 (1971)

[10] J.M. RADCLIFFE
Journ. Phys. A4, 313 (1971)

[11] P.W. ATKINS and J.C. DOBSON
Proc. Roy. Soc. A 321, 321 (1971)

[12] F. GÜRSEY and S. ORFANIDIS
Phys. Rev. D7, 2414 (1973)

[13] W.G. TAM
in Proceedings 1st. Colloquium on Group Theoretical Methods Physics, Marseille 1972 (H. Bacry, Editor), and references therein.

[14] H. BACRY and N.P. CHANG
Phys. Lett. B 44, 286 (1973)

[15] H. BACRY
in Proceedings 3rd International Colloquium on Group Theoretical Methods in Physics, Marseille 1974 (H. Bacry and A. Grossmann, Eds.)

[16] R. STREATER
Commun.Math.Phys. 4 , 217 (1967)

[17] F.T. ARECCHI, E. COURTENS, R. GILMORE and H. THOMAS
Phys. Rev. A6 , 2211 (1972)

[18] J. KUTZNER
Phys. Lett. A 41, 475 (1972)

[19] E.H. LIEB
Commun.Math.Phys. 31, 327 (1973)

[20] J. BELLISSARD and R. HOLTZ
J. Math.Phys. 15 , 1275 (1974)

[21] W. WITSCHEL
J. Phys. A7, 1847 (1974)

[22] J. SCHWINGER
in L.C. BIEDENHARN and VAN DAM Eds., Quantum Theory of Angular Momentum (Academic Press, 1965)

[23] V. BARGMANN
Rev. Mod. Phys. 34 , 300 (1962)

[24] J. VON NEUMANN
Mathematical Foundations of Quantum Mechanics, (Princeton 1955)

[25] L. MICHEL
in the present proceedings

[26] H. BACRY
J. Math. Phys. 15, 1686 (1974)

[27] J. MICKELSSON and J. NIEDERLE
Commun.Math.Phys. 16, 191 (1970)

[28] F.A.E. PIRANI
in Lectures at Brandeis Summer School 1964, vol. I (Prenctice Hall, 1965)

[29] N.J. VILENKIN
Special Functions and the Theory of Group Representations (American Math. Society, 1968), chapter III, section 2

[30] E. INÖNU and E.P. WIGNER
Proc. Nat. Acad. Sci. 39, 510 (1953)

[31] E. SALETAN
J. Math. Phys. 2, 1 (1961)

[32] H. POINCARE
Théorie mathématique de la lumière (vol. 2, Paris, 1892)

[33] W. MILLER
Lie Theory and Special Functions (Acad. Press 1968)

[34] V. BARGMANN, P. BUTERA, L. GIRARDELLO and J.R.KLAUDER
Reports on Math. Phys. 2 , 221 (1971)

[35] A.M. PERELOMOV
Theoreticheskaya i Matematicheskaya Fizika, 6, 213 (1971)

[36] H. BACRY, A. GROSSMANN and J. ZAK
Phys. Rev. B (to appear)

[37] H. BACRY
Classical Hamiltonian for a Spinning Particle (unpublished)

[38] H. BACRY
Commun.Math.Phys. 5, 97 (1967)

[39] H. BACRY
in Proceedings 1st International Colloquium on Group Theoretical Methods in Physics, Marseille 1972 (H. Bacry, Editor)

[40] R. ARENS
Commun.Math.Phys. 21, 139 (1971)

[41] R. ARENS
J.Math.Phys. 12, 2415 (1971)

[42] D. SIMMS
Talk given at the present colloquium and references therein.

[43] N. WOODHOUSE
Talk given at the present colloquium.