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**Murcia, Spain**
**September 20–22, 2004**

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# Preface

The professional dialogue between Mathematicians and Physicists is something as natural as enriching in the daily life of a secondary school teacher. It seems that it is in the following educational step, the university, where the distance begins to be significant, situation that becomes usual as the specialization level grows. Identical situation occurs in the realm of own Physicists or Mathematicians of the different areas, a classical characteristic of the Spanish people that we already see almost natural that a topologist and a statistician nothing must say themselves mathematically speaking. Perhaps it is exaggerated, but it is what we daily observe.

Luckily, who we define ourselves as geometricians or physicists had been thirteen years already trying to be the exception to so displeased rule and in that time giant steps have occurred. We have no doubt that we continue undergoing wide gaps of understanding, but every year a shortening of that ancestral distance is clearly observed. We think that the convergence is going to be achieved, essentially, through the Differential Geometry tools, mainly with theoretical physicists. It turns also out to be true that any other area of Mathematics can contribute in the same way to the progress, not only of Physics, but of any other science.

The participants in these series of workshops have bet to foster a better understanding between Mathematicians and Physicists in order to undertake joint projects of greater scope, such as the society and the current research are demanding from them. We think that this is the right way to promote the Spanish science and these Proceedings will contribute to that aim. Furthermore, they will be the best test of our commitment and support to Spanish quality publications.

In Oviedo, where Murcia was nominated to organize this Workshop, and later in Coimbra, they left the strip in the highest, but we accepted the challenge to surpass it and we spared no effort to get it. We have done one's bit to shorten the distance between geometry and physics; and what is better, to increase the quality, all over the world recognized, of those working in these areas.

We take one's chance to reiterate our public gratefulness, by its commendable perseverance and example, to those who were very deserving of a ham, the colleagues Eduardo Martínez Fernández, Miguel Carlos Muñoz Lecanda and Narciso Román Roy.

The University of Murcia, the Séneca Foundation, the Spanish Ministry of Education and Science, the Cajamurcia Foundation, the Caja de Ahorros del Mediterráneo and the Real Spanish Mathematical Society made the Workshop and these Proceedings possible, so that they deserve our more sincere gratefulness. As well as to all those that participated either learning, or teaching or, simply, engaging in a dialogue.

Murcia, September 2005

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DE MATEMATICAS
AULARIO

# Mini-Course

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# Introduction to general relativity

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**Abstract.** The following is an attempt to provide an introduction to relativity for geometers with an emphasis on the physical side of theory. The content of the following article corresponds to a three hours minicourse consisting of a first session devoted to special relativity, a second one discussing how everyday gravitational phenomena can be described by a curved Lorentzian space-time metric and a last part where the Einstein field equations that govern the curving of space-time by matter are introduced.

*Keywords:* Relativity, gravitation

*2000 Mathematics Subject Classification:* 83A05, 83Cxx

## 1. Introduction

This is not a run of the mill introductory course on relativity because it cannot possibly be. The reason is not that the author has some exceptional insight or a deeply original point of view on the matter but rather the unusual fact that you, the reader, and the audience to whom this set of lectures was delivered has a very special and uncommon feature: a deep and working knowledge of geometry. One of the main obstacles that must be overcome to explain relativity, at least to the average physics students, is the fact that the geometrical language in which the formalism of general relativity is written is usually unfamiliar and only partially understood by them. This was certainly the case when I took my first courses on the subject at the University some time ago. This state of affairs has probably improved in recent years but my impression is that, still, one has to devote some of the (scarce) available time to define and introduce the relevant mathematical concepts, discuss such issues as the intrinsic meaning of geometrical objects irrespective of their coordinate representations, and dispel some common misconceptions (think about the infamous, and very often misunderstood, invariance under general coordinate transformations!). It is certainly relieving to know that I will not have to do this here.

Traditionally relativity has been considered as something quite separated from the common experience; a theory riddled with difficult paradoxes and counterintuitive phenomena. This is apparent in the popular names still given to some of the more spectacular consequences of the theory. I do not share this point of view and I hope that, after reading these pages, you will not either. The message that I want to convey is that once the operational meaning of the relevant physical concepts is clear, relativistic phenomena can be explained, derived, and understood in a natural and consistent way. I concede that some intuition may always be lacking, much in the same way as a blind person may find difficult to understand what perspective means (or the difficulties that almost all of us can have when thinking about four dimensional geometry) but, in my opinion, it is possible to gain a working and systematic ability to precisely derive and understand relativistic phenomena.

Special and General Relativity are usually presented as two completely separated subjects dealing with very different kinds of phenomena: the former is used to describe dynamics at very high relative velocities and energies whereas the latter is the theory of gravity, in fact, *the relativistic theory of gravity*. This is most evident at the level of ordinary undergraduate courses. Special relativity follows a coordinate based approach where inertial systems and Lorentz transformations play a central role in the derivation and description of phenomena whereas courses in general relativity have a more geometrical flavor, or at least use a very different type of language.

It is, nevertheless, worth noting that even within the realm of general relativity usual presentations fall into two broad categories: what I call the geometrical and field theoretical points of view. The reader can get a good idea of what I mean by looking at some popular text books such as Misner, Thorne and Wheeler [1] on one side and Weinberg [2] on the other. Most of the texts on the subject, as well as practitioners in the field, can be classified in most cases in one of these two categories. The differences in their approaches are evident; in the geometric side the emphasis is made on the geometrical, intrinsic, and coordinate independent meaning of the concepts involved. This is in line with the spirit of differential geometry as taught and learned by mathematicians. On the field theoretic side, instead, gravity is considered as a somewhat exotic field theory with peculiar features such as diffeomorphism invariance (treated as a *gauge symmetry*) and an intrinsic nonlinear character that precludes any direct solution to the field equations and renders the traditional field theoretical approaches almost useless.

I certainly do not share this last point of view even though I consider myself as a field theorist; in my opinion a genuine understanding of relativity, both general and special, can only be achieved by using a geometrical approach. Does this mean that relativity is just an application of some concepts of differential geometry? The answer to this question is a resounding **no** because relativity is a physical theory, in fact, the physical theory of space and time. One of the leitmotifs of the following notes is that there is a physical level beyond geometry. In a rough sense geometry refers to distances, angles, and their relationship but in physics these magnitudes must be defined, suitable prescriptions must be given in order to measure them, and the appropriate measuring apparatuses must be used. This is the first level in which our lack of intuition for phenomena occurring at high energies or relative velocities can fool us. Our intuitive way of perceiving the real world around us leans on several implicit assumptions that must be clearly isolated and understood before we can think in relativistic terms. The information that we obtain from our surroundings is carried by physical agents such as sound, heat, and crucially *light*. What we see around us is treated by our brains as an instantaneous picture of the world as it is right now. We think of what we see as something happening in this moment at the different places of our neighborhood. The fact that this is not an accurate physical picture because of the finite speed of light and the necessity to define simultaneity in an appropriate way was one of the main obstacles that had to be overcome in order to arrive at the correct relativistic picture of space-time.

One of the most important physical pillars of relativity has to do precisely with the character of light as the primary carrier of information. It is impossible to overemphasize the importance of this fact because light will play a central role in the operational definitions of physical quantities in relativistic physics. In my view (certainly shared by many others) the simplest way to define physical magnitudes uses light rays in a crucial way. The reason, as we will be discussed in detail later, has to do with the observational fact that the motion of light reflects an intrinsic property of space-time and is independent of the state of motion of light sources.

The first defining feature of these notes has to do with this: I will emphasize the operational physical description above other things and try to justify that this leads to differential geometry as the appropriate language to describe relativistic physics. The first part of these notes will be devoted to a presentation of special relativity that follows this philosophy pioneered by Bondi [3] and updated in the beautiful book by Geroch [4].

Once we recognize the convenience and physical accuracy of the relativistic description of space-time as a manifold endowed with the Minkowski metric it is only natural to generalize the idea and accept the possibility of using other metrics. The great insight of Einstein was to realize that this generalization provides a way to understand relativistic gravity. In the second part I will show you how we can accurately describe the gravitational phenomena of everyday life by using the Schwarzschild metric whose features are central both to the description and the experimental verification of the theory. I will finish by giving a quick (and arguably rather standard) "derivation" of the dynamical equations of the theory, the Einstein field equations. Here I will closely follow the presentation appearing in the classic text of Wald [5]. My purpose in this last section is to provide you with the main ingredients of the theory and give a rough but, hopefully, sufficient picture of how general relativity works.

## 2. Special Relativity

Relativity is the theory of space and time, or rather of space-time; the very first change in our usual perception of reality is the fact that space and time are not independent but intimately related concepts. Space-time is defined as an ensemble of events, that is, things that happen somewhere: places in space and instants of time.

It is important to think a little bit about how these events are organized and interrelated. With some hindsight one can recognize several viewpoints on space and time that I will refer to as the Aristotelian, Galilean and Einstenian points of view (where the attribution to these authors should not be taken as a completely accurate historical fact but rather as a convenient way to identify the thinking prevailing in different epochs of history).

In the Aristotelian point of view space and time are not interrelated but independent notions. Events happen at a certain well defined place and in some absolutely defined instant of time. The absolute character of time and position imply that velocities, or rather, the state of motion of bodies are absolute. Among all the possible reference frames there is a privileged one, at rest, to which the motions of things are referred. A way to somewhat summarize the physics in this framework is to give a list of questions that make sense and can be answered within it, such as: where does a certain event happen?, when does it happen?, are two events simultaneous?, do they occur at the same place, what is the distance between them?, what is the velocity of a moving object? is it at rest?...

In the Galilean point of view there is no privileged rest frame but a class of equivalent reference systems called inertial that work as a set of "personalized" aristotelian frameworks. This is the main and most important departure from the previous point of view and has some important consequences because now positions (and hence velocities) are no longer absolute. Still some features of the Aristotelian view are retained, in particular time and simultaneity keep their absolute character and space is separate from time. Some of the questions that previously made sense cannot be asked now but there are others that we may still pose to ourselves such as those referring to the time difference between events, the distance between *simultaneous* events and relative velocities between moving objects.

The Einstenian point of view, that will be discussed at length in the following, is a dramatic departure from the pictures described in the two previous frameworks. This is particularly so as far as the very structure of space-time is concerned because now space and time cease to be independent objects and melt into a single structure. This shows up in the appearance of the space-time interval as a kind of distance (or time lapse) that relates non necessarily simultaneous events and allow pairs of them to be classified into different types (space-like, time-like or null) with distinct physical meaning. Space-like intervals are suitable to describe simultaneity and distance, time-like intervals correspond to events observed *in situ* by a physical observer and the null ones have to do with light signals. One of the most striking features of this new framework is the fact that absolute simultaneity is lost and becomes observer dependent.

## 2.1. Clocks and Light Rays

I will introduce now the main tools used to define physical space-time concepts: clocks and light rays. A clock is a device that measures the proper time of the physical observer that carries it. Implicit in this definition is the fact that the actual recording of time is carried out *in situ*; mathematically proper time will be defined as a certain kind of length associated to the curve that describes the history of the observer in space-time. It is important to realize that the only time that applies to a given observer is the one measured by his or her own clock. When considering situations involving several observers we will make the assumption that these clocks are identical; i.e. if they were all carried by the same observer they would record the same time and tick at the same pace. This does not mean that the time measurements carried out by different observers at different places must be the same; in fact one of the most important features of relativity is the recognition of the fact that different observers may measure different time intervals between events depending on their state of motion.

The second device that we need to introduce in order to define, at least in principle, physical space-time concepts is the *light ray*. The light rays that I am talking about are, actually, special trajectories in space-time (I am not really thinking about electromagnetic waves although Maxwell's theory actually tells us that real light follows precisely these type of trajectories). The most important fact about light rays originating at a given event is that they move in a way that is independent of the state of motion of the source. This is so central to the following that I will give it a special name and call it the **l-hypothesis**. Though this goes squarely against our experience about how the velocities of objects and reference frames add, it is an experimental fact that has its reflection also on the way we see the universe. It was recognized by de Sitter and others that the shape of astronomical objects such as the orbits of binary stars would appear severely distorted were it not for the fact that light motion is not affected by that of the emitting source [6].

Given a certain event there is a single ray originating or arriving there for each spatial direction. The locus of the events reached by all the light rays arriving and originating at a certain event $O$ is called the light cone at $O$. As we can see, light trajectories encode some intrinsic information about space-time (for example about causality) independent of the objects and processes that produce them. Owing to this they are especially useful to explore space-time around a physical observer and to define geometrical magnitudes and properties in an operational way. The idea of using light rays both as measuring devices and carriers of information can be summarized in a master law of space-time perception: what we see is rays of light.

## 2.2. Inertial Observers

When considering observers it is very useful to introduce a special class of them, called *inertial*, characterized by the fact that they are in the simplest state of motion. This is defined by the condition that they do not perform or suffer any physical action that might change how they move. They are all completely equivalent (there is not a privileged one) but this equivalence does not make them all the same because they can be in motion with respect to each other. The magnitude of this relative motion can be defined and measured by a suitable relative velocity as we will discuss later.

A very important property of inertial observers is that the proper time between the reception of two light rays by an inertial observer $Obs_2$ emitted by another one $Obs_1$ (that happens to coincide with the other at a certain event) is proportional to the proper time interval of emission as measured by the latter. For future reference I will refer to this as the **k-hypothesis**. It is worthwhile pointing out that, as none of these two inertial observers is privileged the proportionality constant is the same if the rays are sent by $Obs_1$ and received by $Obs_2$.

In the following I will make extensive use of space-time diagrams. These are representations of space-time and their content that are very useful to depict and describe physical processes. Used with due care they provide an accurate representation of relativistic phenomena. I will suppose that I have an affine space with points in $\mathbb{R}^4$ and $\mathbb{R}^4$ as the associated (real) vector space. Light rays are represented by straight lines “at a fixed angle” with respect to the direction that represents time; they lie on a cone (the light cone in this representation). Those trajectories passing through the vertex and inside the cone correspond to the histories of physical observers (and objects in general) and will be called world lines. Notice that not every curve in $\mathbb{R}^4$ may be associated to a physical object; it is not possible, for example, to have physical objects that move back and forth in time. In this representation inertial observers are simply described by straight lines, arguably the simplest ones in an affine space.

Even though our physical space-time is four dimensional we will mainly consider here a simplified situation where we have only one spatial and one time dimension. There are certainly some interesting relativistic phenomena that arise when we consider motions in a two or three dimensional space (such as the Thomas precession) but most of the important ones can be understood in the 1+1 dimensional simplified setting that I describe in the following.

## 2.3. The Relativistic Interval

Let us start by discussing the definition and measurement of proper time differences and introduce the all-important concept of relativistic interval. Let us the ask ourselves the following question: Can a given inertial observer find out the proper time interval between events that do not lie on his own world line? To answer this we will use the k-hypothesis. Let us consider (see figure 1) the difference between proper emission and reception times by $Obs_1$, $T_-$ and $T_+$ respectively. Taking into account that $T_{AB} = kT_-$ and by symmetry $T_+ = kT_{AB}$ we immediately find $T_{AB} = \sqrt{T_-T_+}$, i.e. the proper time measured by an inertial observer passing through events $A$ and $B$ is given by the the geometric mean of the time difference between emission and reception of light rays reflected at these two events. Notice that there is also a simple relation between $T_+$ and $T_-$, namely $T_+ = k^2T_- \equiv (1+z)T_-$ where $z > -1$ is called the redshift. This name is borrowed form observational cosmology and refers to a measure of the relative values of the observed frequencies of known atomic transitions compared with values in terrestrial laboratories. Under the assumptions of the standard cosmological model it can be used to determine

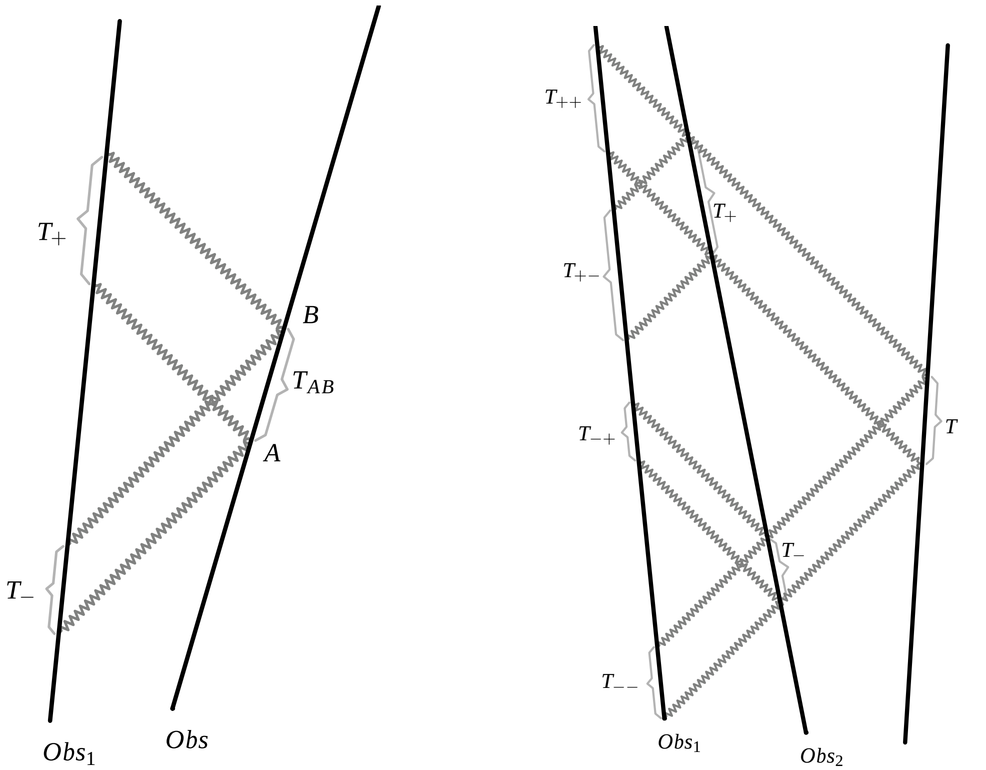



Figure 1: Left: Determination of the proper time interval between $A$ and $B$ by an observer $Obs_1$. Right: Observer independence of this determination.

cosmological distances. In the present context it provides a very convenient way to measure the relative state of motion of the two observers and can be obtained by any of them by simple measurements. It is important to point out that the events $A$ and $B$ cannot be anything but they must be such that the last light ray to leave $Obs_1$ arrives last (in which case we say that the space-time interval between these two events is time-like). The attentive reader may wonder about the intrinsic character of the proper time interval that I have just introduced, is this prescription observer-independent? or more specifically, would the proper time difference measured by a second observer $Obs_2$ coincide with the one obtained by $Obs_1$? The answer to both these questions is in the positive as long as the observers involved are inertial. To see this consider the situation depicted in the right half of figure 1. What we want to figure out now is whether or not $\sqrt{T_+T_-} = \sqrt{T_{++}T_{--}}$ is correct whenever the conditions $T_- = \sqrt{T_{--}T_{-+}}$ and $T_+ = \sqrt{T_{+-}T_{++}}$ are met. It is straightforward to see that this happens if

$$\frac{T_{++}}{T_{+-}} = \frac{T_{-+}}{T_{--}},$$

where the left hand side is written in terms of the time measurements performed by $Obs_1$ to determine $T_+$ and the right hand side in terms of those used to get $T_-$. We conclude that it is possible to define the proper time between two events (in the appropriate time-like relationship) in such a way that it can be measured by any inertial observer following a simple procedure. One of the most important consequences of the fact that the behavior of proper times is the one described above is the so called twins paradox that refers, more precisely, to the observer-dependent character of proper time. If we synchronize two clocks at a certain space-time event and let them follow different world-lines afterwards they will (generically) no longer signal the same time if the are brought together again. In the particular case when one of them measures proper time for an inertial observer and the other moves away and back with constant relative velocity this is a simple and straightforward consequence of the fact that the geometric mean is always less or equal than the arithmetic mean because

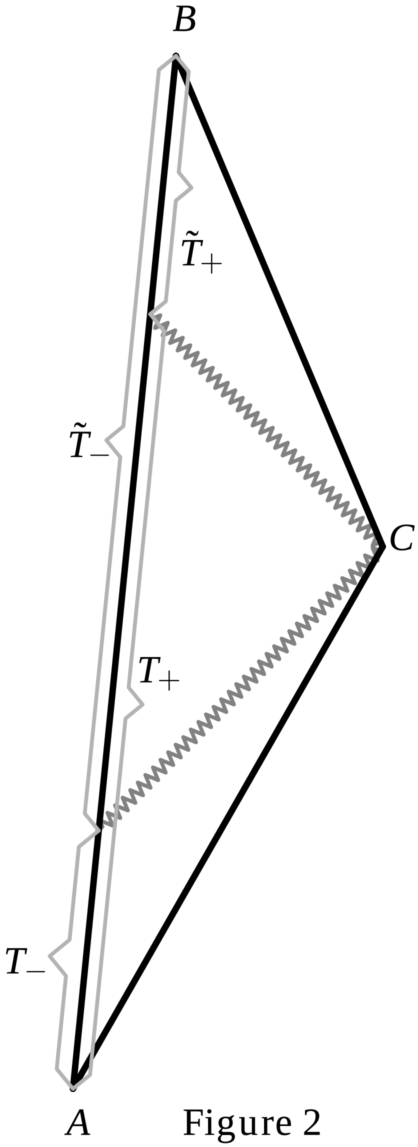



Figure 2

$$\sqrt{T_+T_-} \leq \frac{T_+ + T_-}{2},$$

$$\sqrt{\tilde{T}_-\tilde{T}_+} \leq \frac{\tilde{T}_- + \tilde{T}_+}{2}$$

implies that

$$\sqrt{T_+T_-} + \sqrt{\tilde{T}_+\tilde{T}_-} \leq \frac{1}{2}\left[T_- + T_+ + \tilde{T}_- + \tilde{T}_+\right] = T_- + \tilde{T}_- = T_+ + \tilde{T}_+.$$

Here the definitions of the different time intervals involved are shown in figure 2. This phenomenon is closely related to the time dilation that is routinely observed in the decay of unstable subatomic particles.

### 2.4. Simultaneity

Another time-related concept that requires a detailed analysis is that of simultaneity. The fact that time loses its absolute character has a reflection on the very definition of simultaneous events. Let us consider first a situation (see figure 3) in which we have an inertial observer, an event $B$ outside its world line and another event $A$ in the observers worldline.

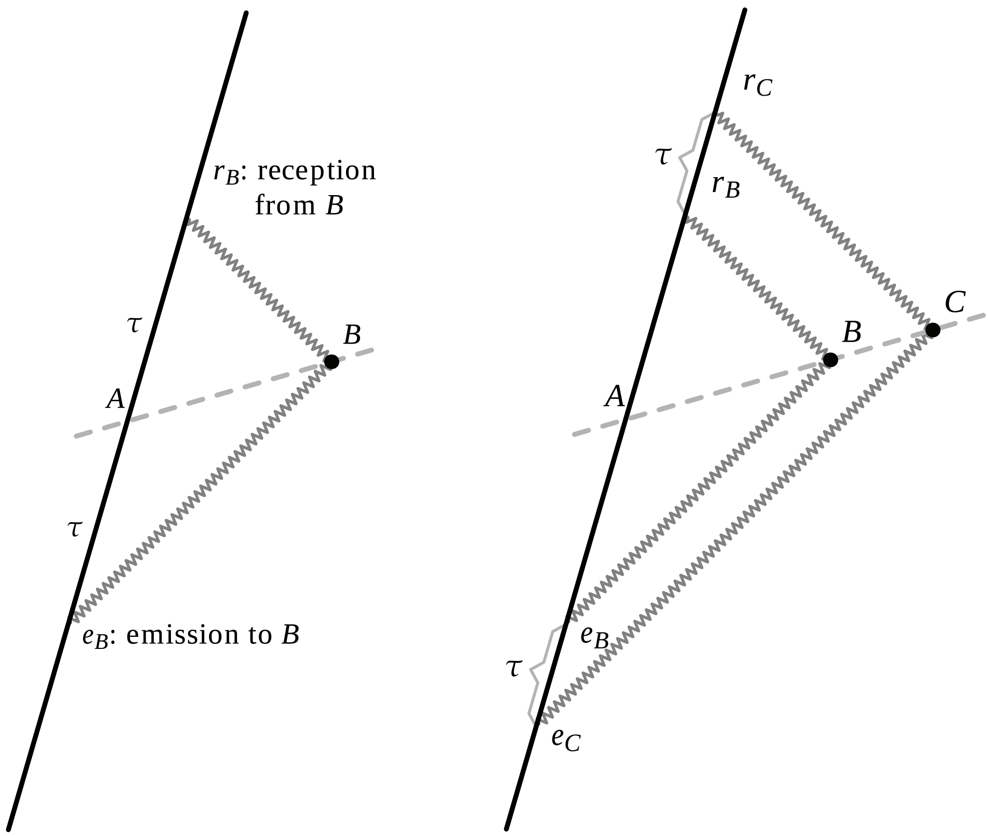



Figure 3: Definition of simultaneity referred to a given inertial observer.

We will say that $A$ and $B$ are simultaneous, *relative to this observer*, if he can send a light ray to $B$ and record its reflection in such a way that the time interval $\tau$ measured by his clock from emission to $A$ is the same as the one from $A$ to reception. Notice that this definition assumes a certain symmetry between departing and returning rays in the sense that the trips that light makes in both directions are considered as identical. It cannot be overemphasized that this definition necessitates of the introduction of an (inertial) observer; it is not possible otherwise to state if two events are simultaneous or not. If instead of a single event we have two, $A$ and $B$, outside the observers world line we can generalize the previous procedure and define their simultaneity. This is done by sending and receiving light rays, as shown in figure 3, and requiring that the time intervals at emission and reception be the same. Again, this definition refers to the particular observer of our choice. Some comments are now in order. This concept of simultaneity defines an equivalence relation whose classes, constituted by mutually simultaneous events are called simultaneity surfaces. The previous definition could be extended to non-inertial observers in which case it becomes a local definition (i.e. valid not far from the observers world line). This is so because, for example, some parts of space-time may not be reachable by light rays emitted by the observer and also because the notions of simultaneity for two different observers may coincide in a neighborhood of their world lines but not everywhere.

### 2.5. Length and Velocity

After discussing the space-time concepts associated with time intervals and simultaneity we introduce now the definition of length or spatial distance. To this end let us consider two events $A$ and $B$ where we place mirrors and an inertial observer $Obs$ such that $A$ and $B$ are simultaneous. We define the spatial length (see figure 4) from $A$ to $B$ as $l_{AB} = c\tau$ where $c$, the speed of light, can be considered as a conversion factor (If we measure time in meters we can put $c = 1$, something that we will often do in the following). A question that we may pose ourselves at this point is the following: Can another inertial observer determine this proper length? The answer is in the positive because $T_- = k\tau$ and $\tau = kT_+$ leads to $\tau = \sqrt{T_- T_+}$. This is precisely the same expression that we found for the proper time interval but in the situation that we are considering now the first emitted ray arrives last in which case we say that the interval between the events $A$ and $B$ is space-like.

Once we have introduced the concepts related to lengths and distances we are ready to define the velocity between inertial observers in the obvious way as a measure of the change of spatial distance with time. This is done in an straightforward way with the setup shown in figure 5.

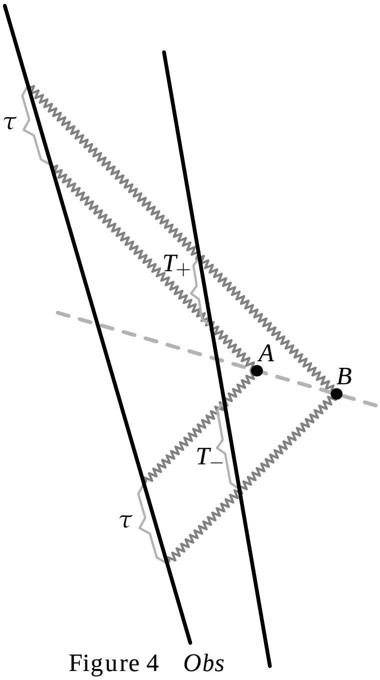



Figure 4

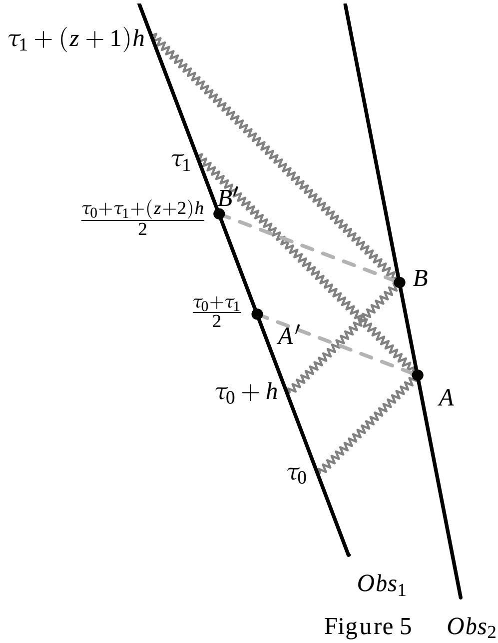



Figure 5

If observer $Obs_1$ is moving with respect to $Obs_2$ there is a change in the reception interval to $(1+z)h$ where $h$ is the (small) proper time delay between the emission of the two light pulses. The redshift $z$ is measured at the time of reception and it is very easy to derive the relative velocity between both observers from its value. If we demand that the second emitted ray must arrive second we have the restriction $z > -1$ and thus the modulus of the velocity is bounded by the value of the speed of light $c$. From figure 5 we find that

$$l_{AA'} = \frac{\tau_1 - \tau_0}{2}c, \quad l_{BB'} = \frac{\tau_1 - \tau_0 + zh}{2}c$$

and, hence

$$v = \lim_{h\to 0} \frac{l_{BB'} - l_{AA'}}{(1+\frac{z}{2})h} = \frac{z}{2+z}c.$$

This expression can be conveniently rewritten as

$$1 + z = \frac{c+v}{c-v}.$$

Instead of two light rays we can consider using periodic waves satisfying the wave equation $\partial_t^2\phi - c^2\triangle\phi = 0$. In this case light rays correspond to characteristic curves and $z$ is measured as a frequency shift. This is just a manifestation of the well known Doppler effect. Notice that this procedure works due to the invariance of the wave equation under Lorentz transformations. The equivalence between the two procedures introduced above explains why the velocity defined in this way is known as the *radar velocity*.

Relativistic lengths and velocities display some interesting an unfamiliar behaviors that we pause to discuss now. As far as lengths are concerned the most striking is the phenomenon of length contraction schematized in figure 6, where the measurement of the length of a ruler by a moving inertial observer is described. This is done by bouncing light rays from the ends of a ruler and recording the arrival times of the reflected rays. The length of the ruler, as determined by the moving observer, is defined as the distance between the two simultaneous (for observer $Obs_1$) events $A$ and $B$. The moving observer can also determine or define the proper length of the ruler (the one that an inertial observer standing in one of the ends would obtain by sending and receiving a light ray from the other end). This is done by recording the reflection of the first ray that he sends from a semitransparent mirror placed at the closer end of the ruler and sending an additional light ray in such a way that its reflection at the near end arrives at the same time as the one bouncing off the far end. From $\tau_1 = T_\alpha(1+z)$ and $T_\beta = \tau_1(1+z)$ we find that

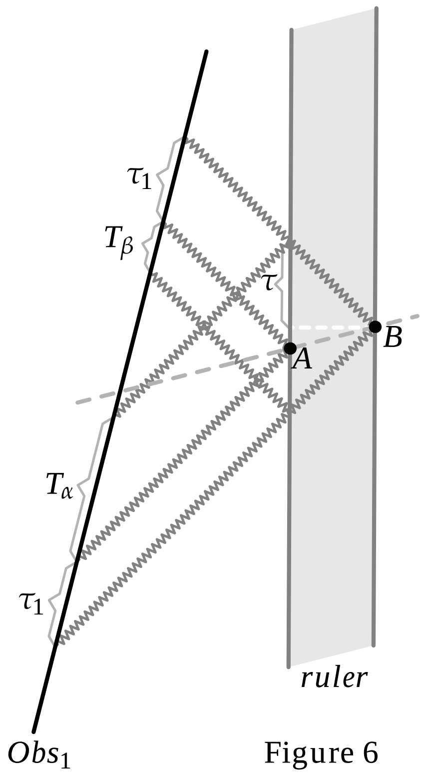


Figure 6

$$\tau_1 + T_\alpha = \tau_1\left(\frac{2+z}{1+z}\right), \quad \tau_1 + T_\beta = \tau_1(2+z)$$

and, hence,

$$2\tau = \sqrt{(\tau_1 + T_\alpha)(\tau_1 + T_\beta)}$$

that allows us to obtain the proper length of the ruler as $c$ times the following value of $\tau$

$$\tau = \tau_1 \frac{2+z}{2\sqrt{1+z}} = \frac{\tau_1}{\sqrt{1-v^2}}.$$

We then conclude that

$$l_1 = l_{prop}\sqrt{1 - v^2} < l_{prop}$$

so that the measured length is smaller than the proper one.

Another interesting phenomenon that can be discussed is the composition of velocities. Let us consider three inertial observers labeled $Obs_0$, $Obs_1$, and $Obs_2$. In order to find the relation among the different relative velocities determined by them let us look at the space-time diagram shown in figure 7. There are four different proper time intervals to be consider by the observer $Obs_0$:

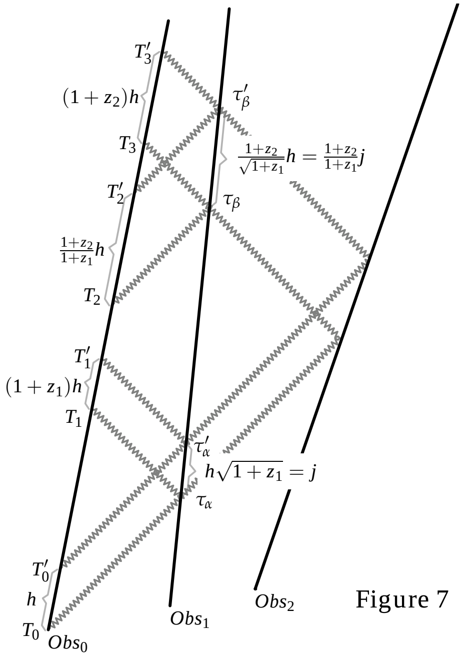


Figure 7

- $T_0' - T_0 = h$, where $h$ is a delay of our choice.
- $T_1' - T_1 = (1 + z_1)h$ which is the "redshifted" reflection by $Obs_1$ of the light rays sent at $T_0$ and $T_0'$.
- $T_2' - T_2 = \frac{1+z_2}{1+z_1}h$ which are two light rays sent by $Obs_0$ to $Obs_1$ in such a way that their reflection at $Obs_1$ coincides with those reflected by $Obs_2$ and emitted at $T_0$ and $T_0'$.
- $T_3' - T_3 = (1 + z_2)h$ which is the "redshifted" reflection by $Obs_2$ of the rays sent by $Obs_0$ at $T_0$, $T_0'$, $T_2$ and $T_2'$.

It is straightforward to obtain now $\tau_\alpha' = \tau_\alpha + h\sqrt{1 + z_1} \equiv \tau_\alpha + j$ and $\tau_\beta' = \tau_\beta + h\frac{1+z_2}{\sqrt{1+z_1}} \equiv \tau_\alpha + j\frac{1+z_2}{1+z_1}$ by using the geometric mean formula for the interval. So finally we find

$$\frac{c + v_{21}}{c - v_{21}} = \frac{1 + z_2}{1 + z_1} = \frac{\frac{c+v_{20}}{c-v_{20}}}{\frac{c+v_{10}}{c-v_{10}}} \to \frac{1 + \beta_{20}}{1 - \beta_{20}} = \frac{(1 + \beta_{21})(1 + \beta_{10})}{(1 - \beta_{21})(1 - \beta_{10})}$$

and hence

$$\beta_{20} = \frac{\beta_{21} + \beta_{10}}{1 + \beta_{21}\beta_{10}}$$

where $\beta_{ij} = v_{ij}/c$. This is a rather strange addition law because it is impossible to go beyond $c$ by composing smaller velocities. If we let $\beta_1$, $\beta_2 \in (-1, 1)$ and

$* : (-1,1)^2 \to (-1,1) : (\beta_1, \beta_2) \mapsto \frac{\beta_1+\beta_2}{1+\beta_1\beta_2}$, then this defines a group. This composition law suggests the introduction of a real, unbounded, parameter $\chi \in (-\infty, \infty)$, the rapidity, defined as $\beta = \tanh \chi$ so that now we have

$$\tanh(\chi_1 + \chi_2) = \frac{\tanh \chi_1 + \tanh \chi_2}{1 + \tanh \chi_1 \tanh \chi_2}.$$

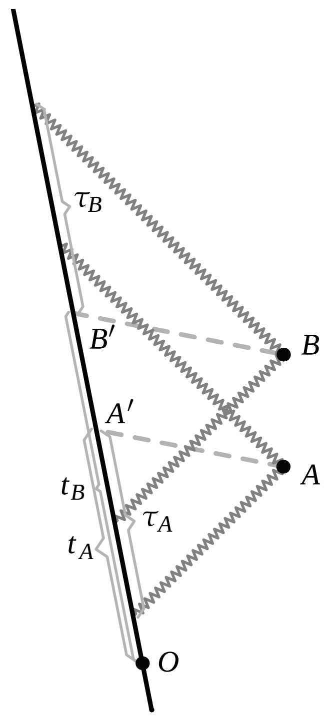
As we see rapidities simply add when composing velocities. It is also interesting to notice also that the k factor can be written as

$$k = \sqrt{\frac{c+v}{c-v}} = e^{\chi}$$

and satisfies a multiplicative composition law $k_{20} = k_{21}k_{10}$.

