The semiclassical approximation to quantum gravity and its observational consequences

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Abstract. We study possible observational effects from the Wheeler-DeWitt equation of quantum geometrodynamics. For this purpose, we perform a semiclassical expansion and derive quantumgravitational correction terms that are inversely proportional to the Planck mass squared. We apply these results to cosmology and calculate the resulting modification of the CMB power spectrum. Although the correction terms are too small to be currently observable, they could provide the key for future tests of quantum gravity.

1. Introduction

The construction of a consistent theory of quantum gravity is among the major open problems of fundamental physics. In addition to conceptual and mathematical difficulties, the main obstacle is the lack of empirical tests so far. It is thus not surprising that many different approaches to quantum gravity exist, which all possess their strengths and weaknesses [1]. A decision between these approaches can eventually be made only on the basis of experiments or observations. For this reason, it is important to calculate as many concrete predictions as possible, even if they are currently too small to be tested.

One example for a concrete prediction is the quantum-gravitational correction to the Newtonian potential between two masses m_1 and m_2 . This was calculated at the one-loop level of quantum gravity as defined by the path integral [2]. The result for the potential is

$$V(r) = -\frac{Gm_1m_2}{r} \left(1 + 3\frac{G(m_1 + m_2)}{rc^2} + \frac{41}{10\pi}\frac{G\hbar}{r^2c^3}\right).$$

The first correction term is a well-known correction from classical general relativity, while the second term is proportional to \hbar and is thus a genuine quantum gravity correction term. One can, of course, not measure this correction in the laboratory, because this would require that the two masses approach each other down to the Planck length. The main merit of this result (and others) is that a concrete prediction can be made.

Predictions can also be made from other approaches to quantum gravity. Here, I will discuss the case of canonical quantum gravity in geometrodynamical variables, on which I have worked myself. The central equation of this approach is the Wheeler–DeWitt equation, which is of the form [1]

$$\hat{H}\Psi = 0,\tag{1}$$

where \hat{H} denotes the full Hamiltonian of gravity and non-gravitational fields, and Ψ depends on the three-dimensional metric and matter variables.

In the following, I shall first briefly review the semiclassical approximation, on which the derivation of the quantum-gravitational effects is based. I then discuss the quantum-gravitational correction terms that lead to the modification of the power spectrum for the anisotropies of the cosmic microwave background (CMB). So far, the observations only lead to upper limits for these terms and I quote some of these limits using the most recent data from the PLANCK satellite mission. These results are based on the research articles [3] and [4], to which I refer the reader for more details.

2. Semiclassical approximation

In the semiclassical approximation to the Wheeler-DeWitt equation, one starts with the ansatz

$$|\Psi[h_{ab}]\rangle = C[h_{ab}] \mathrm{e}^{\mathrm{i}m_{\mathrm{P}}^2 S[h_{ab}]} |\psi[h_{ab}]\rangle \tag{2}$$

and performs an expansion with respect to the inverse Planck mass squared $m_{\rm P}^{-2}$. This is inserted into (1), and consecutive orders in this expansion are considered [5, 6]. This is close to the Born–Oppenheimer (BO) approximation scheme in molecular physics. One may also employ the direct BO scheme, which leads to analogous, but not equivalent results [7]. We shall here restrict ourselves to the $m_{\rm P}^2$ -expansion scheme. In (2), h_{ab} denotes the three-metric, and the Dirac bra-ket notation refers to the non-gravitational fields, for which the usual Hilbert-space structure is assumed [6].

The highest orders of the BO scheme lead to the following picture. One evaluates $|\psi[h_{ab}]\rangle$ along a solution of the classical Einstein equations, $h_{ab}(\mathbf{x}, t)$, which corresponds to a solution $S[h_{ab}]$ of the Hamilton–Jacobi equations. One can then define

$$\dot{h}_{ab} = NG_{abcd} \frac{\delta S}{\delta h_{cd}} + 2D_{(a}N_{b)},$$

where N is the lapse function and N^a is the shift vector. In this way, one can recover a classical spacetime as an approximation, a spacetime that satisfies Einstein's equations in this limit. The time derivative of the matter wave function is then defined by

$$\frac{\partial}{\partial t} |\psi(t)\rangle := \int \mathrm{d}^3x \, \dot{h}_{ab}(\mathbf{x},t) \, \frac{\delta}{\delta h_{ab}(\mathbf{x})} |\psi[h_{ab}]\rangle.$$

