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GRAVITATIONAL RADIATION WITH A POSITIVE COSMOLOGICAL CONSTANT

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Abstract

Gravitational radiation is well-understood in spacetimes that are asymptotically flat. However, our Universe is currently expanding at an accelerated rate, which is best described by including a positive cosmological constant, Λ, in Einstein's equations. Consequently, no matter how far one recedes from sources generating gravitational waves, spacetime curvature never dies and is not asymptotically flat. This dissertation provides first steps to incorporate Λ in the study of gravitational radiation by analyzing linearized gravitational waves on a de Sitter background.

Since the asymptotic structure of de Sitter is very different from that of Minkowski spacetime, many conceptual and technical difficulties arise. The limit $\Lambda \to 0$ can be discontinuous: Although energy carried by gravitational waves is always positive in Minkowski spacetime, it can be arbitrarily negative in de Sitter spacetime. Additionally, many of the standard techniques, including $1/r$ expansions, are no longer applicable.

We generalize Einstein's celebrated quadrupole formula describing the power radiated on a flat background to de Sitter spacetime. Even a tiny Λ brings in qualitatively new features such as contributions from pressure quadrupole moments. Nonetheless, corrections induced by Λ are $\mathcal{O}(\sqrt{\Lambda}t_c)$ with t_c the characteristic time scale of the source and are negligible for current gravitational wave observatories. We demonstrate this explicitly for a binary system in a circular orbit.

Radiative modes are encoded in the transverse-traceless part of the spatial components of a gravitational perturbation. When $\Lambda = 0$, one typically extracts these modes in the wave zone by projecting the gravitational perturbation onto the two-sphere orthogonal to the radial direction. We show that this method for waves emitted by spatially compact sources on Minkowski spacetime generically does not yield the transverse-traceless modes; not even infinitely far away. However, the difference between the transverse-traceless and projected modes is non-dynamical and disappears from all physical observables. When one is interested in 'Coulombic' information not captured by the radiative modes, the projection method does not suffice. This is, for example, important for angular momentum carried by gravitational waves. This result relies on Bondi-type expansions for asymptotically flat spacetimes. Therefore, the projection method is not applicable to de Sitter spacetimes.

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Preamble

> "At this point we must say a few words about the famous *lambda*. The field-equations, in their most general form, contain a term multiplied by a constant, which is denoted by the Greek letter λ (lambda), and which is sometimes called the "cosmological constant". This is a name without any meaning, which was only conferred upon because it was thought appropriate that it should have a name, and because it appeared to have something to do with the constitution of the universe; but it must not be inferred that, since we have given it a name, we know what it means. We have, in fact, not the slightest inkling of what its real significance is. It is put in the equations in order to give the greatest possible degree of mathematical generality, but, so far as its mathematical function is concerned, it is entirely undetermined: it may be positive or negative, it might also be zero. Purely mathematical symbols have no meaning by themselves; its is the privilege of pure mathematician, to quote Bertrand Russell, not to know what they are talking about. They — the symbols — only get a meaning by the interpretation that is put on the equations when they are applied to the solution of physical problems. It is the physicist, and not the mathematician, who must know what he is talking about. "

Willem de Sitter, in *Kosmos, a course of six lectures on the development of our insight into the structure of the Universe* delivered for the Lowell Institute in Boston in November 1931 [1]

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Another collaborator whom I ought to thank is Aruna Kesavan. I have had the good fortune of working with her extensively and I fondly look back at the great team we formed. Aruna was dearly missed during the final year of my PhD. In the first year of my PhD, I connected with Brajesh Gupt during a conference in Warsaw. Initially we were merely acquaintances, but due to a twist in fate he became my next-door-office member and as result of many enticing discussions one of my collaborators. I am not only thankful for the many physics discussions we had, but I am also grateful for the times we spend talking about travel, food, culture and brewing beer. In a later stage, we were joined by Nelson Yokomizo whose broad knowledge helped to a better understanding of many physics concepts. Yet, my time at Penn State most certainly would not have been the same without the interesting interactions with my dear colleagues Anne-Sylvie Deutsch and Suddho Brahma.

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During the final year of my PhD, I got to visit the Perimeter Institute in Canada and the Horace Hearne Jr. Institute for Theoretical Physics at Louisiana State University, each for two months. The people at both institutions deserve a shout-out for their hospitality. In addition, I would like to extend my gratitude to the general relativity, cosmology and quantum gravity communities. They made me feel welcome from the very first instant and the intellectually stimulating environment they created through discussions and conferences has been one of a kind.

Last but not least, I would like to thank those who are closest to me for their unconditional love and support. My parents have always given me the freedom to choose my own path and I can only say that I am eternally grateful for that. These last two months of my PhD were made even more special by my sister Caroline visiting, whom brought more color (and desserts :)) into my life in her own wonderfully chaotic and lively manner. I would also like to thank my brother Rutger for 'dropping me off' at the start of my PhD and sharing many fun facts about the antiquities. It goes without saying that I will forever cherish the weekly Sunday Skype sessions with my grandmothers. They always brightened my day with their anecdotes, love and genuine interest. On a final note, I would like to thank Makoto-chan for supporting me to pursue my dreams. You have been my rock from the other side of the ocean and I will forever cherish all the travel we undertook together during my PhD.

Chapter 1 Introduction

1.1 Gravitational waves in our expanding Universe

We live in exciting times: the recent gravitational wave detections by LIGO mark the beginning of the era of gravitational wave astronomy [2–4]. LIGO's first observation of gravitational waves took place on September 14, 2015. The observed gravitational waves are believed to have been emitted during the coalescence of two back holes that took place 1.34 billion years ago. This observation was not just the first direct detection of gravitational waves, but was also the first observation of two black holes orbiting each other, the first observation of the merger of two black holes and the first observation of a black hole with a mass of $62M_\odot$ (which is much heavier than the masses of stellar-mass black holes determined using X-ray observations, which all fall in the range $5-30M_\odot$ [5]). The second and third events LIGO detected were also a result of the coalescence of two stellar-mass black holes. As the number of gravitational wave observations resulting from the merger of two black holes increases, we will gain valuable insights about the astrophysical formation process of black holes, the progenitors of (stellar-mass) black hole binaries and their abundance in our Universe [6, 7]. The expectation is that gravitational wave observations are not limited to those produced during a black hole binary coalescence. The merger of two neutron stars or a neutron star with a black hole also fall within the observable range of LIGO and future gravitational wave detectors. Once these events are observed, they will likely inform us about various interesting properties of neutron stars such as its equation of state [8–12]. This in turn will provide insight on how matter behaves under extreme densities and pressures. Furthermore, other events such as supernovae and events not yet known to us may potentially be observed with gravitational waves. Thus, gravitational wave observations will contribute to our understanding of the Universe we inhabit.

Given the recent detections, it is difficult to believe that the reality of gravitational waves was debated over several decades! A year after Einstein published the equations of general relativity [13], he provided the first relativistic description of gravitational waves by linearizing these equations off a Minkowski background in the presence of an external, time-changing source [14].[1] However, about twenty years later, Einstein wrote to Max Born that gravitational waves are artifacts of the linear approximation and do not exist in full, non-linear general relativity:

> "Together with a young collaborator [Nathan Rosen], I arrived at the interesting result that gravitational waves do not exist, though they had been assumed a certainty to the first approximation. This shows that the non-linear general relativistic field equations can tell us more or, rather, limit us more than we have believed up to now." [16]

Einstein came to this conclusion based on work he had done with Rosen in which they thought that they had shown that for some gravitational waves that are exact solutions in general relativity, one was always faced with singularities. However, Einstein and Rosen had made a mistake. The 'singularities' they found were coordinate artifacts and disappeared once the appropriate coordinates were used. Although this mistake was pointed out by a referee and the claim that gravitational waves do not exist did not appear in the final published version of the paper, confusion persisted. Indeed Rosen continued to believe that gravitational waves do not exist for several more decades. This confusion was finally removed by the work of Bondi and Sachs in the early 1960s and the community reached a consensus on the reality of gravitational waves [17–21]. Bondi, Sachs and others developed rigorous mathematical methods to show that gravitational wave are indeed physical as they radiate energy out to infinity. Their key insight was to use an expansion along null directions. Some fifteen years after their seminal work, gravitational waves were observed, albeit indirectly. Hulse and Taylor measured the orbital decay rate of the period of a binary system composed of a pulsar (i.e., a highly magnetized neutron star) and a neutron star orbiting a common center of mass [22]. The

[1] The idea of gravitational waves was explored before by Lagrange and Poincaré among others (for a review, see [15]).

observed orbital decay rate matched very accurately the prediction from general relativity based on the emission of gravitational radiation (today the agreement is to an accuracy of 0.3 %! [23]). The recent *direct* detections by LIGO completely settles the issue and unequivocally shows that gravitational waves are physical.

This is not just an exciting time for gravitational wave science. In the last century, many revolutionary ideas replaced incorrect preconceptions about our Universe as technology advanced and cosmological observations became possible. Until the early 1950s, the steady state model – in which the observable Universe remains unchanged – was popular among many leading physicists. However, observations of quasars, abundances of light elements and the cosmic microwave background refuted the steady state model and instead favored the hot Big Bang model, which describes the Universe as continuously expanding. In fact, in 1998, supernovae measurements showed that today the Universe is even expanding in an accelerated fashion [24, 25]. The current accelerated expansion has now also been supported by cosmic microwave background observations [26]. The simplest and most successful explanation of this acceleration is to assume that there is a positive but small cosmological constant Λ, which in geometrized units is given by $\Lambda \sim 10^{-52}$ m^{-2}. This is potentially problematic for gravitational wave science as current gravitational wave theory is based on a framework with $\Lambda = 0$. Given the smallness of Λ, one may expect that the $\Lambda = 0$ framework should suffice and that Λ only introduces small corrections. However, the rich structure of the $\Lambda = 0$ framework is lost when Λ is no longer zero, *irrespective of how small* Λ *is!* For instance, there is no analog of the Bondi news tensor [27–29] that describes gravitational radiation in a gauge invariant way in full, non-linear general relativity with $\Lambda = 0$ [30, 31]. Additionally, the radiation field Ψ_4^0 that is heavily used in both analytical discussions of gravitational radiation and numerical simulations in the $\Lambda = 0$ context acquires an ambiguity in the $\Lambda > 0$ case, called the 'origin dependence' by Penrose [32, 33] and 'direction-dependence' by Krtouš and Podolský [34]. Furthermore, the peeling theorem that describes the asymptotic behavior of the Weyl tensor for a large class of solutions to Einstein's equations with $\Lambda = 0$ does not apply when $\Lambda > 0$. Thus, a new theoretical platform to study gravitational radiation in spacetimes with $\Lambda > 0$ is needed.

The extension of the framework to study gravitational radiation with $\Lambda = 0$ to $\Lambda > 0$ is the main theme of this dissertation. In view of the smallness of the observed value of Λ – despite the dramatic conceptual differences and the lack of the rich structure of the $\Lambda = 0$ framework – the effect of Λ is expected to be small for gravitational wave observations. For instance, the expectation is that Λ does not significantly alter the radiation emitted by the Hulse-Taylor system and thereby its orbital decay rate because of a natural separation of scales: Its orbital period is small (7.75 hours) compared to the time scale associated with Λ (10^{21} hours) and at a luminosity distance of 6.4 kpc, it is nearby from a cosmological perspective. But a clean framework is needed to say with confidence how small the corrections are that Λ introduces, especially, when the sources are at cosmological separations and gravitational waves have been traveling distances of the order of a Gpc before reaching us. Much of the work that addresses the inclusion of $\Lambda > 0$ in the study of gravitational radiation has already been published in a series of papers with my advisor, Professor Abhay Ashtekar, and Dr. Aruna Kesavan [31, 35–37] as well as in Dr. Kesavan's dissertation [38]. In this dissertation, we only discuss the third paper of this series in detail.[2] The main results from the first two papers will be summarized in the remaining part of this introduction. The third paper appears in Chapter 2 and describes at the linearized level gravitational radiation generated by a time-varying source using a multipole expansion for $\Lambda > 0$. Various new techniques are needed: the standard $1/r$ expansion does not apply when Λ is no longer zero nor can one extract the radiative degrees of freedom by projecting the gravitational wave onto the two-sphere orthogonal to the radial direction, as is typically done in gravitational wave theory. Using newly developed methods, we also derive a general formula for power radiated by an arbitrary source. This general formula for power radiated is applied to a concrete physical example in Chapter 3, where we study the power radiated by a binary system in a circular orbit. The power radiated by this system on a de Sitter background is the same in the high-frequency limit as the power radiated in Minkowski spacetime, despite the fact that the two notions of power refer to different Killing vector fields. This chapter is based on work in collaboration with Dr. Jeffrey Hazboun [39]. In Chapter 4, we discuss why in the context of linearized perturbations on a Minkowski background one *is* able to extract the radiative degrees of freedom very far away from the source by projecting the gravitational perturbation onto the two-sphere. A priori, there

[2]One extra section, Section 2.3.3, has been added to the published version that includes additional insights regarding the tail term that have not been published before.

is no reason why this method should work: The radiative degrees of freedom are encoded in the gauge-invariant part of the gravitational perturbation, which is the transverse-traceless part of the spatial components of the gravitational perturbation and not just the components on the spatial two-sphere. This is easily understood by realizing that taking the transverse-traceless part of any spatial tensor is a local operation in momentum space, but *global* in real space. Projecting the gravitational perturbation on the two-sphere orthogonal to the radial direction, on the other hand, is a *local* operation in real space. Nonetheless, this method has been used in the literature repeatedly while a proof of the validity of this method existed only for plane waves. Waves emitted by astrophysical sources such as neutron stars or binary systems composed of two black holes, are not plane waves. Thus, this specific equivalence of the two notions of transversality is not physically relevant. For waves emitted by compact sources, we show in fact that the two notions of transversality in general do not agree – not even very far away from the sources! In particular, this is true for the radiative modes extracted by both methods. However, these radiative modes only differ by a function on the 2-sphere far away from the sources and thus they contain the same dynamical information. As a result, all physical quantities extracted from only the radiative modes using either notion of transversality agree in the limit that one is infinitely far away from the sources. Yet there is one surprising feature that arises due to the presence of sources. Angular momentum carried by gravitational waves *cannot* be expressed using merely the radiative modes; additional 'Coulombic information' is needed. Since projecting the gravitational perturbation onto the two-sphere only captures the two radiative degrees of freedom, this projection method is not sufficient to determine angular momentum radiated. On the other hand, the transverse-traceless part of the spatial components of the gravitational perturbation provides additional fields that carry the relevant information needed to determine angular momentum radiated. We will also comment why the two notions of transversality do not agree for spacetimes with a positive Λ. This is based on joint work with Prof. Ashtekar [40]. This dissertation concludes with a summary and outlook in Chapter 5. In Appendix A and B, some of the details of the calculations are provided.

During my PhD, I have also worked on several issues in early Universe cosmology. The publications that resulted from these projects address an entirely different class of issues and are not included in this dissertation. A summary of these papers is provided in Appendix C. For a list of all publications during my PhD, see the vita attached at the end of this dissertation.

We use the following conventions. Throughout we assume that the underlying spacetime is 4-dimensional and set c=1. The spacetime metric has signature -,+,+,+. The curvature tensors are defined via: $2\nabla_{[a}\nabla_{b]}k_c = R_{abc}{}^d k_d$, $R_{ac} = R_{abc}{}^b$ and $R = g^{ab}R_{ab}$. We use Penrose's abstract index notation [33, 41]: $a, b, \ldots$ will be the abstract indices labeling tensors while indices $\bar{a}, \bar{b}, \ldots$ will be numerical indices. In particular, components of a tensor field T_{ab} (in a specified chart) are denoted by $T_{\bar{a}\bar{b}}$.

1.2 Even a tiny Λ can cast a long shadow

The influential work by Bondi, Sachs and collaborators used a systematic expansion of the metric as one moves in null directions away from the sources generating gravitational waves [17–21]. The idea being that as one moves far away spacetime becomes approximately flat and one can disentangle gravitational radiation from the background curvature. One cannot make this distinction between radiation and background curvature *near* the source(s), as general relativity does not provide a canonical decomposition of spacetime into a background and a dynamical perturbation. This interplay between geometry and the gravitational field makes general relativity both the challenging and interesting theory that it is. It is also one of the main reasons for the long standing confusion on whether gravitational waves are true physical phenomena or mere coordinate artifacts that can be gotten rid off by a coordinate transformation.

For a large class of spacetimes with $\Lambda = 0$ satisfying boundary conditions that capture the idea that sources decay, the metric g_{ab} approaches a flat Minkowski metric η_{ab} as one recedes from the sources along null directions:

$$g_{ab} = \eta_{ab} + \mathcal{O}\left(\frac{1}{r}\right) \tag{1.1}$$

where the fall-off behavior is with respect to the Cartesian chart of the flat metric η_{ab}. However, this η_{ab} is not canonical in the presence of gravitational waves. In fact, there are *infinitely* many Minkowskian metrics that g_{ab} approaches to. For instance, instead of eq. (1.1), as one moves far away from the sources along a null direction, g_{ab} could also be written as

$$g_{ab} = \eta'_{ab} + \mathcal{O}\left(\frac{1}{r'}\right) \tag{1.2}$$

where $\eta'_{ab} \neq \eta_{ab}$. The two Minkowski metrics are not the same, but differ only by $\mathcal{O}(\frac{1}{r})$ (or equivalently, $\mathcal{O}(\frac{1}{r'})$). To make this explicit, let us label the Cartesian coordinates of η_{ab} by (t, x, y, z) and those of η'_{ab} by (t', x', y', z'). A possible relation between the two coordinate systems is: $t' = t + a(\theta, \varphi)$ with $a(\theta, \varphi)$ an arbitrary function on the two-sphere and $x' = x, y' = y$ and $z' = z$. For instance, take $a(\theta, \varphi) = \cos\theta$. Then a straightforward calculation shows that $\eta_{ab} = \eta'_{ab} + \mathcal{O}(\frac{1}{r'})$. Spacetimes with this fall-off behavior are called asymptotically flat spacetimes. When Λ is no longer zero, but positive, no matter how far one moves away from the source, the spacetime curvature never dies off. Ergo, the metric does not approach a Minkowski metric. Instead it seems natural to assume that far away from the source the metric approaches a de Sitter metric (which is the maximally symmetric vacuum solution of Einstein's equations when $\Lambda > 0$, just as the Minkowski metric is the maximally symmetric vacuum solution when $\Lambda = 0$). However, using a Bondi-type expansion for axi-symmetric solutions to Einstein's equations, it was shown in [42] that the metric only approaches the de Sitter metric in the absence of gravitational radiation. Thus, if we allow gravitational waves, the metric differs from the de Sitter metric far away from the sources already at leading order in the Bondi-type expansion! This illustrates how drastically different the study of gravitational radiation in the presence of a positive Λ is in comparison to when $\Lambda = 0$.

Penrose removed the coordinate dependent description of Bondi et al by reformulating the metric expansions along null directions in more geometrical terms using conformal techniques [43]. He showed that by rescaling the physical metric g_{ab} by a conformal factor Ω such that the rescaled metric is $\tilde{g}_{ab} = \Omega^2 g_{ab}$, one can attach a boundary to spacetime. This boundary is the surface on which $\Omega = 0$ and is denoted by $\mathcal{I}$ (pronounced as 'scri'). Physically, it is the place where all null geodesics end and it is the natural arena to study gravitational radiation. On $\mathcal{I}$, gravitational radiation can be extracted unambiguously in a gauge-invariant manner and one cannot confuse the 'background' with gravitational waves. For asymptotically flat spacetimes, Einstein's equations tell us that the normal vector to $\mathcal{I}$, given by $n_a := \nabla_a \Omega$, has a vanishing norm. Hence, $\mathcal{I}$ is a null surface in this scenario. However, for asymptotically de Sitter spacetimes (the analog of asymptotically flat spacetimes but with $\Lambda > 0$), one finds that $\mathcal{I}$ is not a null boundary, but a spatial boundary. This changes the standard framework of gravitational radiation with $\Lambda = 0$ drastically when incorporating a positive cosmological constant.

Most importantly, since $\mathcal{I}$ of asymptotically de Sitter spacetimes is space-like, its normal vector is time-like and does not induce any structure on this surface. This is in contrast to asymptotically flat spacetimes for which $\mathcal{I}$ is ruled by its normal vector n^a that endows $\mathcal{I}$ with additional structure (for null surfaces, its normal vector is also tangential to the surface). The null normal is one of the two key players in the universal structure of asymtotically flat spacetimes. The other being the induced metric on $\mathcal{I}$. To be precise, the universal structure of any asymptotically flat spacetime is given by a conformal class of 3-dimensional, null metrics on $\mathcal{I}$, q_{ab}, and its normal: $\{q_{ab}, n^a\} = \{\omega^2 q_{ab}, \omega^{-1} n^a\}$. A transformation that preserves this universal structure is considered an asymptotic symmetry. The group of all these transformations is the asymptotic symmetry group. In other words, the asymptotic symmetry group is the quotient of the group of diffeomorphisms on the physical spacetime that preserve the boundary conditions by its subgroup of diffeomorphisms that are asymptotically identity. If $\mathcal{I}$ had no special structure, this group would simply be the infinite-dimensional group of all diffeomorphisms on $\mathcal{I}$, Diff($\mathcal{I}$). Due to the structure induced by the normal vector n^a for asymptotically flat spacetimes, its asymptotic symmetry group is not Diff($\mathcal{I}$) but the Bondi-Metzner-Sachs (BMS) group.[3] The BMS group mimics the structure of the Poincaré group: The Poincaré group is the semi-direct product of the group of translations with the Lorentz group and the BMS group is the semi-direct product of the group of *supertranslations* with the Lorentz group. Supertranslations can be thought of as angle dependent translations. Thus, the BMS group is larger than the Poincaré group. This enlargement of the group is due to the non-linearities of Einstein's equations and does not appear if one would work with the linearized approximation. This is closely related to the fact that there is no canonical flat metric η_{ab} on $\mathcal{I}$. The BMS group has a four-dimensional normal subgroup and therefore admits a unique translation subgroup [21]. Asymptotic symmetries play a crucial role in general relativity

[3] If a spacetime possesses a Killing vector field, this Killing vector field naturally extends to a BMS vector field on $\mathcal{I}$.

because they give a physical interpretation to certain conserved quantities on $\mathcal{I}$. For instance, the conserved quantities associated with the translation subgroup of the BMS group correspond to energy and momentum radiated by gravitational waves. An application of the calculation of momentum radiated, for instance, is black hole kicks that have been studied in detail in numerical relativity and are of astrophysical interest [44, 45].

When considering asymptotically de Sitter spacetimes, the loss of the additional structure provided by the normal to $\mathcal{I}$ is severe. The asymptotic symmetry group is no longer reduced from $\mathrm{Diff}(\mathcal{I})$ to a smaller group like the BMS group. Instead, the asymptotic symmetry group remains the infinite-dimensional group of all diffeomorphisms on $\mathcal{I}$. Consequently, one cannot extract physics from the asymptotic behavior of the gravitational field at $\mathcal{I}$ in the same way as for asymptotically flat spacetimes. It is unclear which fluxes to interpret as energy (or momentum or angular momentum) as there are no special vector fields. A natural solution to this problem would be to strengthen the boundary conditions to remedy this lack of structure. This approach has been applied successfully for asymptotically anti-de Sitter spacetimes (for which $\Lambda < 0$). For asymptotically anti-de Sitter spacetimes, in addition to the standard boundary conditions that mimic those for asymptotically flat spacetimes, one also requires the intrinsic metric on $\mathcal{I}$ to be conformally flat. This reduces the asymptotic symmetry group from $\mathrm{Diff}(\mathcal{I})$ to the anti-de Sitter group. Implementing the same condition for asymptotically de Sitter spacetimes, seems promising at first as this nicely reduces the asymptotic symmetry group to the de Sitter group $SO(1,4)$.[4] In addition, various exact solutions with a positive Λ satisfy this condition such as Kerr-de Sitter and Friedmann-Lemaître-Robertson-Walker (FLRW) spacetimes. However, this strengthening of the boundary condition throws out the baby with the bath water. If the intrinsic metric on $\mathcal{I}$ is conformally flat, then the leading order part of the magnetic part of the Weyl tensor, $\mathcal{B}_{ab}$, vanishes on $\mathcal{I}$. We showed that the vanishing of $\mathcal{B}_{ab}$ implies that all fluxes across $\mathcal{I}$ vanish as well. Thus, this extra boundary condition does not rule out gravitational radiation entirely, but severely restricts it as the radiation that reaches $\mathcal{I}$ does not carry any flux of energy, momentum or angular momentum.[5] Thus, this condition

[4] Or a subgroup of $SO(1,4)$ depending on the topology of $\mathcal{I}$. For details on this technical point, see Section 4.3 in [31].

[5] This is explicitly illustrated by studying the solutions homogeneous perturbations on a de Sitter background. These solutions have two independent modes that are, in the cosmology

is not viable as it does not allow one to study properties of the gravitational field on $\mathcal{I}$.

Why is this condition that is so advantageous for asymptotically anti-de Sitter spacetimes not useful for asymptotically de Sitter spacetimes? The answer has (again) to do with the nature of $\mathcal{I}$. Asymptotically anti-de Sitter spacetimes have a time-like $\mathcal{I}$. Therefore, to make the evolution well-defined additional input on $\mathcal{I}$ is needed. Requiring the intrinsic metric on $\mathcal{I}$ to be conformally flat, or equivalently $\mathcal{B}_{ab}$ to be zero on $\mathcal{I}$, makes the evolution well-defined. It can be regarded as a reflective boundary condition. (Gravitational waves in this case also do not carry away energy-momentum or angular momentum across $\mathcal{I}$, which is not surprising given the reflective nature of this boundary condition.) Thus, the space-like nature of $\mathcal{I}$ is what makes asymptotically de Sitter spacetimes so unique. This point is also illustrated by evolving Maxwell's equations on a de Sitter background. Requiring that the magnetic field B_a vanishes on any arbitrary space-like slice (including $\mathcal{I}$) is a severe restriction on the set of solutions: It cuts the set of allowable solutions in half. In contrast, setting B_a to zero on a time-like slice is a mild condition.

In summary, at this point, the study of gravitational radiation with a positive Λ in full, non-linear general relativity is at an impasse. Not imposing any additional structure on $\mathcal{I}$ leaves one with an asymptotic symmetry group that is so large that one cannot extract any physics from it. Whereas imposing a natural extra boundary condition, which also has successfully been used for asymptotically anti-de Sitter spacetimes, is so strong that it rules out any fluxes across $\mathcal{I}$. A new framework is being constructed to overcome this difficulty and an overview of the main ideas can be found in [46]. Especially given the past controversy regarding the physical reality of gravitational waves, a thorough understanding of gravitational radiation when $\Lambda > 0$ in non-linear general relativity is important.

1.2.1 Linearized gravity off de Sitter spacetime

The issue regarding the lack of a special group on $\mathcal{I}$ is naturally circumvented in the linearized approximation. In this case, one can use the isometries of the de Sitter background metric and introduce gravitational waves as perturbations

literature, often referred to as the 'decaying' and 'growing' mode. The decaying mode vanishes on $\mathcal{I}$, while the growing mode is non-zero on $\mathcal{I}$. The magnetic part of the Weyl tensor vanishes on $\mathcal{I}$ only if the growing mode is set to zero. This demonstrates the gravity of this condition: half the degrees of freedom of the gravitational perturbation are killed *by fiat*!

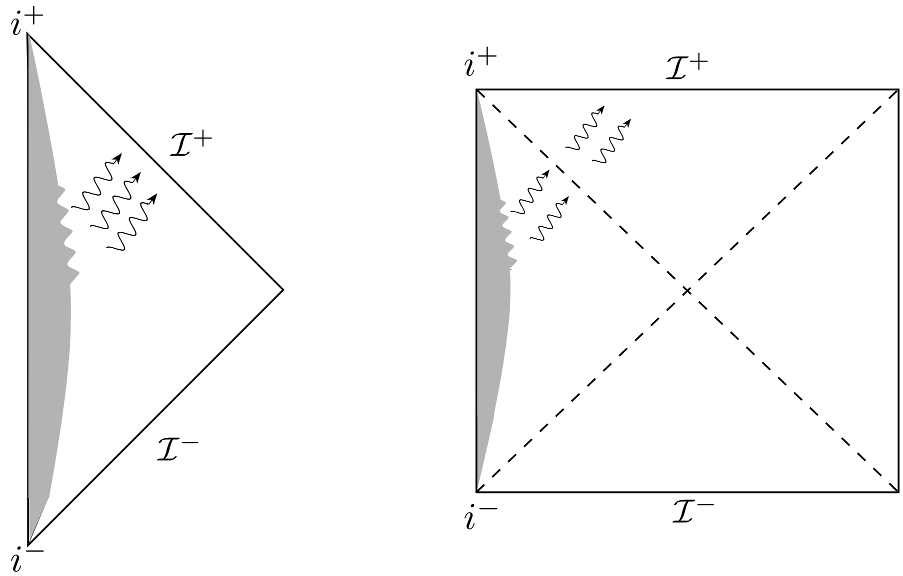

Figure 1.1. On the left, the conformal diagram representing a star when $\Lambda = 0$ is illustrated. The right diagram shows a spacetime with a star when $\Lambda > 0$. It is immediate from these diagrams that when $\Lambda = 0$, $\mathcal{I}$ is null and when $\Lambda > 0$, $\mathcal{I}$ is space-like.

on this background metric. Therefore, we have analyzed linearized fields on a de Sitter background in detail. The lessons learned from this analysis serve as important checks in the final construction of the framework that allows the study of gravitational radiation when $\Lambda > 0$. This is a similar approach to the development of the framework of asymptotically flat spacetimes that was only built after one had a strong intuition from working with linearized fields on a Minkowski background. Here, I will summarize the results from our study of linearized *homogeneous* gravitational perturbations on a de Sitter background. The rest of this dissertation is (mainly) dedicated to the study of gravitational waves generated by a source.

First, before studying perturbations on a de Sitter background, let us contrast the causal structure of Minkowski with de Sitter spacetime. The ultimate aim is to develop a framework to study gravitational radiation emitted by an isolated system such as a single star or binary system. Therefore, let us focus on contrasting an isolated system, confined to a spatially bounded world tube for all time, in spacetimes with $\Lambda = 0$ and $\Lambda > 0$. The conformal diagram of both these spacetimes is depicted in Figure 1.1. The matter world tube has future and past end-points in both situations, denoted by $i^{\pm}$. However, while the future of i^- in the $\Lambda = 0$ case is the entire Minkowski spacetime, for $\Lambda > 0$, it is only the future Poincaré patch of de Sitter. Thus, the causal future of this source covers only the future Poincaré patch, $M_{\rm P}^{+}$. No observer whose world line is confined to the past Poincaré patch can see the isolated system or detect the radiation it emits. Therefore, to study this system, it suffices to restrict oneself to the future Poincaré patch $M_{\rm P}^{+}$ rather than the full de Sitter spacetime. In conformal coordinates (η, x, y, z) adapted to this patch, the de Sitter metric $\bar{g}_{ab}$ takes the form

$$\bar{g}_{ab} = a^2(\eta)\mathring{g}_{ab} \quad \text{with} \quad \mathring{g}_{\bar{a}\bar{b}}\mathrm{d}x^{\bar{a}}\mathrm{d}x^{\bar{b}} = (-\mathrm{d}\eta^2 + \mathrm{d}\vec{x}^2)\,, \tag{1.3}$$

where the scale factor is $a(\eta) = -(H\eta)^{-1}$ and the Hubble parameter H is related to the cosmological constant Λ by $H := \sqrt{\Lambda/3}$. The comoving spatial coordinates span the entire real line $(-\infty,\, \infty)$, while the conformal time coordinate η takes values in $(-\infty,\, 0)$. From this form, it is obvious that the de Sitter spacetime is *locally* conformally related to Minkowski spacetime with the scale factor playing the role of the conformal factor. Hence, locally the null cone structure is in the same in both spacetimes. *Globally*, the two spacetimes are very different as is evident from their conformal diagrams.

One of the important lessons from our study of gravitational perturbations is that the limit $\Lambda \to 0$ is subtle. This subtlety can already be illustrated by examining the de Sitter background metric: Taking $\Lambda \to 0$ of the metric in (1.3) is not well-defined. One needs to use the differential structure induced on the Poincaré patch $M_{\rm P}^{+}$ by the $(t,\, \vec{x})$ coordinates rather than the $(\eta,\, \vec{x})$ coordinates, where the proper time t is related to the conformal time η via $H\eta = -e^{-Ht}$.[6] In the $(t,\, \vec{x})$ chart, we obtain in the limit $\Lambda \to 0$

$$\bar{g}_{\bar{a}\bar{b}}\mathrm{d}x^{\bar{a}}\mathrm{d}x^{\bar{b}} = -\mathrm{d}t^2 + e^{2Ht}\,\mathrm{d}\vec{x}^2 \quad \to \quad -\mathrm{d}t^2 + \mathrm{d}\vec{x}^2 =: \mathring{\eta}_{\bar{a}\bar{b}}\mathrm{d}x^{\bar{a}}\mathrm{d}x^{\bar{b}}\,. \tag{1.4}$$

Note that the Minkowski metric $\mathring{\eta}_{ab}$ is distinct from $\mathring{g}_{ab}$ in (1.3), which is also flat, because each of the Cartesian coordinates of $\mathring{\eta}_{ab}$ takes the full range of values, while η in $\mathring{g}_{ab}$ is restricted to the (negative) half plane.

In addition, gravitational perturbations on a de Sitter background acquire a

[6] The $(t, \vec{x})$ chart is not the only chart in which the limit $\Lambda \to 0$ is well-defined. In fact, any de Sitter coordinate system that is smoothly related to the $(t, \vec{x})$ chart in the limit $\Lambda \to 0$ is a valid chart to take the limit. The differentiable structure of the coordinates determines whether the limit is well-defined. For instance, the well-known static coordinate chart is another example in which the limit $\Lambda \to 0$ is well-defined.

hereditary term and propagate not only sharply along the null cone but also inside the null cone. This is different from linearized perturbations on a Minkowski background, which propagate strictly along the null cone. How does this come about? Although the de Sitter metric is conformally flat, the equation satisfied by the metric perturbation is *not* conformally invariant. As a result, the expression for gravitational perturbations obtains a tail term due to the backscattering off the de Sitter curvature.[7]

Furthermore, we derived formulas for energy-momentum and angular momentum carried by gravitational waves on a de Sitter background. Conserved quantities for test matter, such as scalar, Maxwell or Yang-Mills fields, can be readily constructed using their stress-energy tensor. Linearized gravitational fields, on the other hand, do not have a gauge invariant, local stress-energy tensor because in general relativity gravity is absorbed into spacetime geometry. Therefore, a new strategy is needed. A convenient route is provided by the covariant Hamiltonian framework. The Hamiltonian framework has also been used effectively for gravitational waves in full, non-linear general relativity with $\Lambda = 0$, where it leads to flux integrals associated with the BMS asymptotic symmetries [47]. Here, the covariant phase space $\Gamma_{\rm Cov}$ – that forms the foundation of this framework – consists of the gauge fixed solutions γ_{ab} to the linearized Einstein's equations in the presence of a positive Λ. Each Killing vector field K on $M_{\rm P}^{+}$ naturally defines a vector field $\mathcal{K}$ on $\Gamma_{\rm Cov}$ via $\mathcal{K} := \mathcal{L}_K \gamma_{ab}$.[8] The flow generated by the vector field $\mathcal{K}$ preserves the natural symplectic structure ω of $\Gamma_{\rm Cov}$ and thus generates a one-parameter family of canonical transformations on $(\Gamma_{\rm Cov}, \omega)$. The associated Hamiltonian is given by

$$\mathcal{H}_K := -\frac{1}{2}\omega(\gamma, \mathcal{L}_K \gamma) \tag{1.5}$$

where the overall factor $-\frac{1}{2}$ is a normalization constant (chosen such that the Hamiltonian generated by a de Sitter time translation reduces to the flat space result in the limit $\Lambda \to 0$). The Hamiltonian can be evaluated on any constant time slice, including $\mathcal{I}$. When K is a translation or rotation Killing vector field, respectively, $\mathcal{H}_K$ is interpreted as the flux of energy-momentum or angular momentum carried

[7]The linearized Weyl tensor, on the other hand, satisfies conformally invariant equations and consequently its propagation *is* sharp.

[8]Since γ_{ab} satisfies the linearized Einstein's equations and gauge conditions that all only refer to the background de Sitter metric $\bar{g}_{ab}$, $\mathcal{L}_K \gamma_{ab}$ is also in $\Gamma_{\rm Cov}$.

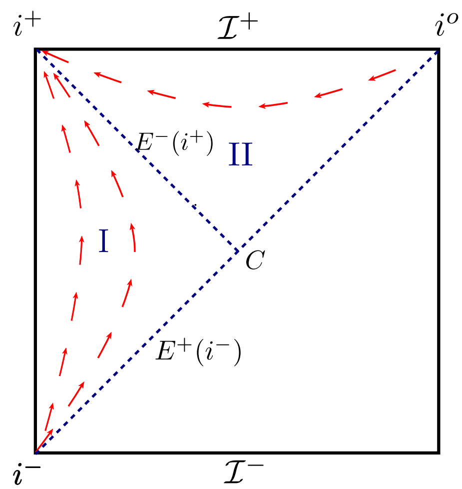

Figure 1.2. The red vectors in this conformal diagram show the integral curves of the time translation Killing field T^a in the Poincaré patch. T^a is future directed and time-like in static patch (region I) and space-like near $\mathcal{I}$ (region II). It is future directed and null on the portion of the event horizon $E^-(i^+)$ to the future of the cross-over 2-sphere (bifurcate horizon) C and on the portion of the null event horizon $E^+(i^-)$ to the past of C. It is *past directed* and null on the portion of $E^+(i^-)$ to the future of C.

by gravitational waves. These expressions are particularly useful to ask physical questions such as how much energy-momentum and angular momentum in the form of gravitational waves is radiated in a given process. These expressions form the foundation for Chapter 2 and 3. We also showed that all fluxes vanish on $\mathcal{I}$ if the magnetic part of the linearized Weyl tensor is set to zero on $\mathcal{I}$ (which is the linearized analog of the condition $\mathcal{B}_{ab} \hat{=} 0$). This explicitly illustrates that the strengthening of the boundary conditions by requiring conformal flatness of the metric on $\mathcal{I}$ is not practicable in the study of gravitational radiation.

Finally, the de Sitter metric admits 10 Killing vector fields. However, since the Poincaré patch is only part of de Sitter spacetime, only those isometries are permissible that map this patch to itself. The subgroup of the 10-dimensional de Sitter group that leaves the Poincaré patch invariant is 7-dimensional, consisting of 4 (de Sitter) translations and 3 rotations. All these de Sitter Killing vector fields are space-like on $\mathcal{I}$. One can check this explicitly for each vector field individually, but it also follows from the following, more general statement: If K^a is a Killing vector field on any asymptotically de Sitter spacetime (M, g_{ab}), then K^a must admit an extension to $\mathcal{I}$ which makes it tangential. The proof is simple. Rewriting the defining property of the Killing vector field, $\mathcal{L}_K g_{ab} = 0$, in terms of quantities that have a well-defined limit to $\mathcal{I}$, we obtain

$$\mathcal{L}_K g_{ab} = \mathcal{L}_{\tilde{K}} \left(\Omega^{-2}\tilde{g}_{ab}\right) = -2\Omega^{-3} \left(\mathcal{L}_{\tilde{K}}\Omega\right) \tilde{g}_{ab} + \Omega^{-2}\mathcal{L}_{\tilde{K}}\tilde{g}_{ab} = 0\,, \tag{1.6}$$

where we used that the Killing vector field on M is identified with the vector field $\tilde{K}^a$ on $\mathcal{I}$, that is, $K^a = \tilde{K}^a$. Multiplying both sides by Ω^3 and evaluating this equation on $\mathcal{I}$ yields $K^a\nabla_a\Omega \,\hat{=}\, 0$. Consequently, K^a is tangential to $\mathcal{I}$ as it is orthogonal to its normal $\nabla_a\Omega$. Thus, any Killing vector field of an asymptotically de Sitter spacetime yields an asymptotic symmetry vector field on $\mathcal{I}$ that is space-like. This also means that the vector generating time translations in de Sitter spacetime is space-like on $\mathcal{I}$ (even though it is time-like in the static patch, see region I in Figure 1.2). Nonetheless, we will refer to this vector as a 'time translation' because (i) it is the limit of the time translation Killing field of the Schwarzschild-de Sitter spacetime as the Schwarzschild mass goes to zero, and (ii) it reduces to a time translation of Minkowski spacetime in the limit $\Lambda \to 0$. In conformal and proper time coordinates, respectively, the time translation vector field T^a is

$$T = T^a\partial_a = -H\left[\eta\frac{\partial}{\partial\eta} + x\frac{\partial}{\partial x} + y\frac{\partial}{\partial y} + z\frac{\partial}{\partial z}\right] \tag{1.7}$$

$$= \frac{\partial}{\partial t} - H\left(x\frac{\partial}{\partial x} + y\frac{\partial}{\partial y} + z\frac{\partial}{\partial z}\right). \tag{1.8}$$

Recall, the limit $\Lambda \to 0$ should not be taken in conformal coordinates (in which T would vanish), but one should use the proper time coordinates. Using the latter set of coordinates, it is immediate that in this limit T^a goes to the Minkowski time translation Killing vector field $\mathring{t}^a = \mathring{\eta}^{ab}\mathring{\nabla}_b t$. Given that T^a is space-like on $\mathcal{I}$, in principle, the energy can be negative on $\mathcal{I}$. In fact, the energy obtained from the Hamiltonian defined by T^a, $\mathcal{H}_T$, is unbounded below. (Unboundedness from below of $\mathcal{H}_T$ is not limited to linearized gravitational fields, but is also true for matter fields in de Sitter spacetime.) Thus, there exist gravitational waves on de Sitter spacetime which carry arbitrarily negative energy, no matter how small the positive Λ is! This is in striking contrast with the $\Lambda = 0$ situation, where the corresponding waves carry strictly positive energy. This yields the following conundrum: if gravitational waves can carry arbitrarily negative energy, what happens to the limit $\Lambda \to 0$ that should recover positive energy in Minkowski spacetime? In other words, what happens to the infinitely many solutions with negative energy in the limit $\Lambda \to 0$? A careful analysis shows that the limit is onto: the solutions with negative energy all get mapped to zero. Geometrically, this occurs because while the Killing field T^a of de Sitter metric for every $\Lambda > 0$ is space-like near $\mathcal{I}$ (region II in Figure 1.2), its limit is the time-like Killing field $\mathring{t}^a$ of $\mathring{\eta}_{ab}$. This comes about as follows. The cosmological horizon $E^-(i^+)$, which bounds the static patch, is at $r = H^{-1}e^{-Ht}$ (with $r^2 = \vec{x} \cdot \vec{x}$). In the limit $\Lambda \to 0$, the static patch in which T^a is time-like fills out the whole Minkowski space. Thus, although $\mathcal{H}_T$ is unbounded below – no matter how small the positive Λ is, the limiting $\mathring{\mathcal{H}}_t$ is strictly positive. This is another nice illustration of the subtlety of the $\Lambda \to 0$ limit and illuminates that even for physical observable quantities the limit can be discontinuous. Thus, it is a priori not necessarily obvious that the effects of positive Λ will be negligible for all gravitational wave observations.

Chapter 2
General formula for power emitted in de Sitter spacetimes

2.1 Introduction

One of the first predictions of general relativity came from Einstein's calculations that demonstrated the existence of gravitational waves in the weak field approximation. Although the idea of gravitational waves was already explored by others including Lagrange and Poincaré (see [15] for a review), Einstein's 1916 paper provided a relativistic description by linearizing field equations of general relativity off Minkowski background, in the presence of an external, time-changing source [14]. Two years later, he also calculated the energy carried by these waves far away from the source. He found that the leading order contribution to the emitted power is proportional to the square of the third time derivative of the mass quadrupole moment [48].