## 2.6. The Interval Revisited

Let us go back now to the space-time interval that we have introduced before.




*Obs* Figure 8

I will start by picking an observer *Obs* and an event $O$ in his world line. In the following I will introduce a way of labeling events by assigning some "coordinates" in a physical way. In principle there may be many ways to do this, for example, one could use the $T_-$ and $T_+$ defined before and measured with the help of $O$ ("null coordinates"). I will choose instead to label each event by assigning a pair of numbers $t$, and $x = c\tau$ to it where $t$ is the proper time measured by the observer from $O$ to the event $B'$ in its world line which is simultaneous to $B$ and $c\tau$ measures the spatial distance between $B'$ and $B$. The reason to do this is because these coordinates have a direct physical interpretation as proper times or spacial distances and are the ones used in the more traditional presentations of the subject. If we have two events $A$ and $B$ it is straightforward to obtain the differences between emission times and reception times and write the square of the interval in terms of them.

$$T_- = (t_B - t_A) - (\tau_B - \tau_A) \equiv \Delta t - \frac{1}{c}\Delta x,\; T_+ = (t_B - t_A) + (\tau_B - \tau_A) \equiv \Delta t + \frac{1}{c}\Delta x$$

$$T_- T_+ = \Delta t^2 - \frac{1}{c^2}\Delta x^2.$$

If the interval from $A$ to $B$ is space-like we get in a completely analogous way that the square of their spatial distance is given by $T_- T_+ = \frac{1}{c^2}\Delta x^2 - \Delta t^2$. Finally if it is lightlike we find again $T_- T_+ = \Delta t^2 - \frac{1}{c^2}\Delta x^2 = 0$. We conclude that intervals can be classified by using a 1+1 dimensional non degenerate quadratic form in space-time and proper times and spatial distances are determined by it! This leads to the introduction of the all-important Minkowski metric

$$ds^2 = dt^2 - \frac{1}{c^2}dx^2$$

and the recognition of the fact that a space-time point of view is much better that the traditional separation of space and time. In fact, according to some authors [7] special relativity cannot be considered as complete until the introduction by Minkowski of the concept of space-time and the realization of the fact that it is completely described by the so called Minkowskian metric.

I will show now that proper time cannot originate in a universal time. In more concrete terms let us ask ourselves the following question: Can we find a scalar function $\Phi(t, x)$ (expressed in the same coordinates introduced above) such that $T_{AB} = \sqrt{(t_A - t_B)^2 - \frac{1}{c^2}(x_A - x_B)^2} = \Phi(t_A, x_A) - \Phi(t_B, x_B)$? The answer to this question is in the negative; the reason is that we would have then

$$\Phi(t, x) - \Phi(0, 0) = \sqrt{t^2 - \frac{x^2}{c^2}}$$

and

$$\Phi(t_A, x_A) - \Phi(t_B, x_B) = \sqrt{t_A^2 - \frac{x_A^2}{c^2}} - \sqrt{t_B^2 - \frac{x_B^2}{c^2}} \neq T_{AB}.$$

As we see the existence of a universal, observer independent, time controlling the ticking of the clocks carried by all the observers is in contradiction with hypotheses **l** and **k** and, hence, if we want to keep any one of them we have to abandon the other. Historically the rejection of the idea of the existence of a universal time was, probably, the single most difficult step that had to be taken to resolve the paradoxical situations posed by the mounting evidence given by experiments; in particular those trying to detect the motion of the earth with respect to the ether.

## 2.7. Lorentz Transformations

At this point the reader may wonder what has happened with the Lorentz transformations that are at the crux of the usual presentations of special relativity. To show how they appear as a consequence of the hypotheses that we have introduced at the beginning let us consider now the following diagram,

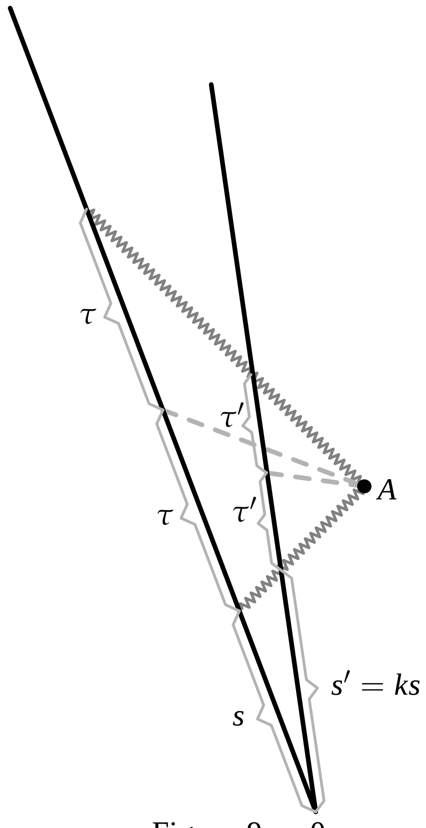



Figure 9

where I discuss a situation in which two inertial observers have worldlines with an event $O$ in common. Let us find out the physical coordinates that they both assign to a given event that we label $A$. To this end we compute

$$\tau' = \frac{(1+k^2)\tau + (1-k^2)t}{2k},$$

$$t' = \tau' + ks = k(t-\tau) + \tau';$$

taking into account that

$$t = \tau + s, \quad t' = \tau' + s',$$

$$k(2\tau' + ks) = 2\tau + s, \quad s = t - \tau,$$

and

$$\frac{1+k^2}{2k} = \frac{1}{\sqrt{1-\frac{v^2}{c^2}}},$$

$$\frac{1-k^2}{2k} = \frac{-\frac{v}{c}}{\sqrt{1-\frac{v^2}{c^2}}}.$$

Hence we see that the relationship between the values of $x$, $x'$, $t$, and $t'$ is given by the familiar expressions that give the Lorentz transformation in 1+1 dimensions

$$x' = \frac{(1+k^2)x + (1-k^2)ct}{2k} = \frac{x - vt}{\sqrt{1-\frac{v^2}{c^2}}}$$

$$t' = \frac{(1+k^2)t + (1-k^2)\tau}{2k} = \frac{t - x\frac{v}{c^2}}{\sqrt{1-\frac{v^2}{c^2}}}.$$

It should be clear at this point the complete equivalence of our presentation and the more familiar ones relying upon the use of these Lorentz transformations.

In the above picture we have viewed space-time as an affine space. It is very useful now to change this point of view and adopt a new one. To this end let us consider space-time as a differentiable manifold $(\mathbb{R}^4, \mathbf{g})$ with the usual differential manifold structure and the Minkowski metric defined on it. This metric can be taken as a twice covariant symmetric tensor with the following form at every point in $\mathbb{R}^4$ in the coordinate basis (in the following I will set $c = 1$)

$$\begin{bmatrix} 1 & 0 & 0 & 0 \\ 0 & -1 & 0 & 0 \\ 0 & 0 & -1 & 0 \\ 0 & 0 & 0 & -1 \end{bmatrix}.$$

All the issues discussed before are easily described in this framework, in particular the classification of vectors and the description of physical observers. At each point in $\mathbb{R}^4$ tangent vectors $\mathbf{v}$ are classified as time-like if $g(\mathbf{v}, \mathbf{v}) > 0$, null if $g(\mathbf{v}, \mathbf{v}) = 0$, and space-like if $g(\mathbf{v}, \mathbf{v}) < 0$. Physical observers and, in particular the inertial ones, are described by curves parametrized by proper time (i.e. with time-like tangent vectors $\mathbf{T}$ satisfying $g(\mathbf{T}, \mathbf{T}) = +1$). Light rays are null geodesics (geodesics with null tangent vectors) and spacelike vectors are suitable to represent measuring rods and certain sets of simultaneous events.

Given a space-like parametrized curve $\gamma(s)$ defined by events that are simultaneous with respect to a given inertial observer its length is

$$\int ds\, \sqrt{-g(\dot{\gamma}, \dot{\gamma})}.$$

Given a time-like parametrized curve $\gamma(s)$ describing an observer that carries a clock the proper time that it measures is given by

$$\int ds\, \sqrt{g(\dot{\gamma}, \dot{\gamma})}.$$

The Minkowski metric has certain symmetries described by Killing vector fields. It is always possible to choose four of these in such a way that they commute and allow the construction of coordinate systems in which the metric takes the form written above. These are the inertial reference systems. Transformations that preserve the form of the metric correspond to the choice of a different set of commuting Killing fields to build the inertial frame; they are the Lorentz transformations. In each of these it is possible to have inertial observers "at rest" (i.e. with constant spatial coordinates). It is very important to realize that some concepts that play an important role in special relativity are, in fact, particular properties of the Minkowski metric due to its very high degree of symmetry.

In the following I will generalize the framework discussed above and introduce general $(+---)$ (Lorentzian) signature metrics. These will not have all of the properties of the Minkowski metric (in particular its symmetries) but will prove to be a very useful and important objects. Now the question is: What kind of physical phenomenon can be described with them?

## 3. Relativistic Gravity

In the following I am going to introduce and discuss a mathematical model for an Einstenian space-time $(\mathcal{M}, \mathbf{g})$ where $\mathcal{M}$ is a four dimensional differentiable manifold and $\mathbf{g}$ is a metric on $\mathcal{M}$ with $(+---)$ signature. We say that $(\mathcal{M}, \mathbf{g})$ and $(\mathcal{M}', \mathbf{g}')$ are isometric if there exists $\theta : \mathcal{M} \to \mathcal{M}'$ a diffeomorphism from one to the other such that $\mathbf{g}' = \theta^* \mathbf{g}$. This defines an equivalence relation whose equivalence classes will be taken as space-time models. In this setting the different frameworks (Aristotelian, Galilean, Einstenian) can be described by introducing different mathematical structures (see a nice discussion in [7]).

I give now the notation and conventions that will be used in the following. Tensors on a vector space $V$ over a field $\mathbb{K}$ (usually $\mathbb{R}$ or $\mathbb{C}$) of type $(k, l)$ are multilinear maps

$$T : \underbrace{V^* \times \cdots \times V^*}_{k} \times \overbrace{V \times \cdots \times V}^{l} \to \mathbb{K}$$

where $V^*$ is the dual vector space of $V$. I will only consider finite dimensional vector spaces (and, hence, $V^{**} \cong V$). Tensor fields are defined at each point $P$ of a differentiable manifold $\mathcal{M}$ by using the tangent and cotangent vector spaces ($T_P$ and $T_P^*$). By taking a basis of $V$ and its dual basis on $V^*$ one can obtain their components; in physics one often works in terms of them. One can introduce the usual operations for tensors such as contractions or exterior products.

In these notes I will use the abstract index notation of Penrose [8] according to which a tensor of type $(k, l)$ will be denoted as $T^{a_1 \ldots a_k}{}_{b_1 \ldots b_l}$. This is a useful notation because it mimics the more traditional component approach but does not refer to any specific coordinate system. Indices are just convenient labels to identify the different types of tensors and their combinations.

A contraction will be represented by repeating indices, for example, $T^{abc}{}_{be}$ (no summing of repeating indices meant!) and tensor products are given by expressions such as $T^{abc}{}_{de}S^{f}{}_{g}$. Symmetries in tensors are taken care of in a straightforward way, for instance, if $T_{ab}v^a w^b = T_{ab}w^a v^b$, $\forall v^a$, $w^a \in V$ then we say that $T_{ab}$ is symmetric and denote it as $T_{ab} = T_{ba}$. Some tensor objects are especially important; in particular the metric $g_{ab}$ [non-degenerate symmetric $(0,2)$ tensor] and its inverse $g^{ab}$ defined to satisfy $g^{ab}g_{bc} = \delta^a{}_c$. As is traditional among physicists I will "raise and lower indices" with the metric (that is, I will use the vector space isomorphism $g : V \to V^* : v^a \mapsto g_{ab}v^b$) whenever I find it appropriate.

## 3.1. Relativistic Free Fall

In the rest of this section I will try to show you how gravity is described in this setting, in particular, I will discuss how and why objects fall. In fact I will do something slightly less trivial and study *relativistic* free fall. In the following I will make use of two hypotheses that can be proved to be true, nonetheless, in some appropriate and physically relevant circumstances. The first one states that test particles (particles that feel the presence of a space-time metric but do not perturb it) move along time-like geodesics. Test particles are important as an idealization that allows us to explore a given space-time metric, physically, without acting on it. The second hypothesis refers to the motion of light rays and states that they follow null geodesics. Test particles are in the simplest state of motion and are the best candidates to become what we can call inertial observers in general relativity.

Let us consider the static, spherically symmetric gravitational field created by a star or a planet. In Newtonian mechanics this field is described in terms of the distance to its center by the potential

$$\Phi = -\frac{GM}{r}$$

and is independent of its angular velocity. In relativistic gravity the situation changes because the gravitational field created by a rotating mass is different from the one created by a non-rotating one. If the rotation is slow it can be accurately described, nevertheless, by the Schwarzschild metric (in the following $G = 1$ so that mass is measured in length units)

$$d\tau^2 = \left(1 - \frac{2M}{r}\right)dt^2 - \left(1 - \frac{2M}{r}\right)^{-1} dr^2 - r^2(d\theta^2 + \sin^2\theta d\phi^2)$$

that is known to represent the space-time geometry outside a static, spherically symmetric distribution of mass $M$. The coordinates in terms of which this is written take values in $r \in (2M, \infty)$, $t \in \mathbb{R}$, $\theta \in (0, \pi)$, and $\phi \in (0, 2\pi)$. As will be discussed later some of the singularities that appear in the components are physical whereas the others are, in a sense, just artifacts of this specific coordinate representation.

The following study of relativistic free fall will require the consideration of radial geodesics. We show some of them (starting in an "outgoing" direction) in figure 10.

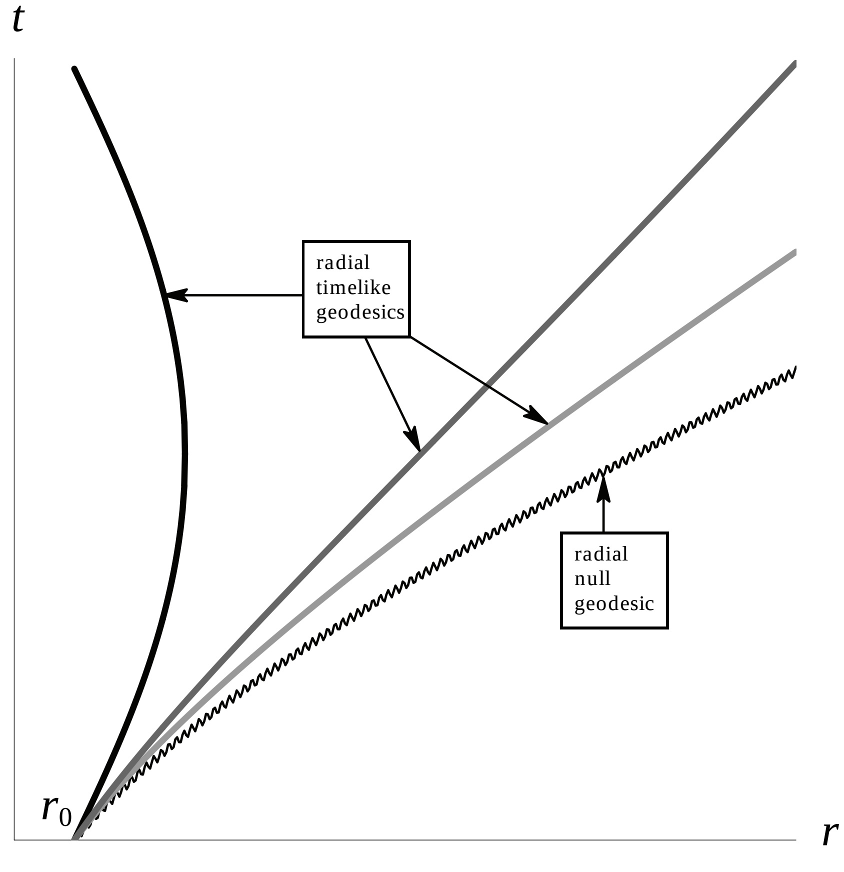



Figure 10: Time-like and null radial geodesics for the Schwarzschild metric.

We see different types of behaviors. There are, for example, radial geodesics that go "up" and then "down" (in the direction of growing $r$) and others that go "up forever". Between these two behaviors there is an intermediate critical situation that separates them. We see also how radial null geodesics look like; in particular we can see that they are, in a certain sense, a limit of the time-like "escaping geodesics". The behavior depicted above is certainly suggestive of the type of situation encountered when we consider ordinary Newtonian free fall. If an object is hurled upwards with a speed below a certain critical value it will eventually fall back to the floor, if, instead, the velocity is above that value it will escape and travel indefinitely with a speed approaching a certain limit. Between these two situations there is a critical one in which the object never falls back but moves with a velocity that gets closer and closer to zero with time.

There are several issues that must be carefully considered before we can accept the previous picture. To begin with it is important to understand that we are plotting $t$ versus $r$ but these are coordinates whose physical (operational) meaning we have not clarified at this point; we do not know what they really measure. There are also other questions, for example, when we ask ourselves why we see things falling when we stand on the ground, how do we know that the ground really corresponds to $r = constant$? This is in fact a non trivial issue that requires the discussion of some important properties of the metric, such as isometries, that I will consider next.

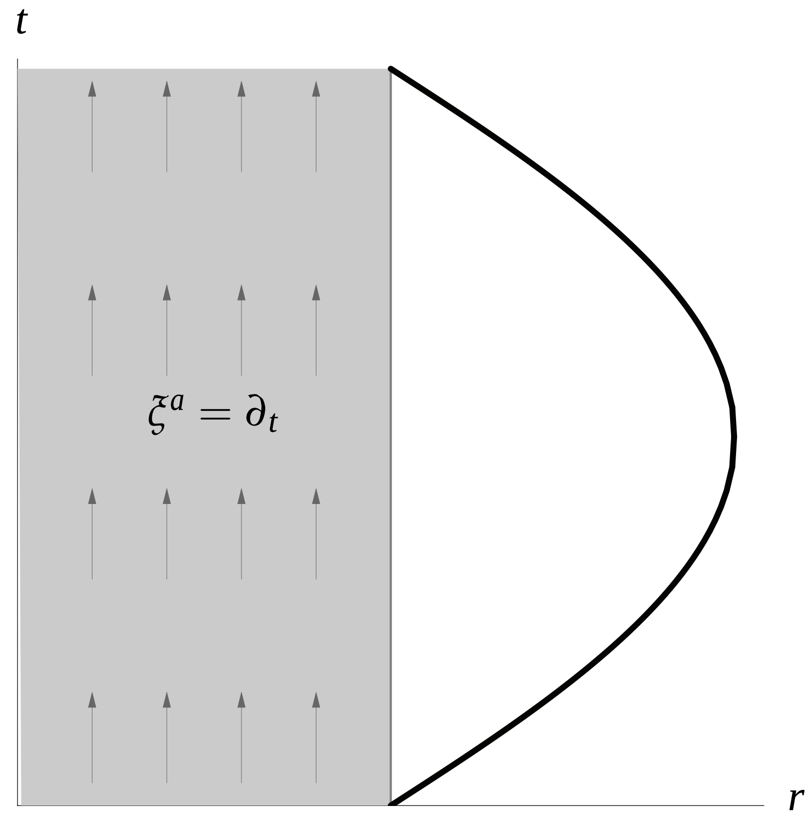



Figure 11: Time-like Killing vector field $\partial_t$. I show an integral curve of this field that represents the *ground* and one of the time-like geodesics.

Given a metric $g_{ab}$ on a manifold $\mathcal{M}$, an isometry is a diffeomorphism $\phi : \mathcal{M} \to \mathcal{M}$ such that $(\phi^* g)_{ab} = g_{ab}$; they are the symmetries of the metric, and by extension of space-time.

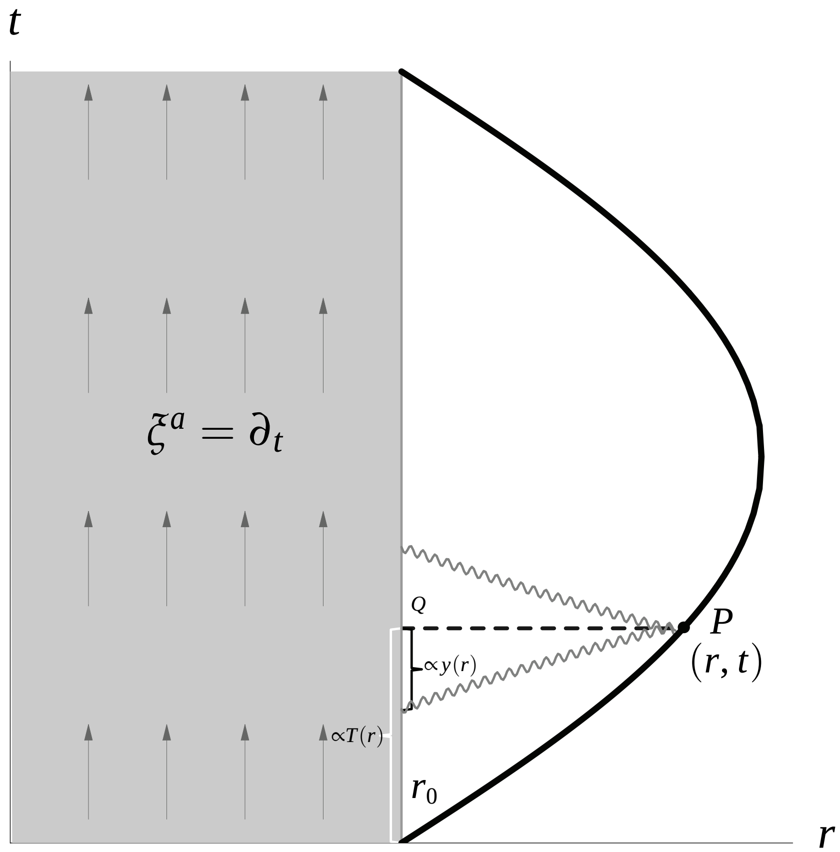



Figure 12: Determination of the *observed* trajectory for a non-escaping time-like geodesic.

If we have a one parameter group of diffeomorphisms $\phi_t$ we can obtain curves on $\mathcal{M}$ by considering the images of points $P \in \mathcal{M}$ under its action. If we take the tangents to all the curves obtained this way for every point in $\mathcal{M}$ we can construct a certain vector field $\xi^a$ that generates $\phi_t$. It is possible to use $\phi_t$ to carry along any smooth tensor field $T^{a\cdots}_{b\cdots}$. By comparing $T^{a\cdots}_{b\cdots}$ with $\phi_{-t}T^{a\cdots}_{b\cdots}$ we can define the Lie derivative of this tensor field. Finding the symmetries of the metric $g_{ab}$ boils down to getting vector fields with the property that the Lie derivatives of $g_{ab}$ along the directions defined by them are zero. This condition can be conveniently rewritten as $\nabla_{(a}\xi_{b)} = 0$ (where $\nabla_a$ is the derivative operator associated with the metric $g_{ab}$ and $\xi_a \equiv g_{ab}\xi^b$).

This is the so called Killing equation whose solutions are known as Killing vector fields.

In the case of the Schwarzschild metric introduced above it is straightforward to show that the time-like vector field $\partial_t$ is a Killing vector field or equivalently that $(t, r, \theta, \phi) \mapsto (t + T, r, \theta, \phi)$ is an isometry for every value of $T$. Metrics for which such time-like (and hypersurface orthogonal) Killing vector fields exist are known as static; the Schwarzschild metric is an example of them. Any observer whose world line is an integral curve of the vector field $\partial_t$ will perceive a non-changing space-time in the sense that any experiment performed by him to explore its properties will yield the same result if repeated at a different instant of time (if the same setup is used). It is important to realize that these are not inertial observers because they are not in free fall; in this respect it is right to say that everyday gravity is a fictitious force very much as the centripetal or Coriolis forces are. In the following I will consider observers with constant $r = r_0$ (and forget about $\theta$ and $\phi$ in the radial case). In a free fall experiment we would say that such an observer stays on the ground.

## 3.2. Radial Free Fall

Let us discuss now the physical meaning of $r$ and $t$. In principle they are just coordinates so it is legitimate to ask ourselves wether we are entitled to assign a physical meaning to them such as vertical distance to the floor or time in flight. Here we can follow the space-time philosophy of the first talk on special relativity and use the definitions that we introduced there for distances and proper times. By adapting them to the present situation we will have natural and easy to understand notions that can be used to describe free fall in physical terms.

Let us pick an event $P$ on the worldline of the falling object and let us trace back to the floor the two light rays (i.e. null geodesics) that arrive and start at $P$ (see figure 12). In this case there is a discrete time symmetry that implies that for an observer with constant $r$ all the events with the same $t$ are simultaneous.

In the following I will get what one could rightly call the equations of the "physical" (or, rather, *observed*) geodesics; that is the equations that give the observed distance in terms of the measured physical time. To this end we need the equations for the time-like radial geodesics.

$$\dot{t} = \frac{E}{1 - \frac{2M}{r}}, \quad \dot{r} = \left[E^2 - \left(1 - \frac{2M}{r}\right)\right]^{1/2}$$

where $E$ is a certain real constant to be interpreted below and the dot denotes derivative with respect to the affine parameter. We also need the equations for the null radial geodesics

$$\dot{t} = \frac{1}{1 - \frac{2M}{r}}, \quad \dot{r} = \pm 1$$

where the plus sign refers to the outgoing null geodesics and the minus to the ingoing ones. From these equations we get

$$\frac{dt}{dr} = \frac{E}{\left(1 - \frac{2M}{r}\right)\left[E^2 - \left(1 - \frac{2M}{r}\right)\right]^{1/2}}$$

for the time-like geodesics, and

$$\frac{dt}{dr} = \frac{\pm 1}{\left(1 - \frac{2M}{r}\right)}$$

for the null ones. The solutions to them are respectively given by

$$t_{time}(r) = t_0 + E\int_{r_0}^{r} \frac{d\rho}{\left(1 - \frac{2M}{\rho}\right)\left[E^2 - \left(1 - \frac{2M}{\rho}\right)\right]^{1/2}}$$
$$t_{null}(r) = t_0 + r - r_0 \pm 2M \log \frac{r - 2M}{r_0 - 2M}.$$

The proper time elapsed at the ground from launch at $r_0$ to the event $Q$ is given by $T(r) = \left(1 - \frac{2M}{r_0}\right)^{1/2} t_{time}(r)$ (as can be easily read from the metric) and the distance to the ground (proportional to the proper time elapsed at the floor from emission to reception of the light ray reaching the free falling object at $P$) is

$$y(r) = \left(1 - \frac{2M}{r_0}\right)^{1/2}\left[r - r_0 + 2M \log \frac{r - 2M}{r_0 - 2M}\right].$$

It is straightforward to get now the velocity and acceleration

$$v(r) \equiv \frac{dy}{dT} = \frac{1}{E}(E^2 - 1 + \frac{2M}{r})^{1/2}, \quad a(r) \equiv \frac{d^2y}{dT^2} = -\frac{M\left(1 - \frac{2M}{r}\right)}{E^2 r^2 \left(1 - \frac{2M}{r_0}\right)^{1/2}}$$

where in these last two equations $r$ should be considered as a function of $y$ obtained by inverting $y(r)$.

We see that $E = \frac{1}{\sqrt{1-v_0^2}}\left(1 - \frac{2M}{r_0}\right)^{1/2}$ $[v_0 \equiv v(r_0)]$ so that for $r_0 >> 2M$ and $v_0 << 1$ we find that $E \simeq 1 - \frac{M}{r_0} + \frac{1}{2}v_0^2$. This can be easily interpreted, in the non-relativistic limit, as the energy per unit mass including the Newtonian gravitational potential energy. For values of $E$ close to one the acceleration is given in a very good approximation by

$$a(y) = -\frac{M}{y^2},$$

which is just the Newtonian result; the relativistic corrections can be easily obtained. At this point the reader may find it strange that the acceleration, in general, depends on $E$ but some thought will convince him that, in fact, this is not so because an outgoing light ray (in a sense described by the $E \to \infty$ limit of a massive particle) moves with no acceleration! This is shown in figure 13.

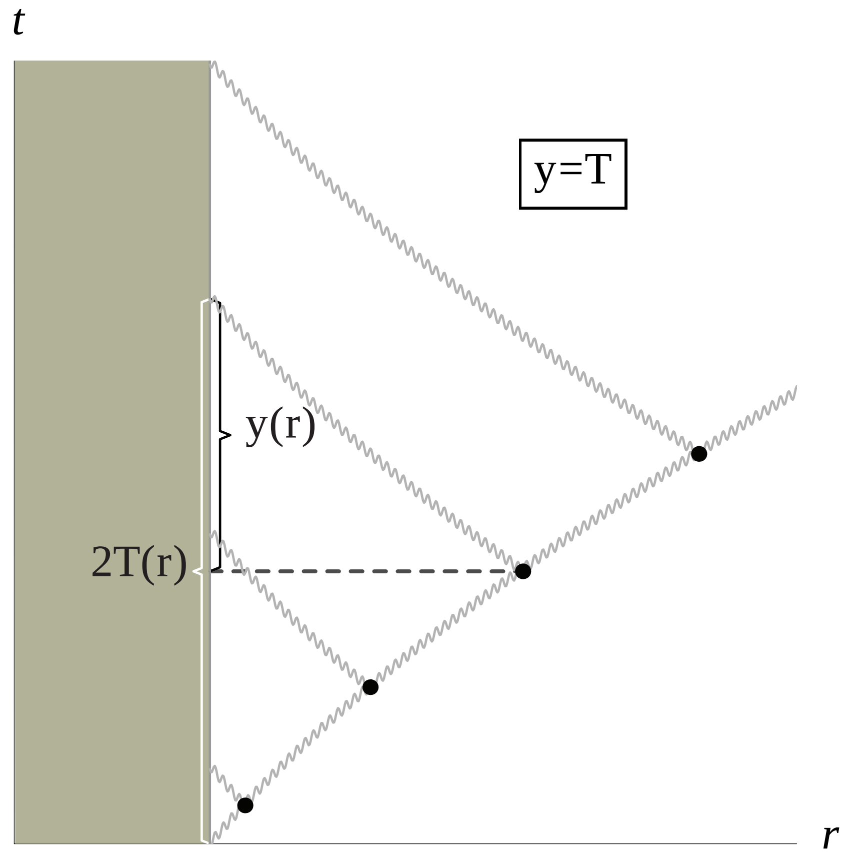



Figure 13: The definitions of distance and proper time that we are using imply that the *observed* trajectory of a light ray ($c = 1$) is $y = T$ and, hence, light moves with no acceleration.

### 3.3. Gravitational Redshift

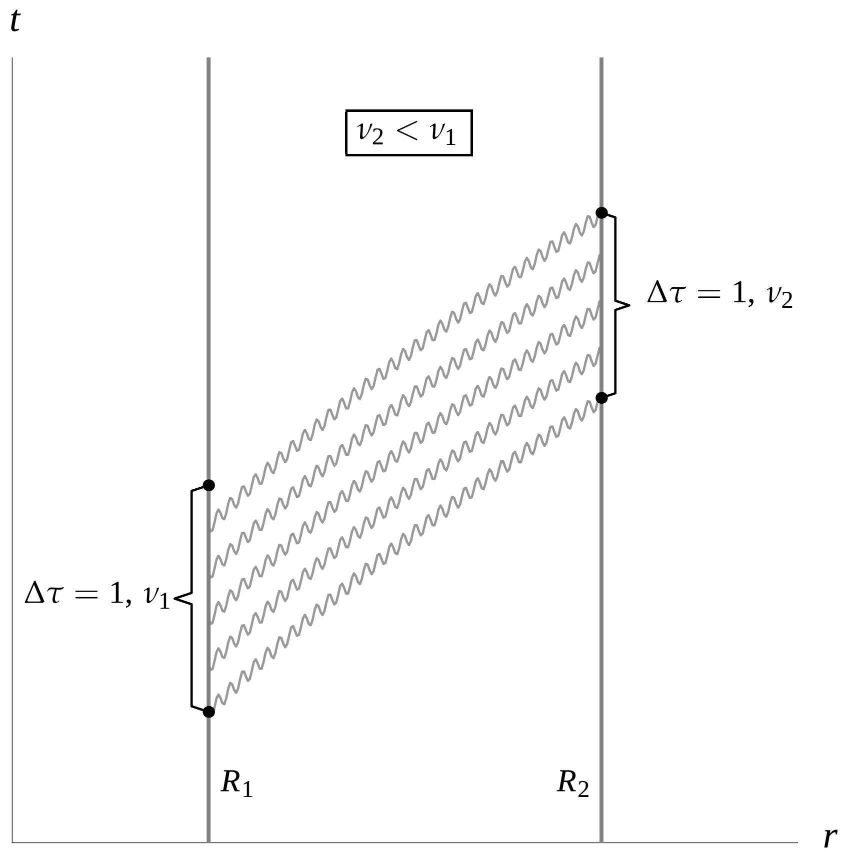



Figure 14: Gravitational redshift

An interesting, and very important from the experimental point of view, physical consequence that can be derived from the previous discussion a is the phenomenon of gravitational redshift (see figure 14). Let us consider two observers at two different heights $R_1$ and $R_2 > R_1$ (radial case) and suppose that $R_1$ emits radiation at a certain frequency $\nu_1$. What is the frequency observed by $R_2$? At $R_1$ the proper time between the emission of two pulses is proportional to $(1-2M/R_1)^{1/2}$ whereas the proper time at $R_2$ (reception) is proportional to $(1-2M/R_2)^{1/2}$. We hence find

$$\frac{\nu_2}{\nu_1} = \frac{(1-2M/R_1)^{1/2}}{(1-2M/R_2)^{1/2}}$$

This has been measured in several important experiments using very different approaches starting from Pound and Rebka (1960) with a setup based on a very sensitive Mössbauer detector used to measure the gravitational redshift on gamma rays emitted from a radioactive cobalt source climbing upwards. These results were later confirmed and refined by Vessot and Levine (1979) in an experiment using hydrogen masers on a sounding rocket known nowadays as Gravity Probe A (with an accuracy of 0.01%). Another interesting verification of this effect was carried out by Hafele and Keating (1972) by carrying atomic clocks in airliners. In fact it is fair to say that gravitational redshift has entered the realm of technology and everyday life as its effects must be taken into account for the correct working of the Global Positioning System.

The non radial geodesics display a rich variety of behaviors. Though I will not discuss them in detail here it is important to point out that they describe Keplerian orbits, at least for values of $r$ much larger than a certain characteristic length (the Schwarzschild radius) and also the corrections to these orbits that are the basis of the classical tests of general relativity (light deflection by compact masses, rotation of perihelia, Shapiro time delay,...).

## 3.4. The Kruskal Extension

The Schwarzschild metric describes the space-time metric in vacuo for a spherically symmetric situation (i.e. outside the earth if one forgets about its rotation). The matching metric inside some spherically symmetric distribution of mass is different and its detailed form depends on the properties of the matter distribution. As long as the value of $r$ that limits the extent of the matter distribution is not smaller or equal than $2M$ (the Schwarzschild radius) nothing dramatic happens, but, what is going on at $r \leq 2M$? Can we extend the Schwarzschild metric beyond $r = 2M$? What happens at $r = 0$?

Answering to these questions requires the discussion of the so called Kruskal extension of the Schwarzschild metric. In order to show how this is done I will consider first a simplified but rather illuminating example. Let us consider the metric [$t \in \mathbb{R}$, $x \in (0, \infty)$]

$$d\tau^2 = x^2 dt^2 - dx^2$$

This metric seems to be singular at $x = 0$ because $\det g_{ab} = 0$ there. Let us however "change coordinates" according to

$$t = \frac{1}{2} \log \left( \frac{T + X}{T - X} \right), \quad x = \sqrt{X^2 - T^2}$$

where now $X > 0$ and $X^2 > T^2$ (see figure 15).