This leads to a functional Schrödinger equation for quantized matter fields in the chosen external classical gravitational field,

$$i\hbar \frac{\partial}{\partial t} |\psi(t)\rangle = \hat{H}^{m} |\psi(t)\rangle, \qquad (3)$$

$$\hat{H}^{\mathrm{m}} := \int \mathrm{d}^{3}x \left\{ N(\mathbf{x})\hat{\mathcal{H}}^{\mathrm{m}}_{\perp}(\mathbf{x}) + N^{a}(\mathbf{x})\hat{\mathcal{H}}^{\mathrm{m}}_{a}(\mathbf{x}) \right\},\tag{4}$$

where \hat{H}^{m} denotes the matter-field Hamiltonian in the Schrödinger picture, which depends parametrically on the (generally non-static) metric coefficients of the curved space-time background recovered from $S[h_{ab}]$. The 'WKB time' t controls the dynamics in this approximation.

3. Quantum-gravitational corrections

The next order in the $m_{\rm p}^{-2}$ -expansion yields quantum-gravitational correction terms [5, 6]. In this way, one finds a 'corrected' Schrödinger equation (3) in which the matter Hamiltonian is replaced according to

$$\hat{H}^{\mathrm{m}} \to \hat{H}^{\mathrm{m}} + \frac{1}{m_{\mathrm{P}}^2} \times (\text{various terms}).$$

The general correction terms can be found in explicit form in [6]. We shall specify them below for the case of our cosmological model. As an example, I mention the quantum-gravitational corrections to the trace anomaly in de Sitter space as calculated in [8],

$$\delta \epsilon \approx -\frac{2G\hbar^2 H_{\rm dS}^6}{3(1440)^2 \pi^3 c^8},$$

where H_{dS} denotes the constant Hubble parameter of de Sitter space. Again, one recognizes that concrete effects can be calculated, even if they are too tiny to be currently observable.

We shall now review the calculations that lead to the quantum-gravitational corrections for the CMB anisotropy power spectrum [3, 4]. We consider the Wheeler–DeWitt equation for small fluctuations in a flat Friedmann–Lemaître universe with scale factor $a \equiv \exp(\alpha)$ and inflaton field ϕ . In fact, we assume the presence of an inflationary regime in the semiclassical limit.

For the inflaton, we choose the simplest potential given by

$$\mathcal{V}(\phi) = \frac{1}{2} m^2 \phi^2 ;$$

however, any other potential obeying at the classical level the slow-roll condition $\dot{\phi}^2 \ll |\mathcal{V}(\phi)|$ should lead to similar results.

The Wheeler-DeWitt equation for the Friedmann-Lemaître ('minisuperspace') background reads

$$\mathcal{H}_0 \Psi_0(\alpha, \phi) \equiv \frac{\mathrm{e}^{-3\alpha}}{2} \left[\frac{1}{m_\mathrm{P}^2} \frac{\partial^2}{\partial \alpha^2} - \frac{\partial^2}{\partial \phi^2} + \mathrm{e}^{6\alpha} m^2 \phi^2 \right] \Psi_0(\alpha, \phi) = 0.$$
(5)

In the following, we choose units with $\hbar = c = 1$ and a modified Planck mass defined by $m_{\rm P} := \sqrt{3\pi/2G} \approx 2.65 \times 10^{19}$ GeV. The scalar field is redefined for convenience by $\phi \to \phi/\sqrt{2\pi}$.

To implement the slow-roll condition at the quantum level, we assume in the following that $\partial^2 \Psi_0 / \partial \phi^2 \ll e^{6\alpha} m^2 \phi^2 \Psi_0$, and we substitute $m\phi$ by $m_{\rm P}H$, where H is the quasistatic Hubble parameter of inflation (this holds in the Born–Oppenheimer approximation). Many investigations in quantum cosmology make use of this quantum slow-roll condition [1].