This prediction was first confirmed by the high precision measurements of the orbital period of the Hulse-Taylor binary [22]. These measurements allowed a direct comparison between the loss of orbital energy and the energy emitted by gravitational waves. Today, observational evidence yields a confirmation of the existence of gravitational quadrupolar radiation to an accuracy of 3 parts in 10^3 [23]. Einstein's calculations and its subsequent refinements and generalizations (due to Eddington [49], Landau and Lifshitz [50], Fock [51], Blanchet and Damour [52] and others), as well as the Bondi-Sachs framework in full general relativity [27, 43], use field equations with a vanishing cosmological constant, Λ. Since the value of Λ is so small, at a practical level it seems natural to just ignore it and use the well-developed $\Lambda = 0$ framework. However, as we discussed in Chapter 1, there are some qualitative differences between the $\Lambda = 0$ and $\Lambda > 0$ cases, making the limit $\Lambda \to 0$ quite subtle.[1] In particular, the limit of observable quantities associated with gravitational waves can be *discontinuous*, whence smallness of Λ does not always translate to smallness of corrections to the $\Lambda = 0$ results. The question then is whether one can reliably justify one's first instinct that Einstein's $\Lambda = 0$ quadrupole formula can receive only negligible corrections, given the smallness of Λ.

To make this concern concrete, let us consider a few illustrations of the qualitative differences. First, while wavelengths of linear fields are constant in Minkowski spacetime, they increase as waves propagate on de Sitter spacetime, and exceed the curvature radius in the asymptotic region of interest. Therefore, the commonly used high-frequency approximation also known as the geometric optics approximation fails in the asymptotic region. Also, one cannot carry over the standard techniques to specify 'near and far wave zones' from the $\Lambda = 0$ case. Second, in Minkowski spacetime one can approach $\mathcal{I}^+$ – the arena on which properties of gravitational waves can be analyzed unambiguously – using $r = r_0$ surfaces with larger and larger values of r_0. Therefore, it is standard practice to use $1/r$ expansions of fields in the analysis of gravitational waves (see, e.g., [53–55]). By contrast, in de Sitter spacetime, such time-like surfaces approach a past cosmological horizon *across which there is no flux of energy or momentum for retarded solutions.* $\mathcal{I}^+$ is now approached by a family of space-like surfaces (on which *time* is constant) whence one cannot use the $1/r$ expansions that dominate the literature on gravitational waves. Third, while $\mathcal{I}^+$ is null in the asymptotically flat case, it is space-like if Λ is positive [43]. Consequently, unfamiliar features can arise as we move from $\Lambda = 0$ to a tiny positive value both in full general relativity and in the linearized limit. In particular, as we saw in Section 1.2.1, energy carried by electromagnetic and linearized gravitational waves can be *negative and of arbitrarily large magnitude* if $\Lambda > 0$ [35]. Since this holds for every $\Lambda > 0$, however tiny, the lower bound on energy carried by these waves has an *infinite* discontinuity at $\Lambda = 0$. Now, if electromagnetic or gravitational waves produced in realistic physical processes

[1]The origin of these subtleties lies in the fact that the observed accelerated expansion makes the asymptotic spacetime geometry in the distant future drastically different from that of asymptotically Minkowski spacetimes. Therefore, although for concreteness and simplicity we will refer to a cosmological constant, as in [31, 35], our main results will not change if instead one has an unknown form of 'dark energy'.

could carry negative energy, we would be faced with a fundamental instability: the source could gain arbitrarily high energy simply by letting the emitted waves carry away negative energy. Thus a positive Λ, however small, opens up an unforeseen possibility, with potential to drastically change gravitational dynamics.[2] Finally, yet another difference is that in the transverse (i.e., Lorenz) traceless gauge the linearized 4-metric field satisfies the *massive* Klein-Gordon equation where the mass is proportional to $\sqrt{\Lambda}$. While the mass is tiny, a priori it is possible that over cosmological distances the difference from the propagation in the $\Lambda = 0$ case could accumulate, creating an $\mathcal{O}(1)$ difference in the linearized metric in the asymptotic region, far way from sources. Since Einstein's quadrupole formula is based on the form of the metric perturbation in this asymptotic region, secular accumulation could then lead to $\mathcal{O}(1)$ departures from that formula, even when Λ is tiny.

These considerations bring out the necessity of a systematic analysis to determine whether the Einstein's quadrupole formula continues to be valid even though many of the key intermediate steps cannot be repeated for the de Sitter background. The goal of this chapter is to complete this task for linearized gravitational waves created by time changing (first order) sources on de Sitter background.

As in the $\Lambda = 0$ case, the calculation involves two steps:

(i) expressing metric perturbations far away from the source in terms of the quadrupole moments of the source, and,

(ii) finding the energy and power radiated by this source in the form of gravitational waves.

However, the extension of the $\Lambda = 0$ analysis introduces unforeseen issues in both steps. In step (i), since the background spacetime is no longer flat, the meaning of 'quadrupole moment' is not immediately clear. The second subtlety concerns both steps. Specifically, due to the curvature of the de Sitter spacetime, the gravitational waves back-scatter. This backscattering introduces a tail term in the solutions to the linearized Einstein's equation already in the first post-Newtonian order. That in and of itself is not problematic. However, if a tail term persisted in the formula for energy loss, one would need to know the history of the source throughout its

[2] As discussed in Chapter 1, we suggested that this possibility will not be realized for realistic sources because the fields they induce on $\mathcal{I}^+$ would be constrained just in the right way for the waves to carry positive energy. However, that argument was meant only as an indication, based on properties of source-free gravitational waves. A detailed analysis of the quadrupole formula for $\Lambda > 0$ is needed to settle this issue in the weak field limit.

evolution in order to determine the flux of energy emitted at any given retarded instant of time.[3] Third, as discussed above, the energy calculated in step (ii) could, in principle, be arbitrarily negative, in which case self-gravitating systems would be drastically unstable to emission of gravitational waves.

Thus, from a conceptual standpoint, the generalization of the quadrupole formula to include a positive Λ is both interesting and subtle. For example, the presence of the tail term opens a door to a new contribution to the 'memory effects' associated with gravitational waves [29, 56, 57]. In addition, as in the asymptotically flat case, it offers guidance in the development of the full, nonlinear framework. Finally, as we will see, this generalization also provides detailed control on the approximations involved in setting Λ to zero.

This chapter is organized as follows. In Section 2.2, we introduce our notation and recall the linearized Einstein's equation on de Sitter background as well as their retarded solutions sourced by a (first order) stress-energy tensor. In Section 2.3, we introduce the late time and post-Newtonian approximations and express the leading terms of solutions in terms of the quadrupole moments of sources. We also discuss in detail the tail term and some of its properties. In Section 2.4, we use these expressions to calculate the energy emitted by the source using Hamiltonian methods on the covariant phase space of the linearized solutions introduced in [35], and then discuss in some detail the novel features that arise because of the presence of a positive Λ. We find that the energy carried away by the gravitational waves produced by a time changing source is necessarily positive. Detailed expressions bear out the expectation that, for sources of gravitational radiation currently under consideration by gravitational wave observatories, the primary modification to Einstein's formula can be incorporated by taking into account the expansion of the Universe and the resulting gravitational redshift. Section 2.5 contains a summary.

2.2 Preliminaries

The isolated system of interest is depicted in the left panel of Figure 2.1 (and specified in greater detail at the beginning of Section 2.3). It represents a matter

[3] In the $\Lambda = 0$ case, backscattering occurs only at higher post-Newtonian orders. These higher order corrections to the quadrupole formula are not needed to compare theory with observations for the Hulse-Taylor pulsar so far because the current observational accuracy is at the 10^{-3} level rather than 10^{-4}.

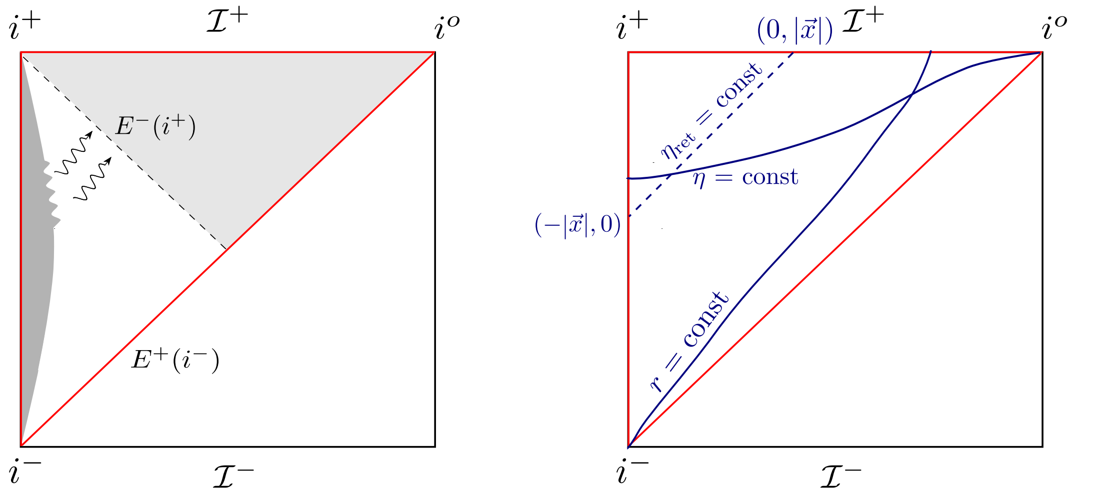

Figure 2.1. *Left Panel:* A time changing quadrupole emitting gravitational waves whose spatial size is uniformly bounded in time. The causal future of such a source covers only the future Poincaré patch $M_{\rm P}^+$ (the upper, red triangle of the figure). There is no incoming radiation across the past boundary $E^+(i^-)$ of $M_{\rm P}^+$ because we use retarded solutions. The light gray shaded region represents a convenient neighborhood of $\mathcal{I}^+$ in which perturbations satisfy a homogeneous equation and the approximation, discussed below, holds everywhere. *Right Panel:* The rate of change of the quadrupole moment at the point $(-|\vec{x}|, \vec{0})$ on the source creates the retarded field at the point $(0, \vec{x})$ on $\mathcal{I}^+$. The figure also shows the cosmological foliation $\eta = \rm const$ and the time-like surfaces $r := |\vec{x}| = \rm const$. As r goes to infinity, the $r = \rm const$ surfaces approach $E^+(i^-)$. Therefore, in contrast with the situation in Minkowski spacetime, for sufficiently large values of r, there is no flux of energy across the $r = \rm const$ surfaces.

source in de Sitter spacetime whose spatial size is uniformly bounded in time. Such a source intersects $\mathcal{I}^\pm$ at single points $i^\pm$. Examples are provided by isolated stars and coalescing binary systems. As discussed in Chapter 1, the causal future of such a source covers only the future Poincaré patch, $M_{\rm P}^+$. Therefore, we restrict ourselves to $M_{\rm P}^+$, which can be coordinatized by the conformal coordinates (η, x, y, z). To study the gravitational radiation emitted by this isolated system in the presence of positive Λ, we consider first order perturbations off the de Sitter background metric $\bar{g}_{ab}$. The perturbed metric is denoted by g_{ab},

$$g_{ab} \;=\; \bar{g}_{ab} + \epsilon\, \gamma_{ab} \;=:\; a^2(\eta)(\mathring{g}_{ab} + \epsilon\, h_{ab})\,, \tag{2.1}$$

where ϵ is a smallness parameter, $a(\eta) = -(H\eta)^{-1}$ the scale factor and $\mathring{g}_{ab}$ the flat metric. While γ_{ab} are the physical first order perturbations off de Sitter spacetime, it is convenient – as will be clear shortly – to use the conformally related mathematical field while solving the linearized Einstein's equation.

In terms of the trace-reversed metric perturbation $\overline{\gamma}_{ab} := \gamma_{ab} - \frac{1}{2}\bar{g}_{ab}\gamma$, the linearized Einstein's equation in the presence of a (first order) linearized source T_{ab} can be written as

$$\overline{\Box}\overline{\gamma}_{ab} - 2\overline{\nabla}_{(a}\overline{\nabla}^{c}\overline{\gamma}_{b)c} + \bar{g}_{ab}\overline{\nabla}^{c}\overline{\nabla}^{d}\overline{\gamma}_{cd} - \frac{2}{3}\Lambda(\overline{\gamma}_{ab} - \bar{g}_{ab}\overline{\gamma}) = -16\pi G\, T_{ab} \tag{2.2}$$

where $\overline{\nabla}$ and $\overline{\Box}$ denote the derivative operator and the d'Alembertian defined by the de Sitter metric $\bar{g}_{ab}$. The solutions to the linearized equation above with sources on the future Poincaré patch $(M_{\rm P}^{+}, \bar{g}_{ab})$ are discussed in detail by de Vega et al. in [58] (see also [59] for a recent discussion). Here we will summarize their results, comment on the physical interpretation, and discuss the limit $\Lambda \to 0$.

Denote by η^a the vector field normal to the cosmological slices $\eta = \text{const}$ satisfying $\eta^a\nabla_a\eta = 1$ and let $n^a := -H\eta\,\eta^a$ denote the future pointing unit normal to these slices. Then, it is convenient to solve (2.2) using the following gauge condition:

$$\overline{\nabla}^{a}\overline{\gamma}_{ab} = 2H\, n^a\, \overline{\gamma}_{ab}\,. \tag{2.3}$$

This is a generalization of the more familiar Lorenz gauge condition and, as with the Lorenz condition, it does not exhaust the gauge freedom. Nonetheless, in this gauge the linearized Einstein's equation (2.2) simplifies significantly when it is rewritten in terms of the field $\bar{\chi}_{ab}$ which is related to the trace-reversed metric perturbations $\overline{\gamma}_{ab}$ via $\bar{\chi}_{ab} := a^{-2}\overline{\gamma}_{ab} = h_{ab} - \frac{1}{2}\mathring{g}_{ab}\,\mathring{g}^{cd}h_{cd}$. Finally, it is easiest to obtain solutions to (2.2) by performing a decomposition of $\bar{\chi}_{ab}$ and T_{ab}, adapted to the cosmological $\eta = \text{const}$ slices:

$$\tilde{\chi} := (\eta^a\eta^b + \mathring{q}^{ab})\,\bar{\chi}_{ab}, \qquad \chi_a := \eta^c\,\mathring{q}_a{}^b\,\bar{\chi}_{bc}, \qquad \chi_{ab} := \mathring{q}_a{}^m\,\mathring{q}_b{}^n\,\bar{\chi}_{mn}, \tag{2.4}$$

$$\tilde{\mathcal{T}} := (\eta^a\eta^b + \mathring{q}^{ab})\,T_{ab}, \qquad \mathcal{T}_a := \eta^c\,\mathring{q}_a{}^b\,T_{bc}, \qquad \mathcal{T}_{ab} := \mathring{q}_a{}^m\,\mathring{q}_b{}^n\,T_{mn}, \tag{2.5}$$

where $\mathring{q}^{ab}$ is the (contravariant) spatial metric on a $\eta = \text{const}$ slice induced by the flat metric $\mathring{g}_{ab}$, i.e., $\mathring{q}^{ab} = \mathring{g}^{ab} + \eta^a\eta^b$. (Note that unlike $\bar{\chi}_{ab}$ in (2.4), the stress energy tensor T_{ab} in (2.5) has neither been rescaled by a^{-2} nor has it been trace-reversed.)

In the chart $(\eta, \vec{x})$, $-(1/4H\eta)\tilde{\chi}$ is the perturbed lapse function and $(H\eta)^{-2}\,\mathring{q}^{ab}\chi_b$ is the perturbed shift field. Thus, as in the linearized theory off Minkowski spacetime, the physical degrees of freedom associated with radiation are encoded in the totally spatial projection χ_{ab}.

It is convenient to regard the fields $\tilde{\chi}, \chi_a$ and χ_{ab}, as living in the flat spacetime $(M_{\rm P}^+, \mathring{g}_{ab})$ because: (i) the gauge condition and field equations have a simple form in terms of the derivative operators defined by $\mathring{g}_{ab}$; and (ii) these gauge conditions and field equations are well-defined also at $\mathcal{I}^+$ because, as we will see in Section 2.4, the metric $\mathring{g}_{ab}$ turns out to provide a viable conformal rescaling of $\bar{g}_{ab}$ that is well-defined at $\mathcal{I}^+$. The gauge conditions (2.3) become

$$\mathring{D}^a\chi_{ab} = \partial_\eta\chi_b - \frac{2}{\eta}\chi_b, \quad \text{and} \quad \mathring{D}^a\chi_a = \partial_\eta(\tilde{\chi} - \chi) - \frac{1}{\eta}\tilde{\chi}\,, \tag{2.6}$$

where $\mathring{D}$ is the derivative operator of the spatial metric $\mathring{q}_{ab}$ and $\chi = \mathring{q}^{ab}\chi_{ab}$. In this gauge, the linearized Einstein's equation (2.2) is split into three as follows

$$\mathring{\Box}\ \left(\frac{1}{\eta}\tilde{\chi}\right) = -\frac{16\pi G}{\eta}\,\tilde{\mathcal{T}}, \tag{2.7}$$

$$\mathring{\Box}\ \left(\frac{1}{\eta}\chi_a\right) = -\frac{16\pi G}{\eta}\,\mathcal{T}_a, \tag{2.8}$$

$$(\mathring{\Box}\ + \frac{2}{\eta}\partial_\eta)\,\chi_{ab} = -16\pi G\,\mathcal{T}_{ab}. \tag{2.9}$$

where $\mathring{\Box}$ is the d'Alembertian operator of the flat metric $\mathring{g}_{ab}$. Using the conservation of the first order stress-energy tensor, $\bar{\nabla}^a T_{ab} = 0$, it is easy to directly verify that the gauge conditions and the field equations are consistent, as they must be.

Since we wish to impose the 'no incoming radiation' boundary conditions, we will seek retarded solutions to these equations. The first two equations, (2.7) and (2.8), can be solved using the scalar retarded Green's function of $\mathring{\Box}$:

$$G_R^{(M)}(x,\, x') = \frac{1}{4\pi|\vec{x} - \vec{x}'|}\ \delta(\eta - \eta' - |\vec{x} - \vec{x}'|) \tag{2.10}$$

to yield

$$\tilde{\chi}(\eta,\, \vec{x}) \quad = 4G\,\eta \int \frac{\mathrm{d}^3\vec{x}'}{|\vec{x} - \vec{x}'|}\,\frac{1}{\eta_{\rm Ret}}\,\tilde{T}(\eta_{\rm Ret},\, \vec{x}'), \ \ \text{and}$$

$$\chi_{\bar{a}}(\eta,\,\vec{x}) \;\; = 4G\,\eta \int \frac{\mathrm{d}^3\vec{x}'}{|\vec{x}-\vec{x}'|}\,\frac{1}{\eta_{\mathrm{Ret}}}\,\mathcal{T}_{\bar{a}}(\eta_{\mathrm{Ret}},\,\vec{x}'), \tag{2.11}$$

where η_{Ret} is the retarded time related to η and $\vec{x}$ by $\eta_{\mathrm{Ret}} := \eta - |\vec{x}-\vec{x}'|$. We could use the scalar Green's function of $\mathring{\Box}$ also in the second equation because $\chi_{\bar{a}}$ refers to the Cartesian components of the vector perturbation. While we will use the solutions (2.11) in an intermediate step, the fluxes of energy, momentum and angular momentum turn out to depend only on χ_{ab} because, as we noted above, the other components correspond to linearized lapse and shift fields.

One can use a *scalar* Green's function also for the Cartesian components of the spatial tensor field χ_{ab}. However, since the operator on the left hand side of (2.9) has the extra term, $(2/\eta)\partial_\eta$, we cannot use the Green's function of the flat space wave operator $\mathring{\Box}$. Instead, [58] provides the retarded Green's function satisfying

$$(\mathring{\Box} + \frac{2}{\eta}\partial_\eta)\, G_R(x,x') \; = -(H^2\eta^2)\,\delta(x,x'). \tag{2.12}$$

In contrast to the flat space scalar Green's function, the solution to this equation has an additional term that extends its support to the region in which x, x' are time-like related:

$$G_R(x,x') = \frac{H^2\,\eta\,\eta'}{4\pi|\vec{x}-\vec{x}'|}\,\delta(\eta-\eta'-|\vec{x}-\vec{x}'|) \;+\; \frac{H^2}{4\pi}\,\theta(\eta-\eta'-|\vec{x}-\vec{x}'|) \tag{2.13}$$

where $\theta(x)$ is the step function which is 1 when $x \geq 0$ and 0 otherwise. Therefore, the solution χ_{ab} is given by

$$\chi_{\bar{a}\bar{b}}(\eta,\vec{x}) = 16\pi G \int \mathrm{d}^3\vec{x}' \int \mathrm{d}\eta'\; G_R(x,x')\,\Big(\frac{1}{H^2\eta'^2}\Big)\,\mathcal{T}_{\bar{a}\bar{b}}(x')\,. \tag{2.14}$$

To simplify the solution, one uses the identity

$$\begin{aligned} &\Big(\tfrac{1}{|\vec{x}-\vec{x}'|}\,\tfrac{\eta}{\eta'}\Big)\;\delta(\eta-\eta'-|\vec{x}-\vec{x}'|) \;+\; \tfrac{1}{\eta'^2}\,\theta(\eta-\eta'-|\vec{x}-\vec{x}'|) \\ =\;& \tfrac{1}{|\vec{x}-\vec{x}'|}\,\delta(\eta-\eta'-|\vec{x}-\vec{x}'|) \;-\; \tfrac{\partial}{\partial\eta'}\Big(\tfrac{1}{\eta'}\,\theta(\eta-\eta'-|\vec{x}-\vec{x}'|)\Big)\,, \end{aligned} \tag{2.15}$$

in (2.14), integrates by parts with respect to η', and shows that the boundary terms do not contribute for any given $(\eta,\vec{x})$. Then everywhere on M_{P}^{+} the solution is given by

$$\begin{aligned}
\chi_{\bar{a}\bar{b}}(\eta,\, \vec{x}) &= 4G \int \frac{\mathrm{d}^3\vec{x}'}{|\vec{x}-\vec{x}'|}\, \mathcal{T}_{\bar{a}\bar{b}}(\eta_{\mathrm{Ret}},\, \vec{x}') \\
&\qquad + 4G \int \mathrm{d}^3\vec{x}' \int_{-\infty}^{\eta_{\mathrm{Ret}}} \mathrm{d}\eta' \,\frac{1}{\eta'}\, \partial_{\eta'} \mathcal{T}_{\bar{a}\bar{b}}(\eta',\, \vec{x}') \qquad (2.16) \\
&\equiv \sharp_{\bar{a}\bar{b}}\,(\eta, \vec{x}) + \flat_{\bar{a}\bar{b}}\,(\eta, \vec{x})\,, \qquad (2.17)
\end{aligned}$$

where $\sharp_{\bar{a}\bar{b}}\,(\eta, \vec{x})$ denotes the *sharp* propagation term and $\flat_{\bar{a}\bar{b}}\,(\eta, \vec{x})$ the hereditary term also known as the *tail* term. Note that this solution relates the Cartesian components of χ_{ab} to those of $\mathcal{T}_{ab}$. Therefore, throughout the rest of the chapter, whenever we use this solution, we will restrict ourselves to components in the Cartesian chart.

The retarded solution (2.16) has an interesting feature. The first term $\sharp_{ab}$ in this expression is identical to the solution for the trace-reversed perturbations which satisfy the linearized Einstein equation (with a first order source $\mathcal{T}_{ab}$) w.r.t. the Minkowski metric $\mathring{g}_{ab}$. The second term $\flat_{ab}$, which is absent in the Minkowski case, depends on the entire history of the behavior of the source up to time η_{Ret}. It results from the backscattering of the perturbation by curvature in the de Sitter background. Thus, in contrast to the $\Lambda = 0$ case, the propagation of the metric perturbation fails to be sharp already at this order.[4] (Although as we shall explicitly show in Section 2.3.3, at first post-Newtonian order, the dynamical part of this tail term does propagate along the null cone.) The retarded solutions (2.11) and (2.16) satisfy the equations of motion (2.7) - (2.9) by construction. However, to obtain a solution to the physical problem at hand, we need to make sure that they also satisfy the gauge conditions (2.6). One can verify that this is the case using conservation of the stress-energy tensor.

Finally, let us discuss the limit $\Lambda \to 0$. From the solution (2.16) it is not obvious that the tail term will disappear in this limit. However, as discussed in Chapter 1, to study this limit one needs to use the differential structure given not by the $(\eta, \vec{x})$ chart, but by the $(t,\, \vec{x})$ chart in which the de Sitter metric $\bar{g}_{ab}$ admits a well-defined

[4]Since the propagation of the linearized Weyl curvature is sharp, one might wonder if the tail term in the retarded solution (2.16) is a gauge artifact. It is not. Since the *gauge invariant part* χ_{ab}^{TT} of the metric perturbation satisfies $(\mathring{\Box} + (2/\eta)\partial_\eta)\chi_{ab}^{\mathrm{TT}} = 16\pi G \mathcal{T}_{ab}^{\mathrm{TT}}$ – i.e., the same equation as χ_{ab} but with T_{ab} on the right hand side replaced by its TT-part – it follows that χ_{ab}^{TT} is given by replacing $\mathcal{T}_{ab}$ with $\mathcal{T}_{ab}^{\mathrm{TT}}$ in (2.16). Therefore it also has a non-trivial tail term.

limit to the Minkowski metric $\mathring{\eta}_{ab}$ as $\Lambda \to 0$. Using the $(t,\, \vec{x})$ chart, it is easy to show that the gauge condition (2.3) and the linearized Einstein's equation (2.2) reduce to the familiar Lorenz gauge condition and linearized Einstein's equation in Minkowski spacetime, respectively,

$$\mathring{\nabla}^b \mathring{\gamma}_{ab} = 0, \quad \text{and} \quad \mathring{\Box}\mathring{\gamma}_{ab} = -16\pi G\, T_{ab}, \tag{2.18}$$

where for notational coherence the metric perturbations off the Minkowski spacetime metric $\mathring{\eta}_{ab}$ are denoted by $\mathring{\gamma}_{ab}$. Note that, while in the de Sitter case different components of the perturbation satisfy different equations, (2.7)-(2.9), in the $\Lambda \to 0$ limit these distinct equations collapse to a single flat space scalar wave equation for all Cartesian components of $\mathring{\gamma}_{ab}$. Consequently, the Green's functions (2.10) and (2.13) used to solve for various components of the de Sitter perturbations, reduce to the scalar Green's function of the flat d'Alembertian operator $\mathring{\Box}$ of $\mathring{\eta}_{ab}$,

$$G_R^{(M)}(x, x') = \frac{1}{4\pi|\vec{x} - \vec{x}'|}\ \delta(t - t' - |\vec{x} - \vec{x}'|). \tag{2.19}$$

Therefore in the $(t, \vec{x})$ chart the retarded solutions of (2.18) are given by

$$\mathring{\gamma}_{\bar{a}\bar{b}}(t,\, \vec{x}) = 4G \int \frac{\mathrm{d}^3\vec{x}'}{|\vec{x} - \vec{x}'|}\, T_{\bar{a}\bar{b}}(t - |\vec{x} - \vec{x}'|, \vec{x}'). \tag{2.20}$$

This also follows directly by first expressing the final solutions (2.11) and (2.16) in the $(t, \vec{x})$ chart and then taking the $\Lambda \to 0$ limit, as it must. Thus, our expectation that the tail term should disappear in the limit $\Lambda \to 0$ is explicitly borne out.

2.3 The retarded solution and quadrupole moments

In full general relativity with positive Λ, spacetimes describing isolated gravitating systems are asymptotically de Sitter. To compute the energy emitted in the form of gravitational waves, one would (numerically) solve Einstein's equations by imposing an appropriate 'no-incoming radiation' boundary condition, find the gravitational fields on $\mathcal{I}^+$, and extract the energy radiated by gravitational waves from these fields. Since at the moment it is unclear how to extract energy in full, non-linear general relativity when $\Lambda > 0$, this dissertation restricts itself to a simplified version of this problem using the first post-de Sitter approximation. This is the same approach Einstein took before gravitational waves were understood in full general relativity in the $\Lambda = 0$ case. We have already incorporated the 'no incoming radiation' boundary condition through retarded Green's functions and our task now is to extract physical information from the emitted gravitational waves by examining these solutions at $\mathcal{I}^+$.

As explained in Section 2.1, the calculation will be performed in two steps. In the first, carried out in this section, we use physically motivated approximations to simplify the retarded solution (2.16) in the asymptotic region near $\mathcal{I}^+$ and relate the leading term to the time variation of the source quadrupole moment. The second step in which we calculate the energy and power will be carried out in Section 2.4.

2.3.1 The late time and post-Newtonian approximations

To extract physical information from eq. (2.16), we need to examine this solution in the asymptotic region near $\mathcal{I}^+$. In linearized gravity off Minkowski spacetime, one can approach $\mathring{\mathcal{I}}^+$ using a family of time-like tubes $r = r_o$, with ever increasing values of the constant r_o. Therefore, one focuses on the form of the retarded solutions at large distances from the source, keeping the leading order $1/r$ contribution, and ignoring terms that fall-off as $1/r^2$. Since the conformal factor used to complete Minkowski spacetime in order to attach the null boundary $\mathcal{I}^+$ falls-off as $1/r$, this approximation is sufficient to recover the asymptotic perturbation on $\mathcal{I}^+$ and extract energy, momentum and angular momentum carried by gravitational waves. In de Sitter spacetime, by contrast, as mentioned in Section 2.1, the $r = r_o$ time-like surfaces approach the cosmological horizon $E^+(i^-)$, rather than $\mathcal{I}^+$ (see the right panel of Figure 2.1). And the flux of energy, momentum or angular momentum across $E^+(i^-)$ vanishes identically for retarded solutions! Indeed, this is precisely the 'no incoming radiation condition'. Thus, $E^+(i^-)$ is analogous to $\mathring{\mathcal{I}}^-$ rather than $\mathring{\mathcal{I}}^+$ in Minkowski spacetime. Therefore, contrary to the strong intuition derived from Minkowski spacetime [53–55], the $1/r$-expansions are now ill-suited to study gravitational waves. In particular, one cannot blindly take over well-understood notions such as the 'wave zone' and the approximation schemes associated with this region. (All these differences occur also for test electromagnetic fields on de Sitter spacetime.)

As explained in [31], $\mathcal{I}^+$ of de Sitter spacetime is space-like and corresponds to the surface $\eta = 0$ (see also Section 2.4.1). Therefore, it can be approached by a family of *space-like* surfaces. The first natural candidate is provided by the cosmological slices $\eta = \text{const}$ used in Section 2.2. Another possibility is to use the family of space-like 3-surfaces which lie in the shaded region of the left panel of Figure 2.1 to which T^a and the three rotational Killing fields of $(M_{\text{P}}^+, \bar{g}_{ab})$ are everywhere tangential. In this section we will use the cosmological slices and in the next section, the 3-surfaces in the shaded region. To summarize, to approach $\mathcal{I}^+$ and extract the radiative part of the solution, we now need a *late time* approximation in place of the Minkowskian 'far field' approximation.

To introduce this approximation, we first need to sharpen our restrictions on the spatial support of the matter source. These conditions will capture the idea that the system under consideration is isolated, e.g., an oscillating star or a compact binary. First, we will assume that the physical size $D(\eta)$ of the system is uniformly bounded by D_o on all $\eta = \text{const}$ slices. A particular consequence of this requirement is that the source punctures $\mathcal{I}^+$ at a single point i^+, and $\mathcal{I}^-$ at a single point i^-, as depicted in Figure 2.1. Physically, this assumption will ensure that a binary, for example, remains compact in spite of the expansion of the Universe. We further sharpen the 'compactness' restriction through a second requirement: $D_o \ll \ell_\Lambda$, where $\ell_\Lambda (= 1/H)$ is the cosmological radius.[5] Finally, for simplicity, we assume that the system is stationary in the distant past and distant future, i.e., $\mathcal{L}_T T_{ab} = 0$ outside a finite η-interval. Such a system is dynamically active only for a finite time interval (η_1, η_2). This simplifying assumption can be weakened substantially to allow $\mathcal{L}_T T_{ab}$ to fall-off at a suitable rate in the approach to $i^\pm$. We use the stronger assumption just to ensure finiteness of various integrals without having to consider the fall-off conditions in detail at each intermediate step. Furthermore, given that we are primarily interested in calculating radiated power at a retarded instant of time, the assumption is not really restrictive.

With these restrictions, we can now obtain an approximate form of the solution (2.16) which is valid near $\mathcal{I}^+$. Consider, then, a cosmological slice, $\eta = \text{const}$, and choose the Cartesian coordinates $\vec{x}$ such that the center of mass of the source lies at the origin. The right side of (2.17) expresses χ_{ab} as a sum of a sharp term and tail term. Let us first simplify the sharp term. As in the standard linearized theory

[5]Given that ℓ_Λ is about 5 Gpc, the condition is easily met by sources of physical interest, such as an isolated oscillating star or a compact binary.

off Minkowski spacetime [53], we first write it as

$$\natural_{\bar{a}\bar{b}}\,(\eta,\vec{x}) = 4G\int \mathrm{d}^3x' \int \mathrm{d}^3y' \, \frac{\mathcal{T}_{\bar{a}\bar{b}}(\eta_{\mathrm{Ret}},\, \vec{y}')}{|\vec{x}-\vec{x}'|}\ \delta(\vec{x}',\vec{y}')\,, \tag{2.21}$$

and Taylor expand the $|\vec{x}-\vec{x}'|$ dependence of the integrand around $\vec{x}'=0$ (recall that the integral over $\vec{x}'$ is over a compact region around the origin, the support of $\mathcal{T}_{\bar{a}\bar{b}}$). In the Taylor expansion, each derivative $\partial/\partial {x'}^{\bar{a}}$ can be replaced by $-\partial/\partial x^{\bar{a}}$ because the $\vec{x}'$-dependence of the integrand of the last integral comes entirely from $|\vec{x}'-\vec{x}|$. The first few terms then become

$$\begin{aligned}
\natural_{\bar{a}\bar{b}}\,(\eta,\vec{x}) &= 4G\int \mathrm{d}^3x' \int \mathrm{d}^3y\, \delta(\vec{x}',\vec{y}) \left[\frac{\mathcal{T}_{\bar{a}\bar{b}}(\eta_{\mathrm{ret}},\vec{y})}{r} - \vec{x}'^i \frac{\partial}{\partial x^i}\left.\frac{\mathcal{T}_{\bar{a}\bar{b}}(\eta_{\mathrm{Ret}},\,\vec{y})}{|\vec{x}-\vec{x}'|}\right|_{\vec{x}'=0}\right.\\
&\qquad \left. + \frac{1}{2}\vec{x}'^i\vec{x}'^j \frac{\partial}{\partial x^i}\frac{\partial}{\partial x^j}\left.\frac{\mathcal{T}_{\bar{a}\bar{b}}(\eta_{\mathrm{Ret}},\,\vec{y})}{|\vec{x}-\vec{x}'|}\right|_{\vec{x}'=0} + \ldots\right]\\
&= 4G\int \mathrm{d}^3x' \int \mathrm{d}^3y\, \delta(\vec{x}',\vec{y}) \left[\frac{\mathcal{T}_{\bar{a}\bar{b}}(\eta_{\mathrm{ret}},\vec{y})}{r} - \vec{x}'^i \frac{\partial}{\partial x^i}\frac{\mathcal{T}_{\bar{a}\bar{b}}(\eta_{\mathrm{ret}},\,\vec{y})}{r}\right.\\
&\qquad \left. + \frac{1}{2}\vec{x}'^i\vec{x}'^j \frac{\partial}{\partial x^i}\frac{\partial}{\partial x^j}\frac{\mathcal{T}_{\bar{a}\bar{b}}(\eta_{\mathrm{ret}},\,\vec{y})}{r} + \ldots\right],
\end{aligned}$$

where the ... denote higher order terms in the Taylor expansion. Note that in the second line we have replaced

$$\eta_{\mathrm{Ret}} = \eta - |\vec{x}-\vec{x}'| \qquad \text{by} \qquad \eta_{\mathrm{ret}} = \eta - r \tag{2.22}$$

because the coefficients of the Taylor expansion are evaluated at $\vec{x}'=0$. Next, we carry out the integral over $\vec{y}'$ and focus only on the first order terms in the Taylor expansion

$$\begin{aligned}
\natural_{\bar{a}\bar{b}}\,(\eta,\vec{x}) \;&=\; \frac{4G}{r}\left[\int \mathrm{d}^3x' \left(\mathcal{T}_{\bar{a}\bar{b}}(\eta_{\mathrm{ret}},\,\vec{x}') + \frac{{x'}^c\hat{r}_c}{r}\,\mathcal{T}_{\bar{a}\bar{b}}(\eta_{\mathrm{ret}},\,\vec{x}')\right.\right.\\
&\quad + \left.\left. ({x'}^c\hat{r}_c)\,\partial_{\eta_{\mathrm{ret}}}\,\mathcal{T}_{\bar{a}\bar{b}}(\eta_{\mathrm{ret}},\,\vec{x}') + \ldots\right)\right] \qquad (2.23)\\
&= \frac{4G}{r}\left[\int \mathrm{d}^3x'\mathcal{T}_{\bar{a}\bar{b}}(\eta_{\mathrm{ret}},\,\vec{x}') + (\frac{{x'}_1^c\hat{r}_c}{r})\int \mathrm{d}^3x'\,\mathcal{T}_{\bar{a}\bar{b}}(\eta_{\mathrm{ret}},\,\vec{x}')\right.\\
&\quad + \left. ({x'}_2^c\,\hat{r}_c)\int \mathrm{d}^3x'\,\partial_{\eta_{\mathrm{ret}}}\,\mathcal{T}_{\bar{a}\bar{b}}(\eta_{\mathrm{ret}},\,\vec{x}') + \ldots\right]. \qquad (2.24)
\end{aligned}$$

In the second step we have used the mean value theorem where $\vec{x}'_1$ and $\vec{x}'_2$ are the points in the support of $\mathcal{T}_{\bar{a}\bar{b}}$ determined by this theorem. Next, using the fact that each of $|x_1'^c \hat{r}_c/r|$ and $|x_2'^c \hat{r}_c/r|$ is bounded by the coordinate radius of the source at $\eta = \eta_{\rm ret}$,

$$d(\eta_{\rm ret}) := D(\eta_{\rm ret})/a(\eta_{\rm ret})\,, \tag{2.25}$$

we conclude

$$\sharp_{\bar{a}\bar{b}}\,(\eta, \vec{x}) \;=\; \frac{4G}{r} \int \mathrm{d}^3x'\, \mathcal{T}_{\bar{a}\bar{b}}(\eta_{\rm ret},\, \vec{x}')\, \Big[\, 1 + \frac{\mathcal{O}(d(\eta_{\rm ret}))}{r} + \frac{\mathcal{O}(d(\eta_{\rm ret}))}{\Delta\eta_{\rm ret}}\, \Big]\,, \tag{2.26}$$

where $\Delta\eta_{\rm ret}$ is the dynamical time scale (measured in the η coordinate) in which the change in the source is of $\mathcal{O}(1)$. It will be clear from Section 2.3.2 that this is the time scale in which the change in the quadrupole moments of the source is $\mathcal{O}(1)$.

Up to this point, the mathematical manipulations are essentially the same as those in the linear theory off Minkowski spacetime [53]. The difference lies in the underlying assumptions and the physical meaning of the approximation scheme. A straightforward calculation relates the second and third terms in the square brackets in (2.26) to physical properties of the source. First, we have

$$\frac{d(\eta_{\rm ret})}{r} \;=\; \frac{D(\eta_{\rm ret})}{\ell_\Lambda}\, \frac{(-\eta_{\rm ret})}{r} \;\leq\; \frac{D_o}{\ell_\Lambda}\, (1 - \frac{\eta}{r})\,. \tag{2.27}$$

Note that to study the asymptotic form of the solution on $\mathcal{I}^+$, unlike in the calculation off Minkowski spacetime, we cannot use a large r approximation. Indeed, in the calculation of the radiated energy in Section 2.4, we will need to integrate over a finite range of r.[6] While $(1 - \eta/r)$ can be large, given *any* $r_o \neq 0$, we can choose a cosmological slice $\eta = \text{const}$ sufficiently close to $\mathcal{I}^+$ such that for all $r > r_o$, $(1 - \eta/r)$ is arbitrarily close to 1, whence $d(\eta_{\rm ret})/r$ is negligible. This is the late-time approximation.[7] In particular, on $\mathcal{I}^+$ (where $\eta = 0$) we can ignore the second term

[6] On $\mathcal{I}^+$ the energy flux will be non-zero in the interval $-\eta_2 < r < -\eta_1$, where (η_1, η_2) is the interval where the source is dynamical, i.e., $\mathcal{L}_T \mathcal{T}_{ab} \neq 0$.

[7] The term 'late-time' approximation should not be confused to mean 'only valid at late times'. On *any* constant time slice one can find a radius r_o such that for $r > r_o$ the approximations used to calculate χ_{ab} are valid. However, at early times this region of validity can be very restricted. At later times, the approximation automatically becomes valid for larger parts of the constant time slice under consideration. Hence, the name 'late-time' approximation.

in the square bracket in (2.26) for all $r > 0$. The third term can be re-expressed as

$$\frac{d(\eta_{\rm ret})}{\Delta\eta_{\rm ret}} = \frac{D(\eta_{\rm ret})}{\Delta t_{\rm ret}} \approx v \tag{2.28}$$

where D is the physical length scale of the source and Δt the interval in proper time in which the source changes by $\mathcal{O}(1)$, and where we have used the standard reasoning from Minkowski spacetime to conclude that the ratio $D(\eta_{\rm ret})/\Delta t_{\rm ret}$ can be identified with the velocity v of the source. We now use the slow motion approximation in which $v \ll 1$ (in our $c = 1$ units). Thus, within our assumptions the sharp term is given by

$$\sharp_{\bar{a}\bar{b}}\,(\eta, \vec{x}) \;=\; \frac{4G}{r}\,\int \mathrm{d}^3x'\, \mathcal{T}_{\bar{a}\bar{b}}(\eta_{\rm ret},\, \vec{x}')\,. \tag{2.29}$$

For the tail term $\flat_{\bar{a}\bar{b}}\,(\eta, \vec{x})$ in (2.17), this procedure only replaces $\eta_{\rm Ret}$ by $\eta_{\rm ret}$.

By adding the two contributions $\sharp_{\bar{a}\bar{b}}$ and $\flat_{\bar{a}\bar{b}}$, we can express χ_{ab} as follows:

$$\begin{aligned} \chi_{\bar{a}\bar{b}}(\eta,\, \vec{x}) \;&=\; \frac{4G}{r}\int \mathrm{d}^3\vec{x}'\; \mathcal{T}_{\bar{a}\bar{b}}(\eta_{\rm ret}, \vec{x}')\Big[\,1 + \mathcal{O}\Big(\frac{D_o}{\ell_\Lambda}\Big(1 - \frac{\eta}{r}\Big)\Big) + \mathcal{O}(v)\,\Big] \\ &+\;\; 4G\int_{-\infty}^{\eta_{\rm ret}}\mathrm{d}\eta'\,\frac{1}{\eta'}\,\partial_{\eta'}\,\int \mathrm{d}^3\vec{x}'\,\mathcal{T}_{\bar{a}\bar{b}}(\eta', \vec{x}')\,. \end{aligned} \tag{2.30}$$

(The error term arising from $\eta_{\rm Ret} \to \eta_{\rm ret}$ in the tail term is included in the square bracket in the first term.) On any $\eta = \eta_o$ slice, the second term in the square bracket can be neglected, in particular, for all $r > -\eta_o$, i.e., beyond the intersection of that slice with the cosmological horizon $E^-(i^+)$. On $\mathcal{I}^+$, it can be neglected for all $r > 0$.