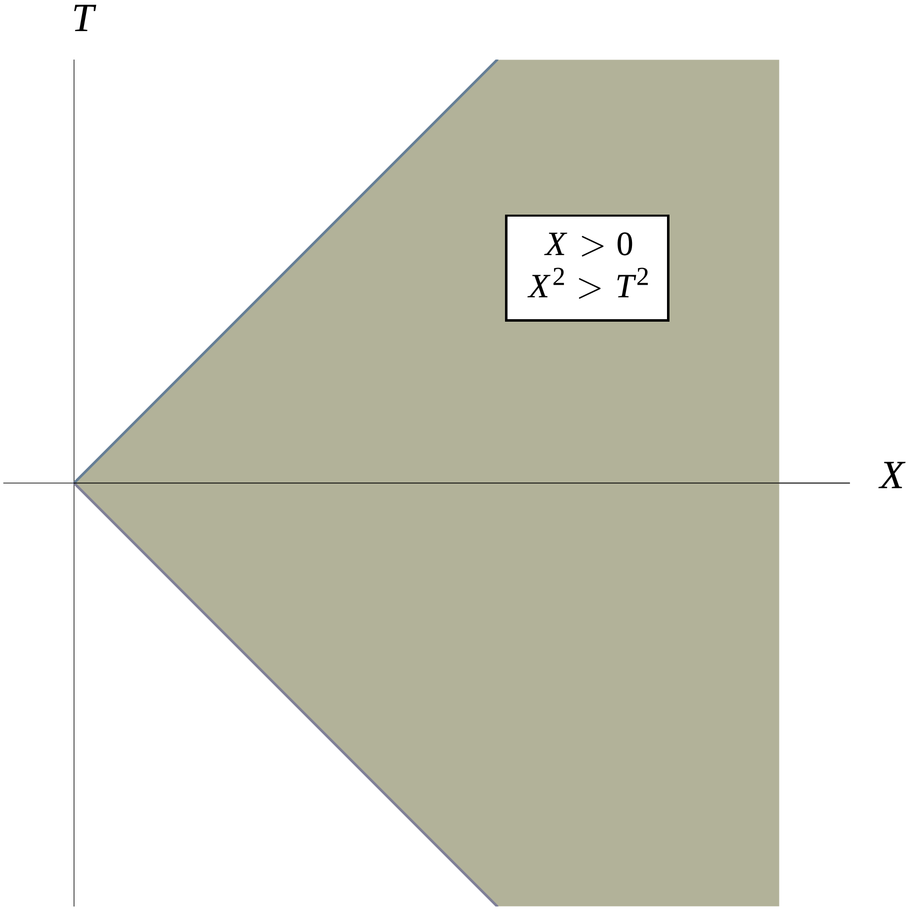



Figure 15: Domain of definition of the Rindler metric.

In these coordinates the previous metric becomes

$$d\tau^2 = dT^2 - dX^2$$

which is just Minkowski defined on a submanifold of $\mathbb{R}^2$. This is known as the Rindler space-time and plays a very important role in the understanding of quantum field theory in curved space-times and phenomena related to black hole evaporation and the presence of horizons. We see then that the initial metric describes just a piece of Minkowskian space-time and suggests that its singularity can be considered as an artifact introduced by a "bad choice of coordinates". By choosing appropriate coordinates, the metric can be extended to a larger manifold in such a way that this extension can be considered as complete in a well defined sense.

There is a way to do something similar with the Schwarzschild solution to construct the Kruskal extension. Consider the coordinate change defined by

$$\left(\tfrac{r}{2M} - 1\right) e^{r/(2M)} = X^2 - T^2$$

$$t = 2M \log\left(\tfrac{T+X}{T-X}\right)$$

where now $X$ and $T$ are constrained to satisfy $X^2 - T^2 > -1$. In these coordinates (Kruskal-Szekeres) the Schwarzschild metric introduced at the beginning of this section becomes

$$d\tau^2 = \frac{32M^3 e^{-r/(2M)}}{r}(dT^2 - dX^2) - r^2(d\theta^2 + \sin^2\theta d\phi^2)$$

A space-time diagram representing the Kruskal extension of the Schwarschild metric (each point represents a full spherical surface) is shown in the figure 16.

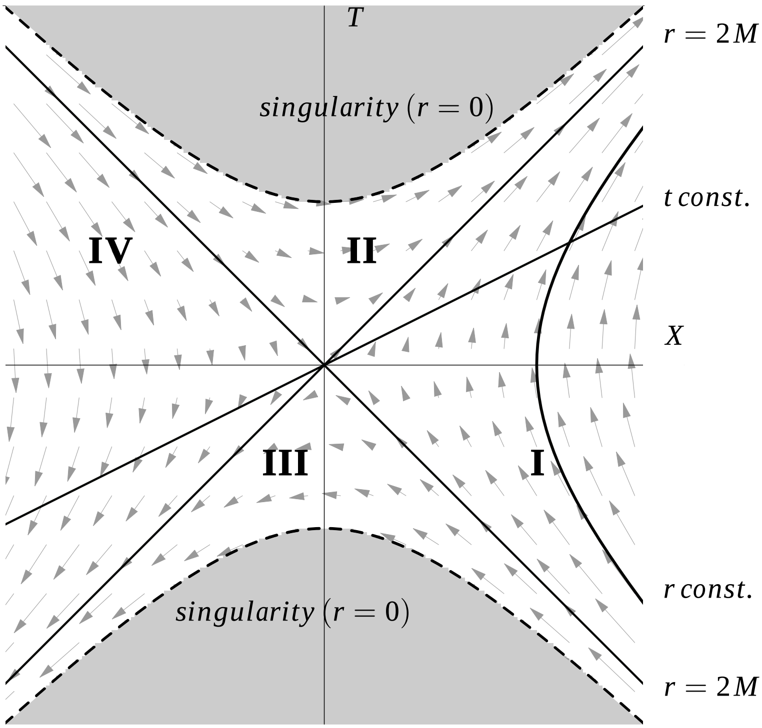



Figure 16: Domain of definition of the Kruskal extension for the Schwarzschild metric. The Killing vector field that reduces to $\partial_t$ in region I is also shown.

The original Schwarzschild metric describes only the region labelled as I in the diagram (and represents the exterior gravitational field of a spherical body). Nothing happens at the boundary of the regions I and II as far as the regularity of the metric is concerned, the metric is well defined there and the singularity at $r = 2M$ is fictitious from the physical point of view. There are, nonetheless, some interesting phenomena to consider when one looks at this extended metric. If, for example, studies the time-like curves in region I it is possible to see that they can be extended to arbitrary values of $r$ with arbitrary large proper time whereas those in II hit the boundary of the diagram in a finite proper time. This means that an object that crosses the horizon will hit the boundary after some time that depends on the trajectory itself and the mass $M$. This boundary corresponds to $r = 0$ and represents a genuine space-time singularity where some quantities, such as the curvature, blow up. It is a very dangerous place because tidal forces in its vicinity become arbitrarily large and, hence, a body that gets too close to it will be torn apart. The fact that $r = 0$ is a singularity implies that the extension obtained is maximal in a precise mathematical sense and there is no way to go beyond it.

One can study the symmetries of the Kruskal extension. It is interesting to look, in particular, at the Killing field that coincides with $\partial_t$ in region I because it has the curious feature of becoming space-like in region II and zero at $X = 0, T = 0$, this means that there is no way of having a "ground" inside region II and, hence, there is a minimum size for a static object that can support a spherically symmetric external geometry. Neutron stars are thought to be very close to this situation. What about regions III, IV and the other singularity? Region III is in a sense the opposite as the black hole II so it is called a white hole, a region of space-time that any object must leave in a finite proper time. Region IV is similar in its properties to region I but it is physically isolated from it in the sense that there is no way to send or receive signals from there (at speeds smaller than the speed of light). There is no realistic astrophysical situation that could give rise to the space-time represented by the full Kruskal extension. In the collapse of a physical object, infalling matter completely "covers" regions III and IV, but region I, part of II and the singularity will still be present.

The previous discussion would lead us to the very important issue of space-time singularities that I can only briefly mention here. Physically they show their presence as the impossibility to arbitrarily extend time-like and null geodesics in their affine parameters (that is if there exist incomplete, time-like and space-like geodesics). This is the property that is proved in the important singularity theorems (of Hawking and Penrose [9]) that show that singularities such as the big bang or black holes are generic physical features and not artifacts of the usual metrics that display them.

# 4. The Einstein Field Equations

We know from Newtonian gravity that the gravitational field is created by the distribution of masses in the universe; in a relativistic setting we would expect that the distribution of matter and energy that creates gravity and determines the space-time metric. How does this come about? What is the detailed mechanism that explains how matter curves space-time?

Let us go back to special relativity and consider something that we completely left aside in the first part of these notes: Relativistic dynamics. Classical mechanics can be very conveniently described by using variational principles. The main idea is to introduce an action functional whose stationary points, determined by differential equations, correspond to the evolution of the system in question. Let us consider here as a particularly relevant example the relativistic particle. The motion of such a particle can be described by a certain worldline that we have to determine dynamically. To this purpose let us fix two space-time events $A$ and $B$ and consider a sufficiently smooth curve $\gamma$ connecting them. We choose as the action the proper time measured along $\gamma$ from $A$ to $B$ multiplied by $-m$ (where $m$ is an attribute of every physical particle known as its rest mass). If we choose inertial coordinates this is

$$S = -m \int_{t_A}^{t_B} \sqrt{1 - \dot{\mathbf{x}}^2}.$$

One can equivalently consider an arbitrary coordinatization of the worldline and work with a reparametrization invariant action. In this case we have gauge symmetries, that manifest themselves as constraints, and a zero Hamiltonian. The Euler-Lagrange equations tell us that the world line of the free relativistic particle is a straight line in space-time. We can, alternatively, use the Hamiltonian formalism to study the resulting dynamics. What we find is that the canonical momenta are given by

$$\mathbf{p} = \frac{m\dot{\mathbf{x}}}{\sqrt{1 - \dot{\mathbf{x}}^2}},$$

and the Hamilton equations imply that these momenta are constant in $t$. Finally the conserved energy is given by $E = \sqrt{m^2 + \mathbf{p}^2}$. It is important to point out that $(E, \mathbf{p})$ can be considered as the components of a (four) vector $p^a = mu^a$ proportional to the so called four-velocity $u^a \equiv \frac{dx^a}{d\tau}$ that satisfies $u_a u^a = +1$. This energy is an observer-dependent concept so that another observer at an event where the particle is present (with four-velocity $v^a$) would measure a different one given by $+v_a p^a$.

## 4.1. The Energy-momentum Tensor

If we have a swarm of non-interacting particles (usually referred to as dust) the action of the system is given by the sum of the actions of the individual particles. We can define a total energy-momentum vector for the system as the sum of the individual contributions of each particle in the obvious way. In the case of continuous matter distributions one can introduce the so called stress-energy-momentum tensor $T_{ab}$ that encodes all the information about the energy and momentum of the system. The different components of this tensor in a given inertial system have simple and intuitive physical meanings. Specifically let us suppose that we have an observer with four-velocity given by $v^a$ and choose three mutually orthogonal space-like vectors $x_1^a$, $x_2^a$, $x_3^a$ satisfying $v_a x_i^a = 0$. The components of $T_{ab}$ for this observer have then the following interpretation:

- $T_{ab}v^a v^b$ is the energy density per unit proper volume.
- $T_{ab}v^a x^b$ is the momentum density of matter in the spatial direction defined by $x^b$.
- $T_{ab}x_i^a x_j^b$ $(i \neq j)$ is the $ij$ component of the stress tensor.

For normal (ordinary, physically observed) matter there are some restrictions on $T_{ab}$ known as energy conditions that constrain its possible form. As we will see later this is very important when we consider the Einstein field equations that describe how matter determines the geometry of space-time because they equate the curvature of space-time to the energy-momentum tensor. Without some appropriate restrictions on the latter *any* metric would be a solution to the field equations for some $T_{ab}$ and the equations themselves would have no predictive power. There are several of them, the most important ones are:

- The weak energy condition stating that for normal matter the energy density measured by an observer with four-velocity $v^a$ must satisfy the inequality $T_{ab}v^a v^b \geq 0$.
- The dominant energy condition: If $v^a$ is the four velocity of an observer we must have that $T_{ab}v^a v^b \geq 0$ and $T^{ab}v_a$ not space-like. Physically this means that the pressure does not exceed the energy density (and hence the velocity of sound is less than the speed of light).
- The strong energy condition: $T_{ab}v^a v^b \geq \frac{1}{2}T$ that is satisfied if no large negative pressures exist. It holds, for example, for the electromagnetic field and massless scalars.

The most important and common types of matter distributions used in general relativity are electromagnetic fields, scalar fields, and perfect fluids. The latter are specially important because they provide an accurate first approximation to study astrophysical systems and the universe as a whole so I will stop to describe them here. Let us consider a Minkowskian space-time (with metric given by $g_{ab}$). A perfect fluid is a matter distribution with

$$T_{ab} = \rho u_a u_b + P(-g_{ab} + u_a u_b)$$

where $\rho$, $P$, and $u^a$ are the mass-energy density, the pressure and the four velocity in the rest frame of (each sufficiently small volume of) the fluid. The dynamics of a perfect fluid subject to no external forces is given by the very important condition

$$\nabla_a T^{ab} = 0$$

Some remarks are now in order. First it is important to point out that in the case of perfect fluids this last equation is completely equivalent (and leads) to the familiar continuity and Euler equations for fluids in a certain inertial coordinate system and in the non relativistic limit in which the fluid velocity is much smaller than the speed of light.

$$\frac{\partial \rho}{\partial t} + \nabla \cdot (\rho \mathbf{v}) = 0$$
$$\rho \left[ \frac{\partial \mathbf{v}}{\partial t} + (\mathbf{v} \cdot \nabla)\mathbf{v} \right] + \nabla P = 0$$

Second it is very important to realize that it implies energy-momentum conservation for the perfect fluid system. To see this let us take a family of observers such that their four velocities satisfy $\nabla_a v^b = 0$ (they are "parallel") and define the (co-)vector field $J_a = T_{ab} v^a$. Then we have

$$\nabla_a J^a = \nabla_a (T^{ab} v_b) = (\nabla_a T^{ab}) v_b + T^{ab} (\nabla_a v_b) = 0$$

and energy momentum conservation follows immediately. Conversely energy momentum conservation for all inertial observers implies that $\nabla_a T^{ab} = 0$. This equation plays a central role in the physical justification of the Einstein field equations that I will present at the end.

At this point it is only natural to wonder how are we supposed to get the expression for the energy-momentum tensor of physically relevant systems satisfying the appropriate conservation conditions. In the specially relativistic case (i.e. when the space-time metric is simply Minkowski) the Noether theorem provides a simple way to obtain it from a Lorentz invariant Lagrangian.

Another way to look at this is to realize that in the presence of symmetries of the metric (represented by Killing vector fields) it is straightforward to obtain conserved densities of the type that we are considering here. For the usual field theories such as the electromagnetic or scalar fields the energy momentum tensors are given by

$$T_{ab} = \frac{1}{4\pi}\left[-F_{ac}F_b{}^c + \frac{1}{4}g_{ab}F_{de}F^{de}\right]$$

where $F_{ab} = \nabla_a A_b - \nabla_b A_a$ and for a scalar field it is straightforward to get

$$T_{ab} = \nabla_a\phi\nabla_b\phi - \frac{1}{2}g_{ab}(\nabla_c\phi\nabla^c\phi - m^2\phi^2).$$

If the metric is not Minkowski then many definitions can be easily adapted but some of the previous statements must be modified. Here, for example, particle motions are still described by time-like curves and the properties of the matter distribution by energy momentum tensors that, in most cases, are simply obtained by substituting partial derivatives for covariant derivatives and the Minkowskian metric by the space-time metric in the familiar special relativistic expressions. They must still satisfy $\nabla^a T_{ab} = 0$ but it is important to realize that in general this cannot longer be considered as an exact conservation law but rather as only an approximate one (valid only locally). Even in the absence of symmetries there are prescriptions to build suitable, divergenceless $T_{ab}$ for virtually any type of matter that we want by writing the generalizations of the familiar specially relativistic matter Lagrangians in tems of general space-time metrics (and the corresponding covariant derivatives). Perfect fluids, for example, continue to be represented in terms of $u^a$, $\rho$, and $P$ and electromagnetic fields are represented by a 2-form field $F_{ab}$.

Before proceeding further let us pause for a moment to discuss in more detail the local conservation of $T_{ab}$. To this end let us concentrate on the case of perfect fluids. In general it may be impossible now to find a family of observers for which $\nabla_{(a}v_{b)} = 0$ (with $v_a v^a = +1$) in which case we cannot argue as we did for Minkowski. If such a family of observers exists nevertheless (the so called stationary case) then $J^a = T^{ab}v_b$ is a conserved current and we still have energy momentum conservation. In fact it can be shown that the symmetries of a metric –described by Killing fields– allow us to define conserved quantities. In the most general situation we have approximate conservation in space-time regions small compared to the curvature radius.

## 4.2. Geodesic Deviation

The next step in this rush to the Einstein equations leads us to consider now the so called geodesic deviation equation. Let us take a smooth 1-parameter family of geodesics $\gamma_s(t)$ such that $\forall s \in \mathbb{R}$, $\gamma_s$ is affinely parametrized by $t$. Let us suppose that the map $f : (t, s) \mapsto \gamma_s(t)$ is sufficiently smooth, one to one, and with smooth inverse. In these conditions we can define a two dimensional submanifold in space-time $\Sigma$ spanned by the points in the geodesics $\gamma_s(t)$ with coordinates given by $(t, s)$. The vector field defined on $\Sigma$ by $T^a = \left(\frac{\partial}{\partial t}\right)^a$ is tangent to each geodesic of the family and satisfies $T^a \partial_a T^b = 0$. A vector $X^a = \left(\frac{\partial}{\partial s}\right)^a$ measuring the deviation between nearby geodesics, can be defined by using the freedom to change affine parameters in each geodesic according to $t \mapsto b(s) + c(s)t$ to get $X_a T^a = 0$ everywhere on $\Sigma$. We can define the rate of change of the displacement to a nearby geodesic as $v^a = T^a \nabla_b X^b$ and the relative acceleration between nearby geodesics as $a^a = T^a \nabla_b v^b$. The geodesic deviation equation simply states that

$$a^a = -R_{cbd}{}^a X^b T^c T^d.$$

where $R_{abc}{}^d$ denotes the Riemann curvature tensor.

## 4.3. The Einstein Equations

Let us get now to the main point in this last part of the paper: The Einstein field equations. Inspired by the Mach principle that suggests that the structure of space-time is influenced by the distribution of matter in the universe, Einstein looked for a set of equations such that the space-time geometry is determined by the distribution of matter and energy. If $v^a$ is the 4- velocity and $x^a$ is the orthogonal deviation vector the tidal acceleration of two nearby particles is $-R_{cbd}{}^a x^b v^c v^d$. In Newtonian gravity the tidal acceleration between particles separated by a vector $\vec{x}$ is $-(\vec{x} \cdot \vec{\nabla})\vec{\nabla}\phi$. This suggests the correspondence $R_{cbd}{}^a v^c v^d \leftrightarrow \partial_b \partial^a \phi$. Now the Newtonian Poisson equation and the fact that the energy density is given by $T_{ab} v^a v^b$ leads us to consider the replacement $\partial_a \partial^a \phi \leftrightarrow 4\pi T_{ab} v^a v^b$ so that putting everything together for any possible observer we are led to write the following equation relating the geometry of space-time and its matter content

$$R_{cd} = 4\pi T_{cd}.$$

This equation was considered by Einstein for some time as the equation of general relativity but, eventually, it had to be discarded because it suffered from a very serious drawback originating in the Bianchi identity $\nabla^c R_{cd} = \frac{1}{2} \nabla_d R$. In this case this would imply that the trace of $T_{ab}$ should be constant throughout space-time, an unacceptable and overly restrictive condition on physical grounds. There is, however, a simple escape from this state of affairs because the combination known as the Einstein curvature tensor $G_{ab} = R_{ab} - \frac{1}{2}g_{ab}R$ satisfies $\nabla^a G_{ab} = 0$. If we consider the equations

$$G_{ab} = 8\pi T_{ab}$$

the previous heuristic arguments would still be true but no inconsistencies arise and, in particular, there is no need to impose unphysical conditions on the matter distribution. These are the famous Einstein field equations. Several comments are in order now. First of all we see that, in some fixed coordinates, these are a system of coupled, nonlinear, second order partial differential equations that are hyperbolic if the metric has Lorentzian signature. For usual choices of matter fields $T_{ab}$ itself depends on the metric, so strictly speaking, geometry shows up also in the "matter side" of the equations. Once they are solved for a certain type of matter fields the dynamics of the matter is completely fixed by the local conservation condition $\nabla^a T_{ab} = 0$, in particular this is true for perfect fluids. In the case of zero pressure (dust) this condition implies that every particle moves along a geodesic. In fact, this is generically true for sufficiently small bodies with "weak enough" self gravity because the condition $\nabla^a T_{ab} = 0$ also implies that they move along geodesics. As we can see it is not necessary now to impose as a separate hypothesis that test bodies move along geodesics because this is a consequence of the Einstein equations; a very non trivial consistency condition is indeed satisfied. It is important to realize, anyway, that for large enough bodies there are deviations from geodesic motion.

The Einstein field equations can be derived from an action principle by using the so-called Einstein-Hilbert action with a Lagrangian proportional to the scalar curvature. If a matter Lagrangian is included, by generalizing the special-relativistic ones in the obvious way, one automatically obtains a locally conserved energy-momentum tensor as a functional derivative of the matter part of the Lagrangian with respect to the metric $g_{ab}$.

$$S = \int \sqrt{|g|}R + S_{mat}$$

It is important to insist on the fact that one can "solve" the Einstein equations by computing $G_{ab}$ for *any* metric and defining $T_{ab}$ as the resulting expression. This would lead generically to very unphysical matter (violating energy conditions or such that no known physical interaction can produce the energy-momentum distribution thus obtained). So beware of exotic solutions! As a final statement I would like to emphasize the enormous difficulties that must be overcome to solve the gravitational field equations in physically realistic situations. This is true even from a purely numerical point of view. In fact, a lot of effort is devoted nowadays to find solutions that are significant from the astrophysical point of view and to obtain precise predictions about the emission of gravitational radiation from astrophysical sources that can be used as templates in the operating gravitational radiation detectors.

Let me wrap up here by merely listing some of the many issues that I did not have time to consider in the space available for these notes: The Newtonian limit, gravitational waves and radiation, cosmological models, homogeneous and isotropic models, the solution of the Einstein equations, algebraically special solutions, perturbation theory, causal structure, well-posedness of the Einstein equations, singularities and singularity theorems, the initial value formulation, asymptotics and asymptotic flatness, gravitational energy, black holes and thermodynamics, the Hamiltonian formulation, numerical solution of the Einstein equations... Considering them here would have extended these notes far beyond the allowed space. These issues are, in many cases, hot research topics, both from the perspective of Physics and Mathematics, deserving close attention and a lot of thinking. I hope that the paper has succeeded in conveying part of the beauty of Relativity and excited the curiosity of the reader about this fascinating subject where both physics and geometry melt in such a harmonious combination.

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# Invited lectures

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# On some probabilistic estimates of heat kernel derivatives and applications

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**Abstract.** We describe how to obtain some probabilistic Bismut formulae for the derivatives of the heat kernel on a Riemannian manifold and give an application to the estimate of the energy in Euclidean Quantum Mechanics.

## 1. Introduction

$M$ shall denote a $d$-dimensional compact complete Riemannian manifold without boundary although generalizations (concerning compactness and boundary) are possible. With respect to the metric $ds^2 = \sum_{i,j} g_{i,j} dm^i dm^j$ the Laplace-Beltrami operator is defined by

$$\Delta = (detg)^{\frac{1}{2}} \frac{\partial}{\partial m^i}(g^{i,j} detg^{-\frac{1}{2}} \frac{\partial}{\partial m^j})$$

where $g^{i,j}$ denotes the inverse of the matrix $g_{i,j}$. Here again we could consider a more general operator by adding a first order term (a vector field) but we are more interested in explaining the ideas rather then consider full generality. There exists a huge number of works concerning estimates of the heat kernel associated with $\Delta$, namely the function $p_t(m_0, m)$ satisfying the p.d.e.

$$\frac{\partial p}{\partial t} = \frac{1}{2}\Delta p \quad \text{with} \quad p_t(m_0, m) \to \delta_{m_0}(m), t \to 0$$

A major insight on these problems is due to Kolmogorov ([10]), who associated with the Laplacian and, more generally, with an elliptic second order linear operator, a stochastic flow of diffeomorphisms, generalizing the well known fact that one can associate a deterministic flow with a vector field. The rôle of partial differential equations was since then made clear in the theory of Markov processes and, "reciprocally", stochastic processes turned out to be a central tool in the study of these equations.

We refer also to two works, that can be seen as landmarks on the subject: Varadhan's results ([17]) that essentially states the behaviour of the heat kernel for small times,

$$lim_{t\to 0}(-2tlog\ p_t(m_0,m)) = d^2(m_0,m)$$

and [11], where the authors prove (analytically) very precise estimates.

We are concerned here with estimates on the derivative of the heat kernel. Such estimates allow, in particular, to deduce the smoothness of the corresponding heat semigroup

$$(e^{\frac{t}{2}\Delta}f)(m_0) = \int_M f(m)p_t(m_0,m)dm$$

where $dm$ denotes the Riemannian volume measure. Here again many authors have considered these type of problems, concerning ponctual or $L^p$ estimates (cf., for example,[1],[7],[15],[16] and also [14], where large deviations arguments are used).

Bismut [1] showed that $t\nabla_{m_0}log\ p_t(m_0,m)$ can be expressed in terms of a conditional expectation of some stochastic process and used such an expression to study the small time assymptotics of the logarithmic derivative of the heat kernel. Other related formulae have been obtained since then (they are far from being unique).

The logarithmic derivative is quite a natural object to investigate. For example in (semiclassical) quantum physics, it has to do with the gradient of the action of the system under consideration.

The heat semigroup can be expressed as an expectation with respect to some stochastic process, in this case the $M$ valued Brownian motion $\rho_w(t)$, starting at $m_0$ at time zero,

$$(e^{\frac{t}{2}\Delta}f)(m_0) = Ef(\rho_w(t))$$

Therefore it is natural to think that derivatives of the heat kernel can be transfered to some derivatives on the path space of the process. And, in order to accomplish such task, one should use the stochastic calculus of variations on the path space or Malliavin calculus ([13]).

This paper is organized as follows: in the next paragraph we recall some notions of Malliavin calculus (in the flat situation, namely on the Wiener space) and, in particular, its integration by parts formula. This one can also be interpreted a rigorous version of Feynman's integration by parts formula (c.f. [2], Part 2) in quantum physics. In paragraph 3 we describe a construction of the Brownian motion on a Riemannian manifold, we consider the Itô map and present an intertwinning formula that allows to transfer derivatives on the path space of a manifold to derivatives on the Wiener space. In paragraph 4 we show how to deduce Bismut formula for the heat kernel derivative and, finally, the last paragraph is devoted to an application: estimating the energy in Euclidean quantum mechanics.

## 2. Malliavin calculus

Let $X$ denote the Wiener space, namely the space of continuous paths $\gamma : [0,1] \to \mathbb{R}^d, \gamma(0) = x_0$ endowed with the Wiener measure $\mu$. This measure is the law of the $\mathbb{R}^d$ valued Brownian motion and is associated to the Laplacian in the sense that the corresponding heat semigroup has the representation

$$(e^{\frac{t}{2}\Delta} f)(x_0) = E_\mu(f(\gamma(t)))$$

Let $H$ be the (Hilbert) subspace of $X$, named after Cameron and Martin, of the paths which are absolutely continuous and whose derivative satisfy $\int_0^1 |\dot{\gamma}(\tau)|^2 d\tau < \infty$. Although dense in $X$, the ($\mu$) measure of $H$ is zero.

For a cylindrical functional $F = f(\gamma(\tau_1),\ ..., \gamma(\tau_m))$ the Malliavin derivative ([13]) is defined by

$$D_\tau F(\gamma) = \sum_{k=1}^{m} \mathbf{1}_{\tau<\tau_k} \partial_k f(\gamma(\tau_1),\ ..., \gamma(\tau_m))$$

The operator $D$ is closed on the completion of the space of cylindrical functionals with respect to the norm $||F||^2 = E_\mu(|F|^2 + \int_0^1 |D_\tau F|^2 d\tau)$ and can therefore be extended to this space. For a "vector field" $z : X \to H$, we define the directional derivatives

$$D_z F = \int_0^1 < D_\tau F, \dot{z}(\tau) > d\tau$$

where $<,>$ denotes the scalar product in $\mathbb{R}^d$. They coincide with the more familiar limit (taken in the a.e.-$\mu$ sense) of $\frac{1}{\epsilon}(F(\gamma + \epsilon z) - F(\gamma))$ when $\epsilon \to 0$.

Girsanov-Cameron-Martin theorem states that, when $z$ is adapted to the increasing filtration $\mathcal{P}_\tau$ generated by the events before time $\tau$ (the Itô filtration), then a shift $\gamma \to \gamma + z$ induces a transformation of the Wiener measure to a measure which is absolutely continuous with respect to $\mu$ and we have an explicit formula for the Radon-Nikodym density (c.f. [18]):

$$E_\mu F(\gamma+z) = E_\mu\big(F(\gamma)exp\{\int_0^1 < \dot{z}, d\gamma(\tau) > -\frac{1}{2}\int_0^1 |\dot{z}|^2 d\tau\}\big)$$

The integral in $d\gamma$ is the Itô integral with respect to Brownian motion.

In the case of deterministic $z$, Cameron and Martin actually proved that belonging to $H$ is a necessary and sufficient condition for the shifted measure to be absolutely continuous with respect to $\mu$. It is therefore natural to consider variations with respect to $H$ valued functionals and therefore consider this space as a tangent space (which explains the terminology "vector fields" used before).

The dual of the derivative with respect to the measure $\mu$ is called the divergence operator. For adapted vector fields $z$ such that $E_\mu \int_0^1 |\dot{z}(\tau)|^2 d\tau < \infty$ it follows from Girsanov-Cameron-Martin theorem that the divergence coincides with the Itô integral and we have the following integration by parts formula:

$$E_\mu(D_z F) = E_\mu\big(F\int_0^1 < \dot{z}(\tau), d\gamma(\tau) >\big)$$

More generally the divergence coincides with an extension of this integral, the so-called Skorohod integral (c.f.[13] and references therein).

We notice that there is, in particular, a class of adapted transformations preserving the Wiener measure: these are the rotations (Levy's theorem). So, if $d\xi^i(\tau) = \sum_j a_{i,j} d\gamma(\tau)$ where $a \equiv a_{i,j}$ is an antisymmetric matrix, $a(0) = 0$, and if $D_\xi F = \int_0^1 D_\tau F d\xi(\tau)$, we have

$$E_\mu(D_\xi(F)) = 0$$

The tangent space to the Wiener space can therefore be extended to include processes $\xi$ satisfying a stochastic differential equation of the form

$$d\xi^i(\tau) = \sum_j a_{i,j} d\gamma^j(\tau) + z^i d\tau$$

with $a$ as above (the so-called tangent processes, c.f. [3]).

## 3. Brownian motion on a Riemannian manifold

If $M$ is a $d$-dimensional Riemannian manifold, let $O(M)$ denote the bundle of orthonormal frames over $M$, namely

$$O(M) = \{(m,r) : m \in M, r : \mathbb{R}^d \to T_m(M) \ \text{ is an Euclidean isometry}\}$$

and $\pi : O(M) \to M,\ \pi(r) = m$ the canonical projection. Let $m_k(t)$ denote the (unique) geodesic starting at $m$ at time zero and having initial velocity $r(e_k)$ with $e_k$,$k = 1,\ ..., d$, a vector in the canonical basis of $\mathbb{R}^d$. The parallel transport of $r$ along $m_k$ defined by the equation $\frac{dr_k}{dt} + \Gamma_{\dot{m}_k} r_k = 0,\ \ r_k(0) = Id$, where $\Gamma$ are the Christoffel symbols of the Levi-Civita connection, determines a vector field on $O(M)$: $A_k(m, r) = \frac{d}{dt}_{|t=0} r_k(t)$. We consider the horizontal Laplacian on $O(M)$, namely the second order differential operator

$$\Delta_{O(M)} = \sum_{k=1}^{d} A_k^2$$

Then

$$\Delta_{O(M)}(f o\pi) = (\Delta f)o\pi$$

where $\Delta$ is the Laplace-Beltrami operator on $M$. These two Laplacians induce two probability measures or two stochastic flows, defined in the path spaces of $O(M)$ and of $M$, respectively; and $\pi$ realizes an isomorphism between these probability spaces. The measure in

$$\mathbb{P}_{m_0}(M) = \{\rho : [0, 1] \to M, \rho \text{ continuous},\ \rho(0) = m_0\}$$

that we shall denote by $\nu$, is the Wiener measure, or the law of the Brownian motion on $M$ and satisfies $(e^{\frac{t}{2}\Delta_M} f)(m_0) = E_\nu(f(\rho(t)))$. The measure in $\mathbb{P}_{m_0}(O(M))$ corresponds to the law of $r(\tau)$, the (Itô stochastic) parallel displacement along the curve $\rho(\tau)$ with respect to the Levi-Civita connection. This lifted curve satisfies an stochastic differential equation of the form $dr_\gamma(\tau) = \sum_k A_k(r_\gamma(\tau)d\gamma^k(\tau),\ \ r_\gamma(0) = r_0$, with $\pi(r_0) = m_0$ (c.f., for example, [9]).

In [12] Malliavin defined the Itô map $I : X \to \mathbb{P}_{m_0}(M)$,

$$I(\gamma)(\tau) = \pi(r_\gamma(\tau))$$

and proved that $I$ is a.s. bijective and provides an isomorphism of measures.

A vector field along the path $\rho$ is a section process of the tangent bundle of $M$, namely a measurable map $Z_\rho(\tau) \in T_{\rho(\tau)}(M)$. We denote by $z$ the image of $Z$ through the parallel transport,

$$z(\tau) = r_0 o[r_\gamma(\tau)]^{-1}(Z(\tau))$$

and assume that $z$ belongs to the Cameron-Martin space $H$.

The derivative of a cylindrical functional $F = f(\rho(\tau_1),\ ..., \rho(\tau_m))$ along a vector field is given by

$$D_Z F(\rho) = \sum_{k=1}^{m} < r_0 o[r_\gamma(\tau_k)]^{-1} \partial_k f, Z(\tau_k) >$$

This derivative can be extended by closure to a suitable Sobolev space of functionals.

The theorem that follows expresses how derivatives in the path space can be transfered to derivatives in the Wiener space. We shall not be precise, here, in the assumptions, namely in the regularity needed for the functionals and the vector fields.

The result is a consequence of the formula for the derivative of the Itô map. Since a (stochastic) parallel transport along the Brownian motion is differentiated, the variation is given in terms of the integral of the curvature tensor along this curve.

**Theorem 1** (Intertwinning formula [3], [4], [6], [8])**.** *A functional $F$ is differentiable along a vector field $Z$ in $\mathcal{P}_{m_0}(M)$ iff $FoI$ is differentiable in $X$ along the process*

$$d\xi(\tau) = [\dot{z} + \frac{1}{2}Ricc(z)]d\tau - (\int_0^\tau \Omega(z, od\gamma))d\gamma(\tau)$$

*where $d\gamma$, $od\gamma$ denote, resp., Itô and Stratonovich stochastic differentiation (c.f.[9]), $\Omega$ and Ricci the curvature and the Ricci tensors in $M$. Furthermore we have:*

$$(D_Z F)oI = D_\xi(FoI)$$

From this theorem we can deduce, in particular, Bismut integration by parts formula:

$$E_\nu(D_Z F) = E_\mu\big((FoI)\int_0^1 [\dot{z}(\tau) + \frac{1}{2}\mathcal{R}_\tau(z(\tau))]d\gamma(\tau)\big)$$

with $\mathcal{R}_\tau$ the Ricci tensor read in the frame bundle. This result follows from the integration by parts formula on the Wiener space and from the fact that $\Omega$ is antisymmetric, the corresponding term in the intertwinning formula having therefore zero divergence.

# 4. Heat kernel derivatives

Given the probabilistic representation of the heat kernel, we differentiate this function by derivating the Brownian motion on $M$ in a convenient direction. Then we apply the intertwinning theorem to tranfer this derivative to the Wiener space (c.f. [5]). The following result can be obtained:

**Theorem 2** (Bismut formula)**.** *Let $f$ be a smooth function on $M$ and $v$ a vector in the tangent space $T_{m_0}(M)$. For fixed $t > 0$ and denoting $P_t f = e^{\frac{t}{2}\Delta} f$,* *we have:*

$$< \nabla P_t f, v >_{T_{m_0}(M)} = \frac{1}{t} E_\mu \big( f(\rho_\gamma(t)) \int_0^t [v + \frac{1}{2}(\tau - t)\mathcal{R}_\tau v] d\gamma(\tau) \big)$$

Idea of the proof:

Let $U$ be the solution of the o.d.e.

$$\frac{dU(\tau)}{d\tau} = -\frac{1}{2}\mathcal{R}_\tau U(\tau), \quad U(0) = Id_{T_{m_0}(M)}$$

Consider $y(\tau) = U(\tau)v - \frac{1}{t}(t - \tau \wedge t)v$, which is a Cameron-Martin vector field. From the intertwinning formula we derive

$$\begin{aligned} < \nabla P_t f, v >_{T_{m_0}(M)} &= E < r_\gamma(t) r_0^{-1} \nabla f, U(t)v >_{T_{m_0}(M)} \\ &= E < r_\gamma(t) r_0^{-1} \nabla f, y(t) >_{T_{m_0}(M)} \\ &= E(D_Y f(\gamma(t))) \end{aligned}$$

where $Y$ denotes the parallel transport of the vector $y$. The result follows from the integration by parts on the path space.