We now introduce inhomogeneities for the scalar field according to

$$\phi \to \phi(t) + \delta \phi(\mathbf{x}, t)$$

and perform a decomposition into Fourier modes with wave vector $\mathbf{k}, k \equiv |\mathbf{k}|$,

$$\delta \phi(\mathbf{x}, t) = \sum_{k} f_k(t) e^{i\mathbf{k}\cdot\mathbf{x}}.$$

A full treatment of the problem should make use of the gauge-invariant variables of cosmological perturbation theory, but this is not required for our purpose.

The Wheeler–DeWitt equation including the fluctuation modes then reads [9]

$$\left[\mathcal{H}_0 + \sum_{k=1}^{\infty} \mathcal{H}_k\right] \Psi(\alpha, \phi, \left\{f_k\right\}_{k=1}^{\infty}) = 0,$$
(6)

with

$$\mathcal{H}_{k} = \frac{1}{2} e^{-3\alpha} \left[-\frac{\partial^{2}}{\partial f_{k}^{2}} + \left(k^{2} e^{4\alpha} + m^{2} e^{6\alpha} \right) f_{k}^{2} \right].$$
(7)

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Since the fluctuations are small, they do not interact among each other, but only with the background. One can thus make the ansatz

$$\Psi(\alpha,\phi,\left\{f_k\right\}_{k=1}^{\infty}) = \Psi_0(\alpha,\phi)\prod_{k=1}^{\infty}\widetilde{\Psi}_k(\alpha,\phi,f_k).$$

The components $\Psi_k(\alpha, \phi, f_k) := \Psi_0(\alpha, \phi) \widetilde{\Psi}_k(\alpha, \phi, f_k)$ then obey the equation

$$\frac{1}{2}e^{-3\alpha}\left[\frac{1}{m_{\rm P}^2}\frac{\partial^2}{\partial\alpha^2} + e^{6\alpha}m_{\rm P}^2H^2 - \frac{\partial^2}{\partial f_k^2} + W_k(\alpha)f_k^2\right]\Psi_k(\alpha,\phi,f_k) = 0$$
(8)

with

$$W_k(\alpha) := k^2 e^{4\alpha} + m^2 e^{6\alpha}.$$

Following the general scheme outlined above, we make the ansatz

$$\Psi_k(\alpha, f_k) = \mathrm{e}^{\mathrm{i}\,S(\alpha, f_k)}$$

and expand $S(\alpha, f_k)$ in terms of powers of $m_{\rm P}^2$,

$$S(\alpha, f_k) = m_{\rm P}^2 S_0 + m_{\rm P}^0 S_1 + m_{\rm P}^{-2} S_2 + \dots$$

We insert this ansatz into the full Wheeler–DeWitt equation (6) and compare consecutive orders of m_P^2 . The highest order is $\mathcal{O}(m_P^4)$ and leads to S_0 being independent of the f_k . The order $\mathcal{O}(m_P^2)$ then demands that S_0 must obey the Hamilton–Jacobi equation

$$\left[\frac{\partial S_0}{\partial \alpha}\right]^2 - V(\alpha) = 0, \quad V(\alpha) := e^{6\alpha} H^2,$$

which is easily solved by $S_0(\alpha) = \pm e^{3\alpha} H/3$. At order $\mathcal{O}(m_P^0)$, we write $\psi_k^{(0)}(\alpha, f_k) \equiv \gamma(\alpha) e^{i S_1(\alpha, f_k)}$ and impose a condition on $\gamma(\alpha)$ that makes it equal to the standard WKB prefactor. After introducing the 'WKB time' according to

$$\frac{\partial}{\partial t} := -e^{-3\alpha} \frac{\partial S_0}{\partial \alpha} \frac{\partial}{\partial \alpha}, \qquad (9)$$

one finds that each $\psi_k^{(0)}$ obeys a Schrödinger equation,

$$i\frac{\partial}{\partial t}\psi_k^{(0)} = \mathcal{H}_k\psi_k^{(0)}.$$
(10)

This corresponds to the general case (3) above as specialized to our model.