Let us conclude by summarizing all the approximations that were made. First, in Section 2.2, we presented the retarded solution to Einstein's equations in the first post-de Sitter approximation. We then assumed that the source is compact in the sense that the physical size $D(\eta)$ of the support of the stress-energy tensor T_{ab} is uniformly bounded by D_o, with $D_o \ll \ell_\Lambda$. Finally, we used the first post-Newtonian approximation to set $v \ll 1$. Note that to obtain (2.31), we did not have to make any assumption relating the dynamical time scale $\Delta t_{\rm ret}$ of the system with the Hubble time $t_H = 1/H$. Astrophysical sources of greatest interest to the current gravitational wave observatories satisfy $\Delta t_{\rm ret} \ll t_H$ (this is the condition that specifies the high-frequency approximation). We will simplify the final results using this approximation in Section 2.4.2, however, throughout the main body of this chapter this assumption is not made and the results are true also for sources that do not satisfy $\Delta t_{\rm ret} \ll t_H$.

To avoid proliferation of symbols, from now on $\chi_{ab}(\eta, \vec{x})$ will stand for the approximate solution obtained by ignoring the $\mathcal{O}\Big((D_o/\ell_\Lambda)(1-\eta/r)\Big)$ and $\mathcal{O}(v)$ terms relative to the $\mathcal{O}(1)$ terms in (2.30). Thus, we will set

$$\chi_{\bar{a}\bar{b}}(\eta,\, \vec{x}) \;=\; \frac{4G}{r}\int \mathrm{d}^3\vec{x}'\; \mathcal{T}_{\bar{a}\bar{b}}(\eta_{\rm ret}, \vec{x}') + 4G\int_{-\infty}^{\eta_{\rm ret}} \mathrm{d}\eta'\,\frac{1}{\eta'}\,\partial_{\eta'}\, \int \mathrm{d}^3\vec{x}'\,\mathcal{T}_{\bar{a}\bar{b}}(\eta', \vec{x}')\,. \quad (2.31)$$

and again denote the sharp and the tail terms by $\sharp_{ab}$ and $\flat_{ab}$ respectively.

2.3.2 Expressing the approximate solutions in terms of quadrupole moments

To make the relation between the energy carried by the gravitational perturbations and the behavior of the source transparent, we will now express the approximate solution in terms of multipole moments of the source. Both terms on the right side of (2.31) involve the integral $\int \mathrm{d}^3\vec{x}'\; \mathcal{T}_{\bar{a}\bar{b}}$ of spatial components of the stress energy tensor of the source. We can rewrite this integral in terms of time derivatives of other components, using the conservation of T_{ab}. Recall that this strategy is used in the $\Lambda = 0$ case to express the integral entirely in terms of the second time derivative of the time-time component of T_{ab}, that is, the energy density. Consequently, for perturbations off flat space, only the mass quadrupole moment is relevant in the far-field approximation. As we will now show, the situation is more complicated in the $\Lambda > 0$ case because the conservation equation, $\overline{\nabla}^a T_{ab} = 0$, has additional terms due to the expansion of the scale factor of the de Sitter background.

In the $(t,\, \vec{x})$ coordinates, projection of the conservation equation along t^a (where, as usual, $t^a\partial_a := \partial/\partial t$) and $\mathring{q}_a{}^b$ yield, respectively,

$$\partial_t \rho - e^{-3Ht}\, \mathring{D}^a \mathcal{T}_a + H\,(3\rho + p_1 + p_2 + p_3) = 0, \quad (2.32)$$

$$\partial_t \mathcal{T}_a - e^{-Ht} \mathring{D}^b \mathcal{T}_{ab} + 2H\, \mathcal{T}_a = 0, \quad (2.33)$$

where the matter density and pressure are defined via

$$\rho = T_{ab} n^a n^b \equiv H^2 \eta^2\, T_{ab}\, \eta^a \eta^b, \quad \text{and} \quad p_{\bar{i}} = T^{ab}\,\partial_a x_{\bar{i}}\;\partial_b x_{\bar{i}}, \quad (2.34)$$

and where $\mathring{D}_a$ is the derivative operator compatible with the flat spatial metric $\mathring{q}_{ab}$. (In the last equation, there is no sum over $\bar{i}$.) In this $(t,\, \vec{x})$ chart it is manifest that when $\Lambda \to 0$ (i.e., $H \to 0$), these equations reduce to the time and space projections of the conservation equation with respect to the Minkowski metric $\mathring{\eta}_{ab}$. Extra terms proportional to H arise in de Sitter spacetime due to the expansion of the scale factor. These, in particular, include all the pressure terms which appear more generally in *any* spatially homogeneous and isotropic spacetime. Consequently, it will turn out that $\int \mathrm{d}^3\vec{x}\ \mathcal{T}_{\bar{a}\bar{b}}$ is related not just to the second time derivative of the mass quadrupole moment of the source as in flat spacetime, but also to the analogous pressure quadrupole moment. The exact dependence on the pressure terms will be derived below. But because they are multiplied by H, it is already clear that these terms will have fewer time derivatives than the corresponding terms involving density.

To recast $\int \mathrm{d}^3\vec{x}\ \mathcal{T}_{\bar{a}\bar{b}}$ in the desired form, our first task is to introduce the notion of mass and pressure quadrupole moments on the de Sitter background. Being a physical attribute of the source, the quadrupole moment at any instant of time should only depend on the physical geometry and coordinate invariant properties of the source. Therefore, we define the mass quadrupole moment as follows:

$$Q^{(\rho)}_{\bar{a}\bar{b}}(\eta) := \int_\Sigma \mathrm{d}^3V\ \rho(\eta)\ \bar{x}_{\bar{a}}\, \bar{x}_{\bar{b}}, \tag{2.35}$$

where Σ denotes any $\eta = \text{const}$ surface with proper volume element d^3V (which in Cartesian coordinates is $a^3(\eta)\, \mathrm{d}x\, \mathrm{d}y\, \mathrm{d}z$) and $\bar{x}_{\bar{a}} := a(\eta)x_{\bar{a}}$ is the *physical* separation of the point $\vec{x}$ from the origin. The pressure quadrupole moment is defined similarly:

$$Q^{(p)}_{\bar{a}\bar{b}}(\eta) := \int_\Sigma \mathrm{d}^3V\ \ (p_1(\eta) + p_2(\eta) + p_3(\eta))\ \bar{x}_{\bar{a}}\, \bar{x}_{\bar{b}}\,. \tag{2.36}$$

We can now use the conservation of stress-energy equations (2.32) and (2.33) to relate the integral $\int \mathrm{d}^3\vec{x}'\ \mathcal{T}_{\bar{a}\bar{b}}$ to these quadrupole moments and their time derivatives.

This derivation follows the same steps as in the standard calculation in Minkowski spacetime. We begin by noting that $\int \mathrm{d}^3\vec{x}'\ \mathcal{T}_{\bar{a}\bar{b}}(t',\, \vec{x}') = -\int \mathrm{d}^3\vec{x}'\ (\mathring{D}^{\bar{m}}\mathcal{T}_{\bar{m}(\bar{a}})x_{\bar{b})}$ because the boundary term that arises in the integration by parts vanishes since the stress-energy tensor has compact spatial support. Using the spatial projection

(2.33) of the conservation equation, we can rewrite the integral as follows:

$$\begin{aligned}\int \mathrm{d}^3\vec{x}'\ \mathcal{T}_{\bar{a}\bar{b}}(t',\,\vec{x}') &= -\int \mathrm{d}^3\vec{x}'\ e^{Ht'}\Big(\partial_{t'}+2H\Big)\,\mathcal{T}_{(\bar{a}}(t',\,\vec{x}')\ x_{\bar{b})} \\ &= \frac{1}{2}\int \mathrm{d}^3\vec{x}'\ e^{Ht'}\,(\partial_{t'}+2H)\,\Big(\mathring{D}^{\bar{m}}\mathcal{T}_{\bar{m}}(t',\,\vec{x}')\Big)\,x_{\bar{a}}\,x_{\bar{b}}. \quad (2.37)\end{aligned}$$

Next, we use (2.32), the projection of the conservation equation along t^a, to eliminate $\mathcal{T}_a$ in favor of the energy density and pressure:

$$\begin{aligned}\int \mathrm{d}^3\vec{x}'\ \mathcal{T}_{\bar{a}\bar{b}}(t',\,\vec{x}') = \frac{1}{2}\int \mathrm{d}^3\vec{x}'\ e^{4Ht'}\Big[&\frac{\partial^2\rho}{\partial t'^2}+H\frac{\partial}{\partial t'}(\,8\,\rho+p_1+p_2+p_3) \\ &+5H^2(3\,\rho+p_1+p_2+p_3)\Big]x_{\bar{a}}x_{\bar{b}}. \quad (2.38)\end{aligned}$$

The last step in this derivation is to express the right side of (2.38) in terms of the quadrupole moments defined in (2.35) and (2.36). A simple calculation yields:

$$\mathring{e}^{\bar{a}}_a\,\mathring{e}^{\bar{b}}_b\ \int \mathrm{d}^3\vec{x}'\,\mathcal{T}_{\bar{a}\bar{b}}(t',\vec{x}') = \frac{1}{2a(t')}\Big[\partial^2_{t'}\,Q^{(\rho)}_{ab}-2H\partial_{t'}\,Q^{(\rho)}_{ab}+H\partial_{t'}Q^{(p)}_{ab}\Big](t')\,, \quad (2.39)$$

where $\mathring{e}^{\bar{a}}_a$ are the basis co-vectors in the $\vec{x}$-chart. Finally, using the fact that Lie derivative of any tensor field Q_{ab} with respect to the time translation Killing vector field is given by $\mathcal{L}_T Q_{ab} = T^c\mathring{\nabla}_c\,Q_{ab} - 2H\,Q_{ab}$, it is straightforward to show that

$$\begin{aligned}\mathring{e}^{\bar{a}}_a\,\mathring{e}^{\bar{b}}_b\ \int \mathrm{d}^3\vec{x}'\,\mathcal{T}_{\bar{a}\bar{b}}(t',\vec{x}') = \frac{1}{2a(t')}\Big[&\mathcal{L}_T\mathcal{L}_T Q^{(\rho)}_{ab}+2H\mathcal{L}_T Q^{(\rho)}_{ab} \\ &+H\mathcal{L}_T Q^{(p)}_{ab}+2H^2 Q^{(p)}_{ab}\Big](t')\,. \quad (2.40)\end{aligned}$$

Since one can readily take the limit $\Lambda \to 0$ in the $(t,\vec{x})$ chart, we see immediately that in this limit one recovers the familiar expression

$$\mathring{e}^{\bar{a}}_a\,\mathring{e}^{\bar{b}}_b\ \int \mathrm{d}^3\vec{x}'\ \mathcal{T}_{\bar{a}\bar{b}}(t',\vec{x}') \quad \to \quad \frac{1}{2}\,[\mathcal{L}_t\,\mathcal{L}_t\,Q^{(\rho)}_{ab}] \quad (2.41)$$

from the discussion of the quadrupole formula in Minkowski spacetime.

Let us return to eq.(2.40). Note that it is an *exact* equality within the post-de Sitter approximation; in Section 2.3.2 we have not used the assumption $D_o \ll \ell_\Lambda$ on the size of the source, nor the post-Newtonian assumption $v \ll 1$. If we invoke, e.g., kinetic theory, then the pressure goes as $p \sim \rho v^2$ and can then be ignored compared to the density ρ. Then (2.40) simplifies to

$$\mathring{e}_a^{\bar{a}}\,\mathring{e}_b^{\bar{b}}\ \int \mathrm{d}^3\vec{x}'\ \mathcal{T}_{\bar{a}\bar{b}}(t',\vec{x}')\ \approx\ \frac{1}{2a(t')}\Big[\mathcal{L}_T\mathcal{L}_T Q^{(\rho)}_{ab} + 2H\mathcal{L}_T Q^{(\rho)}_{ab} + 2H^2 Q^{(p)}_{ab}\Big](t')\,. \quad (2.42)$$

where we have retained the last term because so far we have not made any assumption on relative magnitudes of the dynamical time scale of the system and Hubble time $1/H$. Now, in the post-Minkowski analysis, one does not have to make the assumption $p \ll \rho$ because the continuity equations (2.32) do not involve pressure terms in that case. Furthermore, in the $\Lambda > 0$ case, it turns out that dropping the pressure term from the exact expression (2.40) obscures certain conceptually important features (see footnote 14). Therefore we will retain the full expression for now.

Finally we can express the solution (2.31) on $(M_{\mathrm{P}}^+, \bar{g}_{ab})$ in terms of the source quadrupole moments (after a simple transformation to the $(\eta,\,\vec{x})$ chart). Denoting by an 'overdot' the Lie derivative with respect to T^a, we obtain:

$$\begin{aligned}
\chi_{ab}(\eta,\,\vec{x}) \ &= \ \frac{2G}{r\,a(\eta_{\mathrm{ret}})}\ \Big[\ddot{Q}^{(\rho)}_{ab} + 2H\dot{Q}^{(\rho)}_{ab} + H\dot{Q}^{(p)}_{ab} + 2H^2 Q^{(p)}_{ab}\Big](\eta_{\mathrm{ret}}) \\
&+ \ 2G\int_{-\infty}^{\eta_{\mathrm{ret}}}\frac{\mathrm{d}\eta'}{\eta'}\,\partial_{\eta'}\,\frac{1}{a(\eta')}\Big[\ddot{Q}^{\rho}_{ab} + 2H\dot{Q}^{(\rho)}_{ab} + H\dot{Q}^{(p)}_{ab} + 2H^2 Q^{(p)}_{ab}\Big](\eta') \\
&=: \ \sharp_{ab}(\eta,\vec{x})\,+\,\flat_{ab}(\eta,\vec{x})
\end{aligned} \quad (2.43)$$

This expression is a good approximation to the exact solution (2.16) everywhere on $\mathcal{I}^+$ (except at $r = 0$).

2.3.3 The tail term and its properties

A qualitative difference between the $\Lambda > 0$ and $\Lambda = 0$ case is the presence of the tail term in the retarded solution at lowest in the post-de Sitter. Before we continue to find the expression for the energy and power radiated, we will discuss some properties of this term. First, we show that although the tail term arises as a result of backscattering of the de Sitter background curvature, the dynamical part of the leading order part in the post-Newtonian expansion of the tail term propagates sharply along the null cone. Second, the $\Lambda \to 0$ limit of the tail term is briefly discussed. Third, we compare the size of the sharp and tail term in the asymptotic region. We address the question whether $\flat_{ab}$ is negligible compared to the sharp term $\sharp_{ab}$ if $\Lambda \neq 0$ but tiny. We will show that the answer is in the *negative*. This is another illustration of the subtlety of the limit $\Lambda \to 0$. Finally, we conclude with a remarkable property of the tail term: the sharp and tail term conspire in such a way that the terms proportional to v^n in the sharp and tail term exactly cancel each other on $\mathcal{I}^+$. Thus, the gravitational perturbation χ_{ab} is true to all order in the slow motion expansion![8]

To discuss these properties, it is most convenient to work in the $(t, \vec{x})$ chart. In this chart, the tail term assumes the form

$$\flat_{ab}(t,\vec{x}) \;=\; -2GH\int_{-\infty}^{t_{\rm ret}}\!\mathrm{d}t'\Big[\dddot{Q}^{(\rho)}_{ab}+3H\ddot{Q}^{\rho}_{ab}+2H^2\dot{Q}^{(\rho)}_{ab}+H\ddot{Q}^{(p)}_{ab}+3H^2\dot{Q}^{(p)}_{ab}+2H^3Q^{(p)}_{ab}\Big](t')\,. \tag{2.44}$$

The integral over t can be performed using $\dot{Q}_{ab} = \partial_t Q_{ab} - 2HQ_{ab}$. This results in

$$\flat_{ab}(t,\vec{x}) \;=\; -2GH\,\Big[\,\ddot{Q}^{(\rho)}_{ab} + H\dot{Q}^{(\rho)}_{ab} + H\dot{Q}^{(p)}_{ab} + H^2 Q^{(p)}_{ab}\Big]_{-\infty}^{t_{\rm ret}}\,. \tag{2.45}$$

As shown in Section 2.4.3, the assumption $\mathcal{L}_T\, T_{ab} = 0$ in the distant past implies $\dot{Q}^{(\rho)}_{ab} = -2HQ^{(\rho)}_{ab}$ there (similarly for the pressure quadrupole). Therefore, we have

$$\flat_{ab}(t,\vec{x}) \;= -2GH\,\Big[\,\ddot{Q}^{(\rho)}_{ab} + H\dot{Q}^{(\rho)}_{ab} + H\dot{Q}^{(p)}_{ab} + H^2 Q^{(p)}_{ab}\Big](t_{\rm ret}) \;+\; 2GH^3\, C_{ab}\,, \tag{2.46}$$

where C_{ab} is just a constant term.[9] In general, a hereditary term can have contributions that propagate along *and* inside the null cone. However, the above expression shows that the dynamical part of the hereditary term to leading order in the multipolar expansion only propagates along the null cone. One could describe this dynamical part as an 'instantaneous tail'. The constant term spoils the instantaneous nature of the tail term. Nevertheless, as we shall see in the next section, the constant term does not play any role in the fluxes radiated through $\mathcal{I}^+$. For the expression of the energy flux in (2.67), this is immediate as only derivatives of χ_{ab} appear. In the expression of the flux of angular momentum in (2.84), χ_{ab} does appear without a derivative but the constant term does not contribute because it is integrated against $\mathcal{E}^{ab}$ which is of compact support and divergence-free on $\mathcal{I}^+$.

Second, in Section 2.2, we had seen that the tail term vanishes in the limit

[8] This latter property was not discussed in the original paper [36] and has not been published elsewhere. The rest of this section is based on Appendix A in [36].

[9] Its expression is $C_{ab} = -2Q^{(\rho)}_{ab}(t=-\infty) + Q^{(P)}_{ab}(t=-\infty)$.

$\Lambda \to 0$. The same is true for the leading order part of $\flat_{ab}$ in the post-Newtonian expansion. Since the 'overdot' tends to the well-defined Lie derivative with respect to the time translation Killing vector field $\mathring{t}^a$ in Minkowski spacetime in the limit $\Lambda \to 0$, the overall multiplicative factor H in (2.44) makes it transparent that also $\flat_{ab}$ in its approximate form does vanish in the $\Lambda \to 0$ limit. This makes the limit to Minkowski spacetime continuous.

Now let us come back to the question of the importance of the tail term. In order to do so, we return to eq. (2.43) and, for $r > -\eta$, write χ_{ab} as

$$\chi_{ab}(\eta, \vec{x}) = \frac{2G}{R(\eta_{\rm ret})} \left[(1 - \frac{r}{r-\eta})\, \ddot{Q}^{(\rho)}_{ab}\right] + \mathcal{O}(H) \tag{2.47}$$

where $R(\eta_{\rm ret}) = a(\eta_{\rm ret})\, r$ is the physical distance between the source and the point $\vec{x}$ at time $\eta = \eta_{\rm ret}$. The factor 1 in the square bracket comes from the sharp term while the factor $r/(r-\eta)$ comes from the tail term. At late times the two contributions are comparable and at $\mathcal{I}^+$ they are in fact equal in magnitude but opposite in sign. This occurs no matter how small Λ is! The remainder – i.e., the $\mathcal{O}(H)$ term – also has contributions from both the sharp and the tail terms:

$$\begin{aligned}\chi^{\rm rem}_{ab}(\vec{x}) = \frac{2GH}{R(\eta_{\rm ret})} \Big[2(1 - \frac{1}{2}\frac{r}{r-\eta})\, \dot{Q}^{(\rho)}_{ab} + (1 - \frac{r}{r-\eta})\dot{Q}^{(p)}_{ab} \\ + 2H(1 - \frac{1}{2}\frac{r}{r-\eta})Q^{(p)}_{ab}\Big] + 2H^3 C_{ab}\,.\end{aligned} \tag{2.48}$$

So that on $\mathcal{I}^+$, the linearized perturbation is:

$$\chi_{ab}(\vec{x}) \hat{=} 2GH^2\, [\dot{Q}^{(\rho)}_{ab} + HQ^{(p)}_{ab}] + 2H^3 C_{ab}\,. \tag{2.49}$$

This analysis provides the precise sense in which the backscattering effects encoded in the tail term – which can also be thought of as arising from the addition of a mass term to the propagation equation of $\bar{\gamma}_{ab}$ – provide an $\mathcal{O}(1)$ contribution to the metric perturbation χ_{ab} near $\mathcal{I}^+$. This is a concrete realization of the non-trivial outcome of secular accumulation of small effects we referred to in Section 2.1. Moreover, as we will see in the next section, the tail term contributes on an equal footing as the sharp term to the expression of energy and angular momentum radiated across $\mathcal{I}^+$. In fact, without the tail term these expressions are ill-defined! This stresses the importance of the tail term for de Sitter spacetime. We will return to this point in the next section. A cautionary remark is in order at this point: One should not take the $\Lambda \to 0$ limit of the above equation (2.49) from which one would erroneously conclude to χ_{ab} vanishes in this limit. As discussed, the $\Lambda \to 0$ limit is subtle and has to be taken in the $(t, \vec{x})$ rather than the $(\eta, \vec{x})$ chart. Since the $(t, \vec{x})$ chart breaks down at $\mathcal{I}^+$ (where $\eta = 0$ but $t = \infty$), we cannot take the limit of our χ_{ab} at $\mathcal{I}^+$ of de Sitter spacetime. Rather, we have to 'pass through' the physical spacetime, as was done at the end of Section 2.2 and in this section.

Finally, let us discuss the slow motion expansion of the tail term. We start by performing similar manipulations as in Section 2.3.1, but now apply these to the tail term. We first write the tail term as

$$\flat_{ab} = 4G \int \mathrm{d}^3x' \int \mathrm{d}^3y \, \delta(\vec{x}', \vec{y}) \int_{-\infty}^{\eta_{\mathrm{Ret}}} \mathrm{d}\eta' \frac{1}{\eta'} \partial_{\eta'} \mathcal{T}_{ab}(\eta', \vec{y})$$

and Taylor expand the integrand around $\vec{x}' = 0$:

$$\begin{aligned}\flat_{ab} = 4G \int \mathrm{d}^3x' \int \mathrm{d}^3y \, \delta(\vec{x}', \vec{y}) \Bigg(& \int_{-\infty}^{\eta_{\mathrm{ret}}} \mathrm{d}\eta' \frac{1}{\eta'} \partial_{\eta'} \mathcal{T}_{ab}(\eta', \vec{y}) \\ & - x'^c \frac{\partial}{\partial x^c} \int_{-\infty}^{\eta_{\mathrm{ret}}} \mathrm{d}\eta' \frac{1}{\eta'} \partial_{\eta'} \mathcal{T}_{ab}(\eta', \vec{y}) + \ldots \Bigg) . \end{aligned} \tag{2.50}$$

where the ... indicate the higher order terms in the Taylor expansion. We will focus first on the first order term in $\vec{x}'$ denoted by the superscript (1) on $\flat_{ab}$ and rewrite it in the following way

$$\flat^{(1)}_{ab} = 4G \int \mathrm{d}^3x' \int \mathrm{d}^3y \, \delta(\vec{x}', \vec{y}) \left(-x'^c \frac{\partial}{\partial x^c} \right) \int_{-\infty}^{\eta_{\mathrm{ret}}} \mathrm{d}\eta' \frac{1}{\eta'} \partial_{\eta'} \mathcal{T}_{ab}(\eta', \vec{y}) \tag{2.51}$$

$$= 4G \int \mathrm{d}^3x' \int \mathrm{d}^3y \, \delta(\vec{x}', \vec{y}) \left(-x'^c \frac{\partial \eta_{\mathrm{ret}}}{\partial x^c} \frac{\partial}{\partial \eta_{\mathrm{ret}}} \right) \int_{-\infty}^{\eta_{\mathrm{ret}}} \mathrm{d}\eta' \frac{1}{\eta'} \partial_{\eta'} \mathcal{T}_{ab}(\eta', \vec{y}) \tag{2.52}$$

$$= 4G \int \mathrm{d}^3x' \, \frac{x'^c \hat{r}_c}{\eta_{\mathrm{ret}}} \partial_{\eta_{\mathrm{ret}}} \mathcal{T}_{ab}(\eta_{\mathrm{ret}}, \vec{x}') \, , \tag{2.53}$$

where in the last step the derivative with respect to η_{ret} was taken and the integral over y performed. Evaluated on $\mathcal{I}^+$, this term is

$$\flat^{(1)}_{ab} \, \hat{=} - 4G \int \mathrm{d}^3x' \, \frac{x'^c \hat{r}_c}{r} \partial_{\eta_{\mathrm{ret}}} \mathcal{T}_{ab}(\eta_{\mathrm{ret}}, \vec{x}') \, . \tag{2.54}$$

Following the same arguments in Section 2.3.1, this term is order v. Comparing this term to the order v term of the sharp term (see the last term in eq. (2.23)), it is obvious that these two terms are exactly equal in magnitude but opposite in sign on $\mathcal{I}^+$. Hence, to first order in the Taylor expansion in $\vec{x}'$, the terms proportional to v cancel each other on $\mathcal{I}^+$. Note that this cancellation *only* occurs on $\mathcal{I}^+$ and *not inside spacetime* due to the $\eta_{\rm ret}^{-1}$ factor in eq. (2.53) as opposed to the r^{-1} factor in eq. (2.23). Curiously, this cancellation does not only happen to the lowest order in the multipolar expansion, but occurs to all order. For any nth-order in the Taylor expansion, we can write:

$$\begin{aligned}
\flat_{ab}^{(n)} &= (-1)^n \frac{4G}{\eta_{\rm ret}} \int \mathrm{d}^3x'\, x'^{c_1}\hat{r}_{c_1}\, x'^{c_2}\hat{r}_{c_2}\, \ldots\, x'^{c_n}\hat{r}_{c_n}\, \partial^n_{\eta_{\rm ret}} \mathcal{T}_{ab}(\eta_{\rm ret}, \vec{x}') + \frac{\mathcal{O}(d(\eta_{\rm ret}))}{r} \\
&\hat{=} -(-1)^n \frac{4G}{r} \int \mathrm{d}^3x'\, x'^{c_1}\hat{r}_{c_1}\, x'^{c_2}\hat{r}_{c_2}\, \ldots\, x'^{c_n}\hat{r}_{c_n}\, \partial^n_{\eta_{\rm ret}} \mathcal{T}_{ab}(\eta_{\rm ret}, \vec{x}') + \frac{\mathcal{O}(d(\eta_{\rm ret}))}{r} \\
&\hat{=} -\sharp_{ab}^{(n)} + \frac{\mathcal{O}(d(\eta_{\rm ret}))}{r}
\end{aligned}$$

so that all terms proportional to v^n in the sharp and tail term cancel each other exactly on $\mathcal{I}^+$. On $\mathcal{I}^+$ the gravitational perturbation does not have any terms proportional to v^n![10] In other words, the gravitational wave perturbation on $\mathcal{I}^+$ expressed in terms of quadrupole moments in eq. (2.31) does not require the matter distribution to move slowly; It is also true for sources that do not satisfy $v \ll 1$.

This leads to the following conundrum. Given that the gravitational wave perturbation does not receive any corrections proportional to v^n on $\mathcal{I}^+$, the gravitational strain on $\mathcal{I}^+$ only contains information about the quadrupolar nature of the source and information about the source's higher multipole moments is lost. Is this information indeed inaccessible to $\mathcal{I}^+$? Fortunately, it is not. Although there are no terms proportional to v^n in the gravitational wave perturbation on $\mathcal{I}^+$, these terms are registered in the time derivative of this perturbation, $\partial_\eta \chi_{ab}$, because $\partial_\eta \flat_{ab}^{(n)}$ does not cancel $\partial_\eta \sharp_{ab}^{(n)}$ on $\mathcal{I}^+$. Since the expressions for radiated energy and angular momentum contain time derivatives of χ_{ab}, as we shall see in the next section, the fluxes presented there are only valid to $\mathcal{O}(v)$.[11]

[10] There are terms proportional to $v^m \frac{\mathcal{O}(d(\eta_{\rm ret}))}{r}^{n-m}$, but these can be argued to be small by the fact that $\frac{\mathcal{O}(d(\eta_{\rm ret}))}{r}$ is small.

[11] This is as anticipated based on the expectation that the $\Lambda \to 0$ should recover the expressions for energy and flux radiated by a source on a Minkowski background and the Minkowski result contains terms proportional to v^n.

2.4 Time-varying quadrupole moment and energy flux

In this section, we will carry out the second main step spelled out in Section 2.1: We will use the approximate solution (2.43) to generalize Einstein's quadrupole formula for the energy E_T carried away by gravitational waves across $\mathcal{I}^+$. Since linearized gravitational fields do not have a gauge invariant, local stress-energy tensor, we employ the covariant Hamiltonian framework used in [35] to compute this energy.

This section is divided into three parts. In the first, we will discuss the asymptotic behavior of the fields that enter the expression of energy E_T, in the second, we will derive the quadrupole formula, and in the third, we will discuss its properties.

2.4.1 $\mathcal{I}^+$ and the perturbed electric part $\mathcal{E}_{ab}$ of Weyl curvature

As in the $\Lambda = 0$ case, it is simplest to obtain manifestly gauge invariant expressions of fluxes of energy-momentum and angular momentum carried away by gravitational waves using fields defined on $\mathcal{I}^+$. Therefore we need to carry out a future conformal completion of the background spacetime $(M_{\rm P}^+, \bar{g}_{ab})$. It is natural to seek a completion that makes $(M_{\rm P}^+, \bar{g}_{ab})$ asymptotically de Sitter in a Poincaré patch in the sense of [31]. Because the physical metric $\bar{g}_{ab}$ has the form,

$$\bar{g}_{ab} = a^2 \,\mathring{g}_{ab} \equiv (H\eta)^{-2}\, \mathring{g}_{ab}\,, \tag{2.55}$$

it is easy to verify that such a conformal completion can be obtained by setting the conformal factor $\Omega = -H\eta$, so that the conformally rescaled 4-metric, which is smooth at $\mathcal{I}^+$, is simply the flat metric $\mathring{g}_{ab}$. We will use this completion because all our equations in the Cartesian chart of $\mathring{g}_{ab}$ and the solution χ_{ab} will then automatically hold on the conformally completed spacetime, including $\mathcal{I}^+$. The final results, of course, will be conformally invariant as in [31, 35].

The formulas for fluxes of energy-momentum and angular momentum involve the so-called *perturbed electric part of the Weyl tensor,* $\mathcal{E}_{ab}$, at $\mathcal{I}^+$ [35]. Therefore, we will first express $\mathcal{E}_{ab}$ in terms of the metric perturbations – for which we already have the explicit expression (2.43) in terms of the quadrupole moments – and then discuss its properties needed in the subsequent discussion.

For any spacetime that is weakly asymptotically de Sitter spacetime, the

Weyl curvature of the conformally rescaled metric at $\mathcal{I}$ vanishes and therefore $\Omega^{-1}C_{abc}{}^{d}$ admits a smooth limit there [31, 43]. De Sitter spacetime with linearized perturbations is an example of a weakly asymptotically de Sitter spacetime and hence $\Omega^{-1}C_{abc}{}^{d}$ – which is the sum of the background Weyl curvature scalar and the linearized perturbations, i.e. $\Omega^{-1}C_{abc}{}^{d} = \Omega\mathring{C}_{abc}{}^{d} + \Omega^{-1}\,{}^{(1)}\mathcal{C}_{abc}{}^{d}$ – also has a smooth limit to $\mathcal{I}$. Our conformally rescaled metric $\mathring{g}_{ab}$ is flat, whence the limit of $\Omega^{-1}\mathring{C}_{abc}{}^{d}$ vanishes. Therefore, not only is the first order perturbation ${}^{(1)}\mathcal{C}_{abc}{}^{d}$ such that $\Omega^{-1}\,{}^{(1)}\mathcal{C}_{abc}{}^{d}$ admits a limit to $\mathcal{I}^{+}$, but furthermore the limit is *gauge invariant.* The field of interest is the limit to $\mathcal{I}^{+}$ of its electric part,

$$\mathcal{E}_{ab} := \Omega^{-1}\,({}^{(1)}\mathcal{C}_{amb}{}^{n}\,\eta^{m}\eta_{n}) \;=\; -\,(H\eta)^{-1}\,({}^{(1)}\mathcal{C}_{amb}{}^{n}\,\eta^{m}\eta_{n})\,, \tag{2.56}$$

where, as before, η^{a} is the unit normal to the cosmological slices ($\eta = \text{const}$) with respect to the conformal metric $\mathring{g}_{ab}$ and the indices are raised and lowered also using $\mathring{g}_{ab}$. We need to express $\mathcal{E}_{ab}$ in terms of the (trace-reversed, rescaled) metric perturbation $\bar{\chi}_{ab}$ produced by the source. This can be accomplished using the expression of ${}^{(1)}\mathcal{C}_{abc}{}^{d}$ in terms of the metric perturbation $\bar{\gamma}_{ab}$, and the equation of motion (2.9). The final result is:

$$\mathcal{E}_{ab} \;=\; \frac{1}{2H\eta}\;(\mathring{q}_{a}{}^{c}\mathring{q}_{b}{}^{d} - \frac{1}{3}\mathring{q}_{ab}\mathring{q}^{cd})\;\Big[\frac{1}{2}\mathring{D}_{c}\mathring{D}_{d}\tilde{\chi} - \mathring{D}_{(c}\mathring{D}^{m}\chi_{d)m} - \mathring{D}_{(c}\,\partial_{\eta}\chi_{d)} + (\partial_{\eta}^{2} - \frac{1}{\eta}\partial_{\eta})\chi_{cd}\Big]. \tag{2.57}$$

Let us discuss the limit of each term to $\mathcal{I}^{+}$. Although we already know from general considerations that the left side of (2.57) admits a smooth limit to $\mathcal{I}^{+}$, some care is needed to evaluate the right hand side because there is a $(1/\eta)$ pre-factor, and $\eta = 0$ at $\mathcal{I}^{+}$. However, because the explicit retarded solutions (2.11) decay as η, one can show that the terms involving $\tilde{\chi}$ and $\chi_{\bar{a}}$ admit a smooth limit to $\mathcal{I}$. A more detailed calculation using (2.31) shows that the fourth term, $(1/\eta)\Big(\partial_{\eta}^{2} - \frac{1}{\eta}\partial_{\eta}\Big)\chi_{ab}$, also has a smooth limit to $\mathcal{I}^{+}$:

$$\frac{1}{\eta}\left(\partial_{\eta}^{2} - \frac{1}{\eta}\partial_{\eta}\right)\chi_{\bar{a}\bar{b}} \;=\; \frac{4G}{r}\Big[\frac{1}{\eta_{\rm ret}}\partial_{\eta}^{2}\int \mathrm{d}^{3}\vec{x}'\;\mathcal{T}_{\bar{a}\bar{b}}(\eta_{\rm ret},\vec{x}') - \frac{1}{\eta_{\rm ret}^{2}}\partial_{\eta}\int \mathrm{d}^{3}\vec{x}'\;\mathcal{T}_{\bar{a}\bar{b}}(\eta_{\rm ret},\vec{x}')\Big]\,. \tag{2.58}$$

Thus, we have expressed $\mathcal{E}_{ab}$ at $\mathcal{I}^{+}$ in terms of the perturbed metric, as required. In particular, in spite of the presence of a $(1/\eta)$-pre-factor in (2.57), each of the four terms in that formula for $\mathcal{E}_{ab}$ has well-defined limits to $\mathcal{I}^{+}$. Note, incidentally, that in this calculation not only does the tail term $\flat_{ab}$ in χ_{ab} contribute but the result would diverge at $\eta = 0$ without it. However, the process of taking derivatives has made the integral over η' in $\flat_{ab}$ disappear, showing that the propagation of the left side of (2.58) sharp. These features and eq. (2.58) in particular will play an important role in Section 2.4.2.

We will now discuss the properties of $\mathcal{E}_{ab}$ that will be needed in subsequent calculations. First, the field equations satisfied by the first order perturbation ${}^{(1)}C_{abc}{}^{d}$ are conformally invariant. Since they are completely equivalent to the field equations satisfied by the first order Weyl tensor in the flat spacetime $(M_{\rm P}^{+}, \mathring{g}_{ab})$, we know that the propagation of ${}^{(1)}C_{abc}{}^{d}$ is sharp along the null cones of $\mathring{g}_{ab}$ (which are the same as the null cones of the de Sitter metric $\bar{g}_{ab}$). Therefore the expression of the field $\mathcal{E}_{ab}$ at $\mathcal{I}^{+}$ in terms of source quadrupole moments is also sharp. Indeed, this is true as the dynamical part of χ_{ab} propagates along the null cone and so does $\mathcal{E}_{ab}$. Second, in any neighborhood of $\mathcal{I}^{+}$ where there are no matter sources, the field $\mathcal{E}_{ab}$ is divergence-free

$$\mathring{D}^{a}\mathcal{E}_{ab} \;=\; 0. \tag{2.59}$$

Thus, $\mathcal{E}_{ab}$ is transverse, traceless on $\mathcal{I}^{+}$. This property will make the gauge invariance of our expression of energy flux transparent. Finally, as one would expect from the fact that $\mathcal{E}_{ab}$ is gauge invariant, only the transverse-traceless (TT) components of χ_{ab} (in its decomposition into irreducible parts) contribute to $\mathcal{E}_{ab}$. Let us begin with a standard decomposition of the ten components of the (rescaled, trace-reversed) metric perturbation $\bar{\chi}_{ab}$:

$$\begin{aligned} \tilde{\chi} \;&:=\; (\eta^{a}\eta^{b} + \mathring{q}^{ab})\,\bar{\chi}_{ab}\,, \qquad\qquad \chi := \mathring{q}^{ab}\chi_{ab}, \qquad\qquad \chi_{a} =: \mathring{D}_{a}A + A^{T}_{a}, \\ \chi_{ab} \;&=:\; \frac{1}{3}\mathring{q}_{ab}\,\mathring{q}^{cd}\chi_{cd} + \left(\mathring{D}_{a}\mathring{D}_{b} - \frac{1}{3}\mathring{q}_{ab}\mathring{D}^{2}\right)B + 2\mathring{D}_{(a}B^{T}_{b)} + \chi^{TT}_{ab}\,, \end{aligned} \tag{2.60}$$

where A^{T}_{a} and B^{T}_{a} are transverse and χ^{TT}_{ab} is transverse-traceless,

$$\mathring{D}^{a}A^{T}_{a} = 0 \qquad \mathring{D}^{a}B^{T}_{a} = 0 \qquad \mathring{D}^{a}\chi^{TT}_{ab} = 0 \qquad \mathring{q}^{ab}\chi^{TT}_{ab} = 0\,, \tag{2.61}$$

and $\tilde{\chi}$, χ, B, $\mathring{D}_{a}A$ are the longitudinal modes. Using the gauge condition (2.3), one can show that, in the expression (2.57) of $\mathcal{E}_{ab}$, all contributions from the longitudinal and trace parts of $\bar{\chi}_{ab}$ cancel out and $\mathcal{E}_{ab}$ depends only on χ_{ab}^{TT}:

$$\mathcal{E}_{ab} \;=\; \frac{1}{2H\eta}\,\Big[\partial_\eta^2 - \frac{1}{\eta}\partial_\eta\Big]\chi_{ab}^{TT}\,, \tag{2.62}$$

Since $\mathcal{E}_{ab}$ and χ_{ab}^{TT} are both gauge invariant, the final relation (2.62) holds in any gauge. The limit to $\mathcal{I}^+$ of this equality will play an important role in the next two sub-sections.

Note that in the study of retarded fields produced by compact sources in asymptotically flat spacetimes, one often uses an entirely different decomposition: Here, the $1/r$-part of χ_{ab} (i.e., the far field approximation) of the full retarded solution is projected into radial and the orthogonal spherical directions in *physical space*. Unfortunately, the projection onto the spherical directions is also referred to as the transverse-traceless parts of χ_{ab}. For concreteness, let us denote by P_a^b the projection operator into the 2-sphere orthogonal to the radial direction in the physical space and set $\chi_{ab}^{tt} = (P_a{}^c P_b{}^d - (1/2)P_{ab}P^{cd})\chi_{cd}$. In the literature, in place of tt, the symbol TT is used also for this projection (see, e.g., Chapter 11 of [53], or Section 4.5.1 in [54], or Section 36.10 in [55]). This is confusing because the two notions of transverse traceless parts are distinct and inequivalent as we discussed briefly in Chapter 1 and will elaborate upon in Chapter 4. The first notion is local in momentum space and the resulting χ_{ab}^{TT} is exactly gauge invariant everywhere in spacetime. The second notion, which we will continue to denote by χ_{ab}^{tt}, is local in physical space and χ_{ab}^{tt} is gauge invariant only in a weaker sense involving $1/r$ fall-offs.

2.4.2 Fluxes across $\mathcal{I}^+$

Let us calculate the flux of energy associated with the time translation T^a across $\mathcal{I}^+$. Since T^a is a Killing field of the background spacetime $(M_{\rm P}^+, \bar{g}_{ab})$ we know that, for any choice of admissible conformal completion, T^a admits a smooth extension which is tangential to $\mathcal{I}^+$. For the choice $\Omega = -H\eta$ of the conformal factor we made above, T^a also serves as the dilation vector field with respect to the intrinsic 3-metric $\mathring{q}_{ab}$ on $\mathcal{I}^+$:

$$T \;\hat{=}\; -H\,\Big[x\frac{\partial}{\partial x} + y\frac{\partial}{\partial y} + z\frac{\partial}{\partial z}\Big]. \tag{2.63}$$

From the detailed analysis of the covariant phase space $\Gamma_{\rm Cov}$ carried out in [35], the total energy flux E_T across $\mathcal{I}^+$ is given by the Hamiltonian generating the time translation T^a on $\Gamma_{\rm Cov}$. (Here, we will denote the energy flux by E_T instead of $\mathcal{H}_T$ as was done in Chapter 1.) The result can be expressed most simply in terms of $\mathcal{E}_{ab}$ and the Lie derivative of the metric perturbation with respect to T^a at $\mathcal{I}^+$:

$$E_T \,\hat{=}\, \frac{1}{16\pi GH} \int_{\mathcal{I}^+} \mathrm{d}^3x \;\mathcal{E}_{cd} \left(\mathcal{L}_T \chi_{ab} + 2H\,\chi_{ab}\right) \mathring{q}^{ac}\mathring{q}^{bd}\,. \tag{2.64}$$

Note that because $\mathcal{E}_{ab}$ is transverse-traceless (TT), the integral automatically extracts the TT part of the term in the bracket and we have

$$\begin{aligned} E_T \;\; &\hat{=} \;\; \frac{1}{16\pi GH} \int_{\mathcal{I}^+} \mathrm{d}^3x \;\mathcal{E}_{cd} \left(\mathcal{L}_T \chi_{ab} + 2H\,\chi_{ab}\right)^{TT} \mathring{q}^{ac}\mathring{q}^{bd} \\ &\hat{=} \;\; \frac{1}{16\pi GH} \int_{\mathcal{I}^+} \mathrm{d}^3x \;\mathcal{E}_{cd} \left(T^m \mathring{\nabla}_m \chi_{ab}\right)^{TT} \mathring{q}^{ac}\mathring{q}^{bd}\,, \end{aligned} \tag{2.65}$$

where in the second step we have used the fact that $(\mathcal{L}_T\chi_{ab} + 2H\,\chi_{ab}) = T^m\mathring{\nabla}_m\chi_{ab}$. We note on the side that, because the derivative $T^m\mathring{\nabla}_m$ commutes with the operation of taking the TT part on $\mathcal{I}^+$ (for proof of this and some of the other properties of the TT part used in this section, see Appendix A), the integral can be rewritten as

$$E_T \,\hat{=}\, \frac{1}{16\pi GH} \int_{\mathcal{I}^+} \mathrm{d}^3x \;\mathcal{E}_{cd} \left(T^m \mathring{\nabla}_m \chi_{ab}^{TT}\right) \mathring{q}^{ac}\mathring{q}^{bd} \tag{2.66}$$

which is manifestly gauge invariant.