*Remark* 1. From this result we may, in particular, obtain $L^p$ estimates for the derivative of the heat semigroup. For example, in the situation where

$$||Ricc||_{L^p(dm)} = C_p < \infty$$

for every $p > 1$, we obtain

$$||\nabla P_t f||_{L^p(dm)} \leq (\frac{2}{t} + \frac{t}{6} C^2_{\frac{pq}{q-p}})^{\frac{1}{2}} ||f||^{\frac{p}{q}}_{L^q(dm)}$$

*Remark* 2. Bismut formula appears sometimes as a probabilistic expression for the logarithmic derivative $\nabla log p_t(m_0, m)$ (c.f. [1]). Such expressions can be obtained from the one in last theorem by taking conditional expectations on the underlying stochastic processes.

*Remark* 3. Formulae for derivatives of the heat kernel with respect to the second variable written in terms of stochastic integrals can also be deduced by similar methods.

## 5. An application

In Euclidean Quantum Mechanics (c.f. [2] and [18]) a family of stochastic processes is associated to the self-adjoint Hamiltonian observable $\mathcal{H} = -\frac{1}{2}\Delta +$

$V$, where $V$ denotes a bounded below scalar potential. These processes solve stochastic differential equations of the form (in local coordinates):

$$dz^i(t) = \hbar^{\frac{1}{2}}\sigma_{i,k}(z(t))d\gamma^k(t) - \frac{\hbar}{2}g^{j,k}\Gamma^i_{j,k}(z(t))dt + \hbar\partial_i log\eta_t(z(t))dt$$

where $\sigma = \sqrt{g}$ and with respect to the (usual) past Itô filtration and

$$d_*z^i(t) = \hbar^{\frac{1}{2}}\sigma_{i,k}d_*\gamma^k_*(t) - \frac{\hbar}{2}g^{j,k}\Gamma^i_{j,k}dt - \hbar\partial_i log\eta^*_t dt$$

with respect to the future filtration. Here $\eta$ and $\eta^*$ are, respectively, positive solutions of final and initial value problems for the heat equation with potential $V$. Considering time running in the interval $[0,T]$,

$$\eta_t(x) = e^{\frac{1}{\hbar}(t-T)\mathcal{H}}\eta_T$$

$$\eta^*_t(x) = e^{-\frac{1}{\hbar}t\mathcal{H}}\eta^*_0$$

The law of $z$ at time $t$ is absolutely continuous with respect to the volume measure and its density is $\eta_t\eta^*_t$.

In this framework the energy is defined (following Feynman) by

$$\mathcal{E} = -\frac{1}{2}|\hbar\nabla log\eta_t|^2 - \frac{\hbar^2}{2}div\nabla log\eta_t + V$$

or, in the other filtration, by

$$\mathcal{E}^* = -\frac{1}{2}|\hbar\nabla log\eta^*_t|^2 - \frac{\hbar^2}{2}div\nabla log\eta^*_t + V$$

We want to estimate the mean value of the energy along the trajectories of the process $z(t)$, namely the quantity

$$e(t) = E(\mathcal{E}(z(t)) = \int_M \mathcal{E}\eta_t\eta^*_t dm$$

which is also equal to $E(\mathcal{E}^*(z(t))$, and correspond to the path space counterparts of $< \psi|\mathcal{H}\psi >_{L^2(dm)}$ in quantum mechanics, for a state $\psi$. Since

$$\int_M (div\nabla log\eta^*_t)\eta_t\eta^*_t dm = -\int_M < \nabla log\eta^*_t, \nabla(\eta_t\eta^*_t) > dm$$

we have

$$e(t) = \frac{\hbar^2}{2}\int_M < \nabla\eta_t, \nabla\eta^*_t > dm + \int_M V\eta_t\eta^*_t dm$$

We observe that the energy (say, its $L^p$ norm) in this framework can be entirely estimated in terms of the heat kernel and its derivatives, together with the initial and final conditions and the assumptions on the potential $V$.

In the absence of the potential (if $V$ is different from zero we should introduce a Feynman-Kac representation for the corresponding semigroups) and in the situation of Remark 2. of the last paragraph, we can obtain, for example, the following estimation:

$$|e(t)|^4 \leq \frac{\hbar^8}{2^4}\Big(\frac{2\hbar}{T-t} + \frac{T-t}{6\hbar}C_2\Big)\Big(\frac{2\hbar}{t} + \frac{t}{2\hbar}C_2\Big)||\eta_T||^2_{L^2(dm)}||\eta_0^*||^2_{L^2(dm)}$$

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# An aperçu of harmonic maps and morphisms

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**Abstract.** We present an overview of the theories of harmonic maps and harmonic morphisms and describe three facets which we hope to be particularly relevant to Mathematical Physics: vector fields, Einstein manifolds and Weyl geometry.

*Keywords:* Harmonic maps, harmonic morphisms, Einstein manifolds, Weyl geometry.

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## 1. Harmonic maps

The aim of the theory of harmonic maps is to single out the "best" maps between Riemannian manifolds, as measured by the energy functional:

$$E(\phi, D) = \frac{1}{2}\int_D |d\phi|^2\, v_g,$$

for a map $\phi : (M^m, g) \to (N^n, h)$ and a compact subset $D$ of $M$. The critical points of $E$ (for all compact subsets $D$ if $M$ is not compact itself) are called *harmonic maps*. The associated Euler-Lagrange equation, $\tau(\phi)$, called the *tension field*, is obtained by considering a one-parameter variation of $\phi$, $\phi_t : ]-\epsilon, +\epsilon[\times M^m \to N^n$ with $\phi_0 = \phi$ and $V = \frac{\partial\phi_t}{\partial t}\Big|_{t=0}$, the section tangent to the variation, and computing:

$$\frac{\partial E(\phi_t)}{\partial t}\Big|_{t=0} = -\int_D < V, \tau(\phi) >\; v_g,$$

with $\tau(\phi) = \mathrm{trace}_g \nabla d\phi$. Clearly a map $\phi$ is harmonic if and only if $\tau(\phi) = 0$. Equipping $M^m$ and $N^n$ with local coordinates $(x^i)_{i=1,\dots,m}$ and $(y^\alpha)_{i=1,\dots,n}$, yields a local expression for $\tau(\phi)$:

$$\tau^\alpha(\phi) = \sum_{i,j=1}^m g^{ij} \frac{\partial^2 \phi^\alpha}{\partial x^i \partial x^j} - {}^M\Gamma_{ij}^k \frac{\partial \phi^\alpha}{\partial x^k} + \sum_{\beta,\gamma=1}^n {}^N\Gamma_{\beta\gamma}^\alpha \frac{\partial \phi^\beta}{\partial x^i} \frac{\partial \phi^\gamma}{\partial x^j} \quad \alpha = 1, \dots, n$$

A lot of ready-made examples are at our disposal: harmonic functions ($N = \mathbb{R}$), harmonic 1-forms with integral periods ($N = \mathbb{S}^1$), geodesics ($M = \mathbb{R}$ or $\mathbb{S}^1$), holomorphic maps between complex linear spaces, totally geodesic maps, i.e. such that $\nabla d\phi = 0$, the Hopf maps $\mathbb{S}^3 \to \mathbb{S}^2$, $\mathbb{S}^7 \to \mathbb{S}^4$ and $\mathbb{S}^{15} \to \mathbb{S}^8$. Note that, in general, the composition of harmonic maps is not a harmonic map. The functional $E$ is invariant by conformal change on a two-dimensional domain, and therefore, harmonic maps from Riemann surfaces are meaningful. Unfortunately for the existence of harmonic maps, $E$ satisfies the Palais-Smale condition only when the domain has dimension one, so Eells and Sampson have had to revert to the flow of the associated heat equation to prove:

**Theorem 1.** [12] *Let $M$ and $N$ be compact Riemannian manifolds, if $N$ has non-positive sectional curvature, then, in each homotopy class, there exists a harmonic map.*

Remark that the curvature condition on $N$ means that its universal cover is diffeomorphic to $\mathbb{R}^n$. When $M$ admits a non-empty boundary and $N$ is compact, Hamilton extended Theorem 1 to homotopy classes of fixed boundary values.

Some years later Eells and Ferreira discovered that allowing conformal changes of the metric lifted the curvature restriction.

**Theorem 2.** [8] *Let $(M, g)$ ($\dim M \geq 3$) and $(N, h)$ be compact and $\phi : M \to N$, then there exists $\tilde{g}$ conformal to $g$ and $\tilde{\phi}$ homotopic to $\phi$, with $\tilde{\phi} : (M, \tilde{g}) \to (N, h)$ harmonic.*

Of course, one attractive feature of harmonic maps is that they do not always exist even in some simple cases, like the homotopy class of Brower degree $\pm 1$ between $\mathbb{T}^2$ and $\mathbb{S}^2$ [13], in fact a topological obstruction, the nontrivial class from $\mathbb{P}^2$ to $\mathbb{S}^2$ [9], or the non-zero sections of vector bundles with Euler characteristic different from zero [19] (we will elaborate this point later on), or from a compact to a non-compact complete manifold of non-negative sectional curvature, like a paraboloid of revolution in $\mathbb{R}^3$.
The case conspicuously missing from the Eells-Sampson theorem is of maps between spheres. In spite of many remarkable works, this question of harmonic representatives in homotopy classes remains open for most dimensions,

the cases of positive answer being (cf. [16]):
$\pi_n(\mathbb{S}^n) = \mathbb{Z}$, $n = 1, 2, \ldots, 7, 9, 10, 11, 17$, $\quad \pi_{n+1}(\mathbb{S}^n) = \mathbb{Z}_2$, $n \geq 3$
$\pi_{n+2}(\mathbb{S}^n) = \mathbb{Z}_2$, $n = 5, 6, 7, 8, 13, n \geq 5$ odd
$\pi_{n+3}(\mathbb{S}^n) = \mathbb{Z}_{24}$, $n = 7, 8$, $\quad \pi_7(\mathbb{S}^3) = \mathbb{Z}_2$, $\quad \pi_{15}(\mathbb{S}^9) = \mathbb{Z}_2$

Among the best known properties of harmonic maps are:

• A continuous weakly harmonic map (i.e. a weak solution to $\tau(\phi) = 0$) is smooth ([9]).
• A weakly harmonic map from a surface is smooth ([17]).
• Strong unique continuation: if, at a point, all the derivatives of a harmonic map vanish then it is constant ([28]).
• For an isometric immersion into a Euclidean space, having constant mean curvature is equivalent to its Gauss map being harmonic ([27]).
• An isometric immersion is minimal if and only if it is harmonic ([12].
• Conservation law: the divergence of the stress energy, $S_\phi = \frac{1}{2}|d\phi|^2 g - \phi^* h$, i.e. the Euler-Lagrange equation of $E$ for variations of the metric, of a harmonic map is zero ([3]).

As any functional, $E$ admits a second variation given by the Jacobi operator:

$$J^\phi(V) = \Delta^\phi V + \mathrm{trace}_g \mathrm{Riem}^N(d\phi, V)d\phi$$

A harmonic map is called stable if $\int_M < J^\phi(V), V > v_g$ is positive for any section tangent to $\phi$. Clearly if $M$ is compact and $\mathrm{Riem}^N$ is negative then any harmonic map from $M$ to $N$ is stable, but a harmonic map to ([29]) or from ([22]) a sphere of dimension at least three, is unstable, for example the Hopf maps or the identity of spheres (but it is stable for $\mathbb{S}^2$).

As to the global question, weakly conformal maps between surfaces are minimisers and, more generally, so are holomorphic maps between compact Kähler manifolds, but, on the other hand, the infimum of the energy of any homotopy class from $\mathbb{S}^m$ ($m \geq 3$) to a compact manifold is zero, ruling out the possibility of a minimum, except in the trivial homotopy class.

For more information, we refer to the surveys [9, 10, 11].

## 2. Harmonic morphisms

Though introduced to generalise holomorphic maps between Riemann surfaces, harmonic morphisms originate from Potential theory on Brelot spaces, and suffered, at first, from a very non-geometrical definition:

**Definition 1.** A map $\phi : (M, g) \to (N, h)$ is a harmonic morphism if whenever $f : U \subset N \to \mathbb{R}$ is harmonic and $\phi^{-1}(U) \neq \emptyset$ then $f \circ \phi : \phi^{-1}(U) \subset M \to \mathbb{R}$ is also harmonic.

The simplest examples of harmonic morphisms are weakly conformal maps between Riemann surfaces, holomorphic maps between complex Euclidean spaces and the Hopf maps.

Fortunately, and rather mysteriously, Fuglede [14] and Ishihara [20] independently showed that such maps are exactly the harmonic maps enjoying a property, geometrical in essence, called *horizontal weak conformality* (HWC), defined by: $d\phi_x \equiv 0$ (critical point) or $d\phi_x : H_x = \ker(d\phi_x)^{\perp} \to T_{\phi(x)}N$ is surjective and conformal, of factor $\lambda^2(x)$ (regular point). This immediately implies that, unless $\phi$ is constant, $\dim M$ must be greater than $\dim N$. This dual nature of harmonic morphisms was later on linked to the minimality of the fibres:

**Theorem 3.** [3] *Let $\phi : (M, g) \to (N, h)$ be a HWC map.*
*If $\dim(N) = 2$ then $\phi$ is harmonic if and only if its fibres are minimal.*
*If $\dim(N) \geq 3$ then two assertions imply the third:*
- *$\phi$ is harmonic.*
- *$d\phi(\nabla\lambda^2) = 0$.*
- *the fibres are minimal.*

Moreover, a simple chain rule shows that harmonic morphisms preserve the tension field i.e. if $\phi : M \to N$ is a harmonic morphism and $\psi : N \to P$ then $\tau(\psi \circ \phi) = \lambda^2 \tau(\psi)$.
The relation between the two faces of harmonic morphisms is more direct for polynomials.

**Theorem 4.** [1] *A HWC polynomial mapping between Euclidean spaces is automatically harmonic.*

This has consequences in non-flat situations as well, since from any harmonic morphism one can construct a polynomial harmonic morphisms between Euclidean spaces, in fact its symbol map, and, using the strong Louiville property, deduce that if $\dim M < 2\dim N - 2$ then a harmonic morphism $\phi : M \to N$ must be a submersion. In particular, this implies the non-existence of (non-constant) harmonic morphisms from $\mathbb{S}^{n+1}$ to $\mathbb{S}^n$ ($n \geq 4$) or $\mathbb{S}^{2k}$ to $N^{2k-1}$ ($k \geq 3$). Another property of global implications is the openness of harmonic morphisms ([14]), since it forces a harmonic morphism $\phi$ from a compact manifold to be a surjection onto another compact manifold, hence $\phi^*$ is injective between the first cohomology classes and the first Betti number of the domain must be greater than the one of the target, ruling out, for example, harmonic morphisms from a sphere to a torus. From a three-dimensional space form to a surface, Baird and Wood obtained a classification (up to a conformal transformation of the target):

**Theorem 5.** [4] *Let $\phi : M^3 \to N^2$ be a globally defined harmonic morphism then:*
*If $M^3 = \mathbb{R}^3$ then $\phi$ is the orthogonal projection $\mathbb{R}^3 \to \mathbb{R}^2$.*
*If $M^3 = \mathbb{S}^3$ then $\phi$ is the Hopf map $\mathbb{S}^3 \to \mathbb{S}^2$.*
*If $M^3 = \mathbb{H}^3$ then $\phi$ is the orthogonal projection $\mathbb{H}^3 \to \mathbb{D}^2$ or the projection to the plane at infinity $\mathbb{H}^3 \to \mathbb{C}$.*

While it is not difficult to see that harmonic morphisms from a compact manifold to a manifold with negative Ricci curvature must be stable, one can also show that their stability is linked to their geometry:

**Theorem 6.** • [24] *If a submersive harmonic morphism from a compact manifold into a surface, has its fibres volume-stable then the map is energy-stable.*
• [7] *A stable harmonic map from a compact manifold to $\mathbb{S}^2$ is a harmonic morphism.*

The first result can be illustrated by the Hopf map $\mathbb{S}^3 \to \mathbb{S}^2$ which is unstable, but, being quadratic, quotients to a stable harmonic morphism $\mathbb{P}^3 \to \mathbb{S}^2$.

In 1996, Fuglede extended, in [15], the characterisation of harmonic morphisms to semi-Riemannian geometry by adding to the definition of horizontal weak conformality, the condition: "if $\ker d\phi_x$ is degenerate, then $(\ker d\phi_x)^\perp \subset \ker d\phi_x$".

## 3. Einstein manifolds

Work on the classification of harmonic morphisms with one-dimensional fibres from an Einstein manifold, i.e. a Riemannian manifold $(M, g)$ such that $\mathrm{Ricci} = cg$, is due to Pantilie and Wood and stems from efforts to generalise a result of Bryant:

**Theorem 7.** [6] *A harmonic morphism $\phi : M^{n+1} \to N^n$, where $M^{n+1}$ is of constant sectional curvature, is of Killing or warped product type.*

Three types of one-dimensional harmonic morphisms can be singled out:
• Killing type: fibres are (locally) tangent to a Killing vector field (equivalently $\mathrm{grad}(\lambda) \in H$).
• Warped product type: they are horizontally homothetic and have totally geodesic fibres and integrable horizontal distribution. Locally, this makes $\phi$ the projection of a warped product.
• Type (T): $|(\mathrm{grad}\lambda)^V| = F(\lambda)\,(F \neq 0)$.
While the first is exclusive of the others, the last two can combine.

In general, integrability of the horizontal distribution is required:

**Theorem 8.** [4] *Let $\phi : (M^{n+1}, g) \to (N^n, h)$ be a horizontally homothetic harmonic morphism with integrable horizontal distribution then:*
*• $M$ is Einstein if and only if $N$ is Einstein, in which case $\Delta \ln \lambda = c^M - \lambda^2 c^N$, $c^M$ and $c^N$ being the Einstein constants of $(M^{n+1}, g)$ and $(N^n, h)$, respectively.*
*• If $M$ is compact, then $c^M = c^N = 0$, $\lambda$ is constant and $\phi$ is the projection of a Riemannian product.*

When restricted to low dimension, the three types are the only possible:

**Theorem 9.** [25, 26] *Let $\phi : (M^4, g) \to (N^3, h)$ be a harmonic morphism between orientable manifolds, with $(M^4, g)$ Einstein, then $\phi$ is of one of the three types.*
*If it is of type (T) then $(M^4, g)$ is Ricci-flat and $(N^3, h)$ has non-negative constant sectional curvature $K^N = k^2$. The metric $g$ has the normal form:*

$$g = \lambda^{-2}\phi^* h + \lambda^2 \theta^2,$$

*where $\lambda$ is the dilation of $\phi$ and $\theta = \frac{1}{2k} d(\lambda^{-2}) + \phi^*\alpha$ with $\alpha$ a 1-form on $N$ satisfying, with respect to some orientation on $N$:*

$$d\alpha + 2k * \alpha = 0.$$

*If $M^4$ is compact then $\phi$ is the projection $\mathbb{T}^4 \to \mathbb{T}^3$.*
*When $(N^3, h)$ is of constant curvature, each of the three types leads to, non-equivalent, constructions of Einstein metrics.*

## 4. Weyl geometry

By the sheer conformality of its nature, the condition HWC begs for Weyl geometry. Moreover, while harmonicity is not conformally invariant, unless mapping between surfaces, among the constituents of the tension field equation, $\mathrm{trace}\nabla d\phi = 0$, the connection is the only one not to be conformally invariant. This can be remedied by replacing the Levi-Civita connection by the (torsion-free) Weyl connection $D$ of a conformal manifold $(M, c)$, i.e. $\forall g \in c, \exists$ real one-form $\alpha_g$, called the Lee form, such that:

$$Dg = -2\alpha_g \otimes g \quad \text{(the } -2 \text{ is a non-canonical convention)}.$$

$D$ is the Levi-Civita connection of some metric if and only if $\alpha_g = df$, and $(M, c, D)$ is called a Weyl manifold. A map $\phi : (M, c, D^M \to (N, D^N)$ is harmonic if $\mathrm{trace}_c D d\phi = 0$, where $D$ is the connection of $\phi^{-1}TN \otimes T^*M$ induced by $D^M$ and $D^N$. Then the definition of a harmonic morphism easily extends

to Weyl geometry by decreeing that $\phi : (M, c_M, D^M) \to (N, c_N, D^N)$ is a harmonic morphism between Weyl manifolds if for any $f : U \subset (N, c_N, D^N) \to \mathbb{R}$ harmonic, then $f \circ \phi : (M, c_M, D^M) \to \mathbb{R}$ is also harmonic. Here $\mathbb{R}$ is equipped with its canonical conformal structure. If $\dim N = 2$ then a function on $(N, c_N, D^N)$ is harmonic if and only if it is harmonic w.r.t. to any local representative of $c^N$, since, on a surface, harmonicity is conformally invariant.
The Fuglede-Ishihara characterisation and the Baird-Eells theorem follow suit:
i) $\phi$ is a harmonic morphism if and only if it is a horizontally weakly conformal harmonic map.
ii) Let $\phi : (M, c_M, D^M) \to (N, c_N, D^N)$ be a horizontally weakly conformal submersion.
If $\dim N = 2$ then $\phi$ is harmonic if and only if the fibres are minimal w.r.t. $D^M$.
If $\dim N \geq 3$ then two assertions imply the third:
- $\phi$ is harmonic (morphism).
- the fibres are minimal w.r.t. $D^M$.
- $HD^M = D^N$.

The picture becomes even more interesting when one observes that the results hold in the complex-Riemannian category (as introduced by LeBrun in [21]) and that twistor structures can be mixed in. By a twistor structure on $M$, we mean an integrable foliation $\mathcal{F}$ on a complex manifold $P$ such that $\pi : P \to M$ is a proper complex analytic submersion and $\mathcal{F} \cap (\ker d\pi) = \{0\}$, while twistorial maps preserve twistorial structures, or, equivalently, pull back twistorial functions (locally of the type "$\pi$ composed with a complex analytic function") onto twistorial functions. These definitions are driven by three cases:
i) [21] Let $M^2$ be a 2-dimensional complex Riemannian manifold and $\pi : P \to M^2$ the bundle of null directions, then the pull-back of local null geodesics is a twistor structure.
ii) [18] Let $(M^3, c, D)$ be a Weyl manifold and $\pi : P \to M^3$ the bundle of null planes, then a twistor structure is given by $\mathcal{F}_p \subset T_pP$ being the horizontal lift of $p$ at $p \in P$.
iii) [2] Let $(M^4, c)$ be a conformally flat manifold and $\pi : P \to M^4$ the bundle of null planes in $M^4$, then again, taking $\mathcal{F}_p \subset T_pP$ as the horizontal lift of $p$ at $p \in P$, gives a twistor structure.

These definitions combine with the preceding results to give:

**Theorem 10.** [23] *Let $(M^4, c_M, D^M)$ be an Einstein-Weyl manifold, $(N^2, c_N)$ a conformal manifold and $\phi : (M^4, c_M, D^M) \to (N^2, c_N)$ a submersive harmonic morphism with nowhere degenerate fibres.*

*If $(M^4, c_M)$ is orientable then $\phi$ is twistorial.*
*If $(M^4, c_M)$ is non-orientable then $\phi$ has totally umbilical fibres.*

**Theorem 11.** [23] *Let $(M^4, c_M, D^M)$ and $(N^3, c_N, D^N)$ be Einstein-Weyl manifolds and $\phi : (M^4, c_M, D^M) \to (N^3, c_N, D^N)$ a submersive harmonic morphism with nowhere degenerate fibres.*
*If $(M^4, c_M)$ is orientable then it is anti-self-dual and $\phi$ is twistorial.*
*If $(M^4, c_M)$ is non-orientable then it is conformally flat, its horizontal distribution is integrable and the fibres of $\phi$ are locally generated by conformal vector fields whose orbits are geodesics with respect to $D^M$.*

## 5. Vector fields

A setting potentially rich in examples is to take the target to be the tangent bundle of the domain, and, more generally, everything in this section can be adapted to Riemannian vector bundles. In this situation three variational problems can be posed for the energy functional $E$, the central question of the choice of the metric on the target will be addressed further down. One can search for critical points of $E$ for:
1) variations of any type; i.e. the harmonic map problem.
2) variations restricted to vector fields, to obtain harmonic vector fields.
3) variations through unit (or constant) vector fields, called unit harmonic vector fields.
To describe the metrics on $TM$, we need to recall the canonical decomposition of the tangent space of the tangent bundle of a Riemannian manifold $(M, g)$:
Let $\pi : TM \to M$ be the natural projection and $V$, the vertical space, be the kernel of $d\pi : TTM \to TM$. Its complement in $TTM$, the horizontal space, $H \subset TTM$, is obtained by considering all the curves on $M$ and the parallel vector fields (w.r.t. the Levi-Civita connection on $(M, g)$) along these curves, these, in turn, form curves on $TM$ whose tangent vectors make up $H$. It can also be seen as the kernel of the connection map $K : TTM \to TM$, characterised by $K(dZ(X)) = \nabla_X Z$.
If $TM$ is equipped with a metric such that $\pi : TM \to M$ is a Riemannian submersion with totally geodesic fibres then from the tension field of the first variational problem (i.e. all possible variations), one can deduce the Euler-Lagrange equations of the other two by, for harmonic vector fields, taking only the vertical part of the tension field (owing to the fact that a vector tangent to a variation through sections must be vertical) and, for unit harmonic vector fields, by asking that the vertical part of the tension field be colinear to the vector field itself, instead of zero.
As to the Riemannian metric on $TM$, the two main suggestions, the Sasaki

and Cheeger-Gromoll metrics, have both the disadvantage that, when the base is compact, harmonic vector fields (and a fortiori vector fields which are harmonic maps) must be parallel, a topological obstruction controlled by the Euler characteristic.

To widen the possibilities, a two-parameter family of metrics has been introduced in [5]:

$$h_{m,r}(A,B) = g(d\pi(A), d\pi(B)) + \frac{1}{(1+|e|^2)^m}\left\{g(KA,KB) + rg(KA,e)g(KB,e)\right\},$$

for $e \in TM$ and $A, B \in T_eTM$. With respect to this metric, $\sigma \in C(TM)$ is a harmonic section if it satisfies:

$$(1+|\sigma|^2)\nabla^*\nabla\sigma + 2m\nabla_{X(\sigma)}\sigma = \left(m|\nabla\sigma|^2 - mr|X(\sigma)|^2 - r(1+|\sigma|^2)\Delta(\frac{1}{2}|\sigma|^2)\right)\sigma$$

If we restrict the metric $h_{m,r}$ to a sphere bundle (i.e. to vectors of constant norm) then $g(KA,e) = 0$ and the condition for $\sigma$ to be a critical point of the energy among maps of constant norm equal to $k$ is rather simply:

$$\nabla^*\nabla\sigma = \frac{1}{k2}|\nabla\sigma|^2\sigma,$$

so there is no restriction on the values of $m$ and $r$ for the existence of unit (or $k^2$) $(m,r)$-harmonic sections.
On the other hand, the rigidity of the Sasaki ($m = r = 0$) and Cheeger-Gromoll ($m = r = 1$) metrics extends to any value of $m \in [0,1]$ and $r \geq 0$, that is any harmonic vector field must be parallel [5], but, as soon as $m \geq 1$, there exist non-parallel harmonic vector fields for any $(m,r)$, namely $\sigma = \zeta/\sqrt{m-1}$, where $\zeta$ is the Hopf vector field on the sphere $\mathbb{S}^{2p+1}$. Besides conformal gradient vector fields defined on $\mathbb{S}^n$ by $\sigma = \nabla\lambda$, where $\lambda : \mathbb{S}^n \to \mathbb{R}$, $\lambda(x) = a.x$, are harmonic vector fields for $m = n+1$ and $r = 2-n$ ($n \geq 3$).

On generic domains, we compensate non-compactness by the harmonicity of the norm:

**Theorem 12.** [5] *Let $\sigma : (M,g) \to (TM, h_{m,r})$ be a harmonic vector field such that $|\sigma|^2$ is harmonic and $(1-m)|\sigma|^2 \geq -1$. If $m > 1$ and $r \geq 2(1-m)$ then $\sigma$ must be parallel.*

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# Tulczyjew's triple associated with a Lie affgebroid and a Hamiltonian section

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**Abstract.** In this paper we introduce the so-called Tulczyjew's triple associated with a Lie affgebroid and a Hamiltonian section.

*Keywords:* Lie affgebroids, prolongation of Lie algebroids, action Lie algebroids, canonical involution, Tulczyjew's triple.

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## 1. Introduction

In very recent paper [7], the authors introduced the notion of a Lagrangian submanifold of a symplectic Lie algebroid and they proved that the Lagrangian (Hamiltonian) dynamics on a Lie algebroid $E$ may be described in terms of Lagrangian submanifolds of symplectic Lie algebroids. In this description, a special geometrical construction plays an important role: the so-called Tulczyjew's triple associated with the Lie algebroid $E$. When $E$ is the standard Lie algebroid $TM$ one recovers some well-known results which were proved by Tulczyjew [11, 12].
On the other hand, in [9] (see also [3]) the authors proposed a possible generalization of the notion of a Lie algebroid to affine bundles in order to build a geometrical model for a time-dependent version of Lagrange (Hamilton) equations on Lie algebroids. The resultant mathematical structures are called Lie affgebroids in the terminology of [3].
In this Note we will introduce the so-called Tulcyjew's triple associated with a Lie affgebroid $A$ and a Hamiltonian section. The Note is organized as follows.

In Section 2, we will recall some definitions and results on Lie algebroids (affgebroids) which will be useful in the rest of the paper. In Section 3, we will introduce the notion of a Lie affgebroid morphism and we will show some relations between Lie algebroid and Lie affgebroid morphisms. In Section 4, we will see that a canonical involution may be associated with an arbitrary Lie affgebroid. Finally, in Section 5, we will introduce the so-called Tulczyjew's triple for a Lie affgebroid and a Hamiltonian section.
Proofs of the the results contained in this Note may be found in a forthcoming paper (see [1]). In this paper, we will also see that Tulczyjew's triple associated with a Lie affgebroid and a Hamiltonian section plays an important role (as in the time-independent case) in some interesting descriptions of Lagrangian (Hamiltonian) dynamics on Lie affgebroids.

# 2. Lie algebroids and Lie affgebroids

## 2.1. Lie algebroids

Let $\tau : E \to M$ be a vector bundle of rank $n$ over the manifold $M$ of dimension $m$. Denote by $\Gamma(\tau)$ the $C^\infty(M)$-module of sections of $\tau : E \to M$. A *Lie algebroid structure* $(\llbracket\cdot,\cdot\rrbracket, \rho)$ on $E$ is a Lie bracket $\llbracket\cdot,\cdot\rrbracket$ on the space $\Gamma(\tau)$ and a bundle map $\rho : E \to TM$, *the anchor map*, such that if we also denote by $\rho : \Gamma(\tau) \to \mathfrak{X}(M)$ the induced homomorphism of $C^\infty(M)$-modules then

$$\llbracket X, fY\rrbracket = f\llbracket X, Y\rrbracket + \rho(X)(f)Y, \text{ for } X, Y \in \Gamma(\tau) \text{ and } f \in C^\infty(M).$$

The triple $(E, \llbracket\cdot,\cdot\rrbracket, \rho)$ is called a *Lie algebroid over* $M$ (see [8]).
If $(E, \llbracket\cdot,\cdot\rrbracket, \rho)$ is a Lie algebroid, one may define *the differential of* $E$, $d^E : \Gamma(\wedge^k\tau^*) \longrightarrow \Gamma(\wedge^{k+1}\tau^*)$ and *the Lie derivative with respect to* $X \in \Gamma(\tau)$, $\mathcal{L}^E_X : \Gamma(\wedge^k\tau^*) \longrightarrow \Gamma(\wedge^k\tau^*)$, as the following operators

$$\begin{aligned}
(d^E\mu)(X_0, \dots, X_k) &= \sum_{i=0}^{k}(-1)^i\rho(X_i)(\mu(X_0, \dots, \widehat{X_i}, \dots, X_k)) \\
&+ \sum_{i<j}(-1)^{i+j}\mu(\llbracket X_i, X_j\rrbracket, X_0, \dots, \widehat{X_i}, \dots, \widehat{X_j}, \dots, X_k), \\
\mathcal{L}^E_X\mu &= i_X(d^E\mu) + d^E(i_X\mu),
\end{aligned}$$

for $\mu \in \Gamma(\wedge^k\tau^*)$ and $X_0, \dots, X_k \in \Gamma(\tau)$.
If $E$ is the standard Lie algebroid $TM$ then the differential $d^E = d^{TM}$ is the usual exterior differential associated with $M$ which we will denote by $d_0$.
Now, suppose that $(E, \llbracket\cdot,\cdot\rrbracket, \rho)$ and $(E', \llbracket\cdot,\cdot\rrbracket', \rho')$ are Lie algebroids over $M$ and $M'$, respectively, and that $(F : E \to E', f : M \to M')$ is a morphism between

the vector bundles $E$ and $E'$, i.e., $\tau' \circ F = f \circ \tau$. Then $(F, f)$ is said to be a *Lie algebroid morphism* if $d^E((F, f)^*\phi') = (F, f)^*(d^{E'}\phi')$, for $\phi' \in \Gamma(\wedge^k(\tau')^*)$. Note that $(F, f)^*\phi'$ is the section of the vector bundle $\wedge^k\tau^* \to M$ defined by

$$((F, f)^*\phi')_x(a_1, \dots, a_k) = \phi'_{f(x)}(F(a_1), \dots, F(a_k)), \text{ for } x \in M \text{ and } a_i \in E_x.$$

If $(F, f)$ is a Lie algebroid morphism, $f$ is an injective inmersion and $F_{/E_x} : E_x \to E'_{f(x)}$ is injective, for all $x \in M$, then $(E, [\![\cdot,\cdot]\!], \rho)$ is said to be a *Lie subalgebroid* of $(E', [\![\cdot,\cdot]\!]', \rho')$.

## The prolongation of a Lie algebroid over a fibration

In this section, we will recall the definition of the Lie algebroid structure on the prolongation of a Lie algebroid over a fibration (see [6, 7]).
Let $(E, [\![\cdot,\cdot]\!], \rho)$ be a Lie algebroid of rank $n$ over a manifold $M$ of dimension $m$ and $\pi : M' \to M$ be a fibration. We consider the subset of $E \times TM'$

$$\mathcal{L}^\pi E = \{(b, v') \in E \times TM'/\rho(b) = (T\pi)(v')\},$$

where $T\pi : TM' \to TM$ is the tangent map to $\pi$. Denote by $\tau^\pi : \mathcal{L}^\pi E \to M'$ the map given by $\tau^\pi(b, v') = \pi_{M'}(v')$, $\pi_{M'} : TM' \to M'$ being the canonical projection. If $\dim M' = m'$, one may prove that $\mathcal{L}^\pi E$ is a vector bundle over $M'$ of rank $n + m' - m$ with vector bundle projection $\tau^\pi : \mathcal{L}^\pi E \to M'$.
A section $\widetilde{X}$ of $\tau^\pi : \mathcal{L}^\pi E \to M'$ is said to be *projectable* if there exists a section $X$ of $\tau : E \to M$ and a $\pi$-projectable vector field $U' \in \mathfrak{X}(M')$ to the vector field $\rho(X)$ such that $\widetilde{X}(x') = (X(\pi(x')), U'(x'))$, for all $x' \in M'$. For such a projectable section $\widetilde{X}$, we will use the following notation $\widetilde{X} \equiv (X, U')$. It is easy to prove that one may choose a local basis of projectable sections of the space $\Gamma(\tau^\pi)$.
The vector bundle $\tau^\pi : \mathcal{L}^\pi E \to M'$ admits a Lie algebroid structure $([\![\cdot,\cdot]\!]^\pi, \rho^\pi)$. Indeed, if $(X, U')$ and $(Y, V')$ are projectable sections then

$$[\![(X, U'), (Y, V')]\!]^\pi = ([\![X, Y]\!], [U', V']), \quad \rho^\pi(X, U') = U'. \tag{1}$$

$(\mathcal{L}^\pi E, [\![\cdot,\cdot]\!]^\pi, \rho^\pi)$ is *the prolongation of the Lie algebroid $E$ over the map* $\pi$ (for more details, see [6, 7]).
Now, let $\tau : E \to M$ be a Lie algebroid over a manifold $M$ and $\mathcal{L}^{\tau^*}E$ be the prolongation of $E$ over the projection $\tau^* : E^* \to M$. $\mathcal{L}^{\tau^*}E$ is a Lie algebroid over $E^*$ and we can define a canonical section $\lambda_E$ of the vector bundle $(\mathcal{L}^{\tau^*}E)^* \to E^*$ as follows. If $a^* \in E^*$ and $(b, v) \in (\mathcal{L}^{\tau^*}E)_{a^*}$ then

$$\lambda_E(a^*)(b, v) = a^*(b). \tag{2}$$

$\lambda_E$ is called the *Liouville section of* $(\mathcal{L}^{\tau^*}E)^*$.
Now, one may introduce the section $\Omega_E$ of the vector bundle $\Lambda^2(\mathcal{L}^{\tau^*}E)^* \to E^*$ given by $\Omega_E = -d^{\mathcal{L}^{\tau^*}E}\lambda_E$. One may prove that $\Omega_E$ is a non-degenerate 2-section and, moreover, it is clear that $d^{\mathcal{L}^{\tau^*}E}\Omega_E = 0$. In other words, $\Omega_E$ is a symplectic section. $\Omega_E$ is called *the canonical symplectic section* associated with the Lie algebroid $E$ (for more details, see [7]).