The next order $\mathcal{O}(m_{\rm P}^{-2})$ then leads to the quantum-gravitational correction terms. We decompose for this purpose $S_2(\alpha, f_k)$ as

$$S_2(\alpha, f_k) \equiv \varsigma(\alpha) + \eta(\alpha, f_k)$$

and demand that $\varsigma(\alpha)$ be the standard second-order WKB correction. The wave functions

$$\psi_k^{(1)}(\alpha, f_k) := \psi_k^{(0)}(\alpha, f_k) e^{i m_{\rm P}^{-2} \eta(\alpha, f_k)}$$

then obey the quantum-gravitationally corrected Schrödinger equation

$$i\frac{\partial}{\partial t}\psi_{k}^{(1)} = \mathcal{H}_{k}\psi_{k}^{(1)} - \frac{e^{3\alpha}}{2m_{P}^{2}\psi_{k}^{(0)}} \left[\frac{\left(\mathcal{H}_{k}\right)^{2}}{V}\psi_{k}^{(0)} + i\frac{\partial}{\partial t}\left(\frac{\mathcal{H}_{k}}{V}\right)\psi_{k}^{(0)}\right]\psi_{k}^{(1)}.$$
(11)

The second correction term leads to a possible unitarity violation, which either may be neglected because it is small or may be absorbed into a redefinition of the wave function [4]. We can now calculate from (3) the modifications to the CMB power spectrum.

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4. Modification of the CMB power spectrum

We look for a solution of the uncorrected Schrödinger equation of the Gaussian form

$$\psi_k^{(0)}(t, f_k) = \mathcal{N}_k^{(0)}(t) \,\mathrm{e}^{-\frac{1}{2}\,\Omega_k^{(0)}(t)\,f_k^2},\tag{12}$$

which corresponds to the assumption that the fluctuations f_k are initially in their ground state. From this, one finds the usual power spectrum which is approximately given by

$$\mathcal{P}^{(0)}(k) := \frac{k^3}{2\pi^2} |\delta_k(t_{\text{enter}})|^2 \propto \frac{H^4}{|\dot{\phi}(t)|_t^2} \tag{13}$$

where 'exit' and 'enter' refer to the times when a particular modes leaves and re-enters the Hubble scale, respectively. One thus recovers also in this framework the (approximate) scale-invariant power spectrum following from inflation.

For the solution of the corrected Schrödinger equation (11), we make again a Gaussian ansatz and consider corrections in the exponent and in the normalization. The technical details are somewhat lengthy and can be found in [3, 4]. As a boundary condition, we assume that the corrections to (12) vanish for late times, corresponding to the idea that quantum-gravitational effects become negligible for the present universe.

When solving the corrected Schrödinger equation with this boundary condition, one encounters a subtlety [4]. There exist two solutions with the same asymptotics, one approaching the value zero for the corrections continuously, the other making a jump on the imaginary axis when approaching zero. The formal origin of this difference is the choice in the definition of the exponential integral that occurs in the solution of this equation. We shall give below the consequences for the power spectrum from both solutions.

The quantum-gravitational corrections to the power spectrum can be described by the ansatz

$$\mathcal{P}^{(1)}(k) = \mathcal{P}^{(0)}(k) C_k^2.$$
(14)

Following [10], we can write

$$C_k^2 = 1 + \delta_{\text{WDW}}^{\pm}(k) + \frac{1}{k^6} \mathcal{O}\left(\left(\frac{H}{m_{\text{P}}}\right)^4\right),\tag{15}$$

where $\delta^{\pm}_{\text{WDW}}(k)$ either takes the form

$$\delta_{\rm WDW}^+(k) = \frac{179.09}{k^3} \left(\frac{H}{m_{\rm P}}\right)^2,$$
(16)

which follows from the continuous solution, or the form

$$\delta_{\rm WDW}^{-}(k) = -\frac{247.68}{k^3} \left(\frac{H}{m_{\rm P}}\right)^2,\tag{17}$$

which follows from the discontinuous solution. One thus recognizes that the continuous solution leads to an *enhancement* of the power at large scales (small k), whereas the discontinuous solution leads to a *suppression* of power at large scales. One also recognizes that the power spectrum is no longer scale-invariant, but exhibits a characteristic k^{-3} -dependence.