Next, we return to (2.65) and use (2.62) to express $\mathcal{E}_{ab}$ in terms of the TT-part of χ_{ab}. Using that fact that the operator $(1/\eta)[\partial_\eta^2 - \frac{1}{\eta}\partial_\eta]$ commutes with the operation of taking the TT part, we have:

$$\begin{aligned} E_T \;\; &\hat{=} \;\; \lim_{\to\mathcal{I}} \frac{1}{32\pi GH^2} \int \mathrm{d}^3x \left[\frac{1}{\eta}\left(\partial_\eta^2 - \frac{1}{\eta}\partial_\eta\right)\chi_{ab}\right]^{TT} \left[T^m\mathring{\nabla}_m\chi_{cd}\right]^{TT} \mathring{q}^{ac}\mathring{q}^{bd} \\ &\hat{=} \;\; \lim_{\to\mathcal{I}} \frac{1}{32\pi GH^2} \int \mathrm{d}^3x \left[\frac{1}{\eta}\left(\partial_\eta^2 - \frac{1}{\eta}\partial_\eta\right)\chi_{ab}\right] \left[T^m\mathring{\nabla}_m\chi_{cd}\right]^{TT} \mathring{q}^{ac}\mathring{q}^{bd} \end{aligned} \tag{2.67}$$

where in the second step we removed the TT on the first square bracket because the second square bracket is already TT and therefore the integral automatically extracts only the TT part of the first square bracket. These expressions hold for any solution χ_{ab} that is source-free in a neighborhood of $\mathcal{I}^+$ (e.g. within the shaded region in the left panel of Figure 2.1).

We now use the approximations $D_o/\ell_\Lambda \ll 1$ and $v \ll 1$ spelled out in Section 2.3.1 and insert in (2.67) the convenient expression of χ_{ab} given in (2.31). For the first square bracket we use (2.58) and $\partial_\eta f(\eta - r) = -\partial_r f(\eta - r)$ and evaluate the expression at $\mathcal{I}^+$ by setting $\eta = 0$. The result is:

$$\frac{1}{\eta}\left[\Big(\partial_\eta^2 - \frac{1}{\eta}\partial_\eta\Big)\chi_{\bar{a}\bar{b}}\right](\vec{x}) \,\hat{=}\, -\frac{4G}{r}\,\partial_r\,\left(\frac{1}{r}\,\partial_r\int \mathrm{d}^3x'\,\mathcal{T}_{\bar{a}\bar{b}}(\eta_{\rm ret},\vec{x}')\right). \qquad (2.68)$$

As we noted after (2.58), although the tail term $\flat_{ab}$ in the expression (2.17) of χ_{ab} does contribute to the result, the process of taking derivatives has made the constant term C_{ab} in $\flat_{ab}$ disappear and the result depends only on the behavior of the source at retarded time.

Next, consider the second square bracket in the integrand of (2.67). Since the term multiplying this bracket has a well-defined limit to $\mathcal{I}^+$, we can replace T^m by its limiting value $-Hr\hat{r}^m$ at $\mathcal{I}^+$. Using (2.31,) we find that

$$\left[T^m\mathring{\nabla}_m\chi_{\bar{c}\bar{d}}\right](\vec{x}) \;=\; \frac{4GH}{r}\int \mathrm{d}^3x'\,\mathcal{T}_{\bar{a}\bar{b}}(\eta_{\rm ret},\vec{x}')\,. \qquad (2.69)$$

Substituting (2.68) and (2.69) in the expression for radiated energy (2.67), performing an integration by parts, and using eq. (2.40) to express the integral over the stress-energy tensor in terms of quadrupole moments, we obtain

$$\begin{aligned} E_T \;\;\hat{=}\;\; & \frac{G}{8\pi H}\int\frac{\mathrm{dr}}{r}\,\mathrm{d}^2S\Big[\Big(\partial_r\,Hr\Big(\ddot{Q}^{(\rho)}_{ab} + 2H\,\dot{Q}^{(\rho)}_{ab} + H\dot{Q}^{(p)}_{ab} + 2H^2 Q^{(p)}_{ab}\Big)\Big)\times \\ & \Big(\partial_r\,Hr\Big(\ddot{Q}^{(\rho)}_{cd} + 2H\,\dot{Q}^{(\rho)}_{cd} + H\dot{Q}^{(p)}_{cd} + 2H^2 Q^{(p)}_{cd}\Big)^{TT}\Big)\Big]\,\mathring{q}^{ac}\mathring{q}^{bd}, \end{aligned} \qquad (2.70)$$

where d^2S is the unit 2-sphere volume element of the flat metric $\mathring{q}_{ab}$ at $\mathcal{I}^+$, and, as before an 'overdot' denotes the Lie derivative with respect to T^a. Finally, using the fact that the operation $r\partial_r$ commutes with the operation of extracting the TT part and that the affine parameter T along the integral curves of T^a satisfies $\mathrm{d}T = \mathrm{d}r/(rH)$ at $\mathcal{I}^+$, we obtain

$$E_T \,\hat{=}\, \frac{G}{8\pi}\int_{\mathcal{I}^+}\mathrm{d}T\,\mathrm{d}^2S\,\Big[\mathcal{R}_{ab}(\vec{x})\,\mathcal{R}^{TT}_{cd}(\vec{x})\,\mathring{q}^{ac}\mathring{q}^{bd}\Big]\,, \qquad (2.71)$$

where the 'radiation field' $\mathcal{R}_{ab}(\vec{x})$ on $\mathcal{I}^+$ is given by

$$\mathcal{R}_{ab}(\vec{x}) \hat{=} \Big[\dddot{Q}^{(\rho)}_{ab} + 3H\ddot{Q}^{(\rho)}_{ab} + 2H^2\dot{Q}^{(\rho)}_{ab} + H\ddot{Q}^{(p)}_{ab} + 3H^2\dot{Q}^{(p)}_{ab} + 2H^3 Q^{(p)}_{ab} \Big](\eta_{\rm ret})\,, \quad (2.72)$$

where, as before, $\eta_{\rm ret} = \eta - r \;\hat{=}\; -r$. Note that $\mathcal{R}_{ab}$ is a *field* on $\mathcal{I}^+$ because, given a point $\vec{x}$ on $\mathcal{I}^+$, the quadrupole moments $Q^{(\rho)}_{ab}$ and $Q^{(p)}_{ab}$ are obtained by performing an integral over the source along the 3-surface $\eta = \eta_{\rm ret}$ and these 3-surfaces change as we change $\vec{x}$ on $\mathcal{I}^+$ (see Figure 2.1). This occurs also in the standard quadrupole formula in flat space. There is, however one difference from the standard formula: (2.71) uses the TT decomposition rather than the tt decomposition. Indeed, since the tt decomposition used in the flat space analysis is tied to the $1/r$-expansion, it is not very useful in the de Sitter context. One consequence is that the TT label appears only on the $\mathcal{R}_{cd}$ term in (2.71); the term $\mathcal{R}_{ab}$ is not automatically TT because the volume element in (2.71) is not d^3x. Finally, while components of individual terms such as $\dddot{Q}^{(\rho)}_{\bar{a}\bar{b}}(0, \vec{x})$ depend only on $r \equiv |\vec{x}|$ at $\mathcal{I}^+$ and not on angles, an angular dependence is introduced while taking the TT part. Therefore, the total integrand of (2.71) has a genuine angular dependence; otherwise one could have trivially performed the angular integral and replaced it just by a 4π factor. Conceptually, this situation is the same as for the standard quadrupole formula in flat spacetime: the tt operation also introduces angular dependence.

Finally, as in the $\Lambda = 0$ calculation, let us extract power P_T radiated by the system at any 'instant of time' T_0 at $\mathcal{I}^+$ (i.e., a 2-sphere cross-section of $\mathcal{I}^+$, orthogonal to the orbits of the 'time-translation' T^a):

$$P_T(T_0) \,\hat{=}\, \frac{G}{8\pi} \int_{T=T_0} \mathrm{d}^2 S \left[\mathcal{R}^{ab}(\vec{x})\, \mathcal{R}^{TT}_{ab}(\vec{x}) \right] \quad (2.73)$$

While the expression (2.64) of radiated energy is completely local in χ_{ab}, a degree of non-locality enters while casting it in terms of sources: (2.73) involves only the TT-part of one of the radiation fields. However, because the TT-part is taken only for one of the two radiation fields, one can show that if $\mathcal{L}_T T_{ab} = 0$ at an instant η_o, then the power at $\mathcal{I}^+$ vanishes at the cross-section $T = T_0$ representing the intersection of $\mathcal{I}^+$ with the null cone with vertex $(\eta_o, \vec{x} = \vec{0})$.

The expression (2.71) of radiated energy is the main result of this section. As in Einstein's quadrupole formula, it has been derived using the first post-Newtonian approximation under the assumption that we have an externally specified, first order stress-energy tensor T_{ab} satisfying the conservation equation with respect to the background metric.

Remark: The covariant phase space $\Gamma_{\rm Cov}$ constructed and used in [35] to obtain flux formulas at $\mathcal{I}^+$ consists of *homogeneous* solutions to linearized Einstein's equations. In this chapter, we are considering retarded solutions with a first order source T_{ab}. However, in the shaded neighborhood of $\mathcal{I}^+$ shown in the left panel of Figure 2.1, all (trace-reversed) metric perturbations $\bar{\gamma}_{ab}$ satisfy the homogeneous equation and there is a family of Cauchy surfaces for this neighborhood that approach $\mathcal{I}^+$. Therefore, one can use the covariant phase space framework in this neighborhood to calculate fluxes of energy, momentum and angular momentum carried by the perturbations $\bar{\gamma}_{ab}$ across $\mathcal{I}^+$. In this calculation, we used the leading-order terms in the expression (2.43) of χ_{ab}, ignoring terms of order $\mathcal{O}\Big((D_o/\ell_\Lambda)(1-\eta/r)\Big)$ and $\mathcal{O}(v)$ compared to terms of order $\mathcal{O}(1)$. However, as noted before, the simplified formula (2.31) for the field χ_{ab} is valid in an entire neighborhood of $\mathcal{I}^+$ (the shaded region in the left panel of Figure 2.1). Finally note that, since the flux formula is gauge invariant, the calculation can be carried out in any gauge.

2.4.3 Properties of fluxes across $\mathcal{I}^+$

Our formula of the energy carried by gravitational waves across $\mathcal{I}^+$ have several interesting features which we now discuss in some detail.

(1) First, the cosmological constant term does survive (through $H = \sqrt{\Lambda/3}$) even at $\mathcal{I}^+$. Nonetheless, we explicitly see that, in this first post-Newtonian approximation, the radiated energy is still quadrupolar.

(2) As we discussed in Section 2.4.1, because of its conformal properties, it is clear that $\mathcal{E}_{ab}$ has sharp propagation. However, the fundamental formula (2.64) for the energy flux, we started out with depends also on χ_{ab} whose expression does contain an integral over all η' that extends all the way back to $\eta' = -\infty$. So, why is there no such integral in the final expressions of radiated energy? The reason is that what features in (2.64) is not χ_{ab} itself but rather, *its derivative,* $(\mathcal{L}_T + 2H)\chi_{ab} \,\hat{=}\, -\, Hr\,\partial_r\chi_{ab}$. The integral over η' disappears while taking this

derivative, as we saw in (2.69). This is why our quadrupole formula (2.71) does not contain an explicit tail term in spite of backscattering due to the background de Sitter curvature. As in the asymptotically flat case, of course, tail terms will arise at higher post-Newtonian orders.

(3) In contrast to the Einstein formula, there is a contribution from the time variation of the *pressure* quadrupole and, furthermore, from the *pressure quadrupole itself.* It is well known from the Raychaudhuri equation in cosmology that pressure contributes to gravitational attraction in any FLRW universe. Eq. (2.71) shows that, if $\Lambda > 0$, it also *sources* gravitational waves already in the leading order post-Newtonian approximation. If $p \ll \rho$ as for Newtonian fluids, then the pressure terms $H\ddot{Q}^{(p)}_{ab} + 3H^2\dot{Q}^{(p)}_{ab}$ can be neglected compared to the density terms $3H\ddot{Q}^{(\rho)}_{ab} + 2H^2\dot{Q}^{(\rho)}_{ab}$ and the expression (2.72) of $\mathcal{R}_{ab}$ simplifies to:

$$\mathcal{R}_{ab}(\vec{x}) = \left[\dddot{Q}^{(\rho)}_{ab} + 3H\ddot{Q}^{(\rho)}_{ab} + 2H^2\dot{Q}^{(\rho)}_{ab} + 2H^3 Q^{(p)}_{ab}\right](\eta_{\rm ret})\,. \tag{2.74}$$

For compact binaries of immediate interest to the gravitational wave detectors, we also have $\Delta t_{\rm ret}/t_H \ll 1$ where $\Delta t_{\rm ret}$ is the dynamical time scale in which the mass and pressure quadrupole change by factors of $\mathcal{O}(1)$ and t_H, the Hubble scale.[12] Then the formula further simplifies and acquires a form similar to that of the $\Lambda = 0$ Einstein formula:

$$\mathcal{R}_{ab}(\vec{x}) = \dddot{Q}^{(\rho)}_{ab}(\eta_{\rm ret}) \tag{2.75}$$

When Λ is as tiny as the observations imply, the de Sitter quadrupole and its 'overdots' are extremely well approximated by those in Minkowski spacetime and the $\Lambda > 0$ first post-Newtonian approximation is extremely well-approximated by the standard one. The full expression (2.72) provides a precise control over the errors one makes while using the Einstein formula in presence of Λ.

(4) Positivity of energy flux is not transparent because the integrand of (2.71) is not manifestly positive, as it is in Einstein's formula for flat space. However, one can establish positivity as follows. First, properties of the retarded Green's function imply that the $\chi^{TT}_{ab}(\eta, \vec{x})$ can be expressed using the TT part $\mathcal{T}^{TT'}_{ab}$ of $\mathcal{T}_{ab}(\eta, \vec{x}')$,

[12]This need not be the case for the *very* long wavelength emission due to the coalescence of supermassive black holes.

where the prime in TT' emphasizes that the transverse traceless part refers to the argument $\vec{x}'$:

$$\chi_{\bar{a}\bar{b}}^{TT}(\eta,\, \vec{x}) = 4G \int \frac{\mathrm{d}^3\vec{x}'}{|\vec{x}-\vec{x}'|}\, \mathcal{T}_{\bar{a}\bar{b}}^{TT'}(\eta_{\mathrm{Ret}},\, \vec{x}') + 4G\int \mathrm{d}^3\vec{x}' \int_{-\infty}^{\eta_{\mathrm{Ret}}} \mathrm{d}\eta'\, \frac{1}{\eta'}\, \partial_{\eta'} \mathcal{T}_{\bar{a}\bar{b}}^{TT'}(\eta',\, \vec{x}')\,. \tag{2.76}$$

(The TT in $\chi_{\bar{a}\bar{b}}^{TT}(\eta,\, \vec{x})$ on the left side refers to $\vec{x}$.) Next, let us rewrite the expression (2.67) in terms of χ_{ab}^{TT}

$$E_T \mathrel{\hat{=}} \lim_{\to \mathcal{I}} \frac{1}{32\pi G H^2} \int \mathrm{d}^3x \left[\frac{1}{\eta}\left(\partial_\eta^2 - \frac{1}{\eta}\partial_\eta\right)\chi_{ab}^{TT}\right]\left[T^m \mathring{\nabla}_m \chi_{cd}^{TT}\right] \mathring{q}^{ac}\mathring{q}^{bd} \tag{2.77}$$

where we have used the fact that ∂_η and $T^m\mathring{\nabla}_m$ commute with the operation of taking the TT part. Finally, let us substitute (2.76) in (2.77) and simplify following the procedure of Section 2.3.1 and steps used to pass from (2.67) and (2.69).[13] We obtain:

$$E_T \mathrel{\hat{=}} \frac{G}{2\pi} \int_{\mathcal{I}^+} \mathrm{d}T\, \mathrm{d}^2S \left[\partial_r \int \mathrm{d}^3x' \mathcal{T}_{ab}^{TT'}(\eta_{\mathrm{ret}}, \vec{x}')\right]\left[\partial_r \int \mathrm{d}^3x' \mathcal{T}_{cd}^{TT'}(\eta_{\mathrm{ret}}, \vec{x}')\right] \mathring{q}^{ac}\mathring{q}^{cd}\,, \tag{2.78}$$

which is manifestly positive.

As we discussed in Chapter 1, de Sitter spacetime admits gravitational waves whose energy can be arbitrarily negative in the linearized approximation because the time translation Killing field T^a is space-like in a neighborhood of $\mathcal{I}^+$. Indeed, for systems under consideration, gravitational waves satisfy the homogeneous, linearized Einstein's equations in a neighborhood of $\mathcal{I}^+$ and there is an *infinite dimensional subspace of these solutions for which the total energy is negative [35]*. What, then, is the physical reason behind the positivity of our E_T? Consider the shaded triangular region in the left panel of Figure 2.1. It is bounded by $\mathcal{I}^+$, upper half of $E^+(i^-)$ and $E^-(i^+)$. The time translation vector field T^a is tangential to all these three boundaries, being space-like on $\mathcal{I}^+$, null and *past directed* on the upper half of $E^+(i^-)$, and null and *future directed* on $E^-(i^+)$. As a result, for *any* solution, the energy flux across the upper half of $E^+(i^-)$ is negative, that across $E^-(i^+)$ is positive, and that across $\mathcal{I}^+$ is the sum of the two, which can have either

[13] We assume that integrals involving $\mathcal{T}_{ab}^{TT'}$ are all well-defined. This is a plausible assumption since $\mathcal{T}_{ab}$ is smooth and of compact support whence its Fourier transform is also in Schwartz space.

sign and arbitrary magnitude. Thus, the potentially negative energy contribution at $\mathcal{I}^+$ can be traced directly to the incoming gravitational waves across the upper half of $E^+(i^-)$. Now, in the present calculation, physical considerations led us to the *retarded* metric perturbation created by the time varying quadrupoles. Therefore there is no flux of energy across the cosmological horizon $E^+(i^-)$; the potential negative energy flux across $\mathcal{I}^+$ is simply absent. The entire energy flux across $\mathcal{I}^+$ equals the energy flux across $E^-(i^+)$ which is always positive because T^a is future directed there. To summarize then, while in general the energy flux across $\mathcal{I}^+$ can have either sign, the metric perturbation $\bar{\chi}_{ab}$ at $\mathcal{I}^+$ created by physically reasonable sources are so constrained that the energy carried by gravitational waves across $\mathcal{I}^+$ is necessarily positive.

(5) The fifth feature concerns time dependence of the source. Equations satisfied by the full (trace-reversed, rescaled) metric perturbation $\bar{\chi}_{ab}$ refer only to the background metric $\bar{g}_{ab}$ and T^a is a Killing field of $\bar{g}_{ab}$ which is time-like in the region in which the source T_{ab} resides. Therefore, it follows that if the source is static, i.e., if $\mathcal{L}_T T_{ab} = 0$, then the retarded solution $\bar{\chi}_{ab}$ must satisfy $\mathcal{L}_T\, \bar{\chi}_{ab} = 0$. Physically, one would expect there to be no flux of energy across $\mathcal{I}^+$. But this is not manifest in eq. (2.71) since it contains a term $Q^{(p)}_{ab}$ that does *not* involve a time derivative. Let us therefore examine the fields that enter the definitions (2.35) and (2.36) of quadrupole moments. A simple calculation shows that, *if* $\mathcal{L}_T\, T_{ab} = 0$, the fields that enter the definitions of quadrupole moments satisfy

$$\mathcal{L}_T\, \rho = 0; \quad \mathcal{L}_T\, p = 0; \quad \mathcal{L}_T\, a(\eta) x_{\bar{b}} = 0; \quad \mathcal{L}_T\, \mathring{e}^{\bar{a}}_a = -H\, \mathring{e}^{\bar{a}}_a; \tag{2.79}$$

and the 3-dimensional volume element $\mathrm{d}V$ is preserved under the isometry generated by T^a. Therefore we have:

$$\mathcal{L}_T\, Q^{(\rho)}_{ab} \;=\; -2H\, Q^{(\rho)}_{ab} \quad \text{and} \quad \mathcal{L}_T\, Q^{(p)}_{ab} \;=\; -2H\, Q^{(p)}_{ab}\,. \tag{2.80}$$

Thus, in contrast to what happens in the Minkowski spacetime calculation, because of the expansion of the de Sitter scale factor, now $\mathcal{L}_T\, T_{ab} = 0$ does not imply that quadrupoles are left invariant by the flow generated by T^a. However, using (2.43), (2.71), (2.73) and (2.80), it immediately follows that

$$\text{if} \quad \mathcal{L}_T\, T_{ab} = 0 \text{ everywhere}, \quad \text{then} \qquad E_T \hat{=} 0, \qquad \text{and} \quad P_T(T_0) \hat{=} 0 \tag{2.81}$$

for all T_0. (In fact, it follows from eq. (2.73) that an 'instantaneous' result also holds: if $\mathcal{L}_T\, T_{ab}\, |_{\eta=\eta_{\rm ret}}= 0$, then $P_T(T_0) \hat{=} 0$ where $\eta_{\rm ret} = \eta - r_0 \hat{=} -r_0$ and $T_0 \hat{=} \frac{1}{H} \ln |r_0 H|$.) Thus, the presence of the term without a time derivative of the pressure quadrupole $Q^{(p)}_{ab}$ is in fact essential to ensure that if $\mathcal{L}_T\, T_{ab} = 0$ then E_T and $P_T(T_0)$ vanish on $\mathcal{I}^+$.[14]

(6) Next, let us consider the limit $\Lambda \to 0$. The procedure is rather long as we have to 'pass through' the physical spacetime as in [35] and use results from the covariant phase space framework relating expressions involving the TT and tt decompositions in Minkowski spacetime. We will only summarize the main steps here.

Consider the 1-parameter family of de Sitter backgrounds $\bar{g}^{(\Lambda)}_{ab}$, parametrized by Λ, with a 1-parameter family $T^{(\Lambda)}_{ab}$ of stress-energy tensors, each satisfying the conservation law with respect to the respective $\bar{g}^{(\Lambda)}_{ab}$ and the condition $\mathcal{L}_T T^{(\Lambda)}_{ab} = 0$ outside a compact time interval. Let $\chi^{(\Lambda)}_{ab}(t, \vec{x})$ denote the retarded solutions (2.31) to the field equations and gauge conditions. For each Λ, one can express this solution in terms of the source quadrupoles as in (2.43). The question is whether as $\Lambda \to 0$ this 1-parameter family of solutions has a well-defined limit $\mathring{\chi}_{ab}(t, \vec{x})$. If so, the analysis in Section IV.B.2 of [35] shows that: i) $\mathring{\chi}_{ab}(t, \vec{x})$ satisfies the dynamical equation and gauge conditions with respect to the Minkowski metric $\mathring{\eta}_{ab}$; and, ii) the expression (2.71) of energy in the gravitational waves has a well-defined limit, which is furthermore precisely the energy in the solution $\mathring{\chi}_{ab}(t, \vec{x})$, calculated in Minkowski spacetime.

We have already shown in Section 2.2 that the exact retarded solutions do tend to the exact retarded solution in Minkowski spacetime. We will now show that this is also the case for the *approximate* solution (2.43). As discussed in Section 2.3.3, the tail term expressed in terms of quadrupole moments vanishes in the limit $\Lambda \to 0$.

[14] This consistency would have been obscured if we had ignored the pressure terms relative to the density terms in (2.40), and used the resulting approximation (2.42) to arrive at the expression of χ_{ab}. That is why we kept all the pressure quadrupole terms even though they can be ignored relative to the analogous density quadrupole terms for Newtonian fluids.

The discussion for the sharp term $\sharp_{ab}(t, \vec{x})$ in (2.43) is more delicate. In the $\Lambda \to 0$ limit, we have $T^a \to \mathring{t}^a$, a time translation in Minkowski metric $\mathring{\eta}_{ab}$; $\mathcal{L}_T \to \mathcal{L}_t$; $a(t) \to 1$ and $Q^{(\rho)}_{ab} \to \mathring{Q}^{(\rho)}_{ab}$, the mass quadrupole moment constructed from the limiting stress-energy tensor $\mathring{T}_{ab}$ using the Minkowski metric $\mathring{\eta}_{ab}$. Therefore, the limiting solution is given by

$$\lim_{\Lambda \to 0} \chi^{(\Lambda)}_{ab}(t, \vec{x}) = \frac{2G}{r} \mathcal{L}_t \, \mathcal{L}_t \, \mathring{Q}^{(\rho)}_{ab}(t_{\text{ret}}) =: \mathring{\chi}_{ab}(t, \vec{x}) \tag{2.82}$$

for all $r \gg d(t)$, where d is the physical size of the source with respect to the Minkowski metric $\mathring{\eta}_{ab}$. Now, since by assumption the source is active for a finite time interval, on a $t = \text{const}$ surface sufficiently in the future, the support of the initial data of $\mathring{\chi}_{ab}(t, \vec{x})$ is entirely in a region where the approximation holds. Let us consider only the future of this slice. In that spacetime region we have a 1-parameter family of solutions $\chi^{(\Lambda)}_{ab}(t, \vec{x})$ to the source-free equations whose total energy is given by (2.71) for each $\Lambda > 0$. The limit $\mathring{\chi}_{ab}(t, \vec{x})$ is well-defined, as required. Therefore, in the $\Lambda \to 0$ limit the energy expression (2.71) goes over to the energy in $\mathring{\chi}_{ab}(t, \vec{x})$ with respect to $\mathring{t}^a$ in Minkowski space (see eq. (4.24) of [35]). And we know that this energy is given by the Einstein formula. Thus, in the limit $\Lambda \to 0$ one recovers the standard quadrupole formula in Minkowski spacetime.

To summarize, our energy expression (2.71) arises as the Hamiltonian on the covariant phase space of linearized solutions on de Sitter spacetime, and using results from [35] we can conclude that it tends to the expression of the Hamiltonian in Minkowski space in the $\Lambda \to 0$ limit, which in turn reduces to the Einstein flux formula at $\mathring{\mathcal{I}}^+$. The argument is indirect at the moment because in linearized gravity off *Minkowski spacetime* we have not specified the relation between the TT and tt decompositions. The equality between the two expressions of energy, the first evaluated on space-like planes in terms of the TT decomposition and the second, evaluated at $\mathcal{I}^+$ in terms of tt is well known. In the next chapter, we will also make te relation between the TT and tt decomposition of the gravitational wave perturbation off Minkowski spacetime explicit. This will close the gap in the indirect argument above and make it transparent that in the $\Lambda \to 0$ limit the power radiated on a de Sitter background reduces to the power on a Minkowski background.

(7) So far we have focussed on the energy carried by gravitational waves. Let us

now discuss the flux of 3-momentum across $\mathcal{I}^+$. The component of the 3-momentum along a space translation $S^a{}_{\bar{i}}$ is given by [35]

$$P_{\bar{i}} \,\hat{=}\, \frac{1}{16\pi GH} \int_{\mathcal{I}^+} \mathrm{d}^3x \; \mathcal{E}_{cd} \left(\mathcal{L}_{S_{\bar{i}}}\, \chi_{ab} \right) \mathring{q}^{ac} \mathring{q}^{bd} \tag{2.83}$$

We can again use (2.62) to express $\mathcal{E}_{cd}$ in terms of χ_{cd}: $\mathcal{E}_{ab} = [\frac{1}{\eta}\,(\partial_\eta^2 - \frac{1}{\eta}\partial_\eta)\chi_{ab}]^{TT}$. Now, it is clear from the expression (2.31) of χ_{ab} that its dependence on $\vec{x}$ comes entirely from $\eta_{\rm ret}$. Therefore, χ_{ab} in invariant under the parity operation $\Pi : \vec{x} \to -\vec{x}$, whence $\frac{1}{\eta}\left(\partial_\eta^2 - \frac{1}{\eta}\partial_\eta\right)\chi_{ab}$ is also invariant. Since the operation of taking the TT-part refers only to the 3-metric $\mathring{q}_{ab}$, it also commutes with Π. Hence $\mathcal{E}_{ab}$ is even under Π. The second term, $S^m{}_{\bar{i}}\, \mathring{D}_m \chi_{ab}$ is manifestly odd under Π since S^a is odd but χ_{ab} is even. Therefore the integral on the right side of (2.83) vanishes. Thus, as in the $\Lambda = 0$ case, the gravitational waves sourced by a time changing quadrupole do not carry 3-momentum in the post-de Sitter, first post-Newtonian approximation so long as $D_o \ll \ell_\Lambda$.

(8) Finally, let us consider angular momentum. The flux of angular momentum in the $\bar{i}$-direction is given by [35]:

$$J_{\bar{i}} \,\hat{=}\, \frac{1}{16\pi GH} \int_{\mathcal{I}^+} \mathrm{d}^3x \; \mathcal{E}_{cd} \left(\mathcal{L}_{R_{\bar{i}}}\, \chi_{ab} \right) \mathring{q}^{ac} \mathring{q}^{bd} \tag{2.84}$$

where $R^m{}_{\bar{i}}$ is the rotational Killing field in the $\bar{i}$-th spatial direction. Now, since the $\vec{x}$-dependence in χ_{ab} is derived entirely through $\eta_{\rm ret}$, we have

$$\mathcal{L}_{R_{\bar{i}}} \chi_{ab} = \chi_{mb}\, \mathring{D}_a R^m{}_{\bar{i}} + \chi_{am}\, \mathring{D}_b R^m{}_{\bar{i}} = -2\chi_{m(b}\, \mathring{\epsilon}_{a)n}{}^m\, \mathring{e}^n{}_{\bar{i}}\,. \tag{2.85}$$

Hence,

$$J_{\bar{i}} \,\hat{=}\, -\frac{1}{8\pi GH} \int_{\mathcal{I}^+} \mathrm{d}^3x \; \mathcal{E}_{cd} \left(\mathring{\epsilon}_{am}{}^n \; \mathring{e}^m{}_{\bar{i}}\; \chi_{nb} \right) \mathring{q}^{ac} \mathring{q}^{bd} \tag{2.86}$$

Since χ_{cd} now appears without a derivative in (2.86), there is a major difference between the calculations of energy and 3-momentum fluxes across $\mathcal{I}^+$: Now the integral over η' in the tail term $\flat_{ab}$ in the expression (2.43) of χ_{ab} persists. To evaluate the right side of (2.86), for the sharp term of χ_{ab} we use eq. (2.43) and for the tail term the integrated expression in eq. (2.46), and for $\mathcal{E}_{ab}$ we use eqs. (2.62) and (2.68) as in the calculation of the energy flux. These simplifications lead to:

$$J_{\bar{i}} \hat{=} \frac{G}{4\pi} \int_{\mathcal{I}^+} \mathrm{d}T\, \mathrm{d}^2S \left[\mathcal{R}^{ab}\right] \left[\mathring{\epsilon}_{am}{}^{n}\ \mathring{e}^{m}{}_{\bar{i}} \left(\ddot{Q}^{(\rho)}_{nb} + H\dot{Q}^{(\rho)}_{nb} + H\dot{Q}^{(p)}_{nb} + H^2 Q^{(p)}_{nb}\right)\right]^{TT}, \tag{2.87}$$

where, as before T is the affine parameter along the integral curves of the 'time translation' Killing field T^a and $\mathcal{R}^{ab}$ is defined in (2.72). Note that if the stress-energy satisfies $\mathcal{L}_T T_{ab} = 0$ at some time $\eta = \eta_o$ then the 'radiation field' $\mathcal{R}_{ab}$ vanishes on the cross-section $r = \eta_o$ on $\mathcal{I}^+$, whence the flux of (energy and) angular momentum vanish on that cross-section. Similarly if $\mathcal{L}_{R_{\bar{i}}} T_{ab}$ vanishes at $\eta = \eta_o$, then the flux of angular momentum vanishes on the cross-section $r = \eta_o$. Finally, in the limit $\Lambda \to 0$, using the same argument as that used for energy, one can show that (2.86) reduces to the standard formula in Minkowski spacetime. (Again the argument is indirect at the moment because the expression of the Hamiltonian generating rotations on the covariant phase space in Minkowski spacetime involves the TT part of the solution while the standard expression of angular momentum at null infinity involves the *tt*-part. To make this argument direct, we show their equivalence on $\mathcal{I}$ in Chapter 4.)

2.5 Discussion

Einstein's quadrupole formula has played a seminal role in the study of gravitational waves emitted by astrophysical sources. His analysis was carried out only to the leading post-Newtonian order, assuming that the time-changing quadrupole is a first order, external source in Minkowski spacetime. In spite of these restrictions, his quadrupole formula sufficed to bring to forefront the extreme difficulty of detecting these waves. However, thanks to the richness of our physical Universe and ingenuity of observers, impressive advances have occurred over the last four decades. First, the careful monitoring of the Hulse-Taylor pulsar has provided clear evidence for the validity of the quadrupole formula to a 10^{-3} level accuracy. Furthermore, gravitational wave observatories, equipped with detectors with unprecedented sensitivity, have directly detected gravitational waves [2, 3]. Therefore it is now all the more important that our theoretical understanding of gravitational waves be sufficiently deep to do full justice to the impressive status of the field on the observational front. The goal of this dissertation is to fill a key conceptual gap that still remains: incorporation of the positive cosmological constant in our understanding of the properties of gravitational waves and dynamics of their sources.

Since the observed value of the cosmological constant is so small, one's first reaction is just to ignore its presence. However, as discussed in Chapter 1 and Section 2.1, even a tiny cosmological constant can cast a long shadow because it abruptly changes the conceptual set-up that is used to analyze gravitational waves. Therefore, without a systematic analysis, one can not be confident that the quadrupole formula would continue to be valid in presence of a positive cosmological constant.

Indeed, our analysis revealed that the presence of a cosmological constant does modify Einstein's analysis in unforeseen ways. In particular:
(i) The propagation equation for metric perturbations in the transverse-traceless gauge is not the wave equation as in Minkowski spacetime, but has an effective mass term (see eq. (2.2)). Although this mass is tiny, there is potential for the differences from Minkowskian propagation to accumulate over cosmological distances to produce $\mathcal{O}(1)$ departures in the value of the metric perturbation in the asymptotic region;
(ii) The retarded field receives contributions from both a sharp term and a tail term due to the backscattering by de Sitter curvature. Cumulative effects make the tail term comparable to the sharp term (which has the same form as in Minkowski spacetime) in the asymptotic region near $\mathcal{I}^+$;
(iii) Since the radial r coordinate goes to infinity $\mathring{\mathcal{I}}^+$ of Minkowski spacetime, the analysis of waves makes heavy use of $1/r$ expansions. These can no longer be used in de Sitter spacetime because r ranges over the entire positive real axis on de Sitter $\mathcal{I}^+$. In particular, the tt-decomposition, that is local in space being tailored to the $1/r$ expansions in Minkowski spacetime, is no longer meaningful near de Sitter $\mathcal{I}^+$. Thus, two of the main techniques used for spacetimes with $\Lambda = 0$ do not apply when analyzing gravitational radiation near/on de Sitter $\mathcal{I}^+$.
(iv) The retarded, first order metric perturbation depends not only on the mass quadrupole as in Einstein's calculation but also on the pressure quadrupole. Also, while only the third time derivative of the mass quadrupole features in Einstein's calculation, now we also have a contribution from lower time derivatives of the two quadrupoles, as well as the pressure quadrupole itself;
(v) The *physical* wavelengths $\lambda_{\rm phys}$ of perturbations grow exponentially as the

wave propagates and vastly exceed the curvature radius $\ell_\Lambda = H^{-1} \equiv \sqrt{3/\Lambda}$ in the asymptotic region near $\mathcal{I}^+$. Since waves 'experience' the curvature, their propagation is quite different from that in flat space. Also, since the expression (2.67) involves the metric perturbation evaluated in the zone where $\lambda_{\rm phys} > \ell_\Lambda$, a priori the effect of Λ on radiated energy could be non-negligible;
(vi) $\mathcal{I}^+$, the arena used to analyze properties of gravitational waves unambiguously changes its character from being a null future boundary of spacetime to a space-like one. As a result, all Killing fields of the background de Sitter spacetime – including the 'time translation' used to define energy – are space-like in a neighborhood of $\mathcal{I}^+$. Consequently, while linearized gravitational waves carry positive energy in Minkowski spacetime, de Sitter spacetime admits gravitational waves carrying arbitrarily large negative energy.

These differences are sufficiently striking to cast a doubt on one's initial intuition that the cosmological constant will have no role in the study of compact binaries. For example, they open up the possibility that Einstein's quadrupole formula could receive significant corrections – e.g., of the order $\mathcal{O}(H\lambda_{\rm phy})$ – even though the observed value of H is so small. Interestingly, the final expression (2.73) of radiated power shows that this does not happen for astrophysical processes such as the Hulse-Taylor binary pulsar, or the compact binary mergers that are of greatest interest to the current ground based gravitational wave observatories. How does this come about? Why do the qualitative differences noted in the last paragraph not matter in the final result for these systems? The physical reasons can be summarized as follows:
(a) First, while the propagation of χ_{ab} indeed has a tail term, the dynamical part of the tail term propagates along the null cone. The non-dynamical part is simply a constant term that does not play a role in radiated energy as only certain *derivatives* of χ_{ab} contribute to energy.
(b) Second, while the final expressions (2.71) and (2.87) of radiated energy and angular momentum are evaluated at $\mathcal{I}^+$, the integrand refers to the time derivatives of quadrupole moments evaluated at *retarded* instants of time. In other words, even though (2.71) and (2.87) involve fields at late times, the time scales in the 'dots' in these expressions are determined by $\lambda_{\rm phy}^{\rm source}$, the wavelength evaluated at the source, and *not* by the exponentially larger physical wavelengths $\lambda_{\rm phy}^{\rm asym}$ in the asymptotic region. Therefore for the sources on which gravitational wave

observatories will focus in the foreseeable future, $H\ddot{Q}^{(\rho)}_{ab}$, for example, is suppressed relative to $\dddot{Q}^{(\rho)}_{ab}$ by the factor $H\,\lambda^{\rm source}_{\rm phy}$ (rather than enhanced by the factor $H\,\lambda^{\rm asym}_{\rm phy}$) and $\dddot{Q}^{(\rho)}_{ab}$ completely dominates over the remaining 5 terms (which have H, H^2 or H^3 as coefficient). In particular, the pressure quadrupole can be neglected for these sources. Had our expression of power referred to time scales associated with the asymptotic values of $\lambda_{\rm phy}$, effects discussed in the previous paragraph would have completely altered the picture. Then, the terms with the highest powers of H – in particular the pressure quadrupole $Q^{(p)}_{ab}$ term – would have dominated and the contribution due to $\dddot{Q}^{(\rho)}_{ab}$ would have been completely negligible!
(c) Third, while a neighborhood of $\mathcal{I}^+$ in the Poincaré patch $(M^+_{\rm P}, \bar{g}_{ab})$ does admit gravitational waves carrying arbitrarily large negative energies, our calculation showed that such waves can not result from time-changing quadrupoles. The reason is simplest to explain using the shaded region in the left panel of Figure 2.1. Negative contribution to the energy at $\mathcal{I}^+$ can come only from the waves that arrive from the upper half of $E^+(i^-)$. But the physics of the problem led us to consider *retarded* solutions with the given T_{ab} as source and for these solutions there is no energy flux at all across $E^+(i^-)$. This is why our energy flux (2.71) across $\mathcal{I}^+$ is necessarily positive.

Because of these reasons, for binary coalescences that are of greatest interest to the current gravitational wave observatories, energy and power are determined essentially by the third time derivative of the mass quadrupole, as in Einstein's formula. This quadrupole moment (2.35) is calculated using the physical de Sitter geometry and the time derivative 'overdot' refers to the Lie derivative with respect to the de Sitter time translation T^a. However, in the limit $\Lambda \to 0$, it goes over the mass quadrupole used in Einstein's formula. Therefore, for compact binaries of interest to the current gravitational wave observatories, the difference is again negligible.

However, there are some circumstances in which the differences between $\Lambda = 0$ and $\Lambda > 0$ could be significant. First, consider the tail term in the expression (2.43) of χ_{ab}. Since it arises because of backscattering due to de Sitter curvature, it is proportional to H. However, it involves an integral over a cosmologically large time interval which could compensate the smallness of H and make the tail term comparable to the one that arises from sharp propagation. The tail term could then yield a significant new contribution to the memory effect [29, 56, 57] for detectors placed near $\mathcal{I}^+$. A second example is provided by mergers of supermassive black holes at the centers of two different galaxies, such as Milky way and Andromeda. Since the time scales associated with such galactic coalescences are cosmological, the various effects discussed above will come into play. Gravitational waves created in this process will have *extremely* long wavelength already at inception, making the departures from Einstein's quadrupole formula significant. While these waves will not be detected directly in any foreseeable future, they provide a background which could have indirect influences. An illustration of this general mechanism is provided by inflationary cosmology, where super-horizon modes can induce non-Gaussiantities in observable modes due to mode-mode coupling resulting from non-linearities of general relativity (see, e.g., [60, 61]).

To conclude, we note that this analysis also provides some hints for the gravitational radiation theory in full, non-linear general relativity with a positive Λ which would be of interest to geometric analysis, because of issues such as the positivity of total energy. First, to describe an isolated gravitating systems such as an oscillating star, or one collapsing to form a black hole, or a compact binary, it would be appropriate to consider only the portion of full spacetime that is bounded in the future by $\mathcal{I}^+$ and in the past by the future cosmological event horizon $E^+(i^-)$, where the point i^- represents the past time-like infinity defined by the source. This is because the isolated system and the radiation it emits would be invisible to the rest of the spacetime. Second, the 'no-incoming radiation' boundary condition will have to be imposed on the past boundary, $E^+(i^-)$. Since this is an event horizon, a natural strategy would be to demand that it be a weakly isolated horizon [62–64]. It would be interesting to analyze if this condition would suffice to ensure that the flux of energy across $\mathcal{I}^+$ is positive, as in the weak field limit discussed here. If so, one would have the desired generalization of the celebrated result due to Bondi and Sachs that gravitational waves carry away *positive* energy, in spite of the fact that the corresponding asymptotic 'time translation' on $\mathcal{I}^+$ would now be space-like for $\Lambda > 0$. Third, results of [31] and [35] suggest that there will be a 2-sphere 'charge integral' – representing the generalization of the notion of Bondi-Sachs energy to the $\Lambda > 0$ case – and the difference between charges associated with two different 2-spheres will equal the energy flux across the region bounded by the two 2-spheres. A natural question is whether this charge is also positive.[15] Fourth, the form (2.43)

[15]These Bondi-type charge integrals will also refer to an asymptotic 'time-translation'. They

of the solution χ_{ab} at $\mathcal{I}^+$ implies that the recently proposed [42] generalization of Bondi-type expansions for full general relativity can describe at most half the desired set of asymptotically de Sitter spacetimes. A further generalization is necessary to capture both polarizations at $\mathcal{I}^+$. Finally, in the linear approximation considered in this chapter, the past cosmological event horizon $E^-(i^+)$ of the point at future time-like infinity could be taken to lie in the 'far zone'. Furthermore, since there is no incoming radiation across $E^+(i^-)$ from (the shaded portion of the left panel of) Figure 2.1 it follows that the flux of energy across $E^-(i^+)$ equals that across $\mathcal{I}^+$ and is, in particular, positive. In full, non-linear general relativity, then, $E^-(i^+)$ may well serve as an 'approximate' $\mathcal{I}^+$ to analyze gravitational waves. Because this surface is null, it may be easier to compare results in the $\Lambda > 0$ case with those in the $\Lambda = 0$ case in full general relativity.

will be distinct from the ADM-type charge-integral associated with a *conformal* – rather than time-translation – symmetry discussed in [65], and the intriguing 2-sphere integral recently discovered [66], both of which are known to be positive.