**Action Lie algebroids**

In this section, we will recall the definition of the Lie algebroid structure of an action Lie algebroid (see [6, 7]).
Let $(E, [\![\cdot,\cdot]\!], \rho)$ be a Lie algebroid over a manifold $M$ and $f : M' \to M$ be a smooth map. Denote by $f^*E \subseteq M' \times E$ the pull-back of $E$ over $f$.
Now, suppose that $\Psi : \Gamma(\tau) \to \mathfrak{X}(M')$ is an action of $E$ on $f$, that is, $\Psi$ is a $\mathbb{R}$-linear map which satisfies the following conditions

$$\Psi(hX) = (h \circ f)\Psi X, \quad \Psi[\![X, Y]\!] = [\Psi X, \Psi Y], \quad \Psi X(h \circ f) = \rho(X)(h) \circ f,$$

for $X, Y \in \Gamma(\tau)$ and $h \in C^\infty(M)$. Then, one may introduce a Lie algebroid structure (*the action Lie algebroid structure*) $([\![\cdot,\cdot]\!]_\Psi, \rho_\Psi)$ on the vector bundle $pr_{1|f^*E} : f^*E \subseteq M' \times E \to M'$ which is characterized as follows

$$[\![X \circ f, Y \circ f]\!]_\Psi = [\![X, Y]\!] \circ f, \quad \rho_\Psi(X \circ f) = \Psi(X), \text{ for } X, Y \in \Gamma(\tau).$$

Next, we will apply the above construction to a particular case. Let $(E, [\![\cdot,\cdot]\!], \rho)$ be a Lie algebroid with vector bundle projection $\tau : E \to M$ and $X \in \Gamma(\tau)$. Then, we can consider *the vertical lift* of $X$ as the vector field on $E$ given by $X^v(a) = X(\tau(a))^v_a$, for $a \in E$, where ${}^v_a : E_{\tau(a)} \to T_a(E_{\tau(a)})$ is the canonical isomorphism between the vector spaces $E_{\tau(a)}$ and $T_a(E_{\tau(a)})$. In addition, there exists a unique vector field $X^c$ on $E$, *the complete lift* of $X$, satisfying the two following conditions: (i) $X^c$ is $\tau$-projectable on $\rho(X)$ and (ii) $X^c(\widehat{\alpha}) = \widehat{\mathcal{L}^E_X\alpha}$, for all $\alpha \in \Gamma(\tau^*)$. Here, if $\beta \in \Gamma(\tau^*)$ then $\widehat{\beta}$ is the linear function on $E$ defined by $\widehat{\beta}(b) = \beta(\tau(b))(b)$, for all $b \in E$ (for more details, see [4, 5, 7]).
On the other hand, it is well-known (see, for instance, [2]) that the tangent bundle to $E$, $TE$, is a vector bundle over $TM$ with vector bundle projection the tangent map to $\tau$, $T\tau : TE \to TM$. Moreover, the tangent map to $X$, $TX : TM \to TE$ is a section of the vector bundle $T\tau : TE \to TM$. We may also consider the section $\hat{X} : TM \to TE$ of $T\tau : TE \to TM$ given by

$$\hat{X}(u) = (T_x 0)(u) + X(x)^v_{0(x)}, \tag{3}$$

for $u \in T_xM$, where $0 : M \to E$ is the zero section of $E$.

If $\{e_\alpha\}$ is a local basis of $\Gamma(\tau)$ then $\{Te_\alpha, \hat{e}_\alpha\}$ is a local basis of $\Gamma(T\tau)$.
The vector bundle $T\tau : TE \to TM$ admits a Lie algebroid structure with anchor map $\rho^T$ given by $\rho^T = \sigma_{TM} \circ T(\rho)$, $\sigma_{TM} : T(TM) \to T(TM)$ being the canonical involution of the double tangent bundle. The Lie bracket $[\![\cdot,\cdot]\!]^T$ on the space $\Gamma(T\tau)$ is characterized by the following equalities (see [7, 8])

$$[\![TX, TY]\!]^T = T[\![X, Y]\!], \;\; [\![TX, \hat{Y}]\!]^T = \widehat{[\![X, Y]\!]}, \;\; [\![\hat{X}, \hat{Y}]\!]^T = 0, \tag{4}$$

for $X, Y \in \Gamma(\tau)$. Furthermore, there exists a unique action $\Psi : \Gamma(T\tau) \to \mathfrak{X}(E)$ of the Lie algebroid $(TE, [\![\cdot,\cdot]\!]^T, \rho^T)$ over the anchor map $\rho : E \to TM$ such that $\Psi(TX) = X^c$, $\Psi(\hat{X}) = X^v$, for $X \in \Gamma(\tau)$ (see [7]). Thus, on the vector bundle $\rho^*(TE)$ we can consider the action Lie algebroid structure $([\![\cdot,\cdot]\!]^T_\Psi, \rho^T_\Psi)$.

## 2.2. Lie affgebroids

Let $\tau_A : A \to M$ be an affine bundle with associated vector bundle $\tau_V : V \to M$. Denote by $\tau_{A^+} : A^+ = Aff(A, \mathbb{R}) \to M$ the dual bundle whose fibre over $x \in M$ consists of affine functions on the fibre $A_x$. Note that this bundle has a distinguished section $1_A \in \Gamma(\tau_{A^+})$ corresponding to the constant function 1 on $A$. We also consider the bidual bundle $\tau_{\widetilde{A}} : \widetilde{A} \to M$ whose fibre at $x \in M$ is the vector space $\widetilde{A}_x = (A^+_x)^*$. Then, $A$ may be identified with an affine subbundle of $\widetilde{A}$ via the inclusion $i_A : A \to \widetilde{A}$ given by $i_A(a)(\varphi) = \varphi(a)$, which is injective affine map whose associated linear map is denoted by $i_V : V \to \widetilde{A}$. Thus, $V$ may be identified with a vector subbundle of $\widetilde{A}$.
A *Lie affgebroid structure* on $A$ consists of a Lie algebra structure $[\![\cdot,\cdot]\!]_V$ on $\Gamma(\tau_V)$, a $\mathbb{R}$-linear action $D : \Gamma(\tau_A) \times \Gamma(\tau_V) \to \Gamma(\tau_V)$ of the sections of $A$ on $\Gamma(\tau_V)$ and an affine map $\rho_A : A \to TM$, the *anchor map*, satisfying

$$D_X[\![\bar{Y}, \bar{Z}]\!]_V = [\![D_X\bar{Y}, \bar{Z}]\!]_V + [\![\bar{Y}, D_X\bar{Z}]\!]_V, \quad D_{X+\bar{Y}}\bar{Z} = D_X\bar{Z} + [\![\bar{Y}, \bar{Z}]\!]_V,$$

$$D_X(f\bar{Y}) = fD_X\bar{Y} + \rho_A(X)(f)\bar{Y},$$

for $X \in \Gamma(\tau_A)$, $\bar{Y}, \bar{Z} \in \Gamma(\tau_V)$ and $f \in C^\infty(M)$ (see [3, 9]).
If $([\![\cdot,\cdot]\!]_V, D, \rho_A)$ is a Lie affgebroid structure on an affine bundle $A$ then $(V, [\![\cdot,\cdot]\!]_V, \rho_V)$ is a Lie algebroid, where $\rho_V : V \to TM$ is the vector bundle map associated with the affine morphism $\rho_A : A \to TM$.
A Lie affgebroid structure on an affine bundle $\tau_A : A \to M$ induces a Lie algebroid structure $([\![\cdot,\cdot]\!]_{\widetilde{A}}, \rho_{\widetilde{A}})$ on the bidual bundle $\widetilde{A}$ such that $1_A \in \Gamma(\tau_{A^+})$ is an 1-cocycle in the corresponding Lie algebroid cohomology, that is, $d^{\widetilde{A}}1_A = 0$. Indeed, if $X_0 \in \Gamma(\tau_A)$ then for every section $\widetilde{X}$ of $\widetilde{A}$ there exists a real $C^\infty$-

function $f$ on $M$ and a section $\bar{X} \in \Gamma(\tau_V)$ such that $\widetilde{X} = fX_0 + \bar{X}$ and

$$\begin{aligned} \rho_{\widetilde{A}}(fX_0 + \bar{X}) &= f\rho_A(X_0) + \rho_V(\bar{X}) \\ [\![fX_0 + \bar{X}, gX_0 + \bar{Y}]\!]_{\widetilde{A}} &= (\rho_V(\bar{X})(g) - \rho_V(\bar{Y})(f) + f\rho_A(X_0)(g) \\ &\quad -g\rho_A(X_0)(f))X_0 + [\![\bar{X}, \bar{Y}]\!]_V + fD_{X_0}\bar{Y} - gD_{X_0}\bar{X}. \end{aligned} \tag{5}$$

Conversely, let $(U, [\![\cdot,\cdot]\!]_U, \rho_U)$ be a Lie algebroid over $M$ and $\phi : U \to \mathbb{R}$ be an 1-cocycle of $(U, [\![\cdot,\cdot]\!]_U, \rho_U)$ such that $\phi_{/U_x} \neq 0$, for all $x \in M$. Then, $A = \phi^{-1}\{1\}$ is an affine bundle over $A$ which admits a Lie affgebroid structure in such a way that $(U, [\![\cdot,\cdot]\!]_U, \rho_U)$ may be identified with the bidual Lie algebroid $(\widetilde{A}, [\![\cdot,\cdot]\!]_{\widetilde{A}}, \rho_{\widetilde{A}})$ to $A$ and, under this identification, the 1-cocycle $1_A : \widetilde{A} \to \mathbb{R}$ is just $\phi$. The affine bundle $\tau_A : A \to M$ is modelled on the vector bundle $\tau_V{:}V = \phi^{-1}\{0\} \to M$.

## 3. Lie affgebroid morphisms

Let $\tau_A : A \to M$ (respectively, $\tau_{A'} : A' \to M'$) be an affine bundle with associated vector bundle $\tau_V : V \to M$ (respectively, $\tau_{V'} : V' \to M'$) and $(F : A \to A', f : M \to M')$ be an affine morphism between $\tau_A : A \to M$ and $\tau_{A'} : A' \to M'$ such that the corresponding morphism between the vector bundles $\tau_V : V \to M$ and $\tau_{V'} : V' \to M'$ is the pair $(F^l : V \to V', f : M \to M')$.
Denote by $\tau_{\widetilde{A}} : \widetilde{A} \to M$ (respectively, $\tau_{\widetilde{A}'} : \widetilde{A}' \to M'$) the bidual bundle to $\tau_A : A \to M$ (respectively, $\tau_{A'} : A' \to M'$) and by $\widetilde{F} : \widetilde{A} \to \widetilde{A}'$ the map given by $\widetilde{F}(\widetilde{a})(\varphi') = \widetilde{a}(\varphi' \circ F)$, for $\widetilde{a} \in \widetilde{A}_x$ and $\varphi' \in (A')^+_{f(x)} = Aff(A'_{f(x)}, \mathbb{R})$, with $x \in M$. Then, a direct computation proves that the pair $(\widetilde{F}, f)$ is a vector bundle morphism and $(\widetilde{F}, f)^* 1_{A'} = 1_A$.
Conversely, suppose that $\tau_U : U \to M$ and $\tau_{U'} : U' \to M'$ are vector bundles and that $\phi$ and $\phi'$ are sections of the vector bundles $\tau_U^* : U^* \to M$ and $\tau_{U'}^* : (U')^* \to M'$, respectively, such that $\phi_{|U_x} \neq 0$, for all $x \in M$, and $(\phi')_{|U'_{x'}} \neq 0$, for all $x' \in M'$. Assume also that the pair $(\tilde{F}, f)$ is a morphism between the vector bundles $\tau_U : U \to M$ and $\tau_{U'} : U' \to M'$ such that $(\tilde{F}, f)^*\phi' = \phi$ and denote by $A$ and $V$ (respectively, $A'$ and $V'$) the subsets of $U$ (respectively, $U'$) defined by $A = \phi^{-1}\{1\}$ and $V = \phi^{-1}\{0\}$ (respectively, $A' = (\phi')^{-1}\{1\}$ and $V' = (\phi')^{-1}\{0\}$). Then, it is easy to prove that $\tilde{F}(A) \subseteq A'$ and $\tilde{F}(V) \subseteq V'$. Thus, the pair $(F, f)$ is a morphism between the affine bundles $\tau_A = (\tau_U)_{/A} : A \to M$ and $\tau_{A'} = (\tau_{U'})_{/A'} : A' \to M'$, where $F : A \to A'$ is the restriction of $\tilde{F}$ to $A$. The corresponding morphism between the vector

bundles $\tau_V = (\tau_U)_{/V} : V \to M$ and $\tau_{V'} = (\tau_{U'})_{/V'} : V' \to M'$ is the pair $(F^l, f)$, $F^l : V \to V'$ being the restriction of $\widetilde{F}$ to $V$.
Now, suppose that $(\tau_A : A \to M, \tau_V : V \to M)$ and $(\tau_{A'} : A' \to M', \tau_{V'} : V' \to M')$ are two Lie affgebroids with Lie affgebroid structures $(\llbracket\cdot,\cdot\rrbracket_V, D, \rho_A)$ and $(\llbracket\cdot,\cdot\rrbracket_{V'}, D', \rho_{A'})$ respectively, and that $((F,f),(F^l,f))$ is a morphism between the affine bundles $\tau_A : A \to M$ and $\tau_{A'} : A' \to M'$. Then, the pair $((F,f),(F^l,f))$ is said to be a *Lie affgebroid morphism* if: $(i)$ The pair $(F^l, f)$ is a morphism between the Lie algebroids $(V, \llbracket\cdot,\cdot\rrbracket_V, \rho_V)$ and $(V', \llbracket\cdot,\cdot\rrbracket_{V'}, \rho_{V'})$; $(ii)$ $Tf \circ \rho_A = \rho_{A'} \circ F$ and $(iii)$ If $X$ (respectively, $X'$) is a section of $\tau_A : A \to M$ (respectively, $\tau_{A'} : A' \to M'$) and $\bar{Y}$ (respectively, $\bar{Y}'$) is a section of $\tau_V : V \to M$ (respectively, $\tau_{V'} : V' \to M'$) such that $X' \circ f = F \circ X$ and $\bar{Y}' \circ f = F^l \circ \bar{Y}$ then $F^l \circ D_X \bar{Y} = (D'_{X'} \bar{Y}') \circ f$.
Using (5), one may deduce the following result.

**Proposition 1.** $(i)$ *Suppose that $\tau_A : A \to M$ and $\tau_{A'} : A' \to M'$ are Lie affgebroids and that $\tau_{\widetilde{A}} : \widetilde{A} \to M$ and $\tau_{\widetilde{A}'} : \widetilde{A}' \to M'$ are the bidual vector bundles to $A$ and $A'$, respectively. If $((F,f),(G,f))$ is a Lie affgebroid morphism and $\tilde{F} : \widetilde{A} \to \widetilde{A}'$ is the corresponding morphism between the vector bundles $\widetilde{A}$ and $\widetilde{A}'$ then the pair $(\tilde{F}, f)$ is a morphism between the Lie algebroids $\widetilde{A}$ and $\widetilde{A}'$.*

$(ii)$ *Conversely, suppose that $\tau_U : U \to M$ and $\tau_{U'} : U' \to M'$ are Lie algebroids and that $\phi \in \Gamma(\tau_U^*)$ and $\phi' \in \Gamma(\tau_{U'}^*)$ are 1-cocycles of $\tau_U : U \to M$ and $\tau_{U'} : U' \to M'$, respectively, such that $\phi_{|U_x} \neq 0$, for all $x \in M$, and $(\phi')_{|U'_{x'}} \neq 0$, for all $x' \in M'$. Then, if the pair $(\tilde{F}, f)$ is a Lie algebroid morphism between the Lie algebroids $\tau_U : U \to M$ and $\tau_{U'} : U' \to M'$ satisfying $(\tilde{F}, f)^* \phi' = \phi$, we have that the corresponding morphism $((F,f),(F^l,f))$ between the Lie affgebroids $\tau_A = (\tau_U)_{/A} : A = \phi^{-1}\{1\} \to M$ and $\tau_{A'} = (\tau_{U'})_{/A'} : A' = (\phi')^{-1}\{1\} \to M'$ is a Lie affgebroid morphism.*

## 4. The canonical involution associated with a Lie affgebroid

Let $(\tau_A : A \to M, \tau_V : V \to M, (\llbracket\cdot,\cdot\rrbracket_V, D, \rho_A))$ be a Lie affgebroid and $\tau_{\widetilde{A}} : \widetilde{A} \to M$ be the bidual bundle to $\tau_A : A \to M$. Denote by $\rho_V : V \to TM$ the anchor map of the Lie algebroid $\tau_V : V \to M$, by $(\llbracket\cdot,\cdot\rrbracket_{\widetilde{A}}, \rho_{\widetilde{A}})$ the Lie algebroid structure on $\tau_{\widetilde{A}} : \widetilde{A} \to M$ and by $1_A : \widetilde{A} \to \mathbb{R}$ the distinguished 1-cocycle on $\widetilde{A}$.
We consider the subset $\mathcal{J}^A A$ of the product manifold $A \times TA$ defined by

$$\mathcal{J}^A A = \{(a,v) \in A \times TA / \rho_A(a) = (T\tau_A)(v)\}.$$

Next, we will see that $\mathcal{J}^A A$ admits two Lie affgebroid structures:

<u>The first structure</u> [9]: Let $(\mathcal{L}^{\tau_A}\widetilde{A}, [\![\cdot,\cdot]\!]^{\tau_A}_{\widetilde{A}}, \rho^{\tau_A}_{\widetilde{A}})$ be the prolongation of the Lie algebroid $(\widetilde{A}, [\![\cdot,\cdot]\!]_{\widetilde{A}}, \rho_{\widetilde{A}})$ over the fibration $\tau_A : A \to M$. $\mathcal{L}^{\tau_A}\widetilde{A}$ is a Lie algebroid over $A$ with vector bundle projection $\tau^{\tau_A}_{\widetilde{A}} : \mathcal{L}^{\tau_A}\widetilde{A} \to A$ given by $\tau^{\tau_A}_{\widetilde{A}}(\tilde{a}, v) = \pi_A(v)$, $\pi_A : TA \to A$ being the canonical projection.
Now, we consider the section $\phi_0 : \mathcal{L}^{\tau_A}\widetilde{A} \to \mathbb{R}$ of the dual bundle to $\tau^{\tau_A}_{\widetilde{A}} : \mathcal{L}^{\tau_A}\widetilde{A} \to A$ defined by

$$\phi_0(\tilde{a}, v) = 1_A(\tilde{a}), \quad \text{for} \quad (\tilde{a}, v) \in \mathcal{L}^{\tau_A}\widetilde{A}. \tag{6}$$

Using (1), (6) and the fact that $1_A$ is an 1-cocycle, it follows that $\phi_0$ is also an 1-cocycle and, since $(1_A)_{/\widetilde{A}_x} \neq 0$, for all $x \in M$, then $(\phi_0)_{/(\mathcal{L}^{\tau_A}\widetilde{A})_a} \neq 0$, for all $a \in A$.
On the other hand, we have that

$$(\phi_0)^{-1}\{1\} = \{(\tilde{a}, v) \in \widetilde{A} \times TA / \rho_{\widetilde{A}}(\tilde{a}) = (T\tau_A)(v), 1_A(\tilde{a}) = 1\} = \mathcal{J}^A A. \tag{7}$$

In addition, if $\mathcal{L}^{\tau_A}V$ is the prolongation of the Lie algebroid $(V, [\![\cdot,\cdot]\!]_V, \rho_V)$ over the fibration $\tau_A : A \to M$ then, it is easy to prove that $(\phi_0)^{-1}\{0\} = \mathcal{L}^{\tau_A}V$.
We will denote by $([\![\cdot,\cdot]\!]^{\tau_A}_V, \rho^{\tau_A}_V)$ the Lie algebroid structure on $\tau^{\tau_A}_V : \mathcal{L}^{\tau_A}V \to A$.
From (7), we conclude that $\tau^{\tau_A}_A : \mathcal{J}^A A \to A$, defined by $\tau^{\tau_A}_A(a, v) = \pi_A(v)$, is the affine bundle projection of the affine bundle $\mathcal{J}^A A$ and that $\mathcal{J}^A A$ admits a Lie affgebroid structure in such a way that the bidual Lie algebroid to $\tau^{\tau_A}_A : \mathcal{J}^A A \to A$ is just $(\mathcal{L}^{\tau_A}\widetilde{A}, [\![\cdot,\cdot]\!]^{\tau_A}_{\widetilde{A}}, \rho^{\tau_A}_{\widetilde{A}})$. Finally, it follows the Lie affgebroid $\tau^{\tau_A}_A : \mathcal{J}^A A \to A$ is modelled on the Lie algebroid $(\mathcal{L}^{\tau_A}V, [\![\cdot,\cdot]\!]^{\tau_A}_V, \rho^{\tau_A}_V)$.

*Remark* 1. Denote by $(Id, Ti_A) : \mathcal{L}^{\tau_A}\widetilde{A} \to \mathcal{L}^{\tau_{\tilde{A}}}\widetilde{A}$ the inclusion defined by $(Id, Ti_A)(\tilde{a}, v_b) = (\tilde{a}, (Ti_A)(v_b))$, with $b \in A$. Then, it is easy to prove that $(\mathcal{L}^{\tau_A}\widetilde{A}, [\![\cdot,\cdot]\!]^{\tau_A}_{\widetilde{A}}, \rho^{\tau_A}_{\widetilde{A}})$ is a Lie subalgebroid of $(\mathcal{L}^{\tau_{\tilde{A}}}\widetilde{A}, [\![\cdot,\cdot]\!]^{\tau_{\tilde{A}}}_{\widetilde{A}}, \rho^{\tau_{\tilde{A}}}_{\widetilde{A}})$ via the map $((Id, Ti_A), i_A)$.

<u>The second structure</u>: As we know, the tangent bundle to $\widetilde{A}$, $T\widetilde{A}$, is a Lie algebroid over $TM$ with vector bundle projection $T\tau_{\widetilde{A}} : T\widetilde{A} \to TM$.
Now, we consider the subset $d_0(1_A)^0$ of $T\widetilde{A}$ given by

$$d_0(1_A)^0 = \{\tilde{v} \in T\widetilde{A} / d_0(1_A)(\tilde{v}) = 0\} = \{\tilde{v} \in T\widetilde{A} / \tilde{v}(1_A) = 0\}. \tag{8}$$

$d_0(1_A)^0$ is the total space of a vector subbundle of $T\tau_{\widetilde{A}} : T\widetilde{A} \to TM$. More precisely, suppose that $\tilde{X} \in \Gamma(\tau_{\widetilde{A}})$ and denote by $T\tilde{X} : TM \to T\widetilde{A}$ the tangent map to $\tilde{X}$ and by $\hat{\tilde{X}} : TM \to T\widetilde{A}$ the section of $T\tau_{\widetilde{A}} : T\widetilde{A} \to TM$ defined

by (3). Then, using (8), we deduce the following facts: if $1_A(\tilde{X}) = c$, with $c \in \mathbb{R}$, we have that $T\tilde{X} : TM \to T\widetilde{A}$ is a section of the vector bundle $(T\tau_{\widetilde{A}})_{/d_0(1_A)^0} : d_0(1_A)^0 \to TM$ and if $1_A(\tilde{X}) = 0$ it follows that $\hat{\tilde{X}} : TM \to T\widetilde{A}$ is a section of the vector bundle $(T\tau_{\widetilde{A}})_{/d_0(1_A)^0} : d_0(1_A)^0 \to TM$.
Indeed, if $\{e_0, e_\alpha\}$ is a local basis of $\Gamma(\tau_{\widetilde{A}})$ such that $1_A(e_0) = 1$ and $1_A(e_\alpha) = 0$, for all $\alpha$, then $\{Te_0, Te_\alpha, \hat{e}_\alpha\}$ is local basis of $\Gamma((T\tau_{\widetilde{A}})_{/d_0(1_A)^0})$. Using these facts, (4) and since $1_A : \widetilde{A} \to \mathbb{R}$ is an 1-cocycle, we deduce the following result.

**Proposition 2.** *The vector bundle $(T\tau_{\widetilde{A}})_{/d_0(1_A)^0} : d_0(1_A)^0 \to TM$ is a Lie algebroid and $(T\tau_{\widetilde{A}})_{/d_0(1_A)^0} : d_0(1_A)^0 \to TM$ is a Lie subalgebroid of $T\tau_{\widetilde{A}} : T\widetilde{A} \to TM$, via the canonical inclusion $i : d_0(1_A)^0 \to T\widetilde{A}$.*

Next, we consider the pull-back $\rho_A^*(d_0(1_A)^0)$ of the vector bundle $(T\tau_{\widetilde{A}})_{/d_0(1_A)^0} : d_0(1_A)^0 \to TM$ over the map $\rho_A : A \to TM$ which is a vector bundle over $A$ with vector bundle projection the map $pr_1 : \rho_A^*(d_0(1_A)^0) \to A$ given by $pr_1(a, \tilde{v}) = a$. On the other hand, we will denote by $(i_A, Id) : \rho_A^*(d_0(1_A)^0) \to \rho_{\widetilde{A}}^*(T\widetilde{A})$ the monomorphism (over the canonical inclusion $i_A : A \to \widetilde{A}$) between the vector bundles $\rho_A^*(d_0(1_A)^0) \to A$ and $\rho_{\widetilde{A}}^*(T\widetilde{A}) \to \widetilde{A}$ defined by $(i_A, Id)(a, \tilde{v}) = (i_A(a), \tilde{v})$, for $(a, \tilde{v}) \in \rho_A^*(d_0(1_A)^0)$. We recall that the vector bundle $\rho_{\widetilde{A}}^*(T\widetilde{A}) \to \widetilde{A}$ is an action Lie algebroid (see Section 2.1). Furthermore, we have (see [1] for a proof of this result)

**Proposition 3.** *(i) The vector bundle $\rho_A^*(d_0(1_A)^0) \to A$ is a Lie algebroid over $A$ and the pair $((i_A, Id), i_A)$ is a monomorphism between the Lie algebroids $\rho_A^*(d_0(1_A)^0) \to A$ and $\rho_{\widetilde{A}}^*(T\widetilde{A}) \to \widetilde{A}$.*
*(ii) If $\varphi_0 : \rho_A^*(d_0(1_A)^0) \to \mathbb{R}$ is the linear map given by*

$$\varphi_0(a, \tilde{v}) = 1_A(\pi_A(\tilde{v})), \tag{9}$$

*then $\varphi_0$ is an 1-cocycle of the Lie algebroid $\rho_A^*(d_0(1_A)^0) \to A$ and $\varphi_{0/(\rho_A^*(d_0(1_A)^0))_a} \neq 0$, for all $a \in A$.*

Denote by $((\llbracket\cdot,\cdot\rrbracket_{\widetilde{A}})^T_{\Psi_0}, (\rho_{\widetilde{A}})^T_{\Psi_0})$ the Lie algebroid structure on $\rho_A^*(d_0(1_A)^0) \to A$. Then,

$$\begin{array}{rclrcl}
(\llbracket T^{\rho_A}e_0, T^{\rho_A}e_\alpha\rrbracket_{\widetilde{A}})^T_{\Psi_0} &=& T^{\rho_A}\llbracket e_0, e_\alpha\rrbracket_{\widetilde{A}}, & (\rho_{\widetilde{A}})^T_{\Psi_0}(T^{\rho_A}e_0) &=& (e_0)^c_{/A}, \\
(\llbracket T^{\rho_A}e_\alpha, T^{\rho_A}e_\beta\rrbracket_{\widetilde{A}})^T_{\Psi_0} &=& T^{\rho_A}\llbracket e_\alpha, e_\beta\rrbracket_{\widetilde{A}}, & (\rho_{\widetilde{A}})^T_{\Psi_0}(T^{\rho_A}e_\alpha) &=& (e_\alpha)^c_{/A}, \\
(\llbracket T^{\rho_A}e_0, \hat{e}^{\rho_A}_\alpha\rrbracket_{\widetilde{A}})^T_{\Psi_0} &=& \widehat{\llbracket e_0, e_\alpha\rrbracket}^{\rho_A}_{\widetilde{A}}, & (\rho_{\widetilde{A}})^T_{\Psi_0}(\hat{e}^{\rho_A}_\alpha) &=& (e_\alpha)^v_{/A}, \\
(\llbracket T^{\rho_A}e_\alpha, \hat{e}^{\rho_A}_\beta\rrbracket_{\widetilde{A}})^T_{\Psi_0} &=& \widehat{\llbracket e_\alpha, e_\beta\rrbracket}^{\rho_A}_{\widetilde{A}}, & (\llbracket\hat{e}^{\rho_A}_\alpha, \hat{e}^{\rho_A}_\beta\rrbracket_{\widetilde{A}})^T_{\Psi_0} &=& 0.
\end{array} \tag{10}$$

Here, if $\tilde{X} \in \Gamma(\tau_{\widetilde{A}})$ then $T^{\rho_A}\tilde{X}$ (respectively, $\hat{\tilde{X}}^{\rho_A}$) is the section of $\rho_A^*(d_0(1_A)^0)$ $\to A$ given by $T^{\rho_A}\tilde{X} = T\tilde{X} \circ \rho_A$ (respectively, $\hat{\tilde{X}} \circ \rho_A$).
Now, from (9), it follows that $\varphi_0^{-1}\{1\} = \mathcal{J}^A A$. Thus, we conclude that the affine bundle $pr_1 : \mathcal{J}^A A \to A$ admits a Lie affgebroid structure in such a way that the bidual Lie algebroid to $pr_1 : \mathcal{J}^A A \to A$ is just $(\rho_A^*(d_0(1_A)^0)$, $(\llbracket\cdot,\cdot\rrbracket_{\widetilde{A}})_{\Psi_0}^T, (\rho_{\widetilde{A}})_{\Psi_0}^T)$.
On the other hand, using (9), we obtain that $\varphi_0^{-1}\{0\} = \rho_A^*(TV)$. Therefore, the affine bundle $pr_1 : \mathcal{J}^A A \to A$ is modelled on the vector bundle $pr_1 : \rho_A^*(TV) \to A$. Furthermore, from (10), we deduce that the corresponding Lie algebroid structure is induced by an action $\Psi_V$ of the Lie algebroid $(TV, \llbracket\cdot,\cdot\rrbracket_V^T, \rho_V^T)$ over the anchor map $\rho_A : A \to TM$. For this action, we have that $\Psi_V(T\bar{X}) = (i_V \circ \bar{X})_{/A}^c$, $\Psi_V(\hat{\bar{X}}) = (i_V \circ \bar{X})_{/A}^v$, for $\bar{X} \in \Gamma(\tau_V)$.

## The canonical involution

Let $\mathcal{L}^{\tau_{\tilde{A}}}\widetilde{A}$ be the prolongation of the Lie algebroid $(\widetilde{A}, \llbracket\cdot,\cdot\rrbracket_{\widetilde{A}}, \rho_{\widetilde{A}})$ over the fibration $\tau_{\widetilde{A}} : \widetilde{A} \to M$ and $\rho_{\widetilde{A}}^*(T\widetilde{A}) \equiv \mathcal{L}^{\tau_{\tilde{A}}}\widetilde{A}$ be the pull-back of the Lie algebroid $T\tau_{\widetilde{A}} : T\widetilde{A} \to TM$ over the anchor map $\rho_{\widetilde{A}} : \widetilde{A} \to TM$. If $(\tilde{a}, \tilde{v}_{\tilde{b}}) \in (\mathcal{L}^{\tau_{\tilde{A}}}\widetilde{A})_{\tilde{b}}$, with $\tilde{b} \in \widetilde{A}_x$ and $x \in M$, then there exists a unique tangent vector $\tilde{u}_{\tilde{a}} \in T_{\tilde{a}}\widetilde{A}$ such that:

$$\tilde{u}_{\tilde{a}}(f \circ \tau_{\widetilde{A}}) = (d^{\widetilde{A}} f)(x)(\tilde{b}), \quad \tilde{u}_{\tilde{a}}(\theta) = \tilde{v}_{\tilde{b}}(\theta) + (d^{\widetilde{A}}\theta)(x)(\tilde{b}, \tilde{a}),$$

for $f \in C^\infty(M)$ and $\theta : \widetilde{A} \to \mathbb{R} \in \Gamma(\tau_{A^+})$ (see [7]). Thus, one may define the map

$$\sigma_{\widetilde{A}} : \mathcal{L}^{\tau_{\tilde{A}}}\widetilde{A} \to \rho_{\widetilde{A}}^*(T\widetilde{A}), \quad \sigma_{\widetilde{A}}(\tilde{a}, \tilde{v}_{\tilde{b}}) = (\tilde{b}, \tilde{u}_{\tilde{a}}), \quad \text{for} \quad (\tilde{a}, \tilde{v}_{\tilde{b}}) \in (\mathcal{L}^{\tau_{\tilde{A}}}\widetilde{A})_{\tilde{b}}.$$

$\sigma_{\widetilde{A}}$ is an isomorphism (over the identity $Id : \widetilde{A} \to \widetilde{A}$) between the Lie algebroids $(\mathcal{L}^{\tau_{\tilde{A}}}\widetilde{A}, \llbracket\cdot,\cdot\rrbracket_{\widetilde{A}}^{\tau_{\widetilde{A}}}, \rho_{\widetilde{A}}^{\tau_{\widetilde{A}}})$ and $(\rho_{\widetilde{A}}^*(T\widetilde{A}), (\llbracket\cdot,\cdot\rrbracket_{\widetilde{A}})_\Psi^T, (\rho_{\widetilde{A}})_\Psi^T)$ and, moreover, $\sigma_{\widetilde{A}}^2 = Id$. $\sigma_{\widetilde{A}}$ is called *the canonical involution associated with the Lie algebroid* $(\widetilde{A}, \llbracket\cdot,\cdot\rrbracket_{\widetilde{A}}, \rho_{\widetilde{A}})$ (for more details, see [7]).
Now, we have the following result.

**Theorem 4.** [1] *The restriction of $\sigma_{\widetilde{A}}$ to $\mathcal{J}^A A$ induces an isomorphism $\sigma_A : \mathcal{J}^A A \to \mathcal{J}^A A$ between the Lie affgebroids $\tau_A^{\tau_A} : \mathcal{J}^A A \to A$ and $pr_1 : \mathcal{J}^A A \to A$ and, moreover, $\sigma_A^2 = Id$. The corresponding Lie algebroid isomorphism $\sigma_A^l : \mathcal{L}^{\tau_A}V \to \rho_A^*(TV)$ between the Lie algebroids $\tau_V^{\tau_A} : \mathcal{L}^{\tau_A}V \to A$ and $pr_1 : \rho_A^*(TV) \to A$ is the restriction of $\sigma_{\widetilde{A}}$ to $\mathcal{L}^{\tau_A}V$, that is, $\sigma_A^l = (\sigma_{\widetilde{A}})_{/\mathcal{L}^{\tau_A}V}$.*

**Definition 1.** The map $\sigma_A : \mathcal{J}^A A \to \mathcal{J}^A A$ is called the canonical involution associated with the Lie affgebroid $A$.

# 5. Tulczyjew's triple associated with a Lie affgebroid and a Hamiltonian section

Let $(\tau_A : A \to M, \tau_V : V \to M, ([\![\cdot,\cdot]\!]_V, D, \rho_A))$ be a Lie affgebroid. Denote by $\rho_A^*(TV)$ (respectively, $\rho_A^*(TV^*)$) the pull-back of the vector bundle $T\tau_V : TV \to TM$ (respectively, $T\tau_V^* : TV^* \to TM$) over the anchor map $\rho_A : A \to TM$ and by $\mathcal{L}^{\tau_A}V$ the prolongation of the Lie algebroid $\tau_V : V \to M$ over the projection $\tau_A : A \to M$.
Then, the aim of this section is to introduce the so-called *Tulczyjew's triple associated with $A$ and a Hamiltonian section.* For this purpose, we will proceed in three steps.