Let us now consider the spectral index and its running. These parameters are defined by

$$n_s - 1 := \frac{\mathrm{d}\log\mathcal{P}}{\mathrm{d}\log k} \approx 2\eta - 4\epsilon - 3\delta_{\mathrm{WDW}}^{\pm} \tag{18}$$

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and

$$\alpha_s := \frac{\mathrm{d}n_s}{\mathrm{d}\log k} \approx 2(5\epsilon\eta - 4\epsilon^2 - \Xi^2) + 9\delta_{\mathrm{WDW}}^{\pm},\tag{19}$$

where we have introduced the slow-roll parameters

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$$\epsilon = -\frac{\dot{H}}{H^2} = \frac{4\pi G \left|\dot{\phi}(t)\right|_{t_{\text{exit}}}^2}{H^2} \tag{20}$$

and

$$\eta := -\frac{\ddot{\phi}}{H\dot{\phi}}, \qquad \Xi^2 := \frac{1}{H^2} \frac{\mathrm{d}}{\mathrm{d}t} \frac{\ddot{\phi}}{\dot{\phi}}.$$
(21)

So far, we have treated the scale k as dimensionless. For the comparison with observations, we introduce a reference wavenumber, which can either correspond to the largest observable scale, $k_{\rm min} \sim 1.4 \times 10^{-4} \,{\rm Mpc}^{-1}$, or to the pivot scale used in the WMAP9 analysis, $k_0 = 0.002 \,{\rm Mpc}^{-1}$ [11]. We now write $k/k_{\rm min}$ or k/k_0 instead of k. Since $H/m_{\rm P}$ has to be smaller than about 3.5×10^{-6} because of the observational bound on the tensor-to-scalar ratio r < 0.11 for $k_0 = 0.002 \,{\rm Mpc}^{-1}$ from the Planck 2013 results [12], we find that for $k \to k/k_0$ the absolute value of the quantum-gravitational corrections is bounded by [4]

$$\left|\delta_{\text{WDW}}^{+}(k_{0})\right| \lesssim 2.2 \times 10^{-9}, \ \left|\delta_{\text{WDW}}^{-}(k_{0})\right| \lesssim 3.0 \times 10^{-9},$$
 (22)

while with $k \rightarrow k/k_{\min}$ this limit becomes even more stringent,

$$\left|\delta_{\text{WDW}}^{+}(k_{0})\right| \lesssim 7.5 \times 10^{-13}, \ \left|\delta_{\text{WDW}}^{-}(k_{0})\right| \lesssim 1.0 \times 10^{-12}.$$
 (23)

These values are too tiny to be currently observable.

5. Outlook

In my contribution, I have tried to emphasize that one must derive concrete predictions from approaches to quantum gravity and compare them with observations. I have limited myself here to one approach: quantum geometrodynamics with the Wheeler–DeWitt equation as its central equation. From this, one can derive quantum-gravitational corrections to the CMB anisotropy power spectrum. Assuming a scenario with an inflationary regime in the early universe, these corrections depend on the ratio $(H/M_P)^2$, where H is the quasi-static Hubble parameter of inflation.

Given the known limits on this ratio, the corrections turn out to be too small to be observable. The situation will also not change with the release of further data from the PLANCK mission, since the main source of uncertainty for large scales comes from cosmic variance, which is a fundamental limit. One might, of course, speculate that the quantum-gravitational correction terms can be seen in other situations, such as the galaxy-galaxy correlation functions.

If one did not have the limit on $(H/M_P)^2$, the situation would be different. From the non-observation of the correction terms, one would then get a constraint that H/M_P must be smaller than a value of the order of 10^{17} GeV. It is evident that the quantum-gravitational effects would be stronger if the inflationary scale were closer to the Planck scale. If one attributed the power suppression at large scales, as reported in [12], to the quantum-gravitational corrections, one would predict for H/M_P a value of this order.

Other approaches to quantum gravity have also been applied to this situation. Investigations in loop quantum cosmology, for example, lead to a prediction that the power at large scales is enhanced [10, 13]. The observation of small effects may thus eventually provide the key for the correct quantum theory of gravity [14].

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