Chapter 3 Power radiated by a binary system in a de Sitter Universe

3.1 Introduction

In this chapter, we apply the general formula for power radiated on a de Sitter background to a concrete physical example. A widely studied example in the $\Lambda = 0$ case is gravitational radiation emitted by a binary system of orbiting bodies. There one uses a Newtonian description for the orbital motion of the binary system to calculate the mass quadrupole moments of the source. Then using the first-order post-Newtonian approximation, one calculates the power radiated by this system at $\mathcal{I}$ of Minkowski spacetime by looking at the third time derivative of the quadrupole moment squared. We will use similar approximations in the $\Lambda > 0$ case. In particular, we will consider the dynamics of the source only on a short time scale compared to the time scale associated with $\sqrt{\Lambda}$ such that the expansion of the scale factor can be neglected in describing the orbital motion and a Newtonian description of the orbital dynamics applies. That is, we will neglect terms that are $\mathcal{O}(\sqrt{\Lambda}t_c)$ where t_c is the characteristic time scale of the system. This is also known as the high-frequency approximation. As a consequence, the stress-energy tensor describing the source is not conserved with respect to the de Sitter background and is only conserved up to $\mathcal{O}(\sqrt{\Lambda}t_c)$ and the resulting gravitational wave is an approximate solution valid to $\mathcal{O}(\sqrt{\Lambda}t_c)$. Given this approximate solution, we can now calculate the power radiated. From the general formula for radiated power on a de Sitter background in eq. (2.73), the only term that survives in this approximation scheme is the third time derivative of the quadrupole moment squared. Although this formula now looks deceivingly similar to the formula for power radiated on a

Minkowski background, it is different in two important aspects. First, the power radiated on a de Sitter background is evaluated on $\mathcal{I}$ of de Sitter spacetime which is space-like, whereas the power radiated on a Minkowski background is evaluated on its $\mathcal{I}$ which is null. Second, the notion of energy and power refers to a time translation symmetry. The world line of the center of mass of the binary follows a Killing trajectory and time derivatives of the quadrupole moment are defined by this Killing field. Therefore it is natural to use it to define energy and power. However, this vector field is space-like near and on de Sitter $\mathcal{I}$, while it is everywhere time-like in Minkowski spacetime (and becoming null only at $\mathcal{I}$). In addition to these two crucial conceptual differences, the calculational tools used in the derivation of both formulas are rather different.

This first application of the general formula for power radiated on a de Sitter background illustrates nicely some of the general properties of gravitational power in a concrete example as well as highlighting aspects unique to the power emitted by a binary system. Here, we restrict ourselves to a binary system that evolves in an orbit that is well-approximated by a circular orbit. A good understanding of the amount of power radiated by such a relatively simple system is important for both indirect and direct observations of gravitational waves emitted by binary systems. If the power radiated in the form of gravitational waves was altered by the cosmological constant, the shrinking of the orbit and the decay in period would be modified as well. This would affect indirect observations of gravitational waves that depend on the time derivative of a binary's rotational period over long periods of time [67, 68], as well as direct observations since the power directly influences the evolution of the waveform [69, 70]. In addition, a thorough comprehension of the power radiated by this system is interesting from a theoretical standpoint.

This chapter is structured as follows. First, we carefully spell out all the approximations made to model a binary system on a de Sitter background in Section 3.2. Given these approximations, the resulting power is presented in Section 3.3 and compared to the power radiated in Minkowski spacetime. This is the heart of this chapter. Details of the calculation of the transverse-traceless part of the radiation field, needed to calculate the power, are included in Appendix A. Section 4.5 contains a summary of the result.

3.2 Preliminaries: Physical set-up

The physical system of interest is a binary and one models this system by a linearized source on a cosmological background spacetime. Ideally, one would like to consider a realistic FLRW model describing the different epochs of the Universe that the gravitational radiation emitted by this system would have traveled through. Here we take the approach that we would like to know the *maximal* possible effect of the cosmological constant on the gravitational radiation. Therefore, we ignore the radiation- and matter-dominated epochs of our cosmological history and consider a de Sitter background.[1]

To dynamically model the world lines of the two bodies with mass m_1 and m_2 making up the binary system, we make similar assumptions as in the standard treatment of sources in expanding spacetimes with $\Lambda = 0$ [71, 72].[2] These assumptions are:

1. The characteristic (proper) time scale of the system t_c and the expansion rate of the background are assumed to be such that the expansion of the Universe can be neglected during the orbital cycles of interest:

$$\frac{\dot{a}}{a} t_c \ll 1 \qquad \Longleftrightarrow \qquad \sqrt{\Lambda} t_c \ll 1. \tag{3.1}$$

   This is the familiar high-frequency (short-wave) approximation [73].[3]

2. The relative physical separation between the two bodies $R_*(= a r_* = a\,|\vec{r}_1 - \vec{r}_2|)$ is such that the bodies are far apart compared to the Schwarzschild radius of either body:

$$\frac{Gm}{R_*} \ll 1 \qquad \text{and} \qquad \frac{G\mu}{R_*} \ll 1\,, \tag{3.2}$$

[1] A purely de Sitter background is also appropriate for the stochastic background of gravitational waves from an inflationary period, but that radiation is source-free.

[2] Thanks to Eric Poisson for clarifications in a private communication.

[3] Technically, the high-frequency approximation is formulated in terms of conformal time η: $\mathcal{H}\eta_c \ll 1$ where $\mathcal{H} = \frac{1}{a}\frac{\partial a}{\partial \eta}$ and η_c is the characteristic conformal time. Given t_c, however, we can relate this to the characteristic conformal time by $\eta_c \simeq \frac{t_c}{a_e}$ (where a_e is the scale factor at the time of emission). Since $\mathcal{H}\eta_c \simeq H\, t_c\, a/a_e$, $\mathcal{H}(\eta)\eta_c \ll 1$ is the same as $Ht_c \ll 1$ when we evaluate expressions at the time of emission. For a de Sitter background, the Hubble parameter is $H = \sqrt{\frac{\Lambda}{3}}$ so that the high-frequency approximation is equivalent to $\sqrt{\Lambda} t_c \ll 1$.

where the total mass of this system is $m = m_1 + m_2$ and the reduced mass $\mu = \frac{m_1 m_2}{m}$.

3. Each body moves slowly, that is, in the center of mass frame $v \ll 1$.

4. The pressure of each body is negligible compared to the energy density.

5. Each body is approximately spherically symmetric.

6. The trajectory of the binary is well approximated by a circular orbit.[4]

The first two assumptions ensure that the time evolution of the scale factor can be neglected during the few cycles the system is studied. Therefore, the time behavior of the physical separation R_* a few orbits before and after the time of emission is governed by the time behavior of the conformal separation r multiplied by the scale factor at the time of emission a_e. Similarly, the physical angular velocity Ω is described by the conformal angular velocity ω multiplied by a_e^{-1}:

$$\begin{aligned} R_*(t) &= a_e r_* + \mathcal{O}(\sqrt{\Lambda} t_c), \\ \Omega(t) &= a_e^{-1} \omega + \mathcal{O}(\sqrt{\Lambda} t_c). \end{aligned}$$

From the first two assumptions, it also follows that by a simple constant rescaling of the coordinates, the spacetime metric is well approximated by a Minkowski metric during the few cycles the system is studied:[5]

$$ds^2 = -d\tilde{\eta}^2 + d\tilde{r}^2 + \tilde{r}^2 \left(d\theta^2 + \sin^2\theta \, d\varphi^2 \right) + \mathcal{O}(\sqrt{\Lambda} t_c) \tag{3.3}$$

where $\tilde{\eta} := a_e \eta$ and $\tilde{r} := a_e r$.

Given these rescaled coordinates, one can now interpret approximation 3-6 as the Newtonian approximation and use Newtonian dynamics to describe the motion of the binary in the rescaled coordinates. Approximation 1-2 are critical for this interpretation. An example of a binary system satisfying all of the above approximations is a binary consisting of two weakly self-gravitating bodies such

[4] The motivation for such a binary is strong since gravitational waves effectively act to make eccentric systems circular over time [74], and sources seen later in their evolution are expected to have small eccentricity [75].

[5] As a side remark: since the scale factor is approximately constant, the Hubble radius is infinite to zeroth approximation during this period.

as main-sequence stars. In addition, other examples are binaries of two compact bodies such as neutron stars or black holes. The internal gravity of each body can be arbitrarily strong but as long as their mutual gravity is weak, the Newtonian description applies.

In Newtonian mechanics, the motion of the center of mass is uniform. Consequently, the description of the motion simplifies when the origin of the coordinate system is chosen such that it is attached to the center of mass; the position of each body can then be determined in terms of the relative separation between the bodies. This is what we will do here as well. From conservation of angular momentum, it follows that the orbital motion proceeds within a fixed orbital plane. Choosing the orbital plane to coincide with the $x-y$ plane and introducing the orbital angle to be φ, the equations of motion are

$$\frac{d^2\tilde{r}}{d\tilde{\eta}^2} = \left(\tilde{r}\left(\frac{d\varphi}{d\tilde{\eta}}\right)^2 - \frac{Gm}{\tilde{r}^2}\right)\left[1 + \mathcal{O}(\sqrt{\Lambda}t_c) + \mathcal{O}\Big(\frac{Gm}{R_*}\Big) + \mathcal{O}\Big(\frac{G\mu}{R_*}\Big)\right], \tag{3.4}$$

$$\frac{d}{d\tilde{\eta}}\left(\tilde{r}^2\frac{d\varphi}{d\tilde{\eta}}\right) = 2Gm\left(2 - \frac{\mu}{m}\right)\frac{d\tilde{r}}{d\tilde{\eta}}\frac{d\varphi}{d\tilde{\eta}}\left[1 + \mathcal{O}(\sqrt{\Lambda}t_c) + \mathcal{O}\Big(\frac{Gm}{R_*}\Big) + \mathcal{O}\Big(\frac{G\mu}{R_*}\Big)\right] \tag{3.5}$$

For a circular orbit $\frac{d\tilde{r}}{d\tilde{\eta}} = 0$ and it follows from eq. (3.5) that $\frac{d\varphi}{d\tilde{\eta}} \equiv a_e^{-1}\omega$ is constant. In other words, the physical angular velocity Ω is constant up to terms proportional to $\mathcal{O}(\sqrt{\Lambda}t_c)$. From eq. (3.4), we obtain Kepler's law:[6]

$$\left(\frac{d\varphi}{d\tilde{\eta}}\right)^2 = \frac{Gm}{\tilde{r}^3}\left[1 + \mathcal{O}(\sqrt{\Lambda}t_c) + \mathcal{O}\Big(\frac{Gm}{R_*}\Big) + \mathcal{O}\Big(\frac{G\mu}{R_*}\Big)\right]. \tag{3.7}$$

In terms of the physical separation vector R_* and angular frequency Ω, Kepler's law is:

$$\Omega^2 = \frac{Gm}{R_*^3}\left[1 + \mathcal{O}(\sqrt{\Lambda}t_c) + \mathcal{O}\Big(\frac{Gm}{R_*}\Big) + \mathcal{O}\Big(\frac{G\mu}{R_*}\Big)\right]. \tag{3.8}$$

[6]This is the same Kepler's law that is used for FLRW spacetimes. Often it is written in a slightly different format in which the above physical quantities defined at the source are related to quantities as measured by an observer at redshift z: $\omega^{\text{obs}} = (1+z)^{-1}a_e^{-1}\omega$, $m_z = (1+z)m$ and $r_z = (1+z)a_e r$. In terms of the observed quantities — given the same approximations — Kepler's law reads:

$$\omega_{\text{obs}}^2 = \frac{Gm_z}{r_z^3}. \tag{3.6}$$

Once truncated this version of Kepler's law is similar in form to the Minkowski version, with the only difference being constant factors of a_e. The trajectory of the reduced mass as a function of proper time is now described by

$$\vec{R}_*(t) = \left(R_*(t)\,\cos\left(\Omega t + \frac{\pi}{2}\right)\hat{x} + R_*(t)\,\sin\left(\Omega t + \frac{\pi}{2}\right)\hat{y}\right) \\ \left[1 + \mathcal{O}(\sqrt{\Lambda}t_c) + \mathcal{O}\Big(\frac{Gm}{R_*}\Big) + \mathcal{O}\Big(\frac{G\mu}{R_*}\Big)\right] \,,$$

where $\hat{x}$ ($\hat{y}$) is the unit vector in the $x-$direction ($y-$direction) with respect to the Euclidean 3-metric and $\frac{dR_*}{dt} = \mathcal{O}(\sqrt{\Lambda}t_c)$ and $\frac{d\Omega}{dt} = \mathcal{O}(\sqrt{\Lambda}t_c)$. The trajectory of the reduced mass is illustrated in Figure 3.1 for various choices of R_*. The energy density of this system is

$$\rho = \mu\,\delta^{(3)}(\vec{R} - \vec{R}_*) = \frac{\mu}{a_e^3}\delta^{(3)}(\vec{r} - \vec{r}_*) + \mathcal{O}(\sqrt{\Lambda}t_c).$$

This choice for the energy density ensures that the reduced mass μ remains constant. Note that the stress-energy tensor T_{ab} constructed from this energy density is only approximately conserved. In other words, the divergence of the stress-energy tensor with respect to the derivative operator compatible with the de Sitter metric is $\mathcal{O}(\sqrt{\Lambda}t_c)$: $\bar{\nabla}^a T_{ab} = \mathcal{O}(\sqrt{\Lambda}t_c)$. With these choices, the (mass) quadrupole moment is given by

$$Q^{(\rho)}_{ab}(t) = \frac{\mu R_*^2(t)}{2}\left[(1 - \cos 2\Omega t)\,\nabla_a x \nabla_b x + (1 + \cos 2\Omega t)\,\nabla_a y \nabla_b y \right. \\ \left. -2\sin 2\Omega t\;\nabla_{(a} x \nabla_{b)} y\right]\left[1 + \mathcal{O}(\sqrt{\Lambda}t_c) + \mathcal{O}\Big(\frac{Gm}{R_*}\Big) + \mathcal{O}\Big(\frac{G\mu}{R_*}\Big)\right]. \quad (3.9)$$

Taking $\Lambda \to 0$ is equivalent to taking $a \to 1$ and the quadrupole moment reduces to the flat space result in this limit [69].

3.3 Power radiated

Given this set-up, we can now proceed to calculate the radiated power of this system and comment on its properties. The details of the calculation of the transverse-traceless part of the quadrupole moments, needed for the calculation of power, can be found in Appendix A.

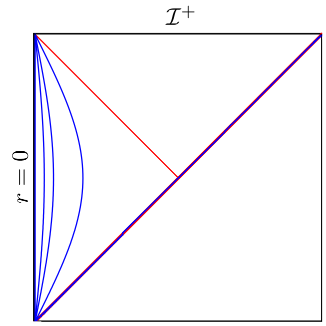

Figure 3.1. This conformal diagram illustrates the physical set-up for a binary system in a circular orbit with respect to physical distances. The straight vertical line on the left indicates the origin $r = 0$ around which the reduced mass of the binary system revolves. The curved (blue) lines are – from left to right – the trajectories of the reduced mass at $R_* = \frac{1}{10}R_\mathrm{c}, \frac{1}{5}R_\mathrm{c}$ and $\frac{1}{2}R_\mathrm{c}$. The values for these trajectories are chosen for illustrative purposes; realistic systems will have a much smaller R_*. The shorter diagonal (red) line denotes the cosmological horizon of the source. (Note that in this figure $\frac{dR_*}{dt}$ is assumed to be exactly zero; not just $\frac{dR_*}{dt} = \mathcal{O}(\sqrt{\Lambda}t_c)$.)

Once the transverse-traceless decomposition of the quadrupole moment $Q^{(\rho)}_{ab}(t_\mathrm{ret})$ in eq. (3.9) and its derivatives, are calculated (for explicit expressions, see eqs.(A.13)-(A.18) in Appendix A), we use the leading order part in the high-frequency approximation of eq.(2.73) to calculate the radiated power:

$$
\begin{aligned}
P &\hat{=} \frac{G}{8\pi}\int d^2S\, \dddot{Q}^{\mathrm{TT}}_{ab}\dddot{Q}^{ab}\left[1+\mathcal{O}(\sqrt{\Lambda}\Omega^{-1})\right] && (3.10)\\
&\hat{=} \frac{32G}{5}\mu^2 R_*^4(t_\mathrm{ret})\Omega^6(t_\mathrm{ret})\left[1+\mathcal{O}(\sqrt{\Lambda}\Omega^{-1})+\mathcal{O}\left(\frac{Gm}{R_*}\right)+\mathcal{O}\left(\frac{G\mu}{R_*}\right)\right]\,, && (3.11)
\end{aligned}
$$

where, as before, $\hat{=}$ denotes equality on $\mathcal{I}$. Since R_* and Ω are constant in time given the approximations made, this equation can simply be written as:

$$
P \hat{=} \frac{32G}{5}\mu^2 R_*^4\Omega^6\left[1+\mathcal{O}(\sqrt{\Lambda}\Omega^{-1})+\mathcal{O}\left(\frac{Gm}{R_*}\right)+\mathcal{O}\left(\frac{G\mu}{R_*}\right)\right]. \tag{3.12}
$$

Let us first comment on three of the main features of this formula. Afterwards, we will contrast this result to the power radiated by a binary system on Minkowski spacetime. First, the radiated power is manifestly gauge-invariant as it solely depends on physical quantities. In addition, the power is clearly positive. Positivity of the power radiated in de Sitter spacetimes was proven for gravitational waves

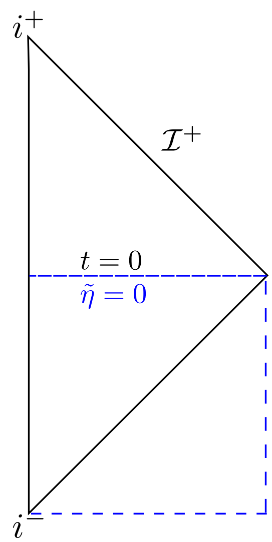

Figure 3.2. This diagram shows the $\tilde{\eta} = 0$ surface of the rescaled flat coordinates. From a Minkowskian perspective, this surface is simply the $t = 0$ surface in the middle of Minkowski spacetime. From a de Sitter perspective, this surface is its future null infinity $\mathcal{I}^+$.

generated by physically realistic sources in Chapter 2. This is the first explicit illustration of that general proof. Third, since there are only two independent scales in this system, the mass and distance scale, the expression for power can be simplified further. Using Kepler's law, see eq. (3.8), the expression for radiated power on a de Sitter background reduces to

$$P \hat{=} \frac{32}{5}\frac{1}{G}\left(GM_c\Omega\right)^{10/3}\left[1 + \mathcal{O}(\sqrt{\Lambda}\Omega^{-1}) + \mathcal{O}\left(\frac{Gm}{R_*}\right) + \mathcal{O}\left(\frac{G\mu}{R_*}\right)\right] \,, \tag{3.13}$$

where $M_c := \mu^{3/5} m^{2/5}$ is the chirp mass.

The result in Minkowski spacetime is identical to eq. (3.12) up to constant factors of a_e. This is surprising for several reasons. At the conceptual level, it seems that de Sitter calculation is essentially carried out on the rescaled Minkowski spacetime in eq. (3.3). However, *the formula for power used is defined on the de Sitter $\mathcal{I}$ and with respect to the de Sitter time translation.* In the rescaled flat coordinates, $\mathcal{I}$ of de Sitter spacetime is at the $\tilde{\eta} = 0$ time slice, which from the Minkowskian perspective is a space-like slice in the middle of spacetime. This is illustrated in Figure 3.2. In addition, the de Sitter time translation is a space-like conformal Killing vector field on the $\tilde{\eta} = 0$ surface in the rescaled flat coordinates.

How then is it possible that this power is equivalent to the power radiated across $\mathcal{I}$ of Minkowski spacetime that is defined with respect to the time translation Killing field of Minkowski space, which is time-like everywhere? Furthermore, the calculational techniques used to derive the power radiated on a de Sitter background are drastically different from those on a Minkowski background. The power radiated in de Sitter spacetime relies on a late time expansion to reach $\mathcal{I}^+$, whereas the power radiated in Minkowski spacetime uses a $1/r$ expansion. In addition, the calculation of the transverse-traceless part on a de Sitter background cannot be done using the algebraic projection operator, instead one is required to solve a set of differential equations to extract the transverse-traceless part. An explicit calculation shows that, for the binary system on a de Sitter background discussed here, Q_{ab}^{TT} and Q_{ab}^{tt} are indeed different (and this difference does *not* vanish in the high-frequency limit):

$$\begin{aligned} Q_{ab}^{TT} - Q_{ab}^{tt} \hat{=} \frac{\mu R_*^2}{2} \Big(& -\frac{1}{6}\left(1+3\cos 2\theta\right)\ \nabla_a r \nabla_b r + r\sin 2\theta\ \nabla_{(a} r \nabla_{b)}\theta \\ & + \frac{r^2}{12}\left(1+3\cos 2\theta\right) \nabla_a\theta\nabla_b\theta + \frac{r^2}{12}\sin^2\theta\left(1+3\cos 2\theta\right)\nabla_a\varphi\nabla_b\varphi \Big) \\ & \left[1+\mathcal{O}(\sqrt{\Lambda}\Omega^{-1})\right] \end{aligned} \tag{3.14}$$

where

$$Q_{ab}^{tt} := \left(P_a{}^c P_b{}^d - \frac{1}{2}P_{ab}P^{cd}\right) Q_{cd} \tag{3.15}$$

with $P_a{}^b = q_a^b - \nabla_a r\ \partial_r^b$ and satisfying $P_a{}^b r_b = 0$, $P_a{}^a = 2$, and $P_a{}^c P_c{}^b = P_a{}^b$. Moreover, the power on a flat background typically makes use of some type of spatial or temporal averaging, such as Brill-Hartle averaging. No such averaging appears in the formula for power radiated on a de Sitter background.

This puzzle is resolved by carefully considering the power radiated across the $t=0$ surface in Minkowski spacetime. Assuming that the source was only dynamic for a finite time interval before $t=0$, the formula for power radiated at the spatial surface given by $t=0$ is

$$P_{\text{Mink}}(r) \hat{=} \frac{G}{8\pi}\int_{t=0} d^2S\ r^2 \left[\left(\frac{\dddot{Q}_{ab}(t_{\text{ret}})}{r}\right)^{TT} \frac{\dddot{Q}_{ab}(t_{\text{ret}})}{r} + \mathcal{O}\Big(\frac{1}{r^3}\Big)\right], \tag{3.16}$$

where the overdots refer to the time translation vector field of Minkowski spacetime. Note that since this formula is not evaluated at $\mathcal{I}$ of Minkowski spacetime, the radiative degrees of freedom in $\left(\frac{\ddot{Q}_{ab}(t_{\rm ret})}{r}\right)$ cannot be obtained by simply projecting it onto the two-sphere. This procedure is only valid on Minkowski $\mathcal{I}$, as we will show in Chapter 4. Therefore, the factor of r^{-1} cannot be pulled out of the TT-operation and the r's in this formula do not cancel at leading order. Together with the fact that there are also higher order terms that cannot be neglected for finite r, this shows that the formula across the $t = 0$ surface in Minkowski spacetime in eq. (3.16) is rather different from the de Sitter formula across the Minkowskian $\tilde{\eta} = 0$ surface in eq. (3.10). Hence, not only are the Killing vector fields used to define the power radiated across the Minkowskian $t = 0$ versus $\tilde{\eta} = 0$ surface different, also the expressions for power are. Only in the limit to $\mathcal{I}$ does the power radiated in Minkowski spacetime become

$$P_{\rm Mink} \hat{=} \frac{G}{8\pi} \int_{\mathcal{I}} d^2S \; \dddot{Q}^{\rm tt}_{ab}(t_{\rm ret}) \dddot{Q}_{ab}(t_{\rm ret}) \tag{3.17}$$

and does the power radiated in Minkowski spacetime agree with the power across $\tilde{\eta} = 0$ of the rescaled flat spacetime using the de Sitter expression. In summary, the power radiated by a binary satisfying approximation 1-6 across $\mathcal{I}$ of de Sitter spacetime is equal to the power radiated by the same binary across $\mathcal{I}$ of Minkowski spacetime.

Remark. Just as on a Minkowski background, for any isolated system, linear momentum radiated in the form of gravitational waves vanishes on a de Sitter background given the approximations made. For the specific case studied here, a binary system restricted to the $x - y$ orbital plane, the flux of angular momentum in the x- and y-direction also vanishes on Minkowski and de Sitter backgrounds. The instantaneous flux of angular momentum in the z-direction, see eq. (2.83), does not vanish and on de Sitter $\mathcal{I}^+$ is given by:

$$\mathcal{L}_T J_z \hat{=} \frac{32G}{5} \mu^2 R_*^4 \Omega^5 \left[1 + \mathcal{O}(\sqrt{\Lambda}\Omega^{-1}) + \mathcal{O}\left(\frac{Gm}{R_*}\right) + \mathcal{O}\left(\frac{G\mu}{R_*}\right)\right]. \tag{3.18}$$

3.4 Discussion

Previous results for linearized perturbations on a de Sitter background have shown certain effects compared to a flat spacetime background. Specifically, using different methods, several groups found that the gravitational memory effect is enhanced by redshift factors [59, 76–78]. This enhancement persists even after taking the high-frequency approximation. Thus, one might anticipate a similar enhancement for power radiated in the form of gravitational waves by a source on a de Sitter background. This expectation is further supported by the striking differences between the calculational techniques used near $\mathcal{I}^+$ of de Sitter spacetimes and those used near $\mathcal{I}^+$ of Minkowski spacetimes: late time versus large r expansions, extracting the radiative degrees of freedom by solving a set of differential equations versus using an algebraic projection operator and in the calculation of power, no averaging versus spatial or temporal averaging.[7] These expectations are not borne out. The power radiated by a binary system on de Sitter spacetime in terms of the reduced mass and angular velocity shows no enhancement (nor decrease) as compared to the result on a Minkowski background for systems we considered. In other words, the standard expression for power radiated in Minkowski spacetimes also applies to de Sitter spacetime in the high-frequency approximation. The high-frequency limit is critical for this equivalence.

This result highlights that in order to probe the cosmological constant by measuring the power, one needs to go beyond the high-frequency approximation. Since the general expression for power radiated in de Sitter spacetimes does not invoke the high-frequency approximation, in principle, there are no obstacles to perform such a calculation. In the current set-up, this regime could not be probed as the dynamics of the binary system could only be determined up to $\mathcal{O}(\sqrt{\Lambda}\Omega^{-1})$. Thus, if one could reliably determine the source dynamics beyond the high-frequency approximation, one could calculate the corrections due to the background curvature on the power. This would allow one to observe Λ through the power emitted by gravitational waves.

The power emitted by the binary system was calculated on $\mathcal{I}^+$ of de Sitter spacetime. A natural question is: how does this result relate to what an observer at a finite time in de Sitter spacetime may detect? Since the dynamical part of the

[7] Albeit, these differences do not indicate whether the power would be enhanced or diminished.

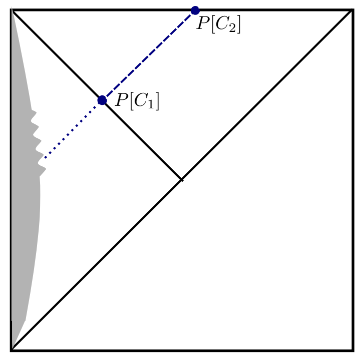

Figure 3.3. There are strong indications, [79], that the power radiated across a two-sphere cross-section on $\mathcal{I}^+$ is the same as the power radiated across any two-sphere along the light-ray emitted from $\mathcal{I}^+$ to the source as long as one remains 'far enough away from the source' and within the high-frequency approximation. Therefore, the power emitted across a two-sphere cross-section on $\mathcal{I}^+$, $P\,[C_2]$, is likely the same as the power emitted on the cosmological horizon, $P\,[C_1]$.

gravitational wave on a de Sitter background propagates sharply (its tail term is an 'instantaneous' tail, see Section 2.3.3), it is likely that the power radiated through a 2-sphere cross-section on $\mathcal{I}^+$ is the same as it is through any two-sphere connected to the two-sphere on $\mathcal{I}^+$ by a null ray [79]. In particular, the power radiated across $\mathcal{I}^+$ is the same as it is through the cosmological horizon. This is illustrated in Figure 3.3. Thus, even though the power presented here was calculated on $\mathcal{I}^+$, there are indications that this power is the same on the cosmological horizon, where an observer may detect such radiation.

Chapter 4
On the conceptual confusion in the notion of transversality

4.1 Introduction

In 1916 Einstein discovered that general relativity admits gravitational waves in the linearized approximation [14]. Almost exactly a century later the LIGO collaboration announced the spectacular detection of gravitational waves produced by coalescing black holes, thereby ushering-in the new field of gravitational wave astronomy [2–4]. There are also ongoing and planned missions to observe primordial gravitational waves [80–82]. Physically, while waves that LIGO detects are produced by astrophysical sources, the origin of primordial radiation is cosmological. Mathematically, in the currently used theoretical paradigm, what LIGO observes is described by retarded solutions of Einstein's equations *sourced by* highly dynamical compact objects in asymptotically flat spacetimes. What cosmological missions hope to observe is described by *source-free* solutions of linearized Einstein's equations on a FLRW background. Thus, not only are their observational techniques very different but the theoretical paradigms that underlie the two missions are also quite different. In particular, they use conceptually distinct notions of *transverse-traceless* modes.

Unfortunately, much of the current literature on gravitational waves in general relativity, including advanced texts as well as review articles, suggests that the two notions are the same and use the same symbol, $h_{ab}^{\rm TT}$, to denote them both. Our reading of several standard references and our discussions with some of the authors led us to conclude that there is genuine conceptual confusion on this issue. The goal of this chapter is to clarify the situation by spelling out what the two notions are, make it explicit that they are distinct, and discuss the relation between them in situations when both notions are available. The main issue arises in linearized gravity and the confusion we referred to is related to gravitational waves produced by isolated bodies. Therefore, it will suffice to restrict ourselves to linearized gravity off Minkowski spacetime. We will do so and briefly return to cosmological perturbations only at the end in Section 4.5.

Consider then linearized gravitational waves in Minkowski spacetime produced by a spatially compact source. One introduces a $t =$ const foliation by space-like planes M_t, and decomposes the spacetime metric perturbation into its irreducible parts. Of immediate interest is the decomposition only of the spatially projected perturbation h_{ab}. If $\mathring{q}_{ab}$ denotes the flat, positive definite 3-metric on the M_t slices, and $\mathring{D}$ its torsion-free connection, then we have the following decomposition of h_{ab} into its irreducible parts:

$$h_{ab} = \frac{1}{3}\mathring{q}_{ab}\, \mathring{q}^{cd} h_{cd} + \left(\mathring{D}_a \mathring{D}_b - \frac{1}{3}\mathring{q}_{ab}\mathring{D}^2\right) S + 2\mathring{D}_{(a} V^T_{b)} + h^{\rm TT}_{ab}, \tag{4.1}$$

where S is a scalar field, $V^{\rm T}_a$ a transverse scalar field and $h^{\rm TT}_{ab}$ a symmetric, transverse-traceless tensor field:

$$\mathring{D}^a V^{\rm T}_a = 0 \qquad \mathring{D}^a h^{\rm TT}_{ab} = 0 \qquad \mathring{q}^{ab} h^{\rm TT}_{ab} = 0\,. \tag{4.2}$$

This is the decomposition of the metric perturbation used in standard cosmology (where one is also interested in the scalar and vector modes). In quantum field theory in Minkowski spacetime, one generally considers source-free solutions to linearized Einstein's equations. Then the gauge invariant information in h_{ab} is contained entirely in $h^{\rm TT}_{ab}$. The vector space of these fields can be naturally endowed with a Poincaré invariant Hermitian inner product. This is the Hilbert space of states of a graviton. It provides irreducible representations of the Poincaré group which have mass zero and helicity ± 2. Thus, this notion of $h^{\rm TT}_{ab}$ is widely used is several areas of physics.

To extract $h^{\rm TT}_{ab}$ from h_{ab} one typically goes to the momentum space where the operation is algebraic and hence local. See, e.g. Box 5.7 in [53], or Section 4.3 in [54], or Section 35.4 of [55]. By contrast the operation is non-local in physical space since it involves inverse power of the Laplacian $\mathring{D}^2$. Thus, if one knows h_{ab} only in a sub-region of the spatial manifold M_t –say the asymptotic region–

one *cannot* determine $h_{ab}^{\rm TT}$ even in that sub-region. Nonetheless, this notion of transverse-traceless modes is widely used because the field $h_{ab}^{\rm TT}$ is *gauge invariant*.

But in the study of retarded fields produced by compact sources, most monographs and review articles switch to *an entirely different* decomposition of the retarded solution to linearized Einstein's equations, which is *local in physical space*. In particular, one introduces a projection operator $P_a{}^b$ into the 2-sphere orthogonal to the radial direction in the *physical space* and extracts a *new* transverse-traceless part of h_{ab} by projecting h_{ab} into the 2-sphere and removing the trace. This is again denoted $h_{ab}^{\rm TT}$ implying that the projection operator $P_a{}^b$ provides just another way to extract what was previously called the transverse-traceless part of h_{ab}. See, e.g., Chapter 11 of [53], or Section 4.5.1 in [54], or Section 36.10 in [55].[1] All subsequent discussion of gravitational waves produced by isolated systems uses the transverse-traceless part of h_{ab} that is extracted using $P_a{}^b$. Consequently, one sees only asymptotic expansions (in powers of $1/r$) of h_{ab} in physical space; Fourier transforms and/or inverse powers of Laplacians are completely absent in actual calculations of radiative modes, wave forms, and expressions of energy carried by gravitational waves.

This is confusing because the two notions of transverse-traceless parts are distinct and inequivalent. To avoid confusion, let us change this notation and set

$$\Big(P_a{}^c P_b{}^d - \frac{1}{2} P_{ab} P^{cd}\Big) h_{cd} =: h_{ab}^{\rm tt}\,. \tag{4.3}$$

As we emphasized, the operation of extracting $h_{ab}^{\rm TT}$ from h_{ab} is highly non-local in physical space and the resulting $h_{ab}^{\rm TT}$ is exactly gauge invariant everywhere in spacetime. On the other hand, the operation of extracting $h_{ab}^{\rm tt}$ is local in physical space and is not gauge invariant. However, we should add that in practice $h_{ab}^{\rm tt}$ is constructed only in the asymptotic region and its $1/r$-part is shown to be gauge invariant under a large class of gauge transformations [53] (but generally it is not explained why it suffices to restrict oneself to that class). As we will see this second notion, $h_{ab}^{\rm tt}$, is tailored to Bondi-Sachs type expansions and behavior of fields near

[1] The underlying logic is illustrated by arguments of Section 1 of [83]. One uses Eq. (4.2) to spell out what one means by $h_{ab}^{\rm TT}$, then says that transversality "requires $h_{ab}^{\rm TT}$ to be orthogonal to each mode's wave vector" and takes the propagation direction to be radial *in space* to conclude that $h_{ab}^{\rm TT}$ defined in Eq. (4.2) is the same as that defined in Eq. (4.3). The reasoning is flawed because the orthogonality $h_{ab}(\vec{k})\,\hat{k}^a = 0$ in momentum space is unrelated to the orthogonality $h_{ab}\hat{r}^a = 0$ in physical space.

null infinity. On the other hand, in the cosmological context, these Bondi-Sachs type expansions are not available and only the h_{ab}^{TT} notion is meaningful.

Still, in the asymptotically flat context, we have two distinct notions of what we mean by transverse-traceless modes. What is the relation between them? Each notion leads to well-defined, leading order asymptotic fields. Do these fields carry the same physical information? One can construct expressions of energy, momentum and angular momentum carried by gravitational waves using either notion. How are they related? To our knowledge these issues have not been spelled out in the literature. The purpose of this chapter is to fill that gap using structure available at null infinity. We will find that while there is a large overlap between physics captured by these notions in the asymptotic region, there are also some differences.

To answer these questions, we will focus on Maxwell fields in Minkowski spacetime, since the central conceptual issues already arise in this technically simpler context. This will allow us to focus on the conceptual issues rather than technical details. The analysis simplifies for a Maxwell field for two reasons. First, for the obvious reason that the potential in Maxwell theory is a rank-1 tensor field whereas in linearized gravity a rank-2 tensor field. Second, using the fact that the theory is conformally invariant, one can pass easily between Minkowski spacetime and its conformal completion that includes null infinity, $\mathcal{I}$. Conceptually, the situation for Maxwell fields is completely parallel to the linearized gravitational case: Again, there are two notions of transverse vector potentials, A_a^{T} and A_a^{t}. The first is gauge invariant but non-local in physical space, while the second is local in space but gauge invariant only in a restricted sense. We will spell out the exact relation between them. In particular, we will show that in presence of sources, *angular momentum carried away by electromagnetic waves is not expressible using just the two radiative modes in* A_a^{t} but also involves longitudinal fields that arise in the discussion of A_a^{T}. These fields are simply not accessible to the second approach to transversality, A_a^{t}, based on the projection operator.

This chapter is structured as follows.[2] In Section 4.2, we use conformal invariance to show that for systems under consideration — where sources are supported on a spatially compact world tube — the Maxwell field F_{ab} satisfies the so-called 'Peeling properties' at null infinity. This discussion will also serve to fix notation and introduce null infinity, $\mathcal{I}$. In Section 4.3, we turn to vector potentials A_a

[2]The material in this chapter is taken almost directly from [40].

since in the gravitational case we are primarily interested in the properties of the linearized metric, not just its curvature. We obtain the minimal fall-off conditions that the potential must satisfy in virtue of field equations as one recedes from sources in null directions. Peeling properties of the Maxwell field simplify this discussion considerably. Next in Section 4.4, we introduce the two notions A_a^{T} and A_a^{t} of transversality. Although A_a^{T} and A_a^{t} differ even at leading order near $\mathcal{I}^+$ (i.e., $\mathcal{O}(1/r)$), we show that the difference is encoded in non-dynamical fields. We then discuss energy-momentum and angular momentum carried by the electromagnetic waves. In the source-free case these quantities can be expressed in terms of the two radiative modes (and the same is true for gravitational waves – even in full non-linear general relativity) [84]. However, in presence of sources *this is no longer the case for angular momentum;* its expression also involves the 'Coulombic information' contained in the (leading order parts of) longitudinal modes, which carry information about the total electric charge in the system. In the discussion, in Section 4.5, we discuss how these results relate to linearized gravitational fields. The overall situation parallels that in the Maxwell case: In presence of sources, the wave forms of h_{ab}^{TT} and h_{ab}^{tt} are in general different even at $\mathcal{I}^+$ but their difference is time-independent. The Bondi mass and flux of angular momentum carried by gravitational waves can not be obtained knowing only the radiative modes encoded in h_{ab}^{tt}; one also needs additional information that can be extracted from h_{ab}^{TT}.

After bulk of this work was completed, Dr. Badri Krishnan drew our attention to an unpublished work of Istávan Rácz [85] which had already pointed out that the presence of sources adds significant complications to the transverse-traceless notion captured in h_{ab}^{TT}. We will discuss the relation with that work at appropriate points.

4.2 Null infinity and the Peeling behavior

As we will see in subsequent sub-sections, presence of sources introduces new features in the asymptotic properties and the physical content of Maxwell potentials, even when the sources are confined to a spatially compact world tube. In that discussion we will assume certain asymptotic behavior of potentials. It is simplest to arrive at these fall-off properties by first noting the immediate consequences of the expressions of retarded solutions and then supplementing them with the implications of the

Peeling properties of the Maxwell field. In the original discussion of Peeling [86], it was assumed that a certain component (Φ_0; see below) of the Maxwell field falls-off as $1/r^3$ and then the fall-off of other components (Φ_1 and Φ_2) was derived. Since it not a priori clear that the initial assumption is satisfied by retarded fields under consideration, for completeness we will now show that the Peeling properties hold also in presence of such sources. This point is probably obvious to experts. We chose to include this discussion because the role of Peeling has been a focus of some of the recent discussions of soft gravitons and photons.[3] Our explicit demonstration will make it clear that new features arise even with the standard Peeling behavior.

4.2.1 Future null infinity $\mathcal{I}^+$

Let us begin with a concrete conformal completion of Minkowski space (M, η_{ab}) that focuses on $\mathcal{I}^+$. In terms of the retarded spherical coordinates $u = t - r, r, \theta, \varphi$, we have

$$\mathrm{d}s^2 = \eta_{ab}\mathrm{d}x^a\mathrm{d}x^b = -\mathrm{d}u^2 - 2\mathrm{d}u\mathrm{d}r + r^2\left(\mathrm{d}\theta^2 + \sin^2\theta\,\mathrm{d}\varphi^2\right). \tag{4.4}$$

Let us conformally rescale η_{ab} with a smooth conformal factor Ω with $\Omega = \frac{1}{r}$ outside the world tube $r = r_0$ for some r_0, and attach to M a boundary $\mathcal{I}^+$ at which Ω vanishes. Then, for $r > r_0$ the conformally rescaled metric $\hat{\eta}_{ab}$ is given by

$$\mathrm{d}\hat{s}^2 = \hat{\eta}_{ab}\mathrm{d}x^a\mathrm{d}x^b = -\Omega^2\mathrm{d}u^2 + 2\mathrm{d}u\mathrm{d}\Omega + \left(\mathrm{d}\theta^2 + \sin^2\theta\,\mathrm{d}\varphi^2\right). \tag{4.5}$$

The rescaled metric $\hat{\eta}_{ab}$ is well-defined everywhere on the manifold with boundary $\hat{M} = M \cup \mathcal{I}^+$. Since Ω vanishes on the boundary $\mathcal{I}^+$, it is coordinatized by $u \in (-\infty, \infty)$ and $\theta, \varphi \in \mathbb{S}^2$. It is thus topologically $\mathbb{S}^2 \times \mathbb{R}$, with a null normal $\tilde{n}^a \hat{=} \hat{\eta}^{ab}\nabla_b\Omega$ that satisfies $\hat{\nabla}_a\tilde{n}_b \hat{=} 0$, where from now onwards $\hat{=}$ *stands for equality restricted to the points of the boundary* $\mathcal{I}^+$. In particular, the conformal frame is 'divergence free'. Furthermore, the pull-back of $\hat{\eta}_{ab}$ to $\mathcal{I}^+$ is the unit 2-sphere metric

[3] In the Christodoulou-Klainerman approach to the non-linear stability of Minkowski spacetime [87], Ψ_1 and Ψ_0 do not peel in the standard manner, given, e.g., in [86]. This feature is sometimes used to argue that failure of the standard Peeling behavior plays a crucial role in some of the discussion of infrared charges and soft gravitons and photons [88]. However, whether standard Peeling holds depends on the boundary conditions that the initial data satisfy. For example, it does hold in the Chrusciel-Delay [89] approach to non-linear stability of Minkowski spacetime. Infrared (or soft) photon charges as well as the new features we discuss arise also with standard Peeling shown in this sub-section.

q_{ab} with $q_{ab}\,\mathrm{d}x^a\mathrm{d}x^b = \mathrm{d}\theta^2 + \sin^2\theta\ \mathrm{d}\varphi^2$. Thus, with this conformal completion we are in a Bondi conformal frame at $\mathcal{I}^+$ (see, e.g., [90]). A general Killing field K^a of η_{ab} has the limit

$$K^a \,\hat{=}\, \Big(\alpha(\theta,\varphi) + u\beta(\theta,\varphi)\Big)\,\tilde{n}^a + h^a \tag{4.6}$$

to $\mathcal{I}^+$. Here $\alpha(\theta,\varphi)$ is a linear combination of the the first four spherical harmonics, Y_{00} and Y_{1m}; $\beta(\theta,\varphi)$ of the three $Y_{1,m}$; and h^a is a 'horizontal' (i.e., tangential to the $u\hat{=}$const 2-sphere cross-sections of $\mathcal{I}^+$) and conformal Killing field of the unit 2-sphere metric, $\mathcal{L}_h q_{ab}\hat{=}2\beta q_{ab}$, satisfying $\mathcal{L}_h\tilde{n}^a\hat{=}0$ (see, e.g., [38, 91]). In particular, the translation Killing fields of η_{ab} are represented by $\alpha\tilde{n}^a$. We will use these facts in Section 4.4.