<u>FIRST STEP</u>: In this first step, we will introduce a canonical isomorphism $A_A : \rho_A^*(TV^*) \to (\mathcal{L}^{\tau_A}V)^*$, over the identity of $A$, between the vector bundles $\rho_A^*(TV^*) \to A$ and $(\mathcal{L}^{\tau_A}V)^* \to A$.
Let $< \cdot,\cdot >: V \times_M V^* \to \mathbb{R}$ be the natural pairing. If $b \in A$, $(b, X_u) \in \rho_A^*(TV)_b$ and $(b, X_\alpha) \in \rho_A^*(TV^*)_b$ then $(X_u, X_\alpha) \in T_{(u,\alpha)}(V \times_M V^*)$ and we may consider the map $T\widetilde{< \cdot,\cdot >} : \rho_A^*(TV) \times_A \rho_A^*(TV^*) \to \mathbb{R}$ given by

$$T\widetilde{< \cdot,\cdot >}((b, X_u), (b, X_\alpha)) = dt_{<u,\alpha>}((T_{(u,\alpha)} < \cdot,\cdot >)(X_u, X_\alpha)),$$

where $t$ is the usual coordinate on $\mathbb{R}$ and $T < \cdot,\cdot >: T(V \times_M V^*) \to T\mathbb{R}$ is the tangent map to $< \cdot,\cdot >: V \times_M V^* \to \mathbb{R}$. Since the pairing $< \cdot,\cdot >$ is non-singular, it follows that the map $T\widetilde{< \cdot,\cdot >}$ is also a non-singular pairing and, thus, it induces an isomorphism (over the identity of $A$) between the vector bundles $\rho_A^*(TV) \to A$ and $\rho_A^*(TV^*)^* \to A$ which we also denote by $T\widetilde{< \cdot,\cdot >} : \rho_A^*(TV) \to \rho_A^*(TV^*)^*$.
Next, we consider the isomorphism of vector bundles $A_A^* : \mathcal{L}^{\tau_A}V \to \rho_A^*(TV^*)^*$ defined by $A_A^* = T\widetilde{< \cdot,\cdot >} \circ \sigma_A^l$, $\sigma_A^l : \mathcal{L}^{\tau_A}V \to \rho_A^*(TV)$ being the linear part of the canonical involution associated with the Lie affgebroid $A$. Then, the isomorphism $A_A : \rho_A^*(TV^*) \to (\mathcal{L}^{\tau_A}V)^*$ is just the dual map to $A_A^* : \mathcal{L}^{\tau_A}V \to \rho_A^*(TV^*)^*$.

<u>SECOND STEP</u>: Denote by $\tau_{\widetilde{A}} : \widetilde{A} \to M$ the bidual Lie algebroid to the Lie affgebroid $\tau_A : A \to M$ and by $\mathcal{L}^{\tau_V^*}\widetilde{A}$ the prolongation of $\widetilde{A}$ over the projection $\tau_V^* : V^* \to M$. Then, in this step, using a Hamiltonian section, we will introduce a cosymplectic structure on the Lie algebroid $\tau_{\widetilde{A}}^{\tau_V^*} : \mathcal{L}^{\tau_V^*}\widetilde{A} \to V^*$.
Let $\tau_{A^+} : A^+ \to M$ be the dual vector bundle to the affine bundle $\tau_A : A \to M$, $\mu : A^+ \to V^*$ be the canonical projection given by $\mu(\varphi) = \varphi^l$, for $\varphi \in A_x^+$, with $x \in M$, where $\varphi^l \in V_x^*$ is the linear map associated with the affine map $\varphi$ and $h : V^* \to A^+$ be a Hamiltonian section, that is, $h \in \Gamma(\mu)$.

Now, we consider the prolongation $\mathcal{L}^{\tau_{A^+}}\widetilde{A}$ of the Lie algebroid $\widetilde{A}$ over the projection $\tau_{A^+} : A^+ \to M$ and the map $\mathcal{L}h : \mathcal{L}^{\tau_V^*}\widetilde{A} \to \mathcal{L}^{\tau_{A^+}}\widetilde{A}$ defined by $\mathcal{L}h(\tilde{a}, X_\alpha) = (\tilde{a}, (T_\alpha h)(X_\alpha))$, for $(\tilde{a}, X_\alpha) \in (\mathcal{L}^{\tau_V^*}\widetilde{A})_\alpha$, with $\alpha \in V^*$, where $T_\alpha h : T_\alpha V^* \to T_{h(\alpha)}A^+$ is the tangent map to $h$ at $\alpha$. It is easy to prove that the pair $(\mathcal{L}h, h)$ is a Lie algebroid morphism between the Lie algebroids $\tau_{\widetilde{A}}^{\tau_V^*} : \mathcal{L}^{\tau_V^*}\widetilde{A} \to V^*$ and $\tau_{\widetilde{A}}^{\tau_{A^+}} : \mathcal{L}^{\tau_{A^+}}\widetilde{A} \to A^+$.
Next, denote by $\lambda_h$ and $\Omega_h$ the sections of the vector bundles $(\mathcal{L}^{\tau_V^*}\widetilde{A})^* \to V^*$ and $\Lambda^2(\mathcal{L}^{\tau_V^*}\widetilde{A})^* \to V^*$ given by

$$\lambda_h = (\mathcal{L}h, h)^*(\lambda_{\widetilde{A}}), \;\; \Omega_h = (\mathcal{L}h, h)^*(\Omega_{\widetilde{A}}),$$

where $\lambda_{\widetilde{A}}$ is the Liouville section of the vector bundle $(\mathcal{L}^{\tau_{A^+}}\widetilde{A})^* \to A^+$ and $\Omega_{\widetilde{A}}$ is the canonical symplectic section associated with the Lie algebroid $\tilde{A}$. Note that $\Omega_h = -d^{\mathcal{L}^{\tau_V^*}\widetilde{A}}\lambda_h$. On the other hand, let $\widetilde{1_A} : \mathcal{L}^{\tau_V^*}\widetilde{A} \to \mathbb{R}$ be the section of $(\mathcal{L}^{\tau_V^*}\widetilde{A})^* \to V^*$ defined by $\widetilde{1_A}(\tilde{a}, X_\alpha) = 1_A(\tilde{a})$, for $(\tilde{a}, X_\alpha) \in \mathcal{L}^{\tau_V^*}\widetilde{A}$. Since $1_A : \widetilde{A} \to \mathbb{R}$ is an 1-cocycle of the Lie algebroid $\tau_{\widetilde{A}} : \widetilde{A} \to M$, it follows that $\widetilde{1_A}$ is also an 1-cocycle of the Lie algebroid $\mathcal{L}^{\tau_V^*}\widetilde{A} \to V^*$. Furthermore, $(\widetilde{1_A})_{/(\mathcal{L}^{\tau_V^*}\widetilde{A})_\alpha} \neq 0$, for all $\alpha \in V^*$.
Finally, we will see that the pair $(\Omega_h, \widetilde{1_A})$ is a cosymplectic structure on the Lie algebroid $\tau_{\widetilde{A}}^{\tau_V^*} : \mathcal{L}^{\tau_V^*}\widetilde{A} \to V^*$. Note that the rank of the vector bundle $\tau_{\widetilde{A}}^{\tau_V^*} : \mathcal{L}^{\tau_V^*}\widetilde{A} \to V^*$ is $2n+1$, $n$ being the rank of the affine bundle $\tau_A : A \to M$. Moreover, if $\mathcal{L}^{\tau_V^*}V$ is the prolongation of the Lie algebroid $\tau_V : V \to M$ over the projection $\tau_V^* : V^* \to M$ and $(i_V, Id) : \mathcal{L}^{\tau_V^*}V \to \mathcal{L}^{\tau_V^*}\widetilde{A}$ is the canonical inclusion, then it is clear that $((i_V, Id), Id)$ is a Lie algebroid morphism. In addition, using (2) and the fact that $\mu \circ h = Id$, one may prove that $(i_V, Id)^*\lambda_h = \lambda_V$, $\lambda_V$ being the Liouville section of $(\mathcal{L}^{\tau_V^*}V)^* \to V^*$. This implies that,

$$(i_V, Id)^*\Omega_h = \Omega_V, \;\; (i_V, Id)^*(\widetilde{1_A}) = 0, \tag{11}$$

where $\Omega_V$ is the canonical symplectic section associated with $V$. Thus,

$$\{\widetilde{1_A} \wedge \Omega_h^n\}(\alpha) = \{\widetilde{1_A} \wedge \Omega_h \wedge \dots^{(n} \dots \wedge \Omega_h\}(\alpha) \neq 0, \;\; \text{for all} \;\; \alpha \in V^*.$$

Therefore, since $d^{\mathcal{L}^{\tau_V^*}\widetilde{A}}\widetilde{1_A} = 0$ and $d^{\mathcal{L}^{\tau_V^*}\widetilde{A}}\Omega_h = 0$, we conclude that the pair $(\Omega_h, \widetilde{1_A})$ is a cosymplectic structure on the Lie algebroid $\mathcal{L}^{\tau_V^*}\widetilde{A} \to V^*$.

Third step: A direct computation proves that $(\widetilde{1_A})^{-1}\{1\} = \rho_A^*(TV^*)$ and $(\widetilde{1_A})^{-1}\{0\} = \mathcal{L}^{\tau_V^*}V$. Consequently, the map $\widetilde{\pi_{V^*}} : \rho_A^*(TV^*) \to V^*$ is the affine bundle projection of the affine bundle $\rho_A^*(TV^*)$ which admits a Lie affgebroid structure in such a way that the bidual Lie algebroid is $\tau_{\widetilde{A}}^{\tau_V^*} : \mathcal{L}^{\tau_V^*}\widetilde{A} \to V^*$.

In addition, the Lie affgebroid $\widetilde{\pi_{V^*}} : \rho_A^*(TV^*) \to V^*$ is modelled on the Lie algebroid $\tau_V^{\tau_V^*} : \mathcal{L}^{\tau_V^*}V \to V^*$.
Now, we introduce the map $\flat_{\Omega_h} : \rho_A^*(TV^*) \to (\mathcal{L}^{\tau_V^*}V)^*$ defined by

$$\{\flat_{\Omega_h}(\alpha)(a, X_\alpha)\}(u, Y_\alpha) = \Omega_h(\alpha)((i_A(a), X_\alpha), (i_V(u), Y_\alpha)), \tag{12}$$

for $\alpha \in V^*$, $(a, X_\alpha) \in \rho_A^*(TV^*)_\alpha$ and $(u, Y_\alpha) \in (\mathcal{L}^{\tau_V^*}V)_\alpha$.
On the other hand, let $\flat_{\Omega_V} : \mathcal{L}^{\tau_V^*}V \to (\mathcal{L}^{\tau_V^*}V)^*$ be the canonical isomorphism given by

$$\{\flat_{\Omega_V}(\alpha)(u, Y_\alpha)\}(v, Z_\alpha) = \Omega_V(\alpha)((u, Y_\alpha), (v, Z_\alpha)), \tag{13}$$

for $(u, Y_\alpha), (v, Z_\alpha) \in (\mathcal{L}^{\tau_V^*}V)_\alpha$, with $\alpha \in V^*$. Then, using (11), (12) and (13), it follows that $\flat_{\Omega_h}$ is an affine isomorphism over the identity of $V^*$ and the corresponding linear isomorphism between the vector bundles $\tau_V^{\tau_V^*} : \mathcal{L}^{\tau_V^*}V \to V^*$ and $(\tau_V^{\tau_V^*})^* : (\mathcal{L}^{\tau_V^*}V)^* \to V^*$ is just the map $\flat_{\Omega_V}$.
In conclusion, we have the following commutative diagram

$$\begin{tikzcd}
(\mathcal{L}^{\tau_A}V)^* \arrow[dr, "\tau_V^{\tau_A}"'] & & \rho_A^*(TV^*) \arrow[ll, "A_A"'] \arrow[rr, "\flat_{\Omega_h}"] \arrow[dl, "pr_1"] \arrow[dr, "\widetilde{\pi_{V^*}}"'] & & (\mathcal{L}^{\tau_V^*}V)^* \arrow[dl, "(\tau_V^{\tau_V^*})^*"] \\
& A & & V^* &
\end{tikzcd}$$

This diagram will be called Tulczyjew's triple associated the Lie affgebroid $A$ and the Hamiltonian section $h$.

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# New solutions of the Nizhnik-Novikov-Veselov model through symmetry transformations

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**Abstract.** Making use of the theory of symmetry transformations in PDEs we construct new solutions of the Nizhnik-Novikov-Veselov model, an integrable system in 2+1 dimensions. These solutions are constructed by applying some elements of the symmetry group to known solutions of the model.

We find that the solutions obtained by means of this technique describe interesting processes. For example, we present source and sink solutions, solutions describing the creation or the diffusion (or both) of a breather or a ring soliton, finite time blow-up processes and finite time source solutions.

*Keywords:* NNV model, symmetries, singularity formation

*2000 Mathematics Subject Classification:* 35Q51, 35Q53

## 1. Introduction

Among the 2+1 dimensional integrable systems that have been found to exhibit solutions describing processes of interaction of exponentially localized structures, one can find the Nizhnik-Novikov-Veselov (NNV) system, i.e.

$$\begin{aligned} q_t &= q_{xxx} + q_{yyy} + 6(qu_1)_x + 6(qu_2)_y, \\ u_{1y} &= q_x, \quad u_{2x} = q_y. \end{aligned} \tag{1}$$

This system was proposed as an extension of the Korteweg de Vries (KdV) equation in dimension two, with symmetry in the spacial variables. On the other hand, (1) [3, 6] is one of the first members of a hierarchy of integrable systems emerging from a bilinear identity related to a Clifford algebra, which is generated by two neutral fermion fields. In [8] it is proved that (1) admits exponentially localized solutions travelling at constant velocity and presenting an internal oscillation, these solutions are referred to as breathers. The exponentially localized dependent variable is $q$, while $u_1$ and $u_2$ play the role of potentials.

Moreover, it is also proved (see [8]) that in solutions describing interaction processes of breathers, they manifest dynamical properties similar to the dromion solutions of the Davey-Stewartson equation [5], for instance they change their form under interaction. More recently, [7] new solutions of this system have been determined, by applying the variable separation method in nonlinear PDEs. Among these solutions one can cite dromions, lumps or ring solitons.

It is also worth noting that the evolution associated to (1) conserves the mass, defined by

$$M = \int_{\mathbb{R}^2} q(x, y, t) dx dy,$$

for localized solutions. It is a consequence of the fact that the right hand side of the first equation in (1) is a divergence.

On the other hand, as most of the 2+1 dimensional integrable systems that have been studied from the point of view of the the theory of symmetry transformations in PDEs (see for example [4, 14] for the KP equation or [2] for the Davey-Stewartson equation), the system (1) admits an infinite dimensional group of symmetries [11]. Making use of these groups, new solutions of the previous systems have been constructed [9, 10, 11]. It worths to be noticed that in many cases, these solutions describe interesting processes of singularity formation.

In this work, we focus out attention in the construction of new solutions of (1) by applying some of the simplest symmetry group elements to known solutions of (1). In this way, we obtain

- solutions that describe processes of singularity formation at finite time:
  - Solutions describing blow-up processes and instantaneous source solutions. In both cases the finite time singularity appears in the solution.
  - Solutions describing finite time creation or annihilation processes. In these cases the singularity appears in the time derivative of the solution.

- Solutions describing creation or diffusion processes, sink and source solutions.

## 2. Symmetry group

In order to find the Lie algebra associated to the Lie group of symmetry transformations of (1), we look for vectorial fields of the form:

$$\begin{aligned} V &= \xi_1(x,y,t,q,u_1,u_2)\frac{\partial}{\partial x} + \xi_2(x,y,t,q,u_1,u_2)\frac{\partial}{\partial y} + \xi_3(x,y,t,q,u_1,u_2)\frac{\partial}{\partial t} + \\ &\quad \phi_1(x,y,t,q,u_1,u_2)\frac{\partial}{\partial q} + \phi_2(x,y,t,q,u_1,u_2)\frac{\partial}{\partial u_1} + \phi_3(x,y,t,q,u_1,u_2)\frac{\partial}{\partial u_2}, \end{aligned}$$

which leave invariant the third prolongation of (1). Using the algorithmic techniques (see for example [1, 12, 13]) we find that the general element of the Lie algebra has the form

$$V_1(f) + V_2(g) + V_3(h)$$

where

$$\begin{aligned} V_1(f) &= \frac{1}{3}f'(t)x\frac{\partial}{\partial x} + \frac{1}{3}f'(t)y\frac{\partial}{\partial y} + f(t)\frac{\partial}{\partial t} - \frac{2}{3}f'(t)q\frac{\partial}{\partial q} \\ &\quad - \left(\frac{2}{3}f'(t)u_1 + \frac{1}{18}f''(t)x\right)\frac{\partial}{\partial u_1} \\ &\quad - \left(\frac{2}{3}f'(t)u_2 + \frac{1}{18}f''(t)y\right)\frac{\partial}{\partial u_2} \qquad (2) \\ V_2(g) &= g(t)\frac{\partial}{\partial x} - \frac{1}{6}g'(t)\frac{\partial}{\partial u_1}, \\ V_3(h) &= h(t)\frac{\partial}{\partial y} - \frac{1}{6}h'(t)\frac{\partial}{\partial u_2}, \end{aligned}$$

with $f$, $g$ and $h$ being arbitrary functions on the time variable $t$.

From the expression of the arbitrary element of the Lie algebra we can obtain the equations of the symmetry transformation group, by solving the system

$$\begin{array}{rclrcl}
\dfrac{dX}{ds} & = & \dfrac{1}{3}Xf'(T)+g(T), & X(0) & = & x,\\[2ex]
\dfrac{dY}{ds} & = & \dfrac{1}{3}Yf'(T)+h(T), & Y(0) & = & y,\\[2ex]
\dfrac{dT}{ds} & = & f(T), & T(0) & = & t,\\[2ex]
\dfrac{dQ}{ds} & = & -\dfrac{2}{3}Qf'(T), & Q(0) & = & q,\\[2ex]
\dfrac{dU_1}{ds} & = & -\dfrac{2}{3}U_1f'(T)-\dfrac{1}{18}Xf''(T)-\dfrac{1}{6}g'(T), & U_1(0) & = & u_1,\\[2ex]
\dfrac{dU_2}{ds} & = & -\dfrac{2}{3}U_2f'(T)-\dfrac{1}{18}Yf''(T)-\dfrac{1}{6}h'(T), & U_2(0) & = & u_2,
\end{array}$$

with $s$ being the parameter of the group. From the solution of this system we obtain that if $q(x,y,t)$, $u_1(x,y,t)$, $u_2(x,y,t)$ is a solution of (1) (we will refer to this solution as the *initial solution*), a new family of solutions of (1) is given by

If $f \not\equiv 0$:

$$\begin{array}{rcl}
Q(X,Y,T;s) & = & \left(\dfrac{f(t)}{f(T)}\right)^{\frac{2}{3}} q(x,y,t),\\[3ex]
U_1(X,Y,T;s) & = & \left(\dfrac{f(t)}{f(T)}\right)^{\frac{2}{3}} u_1(x,y,t) - \dfrac{X}{18}\dfrac{f'(T)-f'(t)}{f(T)} - \varphi_1(T),\\[3ex]
U_2(X,Y,T;s) & = & \left(\dfrac{f(t)}{f(T)}\right)^{\frac{2}{3}} u_2(x,y,t) - \dfrac{Y}{18}\dfrac{f'(T)-f'(t)}{f(T)} - \varphi_2(T)
\end{array} \tag{3}$$

where

$$x = \left(\frac{f(t)}{f(T)}\right)^{\frac{1}{3}}(X-\psi_1(T)), \quad y = \left(\frac{f(t)}{f(T)}\right)^{\frac{1}{3}}(Y-\psi_2(T)), \quad t = \Phi^{-1}(\Phi(T)-s), \tag{4}$$

with

$$\Phi(T) = \int^T \frac{1}{f(\xi)}d\xi, \tag{5}$$

and with $\varphi_1$, $\varphi_2$, $\psi_1$ and $\psi_2$ depending on $f$, $g$, $h$ in such a way that all these functions vanish in the case that $g \equiv h \equiv 0$.

If $f \equiv 0$:

$$\begin{aligned} Q(X,Y,T;s) &= q(X - sg(T), Y - sh(T), T), \\ U_1(X,Y,T;s) &= u_1(X - sg(T), Y - sh(T), T) - \frac{s}{6} g'(T), \\ U_2(X,Y,T;s) &= u_2(X - sg(T), Y - sh(T), T) - \frac{s}{6} h'(T). \end{aligned} \tag{6}$$

From (3), (4) and (6), it is clear that the arbitrary functions $g$ and $h$, just move globally the solution, at an arbitrary velocity and on an arbitrary curve in the plane. In view of this, we take in what follows $g \equiv h \equiv 0$ and consider the solution (3)-(4) with $\varphi_1 \equiv \varphi_2 \equiv \psi_1 \equiv \psi_2 \equiv 0$.

## 3. New solutions

According to the last paragraph, we are going to analyze the behavior of the solution (3)-(4) with $g \equiv h \equiv 0$. We focus our attention in the dependent variable $Q$ (the dependent variable associated to the exponentially localized structures). From (3)-(4) realize that:

- If there exists $T_1$ such that:

$$\lim_{T \to T_1} \frac{f(t)}{f(T)} = 0, \tag{7}$$

  then we find that

$$\max_{(X,Y) \in \mathbb{R}^2} |Q(X,Y,T;s)| \to 0, \quad \text{as} \quad T \to T_1.$$

  Thus, we can find the following types of solutions

  – If $T_1 \in \mathbb{R}$ and (7) is satisfied as a left-limit, the solution describes the annihilation of a structure or set of structures.

  – If $T_1 \in \mathbb{R}$ and (7) is satisfied as a right-limit, the solution describes the instantaneous creation of a structure or set of structures.

  – If $T_1 = -\infty$ the solution describes a creation process.

  – If $T_1 = \infty$ the solution describes a diffusion process.

- If there exists $T_2$ such that:

$$\lim_{T \to T_2} \frac{f(t)}{f(T)} = 1, \tag{8}$$

then we find that

$$Q(X, Y, T; s) \sim q(X, Y, T), \quad \text{as} \quad T \to T_2.$$

Thus, the new solution behaves as our starting solution as $T \to T_2$.

- If there exists $T_3$ such that:

$$\lim_{T \to T_3} \frac{f(t)}{f(T)} = \infty, \tag{9}$$

then we find that

$$Q(X, Y, T; s) \to c\,\delta(X, Y), \quad \text{as} \quad T \to T_3,$$

where $c$ is a certain constant related to the mass of our initial solution and $\delta$ stands by the Dirac delta distribution. Its meaning can be understood as the whole mass of the solution is, in the limit, concentrated in the origin. Thus, we can interpret the new solution depending on $T_3$ as:

  - If $T_3 \in \mathbb{R}$ and (9) is satisfied as a left-limit, the solution describes a finite time blow-up process.
  - If $T_3 \in \mathbb{R}$ and (7) is satisfied as a right-limit, the solution is an instantaneous source solution.
  - If $T_3 = -\infty$ the solution is a source solution.
  - If $T_3 = \infty$ the solution is a sink solution.

Taking into account all these facts we are going to describe next some examples of solutions exhibiting the previous behaviors

## 3.1. Example 1: $f(t) = e^{\alpha t}$, $\alpha \neq 0$

We have for this choice of the arbitrary function $f$ that

$$\frac{f(t)}{f(T)} = \frac{1}{1 + \alpha s e^{\alpha T}}.$$

Thus, by introducing

$$T_0 = -\frac{1}{\alpha} \ln(-\alpha s), \quad \text{for the cases in which} \quad \alpha s < 0,$$

we find

- If $\alpha > 0$, $s < 0$, $T \in (-\infty, T_0)$, our solutions satisfies:

$$Q(X,Y,T) \sim q(X,Y,T) \quad \text{as} \quad T \to -\infty,$$

$$Q(X,Y,T) \to c\,\delta(X,Y) \quad \text{as} \quad T \to T_0^-,$$

Consequently, the new solution describes a finite time blow-up process.

- If $\alpha > 0$, $s < 0$, $T \in (T_0, \infty)$, we have that:

$$Q(X,Y,T) \to c\,\delta(X,Y) \quad \text{as} \quad T \to T_0^+,$$

$$Q(X,Y,T) \to 0 \qquad \text{as} \quad T \to \infty,$$

Then, the solution is an instantaneous source solution in which the structures created from the source suffer a diffusion process.

- If $\alpha < 0$, $s > 0$, $T \in (-\infty, T_0)$, we have that:

$$Q(X,Y,T) \to 0 \qquad \text{as} \quad T \to -\infty,$$

$$Q(X,Y,T) \to c\,\delta(X,Y) \quad \text{as} \quad T \to T_0^-,$$

then, the solution describes a creation process, and after the creation of the structures a finite-time blow-up takes place.

- If $\alpha < 0$, $s > 0$, $T \in (T_0, \infty)$, our solutions satisfies:

$$Q(X,Y,T) \to c\,\delta(X,Y) \quad \text{as} \quad T \to T_0^+,$$

$$Q(X,Y,T) \sim q(X,Y,T) \quad \text{as} \quad T \to \infty,$$

consequently, it is an instantaneous source solution.

- If $\alpha > 0$, $s > 0$, the solution is defined for $T \in \mathbb{R}$ and satisfies:

$$Q(X,Y,T) \sim q(X,Y,T) \quad \text{as} \quad T \to -\infty,$$

$$\max_{(X,Y)\in\mathbb{R}^2} |Q(X,Y,T)| \to 0 \quad \text{as} \quad T \to \infty,$$

thus, we have a regular solution describing a diffusion process.

- Finally, if $\alpha < 0$, $s < 0$, the solution is defined for $T \in \mathbb{R}$ and satisfies:

$$\max_{(X,Y)\in\mathbb{R}^2} |Q(X,Y,T)| \to 0 \quad \text{as} \quad T \to -\infty,$$

$$Q(X,Y,T) \sim q(X,Y,T) \qquad \text{as} \quad T \to \infty,$$

and the solution describes a creation process.

### 3.2. Example 2: $f(t) = t^a$, $a \neq 0, 1$, $t \in (0,\infty)$

In this case we have that

$$\frac{f(t)}{f(T)} = \left[1 - s(1-a)T^{a-1}\right]^{\frac{a}{1-a}},$$

now, by introducing

$$T_0 = \left(\frac{1}{s(1-a)}\right)^{\frac{1}{a-1}}, \quad \text{for the cases in which} \quad s(1-a) > 0,$$

and discussing as in the previous example we have that:

- If $a < 0$, $s > 0$, the solution is only defined when $T \in (T_0, \infty)$, and satisfies

$$Q(X,Y,T) \to c\,\delta(x,y) \qquad \text{as} \quad T \to T_0^+,$$

$$Q(X,Y,T) \sim q(X,Y,T) \quad \text{as} \quad T \to \infty,$$

thus, our solution is an instantaneous source solution.

- If $a < 0$, $s < 0$, we have the solution defined for $T \in (0,\infty)$ and the asymptotic behavior is given by:

$$\max_{(X,Y)\in\mathbb{R}^2} |Q(X,Y,T)| \to 0 \quad \text{as} \quad T \to 0^+,$$

$$Q(X,Y,T) \sim q(X,Y,T) \qquad \text{as} \quad T \to \infty.$$

As we have previously discussed, that means that the structures in the solution are instantaneously created at $T = 0$.

- If $a \in (0, 1),\ s > 0$, the solution has sense for $T \in (T_0, \infty)$ and satisfies

$$\max_{(X,Y)\in\mathbb{R}^2} |Q(X,Y,T)| \to 0 \quad \text{as} \quad T \to T_0^+,$$

$$Q(X,Y,T) \sim q(X,Y,T) \qquad \text{as} \quad T \to \infty.$$

  In this case, the solution describes an instantaneous creation process (the creation takes place at $T = T_0$). We illustrate this solution in figure 1 (in the next section) taking as the initial solution the breather solution. We have choose $a = \frac{1}{2}$ and $s = 1$, thus $T_0 = \frac{1}{4}$. We can see the creation process for $T \gtrsim T_0$, thus for $T = 0.26$ the structure is very small and it can not be appreciated that it is an exponentially localized structure. We can see its creation by plotting the graphic for $T = 0.4$ and $T = 0.6$. In the last three figures ($T = 1, 2, 3$) we observe the usual evolution of the breather solution (inner oscillation and movement at a constant velocity).

- If $a \in (0, 1),\ s < 0$, the solution is defined for $T \in (0, \infty)$ and verifies

$$Q(X,Y,T) \to c\,\delta(x,y) \qquad \text{as} \quad T \to 0^+,$$

$$Q(X,Y,T) \sim q(X,Y,T) \quad \text{as} \quad T \to \infty,$$

  thus, our solution is an instantaneous source solution.

- If $a > 1,\ s > 0$, we also have a regular solution for $T \in (0, \infty)$ and it is clear that

$$Q(X,Y,T) \sim q(X,Y,T) \qquad \text{as} \quad T \to 0^+,$$

$$\max_{(X,Y)\in\mathbb{R}^2} |Q(X,Y,T)| \to 0 \quad \text{as} \quad T \to \infty,$$

  thus, the solution describes a diffusion process.

- If $a > 1,\ s < 0$, our solution only has sense for $T \in (0, T_0)$ and we have that

$$Q(X,Y,T) \sim q(X,Y,T) \quad \text{as} \quad T \to 0^+,$$

$$Q(X,Y,T) \to c\,\delta(x,y) \qquad \text{as} \quad T \to T_0^-,$$

  i.e. the solution blows-up at finite time $T_0$.

Finally, we recall that we have taken the arbitrary functions $g$ and $h$ as zero. We have seen at the end of section 2, that the only effect of these functions in the solution (dependent variable $Q$) is a global movement of the solution on the plane. Thus, by taking appropriate functions $g$ and $h$, all the processes previously described can take place at arbitrary and not necessarily constant velocities in $\mathbb{R}^2$. For example, a source solution can emerge in any point in the plane, and not only in the origin as in our examples above.

## 4. Figures

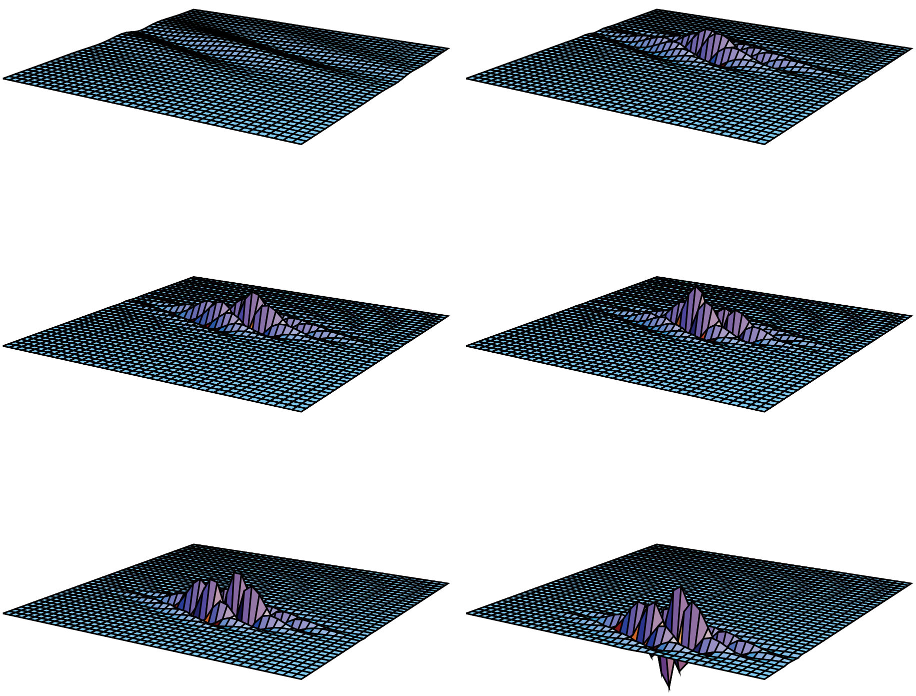

Figure 1: Deformation of the breather solution with $f(t) = t^{\frac{1}{2}}$, $s = 1$ and $T = 0.26, 0.4, 0.6, 1, 2, 3$, respectively

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# Invariant surfaces in $\mathbb{H}^2 \times \mathbb{R}$ with constant (mean or Gauss) curvature

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## 1. Introduction

Surfaces of revolution in the Euclidean three dimensional space are the first examples of invariant surfaces. They are invariant under the action of the one-parameter subgroup $SO(2)$ of the isometry group of $\mathbb{R}^3$. Since the beginning of differential geometry of surfaces much attention has been given to the surfaces of revolution with constant Gauss curvature or constant mean curvature. The surfaces of revolution with constant Gauss curvature seem to be known to Minding (1839), while those with constant mean curvature have been classified by Delaunay (1841). Aside from rotations, the isometry group of $\mathbb{R}^3$ includes the one-parameter subgroup of translations and that of helicoidal motions. It is rather interesting to note that the classification and explicit parametrisations of helicoidal surfaces in $\mathbb{R}^3$ (invariant under the action of helicoidal groups) with constant mean curvature have been achieved only recently, in 1982, by M.P. Do Carmo and M. Dajczer in [3]. In the last decades, due to the development of suitable reduction techniques [1, 4, 5, 7], appeared many works on the study of surfaces in a three dimensional manifold which are invariant under the action of a one-parameter subgroup of the isometry group, see, for example, [2, 5, 6, 7, 8, 9, 10, 14]. In this paper we consider the three dimensional manifold $\mathbb{H}^2 \times \mathbb{R}$, where by $\mathbb{H}^2$ we denote, as usual, the half-plane model of the hyperbolic space endowed with the standard metric of constant Gauss curvature $-1$. The space $\mathbb{H}^2 \times \mathbb{R}$ is one of the eight Thurston's geometries and its isometry group is of dimension 4 which is, among the 3-dimensional spaces with non-constant sectional curvature, the greatest possible. This means that the space $\mathbb{H}^2 \times \mathbb{R}$ is sufficiently symmetric to motivate the study of surfaces which are invariant under the action of a one-parameter subgroup of the isometry group. The paper is divided as follows. In section 2 we summarize the reduction techniques for invariant surfaces with constant mean curvature (CMC) or constant Gauss curvature; in section 3 we describe the isometry group of $\mathbb{H}^2 \times \mathbb{R}$ and the one-parameter groups of isometries; in section 4 we give the classification of invariant surfaces with constant mean curvature and in section 5 we analyze the invariant surfaces with constant Gauss curvature.

## 2. Reduction Techniques

Let $(N^3, g)$ be a three dimensional Riemannian manifold and let $X$ be a Killing vector field on $N$. Then $X$ generates a one-parameter subgroup $G_X$ of the group of isometries of $(N^3, g)$. For $x \in N$, the isotropy subgroup $G_x$ of $G_X$ is compact and the quotient space $G_X/G_x$ is diffeomorphic to the orbit $G(x) = \{gx \in N : g \in G\}$. An orbit $G(x)$ is called principal if there exists an open neighbourhood $U \subset N$ of $x$ such that for all orbits $G(y)$, $y \in U$, the isotropy subgroups $G_y$ are conjugate. If $N/G_X$ is connected, from the Principal Orbit Theorem ([13]), the principal orbits are all diffeomorphic and the regular set $N_r$, consisting of points belonging to principal orbits, is open and dense in $N$. Moreover, the quotient space $N_r/G_X$ is a connected differentiable manifold and the quotient map $\pi : N_r \to N_r/G_X$ is a submersion.

**Definition 1.** Let $f : M^2 \to (N^3, g)$ be an immersion from a surface $M^2$ into $N^3$ and assume that $f(M) \subset N_r$. We say that $f$ is a $G_X$-**equivariant** immersion, and $f(M)$ a $G_X$-**invariant** surface of $N$, if there exists an action of $G_X$ on $M^2$ such that for any $x \in M^2$ and $g \in G_X$ we have $f(gx) = gf(x)$.

A $G_X$-equivariant immersion $f : M^2 \to (N^3, g)$ induces on $M^2$ a Riemannian metric, the pull-back metric, denoted by $g_f$ and called the $G_X$-*invariant induced metric*. Let $f : M^2 \to (N^3, g)$ be a $G_X$-equivariant immersion from a surface $M^2$ into a Riemannian manifold $(N^3, g)$ and let endow $M^2$ with the $G_X$-invariant induced metric $g_f$. Assume that $f(M^2) \subset N_r$ and that $N/G_X$ is connected. Then $f$ induces an immersion $\tilde{f} : M/G_X \to N_r/G_X$ between the orbit spaces; moreover, the space $N_r/G_X$ can be equipped with a Riemannian metric, the *quotient metric*, so that the quotient map $\pi : N_r \to N_r/G_X$ is a Riemannian submersion. Following [8] we shall describe the quotient metric of the regular part of the orbit space $N/G_X$. It is well known (see, for example [11]) that $N_r/G_X$ can be locally parametrized by the invariant functions of the Killing vector field $X$. If $\{f_1, f_2\}$ is a complete set of invariant functions on a $G_X$-invariant subset of $N_r$, then the quotient metric is given by $\tilde{g} = \sum_{i,j=1}^{2} h^{ij} df_i \otimes df_j$ where $(h^{ij})$ is the inverse of the matrix $(h_{ij})$ with entries $h_{ij} = g(\nabla f_i, \nabla f_j)$. If we denote by $\omega(y) = \|X(y)\|$ the volume function of the principal orbit $G(y) = \{gy : g \in G\}$, then the mean curvature function of $f$ can be expressed in terms of the geodesic curvature of $\tilde{f}$ and of the function $\omega(y)$ as it is shown in the following

**Theorem 1** (Reduction Theorem [1]). *Let $H$ be the mean curvature function of $f : M^2 \to N^3$ and $k_g$ the geodesic curvature of $\tilde{f} : M/G_X \to N_r/G_X$. Then*

$$H = k_g - \partial_n(\ln \omega),$$

*where $n$ is the unit normal vector to $M/G_X$ in $N_r^3/G_X$.*

## 2.1. Invariant surfaces with constant Gauss curvature

We first give a local description of the $G_X$-invariant surfaces of $N^3$. Let $\gamma : (a,b) \subset \mathbb{R} \to (N_r^3/G_X, \tilde{g})$ be a curve parametrized by arc length and let $\tilde{\gamma} : (a,b) \subset \mathbb{R} \to N_r^3$ be a lift of $\gamma$, such that $d\pi(\tilde{\gamma}') = \gamma'$. If we denote by $\phi_r$, $r \in (-\epsilon, \epsilon)$, the local flow of the Killing vector field $X$, then the map

$$\psi : (a,b) \times (-\epsilon, \epsilon) \to N^3, \quad \psi(t,r) = \phi_r(\tilde{\gamma}(t)),$$

defines a parametrized $G_X$-invariant surface. Conversely, if $f : M^2 \to N_r^3$ is a $G_X$-equivariant immersion, then $\tilde{f}$ defines a curve in $(N_r^3/G_X, \tilde{g})$ that can be, locally, parametrized by arc length. The curve $\gamma$ is generally called the *profile* curve. The following theorem describe (locally) the invariant surfaces with constant Gauss curvature.

**Theorem 2** ([10]). *Let $f : M^2 \to (N^3, g)$ be a $G_X$-equivariant immersion, $\gamma : (a,b) \subset \mathbb{R} \to (N_r^3/G_X, \tilde{g})$ a parametrisation by arc length of $\tilde{f}$ and $\tilde{\gamma}$ a lift of $\gamma$.*

*(i) If the $G_X$-invariant induced metric $g_f$ is of constant Gauss curvature $K$, then the function $\omega(t) = \|X(\tilde{\gamma})\|_g$ satisfies the following differential equation*

$$\frac{d^2}{dt^2}\omega(t) + K\omega(t) = 0. \tag{1}$$

*(ii) Vice versa, suppose that Equation* (1) *holds with $K$ a real constant. Then, in all points where $d(\omega^2)/dt \neq 0$, the $G_X$-invariant induced metric $g_f$ has constant Gauss curvature.*

By integration of (1) we have

**Corollary 3.** *Let $f : M^2 \to (N^3, g)$ be a $G_X$-equivariant immersion which induces a $G_X$-invariant metric $g_f$ on $M^2$ of constant Gauss curvature $K$. Then the norm $\omega(t)$ of the Killing vector field $X$ along a lift of the profile curve is:*

*for $K = 0$ given by $\omega(t) = c_1 t + c_2$;*

*for $K = 1/R^2 > 0$ given by $\omega(t) = c_1 \cos(t/R) + c_2 \sin(t/R)$;*

*for $K = -1/R^2 < 0$ given by $\omega(t) = c_1 \cosh(t/R) + c_2 \sinh(t/R)$,*

*with $c_1, c_2 \in \mathbb{R}$.*

As we shall show in Section 5 the profile curve of a $G_X$-invariant surface can be parametrized as a function of $\omega$. Thus, using Corollary 3, we can give the explicit parametrisation of the profile curve.

*Remark* 1. If $(N^3, g) = (\mathbb{R}^3, can)$ is the Euclidean three dimensional space, then the Killing vector fields generate either translations or rotations. In the case of translations the quotient space $\mathbb{R}^3/G_X$ is $\mathbb{R}^2$ with the flat metric and $\omega$ is constant. Thus, from Equation 1, we see that any curve in the quotient space generates a flat right cylinder. In the case of rotations we can assume, without loss of generality, that the rotation is about a coordinate axis, say $x_3$. Then the Killing vector field is $X = -x_2 \frac{\partial}{\partial x_1} + x_1 \frac{\partial}{\partial x_2}$ and the regular part of the quotient space is $\mathbb{R}^3_r/G_X = \{(x_1, x_2, x_3) \in \mathbb{R}^3 \ : \ x_2 = 0, \ x_1 > 0\}$ with the flat metric. If $\gamma(t) = (u(t), 0, v(t)) \in \mathbb{R}^3_r/G_X$ is a arc length parametrized profile curve of a $G_X$-invariant surface, then the norm of $X$ restricted to the profile curve is $\omega = u(t)$ and, using Corollary 3, we find the classical explicit parametrisation of surfaces of revolution with constant Gauss curvature.

## 3. One-parameter subgroups of isometries of $\mathbb{H}^2 \times \mathbb{R}$

Let $\mathbb{H}^2 = \{(x, y) \in \mathbb{R}^2 \ : \ y > 0\}$ be the half plane model of the hyperbolic plane endowed with the metric, of constant Gauss curvature $-1$, given by

$$g_{\mathbb{H}} = \frac{dx^2 + dy^2}{y^2}.$$

The hyperbolic plane $\mathbb{H}^2$, with the group structure derived by the composition of proper affine transformations, is a Lie group and the metric $g_{\mathbb{H}}$ is left-invariant. Then the product space $\mathbb{H}^2 \times \mathbb{R}$ is a Lie group with the product structure

$$L_{(x,y,z)}(x',y',z') = (x,y,z) * (x',y',z') = (x'y + x, yy', z + z')$$

and the left invariant metric given by the product metric

$$g = \frac{dx^2 + dy^2}{y^2} + dz^2.$$

From a direct integration of the Killing equation $L_X g = 0$ we have

**Proposition 4.** *The Lie algebra of the infinitesimal isometries of the product* $(\mathbb{H}^2 \times \mathbb{R}, g)$ *admits the following bases of Killing vector fields*

$$X_1 = \frac{(x^2 - y^2)}{2}\frac{\partial}{\partial x} + xy\frac{\partial}{\partial y}; \quad X_2 = \frac{\partial}{\partial x}; \quad X_3 = x\frac{\partial}{\partial x} + y\frac{\partial}{\partial y}; \quad X_4 = \frac{\partial}{\partial z}.$$

Let denote by $G_i$ the one-parameter subgroup of isometries generated by $X_i$, by $G_{ij}$ the one-parameter subgroup of isometries generated by linear combinations of $X_i$ and $X_j$ and so on. Explicitly we have that

$$\begin{aligned}
G_1 &= \{\mathcal{L}_{(t,0,0,0)} \,|\, t \in \mathbb{R}\} \qquad \text{with} \\
&\mathcal{L}_{(t,0,0,0)}(x,y,z) = \Big(\frac{-2[t(x^2+y^2) - 2x]}{(tx-2)^2 + t^2y^2}, \frac{4y}{(tx-2)^2+t^2y^2}, z\Big); \\
G_2 &= \{\mathcal{L}_{(0,t,0,0)} \,|\, t \in \mathbb{R}\} \qquad \text{with} \qquad \mathcal{L}_{(0,t,0,0)}(x,y,z) = (x+t, y, z); \\
G_3 &= \{\mathcal{L}_{(0,0,t,0)} \,|\, t \in \mathbb{R}\} \qquad \text{with} \qquad \mathcal{L}_{(0,0,t,0)}(x,y,z) = (e^t x, e^t y, z); \\
G_4 &= \{\mathcal{L}_{(0,0,0,t)} \,|\, t \in \mathbb{R}\} \qquad \text{with} \qquad \mathcal{L}_{(0,0,0,t)}(x,y,z) = (x, y, z+t).
\end{aligned}$$

*Remark* 2. The integral curves of $X_2$, $X_3$ and $X_4$ are easy to picture out. In fact, for $t$ fixed the isometries $\mathcal{L}_{(0,t,0,0)} \equiv L_{(t,1,0)}$, $\mathcal{L}_{(0,0,t,0)} \equiv L_{(0,e^t,0)}$ and $\mathcal{L}_{(0,0,0,t)} \equiv L_{(0,1,t)}$ are left translations. The integral curve of $X_1$, through the point $p_0 = (x_0, y_0, z_0) \in \mathbb{H}^2 \times \mathbb{R}$, is $\mathcal{L}_{(t,0,0,0)}(x_0,y_0,z_0) = (x(t), y(t), z_0)$, where

$$x(t)^2 + y(t)^2 - \Big(\frac{x_0^2 + y_0^2}{y_0}\Big)y(t) = 0.$$

Therefore, it is a horocycle, in the plane $z = z_0$, with radius $(x_0^2 + y_0^2)/2y_0$ and centered at $(0, (x_0^2 + y_0^2)/2y_0, z_0)$. In Figure 1 there is a plot of the integral curves of $X_1$ through three different points.

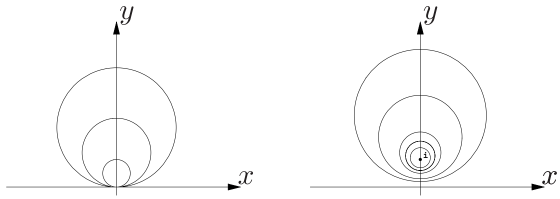


Figure 1: Integrale curves of $X_1$ (left) and of $X_{12}^*$ (right).

Two groups $G_X$ and $G_Y$, generated by two Killing vector fields $X$ and $Y$, are *conjugate* if there exists an isometry $\varphi$ of $\mathbb{H}^2 \times \mathbb{R}$ such that $G_Y = \varphi^{-1} G_X \varphi$. If $G_X$ and $G_Y$ are conjugate, then the respectively invariant surfaces are congruent, i.e. isometric with respect to the isometry $\varphi$ of the ambient space. Therefore, we can reduce the study of the invariant surfaces by analyzing all the conjugate one-parameter groups of isometries. In [12] there is the complete list of the conjugate groups of isometries in $\mathbb{H}^2 \times \mathbb{R}$ which gives the following

**Lemma 5** ([12])**.** *Any surface in $\mathbb{H}^2 \times \mathbb{R}$ which is invariant under the action of a one-parameter subgroup of isometries $G_X$, generated by a Killing vector field $X = \sum_i a_i X_i$, is isometric to a surface invariant under the action of one of the following groups*

$$G_{24}, \quad G_{34}, \quad G_{12}^*, \quad G_{124}^*,$$

*where $G_{12}^*$ is the one-parameter group generated by $X_{12}^* = X_1 + (X_2)/2$ and $G_{124}^*$ is the one-parameter group generated by $X_{12}^*$ and $X_4$.*

*Remark* 3. The integral curve of $X_{12}^* = X_1 + X_2/2$ through the point $p_0 = (x_0, y_0, z_0) \in \mathbb{H}^2 \times \mathbb{R}$ is $\mathcal{L}_{(t,t/2,0,0)}(x_0, y_0, z_0) = (x(t), y(t), z_0)$, where

$$x(t)^2 + y(t)^2 - \Big(\frac{1 + x_0^2 + y_0^2}{y_0}\Big) y(t) + 1 = 0.$$

An easy computation shows that the hyperbolic distance from a point of the integral curve to the point $(0, 1, z_0)$ is constant. Therefore, the integral curves of $X_{12}^*$ are geodesic circles centred at $(0, 1, z_0)$ (see Figure 1).

## 4. Invariant surfaces with constant mean curvature

In this section we shall consider only the actions of $G_4$ and $G_{124}^*$ which lead to invariant surfaces with a nice geometric description. For a detailed account of the classifications presented in this section and for the other actions, we refer the reader to [12].

### 4.1. CMC surfaces invariant under the action of the group $G_4$

The group $G_4$, generated by the Killing vector field $X_4 = \frac{\partial}{\partial z}$, acts freely on $\mathbb{H}^2 \times \mathbb{R}$, thus the regular part is the whole space. A complete set of invariant functions of $X_4$ is

$$u(x, y, z) = x \qquad \text{and} \qquad v(x, y, z) = y.$$

Thus the orbit space is $\mathbb{H}^2 = \{(u, v) \in \mathbb{R}^2 \mid v > 0\}$ and the orbital metric is given by $g_{\mathbb{H}} = \frac{du^2+dv^2}{v^2}$. From the Reduction Theorem 1 we have that a curve $\gamma(s) = (u(s), v(s))$ in the orbit space $\mathbb{H}^2$, parametrized by arc length, generates a CMC surface if $u$ and $v$ satisfy the following system

$$\begin{cases} \dot{u} = v\,\cos\sigma, \quad \dot{v} = v\,\sin\sigma, \\ H = \dot{\sigma} + \cos\sigma = k_g, \end{cases} \tag{2}$$

where $\sigma = \sigma(s)$ is the angle between $\dot{\gamma}$ and the positive $u$ direction, while $k_g$ is the geodesic curvature of $\gamma$. Now, assuming that the mean curvature $H$ is constant and non negative, we have that the function $J(s) = \dot{\sigma}/v$ is constant along any curve $\gamma(s)$ which is a solution of system (2). Thus the solutions of (2) are given by $J(s) = k$, for some $k \in \mathbb{R}$. By a qualitative analysis of the equation $J(s) = k$, we can prove the following

**Theorem 6.** *The CMC surfaces in $\mathbb{H}^2 \times \mathbb{R}$, which are invariant under the action of the subgroup $G_4$, are vertical cylinders over curves of $\mathbb{H}^2$ with constant geodesic curvature. Moreover:*

- *if $H = 0$, they are geodesics of $\mathbb{H}^2$. In particular, if*
  1. *$k = 0$, the curve is an Euclidean ray normal to the line $v = 0$;*
  2. *$k \neq 0$, the curve is an Euclidean semicircle with center on the line $v = 0$;*
- *if $H > 0$ they are:*
  1. *for $H > 1$ Euclidean circles;*
  2. *for $H = 1$ horocycles;*
  3. *for $H < 1$ hypercycles.*

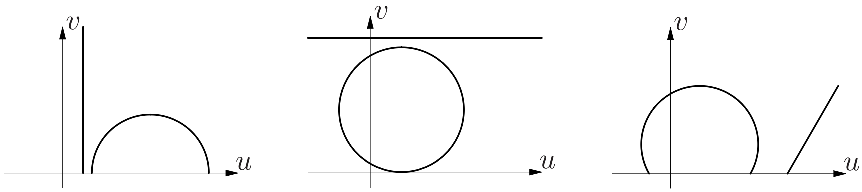


Figure 2: Profile curves of $G_4$-invariant CMC surfaces: geodesics (left), horocycles (center) and hypercycles (right).

### 4.2. Helicoidal CMC surfaces: invariant under the action of $G^*_{124}$

Introducing the cylindrical coordinates $(r, \theta, z)$ into $(\mathbb{H}^2 \times \mathbb{R}, g)$, with $r > 0$ and $\theta \in (0, \pi)$, the metric $g$ takes the form

$$g = \frac{dr^2}{r^2 \sin^2 \theta} + \frac{d\theta^2}{\sin^2 \theta} + dz^2,$$

and the Killing vector field $X^*_{124}$ becomes

$$X^*_{124} = X^*_{12} + aX_4 = \frac{r^2+1}{2} \cos\theta \frac{\partial}{\partial r} + \frac{r^2-1}{2r} \sin\theta \frac{\partial}{\partial \theta} + a\frac{\partial}{\partial z}.$$

Choosing the invariant functions

$$u(r,\theta,z) = \frac{r^2+1}{r\sin\theta} \qquad \text{and} \qquad v(r,\theta,z) = z + a \arctan\left(\frac{2r\cos\theta}{r^2-1}\right),$$

the orbit space is $\mathcal{B} = \{(u, v) \in \mathbb{R}^2 |\ u \geq 2\}$ and the quotient metric reduces to

$$\widetilde{g} = \frac{du^2}{(u^2-4)} + \frac{u^2-4}{u^2+4(a^2-1)} dv^2.$$

The system of ODE's that characterizes the profile curve $\gamma(s)$ of a $G^*_{124}$-surface is:

$$\begin{cases} \dot{u} = \sqrt{u^2-4}\cos\sigma, \quad \dot{v} = \sqrt{\dfrac{u^2+4(a^2-1)}{u^2-4}} \sin\sigma, \\ \dot{\sigma} = H - \dfrac{u}{\sqrt{u^2-4}} \sin\sigma. \end{cases} \tag{3}$$

*Remark* 4. The Equation (3) for $\sigma$ has a singularity at the boundary of $\mathcal{B}$. This type of singularity has been dealt extensively in the literature (see, for example, [4, 6]). In particular, solutions that go to the boundary must enter *orthogonally*, which means that the generated surface will be regular at those points.

If $H$ is constant, the function $J(s) = \sqrt{u^2-4}\,\sin\sigma - Hu$ is constant along any curve $\gamma(s)$ which is a solution of System (3). Thus the solutions of this system are given by $\sqrt{u^2-4}\,\sin\sigma - Hu = k$, $k \in \mathbb{R}$. As in the previous section a qualitative analysis of the equation $J(s) = k$, gives the following characterization of the profile curves of CMC helicoidal surfaces.

**Theorem 7.** *Let $\Sigma \subset \mathbb{H}^2 \times \mathbb{R}$ be a CMC helicoidal surface and let $\gamma$ be the profile curve in the orbit space. Then we have the following characterization of $\gamma$ according to the value of the mean curvature $H$ and of $k$.*

1. $(\mathbf{H > 1})$ - *The profile curve is of Delaunay type. Moreover if*
   - $k < -2H$ *is of* **nodary-type**;
   - $k = -2H$ *is of* **circle-type**;
   - $k > -2H$ *is of* **undulary-type**.
2. $(\mathbf{H = 1})$ - *The profile curve is*
   - *for* $k < -2$ *of* **folium-type**;
   - *for* $k = -2$ *of* **conic-type**;
   - *for* $k > -2$ *of* **bell-type**.
3. $(\mathbf{0 < H < 1})$ - *The profile curve is*
   - *for* $k < -2H$ *of* **bounded folium-type**;
   - *for* $k = -2H$ *of* **helicoidal-type**;
   - *for* $k > -2H$ *of* **bounded bell-type**.
4. $(\mathbf{H} = 0)$ - *The profile curve is*
   - *for* $k = 0$ *a horizontal straight line;*
   - *for* $k \neq 0$ *of* **catenary-type**.

*Remark* 5. (i) The plots of the profile curves in Figure 3 and Figure 4 are drawn using the qualitative analysis of the angle $\sigma$ with respect to the metric $\widetilde{g}$ of the orbit space.
(ii) The plots of the profile curves of the helicoidal surfaces with $0 < H < 1$ are similar to those with $H = 1$; the only different is that for $0 < H < 1$ the limit of the angle $\sigma$, for $u$ that goes to infinity, is between 0 and $\pi/2$, instead of $\pi/2$ as for the helicoidal surfaces with $H = 1$.
(iii) We note that some of the invariant surfaces described in Theorem 7 are complete, for example the minimal surface of revolution ($G_{12}^*$-invariant) generated by a curve of catenary-type. Moreover, there are interesting examples

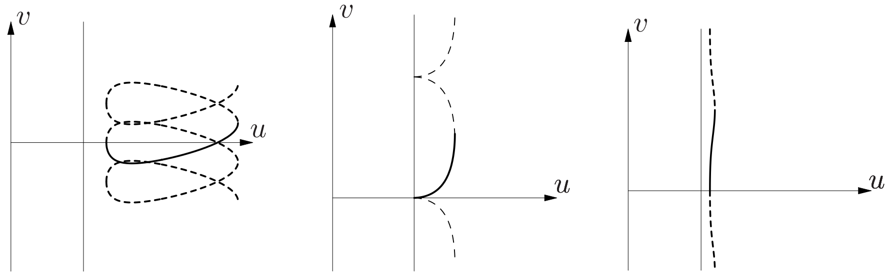


Figure 3: Profile curves of helicoidal CMC surfaces with $H > 1$: nodary-type (left), circle-type (center) and undulary-type (right).

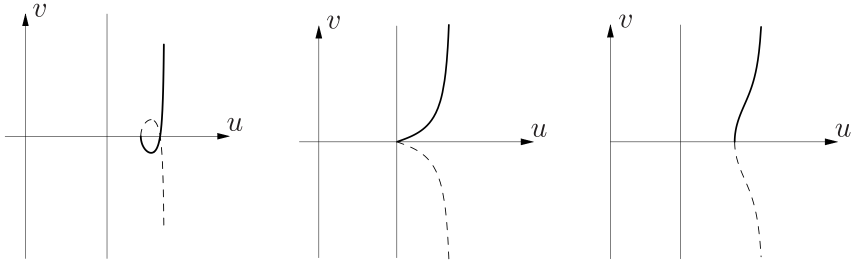


Figure 4: Profile curves of helicoidal CMC surfaces with $H = 1$: folium-type (left), conic-type (center) and bell-type (right).

of complete minimal surfaces which are invariant under the action of the group $G_{34}$. It is proved in [12] that the function $f : \mathbb{H}^2 \to \mathbb{R}$, given by $f(x, y) = \ln(x^2 + y^2)$, defines a complete minimal graph of $\mathbb{H}^2 \times \mathbb{R}$ which is $G_{34}$ invariant; thus the Berstein Theorem in $\mathbb{H}^2 \times \mathbb{R}$ does not hold. In Figure 5 there is a plot of such a surface.

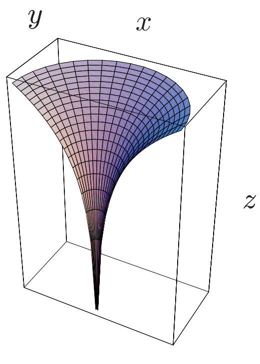


Figure 5: A complete minimal graph of $\mathbb{H}^2 \times \mathbb{R}$.

## 5. Invariant surfaces with constant Gauss curvature

Let $G$ be a one-parameter group of isometries among those described in Lemma 5. If we denote, as before, by $\omega$ the volume function of the principal orbits, we can give the following local description of the $G$-invariant surfaces of $\mathbb{H}^2 \times \mathbb{R}$.

**Theorem 8** ([10]). *Let $\gamma = (u(s), v(s))$ be a curve in the orbit space $(\mathbb{H}^2 \times \mathbb{R}/G, \tilde{g})$, parametrized by arc length, which is the profile curve of a $G$-invariant surface in $(\mathbb{H}^2 \times \mathbb{R})$. Then:*

- *if $G = G_4$, the orbit space is $\mathbb{H}^2$ and any curve parametrized by arc length is the profile curve of a flat $G_4$-invariant cylinder;*
- *if $G = G_{24}$, the orbit space is $\mathcal{B} = \{(u, v) \in \mathbb{R}^2 \ : \ u > 0\}$ and the profile curve can be parametrized by*

$$\begin{cases} u(s) = |a|/\sqrt{\omega^2 - b^2}, \quad a, b \in \mathbb{R} \\ v(s) = \int_{s_0}^{s} \sqrt{\frac{a^2\omega^2}{\omega^2 - b^2}\left[1 - \left(\frac{\omega\omega'}{\omega^2 - b^2}\right)^2\right]}dt; \end{cases}$$

- *if $G = G_{34}$, then the orbit space is $\mathcal{B} = \{(u, v) \in \mathbb{R}^2 \ : \ 0 < u < \pi\}$ and the profile curve can be parametrized by*

$$\begin{cases} u(s) = \arcsin\left(|a|/\sqrt{\omega^2 - b^2}\right), \quad a, b \in \mathbb{R} \\ v(s) = \int_{s_0}^{s} \sqrt{\frac{a^2\omega^2}{\omega^2 - b^2}\left[1 - \frac{(\omega\omega')^2}{(\omega^2 - b^2)(\omega^2 - a^2 - b^2)}\right]}dt; \end{cases}$$

- *if $G = G^*_{124}$, then the orbit space is $\mathcal{B} = \{(u, v) \in \mathbb{R}^2 \ : u \geq 2\}$ and the profile curve can be parametrized by*

$$\begin{cases} u(s) = 2\sqrt{\omega^2 + 1 - a^2}, \quad a \in \mathbb{R} \\ v(s) = \int_{s_0}^{s} \sqrt{\frac{\omega^2}{\omega^2 - a^2}\left[1 - \frac{(\omega\omega')^2}{(\omega^2 - a^2)(\omega^2 + 1 - a^2)}\right]}dt. \end{cases}$$

Now if $\gamma$ is the profile curve of a $G$-invariant surface in $\mathbb{H}^2 \times \mathbb{R}$ with constant Gauss curvature, then the explicit parametrisation of $\gamma$ can be obtained by replacing in Theorem 8 the corresponding expression of the function $\omega$, according to the value of the Gauss curvature $K$, as we have described in Corollary 3. For example, in the case of the $G^*_{12}$-invariant surfaces of $\mathbb{H}^2 \times \mathbb{R}$, for some values of $\omega$ we have:

1) if $K = 0$, choosing $\omega(s) = s$, we have the following parametrisation for the profile curve
$$\gamma(s) = (2\sqrt{s^2+1}, \sqrt{s^2+1});$$

2) if $K > 0$, choosing $\omega(s) = \cos s$, the corresponding profile curve is
$$\gamma(s) = (2\sqrt{\cos^2 s + 1}, \frac{\sqrt{2\cos^2 s}}{\cos s} \arctan\Big(\frac{\sin s}{\sqrt{\cos^2 s + 1}}\Big));$$

3) if $K < 0$, taking the function $\omega(s) = \sinh s$, we obtain
$$\gamma(s) = (2\cosh s, 0).$$

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# Classical and quantum geometry of moduli spaces in three-dimensional gravity

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**Abstract.** We describe some results concerning the phase space of 3-dimensional Einstein gravity when space is a torus and with negative cosmological constant. The approach uses the holonomy matrices of flat $SL(2,\mathbb{R})$ connections on the torus to parametrise the geometry. After quantization, these matrices acquire non-commuting entries, in such a way that they satisfy $q$-commutation relations and exhibit interesting geometrical properties. In particular they lead to a quantization of the Goldman bracket.

*Keywords:* quantum geometry, moduli spaces, 3-dimensional spacetime, gravity, holonomy, Goldman bracket

*2000 Mathematics Subject Classification:* 83C45

## 1. Introduction

From the point of view of geometry, the theory of classical general relativity (see Fernando Barbero's lectures in this volume) is the study of Riemannian or semi-Riemannian geometries (depending on the choice of Euclidean or Lorentzian signature) which satisfy the Einstein equations. In 3-dimensional spacetime these equations for the components $g_{\mu\nu}$ of the metric tensor are derived from the Einstein–Hilbert action

$$\int \sqrt{|g|}(R+\Lambda)d^3x \tag{1}$$

where integration is over the spacetime manifold, and we have included a cosmological constant $\Lambda$. In the first term of (1) the Ricci scalar, a contraction of the Riemann tensor, appears. This term may be written as follows:

$$\frac{1}{2}\int R^{\mu\sigma}{}_{\rho\nu}\epsilon_{\mu\sigma\alpha}\epsilon^{\rho\nu\beta}\delta^{\alpha}_{\beta}d^3x \tag{2}$$

where the usual summation convention over repeated indices is used and indices on the totally antisymmetric tensor $\epsilon_{\mu\nu\rho}$ are raised with the inverse metric tensor $g^{\mu\nu}$.

It is convenient to rewrite the action (1) in terms of orthonormal dreibeins or triads $e^a$. These are a local basis of 1-forms

$$e^a = e^a_\mu(x)\ dx^\mu, a = 1, 2, 3 \tag{3}$$

such that

$$g_{\mu\nu}dx^\mu \otimes dx^\nu = e^a \otimes e^b \eta_{ab} \tag{4}$$

where $\eta_{ab} = \mathrm{diag}(-1, 1, 1)_{ab}$. Then the action (1) takes the form

$$\int (R^{ab} \wedge e^c + \frac{\Lambda}{3}e^a \wedge e^b \wedge e^c)\epsilon_{abc} \tag{5}$$

where $R^{ab}$ are the curvature 2-forms

$$R^{ab} = \frac{1}{2}R^{ab}{}_{\mu\nu}dx^\mu \wedge dx^\nu \tag{6}$$

and $R^{ab}{}_{\mu\nu}$ is the Riemann tensor that appears in (2) contracted with the dreibein components (3).

In the dreibein formulation, there is an extra gauge symmetry of local Lorentz transformations $e^a \mapsto M^a{}_b e^b$ where $M \in SO(2, 1)$ (local, since $M$ depends on the point of spacetime). This extra freedom arises since one may simultaneously rotate the three fields $e^a$, whilst preserving the metric and the condition (4).

There is a striking similarity between the action in the form (5) and the Chern-Simons action for a connection $A$ in a principal $G$-bundle, which has the structure

$$\int \mathrm{tr}(F \wedge A - \frac{1}{3}A \wedge A \wedge A).$$

Indeed, it was shown by Witten [1] that the action (5) may be interpreted as a Chern-Simons action for $G = SO(2, 2)$, when $\Lambda < 0$ (and for $G = SO(3, 1)$ when $\Lambda > 0$). The connection in the Chern-Simons theory is given in terms of the dreibein $e^a$ and spin connection (or Ricci rotation coefficient) $\omega^{ab}$ by:

$$A = \frac{1}{2}\omega^{ab}M_{ab} + e^a M_{a4}, \tag{7}$$

where the indices $a, b$ run from 1 to 3, and $\{M_{AB}\}_{A,B=1,\ldots,4}$ is a basis of the Lie algebra of $SO(2,2)$. Note that in this so-called first-order formalism, the dreibein $e^a$ and spin connection $\omega^{ab}$ are independent fields.

We conclude this introduction with a short discussion of the relation between connections and holonomy. Given a connection on a principal $G$-bundle, a holonomy is an assignment of an element $H(\gamma)$ of $G$ to each (based) loop $\gamma$ on the manifold, obtained by lifting the loop into the total space of the bundle and comparing the starting and end points of the lifted loop in the fibre over the basepoint. Holonomy is, in a suitable sense, equivalent to the connection it is derived from. When the connection is flat, i.e. has zero fieldstrength $F$, the holonomy of $\gamma$ only depends on $\gamma$ up to homotopy. Thus an efficient way of describing flat connections is to specify a group morphism from the fundamental group of the manifold to the group $G$.

## 2. Equations of motion and the classical phase space

Consider the Chern-Simons action

$$\int_{\Sigma\times\mathbb{R}} \mathrm{tr}(A\wedge dA + \frac{2}{3}A\wedge A\wedge A) \tag{8}$$

on a spacetime of the form $\Sigma\times\mathbb{R}$, where $\Sigma$ is a closed surface representing space and $\mathbb{R}$ represents time. The connection 1-form $A$ may be written as

$$A = A_i dx^i + A_0 dx^0$$

where $x^i$, $i = 1, 2$ are coordinates on $\Sigma$, and $x^0 = t$ is the time coordinate. Imposing the gauge fixing condition

$$A_0 = 0$$

and the corresponding constraint

$$F_{ij} dx^i \wedge dx^j = 0$$

we see that the connections are flat. The action (8) now has the structure

$$\int A_2 \partial_0 A_1 + \ldots$$

and therefore "$A$ is its own conjugate momentum". The Poisson brackets for the components of $A$ (see equation (7)) have the following form:

$$\left\{ {A_1}^a(x), {A_j}^b(y) \right\} = \delta^{ab}\epsilon_{ij}\delta^2(x-y) \quad \epsilon_{12} = 1. \tag{9}$$

We now choose the space manifold to be the torus $\mathbb{T}^2$, and since the group $SO(2,2)$ is isomorphic to $SL(2,\mathbb{R}) \times SL(2,\mathbb{R})/\mathbb{Z}_2$, we restrict ourselves to studying the phase (moduli) space $P$ of flat $SL(2,\mathbb{R})$ connections on the torus $\mathbb{T}^2$, modulo gauge transformations. Note that $P$ is in principle a complicated space to describe, being an infinite-dimensional space divided by an infinite-dimensional group, but in the holonomy picture there is a very simple finite-dimensional description, as we will see shortly. In this picture the moduli space of flat $SO(2,2)$ connections can then be constructed from the product of two copies of $P$ by identifying equivalent elements under the $\mathbb{Z}_2$ action.

Since $\pi_1(\mathbb{T}^2) = \langle \gamma_1, \gamma_2 | \gamma_1\gamma_2\gamma_1^{-1}\gamma_2^{-1} = 1 \rangle$ where $\gamma_1$ and $\gamma_2$ are a pair of generating cycles, a holonomy

$$H : \pi_1(\mathbb{T}^2) \to SL(2,\mathbb{R})$$

is given by $U_1 := H(\gamma_1)$ and $U_2 := H(\gamma_2)$, since this determines $H$ on any other homotopy class of loops. The phase space $P$ is then

$$P = \{(U_1, U_2) | U_1U_2 = U_2U_1\} / \sim$$

where $\sim$ denotes the remaining gauge freedom, namely

$$(U_1, U_2) \sim (S^{-1}U_1S, S^{-1}U_2S)$$

for any $S \in SL(2,\mathbb{R})$.

For a single matrix $U \in SL(2,\mathbb{R})$ there are four possibilities for how $U$ can be conjugated into a standard form:

A) $U$ has 2 real eigenvalues:

$$S^{-1}US = \begin{pmatrix} \lambda & 0 \\ 0 & \lambda^{-1} \end{pmatrix}$$

B) $U$ has 1 real eigenvalue with an eigenspace of dimension 2:

$$S^{-1}US = \pm \begin{pmatrix} 1 & 0 \\ 0 & 1 \end{pmatrix}$$

C) $U$ has 1 real eigenvalue with an eigenspace of dimension 1:

$$S^{-1}US = \begin{pmatrix} \pm 1 & 1 \\ 0 & \pm 1 \end{pmatrix}$$

D) $U$ has no real eigenvalues:

$$S^{-1}US = \begin{pmatrix} \cos\theta & -\sin\theta \\ \sin\theta & \cos\theta \end{pmatrix}$$

A similar analysis for a pair of commuting $SL(2,\mathbb{R})$ matrices led in [2] to an explicit parametrization of the classical phase space $P$, whose elements are pairs $(U_1, U_2)$ of commuting $SL(2,\mathbb{R})$ matrices, identified up to simultaneous conjugation by an $SL(2,\mathbb{R})$ matrix. The parametrization is based on putting the pairs of matrices into a standard form, analogous to e.g. Jordan canonical form, but for pairs rather than a single matrix. The structure of $P$ resembles that of a cell complex with, for instance 2-dimensional cells consisting of pairs of diagonal matrices, or pairs of rotation matrices. However there are also 1-dimensional cells which consist of e.g. pairs of non-diagonalisable matrices of the form:

$$U_1 = \begin{pmatrix} 1 & \cos\alpha \\ 0 & 1 \end{pmatrix} \quad U_2 = \begin{pmatrix} 1 & \sin\alpha \\ 0 & 1 \end{pmatrix},\ 0 < \alpha < \frac{\pi}{2}. \tag{10}$$

For further details, depictions of $P$ and a discussion of its topology see [2].

## 3. Quantization via quantum matrices

The Poisson brackets (9) are for non-gauge-invariant variables so it is convenient to change to gauge-invariant variables, and an obvious choice are the traced holonomies

$$T(\gamma) = \frac{1}{2}\ \mathrm{tr}\ H(\gamma)$$

which are gauge-invariant due to the conjugation invariance of the trace. The holonomy is sometimes written as a path-ordered exponential, or Chen integral,

$$H(\gamma) = \mathcal{P}\exp\int_\gamma A$$

and from equation (9) the Poisson brackets between the $T(\gamma)$ are only non-vanishing if the loops intersect transversally. From trace identities for $2\times 2$ matrices it is enough to consider the following three variables:

$$T_1 := T(\gamma_1) \quad T_2 := T(\gamma_2) \quad T_3 := T(\gamma_1\gamma_2)$$

(which are not independent since they satisfy the identity $T_1^2 + T_2^2 + T_3^2 - 2T_1T_2T_3 = 1$). Their Poisson bracket relations are [3]

$$\{T_i, T_j\} = \epsilon_{ij}{}^k T_k + T_iT_j, \quad i,j,k = 1,2,3. \tag{11}$$

(here we have rescaled the variables compared to [3] to absorb the coupling constants).