4.2.2 Peeling

Consider a retarded solution F_{ab} of Maxwell's equation on Minkowski spacetime $(M,\, \eta_{ab})$ with a source current J^a which is smooth and of compact spatial support,

$$\mathrm{d}\,F = 0, \qquad \mathrm{d}\,{}^\star F = 4\pi\ {}^\star J, \tag{4.7}$$

where ${}^\star\hat{J}_{abc} = \epsilon_{dabc}J^d$. The question is whether the limit to $\mathcal{I}^+$ — i.e., the limit $r\to\infty$ keeping u,θ,φ constant — of the retarded solution F_{ab} to these equations satisfies the standard Peeling properties.

To answer this question, one can directly calculate the retarded solution F_{ab} and examine its asymptotic behavior in detail. However, it is much simpler to use conformal invariance of Maxwell's equations. By inspection it follows that $\hat{F}_{ab} := F_{ab}$ and ${}^\star\hat{F}_{ab} = \frac{1}{2}\hat{\epsilon}_{ab}{}^{cd}\hat{F}_{cd} = {}^\star F_{ab}$ also satisfy Maxwell's equations on $(\hat{M}, \hat{\eta}_{ab})$:

$$\mathrm{d}\,\hat{F} = 0, \qquad \mathrm{d}^\star\,\hat{F} = 4\pi\ {}^\star\hat{J} \tag{4.8}$$

with ${}^\star\hat{J} = {}^\star J$. Since $\mathcal{I}^+$ is just a sub-manifold of the conformally completed spacetime $(\hat{M},\, \hat{\eta}_{ab})$ that is 'a finite distance away from sources', it follows that $\hat{F}_{ab}$ is smooth on $\mathcal{I}^+$. We will now show that this fact implies that F_{ab} satisfies the standard Peeling properties.

For this, let us use the Newman-Penrose null co-tetrad on Minkowski spacetime

$(M,\, \eta_{ab})$, given by

$$
\begin{aligned}
n_a &= -\frac{1}{\sqrt{2}}\,(\nabla_a t + \nabla_a r), \qquad \ell_a = -\frac{1}{\sqrt{2}}\,(\nabla_a t - \nabla_a r), \\
m_a &= \frac{r}{\sqrt{2}}\,(\nabla_a \theta + \frac{i}{\sin\theta}\,\nabla_a \phi), \quad \bar{m}_a = \frac{r}{\sqrt{2}}\,(\nabla_a \theta - \frac{i}{\sin\theta}\,\nabla_a \phi)\,. \qquad (4.9)
\end{aligned}
$$

The Newman-Penrose null tetrad is obtained simply by raising indices of these 1-forms with η^{ab}.[4] It is straightforward to check that

$$\hat{n}^a := n^a,\ \hat{\ell}^a := r^2\,\ell^a,\ \hat{m}^a := r\,m^a,\ \text{and}\ \hat{\bar{m}}^a := r\,\bar{m}^a \qquad (4.10)$$

have smooth, non-vanishing limits to $\mathcal{I}^+$ and define a null tetrad there. Therefore, components of the Maxwell fields $\hat{F}_{ab}$ in the hatted null tetrad have smooth limits to $\mathcal{I}^+$. This in turn implies that components of F_{ab} in the Newman-Penrose tetrad in Minkowski spacetime have the following asymptotic behavior

$$
\begin{aligned}
\Phi_2 &= F_{ab} n^a \bar{m}^b = \frac{1}{r}\ \hat{F}_{ab}\hat{n}^a \hat{\bar{m}}^b = \frac{\Phi_2^0(u,\theta,\varphi)}{r} + \mathcal{O}\Big(\frac{1}{r^2}\Big), \qquad (4.11) \\
\Phi_1 &= \frac{1}{2} F_{ab}\left(n^a l^b + m^a \bar{m}^b\right) \\
&= \frac{1}{2r^2}\ \hat{F}_{ab}\left(\hat{n}^a \hat{l}^b + \hat{m}^a \hat{\bar{m}}^b\right) = \frac{\Phi_1^0(u,\theta,\varphi)}{r^2} + \mathcal{O}\Big(\frac{1}{r^3}\Big), \qquad (4.12) \\
\Phi_0 &= F_{ab} m^a l^b = \frac{1}{r^3}\ \hat{F}_{ab}\hat{m}^a \hat{l}^b = \frac{\Phi_0^0(u,\theta,\varphi)}{r^3} + \mathcal{O}\Big(\frac{1}{r^4}\Big)\,, \qquad (4.13)
\end{aligned}
$$

since the hatted fields have well-defined limits as $r \to \infty$, keeping u, θ, φ constant. These are precisely the standard Peeling properties of the Maxwell field [86]. Presence of sources does not introduce any new element in this result. Fields Φ_2^0, Φ_1^0 and Φ_0^0 on $\mathcal{I}^+$ encode the leading order, asymptotic Maxwell field. Maxwell's equations leave Φ_2^0 unconstrained. Because of this property, and its $1/r$ fall-off, Φ_2^0 is called the 'radiation field'. The time derivatives of Φ_1^0 and Φ_0^0 are determined by

[4] Note that, with these conventions, at $\mathcal{I}^+$ we have $\tilde{n}^a \hat{=} \frac{1}{\sqrt{2}}\hat{n}^a$, and $\tilde{n}^a$ is the limit to $\mathcal{I}^+$ of the unit time translation $t^a = -\eta^{ab}\nabla_b t$ of the Minkowski metric η_{ab}.

angular derivatives of Φ_2^0:[5]

$$\partial_u \Phi_1^0 = \eth \Phi_2^0, \qquad \text{and} \qquad \partial_u \Phi_0^0 = \eth \Phi_1^0\,. \tag{4.14}$$

In this sense Φ_1^0 and Φ_0^0 do not have any dynamical degrees of freedom of their own: given Φ_2^0, they are determined by their values at spatial infinity i^o (i.e., $u = -\infty$), or, time-like infinity i^+ (i.e. $u = \infty$.). In the next sub-section we will use these fall-off properties to find useful relations between components of the vector potential that hold for any choice of gauge.

4.3 Asymptotic conditions on potentials

We will need asymptotic conditions that are satisfied by vector potentials A_a of our retarded Maxwell fields. Unlike for the Maxwell field F_{ab}, here we cannot just invoke conformal invariance because the gauge conditions of interest generally fail to be conformally invariant. A seemingly natural strategy would be to assume that, since F_{ab} admits a smooth limit to $\mathcal{I}^+$, one should simply require that the potential A_a is also smooth there. However, as we show below, in presence of sources this requirement is *not* met in gauges that are commonly used. Therefore, a more systematic approach is needed to arrive at an appropriate set of boundary conditions, compatible with gauges that are relevant to the discussions of the two notions of transversality, A_a^{T} and A_a^{t}.

Given a slicing of Minkowski space by hyperplanes $t = \text{const}$, we can decompose the 4-potential A_a as follows:

$$\begin{aligned} A_a &= -\phi \nabla_a t + \vec{A}_a \\ &= -\phi \nabla_a u + (-\phi + A_1)\, \nabla_a r + A_2\, m_a + \bar{A}_2\, \bar{m}_a. \end{aligned} \tag{4.15}$$

We will first examine the asymptotic behavior of $\vec{A}_a$ in the Coulomb gauge since it selects for us the transverse part $\vec{A}_a^{\mathrm{T}}$ of the vector potential we are interested in. Thus, we just start with a vector potential $\vec{A}_a$ that satisfies $\mathring{D}^a \vec{A}_a = 0$ but make

[5] Fields such as Φ_2, Φ_1 and Φ_0 depend on the choice of the dyad $\hat{m}^a, \hat{\bar{m}}_a$ on the $(\theta - \varphi)$ 2-sphere. A field f is said to have spin weight s if it transforms as $f \to e^{is\chi} f$ under the dyad rotation $\hat{m}^a \to e^{i\chi} \hat{m}^a$. The angular derivative $\eth$ of a spin s weighted field is defined by $\eth f = \frac{1}{\sqrt{2}}\,(\hat{m}^a D_a f - \frac{s}{\sqrt{2}}\, \cot\theta\, f) \equiv \frac{1}{2}\,(\partial_\theta f + \frac{i}{\sin\theta} \partial_\phi f - s \cot\theta f)$, where D is the derivative operator on a unit 2-sphere.

no a priori assumptions on its fall-off. The fall-off conditions will be arrived at by using the fact that we are interested in retarded solution to the wave equation that $\vec{A}_a$ satisfies (see below). For ϕ we assume that it tends to zero at infinity. Our primary goal is to motivate the rather weak boundary conditions we wish to impose. Therefore we will sketch the argument without entering into the details of functional analysis.

Let us begin by noting a few consequences of our assumptions that will be useful in determining the asymptotic properties of the solution. Since the source current j_a is smooth and of compact spatial support, on each $t = \text{const}$ slice its spatial projection $\vec{j}_a$ is in particular in the Schwartz space $\mathcal{S}$ of C^∞ functions which, together with all its derivatives falls-off faster than the inverse of any polynomial in r as $r \to \infty$ on $t = \text{const}$ surfaces. Since $\mathcal{S}$ is stable under Fourier transformations, the Fourier transform of $\vec{j}_a$ is also in the Schwartz space in the momentum space. Because the operation of extracting the transverse part is algebraic in the momentum space, it follows immediately that $\vec{j}^{\,\mathrm{T}}_a(\vec{k})$ decays faster than any inverse polynomial in $|\vec{k}|$ as $|\vec{k}| \to \infty$. However, since the projection operator into the transverse part projects $\vec{j}_a(\vec{k})$ into 2-spheres centered at the origin, it fails to be smooth at the origin in the momentum space, whence $\vec{j}^{\,\mathrm{T}}_a(\vec{k})$ also fails to be smooth there. Therefore, although it is smooth everywhere else, bounded at the origin, and decays rapidly as $|\vec{k}| \to \infty$, in general $\vec{j}^{\,\mathrm{T}}_a(\vec{k})$ is not in $\mathcal{S}$. So neither is $\vec{j}^{\,\mathrm{T}}_a(\vec{x})$ in the physical space in the Schwartz space $\mathcal{S}$. Nonetheless, properties of $\vec{j}^{\,\mathrm{T}}_a(\vec{k})$ we just summarized imply that $\vec{j}^{\,\mathrm{T}}_a(\vec{x})$ is smooth and integrable: $\int \vec{j}^{\,\mathrm{T}}_a(\vec{x})\, \mathrm{d}^3x < \infty$. (In the terminology one often uses in the physics literature, the boundedness of the integral can be taken to mean that the Cartesian components of $\vec{j}^{\,\mathrm{T}}_a(\vec{x})$ fall-off faster than $1/r^3$.) Thus decomposition of j_a into Transverse and Longitudinal parts is well-defined on each $t = \text{const}$ surface:

$$j_a = -\rho\, \nabla_a t + \vec{j}_a \;= -\rho\, \nabla_a t + \vec{j}^{\,\mathrm{T}}_a + D_a L$$

with $\mathring{D}^a \vec{j}^{\,\mathrm{T}}_a = 0$.

Now, *in the Coulomb gauge* where $\vec{A}_a = \vec{A}^{\mathrm{T}}_a$, Maxwell's equations become

$$\mathring{D}^2 \phi = -4\pi\rho \quad \text{and} \tag{4.16}$$

$$\Box \vec{A}_a = -4\pi \left(\vec{j}_a - \frac{1}{4\pi} D_a \dot{\phi} \right) = -4\pi \vec{j}^{\,\mathrm{T}}_a \tag{4.17}$$

where in the second step we have used (4.16) and the conservation of 4-current. The equation for ϕ can be solved on each $t = \text{const}$ slice and we consider the retarded solution for $\vec{A}_a$:

$$\phi(t, \vec{x}) = \int \mathrm{d}^3x' \; \frac{\rho(t, \vec{x}')}{|\vec{x} - \vec{x}'|} \tag{4.18}$$

$$\vec{A}_a(t, \vec{x}) = \int \mathrm{d}^3x' \; \frac{\vec{j}_a^{\mathrm{T}}(t - |\vec{x} - \vec{x}'|\,, \vec{x}')}{|\vec{x} - \vec{x}'|}\,. \tag{4.19}$$

Since ρ has compact support and $\vec{j}_a^{\mathrm{T}}$ is bounded and integrable, an examination of the integral on the right shows that the solutions ϕ and (the Cartesian components of) $\vec{A}_a$ fall-off at least as fast as $\mathcal{O}(\frac{1}{r})$ as $r \to \infty$ along constant u, θ, ϕ directions. Consequently, in the Coulomb gauge, our solutions ϕ and $\vec{A}_a$ have the following asymptotic behavior:

$$\phi = \frac{\phi^0(u, \theta, \varphi)}{r} + \frac{\phi^1(u, \theta, \varphi)}{r^2} + \dots \tag{4.20}$$

$$A_1 = \frac{A_1^0(u, \theta, \varphi)}{r} + \frac{A_1^1(u, \theta, \varphi)}{r^2} + \dots \tag{4.21}$$

$$A_2 = \frac{A_2^0(u, \theta, \varphi)}{r} + \frac{A_2^1(u, \theta, \varphi)}{r^2} + \dots \tag{4.22}$$

Although we decomposed the 4-potential using a specific foliation by space-like hyperplanes, the asymptotic conditions (4.20) - (4.22) are insensitive to this choice.

Using this result in the Coulomb gauge as motivation, *from now on we will restrict ourselves to gauges in which 4-potential has the fall-off behavior given above.* Explicit examples involving point charges and dipoles listed in Appendix B show that these boundary conditions are physically reasonable. These fall-off requirements are quite weak. (For example, our arguments in the Coulomb gauge go through even if the sources were not of spatial compact support but were only in the Schwartz space $\mathcal{S}$.) As a consequence, by themselves (4.20) - (4.22) will not be sufficient for our analysis. However, as we will see, field equations and gauge conditions imply further restrictions that supplement these minimal fall-off conditions, and the sum total of these requirements is sufficient to arrive at physically interesting results.

Remark: Although $\vec{j}_a$ is of compact support, the support of its transverse part $\vec{j}_a^{\mathrm{T}}$ extends to spatial infinity (although, as we saw, $\vec{j}_a^{\mathrm{T}}$ falls-off sufficiently fast for its integral to be finite). Rácz pointed out in [85] that this fact introduces a number of complications that have been generally overlooked in the literature.

Next, let us examine the limit of the vector potential to $\mathcal{I}^+$. For this we have to express ϕ and $\vec{A}_a$ of (4.15), in terms of basis vectors in the conformally completed spacetime $(\hat{M}, \hat{\eta}_{ab})$ which are well-behaved at $\mathcal{I}^+$. The result is:

$$\begin{aligned} A_a &= \sqrt{2}\phi\hat{\ell}_a + \Omega^{-2}(\phi - A_1)\tilde{n}_a + \Omega^{-1}A_2\hat{m}_a + \Omega^{-1}\bar{A}_2\hat{\bar{m}}_a \\ &= \sqrt{2}\Big(\Omega\phi^0 + \Omega^2\phi^1 + \dots\Big)\hat{\ell}_a + \Big(\Omega^{-1}\phi^0 + \phi^1 + \dots - \Omega^{-1}A_1^0 - A_1^1 - \dots\Big)\tilde{n}_a \\ &+ \Big(A_2^0 + \Omega A_2^1 + \dots\Big)\hat{m}_a + \Big(\bar{A}_2^0 + \Omega\bar{A}_2^1 + \dots\Big)\hat{\bar{m}}_a\,. \end{aligned} \tag{4.23}$$

Thus, in spite of the fall-offs (4.20) - (4.22) in Minkowski space, because of the presence of Ω^{-1} terms the 4-potential A_a diverges at $\mathcal{I}^+$ in the conformally completed spacetime, unless ϕ and A_1 fall-off faster than $1/r$. However even for a static point charge in Minkowski space, the stronger fall-off conditions are not met. Thus, in presence of sources, one cannot assume that the 4-potential admits a limit to $\mathcal{I}^+$ if we are interested in extracting the transverse modes $\vec{A}_a^{\mathrm{T}}$. The leading order asymptotic fields, ϕ^0, A_1^0 and A_2^0 are given by limits to $\mathcal{I}^+$ of $\Omega^{-1}A_a\hat{n}^a$, $\Omega A_a\hat{\ell}^a$ and $A_a\hat{\bar{m}}^a$ respectively. Finally, as we will see, the transverse field A_a^{t} at $\mathcal{I}^+$ knows only about A_2^0, $\bar{A}_2^0$. This information can also be extracted from A_a by first taking the pull-back $\underleftarrow{A}_a$ to the $\Omega = \text{const}$ surfaces of A_a and then taking the limit to $\mathcal{I}^+$: $\lim_{\mathcal{I}^+}\underleftarrow{A}_a = A_2^0\hat{m}_a + \bar{A}_2^0\hat{\bar{m}}^a$.

As noted aboves, Maxwell's equations — and, in particular, the Peeling properties they imply — can be used to further restrict this asymptotic behavior of A_a *irrespective of the gauge choice* so long as it meets the requirements (4.20) - (4.22). Let us begin by expressing the Newman-Penrose components of the asymptotic Maxwell field in terms of the vector potential:

$$\begin{aligned} \Phi_2 &= \frac{\sqrt{2}}{r}\partial_u A_2^0 + \mathcal{O}\Big(\frac{1}{r^2}\Big) \equiv \frac{\Phi_2^0}{r} + \mathcal{O}\Big(\frac{1}{r^2}\Big) \\ \Phi_1 &= \frac{1}{2r}\underbrace{\partial_u(A_1^0 - \phi^0)}_{=0 \text{ by } (4.12)} + \frac{1}{2r^2}\Big[-\phi^0 + \partial_u(A_1^1 - \phi^1) + \sqrt{2}\eth A_2^0 - \sqrt{2}\bar{\eth}\bar{A}_2^0\Big] + \mathcal{O}\Big(\frac{1}{r^3}\Big) \\ &\equiv \frac{\Phi_1^0}{r^2} + \mathcal{O}\Big(\frac{1}{r^3}\Big) \end{aligned} \tag{4.24}$$

$$\Phi_0 = \frac{1}{r^2}\,\underbrace{\eth(A_1^0 - \phi^0)}_{=0 \quad \text{by (4.13)}} + \frac{1}{r^3}\left[\eth(A_1^1 - \phi^1) + \frac{1}{\sqrt{2}}\bar{A}_2^1\right] + \mathcal{O}\Big(\frac{1}{r^4}\Big) \equiv \frac{\Phi_0^0}{r^3} + \mathcal{O}\Big(\frac{1}{r^4}\Big)\,.$$

These equations have several noteworthy features.

(i) First, the asymptotic 'radiation field' Φ_2^0 is determined by the time derivative of A_2^0: $\Phi_2^0 = \sqrt{2}\partial_u A_2^0$. In addition, angular derivatives of A_2^0 determine $\mathrm{Im}\Phi_1^0$. Therefore, the complex field A_2^0 at $\mathcal{I}^+$ represents the *two radiative modes* of the Maxwell field at $\mathcal{I}^+$.

(ii) The fields $\mathrm{Re}\Phi_1^0$ and Φ_0^0 carry the additional 'Coulombic information' in the Maxwell field at $\mathcal{I}^+$. For example, in the source-free region, the projection along $\hat{n}^b$ of the Maxwell equation $\hat{\nabla}^a \hat{F}_{ab} \hat{=} 0$ reads [86]

$$\partial_u\,\mathrm{Re}\Phi_1^0 \hat{=} \mathrm{Re}\Big(\eth\,\Phi_2^0\Big) \hat{=} \sqrt{2}\partial_u\,\mathrm{Re}\Big(\eth\,A_2^0\Big) \tag{4.25}$$

From the first equality it follows that $Q \hat{=} -\frac{1}{2\pi}\oint \mathrm{Re}\Phi_1^0\,\mathrm{d}^2S$ is conserved, where the integral is taken over a 2-sphere cross-section of $\mathcal{I}^+$. Q is of course the the total electric charge of the source. The second equality implies

$$\mathrm{Re}\Phi_1^0 \hat{=} \sqrt{2}\,\mathrm{Re}\eth A_2^0 + G(\theta,\varphi)\,. \tag{4.26}$$

$G(\theta,\varphi)$ is the 'integration constant'. It is this non-dynamical, real function $G(\theta,\varphi)$ that carries the 'Coulombic information' in $\mathrm{Re}\Phi_1^0$ that escapes in the radiative modes A_2^0. Note in particular, that the electric charge Q can also be expressed as $Q \hat{=} -\frac{1}{2\pi}\oint G(\theta,\varphi)\mathrm{d}^2S$. *From Eq. (4.12) it follows that $G(\theta,\varphi)$ is encoded in the 'non-radiative parts' ϕ^0 and $\partial_u(A_1^1 - \phi^1)$ of the vector potential* A_a in any of the gauges in our class.

(iii) From Peeling properties (4.12) and (4.13) it follows that the field $A_1^0 - \phi^0$ is constant on $\mathcal{I}^+$ and its value does not enter any of the physical quantities one normally considers, such as energy, momentum, angular momentum, electric charge, infra-red charges [90] and electromagnetic memory effect [92]. This property holds in any choice of gauge so long as the potential A_a satisfies the fall-off conditions (4.20) - (4.22). Therefore, if one includes ϕ^0 in the list of components of A_a of interest at $\mathcal{I}^+$, one can drop A_1^0 from the list. This conclusion can also reached by imposing Maxwell equations. But the reasoning is cumbersome. Moreover, the argument in terms of Peeling brings to the forefront the fact that this property must hold in any gauge so long as our fall-off conditions (4.20) - (4.22) are satisfied. (iv) From Maxwell's equation (4.16) in the Coulomb gauge it follows that the total electric charge of the system is directly determined by the scalar potential ϕ^0 on $\mathcal{I}^+$: $Q = \frac{1}{4\pi}\oint \phi^0(u,\theta,\varphi)\mathrm{d}^2S$ where the integral is taken over a cross-section of $\mathcal{I}^+$. (v) Finally, we observe that several simplifications occur in absence of sources. Let us work in the Coulomb gauge. Then, Eq. (4.16) leads us to set $\phi = 0$ everywhere in Minkowski space. In particular, this implies that $A_a\hat{n}^a \hat{=} 0$ and we can set $\phi^0 = 0$, and $\phi^1 = 0$ in Eq. (4.24). Furthermore now, the Coulomb gauge and the Lorenz gauge coincide. Since $\mathring{D}^a A_a = 0$ everywhere in Minkowski space, by multiplying this equation by Ω^{-1} and Ω^{-2} and taking the limit to $\mathcal{I}^+$, we obtain, respectively,

$$\partial_u A_1^0 \hat{=} 0 \qquad \text{and} \quad \partial_u A_1^1 \hat{=} 2\sqrt{2}\mathrm{Re}(\eth A_2^0) + A_1^0\,. \tag{4.27}$$

(The first of these equations is in fact already implied by $\phi^0 \hat{=} 0$ since $\phi^0 - A_1^0$ is constant on $\mathcal{I}^+$.) In the source-free case, now under consideration, the limit to i^o along $\mathcal{I}^+$ of A_1^0 vanishes if the initial data for $\vec{A}_a$ falls off at spatial infinity sufficiently rapidly. Since we already know that A_1^0 is a constant on $\mathcal{I}^+$, it is identically zero. Therefore, the second equation in (4.24) simplifies:

$$\Phi_1 = \frac{\sqrt{2}}{r^2}\eth A_2^0 \qquad \text{i.e.,} \quad G(\theta,\varphi) \hat{=} 0 \tag{4.28}$$

in (4.26). This simplification makes a key difference in the consideration of angular momentum with and without source currents.

Let us summarize. In absence of sources, there are natural gauges in which the 4-potential A_a admits a smooth limit to $\mathcal{I}^+$. Furthermore, one can choose a gauge with $A_a n^a \hat{=} 0$. Then the 2 remaining components of the pull-back $\underleftarrow{A}_a$ of A_a to $\mathcal{I}^+$ are encoded in A_2^0. These radiative modes encode full information in the source-free solution. In particular, the energy, momentum and angular momentum carried by electromagnetic fields can be expressed using the pull-back $\underleftarrow{A}_a$ to $\mathcal{I}^+$ or equivalently, A_2^0 [84]. The situation is quite different once we have sources. The 4-potential can diverge at $\mathcal{I}^+$ in well-motivated gauges (such as the Coulomb and Lorenz gauges). Now, the components A_2^0 at $\mathcal{I}^+$ carry information only about the radiative degrees of freedom in the solution. One needs additional components of the potential encoded, e.g., in $G(\theta, \phi)$, to capture the 'Coulombic aspects' of the solution.

4.4 A_a^{T} versus A_a^{t}

Sections 4.2 and 4.3 provide the necessary platform to compare the two notions of transversality. We will first introduce these notions, emphasizing their asymptotic behavior, and then contrast them.

4.4.1 The two notions

By definition A_a^{T} satisfies the Coulomb gauge condition $\mathring{D}^a \vec{A}_a^{\mathrm{T}} = 0$ everywhere in Minkowski space. Since by assumption A_a^{T} must satisfy (4.20) - (4.22), it follows that there is no residual gauge freedom. Hence all expressions constructed from A_a^T are gauge invariant on the entire spacetime.

For notational clarity, we will use an underbar to denote vector potentials in the Coulomb gauge. The asymptotic expansions (4.20) - (4.22) show that the leading-order part of $\underline{\mathrm{A}}_a$ is captured in 4 functions on $\mathcal{I}^+$, namely, two real functions $\underline{\phi}^0, \underline{\mathrm{A}}_1^0$ and a complex function $\underline{\mathrm{A}}_2^0$. As we saw, A_2^0 represents the radiative degrees of freedom of the Maxwell field. At first, it seems surprising that the leading-order part of the 3-potential, $\underline{\vec{\mathrm{A}}}_a$, has an additional component $\underline{\mathrm{A}}_1^0$. Shouldn't the requirement of transversality leave us with just two? Recall, from the first equation of (4.27), that this requirement does lead to a non-trivial restriction: $\underline{\mathrm{A}}_1^0$ is non-dynamical; it is a function only of θ and φ on $\mathcal{I}^+$. That is, $\underline{\vec{\mathrm{A}}}$ carries information worth two (real) functions $\underline{\mathrm{A}}_2^0(u, \theta, \varphi)$ of three variables *and* one function $\underline{\mathrm{A}}_1^0(\theta, \varphi)$ only of two variables. Thus, in terms of asymptotic fields on $\mathcal{I}^+$, implications of the requirement $\mathring{D}^a \vec{A}_a^{\mathrm{T}} = 0$ are subtle: in addition to the two degrees of freedom $\underline{\mathrm{A}}_2^0$ one would expect, we are left with an unanticipated, additional function $\underline{\mathrm{A}}_1^0$ of two variables. Now, as we saw in Section 4.3, because of Maxwell's equations (which in particular imply the Peeling properties), $(\underline{\phi}^0 - \underline{\mathrm{A}}_1^0)$ is a constant on $\mathcal{I}^+$ in any gauge. Hence in the Coulomb gauge now under consideration, $\underline{\phi}^0$ is also a non-dynamical function only of (θ, ϕ) and we can trade $\underline{\mathrm{A}}_1^0$ for $\underline{\phi}^0$. We will do so because, in view of (4.16), the interpretation of $\underline{\phi}^0$ is more direct: it captures the 'Coulombic aspects' of the asymptotic Maxwell field. In particular, as we saw, the electric charge is given by $Q = \frac{1}{4\pi} \oint \phi^0 \mathrm{d}^2 S$. Finally, since A_a^T is gauge invariant on the entire spacetime, higher order fields such as $\underline{\phi}^1$ and $\underline{\mathrm{A}}_1^1$ also have an invariant meaning. They feature in the expression (4.24) of $\mathrm{Re}\Phi_1^0$ and, together with $\underline{\phi}^0$ and $\underline{\mathrm{A}}_2^0$, suffice to determine it. Therefore, the function $G(\theta, \varphi)$ in (4.26) is also determined by these fields. This fact will play an important role in our discussion of angular momentum.

The second notion of transversality A_a^{t} that is widely used in the literature (to motivate constructions in the gravitational case) is local in physical space (see, e.g. [53]). One sets

$$A_a^{\mathrm{t}} := P_a^{\ b} A_b, \quad \text{where} \quad P_a^{\ b} = m_a \bar{m}^b + \bar{m}_a m^b \ \equiv \ \hat{m}_a \hat{\bar{m}}^b + \hat{\bar{m}}_a \hat{m}^b \tag{4.29}$$

is the projection operator that projects fields into the 2-spheres $r = \text{const}$, $t = \text{const}$ in Minkowski spacetime. Using the expansion (4.15) of A_a^{t} in terms of its components and the assumed fall-off (4.20) - (4.22) of these components, we obtain the following expansion of A_a^{t} in a neighborhood of $\mathcal{I}^+$:

$$A_a^{\mathrm{t}} = \Big(A_2^0 + \Omega A_2^1 + \dots\Big)\hat{m}_a + \Big(\bar{A}_2^0 + \Omega \bar{A}_2^1 + \dots\Big)\hat{\bar{m}}_a \hat{=} A_2^0\, \hat{m}_a + \bar{A}_2^0\, \hat{\bar{m}}_a. \tag{4.30}$$

Note that A_a^{t} automatically satisfies $A_a^{\mathrm{t}} \hat{n}^a \hat{=} 0$ at $\mathcal{I}^+$.

Often A_a^{t} is defined via (4.29) without specifying any gauge conditions even though the result is obviously not gauge invariant. What would happen if we use the Lorenz gauge in Minkowski spacetime so that the dynamical equation satisfied by A_a is just the wave equation? Indeed, more careful treatments make this choice. Then the residual gauge freedom is restricted to $A_a \to A_a + \nabla_a \Lambda$ with $\Box \Lambda = 0$. Since this gauge transformation also needs to preserve the fall-off conditions (4.20) - (4.22), the solution to the wave equation has the form[6]

$$\Lambda = \frac{\Lambda^0(u, \theta, \varphi)}{r} + \mathcal{O}\Big(\frac{1}{r^2}\Big)\,, \tag{4.31}$$

[6]Note that if Λ had a leading order term of the type $\Lambda = \Lambda_0^0(\theta, \varphi) + \mathcal{O}(\frac{1}{r})$, the Cartesian components of $\nabla\Lambda$ *would* fall-off as $1/r$ as needed. But this possibility is ruled out by the fact that $\Box\Lambda$ would then not vanish to leading order. If Λ had terms of the form $\Lambda = (\ln r/r)\, \Lambda^0(u, \theta, \varphi) + \mathcal{O}(\frac{1}{r^2})$, or $\Lambda = \Lambda^0(u, \theta, \varphi) + \mathcal{O}(\frac{1}{r})$, the gauge transformation would fail to preserve the fall-off conditions on A_a.

leading to gauge transformation on A_a of the form

$$A_a \;\to\; A_a + \frac{\partial_u \Lambda^0(u,\theta,\varphi)}{r}\nabla_a u + \frac{\partial_\theta \Lambda^0(u,\theta,\varphi)}{r}\nabla_a \theta + \frac{\partial_\varphi \Lambda^0(u,\theta,\varphi)}{r}\nabla_a \varphi + \mathcal{O}\Big(\frac{1}{r^2}\Big)\,.$$

Hence,

$$\phi^0 \to \phi^0 - \partial_u \Lambda^0(u,\theta,\varphi), \qquad A^0_1 \to A^0_1 - \partial_u \Lambda^0(u,\theta,\varphi), \qquad A^0_2 \to A^0_2 \tag{4.32}$$

and at $\mathcal{I}^+$ we have: $A^t_a \hat{=} A'^t_a$. Thus, $A^{\rm t}_a$ is gauge invariant on $\mathcal{I}^+$ if A_a satisfies the Lorenz gauge near $\mathcal{I}^+$ and A_a has the desired fall-off behavior. Note however that, even with the Lorenz gauge imposed, ϕ^0 and higher order fields including A^1_1 are not gauge invariant. For notational clarity, from now on we will denote fields associated with the vector potential in the Lorenz gauge by an undertilde. Thus, $\utilde{A}^0_2$ will denote the radiative modes extracted from $A^{\rm t}_a$ in the Lorenz gauge.

4.4.2 Comparison

We can now compare the two notions of transversality. First, even with the Lorenz gauge condition, $A^{\rm t}_a$ is *not* gauge invariant beyond the leading asymptotic order. By contrast, $A^{\rm T}_a$ is fully gauge invariant. How do the leading order fields compare? As we saw, each notion of transversality enables one to single out two radiative modes at $\mathcal{I}^+$: $\underline{\rm A}^0_2$ from $A^{\rm T}_a$, and $\utilde{A}^0_2$ from $A^{\rm t}_a$. Both are gauge invariant. Furthermore, we know that the gauge invariant field Φ^0_2 is related to the vector potential via $\Phi^0_2 = \sqrt{2}\,\partial_u A^0_2$, *in any gauge* satisfying the fall-off conditions. Therefore, the radiative modes in the two notions are related by a non-dynamical function on $\mathcal{I}^+$:

$$\underline{\rm A}^0_2(u,\theta,\varphi) - \utilde{A}^0_2(u,\theta,\varphi) \hat{=} f(\theta,\varphi)\,. \tag{4.33}$$

(Since $\mathrm{Im}\Phi^0_1 \hat{=} \sqrt{2}\;\mathrm{Im}(\eth A^0_2)$ in any gauge, it follows that $\mathrm{Im}(\eth f) \hat{=} 0$, which in turn implies that $f = \bar{\eth} h$ where h is real, i.e., f is 'purely electric' [86].) There is no a priori guarantee that the functions representing radiative modes in $A^{\rm T}_a$ are the same as those in $A^{\rm t}_a$ *even at* $\mathcal{I}^+$. However, the difference between them is a non-dynamical function. As we show below, it drops out of all physical quantities that can be constructed from the radiative modes –including those associated with soft (i.e. infrared) 'charges' and the memory effect.

As far as fields at $\mathcal{I}^+$ are concerned, the main difference in the two approaches to transversality of the vector potential is the following. In the first approach leading to A_a^{T}, as we saw, $\underline{\vec{\mathrm{A}}}_a$ has *three* independent components at $\mathcal{I}^+$ even to leading order: In addition to $\underline{\mathrm{A}}_2^0$, the leading order part of $\underline{\vec{\mathrm{A}}}_a$ provides us with $\underline{\mathrm{A}}_1^0$, which as we saw can be traded for $\underline{\phi}^0$. This additional field is non-dynamical on $\mathcal{I}^+$, a hallmark of fields carrying the 'Coulombic information' in the asymptotic Maxwell field. Furthermore, in this approach, higher order fields such as $\underline{\phi}^1$ and $\underline{\mathrm{A}}_1^1$ are also accessible and gauge invariant. By contrast, in the second approach leading to A_a^{t}, only the leading order fields $\underset{\sim}{A}_2^0$ are gauge invariant. While we can introduce $\underset{\sim}{\phi}^0$ and higher order fields, they are not invariant under the restricted gauge transformations even after imposing the Lorenz gauge.

How does this difference manifest itself? First, as we saw, the total electric charge in the system is not encoded in the radiative modes. However, it can be expressed as an integral of ϕ^0 over a 2-sphere cross-section of $\mathcal{I}^+$, accessible in the A_a^{T} approach but not in the A_a^{t} approach. Thus, the information about leading order 'Coulombic properties' of the solution — in particular, the total electric charge — is accessible to the first approach but not to the second. In the gravitational case, the situation is parallel, with electric charge replaced by the linearized Bondi 4-momentum.

Based on this fact, one's first intuition could be that the A_a^{t}-approach would not be adequate to handle issues such as the soft charges and memory effect which are related to the 'charge aspect' Φ_1^0 at $\mathcal{I}^+$, but should be adequate for calculating physical quantities associated with electromagnetic waves such as the energy, momentum and angular momentum they carry. However, it turns out that neither of these expectations is borne out; the situation is more subtle.

4.4.3 Fluxes of energy-momentum and angular momentum

Let us begin with the expressions fluxes of energy-momentum and angular momentum across $\mathcal{I}^+$. Since every Killing field K^a admits a smooth limit (4.6) to $\mathcal{I}^+$, using conformal invariance of the Maxwell field, the flux $\mathcal{F}_K$ can be expressed as:

$$\begin{aligned}\mathcal{F}_K &= \int_{\mathcal{I}^+} \hat{T}_{ab}\, K^a \tilde{n}^b \,\mathrm{d}u\, \mathrm{d}^2S \\ &= \int_{\mathcal{I}^+} \Big(\hat{F}_{ac}\hat{F}_{bd}\,\hat{\eta}^{cd} - \frac{1}{4}\hat{\eta}_{ab}\hat{F}_{cd}\hat{F}^{cd}\Big) K^a \tilde{n}^b \,\mathrm{d}u\, \mathrm{d}^2S\,. \end{aligned} \tag{4.34}$$

Recall from (4.6) that for translation Killing fields, $K^a \hat{=} \alpha(\theta,\varphi)\tilde{n}^a$, with $\alpha(\theta,\varphi)$ characterizing that the translation is a linear combination of the first four spherical harmonics. Therefore, the corresponding flux becomes:

$$\begin{aligned} \mathcal{F}_\alpha &= \int_{\mathcal{I}^+} |\Phi_2^0|^2\, \alpha(\theta,\varphi)\, \mathrm{d}u\, \mathrm{d}^2 S \\ &= \int_{\mathcal{I}^+} 2\, |\partial_u A_2^0|^2\, \alpha(\theta,\varphi)\, \mathrm{d}u\, \mathrm{d}^2 S\,. \end{aligned} \tag{4.35}$$

Thus the energy-momentum flux is expressible entirely in terms of the radiative modes A_2^0. Although we focused on the total flux of energy-momentum across $\mathcal{I}^+$, since no integration by parts was involved, it is clear from the calculation that the integrand, representing the *local flux*, can also be expressed in terms of A_2^0. Furthermore, the expression holds in any gauge.

Next, let us consider the component of angular momentum along a spatial rotation h^a which is tangential to the $u = \text{const}$ 2-sphere cross- sections of $\mathcal{I}^+$ and satisfies $\mathcal{L}_h q_{ab} \hat{=} 0$ and $\mathcal{L}_h \tilde{n}^a \hat{=} 0$ (see (4.6)). Then we can expand h^a as $h^a \hat{=} g(\theta,\varphi)\hat{m}^a + \bar{g}(\theta,\varphi)\hat{\bar{m}}^a$ (where g satisfies $\bar{\eth} g \hat{=} 0$ [38, 91]). Substituting $K^a = h^a$ in (4.34), we obtain the flux of the h^a-component of angular momentum:

$$\mathcal{F}_g = \sqrt{2} \int_{\mathcal{I}^+} \mathrm{Re}\Big[\bar{\Phi}_2^0 \Phi_1^0\, g(\theta,\varphi)\Big]\, \mathrm{d}u\, \mathrm{d}^2 S \tag{4.36}$$

$$= 2 \int_{\mathcal{I}^+} \mathrm{Re}\Big[(\partial_u \bar{A}_2^0)\, (\sqrt{2}\eth A_2^0 + G(\theta,\varphi))\, g(\theta,\varphi)\Big]\, \mathrm{d}u\, \mathrm{d}^2 S \tag{4.37}$$

where in the second step we have used (4.26) to express the real part of Φ_1^0 in terms of A_2^0 and $G(\theta,\varphi)$. Again, these expressions hold in any gauge. The integrand in (4.36) involves only Φ_2^0 and Φ_1^0 which are both manifestly gauge invariant[7] and the passage to (4.37) featuring A_2^0 and $G(\theta,\varphi)$ did not involve any integration by parts. Therefore the integrands in each of these expressions represents the *local flux* of angular momentum. In contrast to energy-momentum, the local as well as the integrated flux of angular momentum depends on the asymptotic 'Coulombic part' of the Maxwell field through $\mathrm{Re}\Phi_1^0$. Eq. (4.37) makes it explicit that the flux of angular momentum *cannot* be expressed purely in terms of the two radiative modes captured in A_2^0; we also need the 'Coulombic information' in $\mathrm{Re}\Phi_1^0$ encoded

[7] While the second expression also holds in any choice of gauge since $\partial_u A_2^0$ and $\mathrm{Re}\Phi_1^0 = \sqrt{2}\mathrm{Re}\eth A_2^0 + G(\theta,\varphi)$ are both gauge invariant, the individual terms, $\mathrm{Re}\eth A_2^0$ and $G(\theta,\varphi)$ are not. In particular $\eth \underline{A}_2^0$ that features in the expression of A_a^{T} need not equal $\eth \underset{\sim}{A}_2^0$ that features in A_a^{t}.

in the function $G(\theta,\varphi)$. In particular, this implies that angular momentum carried away by electromagnetic waves cannot be expressed using only the component of the electromagnetic vector potential in A^{t}_a. This fact appears not to have been noticed in the literature. By contrast, A^{T}_a provides additional asymptotic fields that are sufficient to obtain $\mathrm{Re}\Phi^0_1$ (and $G(\theta,\varphi)$) and hence the local and total flux of angular momentum at $\mathcal{I}^+$.

4.4.4 Soft charges and electromagnetic memory

The 'soft' or 'infrared charges' $q_{\tilde{\alpha}}$ are obtained by integrating the charge aspect $\mathrm{Re}\Phi^0_1$ against real-valued test fields $\tilde{\alpha}$ [90]:

$$q_{\tilde{\alpha}} = \frac{1}{4\pi}\left(\oint_{u=\infty} - \oint_{u=-\infty}\right)\Big[\tilde{\alpha}\ \mathrm{Re}\Phi^0_1(u,\theta,\varphi)\Big]\,\mathrm{d}^2S. \tag{4.38}$$

For any Maxwell field $q_{\tilde{\alpha}}$ vanishes if $\tilde{\alpha} = \text{const}$ by charge conservation. However, for a general $\tilde{\alpha}$ it carries non-trivial information about the 'Coulombic aspect' of the asymptotic Maxwell field. Since the integrand features $\mathrm{Re}\Phi^0_1$, one might first expect that the soft charge would not be expressible purely in terms of radiative modes. However, using (4.26), it follows that

$$q_{\tilde{\alpha}} = -\frac{\sqrt{2}}{4\pi}\left(\oint_{u=\infty} - \oint_{u=-\infty}\right)\mathrm{Re}\Big[A^0_2(u,\theta,\varphi)\ \eth\tilde{\alpha}(\theta,\varphi)\Big]\,\mathrm{d}^2S\,, \tag{4.39}$$

since G is u-independent. Thus, the soft charges can in fact be expressed using only the radiative modes in A^0_2. (Since the kernel of $\eth$ on spin-weight zero functions consists only of constants, in general $q_{\tilde{\alpha}}$ vanishes only if $\tilde{\alpha} = \text{const.}$) Indeed, using the Maxwell equation $\partial_u\Phi^0_1 = \eth\Phi^0_2$ we can recast the soft charges in terms only of the radiation field Φ^0_2:

$$\begin{aligned} q_{\tilde{\alpha}} &= -\frac{1}{4\pi}\int_{\mathcal{I}^+}\Big(\Phi^0_2(u,\theta,\varphi)\ \eth\tilde{\alpha}\Big)\,\mathrm{d}u\,\mathrm{d}^2S \\ &= -\frac{1}{4\pi}\oint \mathrm{d}^2S\,\eth\tilde{\alpha}\,\left(\int_{-\infty}^{\infty}\mathrm{d}u\,\Phi^0_2(u,\theta,\varphi)\right). \end{aligned} \tag{4.40}$$

Thus, even though the soft charges are associated with $\mathrm{Re}\Phi^0_1$ that one normally thinks of as carrying 'Coulombic information', it can be expressed entirely in terms of radiative modes and is therefore accessible in the A^t_a-framework. We will now show that the situation is similar for electromagnetic memory.

The electromagnetic analog of the memory effect is a 'kick' — that is, change in velocity v that a test particle with charge $\mathfrak{q}$ in the asymptotic region undergoes after the passage of an electromagnetic wave. If the particle is initially following a trajectory of the time-translation Killing field t^a in the asymptotic region with $\theta = \theta_o, \varphi = \varphi_o, r = r_o$, it would acquire a velocity of magnitude Δv given by [92]

$$\Delta v = \frac{\mathfrak{q}}{m}\left[\int_{-\infty}^{\infty} \mathrm{d}u \vec{E}(u, r_o, \theta_o, \varphi_o) \cdot \int_{-\infty}^{\infty} \mathrm{d}u \vec{E}(u, r_o, \theta_o, \varphi_o)\right]^{\frac{1}{2}}, \tag{4.41}$$

where E^a is the electric field $E^a = F^{ab}t_b$ in the rest frame of the particle. We can rewrite this expression using the asymptotic expansion of the Maxwell field as:

$$\begin{aligned}
\Delta v &= \frac{\mathfrak{q}}{m r_o}\left|\int_{-\infty}^{\infty} \mathrm{d}u \; \Phi_2^0(u, \theta_0, \varphi_0)\right| + \mathcal{O}\Big(\frac{1}{r_o^2}\Big) \\
&= \frac{\sqrt{2}\mathfrak{q}}{m r_o}\left|A_2^0(u = \infty, \theta_o, \varphi_o) - A_2^0(u = -\infty, \theta_o, \varphi_o)\right| + \mathcal{O}\Big(\frac{1}{r_o^2}\Big).
\end{aligned} \tag{4.42}$$

Thus, the leading term in the electromagnetic memory is proportional to absolute value $|q_{\tilde{\alpha}}|$ of the soft charge, for $\tilde{\alpha}$ such that $\eth\tilde{\alpha}$ is the Dirac distribution centered at $\theta = \theta_o$ and $\varphi = \varphi_o$. Although the kick is closely related to the charge aspect, it can can be expressed entirely in terms of the radiation field Φ_2^0, or, radiative modes A_2^0 of the vector potential.

4.4.5 Summary

A_a^{T} is gauge invariant everywhere in spacetime while only the limit to $\mathcal{I}^+$ of A_a^{t} is gauge invariant. At $\mathcal{I}^+$, the A_a^{T}-framework provides us two leading order fields $\underline{\mathrm{A}}_2^0$ and $\underline{\phi}^0$ (or, equivalently, $\underline{\mathrm{A}}_1^0$), as well a hierarchy of higher order fields, such as $\underline{\phi}^1$, $\underline{\mathrm{A}}_1^1$, … that are all gauge invariant. By contrast, only the gauge invariant fields that the A_a^{t}-framework provides at $\mathcal{I}^+$ is $\utilde{A}_2^0$. The two radiative modes of the electromagnetic field are encoded in $\underline{\mathrm{A}}_2^0$ and $\utilde{A}_2^0$. However, in general $\underline{\mathrm{A}}_2^0$ and $\utilde{A}_2^0$ do not agree even asymptotically, i.e., at $\mathcal{I}^+$, but their difference is encoded in a non-dynamical function $f(\theta, \varphi)$ that is irrelevant for all physical observables. We can express energy momentum, soft charges and electromagnetic memory using only the radiative modes $\underline{\mathrm{A}}_2^0$ or $\utilde{A}_2^0$. However, in presence of sources, angular momentum carried by electromagnetic waves cannot be expressed using just the radiative modes; we also need 'Coulombic information'. The additional fields at $\mathcal{I}^+$ provided by the A_a^{T}-framework carry this information.

Remarks:

(i) It is instructive to rewrite the angular momentum flux (4.37) by performing an integration by parts as:

$$\mathcal{F}_g = 2\int_{\mathcal{I}^+} \mathrm{Re}\Big[(\partial_u \bar{A}_2^0)(\sqrt{2}\eth A_2^0)\, g(\theta,\varphi)\Big] - \Big(\oint_{u=\infty} - \oint_{u=-\infty}\Big)\, \mathrm{Re}[\bar{A}_2^0\, G(\theta,\varphi)\, g(\theta,\varphi)]\,. \tag{4.43}$$

This form brings out the fact that while the flux does depend on the 'Coulombic information' in $\mathrm{Re}\Phi_1^0$, it can be made to enter only through boundary terms at the two ends, i^+ and i^o, of $\mathcal{I}^+$.

(ii) In the source-free case, we know from (4.28) that $G(\theta,\varphi)$ vanishes and hence, like the energy-momentum flux, the angular momentum flux is now expressible only in terms of the two radiative modes A_2^0. For the Maxwell field, we obtained expressions of these fluxes, using the stress-energy tensor. For linearized gravity, there is no local, gauge invariant stress-energy tensor. However, already in the Maxwell case one can use a covariant phase space framework and recover these fluxes as Hamiltonians generating canonical transformations induced by the action of Killing vectors K^a on solutions of Maxwell equations [84]. In the source-free case, this method extends to linearized gravitational waves [35] and, indeed, also to gravitational waves in full general relativity [84].

(iii) In the absence of sources, the covariant phase space can be coordinatized by the two radiative modes of the Maxwell [84] or gravitational fields [93]. But in presence of sources, this coordinatization fails to be faithful; for example, it fails to distinguish different stationary solutions. As we just saw in this section, in presence of sources, one needs additional 'Coulombic information' already to obtain the correct expression of the flux of angular momentum. This requires an extension of the covariant phase space method. We have already carried out this extension for the Maxwell field and the main steps have been laid down for linearized gravity. Eq. (4.43) tells us that the bulk term in the expression of angular momentum is the same as that in the source-free case, but we now have to supplement it with boundary terms that encode the relevant 'Coulombic information.' This structure extends to the generalized phase space constructed at $\mathcal{I}^+$ to accommodate the presence of sources in the interior. The symplectic structure of this phase space has a bulk term that is the same as that in the source-free case but it is now supplemented by boundary terms at i^+ and i^o that involve appropriate non-radiative fields.

4.5 Discussion

Somewhat surprisingly, in many of the widely used monographs and review articles, two different notions are denoted by the same term, *transverse-traceless modes* of (linearized) gravitational waves, without realizing that they are conceptually distinct. These treatments generally begin with the fully gauge invariant notion that is *local in the momentum space*, and hence non-local in physical space, and label the modes with a superscript TT. This is the notion that features in the decomposition of metric perturbations into scalar, vector and tensor modes. However, while discussing gravitational waves emitted by an isolated system — typically in a later section — the relevant modes are extracted using a 'projection operator' that is *local in physical space.* These modes are also called transverse-traceless and again labeled by the same superscript TT, implying that this is just a reformulation of the previous notion. But as we shown in this chapter, the two notions are distinct and the difference persists even in the asymptotic region.

There is one exception to this statement that we have not yet discussed: for plane waves, the two notions of transversality in fact coincide. This is clear from the Fourier transform of the spatial components of the linearized gravitational perturbation h_{ab} on a $t = \text{const}$ slice:

$$h_{ab}(t, \vec{x}) = \int \frac{\mathrm{d}^3 k}{(2\pi)^3} \left(\tilde{h}_{ab}(t, \vec{k}) e^{i\vec{k}\cdot\vec{x}} + \text{c.c.} \right). \tag{4.44}$$

If h_{ab} is transverse and traceless in the usual sense, that is, $h_{ab} = h^{\mathrm{TT}}_{ab}$ and satisfies $\mathring{q}^{ab} h^{\mathrm{TT}}_{ab} = 0 = \mathring{D}^a h^{\mathrm{TT}}_{ab}$, this translates to the following conditions on the Fourier transformed potential $\tilde{h}_{ab}$:

$$\hat{k}^a \tilde{h}^{\mathrm{TT}}_{ab} = 0 \qquad \text{and} \qquad \mathring{q}^{ab} \tilde{h}^{\mathrm{TT}}_{ab} = 0\,, \tag{4.45}$$

where $\hat{k}^a$ is the unit wave-vector. When considering a plane wave traveling along

$\hat{k}^a$, integrating the first condition in (4.45) over d^3k and using the isomorphism between real and Fourier space to identify $\hat{k}^a$ with the radial unit vector $\hat{r}^a$, one can conclude that this condition in real space becomes $\hat{r}^a h_{ab}^{\mathrm{TT}} = 0$. Hence, for plane waves $h_{ab}^{\mathrm{TT}} = h_{ab}^{\mathrm{tt}}$. This is *only true for plane waves*, because the integral over d^3k prohibits the identification of $\hat{k}^a$ with $\hat{r}^a$ whenever one considers a superposition of waves with wave-vectors that are not aligned. Since waves emitted by astrophysical sources are not plane waves, the equivalence for this particular case is not directly physically relevant.[8]

Generically, the two notions of transversality are not equivalent. How do these differences manifest themselves? To investigate this, we studied this issue in the technically simpler context of Maxwell theory in which two similar notions of transversality exist: A_a^{T} and A_a^{t}. Null infinity provides the appropriate platform to compare these two notions. The leading order terms of the Newman-Penrose scalars Φ_2^0 and $\mathrm{Im}\Phi_1^0$ capture the radiative modes and are determined by time and angular derivatives of A_2^0 (and $\bar{A}_2^0$). Hence, the radiative degrees of freedom are encoded in A_2^0, which is the leading order part of the angular components of A_a. For these radiative degrees of freedom, the two notions of transversality are the same up to a non-dynamical function on the two-sphere. This non-dynamical function is not physically relevant as it does not appear in any physical observable quantities such as energy-momentum flux radiated across $\mathcal{I}$, electromagnetic memory or soft charges. There are important physical differences, however, between the two notions. A_a^{T} contains more information than A_a^{t}; it also contains 'Coulombic information' that is not available to A_a^{t}. Physically, this information manifest itself in angular momentum.

We discussed these issues in detail in the technically simpler context of Maxwell theory because the structure is completely parallel to that for linearized gravity. Of course, in the Maxwell theory, one can forego potentials altogether and express all physical observables directly in terms of the Maxwell field F_{ab}, which is accessible in both approaches. But for linearized gravitational fields, we cannot express even

[8]One may argue (see, for instance, [72]) that the direction of propagation of waves produced by astrophysical objects as observed by gravitational wave observatories on earth is well-defined. Consequently, for all practical purposes, the observed waves behave effectively as plane waves. This heuristic argument, however, does not provide much insight. It does not bring a deeper conceptual understanding of the relation between h_{ab}^{TT} and h_{ab}^{tt} discussed in this chapter, nor does it give quantitative control on the difference between h_{ab}^{TT} and h_{ab}^{tt}. Moreover, it completely misses the important observation that radiated angular momentum is captured by h_{ab}^{TT} but not by h_{ab}^{tt}.

the basic physical quantities such as the flux of energy, momentum and angular momentum in terms of linearized curvature; we have to work with potentials. Therefore, we studied the Maxwell case in terms of the vector potential in detail. Analogously to the electromagnetic case, the Newman-Penrose scalars Ψ_4^0, Ψ_3^0 and $\mathrm{Im}\Psi_2^0$ capture the radiative modes in the context of linearized gravitational fields (and in full general relativity — although that is not relevant to this discussion). Just as A_2^0 in Maxwell theory determined the radiative aspects through its time and angular derivatives, so does the linearized shear encoded in the angular components of the metric perturbation determine Ψ_4^0, Ψ_3^0 and $\mathrm{Im}\Psi_2^0$. The linearized shear for h_{ab}^{TT} and h_{ab}^{tt} is generically not the same, not even at $\mathcal{I}^+$, but it differs only by a non-dynamical function on the two-sphere that does not appear in any physical observables typically considered. Futhermore, similar to A_a^{T} in Maxwell theory, h_{ab}^{TT} provides additional fields at $\mathcal{I}$ that contain information about the 'Coulombic aspect' of the field. Preliminary results indicate that these additional fields play an important role for angular momentum radiated by gravitational perturbations. The analysis of angular momentum radiated is rather technical as for linearized gravity no stress-energy tensor exists. To by-pass this issue, we use the covariant phase space framework to calculate fluxes radiated by considering the Hamiltonians generating canonical transformations induced by the action of Killing vectors K^a on solutions of the linearized Einstein's equations [84]. However, the current covariant phase space framework is only valid in the absence of sources. The main steps for the extension of this framework to include sources have been carried out and will appear elsewhere.

To conclude, we do not claim that there is a major error in the final expressions in standard gravitational wave theory literature that heavily uses the second notion of transversality. But it can lead to physically incorrect results for a few quantities such as angular momentum carried by gravitational waves. The primary goal of this chapter was to clarify the conceptual differences between the two notions. This clear distinction between the two notions of transversality is especially important when considering stochastic backgrounds of gravitational radiation or when one wants to place an isolated system in a realistic cosmology, where asymptotically flat spacetimes are replaced by asymptotically de Sitter spacetimes. In neither of these scenarios can one use the second notion of transversality, h_{ab}^{tt}, that relies heavily on the Bondi expansions near null infinity $\mathcal{I}^+$ in asymptotically flat spacetimes.

Chapter 5
Conclusion and outlook

With the recent gravitational wave detections by LIGO, a new era has started: the era of gravitational wave astronomy [2–4]. As a qualitatively new channel to study astrophysical phenomena, gravitational wave observations are likely to revolutionize our understanding of the Universe. In cosmology, they will allow us to look deeper into the past of our Universe than has been possible with CMB observations and study the earliest stages just after the big bang. However, in order to use gravitational waves for cosmology, we need to go beyond our present understanding of these waves.

The current theoretical framework was developed with nearby astrophysical sources in mind: it treats the source that emits gravitational waves as an isolated system in an asymptotically flat universe. In this set-up, spacetime curvature decays as we recede from sources. However, this is not what happens in our physical Universe. Not only does the spacetime curvature prevail because of approximate spatial homogeneity but there is also an accelerated expansion that is best described by a positive cosmological constant Λ. Somewhat surprisingly, the global structure of solutions of Einstein's equations with a positive Λ is very different from that with zero Λ, *no matter how small* Λ *is.* Since radiative properties of solutions can be extracted only by moving far away from sources — to the 'radiation zone' — differences in the global structure become very important in the discussion of properties of gravitational radiation. In particular, the rich framework used to describe gravitational waves in the asymptotically flat context is no longer applicable, nor can it be simply tweaked. One needs completely new methods to incorporate the global issues brought in by a positive Λ. This dissertation provided first steps to overcome these limitations of the current theory of gravitational waves by addressing issues that arise in presence of a positive Λ for linearized gravitational waves. To construct a theory that can adequately describe gravitational waves in our Universe, one would have to incorporate cosmological matter in addition to Λ and extend the framework to full, non-linear general relativity.

5.1 Summary

In the asymptotically flat case, by going far away from sources in null directions one can disentangle the gravitational waves from the background curvature. This insight by Bondi allowed him to show that gravitational waves are physical since they carry energy. Penrose formalized these ideas using conformal methods and showed that the boundary of spacetime, $\mathcal{I}$, is the natural arena on which gravitational radiation can be studied unambiguously. On $\mathcal{I}$ of asymptotically flat spacetimes, a rich structure exists. For instance, there is an asymptotic symmetry group, called the BMS group, that leaves the asymptotic structure invariant. In addition, one can define the Bondi news tensor that describes gravitational radiation in a gauge invariant manner in full, non-linear general relativity. A key difference between asymptotically flat spacetimes and asymptotically de Sitter spacetimes is the nature of $\mathcal{I}$. As a consequence of Einstein's equations, $\mathcal{I}$ is a null surface when with $\Lambda = 0$, while space-like when $\Lambda > 0$. Many conceptual difficulties arise as a result, as we elaborated upon in Chapter 1. For instance, we do not yet have an analog in asymptotically de Sitter spacetimes of the Bondi news tensor. Nor is there an analog of the BMS group that allows for an interpretation of energy-momentum or angular momentum emitted in the form of gravitational radiation.

These issues were circumvented by studying linearized gravitational fields on a de Sitter background. The results obtained by studying the linearized fields serve as important checks in the final construction of the framework that allows the study of gravitational radiation in full non-linear general relativity when $\Lambda > 0$. In addition, they are interesting on their own since already at the linearized level subtleties arise.

First of all, in asymptotically flat spacetimes the properties of gravitational waves are best understood in the region far away from the sources that create them. Therefore, one often employs an $1/r$ expansion to study gravitational waves. However, when $\Lambda \neq 0$ this expansion is not physically relevant. Instead a late time expansion is needed because $\mathcal{I}$ is no longer a null surface but is space-like.

Second, while wavelengths of fields propagating on a flat spacetime are constant, on a de Sitter spacetime they increase and even exceed the curvature radius in the asymptotic region. Consequently, the commonly used high-frequency approximation is not valid near de Sitter $\mathcal{I}$. The breakdown of the high-frequency limit near $\mathcal{I}$ is further illustrated by the appearance of a hereditary term in the expression for gravitational wave perturbations. This term vanishes in the high-frequency limit but its value near $\mathcal{I}$ is comparable to the sharply propagating term and thus should not be neglected. Another interesting lesson learned from the study of linearized fields on a de Sitter background is that the $\Lambda \to 0$ can be discontinuous. A worrisome example is the discontinuity in energy: for gravitational perturbations in the presence of a positive Λ the energy is unbounded below and thus can be arbitrarily negative, whereas for perturbations in flat spacetime the energy is always positive because of the positive mass theorem [94, 95]. Nevertheless, we showed that the energy of gravitational waves emitted by physical sources cannot be negative. For *non-linear* gravitational wave theory, this is still an important open question and can only be addressed after the development of the framework that addresses gravitational radiation in full general relativity for spacetimes with $\Lambda > 0$.

Furthermore, many subtleties arise in the generalization from Einstein's celebrated quadrupole formula describing the power emitted in the form of gravitational radiation on a flat spacetime to de Sitter spacetime. This is described in Chapter 2. This generalization occurred exactly a 100 years after Einstein's initial formula and the long time span between these results highlights the conceptual subtleness and technical difficulty of this generalization. At the technical level, standard techniques to study gravitational waves on flat spacetimes such as the $1/r$ expansion, the high-frequency approximation and the algebraic projection operator to extract the radiative degrees of freedom do not apply to gravitational waves on a de Sitter background. At the conceptual level, we found that power radiated introduces new features. Although the power radiated is still quadrupolar in nature within our post-de Sitter, first order post-Newtonian approximation, it receives contributions from both mass quadrupole moments as well as pressure quadrupole moments. From cosmology, it is well known that pressure gravitates. This result shows that it also directly sources gravitational waves — already at this leading order approximation. In addition, not only third time derivatives of the quadrupole moments contribute to the power radiated but also lower order time derivatives contribute. This is in contrast to the result on a Minkowski background that at this order depends only on the third time derivative of the mass quadrupole moment. The lower order time derivatives and pressure quadrupole moments are all accompanied by factors of $\sqrt{\Lambda}$ so that in the limit $\Lambda \to 0$ the expression for power radiated on a Minkowski background is recovered. Moreover, despite the fact that the gravitational wave itself has a tail term, the power at any given time only depends on the dynamics of the source at the corresponding retarded time. This is due to the fact that the energy radiated depends not on the gravitational perturbation directly but on its derivatives and the derivatives of the tail term propagate sharply. As a result, the tail term does not explicitly appear in the power radiated. (This should not be mistaken to mean that the tail term itself does not contribute to the energy radiated on $\mathcal{I}$, it certainly does.) Thus, while even a small cosmological constant requires entirely new techniques and introduces qualitatively new features to the power radiated, in the end, the corrections to Einstein's formula due to Λ are negligible for current gravitational wave observatories.

There are situations in which the differences between the $\Lambda > 0$ and $\Lambda = 0$ case could be significant. For instance, one of the most powerful sources of gravitational waves is the interaction of two supermassive black holes following the merger of their host galaxies. The time scales associated with this system are cosmological and thus one may hope to detect effects due to $\Lambda > 0$. Therefore, in Chapter 3, we applied the formula for power radiated by an arbitrary source on a de Sitter background to a binary system in a circular orbit. The dynamics of the binary could only reliably be determined up to $\mathcal{O}(\sqrt{\Lambda}\Omega^{-1})$ with Ω the angular velocity of the binary. Consequently, the resulting power could also only be reliably determined up to $\mathcal{O}(\sqrt{\Lambda}\Omega^{-1})$ and the power radiated in terms of the reduced mass and angular velocity is identical to the standard expression for power radiated by the same system on a Minkowski background (up to constant factor of the scale factor at the time of emission). By construction, no corrections due to Λ appear. From a physical perspective, this result is as expected: the effect of Λ is small due to a natural separation of scales. However, from a mathematical perspective, the final answer is rather surprising given that the limit $\Lambda \to 0$ can be discontinuous and the different (physical) approximations made and calculational tools used.

One of the key difference in the calculational tools is the $1/r$ expansion versus the late time expansion. Another important difference between gravitational waves on a Minkowski and de Sitter background is how the radiative degrees of freedom are extracted. When $\Lambda = 0$, one extracts the radiative degrees of freedom of the gravitational waves in the wave zone using an algebraic projection operator. This algebraic projection operator projects the spatial part of the gravitational perturbation onto the two-sphere orthogonal to the radial direction. This has become standard in gravitational wave theory. When $\Lambda > 0$, however, one needs to extract the transverse-traceless part of the spatial components of the linearized potential. In Chapter 4, we studied the relation between the two methods for extracting the radiative modes in the context of asymptotically flat spacetimes. We showed that the radiative degrees of freedom extracted by the two methods generically do not agree; not even at $\mathcal{I}$. Their difference is, however, not physically important as it can be completely encoded by a function of the two angular coordinates $f(\theta, \varphi)$ and does not contain any dynamical information. The difference $f(\theta, \varphi)$ disappears from physical observable quantities typically considered such as energy carried by the gravitational waves. Hence, for questions for which only the radiative degrees of freedom are relevant, both methods yield the same result. The difference between the two methods becomes important when one is also interested in 'Coulombic' information. This information is not available to the algebraic projection method, whereas it is to h_{ab}^{TT}. For instance, angular momentum carried by gravitational waves cannot be calculated from h_{ab}^{tt} alone. We also showed that the algebraic projection operator does not extract the radiative degrees of freedom for (asymptotically) de Sitter spacetimes, since the algebraic projection method is tailored to Bondi-Sachs type expansions and behavior of fields near $\mathcal{I}$ in asymptotically flat spacetimes and does not generalize to cases for which $\mathcal{I}$ is space-like.

5.2 Looking forward

The primary goal of this dissertation was conceptual. Given the observed smallness of the cosmological constant, it is natural to expect that the observational consequences of Λ on gravitational radiation will also be small. However, without a framework to calculate the corrections induced by Λ one cannot say with confidence how small these corrections are. The formula for power radiated in the presence of $\Lambda > 0$ derived in Chapter 2 is a first step towards answering this question. The expression shows explicitly that the effects of Λ are suppressed by the characteristic time scale associated with the source. This result is non-trivial given the fact that the limit $\Lambda \to 0$ can be discontinuous.

Nonetheless, much is still to be understood both with regard to calculational tools as well as fundamental questions. In particular, all results at the linearized level so far have been for a de Sitter background. To be directly relevant for observations, these results need to be extended to more realistic cosmological spacetimes such as FLRW models with positive Λ that include matter. Tied to this generalization to spacetimes with matter, is the question what the origin of the $\mathcal{O}(\sqrt{\Lambda}t_c)$ corrections is. The cosmological constant is a measure of curvature that plays two roles: it is present at the time of generation of the gravitational waves and it influences their propagation. Is one of these a dominating factor in giving rise to the $\mathcal{O}(\sqrt{\Lambda}t_c)$ terms in power radiated? Given that in the formula for power radiated all terms are to be evaluated at the retarded instant of time, it is plausible that the curvature at the time of the wave generation is the key player. If this is indeed the case *and* one assumes that the result for power radiated generalizes in a straightforward manner to spacetimes with matter and $\Lambda > 0$, then it seems plausible that the value of $\sqrt{\Lambda}$ in the power radiated should be replaced by that of $\sqrt{3}H$ at the moment the waves are generated. The value of $\sqrt{3}H$ was much larger in the past than the value of $\sqrt{\Lambda}$. This could make the corrections to the power radiated due to the expansion of the Universe larger and easier to observe. We will conclude by providing some estimates.

Let us suppose that one percent corrections to the power radiated could be readily observed and focus on a binary formed at $z \sim 100$, the epoch when the first stars and black holes were formed. When the period of the orbital motion is $\sim 10^7$ years, the corrections due to the curvature are then observable. For a binary black hole system with a total mass of about two solar masses, this implies, in the Newtonian approximation, a physical separation between the two bodies of 0.4 pc (and a velocity of the individual black holes of ~ 150 m s^{-1}). A binary black hole system with total mass $\sim 10^6 M_\odot$ has to be separated by 30 pc (and the internal velocity is approximately 10^4 m s^{-1}). Such systems should exist among the first stars and black holes, but we do not know how abundant they would be. To measure these effects, we would need gravitational wave detectors sensitive to wavelengths $\lambda \sim 5$ Mpc. For comparison, the pulsar timing array is sensitive to $\lambda \sim 3$ pc [96].

On the other hand, if the terms containing Λ in the power radiated are mainly due to the waves propagating on a curved background and again one assumes that the result for power radiated generalizes trivially to expanding spacetimes with matter, it will be more difficult to observe the effect of Λ on gravitational radiation. Concretely, if one assumes again that one can resolve the power to a precision of one percent then the period of a binary system has to be roughly 10^{10} years. The black holes making up a binary with total mass equaling $\sim 2M_\odot$ are now separated by 26 pc (and $v \sim 18$ m s^{-1}), while for a binary with total mass $10^6 M_\odot$ the separation is 10 kpc (and $v \sim 10^3$ m s^{-1}). Our current methods that focus on $\mathcal{I}$ do not address what the dominating factor is: the curvature at the source or during propagation. New tools may be needed to satisfactorily answer this question. In addition, if the terms of $\mathcal{O}(\sqrt{\Lambda} t_c)$ in the radiated power can be attributed to the wave propagation effect, it is critical to gain a better understanding of the source dynamics on longer time scales. In Chapter 3, the source dynamics could only reliably be determined in the approximation that the scale factor is constant during the time of interest. This assumption will then need to be relaxed. If the effects due to Λ have their origin in the wave generation process, the source dynamics on long time scales is not pivotal in understanding the corrections due to the cosmological background on the power radiated.

From a more practical point of view, it is worthwhile to investigate whether there is an efficient way to extract the radiative degrees of freedom from a gravitational wave perturbation. As mentioned before, in Minkowski spacetime, one can extract the radiative modes on $\mathcal{I}$ using the algebraic projection operator. This projection operator is remarkably simple from a computational perspective and avoids solving a cumbersome set of differential equations otherwise needed to obtain the radiative modes. Applying this algebraic projection operator to gravitational perturbations on a de Sitter background, however, does *not* yield the radiative degrees of freedom. Are there other methods that allow a simple extraction of the radiative modes of the gravitational perturbations on a de Sitter spacetime?

A more fundamental issue is related to the gravitational memory effect. This effect is well understood in asymptotically flat spacetimes and describes the phenomenon that a passing gravitational wave displaces test masses permanently. Several authors have also studied this effect in cosmological spacetimes with a positive Λ and found an enhancement of this effect as compared to flat spacetimes [77, 78]. In addition, since already at the lowest order in the approximations for gravitational waves on de Sitter spacetime a tail term appears — which is absent for flat spacetime — it has been argued that also a new type of linear memory exists [59, 76]. These studies have some important limitations as they employ a $1/r$ expansion which is not appropriate. The connection between these ideas and how they may relate to the further development of the framework of asymptotically de Sitter spacetimes in full, non-linear general relativity is interesting.

Finally, it is important to understand gravitational radiation in non-linear general relativity especially given the historical confusion on the reality of gravitational waves.

Appendix A
Transverse-traceless decomposition of the quadrupole moment

In this appendix,[1] we first briefly recall the decomposition of a spatial tensor into its irreducible parts and outline a generic prescription to extract the transverse-traceless part of this decomposition. In addition, we discuss some important properties of this decomposition. Next, we apply this method to calculate the transverse-traceless part of the quadrupole moment in eq. (3.9) and comment on the calculation of the transverse-traceless part of the radiation field $\mathcal{R}_{ab}$. For this calculation, the algebraic computing software *Maple* was used extensively for solving the differential equations and *Mathematica* for the tensor analysis.

A.1 General properties TT decomposition

As discussed in chapter 2 and 4, any spatial symmetric rank-2 tensor S_{ab} can be decomposed into its irreducible parts in the following form [53, 97]:

$$S_{ab} = \frac{1}{3}q_{ab}S + \left(D_a D_b - \frac{1}{3}q_{ab}D^2\right)B + 2D_{(a}B^T_{b)} + S^{TT}_{ab} \tag{A.1}$$

where q_{ab} is the spatially flat metric and D_a is the covariant derivative compatible with q_{ab}.[2] S is the spatial trace of S_{ab} ($S := q^{ab}S_{ab}$), B^T_a is a transverse vector so that $D^a B^T_a = 0$, and S^{TT}_{ab} is a transverse and traceless tensor.[3] In order to

[1]This appendix is an expanded version of appendix A in [39].

[2]The transverse-traceless decomposition refers to the flat spatial metric and not to the spatial de Sitter metric. Since only the flat metric plays a role, this metric will be denoted by q_{ab} instead of $\mathring{q}_{ab}$ in this appendix to avoid notational clutter.

[3]York, [98], uses a similar decomposition using only the vector W_a, related to our components via $W_a = B^T_a + \frac{1}{2}D_a B$.

extract the radiative degrees of freedom of the gravitational field that are encoded in the transverse-traceless part of the spatial components, one often uses an algebraic projection operator in gravitational wave theory. However, as the above decomposition indicates, this generically will not extract the transverse-traceless part S_{ab}^{TT}. Instead one needs to solve a set of differential equations. The 'recipe' to obtain the TT part of a tensor S_{ab} is as follows:

1. First, by transvecting the above decomposition with two covariant derivatives, one obtains a Poisson equation for the Laplacian of one the longitudinal modes, i.e., for $D^2 B$:

$$D^2(D^2 B) = \frac{3}{2} D^a D^b S_{ab} - \frac{1}{2} D^2 S \tag{A.2}$$

   Denote $D^2 B$ by A and solve for A.

2. Next, solve for the longitudinal mode B itself using $D^2 B = A$.

3. Now, solve for the two independent transverse modes – encoded in B_a^T – satisfying

$$D^2 B_a^T = D^b S_{ab} - \frac{1}{3} D_a S - \frac{2}{3} D^2 D_a B \,, \tag{A.3}$$

$$D^a B_a^T = 0 \,, \tag{A.4}$$

   where the first equation was obtained by contracting eq. (A.1) with D^b.

4. Lastly, the transverse-traceless part of S_{ab} is simply obtained by subtracting the transverse and longitudinal modes from S_{ab} in the following way: $S_{ab}^{TT} = S_{ab} - \frac{1}{3} q_{ab} S - \left(D_a D_b - \frac{1}{3} q_{ab} D^2 \right) B - 2 D_{(a} B_{b)}^T$.

Note that the order of step 2 and 3 can be interchanged.

Before we use apply this algorithm, let us discuss some important properties of this decomposition that are used in the derivation of the expression for power on a de Sitter background in section 2.4.2 and in the explicit calculation of the transverse-traceless part of the radiation field of the binary system discussed in chapter 3. First, if a spatial transverse-traceless tensor S_{ab}^{TT} is contracted with a generic, symmetric spatial tensor X_{ab} and integrated over the proper volume element on a constant time slice, then only the transverse-traceless part of X_{ab}

contributes to the integral. In other words, S_{ab}^{TT} extracts the transverse-traceless part of X_{ab} if integrated:

$$\int d^3V\ X^{ab} S_{ab}^{TT} = \int d^3V\ X_{TT}^{ab} S_{ab}^{TT}. \tag{A.5}$$

This equivalence is not true without the (proper) volume integral. The proof is rather straightforward:

$$\begin{aligned}
\int d^3V\ X^{ab} S_{ab}^{TT} &= \int d^3V\ \left[\frac{1}{3} q_{ab} X + \left(D_a D_b - \frac{1}{3} q_{ab} D^2\right) B_X + 2 D_{(a} Y_{b)}^T + X_{ab}^{TT}\right] S_{TT}^{ab} \\
&= \int d^3V\ \left[D_a D_b B_X + 2 D_a Y_b^T + S_{ab}^{TT}\right] S_{TT}^{ab} \\
&= \int d^3V\ \left[-D^b B_X\, D^a S_{ab}^{TT} - Y_T^b D^a S_{ab}^{TT} + X_{ab}^{TT} S_{TT}^{ab}\right] \\
&= \int d^3V\ X_{TT}^{ab} S_{ab}^{TT}\,,
\end{aligned}$$

where notation is analogous to eq. (A.1) with X the spatial trace of X_{ab}, B_X the other longitudinal mode and Y_a^T describing the transverse vector modes. In going from the first to the second line, the tracelessness of S_{ab}^{TT} automatically cancels two terms. In the next line, integration by parts is performed and the boundary terms are assumed to vanish (which is, for instance, the case if S_{ab}^{TT} has compact support). In the final line, we used that S_{ab}^{TT} is transverse.

Second, since the decomposition in eq. (A.1) is done on a spatial slice, it commutes with time derivatives so that $(\partial_t S_{ab})^{TT} = \partial_t S_{ab}^{TT}$, or equivalently $(\partial_\eta S_{ab})^{TT} = \partial_\eta S_{ab}^{TT}$. However, extracting the transverse-traceless part of a tensor generically does not commute with taking *spatial* derivatives. A notable exception to this is the derivative along the 'dilation' vector field, $r\partial_r$. The time translation vector field of de Sitter spacetime is composed of this dilation vector field and a time derivative (see chapter 1), consequently, $(\mathcal{L}_T Q_{ab})^{TT} \hat{=} \mathcal{L}_T Q_{ab}^{TT}$. Using this property, one can easily obtain the transverse-traceless part of the radiation field $\mathcal{R}_{ab}$ if one knows Q_{ab}^{TT} by simply calculating the appropriate Lie derivatives of Q_{ab}^{TT}. Given the importance of this property, let us elaborate upon its proof. We need to show that if S_{ab}^{TT} is transverse and traceless, $\mathcal{L}_T S_{ab}^{TT}$ is also transverse and traceless. In other words, $D^a \mathcal{L}_T S_{ab}^{TT}$ and $q^{ab} \mathcal{L}_T S_{ab}^{TT}$ have to be zero. Rewriting both conditions, we obtain:

$$D^a \mathcal{L}_T S^{TT}_{ab} = D^a \Big(T^c D_c S^{TT}_{ab} - 2H S^{TT}_{ab} \Big) = D^a T^c \, D_c S^{TT}_{ab} + T^c \, D_c D_a S^{TT}_{ab} - 2H D^a S^{TT}_{ab}$$
$$q^{ab} \mathcal{L}_T S^{TT}_{ab} = \mathcal{L}_T \left(q^{ab} S^{TT}_{ab} \right) - S^{TT}_{ab} \mathcal{L}_T q^{ab}.$$

Using that $D^a T^c = -H q^{ac}$, flat derivatives operators commute, $\mathcal{L}_T q^{ab} = 2H q^{ab}$ and the transverse-traceless property of S^{TT}_{ab}, it follows immediately that indeed $D^a \mathcal{L}_T S^{TT}_{ab} = 0$ and $q^{ab} \mathcal{L}_T S^{TT}_{ab} = 0$.

Lastly, taking the limit to $\mathcal{I}$ commutes with extracting the transverse-traceless part of any symmetric, spatial tensor. This is clear from the procedure to calculate the transverse-traceless part and we have also checked this explicitly for a simplified example.

A.2 Transverse-traceless decomposition of the quadrupole moment

Now we apply the algorithm to calculate the transverse-traceless part of the quadrupole moment in eq. (3.9) evaluated at retarded time on $\mathcal{I}$. Using that the order of taking the limit to $\mathcal{I}$ and extracting the transverse-traceless part is not important, we first take the limit to $\mathcal{I}$ of the quadrupole moment. This simplifies the process of calculating the transverse-traceless part. In each step, we only keep the particular solutions to the Poisson equations as we are only interested in the solution due to the source of the gravitational waves. In the first and second step, normal techniques for solving the Poisson equation can be used. Since the source is quadrupolar finding solutions is straightforward and the solutions are

$$A(r,\theta,\varphi) = -\frac{6\mu R_*^2 \Omega^2 \sin^2\theta}{18H^2 + 8\Omega^2} \Big[\cos\left(2\Omega t_{\mathrm{ret}} + 2\varphi\right) - \frac{3}{2}\frac{H}{\Omega} \sin\left(2\Omega t_{\mathrm{ret}} + 2\varphi\right) \Big] \tag{A.6}$$
$$\begin{aligned} B(r,\theta,\varphi) = \frac{12\mu R_*^2 \Omega^2 H^2 \, r^2 \sin^2\theta}{900H^4 + 544H^2\Omega^2 + 64\Omega^4} \Big[& \Big(1 - \frac{15}{4}\frac{H^2}{\Omega^2} \Big) \cos\left(2\Omega t_{\mathrm{ret}} + 2\varphi\right) \\ & - \frac{4H}{\Omega} \sin\left(2\Omega t_{\mathrm{ret}} + 2\varphi\right) \Big], \end{aligned} \tag{A.7}$$

where t_{ret} on $\mathcal{I}$ is $t_{\mathrm{ret}} = -H^{-1} \ln Hr$. Thus, these solutions do not dependent on time (as should be the case given that Q_{ab} is evaluated on $\mathcal{I}$) and additional $\ln Hr$ dependence hides in t_{ret}. In spherical coordinates the radial component of the vector-Poisson equation in eq. (A.3) is simplified by using the transverse condition to substitute out all but the radial components of the vector-Laplacian operator, giving an almost-standard scalar Helmholtz equation:

$$\left[D^2 + \frac{2}{r}\partial_r + \frac{2}{r^2}\right]B_r^T = q^{ab}D_aQ_{br} - \frac{1}{3}\partial_r(q^{ab}Q_{ab}) - \frac{2}{3}\partial_r D^2 B$$

where the explicit expression for the source term on the right hand side is $\frac{2}{r}A$. Here the index r refers to the orthonormal component of B_a^T in the direction of r. (Similary, we shall use θ and φ indices on B to denote the orthonormal components proportional to $\nabla_a\theta$ and $\nabla_a\varphi$, respectively.) Solving this equation for B_r^T yields the following solution:

$$B_r^T(r,\theta,\varphi) = \frac{2}{r}B(r,\theta,\varphi) \tag{A.8}$$

The θ-component of the vector-Poisson equation is the most challenging, since the source term for the component that appears on the right hand side of eq. (A.3) is not purely quadrupolar:

$$\left[D^2 + \frac{2\cot\theta}{r^2}\partial_\theta + \frac{\cot^2\theta - 1}{r^2}\right]B_\theta^T = q^{ab}D_aQ_{b\theta} - \frac{1}{3r}\partial_\theta(q^{ab}Q_{ab}) - \frac{2}{3r}\partial_\theta D^2 B \tag{A.9}$$

as is clear from the θ dependence in the explicit expression for the right hand side:

$$\text{RHS} = \frac{\mu R_*^2\Omega\sin(2\theta)}{2Hr}\left[\sin\left(2\Omega t_{\text{ret}} + 2\varphi\right) + \frac{H}{\Omega}\cos\left(2\Omega t_{\text{ret}} + 2\varphi\right)\right]. \tag{A.10}$$

Fortunately, the angular dependence of the solution follows from the source term and using an ansatz of the form

$$B_\theta^T = \mu R_*^2 \sin 2\theta\left[f_1(r)\sin\left(2\Omega t_{\text{ret}} + 2\varphi\right) + f_2(r)\cos\left(2\Omega t_{\text{ret}} + 2\varphi\right)\right] \tag{A.11}$$

allows one to solve for the two functions of the radial coordinate, $f_1(r)$ and $f_2(r)$.[4] This gives the following solution:

$$B_\theta^T = -\frac{\mu R_*^2\Omega H\, r\, \sin(2\theta)}{50H^2 + 8\Omega^2}\left[\sin\left(2\Omega t_{\text{ret}} + 2\varphi\right) + \frac{5}{2}\frac{H}{\Omega}\cos\left(2\Omega t_{\text{ret}} + 2\varphi\right)\right] \tag{A.12}$$

[4] In this case the functional form of $f_1(r)$ and $f_2(r)$ is rather simple, but this method generalizes nicely for similar, yet more complicated scenarios.

Once these two components of B^T_a are known, the transverse condition, eq. (A.4), can be integrated to find the third component, B^T_φ:

$$B^T_\varphi = -\frac{H\mu R_*^2 \Omega\, r}{25H^2+4\Omega^2}\left[\cos\left(2\Omega t_{\rm ret}+2\varphi\right) - \frac{5}{2}\frac{H}{\Omega}\sin\left(2\Omega t_{\rm ret}+2\varphi\right)\right].$$

Using these results, we obtain the following transverse-traceless components of the quadrupole moment Q_{ab} evaluated at retarded time on de Sitter $\mathcal{I}^+$ (again, written in an orthonormal frame):

$$\begin{aligned}
Q^{TT}_{rr} \hat{=} & -\frac{3\mu R_*^2 H^2\sin^2\theta}{(9H^2+4\Omega^2)(25H^2+4\Omega^2)}\Big[\left(15H^2-4\Omega^2\right)\cos\left(2\Omega t_{\rm ret}+2\varphi\right) \\
& + 16H\Omega\sin\left(2\Omega t_{\rm ret}+2\varphi\right) + \left(9H^2+4\Omega^2\right)\left(25H^2+4\Omega^2\right) \\
& + \cos(2\theta)\left(675H^4+408H^2\Omega^2+48\Omega^4\right)\Big] \qquad (A.13)
\end{aligned}$$

$$\begin{aligned}
Q^{TT}_{r\theta} \hat{=} & \frac{\mu R_*^2\sin(2\theta)}{4\left(25H^2+4\Omega^2\right)}\Big[-4H\Omega\sin\left(2\Omega t_{\rm ret}+2\varphi\right) - 10H^2\cos\left(2\Omega t_{\rm ret}+2\varphi\right) \\
& + 25H^2+4\Omega^2\Big] \qquad (A.14)
\end{aligned}$$

$$Q^{TT}_{r\varphi} \hat{=} \frac{\mu R_*^2 H\sin\theta}{25H^2+4\Omega^2}\left[5H\sin\left(2\Omega t_{\rm ret}+2\varphi\right) - 2\Omega\cos\left(2\Omega t_{\rm ret}+2\varphi\right)\right] \qquad (A.15)$$

$$\begin{aligned}
Q^{TT}_{\theta\theta} \hat{=} & \frac{\mu R_*^2}{12\left(9H^2+4\Omega^2\right)\left(25H^2+4\Omega^2\right)}\Big[-16\Omega^4-136H^2\Omega^2-225H^4 \\
& + \cos(2\theta)\left(675H^4+408H^2\Omega^2+48\Omega^4\right) - 6\Big(45H^4+87H^2\Omega^2 \\
& + \cos(2\theta)\left(45H^4+21H^2\Omega^2+4\Omega^4\right)+12\Omega^4\Big)\cos\left(2\Omega t_{\rm ret}+2\varphi\right) \\
& + 6H\Omega\left(\cos(2\theta)\left(4\Omega^2-3H^2\right)+87H^2+12\Omega^2\right)\sin\left(2\Omega t_{\rm ret}+2\varphi\right)\Big] \qquad (A.16)
\end{aligned}$$

$$\begin{aligned}
Q^{TT}_{\theta\varphi} \hat{=} & \frac{\mu R_*^2\cos(\theta)}{225H^4+136H^2\Omega^2+16\Omega^4}\Big[\left(42H^3\Omega+8H\Omega^3\right)\cos\left(2\Omega t_{\rm ret}+2\varphi\right) \\
& + \left(45H^4+54H^2\Omega^2+8\Omega^4\right)\sin\left(2\Omega t_{\rm ret}+2\varphi\right)\Big] \qquad (A.17)
\end{aligned}$$

$$\begin{aligned}
Q^{TT}_{\varphi\varphi} \hat{=} & \frac{\mu R_*^2}{12\left(9H^2+4\Omega^2\right)\left(25H^2+4\Omega^2\right)}\Bigg[450H^4+272H^2\Omega^2+32\Omega^4 \\
& - 6H\Omega\left(\cos(2\theta)\left(45H^2+4\Omega^2\right)+39H^2+12\Omega^2\right)\sin\left(2\Omega t_{\rm ret}+2\varphi\right)\Bigg)
\end{aligned}$$

$$6\left(90H^4+\Omega^2\cos(2\theta)\left(33H^2+4\Omega^2\right)+75H^2\Omega^2+12\Omega^4\right)\cos\left(2\Omega t_{\mathrm{ret}}+2\varphi\right)\Bigg]\,. \tag{A.18}$$

Let us comment on a few properties of the above expressions. First, recall that the quadrupole moment is evaluated on $\mathcal{I}^+$ and that $t_{\mathrm{ret}}=-\frac{1}{H}\ln(Hr)$. Second, although in chapter 3, we are interested in the high-frequency limit of Q_{ab}^{TT}, the above expressions are true to all order in $\frac{H}{\Omega}$ (assuming $\frac{dR_*}{dt}=0$, not just $\frac{dR_*}{dt}=\mathcal{O}(\sqrt{\Lambda}t_c)$). The differential equations mix powers of $\frac{H}{\Omega}$. This mixing is responsible for the complicated form of Q_{ab}^{TT} as compared to Q_{ab} itself. To obtain the high-frequency limit, we truncate the result in the end. As shown in eq. (3.14), taking the high-frequency limit of Q_{ab}^{TT} does *not* reduce Q_{ab}^{TT} to Q_{ab}^{tt}.

Appendix B
Point charges and dipoles

In chapter 4, the asymptotic behavior of the vector potential A_a in Maxwell theory was investigated. We showed that for retarded solutions to Maxwell's equations in the Coulomb gauge with sources confined to a spatially compact world tube, the Cartesian components of A_a fall-off as $\mathcal{O}(r^{-1})$ in the limit to $\mathcal{I}^+$. This fall-off behavior in the Coulomb gauge was used as motivation to restrict ourselves to gauges for which the vector potential has this fall-off behavior. Here, we show the leading order behavior of the vector potential in the Coulomb gauge for several physical examples including a static charge and oscillating dipole. We discuss the oscillating dipole also in the Lorenz gauge. It also exhibits the desired fall-off behavior in that gauge, thereby illustrating that the asymptotic conditions satisfied by the vector potential in the Coulomb gauge also accommodates other often chosen gauges.

The simplest example in the Coulomb gauge is provided by a point charge $\mathfrak{q}$ moving along the origin of the rest-frame in which the Coulomb gauge is imposed. In other words, consider a source described by $j_a = -\mathfrak{q}\, \delta^{(3)}(\vec{r})\ \nabla_a t$. Then the only non-zero component of the retarded solution for the vector potential A_a is $\underline{\phi}^0 = \mathfrak{q}$. A slightly more interesting case is obtained by boosting this charge with respect to the rest-frame in which the Coulomb gauge is imposed [85]. For instance, boosting $\mathfrak{q}$ in the x-direction with velocity v is described by a source with charge density $\rho = \mathfrak{q}\, \delta^{(3)}(\vec{r} - vt\, \hat{x})$ with $\hat{x}$ the unit vector in the x-direction. This yields the following leading order behavior for the parts of A_a containing 'Coulombic information':

$$\underline{\phi}^0 = \frac{\mathfrak{q}}{\sqrt{1 - 2v \sin\theta \cos\phi + v^2}}$$

$$\underline{A}_1^0 = -\mathfrak{q} + \frac{\mathfrak{q}}{\sqrt{1 - 2v\sin\theta\cos\phi + v^2}}\,.$$

The radiative modes vanish, that is, $\underline{A}_2^0 = \underline{\bar{A}}_2^0 = 0$, as this source does not radiate. Instead of boosting the charge, one can oscillate the charge around the origin along the x-axis of the reference frame in which the Coulomb gauge is imposed [85]. When the charge is oscillates harmonically with amplitude a and angular frequency ω, the charge density is given by $\rho = \mathfrak{q}\,\delta^{(3)}(\vec{r} - a\,\sin(\omega t)\hat{x})$. The only non-vanishing leading order potential in this case is $\underline{\phi}^0 = \mathfrak{q}$.

Dipole sources are also physically interesting sources and we will show for a particular example that they exhibit the desired fall-off behavior in both the Coulomb and Lorenz gauge. Take, for example, an oscillating dipole situated at the origin with strength $\mathfrak{p}$ and angular frequency ω so that the source is $j_a = \frac{\mathfrak{p}}{4\pi}\Big(\cos\omega t\,\delta(x)\delta(y)\delta'(z)\nabla_a t - \omega\sin\omega t\,\delta^{(3)}(\vec{x})\nabla_a z\Big)$. If the Coulomb gauge is implemented on any of the constant t-slices, the scalar potential ϕ is obtained by solving the Poisson equation with the charge density ρ as a source:

$$\vec{\nabla}^2\underline{\phi} = -4\pi\rho = -\mathfrak{p}\,\cos(\omega t)\delta(x)\delta(y)\delta'(z). \tag{B.1}$$

The Green's function for the Laplace equation satisfies

$$\vec{\nabla}^2 G^3(\vec{x},\vec{x}') = \delta^{(3)}(\vec{x},\vec{x}')$$

and is given by

$$G^3(\vec{x},\vec{x}') = -\frac{1}{4\pi|\vec{x}-\vec{x}'|}.$$

Therefore, $\underline{\phi}$ in the Coulomb gauge is

$$\underline{\phi} = \int d^3\vec{x}'\,\frac{\mathfrak{p}\,\cos\omega t}{4\pi}\frac{\delta(x')\delta(y')\delta'(z')}{|\vec{x}-\vec{x}'|} \tag{B.2}$$

$$= -\frac{\mathfrak{p}\,\cos\omega t}{4\pi}\int dz'\,\frac{\partial}{\partial z'}\left(\frac{1}{\sqrt{x^2+y^2+(z-z')^2}}\right)\delta(z') \tag{B.3}$$

$$= -\frac{\mathfrak{p}\,\cos\omega t}{4\pi}\frac{\cos\theta}{r^2}. \tag{B.4}$$

The solution for the 3-vector potential in the Coulomb gauge to leading order is [99]

$$\underline{\vec{A}}_a = \frac{\mathfrak{p}\,\omega}{4\pi r}\left(-\sin\omega t\,\cos\omega r + \cos\omega t\,\sin\omega r\right)\left[\nabla_a z - \hat{r}^b\nabla_b z\,\nabla_a r\right] + \mathcal{O}(r^{-2}) \tag{B.5}$$

$$= \frac{\mathfrak{p}\,\omega}{4\sqrt{2}\pi r}\sin\omega(t-r)\,\sin\theta\,[m_a + \bar{m}_a] + \mathcal{O}(r^{-2}). \tag{B.6}$$

Hence, the leading order potentials are given by

$$\underline{\phi}^0 = 0 \qquad \underline{A}_1^0 = 0$$
$$\underline{A}_2^0 = \underline{\bar{A}}_2^0 = \frac{\mathfrak{p}\,\omega}{4\sqrt{2}\pi}\sin\theta\,\sin\omega u\,.$$

Now let us compare the potentials of this physical system in the Coulomb gauge with those in the Lorenz gauge. In this case, the scalar vector potential satisfies the inhomogeneous wave equation:

$$\Box\underset{\sim}{\phi} = \left(-\partial_t^2 + \vec{\nabla}^2\right)\underset{\sim}{\phi} = -4\pi\rho = \mathfrak{p}\,\cos(\omega t)\delta(x)\delta(y)\delta'(z).$$

Since the Green's function in Minkowski spacetime satisfies

$$\left(-\partial_t^2 + \vec{\nabla}^2\right)G(x,x') = \delta^{(4)}(x,x')$$

and has the following retarded solution

$$G_{\text{ret}}(x,x') = -\frac{\delta(t-t'-|\vec{x}-\vec{x}'|)}{4\pi|\vec{x}-\vec{x}'|}, \tag{B.7}$$

the scalar potential $\underset{\sim}{\phi}$ in the Lorenz gauge is

$$\underset{\sim}{\phi}(t,\vec{x}) = -\mathfrak{p}\int dt'\int d^3\vec{x}'\,\frac{\delta(t-t'-|\vec{x}-\vec{x}'|)}{4\pi|\vec{x}-\vec{x}'|}\cos(\omega t')\delta(x')\delta(y')\delta'(z') \tag{B.8}$$

$$= -\frac{\mathfrak{p}\,\omega\,\cos\theta\,\sin(\omega u)}{4\pi r} + \frac{\cos\theta\,\mathfrak{p}\,\cos(\omega u)}{4\pi r^2}\,. \tag{B.9}$$

The retarded solution for the spatial part of the vector potential is [99]

$$\underset{\sim}{\vec{A}}_a = -\frac{\mathfrak{p}\,\omega}{4\pi r}\sin(\omega(t-r))\,\nabla_a z \tag{B.10}$$

$$= -\frac{\mathfrak{p}\,\omega}{4\pi}\sin(\omega u)\left(\cos\theta\nabla_a r - \frac{1}{\sqrt{2}}\sin\theta\,m_a - \frac{1}{\sqrt{2}}\sin\theta\,\bar{m}_a\right). \tag{B.11}$$

Thus, in the Lorenz gauge the leading order potentials are

$$\begin{aligned}
\underset{\sim}{\phi}^0 &= -\frac{\mathfrak{p}\,\omega\cos\theta\,\sin\omega u}{4\pi}\\
\underset{\sim}{A}{}^0_1 &= -\frac{\mathfrak{p}\,\omega\cos\theta\,\sin\omega u}{4\pi}\\
\underset{\sim}{A}{}^0_2 &= \bar{A}^0_2 = \frac{\mathfrak{p}\,\omega}{4\sqrt{2}\pi}\sin\theta\,\sin\omega u\,.
\end{aligned}$$

The examples in this appendix nicely illustrate three general properties discussed in chapter 4. First, the 4-vector potential does not always have a well-defined limit to $\mathcal{I}$. Specifically, whenever $A^0_1 \neq 0$, A_a does not have a limit. Second, Peeling (which follows from Maxwell's equations) implies that $\phi^0 - A^0_1 = \text{const}$. This is true for all of the above examples while there are several cases in which neither ϕ^0 nor A^0_1 is independently constant. Third, the radiative modes in the Coulomb and Lorenz gauge, respectively, encoded in $\underline{A}^0_2$ and $\underset{\sim}{A}{}^0_2$, differ only by a function on the two-sphere $f(\theta,\varphi)$. For the oscillating dipole, this function is trivial: $f(\theta,\varphi) = 0$.

Appendix C
Various investigations in early Universe cosmology

Inflation is the leading paradigm of the early Universe, according to which the tiny temperature fluctuations observed in the cosmic microwave background (CMB) originate from quantum vacuum fluctuations at very early times [100–106]. Specifically, starting with generic initial conditions, inflation produces a nearly scale-invariant primordial spectrum for scalar and tensor fluctuations at the end of inflation. Evolving these primordial power spectra with Boltzmann equations in conjunction with Einstein's equations to the time of last scattering reproduces the temperature power spectrum observed in the CMB remarkably well. However, several unexpected features have been observed at large angular scales by both WMAP and *Planck* . For instance, an amplitude deficit for the temperature anisotropies in the spherical harmonics decomposition has been observed for $\ell \lesssim 30$ as well as a hemispherical power asymmetry [107]. Although each of these anomalies by itself has low statistical significance (less than 3σ), taken together these anomalies imply a violation of statistical isotropy and scale invariance of the inflationary perturbations. There are several possible explanations for these anomalies: (i) They may be due to foreground residuals and/or systematic effects, (ii) They are merely are a statistical fluctuation, or (iii) They are a result of primordial physics yet to be understood. Hence, understanding their origin is fundamental to either validate the standard model of cosmology or to explore new physics.

During my PhD, I have studied two different extensions to the standard vanilla slow-roll single field model of inflation that may (partially) explain at least one of the observed anomalies, that is, the power suppression anomaly. In particular, in collaboration with Dr. Brajesh Gupt and Dr. Nelson Yokomizo, we have studied the effect of positive spatial curvature on the primordial scalar and tensor power spectra and its effects on the temperature anisotropy spectrum $C_{\ell\ell}^{TT}$ and the B-mode polarization spectrum [108, 109]. Spatial curvature introduces a length scale that breaks scale invariance in a natural way. Previously, the effect of positive spatial curvature during inflation had only been studied at the level of the background evolution and in a simplified setting for the primordial fluctuations in which the inflationary background was replaced with an exact de Sitter background [110–112]. We carried out the first systematic study of the effect of spatial curvature on the temperature anisotropies observed in the CMB by performing a detailed analysis of the inflationary evolution of gauge-invariant perturbations in a closed FLRW Universe. The results are summarized in section C.1 and show that spatial curvature might partially explain the observed power suppression. Together with Dr. Gupt, we also studied an extension of the standard inflationary paradigm to the pre-inflationary era in the context of loop quantum cosmology [113, 114]. Earlier work had used the framework of quantum fields on quantum spacetimes to extend the inflationary phase all the way to the Planck scale and study the observational imprints of the quantum gravity dominated regime on the inflationary perturbations [115]. Interestingly, the ultraviolet modifications of the background dynamics that resolve the big bang singularity influence the infrared modes of the perturbations: The infrared modes are not in a Bunch-Davies vacuum at the onset of inflation. As a result, the primordial power spectrum deviates from the nearly scale-invariant power spectrum predicted by inflation. Depending on the initial state chosen in the deep quantum gravity regime, the resulting primordial power spectrum shows either enhancement or suppression on large scales that subsequently leads to power enhancement or suppression on large angular scales in the temperature anisotropies observed in the CMB. Thus, this opens up the possibility that quantum gravity effects may account for the power suppression anomaly. These results, however, were only obtained for single field inflation with a quadratic potential. We showed that the type of observable modifications due to the pre-inflationary phase are similar for the Starobinsky potential and argued that in general these modifications are robust under the change of the inflationary potential. This work is summarized in section C.2.

Both scenarios discussed above only leave observable imprints on the CMB if the number of e-folds is not much larger than ~ 70, the minimum number of e-folds required for inflation to successfully explain the nearly scale invariant primordial spectra. When the number of e-folds is larger than this minimum, our observable Universe is only part of the entire Universe and new effects need to be accounted for. This is the third project I worked on that I will describe in this appendix. This is joint work with Dr. Suddhasattwa Brahma, Anne-Sylvie Deutsch and Prof. Sarah Shandera. Specifically, if there is coupling between modes of different wavelengths, the local statistics in our observable Universe is generically different from the global statistics. Depending on the realization of the long wavelength modes in the observable patch and the type of mode-mode coupling, the amplitude of the locally observed power spectrum can be shifted and an additional scale dependence can be introduced thereby modifying the spectral index [116]. Thus, even if one in principle would be able to observe all n-point correlation functions, one would only be able to uniquely construct a Langrangian for the fluctuations in our observable Universe but *not* for the global Universe. Nonetheless, we showed that the mass of one of the inflationary fields in quasi-single field inflation can be reliably determined from measurements of the scaling of the bispectrum despite mode-mode coupling due to non-derivative self-interactions [117]. In addition, we showed that while the spectral index n_s in the locally observed volume is generically shifted away from the global mean, this shift is small. This is in contrast to the tensor-to-scalar ratio r that can be significantly different in the locally observed volume as compared to the global volume. Our results are discussed in section C.3.

C.1 Spatially closed Universe

One of the compelling features of the inflationary scenario is that it leads to an almost flat, homogeneous and isotropic Universe today starting from generic initial conditions. In particular, inflation 'dilutes' away any effects of spatial curvature and the spacetime is extremely well approximated by a spatially flat FLRW model at the end of inflation. In other words, even though spatial curvature effects are subdominant today, they could have been important in the early phases of inflation. Modes of metric perturbations exiting the Hubble horizon at these early times during which spatial curvature could be non-negligible can carry imprints of this in their primordial power spectra. Specifcally, spatial curvature can affect observations in two ways: (i) by affecting the background spacetime evolution in the post-inflationary era [118]; (ii) by introducing corrections to the primordial power spectrum at the end of inflation [110, 111, 119–122].

Recent cosmological observations determine the density parameter due to the spatial curvature to be $\Omega_\mathrm{k} = -0.005^{+0.016}_{-0.017}$ from *Planck* data alone and $\Omega_\mathrm{k} = 0.000^{+0.005}_{-0.005}$ by combining *Planck* measurements with BAO data [123]. While these values for Ω_k are compatible with a flat FLRW model, they allow the possibility that our Universe is spatially closed with spatial topology of a 3-sphere. For $\Omega_\mathrm{k} = -0.005$ (which is the value we shall work with throughout), the physical radius of this 3-sphere today is approximately 64 Gpc, which is ~ 4.5 times the physical radius of the CMB sphere. This indicates that the effects of spatial curvature, if present, would be most prominent for length scales similar to or larger than that of the CMB sphere. A detailed analysis shows that this is indeed the case. The evolution of the background geometry and gauge-invariant scalar and tensor perturbations in a spatially closed FLRW universe where inflation is driven by a single scalar field in the presence of an inflationary potential (taken to be either quadratic or Starobinsky potential) show that effects of the non-zero spatial curvature become important at the early phases of inflation (~ 60 e-folds before the end of inflation). The resulting *scalar* power spectrum is different from the almost scale invariant power spectrum obtained in a spatially flat inflationary FLRW spacetime for long wavelength modes. The ratio of the scalar power spectrum in the spatially closed and spatialy flat model is shown in the left panel of figure C.1. The power spectrum in the closed model shows oscillatory behavior that is most prominent for the long wavelength modes. In addition, when these oscillations are averaged out, there is suppression of power at such scales as compared to the flat model. For $\Omega_\mathrm{k} = -0.005$, the power deficit in the primordial power spectrum leads to suppression of power in the temperature anisotropy spectrum C_ℓ^{TT} at $\ell < 20$ [108]. For small wavelength modes, the power spectrum approaches the nearly scale invariant power spectrum and the resulting C_ℓ^{TT} agrees extremely well with that in the flat model for $\ell \gtrsim 20$. Similarly to the scalar perturbations, the tensor power spectrum also shows power suppression for long wavelength modes (see right panel of figure C.1). However, we find that the relative suppression for tensor modes is smaller than that for the scalar modes. The suppression in the tensor power spectrum is weak and limited to very long wavelength modes, which correspond to super-horizon modes. Consequently,

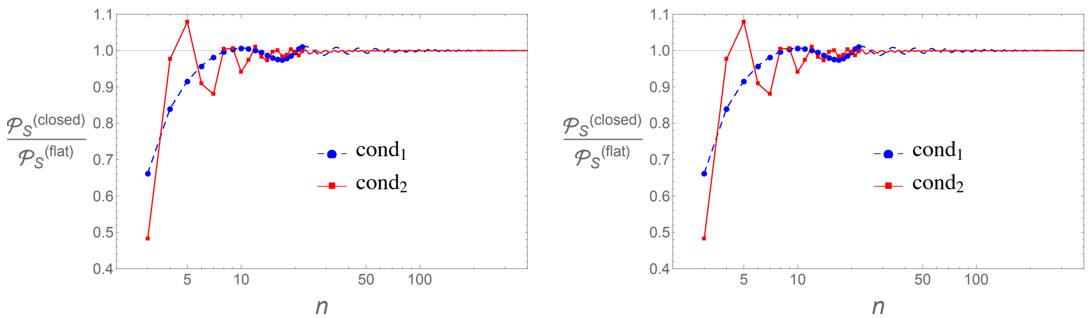

Figure C.1. *Left panel:* Ratio of the scalar power spectrum in the closed FLRW model with $\Omega_{\mathrm{k}} = -0.005$ to that in the flat model as a function of the spherical wavenumber n. *Right panel:* Ratio of the tensor power spectrum in the closed FLRW model with $\Omega_{\mathrm{k}} = -0.005$ to that in the flat model. Both spectra show suppression of power at small n. However, the scalar modes are significantly more suppressed in the closed Universe as compared to the tensor modes. The two curves in each figure correspond to two different initial conditions for the background geometry: 'cond$_1$' corresponds to initial conditions matching the best fit values for the amplitude of the scalar power spectrum A_{s} and its running n_{s} as measured by *Planck* and 'cond$_2$' is chosen to generate maximal suppression while still being consistent with the measured allowed range in A_{s} and n_{s}. In both figures the spectra are discrete in n (for the clarity of presentation discrete points are shown only for $n < 20$).

the resulting observable polarization anisotropy spectrum C_ℓ^{BB} in closed FLRW model differs less than a percent from the flat FLRW model.

Since the tensor and scalar spectra are modified differently by the spatial curvature, the tensor-to-scalar ratio at the long wavelength modes acquires scale dependent corrections which further leads to the violation of the standard slow-roll consistency relation obtained for the flat FLRW model. This is shown in figure C.2. Hence, although the modifications in the tensor spectrum due to positive spatial curvature do not have a direct observable imprint in the B-mode polarization signal, they lead to violation of the standard slow-roll consistency relation for long wavelength modes. This deviation from the standard relation, if observed, can be used to further refine the constraints on spatial curvature of the Universe.

In our work, we focused on the analysis of *geometric* effects due to the presence of spatial curvature, but our framework can also be applied to study observational signatures of spatial topology in locally spherical universes. We restricted ourselves to spacetimes with spatial sections isometric to the three-sphere, the simplest global geometry compatible with a constant positive spatial curvature, but solutions for the equations of motion for perturbations on nontrivial topologies with positive

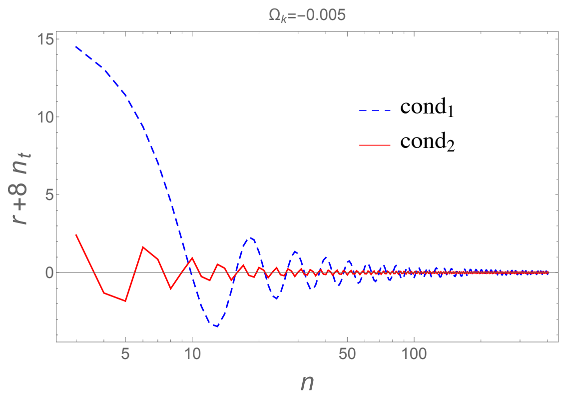

Figure C.2. The deviation from the consistency relation ($r + 8n_{\mathrm{t}} = 0$) for $\Omega_{\mathrm{k}} = -0.005$ plotted with respect to the spherical wavenumber n. It is evident that for small n deviation from $r + 8n_{\mathrm{t}} = 0$ is more prominent. These deviations approach zero as n increases. Similar to figure C.1, cond$_1$ and cond$_2$ correspond to two different choices for the background initial conditions.

curvature can always be written as linear superpositions of normal modes on the three-sphere. Consequently, the evolution of the inflationary perturbations and the primordial power spectrum for arbitrary topologies with positive curvature can be determined from our results for the dynamics of the perturbations on the three-sphere.

C.2 Robustness of predictions of a quantum gravity extension to inflation

The inflationary scenario is highly successful in explaining, with minimal assumptions, the primordial origin of structure formation and small inhomogeneities observed in the cosmic microwave background (CMB). However, the standard inflationary models are based on general relativity and therefore inherit the big bang singularity and cannot be extended too far into the past. Using techniques from loop quantum gravity, it was shown that there is a self-consistent pre-inflationary extension all the way to the Planck scale [115, 124, 125]. This extension has a finite quantum gravity regime and confronts the problem of a past singularity. At the background level, the underlying quantum geometric effects of loop quantum cosmology (LQC) modify the Planck scale physics leading to the resolution of classical singularities in a variety of homogeneous and inhomogeneous cosmological settings [126, 127] (see, e.g., [128] for a review). In all these models, the big bang is replaced by a quantum bounce, evolution is deterministic and all curvature scalars remain finite throughout the evolution.[1] Perturbations propagating on these quantum cosmological spacetimes can inherit potentially observable imprints of the quantum gravity regime. This had been studied in great detail for single field inflation with a quadratic potential [124, 125, 134]. Although the quadratic potential is attractive due to its simplicity, recent data from Planck show that, for single field inflation, the Starobinsky potential is favored [135]. We studied the predictions of the pre-inflationary dynamics of single field inflation with a Starobinsky potential and the robustness of the observational predictions of LQC under a change of inflationary potential.

To understand the evolution of quantum perturbations, we used the formalism of quantum fields on a *quantum* cosmological spacetime. Within the test field approximation (that is, the backreaction of the perturbations on the background quantum geometry is negligible), the dynamics of the perturbations on the quantum FLRW geometry is remarkably simple: It is completely equivalent to that of perturbations evolving on a 'dressed' FLRW metric, where 'dressing' refers to quantum corrections [124, 134]. Interestingly, for sharply peaked states, the dressed metric is extremely well described by the effective description of LQC, given by the following modified Friedmann and Raychaudhuri equations [136–138]:

$$H^2 := \left(\frac{\dot{a}}{a}\right)^2 = \frac{8\pi G}{3}\rho\left(1-\frac{\rho}{\rho_{\rm max}}\right), \tag{C.1}$$

$$\dot{H} = -4\pi G\,(\rho+P)\left(1-2\frac{\rho}{\rho_{\rm max}}\right), \tag{C.2}$$

where H is the Hubble rate, a is the scale factor, $\rho_{\rm max} = 18\pi/\Delta_o^3\,\rho_{\rm Pl} \approx 0.41\,\rho_{\rm Pl}$ is the universal maximum of the energy density and $\Delta_o \approx 5.17$ is the minimum eigenvalue of the area operator, whose value is fixed via black hole entropy calculations in loop quantum gravity [139, 140]. Note that the LQC modifications are dominant only in the quantum gravity regime, where the energy density of the matter field is Planckian. When the spacetime curvature is sub-Planckian ($\rho \ll \rho_{\rm max}$), equations (C.1) and (C.2) reduce to the classical Friedmann and Raychaudhuri equations.

[1] For other approaches to bouncing models see, e.g. [129–132]. Also see [133] (and references therein) for a recent review.

The evolution of the Fourier transform of the gauge invariant Mukhanov-Sasaki scalar mode q_k and two tensor modes e_k on the dressed quantum geometry is given by [124, 125]:

$$q_k''(\tilde{\eta}) \quad + 2\frac{\tilde{a}'}{\tilde{a}} q_k'(\tilde{\eta}) + \left(k^2 + \tilde{\mathcal{U}}(\tilde{\eta})\right) q_k(\tilde{\eta}) = 0, \tag{C.3}$$

$$e_k''(\tilde{\eta}) \quad + 2\frac{\tilde{a}'}{\tilde{a}} e_k'(\tilde{\eta}) + k^2 e_k(\tilde{\eta}) = 0, \tag{C.4}$$

where the prime denotes the derivative with respect to the dressed conformal time $\tilde{\eta}$ and $\tilde{\mathcal{U}}$ is the dressed effective scalar potential that is completely determined by background quantities [124, 125].

Given this set-up, a priori there is no reason to believe that the pre-inflationary dynamics of the Starobinsky potential and quadratic potential are similar because: (i) already their inflationary dynamics is quite different; and (ii) the evolution equation for the scalar perturbations depends explicitly on the potential. Despite these differences, we found that the observationally relevant initial conditions are all kinetic energy dominated in the quantum gravity regime and therefore the inflaton behaves essentially as a massless scalar field during this regime. In other words, the inflaton evolves as a free particle in the deep quantum gravity dominated regime. Thus, the details of the potential do not affect the quantum gravity dominated phase significantly. Consequently, the occurrence of desired inflationary phase and corrections to the primordial power spectrum are robust features of LQC.

In particular, we showed that for the Starobinsky potential there are natural initial conditions for both the background spacetime and perturbations that lead to the desired slow-roll phase compatible with observations. In fact, almost all initial conditions starting at the bounce lead to the desired slow-roll phase. The quantum gravity phase excites the modes of the perturbations and as a result they carry excitations over the Bunch-Davies state at the onset of inflation, giving corrections to the standard inflationary power spectrum. There exist initial conditions for which these LQC induced corrections to the standard inflationary predictions at large angular scales are observable while being in complete agreement with observations at small angular scales!

Furthermore, just as in the standard inflationary paradigm, evolution of the scalar and tensor modes is different due to the presence of the effective potential $\tilde{\mathcal{U}}$ for the scalar evolution equation. Therefore, the particle density for scalar and

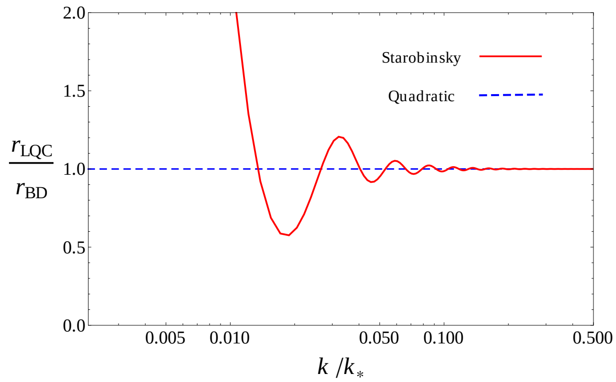

Figure C.3. Comparison of LQC corrections to tensor-to-scalar ratio between the Starobinsky and quadratic potential. Since the pre-inflationary dynamics extends to the deep Planck regime where the spacetime metric cannot be approximated by a de Sitter metric, it is not meaningful to require the modes to be in the Bunch-Davies vacuum at the bounce. Here, we choose the quantum perturbations to be in a 4th order adiabatic 'vacuum' that respects the symmetry of the background spacetime and whose expectation value of the renormalized energy density of the perturbations is negligible with respect to the background energy density at the bounce. In addition, we required the state to give maximum power suppression in the primordial power spectrum.

tensor modes should be different from each other for $k^2 \lesssim \tilde{\mathcal{U}}$. It turns out that the numerical value of $\tilde{\mathcal{U}}$ is typically very small compared to even the smallest observable wavenumber $k_{\rm min}$ and consequently negligible. However, a small subset of the initial data surface exists for which the effect of $\tilde{\mathcal{U}}$ on the scalar modes is no longer negligible. The tensor and scalar particle densities are therefore different for these modes and $r_{\rm LQC} \neq r_{\rm BD}$, where r is the tensor-to-scalar ratio and the subscript 'BD' refers to the predictions from the standard inflationary paradigm. This is distinct from the quadratic potential, where this difference is negligible for all $k \gtrsim 10^{-5} k_{\rm min}$. This is shown in figure C.3, where $r_{\rm LQC}/r_{\rm BD}$ is plotted for the Starobinsky and the quadratic potential. The quantum gravity induced deviations from $r_{\rm BD}$ are not directly relevant for observational modes, but the altered behavior of these super horizon modes can change the observed three-point functions (and higher order correlation functions) through mode-mode coupling as well as play an important role for tensor fossils. This difference between the scalar and tensor modes as compared to the standard picture, could thus lead to signatures in the non-Gaussian modulation of the power spectrum due to super horizon modes [141, 142]. It is noteworthy that the set of initial conditions for these effects is very small. Nonetheless, it is interesting as these initial conditions fall nicely into the regime for which the LQC corrections to the power spectrum are in the observable window. Thus, if these LQC corrections are observed, there is also hope to observe this effect. Furthermore, these initial conditions fall on the exponential side of the potential and not on the plateau side. Therefore, they are very likely to be present for potentials which have exponential walls such as the Higgs potential. This potentially opens new avenues to explore the origin of CMB anomalies in quantum gravity.

C.3 Cosmic variance in quasi-single field inflation

In the previous two sections, the focus was the primordial power spectra and the resulting power spectrum of the temperature anisotropies. However, the statistics of the (primordial) fluctuations beyond the power spectrum contain a wealth of information beyond what one could learn from the power spectrum alone and are collectively known as non-Gaussianities. For instance, these higher order correlations functions could inform us of the particles present during inflation and their interactions. Although the *Planck* satellite bounds on non-Gaussianity are excellent ($\sigma(f_{\rm NL}) \sim \mathcal{O}(10)$ [143]), they do not yet cross even the highest theoretically interesting (and roughly shape-independent [144]) threshold to rule out $f_{\rm NL} \sim \mathcal{O}(1)$. Future data will help probe the remaining parameter space, and either stronger constraints or detection of non-Gaussianity would provide an important clue about physics at the inflationary scale.

When the primordial fluctuations are entirely or partly sourced by a light field other than the inflaton, the correlation functions can have the interesting property that locally measured statistics depend on the long wavelength modes. For example, this occurs in curvaton scenarios in which the "local" bispectrum correlates the amplitude of short wavelength fluctuations with the amplitudes of much longer wavelength modes [145–150].

More generally, in the presence of a local ansatz non-Gaussianity, the power spectrum measured in sub-volumes will vary depending on how much the background fluctuations deviate from the mean [151–154]. While the bispectrum (three-point function) can cause the power spectrum to vary in sub-volumes, the trispectrum (four-point correlation function) can also generate shifts to the bispectrum and the power spectrum in biased sub-volumes (that is, sub-volumes whose long wavelength background is not the mean). More generally, long wavelength modes in the n-point correlation function can shift any lower order correlation function [151, 155]. Since nearly all sub-volumes will have a long wavelength background that is not zero, statistics measured in any small region are likely to be biased compared to the mean statistics of the larger volume. Schematically, the shift to the power spectrum of a non-Gaussian field ζ in a sub-volume due to the long wavelength background can be written as:

$$\begin{aligned}\langle\zeta(\vec{k}_1)\zeta(\vec{k}_2)\rangle_{V_S} &= \langle\zeta^G(\vec{k}_1)\zeta^G(\vec{k}_2)\rangle_{V_S} + \langle\zeta^G(\vec{k}_1)\zeta^G(\vec{k}_2)\rangle_{V_S}\,\langle\zeta^G(\vec{k}_L)\rangle_{V_S} + \ldots \\ &= \langle\zeta^G(\vec{k}_1)\zeta^G(\vec{k}_2)\rangle_{V_S}\rangle\,(1 + \zeta_L + \ldots)\end{aligned}$$

where ζ^G is a Gaussian field, V_S is the sub-volume, and the averages are performed with respect to the sub-volume. The wave number $\vec{k}_1$ and $\vec{k}_2$ correspond to wavelengths within the sub-volume. The Gaussian field averaged over the total volume V_L vanishes, i.e. $\langle\zeta^G(\vec{k}_L)\rangle_{V_L} = 0$, but its average over the sub-volume is non-zero and denoted by $\langle\zeta^G(\vec{k}_L)\rangle_{V_S} := \zeta_L$.[2] In the limit that the sub-volume covers the entire volume V_L the terms proportional to ζ_L vanish.

This correlation between local statistics and the long wavelength background can be used to detect non-Gaussianity (e.g., through the non-Gaussian halo bias [156]) when applied to sub-volumes where at least some long wavelength modes are observable. But it may also be relevant to the conclusions we can draw about inflationary physics: our entire observable universe is possibly a sub-volume of a larger, unobservable space. This is a true for any scenario for which the number of e-folds during inflation is larger than the minimum number required. If the perturbations we observe turn out to have any form of long-short mode coupling, we must assume there is a 'super cosmic variance' uncertainty in comparing observations in our Hubble volume with the mean predictions of inflationary models with more than the minimum number of e-folds [116, 151–155, 157–159]. In that case, there is not necessarily a one-to-one map between properties of the correlation functions we measure and parameters in an inflationary Lagrangian. We studied an explicit model where some, but not all, of the parameters of the Lagrangian are obscured

[2]Note, the definition of ζ_L here differs from the one used in [117] that is defined in momentum-space.

by this cosmic variance.

An interesting example that naturally interpolates between the single-clock inflationary models in which there is no significant coupling between modes of very different wavelengths [160–164] and curvaton inflationary models is quasi-single field inflation [165]. Quasi-single field inflation has an additional scalar field that is coupled to the inflaton during inflation. This additional field does not contribute to the background expansion and its self-interactions are not restricted by the approximate shift symmetry that the inflaton is subject to. Observational evidence for this 'hidden sector' field would be found in the non-Gaussianity it indirectly sources in the adiabatic mode. Previous studies have shown that when the hidden sector field has a cubic self-interaction, the degree of long-short coupling in the observed bispectrum is determined by the mass of the second field (with the strongest coupling coming from a massless field). The coupling between modes of very different wavelengths is captured by the squeezed limit of the bispectrum, where all momenta have wavelengths that are within the local sub-volume but with one momenta having a much longer wavelength than the others (e.g., $k_1 \ll k_2 \approx k_3$). Measuring the dependence on the long wavelength mode (k_1) in the squeezed limit of the bispectrum would reveal the mass of this spectator field [165–168]. We investigated how robust the quasi-single field bispectrum shape is to cosmic variance when higher order correlations are included in the model. We performed the calculations in two different ways. The conceptually straightforward, but calculationally more difficult method is to take soft limits of the full in-in calculations. Our newly developed techniques allowed a more straightforward calculation by performing a long-short wavelength split in the late-time correlation functions. Both methods are in perfect agreement with each other.

When the hidden sector field is sufficiently light, any additional correlations may bias the statistics observed in sub-volumes. Here, we considered correlations generated by additional (i.e., quartic and beyond) non-derivative self-interactions of the hidden sector fluctuations. Any degree of long-short mode coupling causes statistics in sub-volumes to differ from the global mean statistics by an amount that depends on the amplitude of fluctuations with wavelengths larger than sub-volume size. Specifically, by computing the power spectrum and bispectrum in sub-volumes with non-zero long wavelength background modes we showed that (similarly to the local model), *any* non-derivative self-interaction of the spectator field leads to the same pattern/shape of correlation functions in sufficiently biased sub-volumes. In particular, the spectral index observed in sub-volumes is generically shifted away from the mean, but the shift is small. The tensor-to-scalar ratio r can be significantly different in the locally observed volume as compared to the global volume. When the hidden sector field is light and the number of e-folds before the largest observable mode today excited the horizon is large, the effect is the largest. This can in fact change the value of r by two orders of magnitude! In addition, non-zero long wavelength fluctuations induce a tree-level bispectrum locally even if the mean theory does not contain one. Furthermore, the squeezed limit scaling of the bispectrum is the same in all sub-volumes, while the local amplitude of fluctuations and amplitude of non-Gaussianity are subject to cosmic variance. So, although the details of a light hidden sector field's potential can be obscured by cosmic variance, its mass can be robustly determined by any post-inflationary observer.

It is important to note that these long wavelength effects are distinct from those that *do* affect the mass of a light scalar in de Sitter space. For example, the loop diagrams for a light scalar with a quartic self-interaction have a naive infrared dependence that can be re-summed into a correction to the mass of the field [169–171]. The mass of the light scalar is corrected everywhere, not just in some sub-volumes, and it is the shifted mass that will be observed in the squeezed limit of the bispectrum. This type of mass correction is a dynamical effect of field theory in (quasi-)de Sitter space; the sub-sampling of non-Gaussian statistics in the post-inflationary world is not sensitive to the dynamics that generated the primordial fluctuations.

Our results are a concrete example of an initial conditions problem for inflation: when cosmological data is only good enough to measure some properties of some correlation functions (e.g., the bispectrum averaged over the full sky), that data only allows us to uniquely construct a Lagrangian for the fluctuations in our observed volume. The local Lagrangian is consistent with a larger set of possible "global" Lagrangians sourcing additional inflation and which may share some features (here the mass of the spectator field) but differ in others (here the spectator field's potential). Of course, in the absence of cosmic variance this is also true, but there is a straightforward order by order map between the dimension of terms in the fluctuation Lagrangian and the order of correlation function measured. Cosmic variance from long-short mode coupling breaks that mapping. Making additional measurements such as any position-dependence of the observed bispectrum or a detection of a trispectrum would remove some degeneracies, but not all. It would be interesting to extend our results to other multi-field scenarios to understand more generally which properties of the particle content and dynamics can be determined regardless of cosmic variance and which cannot.

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Vita

Béatrice Bonga

Béatrice Bonga received her Bachelor of Arts degree in Psychology and Bachelor of Science degree in Physics, both summa cum laude, from Utrecht University in The Netherlands in 2010. During these studies, she also spent a semester at the University of California at Berkeley. Her passion for physics led her to pursue a masters in Theoretical Physics at Utrecht University, during which she conducted research on linearized quantum gravity under the supervision of Dr. Igor Khavkine and Prof. Renate Loll.

After obtaining her masters degree cum laude in 2012, she started her PhD at the Pennsylvania State University under the supervision of Prof. Abhay Ashtekar. She has worked on various topics including asymptotics with a positive cosmological constant in the context of gravitational radiation and topics in early universe cosmology.

Béatrice has received the Professor Stanley Shepherd Graduate Teaching Assistant Award in 2017 as well as the Graduate Teaching Assistant Award in 2014 from the Physics Department at Penn State. She has been invited to give seminars at various universities including Cambridge and Caltech and has been an invited speaker at two conferences. Her presentations have been recognized with several awards including the Hartle Award for best talk by a student during the GR21 conference in 2016 and the Peter Eklund Memorial Lectureship in 2015. Being passionate about physics outreach and communication, Béatrice co-founded the Physics Outreach Program at Penn State, gave talks at the local high school and nearby colleges and was selected to participate in ComSciCon (a conference about science communication) at Harvard University in 2015.

List of publications

A. Ashtekar, B. Bonga, A. Kesavan, "Asymptotics with a positive cosmological constant. I. Basic framework" (*Class. Quantum Grav.* 32, 2015)

——, "Asymptotics with a positive cosmological constant: II. Linear fields on de Sitter space-time" (*Phys. Rev. D* 92, 2015)

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B. Bonga and B. Gupt, "Phenomenological investigation of a quantum gravity extension of inflation with the Starobinsky potential" (*Phys. Rev. D* 93, 2016)

——, "Inflation with the Starobinsky potential in Loop Quantum Cosmology" (*GRG* 48, 2016)

B. Bonga, S. Brahma, A.-S. Deutsch and S. Shandera, "Cosmic variance in inflation with two light scalars" (*JCAP* 05, 2016)

B. Bonga, B. Gupt and N. Yokomizo, "Inflation in the closed FLRW model and the CMB" (*JCAP* 10, 2016)

——, "Tensor perturbations during inflation in a spatially closed Universe" (*JCAP* 05, 2017)

B. Bonga and J. Hazboun, "Power radiated by a binary system in a de Sitter Universe" (in preparation)

A. Ashtekar and B. Bonga, "On the conceptual confusion in the notion of transverse-traceless modes" (in preparation)