The first term on the right-hand-side of equation (11) has a geometric interpretation in terms of rerouted loops: e.g. for $i = 1$, $j = 2$ the two cycles $\gamma_1$ and $\gamma_2$ intersect transversally at one point, and from homotopy invariance of the holonomy $T_3$ is the traced holonomy corresponding to the loop $\gamma_1 S \gamma_2$ obtained by starting at the basepoint, following $\gamma_1$ to the intersection point $S$, rerouting along the loop $\gamma_2$ back to the intersection point, and finally continuing again along $\gamma_1$ back to the basepoint. We will see more of these rerouted loops shortly.

We observe that by parametrising the variables as follows:

$$T_1 = \cosh r_1 \quad T_2 = \cosh r_2 \quad T_3 = \cosh(r_1 + r_2)$$

equation (11) is solved by setting:

$$\{r_1, r_2\} = 1.$$

On quantization, replacing $T_i$, $r_j$ by operators $\hat{T}_i$, $\hat{r}_j$ respectively implies the corresponding commutation relation:

$$[\hat{r}_1, \hat{r}_2] = i\hbar. \tag{12}$$

The operators $\hat{T}_i$ satisfy a $q$-deformed ($q = e^{i\hbar}$) cubic relation, which can be interpreted in terms of a quantum Casimir operator for the quantum group $SU(2)_q$, see [4].

We note that e.g.

$$T_1 = \frac{1}{2}\ \mathrm{tr}\ U_1 = \cosh r_1 = \frac{1}{2}(e^{r_1} + e^{-r_1})$$

so that by introducing the *quantum* matrices

$$\hat{U}_i = \begin{pmatrix} e^{\hat{r}_i} & 0 \\ 0 & e^{-\hat{r}_i} \end{pmatrix} \; i = 1, 2 \tag{13}$$

we have the analogous relation between $\hat{T}_i$ and $\hat{U}_i$, namely

$$\hat{T}_i = \frac{1}{2}\ \mathrm{tr}\ \hat{U}_i \quad i = 1, 2.$$

We also notice that these quantum matrices satisfy the following fundamental relation:

$$\hat{U}_1 \hat{U}_2 = q\ \hat{U}_2 \hat{U}_1, \tag{14}$$

where we are using matrix multiplication of operator-valued matrices (the usual algebraic rule, but paying strict attention to the order of the symbols). For example, the relation

$$e^{\hat{r}_1} e^{\hat{r}_2} = q \; e^{\hat{r}_2} e^{\hat{r}_1}$$

follows from the commutation relation (12) between the operators $\hat{r}_i$.

The cubic constraint satisfied by the quantum variables $\hat{T}_i$ is rather complicated, so instead we work with the quantum holonomy matrices $\hat{U}_i$ themselves rather than with the trace functions $\hat{T}_i$. It is important to note that even though the quantum matrices $\hat{U}_i$ are not gauge-invariant, i.e.

$$\hat{U}_i \neq S^{-1} \hat{U}_i S$$

for general $S$, the fundamental equation (14) is gauge-covariant, and is also covariant under the modular symmetry of the theory, i.e. the group of large diffeomorphisms of the torus, see [5]. Thus our idea is to substitute invariant variables obeying complicated equations by non-invariant matrix variables satisfying natural $q$-commutation relations like the fundamental relation (14). Certainly for the case of diagonal matrices these two viewpoints are entirely equivalent.

We have also studied, in [5], what happens when one imposes the fundamental equation for a pair of upper-triangular quantum matrices, which should correspond, in some sense, to the quantization of the 1-dimensional upper-triangular cell of the classical phase space mentioned in section 2.

If one parametrizes the quantum matrices $\hat{U}_i$ as follows:

$$\hat{U}_i = \begin{pmatrix} \hat{\alpha}_i & \hat{\beta}_i \\ 0 & \hat{\alpha}_i^{-1} \end{pmatrix}, \tag{15}$$

where the $\hat{\alpha}_i$, $\hat{\beta}_i$ are operators to be determined, a solution to equation (14) is given by:

$$\begin{aligned} \hat{\alpha}_1 \psi(b) &= \exp \frac{d}{db} \psi(b) \\ \hat{\alpha}_2 \psi(b) &= \exp i\hbar b \, \psi(b) \\ \hat{\beta}_i \psi(b) &= \hat{\alpha}_i \psi(-b) \end{aligned} \tag{16}$$

Note the change of sign in the argument of $\psi$ in the last of equations (16). It can be checked, from (16) that

$$\hat{\alpha}_1 \hat{\alpha}_2 = q \; \hat{\alpha}_2 \hat{\alpha}_1$$

as required, but we also get an *internal* commutation relation

$$\hat{\alpha}_1 \hat{\beta}_1 = \hat{\beta}_1 \hat{\alpha}_1^{-1}$$

for the elements of $\hat{U}_1$ and similarly for $\hat{U}_2$, which curiously does not involve the quantum parameter $\hbar$.

Note that it is impossible to find solutions to (14) with $\hat{\alpha}_i = \mathbb{I}$, the unit operator, by naive analogy with equation (10), since $\hat{\beta}_1 + \hat{\beta}_2 \neq q(\hat{\beta}_1 + \hat{\beta}_2)$. Thus in terms of the number of quantum parameters, this upper-triangular sector would appear to be as substantial as the triangular sector, unlike the classical case.

Finally we remark that in [6], we studied equations like (15) from an algebraic point of view, and found that their solutions have several interesting properties analogous to quantum groups.

## 4. Reroutings and the quantized Goldman bracket

Here we briefly describe our most recent work, for a full treatment see [7]. In section 3 we only considered the quantum matrices assigned to $\gamma_1$ and $\gamma_2$, so it is natural to try and understand how to assign quantum matrices to other loops, and to study the relationships between them. A useful way of doing this, proposed in [8], is to introduce a constant quantum connection

$$\hat{A} = (\hat{r}_1 dx + \hat{r}_2 dy) \begin{pmatrix} 1 & 0 \\ 0 & -1 \end{pmatrix},$$

where constant means that the $\hat{r}_i$ do not depend on the spatial coordinates $x$, $y$ of the torus. Then the assignment of a quantum matrix to any loop is given by the holonomy of this connection along the loop:

$$\gamma \mapsto \hat{U}_\gamma = \exp \int_\gamma \hat{A}. \tag{17}$$

It can easily be seen that (17) reproduces the quantum matrices $\hat{U}_i$ of equation (13), if $\gamma_1$ is the loop with $y$ coordinate constant and $x$ running from 0 to 1, and $\gamma_2$ is the loop with $x$ constant and $y$ running from 0 to 1.

It is convenient to identify the torus $\mathbb{T}^2$ with $\mathbb{R}^2/\mathbb{Z}^2$, where $\mathbb{Z}^2$ consists of points with integer $x$ and $y$ coordinates. We consider all loops on the torus represented by piecewise-linear (PL) paths between integer points on the $(x, y)$ plane, and work with this description, keeping in mind that paths represent loops. In particular the integer paths denoted $(m, n)$ are straight paths between $(0, 0)$ and $(m, n)$ with $m$, $n$ integers. Thus for example we assign to the integer path $(2, 1)$ the quantum matrix

$$\hat{U}_{(2,1)} = \begin{pmatrix} e^{2\hat{r}_1 + \hat{r}_2} & 0 \\ 0 & e^{-2\hat{r}_1 - \hat{r}_2} \end{pmatrix}.$$

Consider two homotopic loops $\gamma_1$ and $\gamma_2$ corresponding to PL paths both starting at $(0,0)$ and ending at the same integer point in the plane. It was shown in [7] that there is the following relationship between the respective quantum matrices:

$$\hat{U}_{\gamma_1} = q^{S(\gamma_1,\gamma_2)}\hat{U}_{\gamma_2}, \tag{18}$$

where $S(\gamma_1,\gamma_2)$ denotes the signed area enclosed between the paths $\gamma_1$ and $\gamma_2$. For example, the exponent (the number 1) of $q$ in the fundamental relation (14) is the signed area between two paths around the perimeter of the unit square, starting at $(0,0)$ and ending at $(1,1)$, the first via $(1,0)$ and the second via $(0,1)$.

The traces of these quantum matrices also exhibit commutation relations with interesting properties. Let

$$\hat{T}(m,n) := \operatorname{tr} \hat{U}_{(m,n)}.$$

(note we have dropped the factor $\frac{1}{2}$ for easier comparison with the Goldman result below). It was shown in [7] that the following commutation relation holds:

$$[\hat{T}(m,n), \hat{T}(s,t)] = (q^{\frac{mt-ns}{2}} - q^{-\frac{mt-ns}{2}})\left(\hat{T}(m+s,n+t) - \hat{T}(m-s,n-t)\right) \tag{19}$$

There are some surprising geometric aspects to equation (19). The number $mt-ns$ appearing in the exponents is the signed area of the parallelogram spanned by the vectors $(m,n)$ and $(s,t)$. The same expression equals the suitably-defined total intersection number (including and counting multiplicities) of the two loops represented by the paths $(m,n)$ and $(s,t)$. Equation (19) can, in fact, be viewed as a quantization of a bracket due to Goldman [9] for the loops corresponding to such integer paths. This bracket is a Poisson bracket for the functions $T(\gamma) = \operatorname{tr} U_\gamma$ given by:

$$\{T(\gamma_1), T(\gamma_2)\} = \sum_{S \in \gamma_1 \sharp \gamma_2} \epsilon(\gamma_1,\gamma_2,S)(T(\gamma_1 S \gamma_2) - T(\gamma_1 S \gamma_2^{-1})) \tag{20}$$

where $\gamma_1 \sharp \gamma_2$ denotes the set of transversal intersection points of $\gamma_1$ and $\gamma_2$ and $\epsilon(\gamma_1,\gamma_2,S)$ is their intersection index for the intersection point $S$. In equation (20) $\gamma_1 S \gamma_2$ and $\gamma_1 S \gamma_2^{-1}$ denote the loops which follow $\gamma_1$ and are rerouted along $\gamma_2$, or its inverse, at the intersection point $S$ as described previously. For the integer loops considered here, all the rerouted loops $\gamma_1 S \gamma_2$ are homotopic to the integer loop $(m+s,n+t)$, with an analogous statement for the loops $\gamma_1 S \gamma_2^{-1}$. It follows that the classical Goldman bracket (20) can be written as

$$\{T(m,n), T(s,t)\} = (mt-ns)(T(m+s,n+t) - T(m-s,n-t)).$$

Therefore the first factor on the right hand side of (19) may be thought of as a quantum total intersection number for the loops $(m, n)$ and $(s, t)$.

We remark that in [7] we also derived a different form of (19) where each rerouted loop appears separately. The different terms are related by the same area phases as in (18). In these proofs we used a classical geometric result [10] namely Pick's Theorem (1899), which expresses the area $A(P)$ of a lattice polygon $P$ with vertices at integer lattice points of the plane in terms of the number of interior lattice points $I(P)$ and the number of boundary lattice points $B(P)$ as follows:

$$A(P) = I(P) + \frac{B(P)}{2} - 1.$$

Full details are given in [7].

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# On warped and double warped space-times

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**Abstract.** Warped and double warped space-times are defined and classified according to the existence of preferred vector and tensor fields on the locally decomposable space-times they are associated to by means of special conformal transformations. An invariant geometric characterization is also presented, as well as a characterization based on the Newman-Penrose formalism.

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## 1. Introduction

In [1] warped product manifolds are obtained by homothetically warping the product metric on a semi-Riemannian product manifold. Warped space-times are then viewed as conformal to locally decomposable ones, the conformal factor depending on the variables defined only on one of the spaces used to build the decomposable space. They are classified and characterized in an invariant manner in [2] and [3], where their geometry is expressed in terms of the warping function and the geometry of the underlying locally decomposable space-times. It is also referred (see references listed in [2]) that there is a variety of exact solutions to Einstein's equations which can be viewed as warped space-times, for example Schwarzchild, Bertotti-Robinson, Robertson-Walker, Reissner-Nordstrom, de Sitter, therefore their study being quite relevant in general relativity. From the point of view of describing solutions to Einstein's equations, it is an interesting problem to investigate double warped space-times. These are also conformal to locally decomposable ones, however the conformal factor is now separable on the coordinates defined on each space used to build the decomposable space-time. It should also be referred that warped and double warped space-times are are special cases of twisted products [4]. In [5] an invariant characterization for double warped space-times is presented in terms of the Newman-Penrose formalism [6] and a classification scheme is proposed, as well as a detailed study of their conformal algebra.

Here warped and double warped space-times are defined and classified according to the existence of preferred vector and tensor fields on the underlying decomposable space-times. Therefore it is convenient to start with a brief summary of results on locally decomposable space-times [7].

Given two metric manifolds $(M_1, h_1)$ and $(M_2, h_2)$, one can build a new metric manifold $(M, g)$ by setting

$$M = M_1 \times M_2 \tag{1}$$

and

$$g = {\pi_1}^{\star} h_1 \oplus {\pi_2}^{\star} h_2, \tag{2}$$

or simply $g = h_1 \otimes h_2$, $\pi_1$ and $\pi_2$ being the canonical projections onto $M_1$ and $M_2$, respectively.

Throughout this text we will be interested in space-time manifolds $(M, g)$, therefore we will consider that $dimM = dimM_1 + dimM_2 = 4$ and that $g$ is a Lorentzian metric. Consequently one of the manifolds $(M_1, h_1)$, $(M_2, h_2)$ is Lorentzian and the other is Riemannian.

Since the considerations to be presented are mainly local, we will assume that, for each point $p$ in $M$, there exists a neighborhood $U$ of $p$ where a coordinate system adapted to the product structure exists, namely $(u = x^0, x^1, x^2, x^3)$. Two main cases then arise, as summarized below.

(A) $(M, g)$ is a $1+3$ locally decomposable space-time, if it admits a global, non-null, nowhere zero covariantly constant vector field. For this case the line element associated with $g$ can be written in the following form

$$ds^2 = \epsilon du^2 + h_{\alpha\beta}(x^\gamma) dx^\alpha dx^\beta, \tag{3}$$

where greek indices take the values $1, 2, 3$ and $\epsilon = \pm 1$. Here $h_1 = \epsilon du \otimes du$ and $h_2 = h_{\alpha\beta}(x^\gamma) dx^\alpha \otimes dx^\beta$. Moreover, when the covariantly constant vector field $\partial/\partial u$ is timelike (respectively spacelike), then $\epsilon = -1$ (respectively $\epsilon = +1$) and the space-time is locally $1+3$ spacelike (respectively timelike) decomposable.

It should be noticed that, if the space-time admits another global covariantly constant non-null vector field, nowhere zero, then the space-time decomposes further and can be referred to as being $1 + 1 + 2$ spacelike or $1 + 1 + 2$ timelike in an obvious notation.

(B) $(M, g)$ is a $2 + 2$ locally decomposable space-time, in which case no global, covariantly constant, nowhere zero vector field exists in $M$, but the space-time admits two global recurrent null vector fields, linearly independent. This is equivalent to saying that in $(M, g)$ there are two global, linearly independent covariantly constant tensor fields of rank 2, namely $P$ and $Q$, such that $g_{ab} = P_{ab} + Q_{ab}$ with $P_{ab;c} = Q_{ab;c} = 0$ $(a, b, c, \cdots = 0, 1, 2, 3)$. The line element associated with $g$ can have, for this case, the following form:

$$ds^2 = h_{1\ \alpha\beta}(x^\gamma)dx^\alpha dx^\beta + h_{2\ AB}(x^D)dx^A dx^B, \tag{4}$$

where $h_1$ and $h_2$ are two 2-dimensional metrics on $M_1$ and $M_2$ respectively such that $\pi_1^* h_1 = P$ and $\pi_2^* h_2 = Q$.

## 2. General Concepts

We now consider the two metric manifolds $(M_1, h_1)$ and $(M_2, h_2)$ introduced above and a smooth function $\theta : M_1 \to \mathbb{R}$, which will be called a *warping function*. We can now build a new manifold $(M, g)$, namely a *warped product manifold* (see [1], [8]) by setting $M = M_1 \times M_2$ and

$$g = h_1 \otimes \mathrm{e}^{2\theta} \pi^* h_2, \tag{5}$$

or simply $g = h_1 \otimes \mathrm{e}^{2\theta} h_2$, where $\pi$ is the canonical projection onto $M_2$ and will be omitted where there is no risk of confusion.

It should be noticed that (5) can also be written as

$$g = \mathrm{e}^{2\theta}(h_1' \otimes h_2) = \mathrm{e}^{2\theta} g', \tag{6}$$

where we have defined $h_1' \equiv \mathrm{e}^{-2\theta} h_1$ as a new metric on $M_1$. Therefore a warped space-time is conformal to a decomposable space-time $(M, g')$, the conformal factor depending only on the coordinates on either $M_1$ or $M_2$.

Similarly, if one considers two smooth functions $\theta_1 : M_1 \to \mathbb{R}$, $\theta_2 : M_2 \to \mathbb{R}$ (*warping functions*), one can build a new metric manifold $(M, g)$ by setting $M = M_1 \times M_2$ and

$$g = \mathrm{e}^{2\theta_2} \pi_1^* h_1 \otimes \mathrm{e}^{2\theta_1} \pi_2^* h_2, \tag{7}$$

where $\pi_1, \pi_2$ are the canonical projections onto $M_1$ and $M_2$ respectively. As before, if no risk of confusion exists we will simply write $g = \mathrm{e}^{2\theta_2} h_1 \otimes \mathrm{e}^{2\theta_1} h_2$.

This metric manifold $(M, g)$ will be called a *double warped product manifold* (see [5]). It is now clear that, if either $\theta_1$ or $\theta_2$ are constant, then $(M, g)$ becomes a warped manifold. If, moreover, both warping functions are constant then $(M, g)$ is a decomposable metric manifold.

It should be noticed that (7) can be written as follows:

$$g = \mathrm{e}^{2(\theta_1+\theta_2)}[\mathrm{e}^{-2\theta_1}h_1 \otimes \mathrm{e}^{-2\theta_2}h_2] = \mathrm{e}^{2(\theta_1+\theta_2)}[h_1' \otimes h_2'] = \mathrm{e}^{2(\theta_1+\theta_2)}g'. \tag{8}$$

Due to the definition of the warping functions, it is clear that $h_1'$ and $h_2'$ are metrics on $M_1$ and $M_2$, respectively. Therefore, a double warped space-time can be thought of as being conformal to a locally decomposable one, say $(M, g')$, the conformal factor being separable on the coordinates defined on those submanifolds used to construct $M$.

On what follows, we will write $h_1$ and $h_2$ instead of $h_1'$ and $h_2'$, whenever there is no risk of confusion.

# 3. Canonical Forms for the Metric

When studying warped and double warped space-times $(M, g)$, it is our goal to express their geometry in terms of the warping functions and the geometries of the underlying spaces $M_1$ and $M_2$. It appears natural to consider two main classes of warped space-times: one class includes space-times conformally related to $1 + 3$ locally decomposable space-times, as shown in (3), the other class containing those space-times which are obtained from $2 + 2$ locally decomposable space-times, as in (4). Analogous considerations apply to double warped metrics.

## 3.1. Warped Space-Times

According to [2], warped space-times $(M, g)$ are always included in one of the classes described below.

*Class A:* $(M, g)$ is conformally related to a $1 + 3$ locally decomposable space-time $(M, g')$. Two different sub-classes are distinguished, namely class $A_1$ corresponding to a $1 + 3$ spacelike $(M, g')$ and class $A_2$ corresponding to a $1 + 3$ timelike $(M, g')$.

*Class B:* $(M, g)$ is conformally related to a $2 + 2$ locally decomposable space-time $(M, g')$.

It is now convenient to write canonical forms for the line element associated with the metric of **warped** space-times in each of the referred classes. If one uses (3), (5) and (6), the following canonical forms arise.

Class $A_1$:

$$ds^2 = \epsilon du^2 + \mathrm{e}^{2\theta(u)} h_{\alpha\beta}(x^\gamma) dx^\alpha dx^\beta \tag{9}$$

Class $A_2$:

$$ds^2 = \epsilon \mathrm{e}^{2\theta(x^\gamma)} du^2 + h_{\alpha\beta}(x^\gamma) dx^\alpha dx^\beta, \tag{10}$$

Class $B$:

$$ds^2 = h_{1\ \alpha\beta}(x^\gamma) dx^\alpha dx^\beta + \mathrm{e}^{2\theta(x^\gamma)} h_{2\ AB}(x^C) dx^A dx^B. \tag{11}$$

Here we have followed the notation used for writing (3) and (4).

Since $h_{\alpha\beta}(x^\gamma)$ is a three-dimensional metric it can always be written in a diagonal form so that the line elements (9) and (10) can be rewritten taking this property into account. Similarly (11) can also be modified if one takes into account that a two-dimensional space is always conformally flat.

## 3.2. Double Warped Space-Times

The similarity between the construction of warped and of double warped space-times induces a similar procedure for writing canonical forms for the line element of double warped space-times. In fact, according to [5], **double warped** space-times $(M, g)$ are always included in one of the following classes:

*Class A:* $(M, g)$ is conformally related to a $1+3$ locally decomposable space-time $(M, g')$. Two different sub-classes are distinguished, namely class $A_1$ corresponding to a locally $1+3$ decomposable spacelike $(M, g')$ and class $A_2$ corresponding to a locally $1+3$ decomposable timelike $(M, g')$.

*Class B:* $(M, g)$ is conformally related to a $2+2$ locally decomposable space-time $(M, g')$.

Canonical forms for the line element associated with the metric in each of these classes are presented, following a procedure similar to the one used in studing warped space-times.

Class $A_1$:

$$ds^2 = \mathrm{e}^{2(\theta_1(u)+\theta_2(x^D))} \left[-du^2 + h_{AB}(x^D) dx^A dx^B\right], \tag{12}$$

Class $A_2$:

$$ds^2 = \mathrm{e}^{2(\theta_1(x^\gamma)+\theta_2(u))}\left[h_{\alpha\beta}(x^\gamma)dx^\alpha dx^\beta + du^2\right], \tag{13}$$

Class $B$:

$$ds^2 = \mathrm{e}^{2(\theta_1(x^\gamma)+\theta_2(x^D))}\left[h_{1\ \alpha\beta}(x^\gamma)dx^\alpha dx^\beta + h_{2\ AB}(x^D)dx^A dx^B\right]. \tag{14}$$

Again the notation used for writing (3) and (4) is applied in these canonical forms. The canonical forms written above can be cast in a different manner if one applies the considerations referred to in the previous section.

## 4. Geometric Characterization

We recall that a vector field $\vec{X}$ on $M$ is a *conformal Killing vector* whenever

$$\mathcal{L}_{\vec{X}} g = 2\phi g.$$

Here $\phi$ (*conformal factor*) is a scalar field on $U$ and, as usual, $\mathcal{L}_{\vec{X}}$ is the Lie derivative operator with respect to the vector field $\vec{X}$. The special cases $\phi =$ constant and $\phi = 0$ correspond respectively to $\vec{X}$ being a *homothetic vector field* (HV) and a *Killing vector field* (KV). Moreover, a CKV is said to be *proper* whenever it is non-homothetic (i.e. $\phi \neq$ const). Similarly, a HV which is not a KV (i.e. $\phi =$ const $\neq 0$) will be designated as 'proper homothetic'. A proper CKV is said to be a *special CKV* (SCKV) whenever its associated conformal factor $\phi$ satisfies $\phi_{a;b} = 0$ in any coordinate chart.

In order to establish an invariant geometric characterization of warped and double warped space-times $(M, g)$ one can look at preferred vector fields living on the underlying decomposable space-time $(M, g')$ and investigate how those vector fields are transformed when the warping functions are introduced.

For example, it is easy to see from (3) that a global, non null, nowhere zero, covariantly constant vector field for a $1+3$ locally decomposable space-time $(M, g')$ can be rescaled so as to become a KV in the associated warped $(M, g)$.

For warped space-times of classes $A_1$ and $A_2$ the characterization can be summarized as follows. The proofs of the following theorem can be found in [2].

**Theorem 1.** *The necessary and sufficient condition for $(M, g)$ to be warped class $A$ is that a global, non-null, nowhere zero, hypersurface orthogonal unit vector field exists that is shearfree* *(a) and such that it is geodesic with expansion* $\Theta$ *satifying* $\Theta_{,c}h^c_a = 0$, *where* $h_{ab} \equiv g_{ab} - \epsilon u_a u_b$ *is the orthogonal projector to* $u^a$, *in which case* $(M, g)$ *is warped class* $A_1$.

*(b) and such that it is non-expanding, its acceleration being a gradient, in which case* $(M, g)$ *is warped class* $A_2$.

The definitions used for expansion and shear can be found in [9], if $u$ is timelike, and in [10], if $u$ is spacelike.

This theorem is proved in [2]. However results of (a) can also be found in [11].

The following theorem was proved in [5] and establishes an invariant characterization for class A double warped space-times.

**Theorem 2.** *The necessary and sufficient condition for* $(M, g)$ *to be a double warped class A spacetime is that it admits a non-null, nowhere vanishing CKV* $\vec{X}$ *which is hypersurface orthogonal and such that the gradient of its associated conformal factor* $\psi$ *is parallel to* $\vec{X}$.

The characterization of warped spacetimes can now be easily obtained, as stated in the following corollary.

**Corollary 3.** *If the CKV* $\vec{X}$ *in theorem 2 is a Killing Vector (KV) then the spacetime is warped of class* $A_2$. *If* $\vec{X}$ *is a proper (non-KV) gradient CKV (i.e. if the associated conformal bivector* $F_{ab} = X_{a;b} - X_{b;a}$ *vanishes) the spacetime is class* $A_1$ *warped.*

It is worthwhile noticing that theorem 2 provides also an invariant characterization of spacetimes conformal to 1+3 locally decomposable spacetimes:

**Corollary 4.** *The necessary and sufficient condition for* $(M, g)$ *to be conformally related to a 1+3 decomposable spacetime* $(M, g')$ *is that it admits a non-null, nowhere vanishing conformal Killing vector (CKV)* $\vec{X}$ *which is hypersurface orthogonal.*

On the other hand, if the underlying space-time $(M, g')$ is locally $2 + 2$ decomposable the characterization of warped and double warped space-times in given in the following theorem. The proof of this theorem is provided in [2] and [5].

**Theorem 5.** *The necessary and sufficient condition for* $(M, g)$ *to be conformally related to a 2+2 decomposable spacetime* $(M, g')$ *with* $g = \mathrm{e}^{2\theta} g'$ *($\theta$ being a real function), is that there exist null vectors* $\vec{l}$ *and* $\vec{k}$ *($l^a k_a = -1$) satisfying*

$$l_{a;b} = A\mathrm{e}^{-\theta} l_a l_b - \theta_{,a} l_b + (\theta_{,c} l^c) g_{ab} \tag{15}$$

$$k_{a;b} = -A\mathrm{e}^{-\theta}k_a l_b - \theta_{,a}k_b + (\theta_{,c}k^c)g_{ab} \tag{16}$$

*for some function A. Further, $(M, g)$ is class B double warped if and only if*

$$H_a^c \left(h_b^d \theta_{,d}\right)_{;c} + 2\left(h_b^d \theta_{,d}\right)(H_a^c \theta_{,c}) = 0, \tag{17}$$

*where*

$$h_{ab} \equiv -2k_{(a}l_{b)} \text{ and } H_{ab} \equiv g_{ab} + h_{ab}. \tag{18}$$

## 5. Newman-Penrose Characterization

The invariant characterization of double warped space-times using the Newman-Penrose (NP) formalism was obtained in [5]. Here we present this classification for class $A_1$, class $A_2$ and class $B$ double warped space-times. However the corresponding classification for warped space-times of those classes can easily be obtained if one restricts the form of the conformal factor in an obvious manner.

With the notation used in Theorem 2 we have that for a class $A_1$ double warped spacetime a coordinate chart $\{u, x^K\}$ exists such that the line element takes the form (12). Then $\vec{X} = \partial_u$ is a timelike hypersurface orthogonal CKV with associated conformal factor $\psi(u) = \theta_{1,u}(u)$, and $\vec{u} = \mathbf{e}^{-U}\partial_u$ is a unit timelike vector field parallel to $\vec{X}$. For convenience we will write $U(u, x^K) = \theta_1(u) + \theta_2(x^K)$. The following theorems were proved in [5].

**Theorem 6.** *$(M, g)$ is a class $A_1$ double warped spacetime if and only if there exist a function $U : M \to \mathbb{R}$ and a canonical complex null tetrad $\{k_a, l_a, m_a, \bar{m}_a\}$ ($k^a l_a = -m^a \bar{m}_a = -1$) in which:*

$$DU = \epsilon + \bar{\epsilon} \tag{19}$$
$$\triangle U = -(\gamma + \bar{\gamma}) \tag{20}$$
$$\delta U = \kappa + \bar{\pi} = -(\tau + \bar{\nu}) \tag{21}$$
$$\sigma + \bar{\lambda} = 0 \tag{22}$$
$$\alpha + \bar{\beta} = 0 \tag{23}$$
$$\epsilon + \bar{\epsilon} + \gamma + \bar{\gamma} = \rho + \bar{\mu} \tag{24}$$
$$D(\rho + \bar{\mu}) = -\Phi \tag{25}$$
$$\triangle(\rho + \bar{\mu}) = \Phi \tag{26}$$
$$\delta(\rho + \bar{\mu}) = \bar{\delta}(\rho + \bar{\mu}) = 0 \tag{27}$$

*where $\Phi = \Phi(u)$ is a real function of the timelike coordinate $u$.*

A similar notation is used to establish the classification of $A_2$ double warped space-times within NP formalism. The result is summarized in the following theorem.

**Theorem 7.** *$(M,g)$ is a class $A_2$ double warped spacetime if and only if there exist a function $U : M \to \mathbb{R}$ and a canonical complex null tetrad $\{k_a, l_a, m_a, \bar{m}_a\}$ ($k^a l_a = -m^a \bar{m}_a = -1$) in which one of the following sets of equations holds:*

*(i)*

$$DU = \epsilon + \bar{\epsilon} \qquad \triangle U = -(\gamma + \bar{\gamma}) \qquad \delta U = -\kappa + \bar{\pi} = \tau + \bar{\nu} \tag{28}$$

$$\sigma - \bar{\lambda} = \alpha + \bar{\beta} = 0 \tag{29}$$

$$\epsilon + \bar{\epsilon} - (\gamma + \bar{\gamma}) = \rho - \bar{\mu} \tag{30}$$

$$D(\rho - \bar{\mu}) = \triangle(\rho - \bar{\mu}) = \Phi \tag{31}$$

$$\delta(\rho - \bar{\mu}) = \bar{\delta}(\rho - \bar{\mu}) = 0 \tag{32}$$

*(ii)*

$$DU = \sigma + \bar{\rho} \qquad \triangle U = -(\bar{\lambda} + \mu) \qquad \delta U = \bar{\alpha} - \beta \tag{33}$$

$$\delta U + \bar{\delta} U = \pi + \bar{\pi} = -(\tau + \bar{\tau}) \tag{34}$$

$$\kappa + \bar{\kappa} = \nu + \bar{\nu} = 0 \tag{35}$$

$$\epsilon - \bar{\epsilon} = 0 \gamma - \bar{\gamma} = 0 \tag{36}$$

$$\delta(\pi + \bar{\pi}) = \bar{\delta}(\pi + \bar{\pi}) = \Phi' \tag{37}$$

$$\triangle(\pi + \bar{\pi}) = D(\pi + \bar{\pi}) = 0 \tag{38}$$

*(iii)*

$$DU = -\sigma + \bar{\rho} \qquad \triangle U = \bar{\lambda} - \mu \qquad \delta U = \bar{\alpha} - \beta \tag{39}$$

$$\delta U - \bar{\delta} U = -\pi + \bar{\pi} = -\tau + \bar{\tau} \tag{40}$$

$$\kappa - \bar{\kappa} = \nu - \bar{\nu} = 0 \tag{41}$$

$$\epsilon - \bar{\epsilon} = \gamma - \bar{\gamma} = 0 \tag{42}$$

$$-\delta(\pi - \bar{\pi}) = \bar{\delta}(\pi - \bar{\pi}) = -\Phi'' \tag{43}$$

$$\triangle(\pi - \bar{\pi}) = D(\pi - \bar{\pi}) = 0 \tag{44}$$

*where $\Phi, \Phi'$ and $\Phi''$ are real functions of the spacelike coordinate $u$.*

In order to establish a characterization of class B double warped space-times, using the NP formalism, a complex null tetrad $\{k_a, l_a, m_a, \bar{m}_a\}$ is chosen such that $\vec{k}$ and $\vec{l}$ are the vectors in (15) and (16), i.e. $k^a l_a = -m^a \bar{m}_a = -1$ all other inner products vanishing. The following theorem then holds.

**Theorem 8.** *The necessary and sufficient condition for $(M, g)$ to be conformally related to a 2+2 decomposable spacetime $(M, g')$, with $g = \mathrm{e}^{2\theta} g'$, is that there exist a function $\theta : M \to \mathbb{R}$ and a canonical complex null tetrad $\{k_a, l_a, m_a, \bar{m}_a\}$ as described above such that*

$$\begin{aligned} \kappa = \sigma = \lambda = \nu = \alpha + \bar{\beta} = \pi + \bar{\tau} = \rho + (\epsilon + \bar{\epsilon}) = 0 \\ A\mathrm{e}^{-\theta} = \mu + (\gamma + \bar{\gamma}) \\ \rho = -D\theta, \quad \mu = \triangle\theta, \quad \tau = -\delta\theta \end{aligned} \tag{45}$$

*where $A$ is the real function appearing in (15) and (16). Furthermore, $(M, g)$ is class B double warped if and only if*

$$\delta\rho = -2\rho\tau, \quad \delta\mu = -2\mu\tau, \quad \rho\mu = 0 \tag{46}$$

The characterization of class A and class B double warped spacetimes given in the results stated here should prove useful in formulating an algorithm for classifying such metrics. This is the case since such characterization is coordinate independent although tetrad dependent. In what follows the tetrads described above will be designated as *double warped tetrads of class A and B*, as appropriate.

Thus, in order to determine whether a given metric $g$ represents a double warped spacetime, one can use the theorems stated either with a coordinate or a tetrad approach through the following scheme:

1. Determine the Petrov type of the Weyl tensor associated with the metric $g$ and choose a canonical tetrad $\{k_a, l_a, m_a, \bar{m}_a\}$ such that $g_{ab} = 2[-l_{(a}k_{b)} + m_{(a}\bar{m}_{b)}]$.

2. Determine the NP spin coefficients and their NP derivatives in the chosen tetrad (1).

3. If the scalars determined in step (2) satisfy the relations of theorem 6 or 7 (respectively 8) for some function $U$ (respectively $\theta$), then the spacetime is double warped of class A (respectively B) and the algorithm stops here, otherwise continue the algorithm.

4. If possible, find the Lorentz transformation of the invariance group that transforms tetrad (1) into a double warped tetrad, i.e. such that the corresponding NP spin coefficients and NP derivatives obey the conditions in theorem 6 or 7 (respectively 8). If such a transformation exists, the spacetime is double warped of class A (respectively B), otherwise it is not double warped.

The Lorentz transformations considered in step (4) must belong to the invariance group of the Petrov type of the metric $g$ since in step (1) one chooses a canonical tetrad. Thus, for instance, if the given metric is of Petrov type D or N, then in step (4) one looks for spin and boost transformations or for null rotations respectively.

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# Brownian motion and symplectic geometry

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**Abstract.** In contrast with Stochastic Riemannian Geometry, a Symplectic one is unknown. This is a survey of a recent approach to the subject involving ideas from Quantum Physics.

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## Introduction

This is an expository introduction to the relations between Brownian motion and Symplectic Geometry. This subject is still in its infancy. As a matter of fact, although the notion of Stochastic Riemannian Geometry is almost as old as the theory of Brownian motion itself, one can observe a remarkable absence of results as far as Symplectic Geometry is concerned. One of our purposes, here, is to argue in favor of an approach where Quantum Mechanics is used as a guide line. For our needs, the free heat equation or its quantum counterpart, the dynamics of the free quantum particle, will be sufficient since the problems we address are basic. Many of the ideas involved in our approach relate to a probabilistic reinterpretation of Feynman's path integral method, known as "Euclidean Quantum Mechanics" (cf. [1]) but the familiarity with this method is not really indispensable, hopefully, to understand the content of the present introduction. We have adopted a style of presentation which should be accessible to a mixed community of mathematicians and physicists. It is our hope that we shall be able to convince them that the method advocated here produces interesting results for both communities, especially if one considers the simplicity of our underlying hypotheses and the various directions of generalization available.

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## 1. The Laplacian in Classical and Quantum Physics

The Laplacian operator is omnipresent in Theoretical Physics. Let us adopt the convention $H_0 = -\frac{\hbar^2}{2}\Delta$, for $\hbar$ a strictly positive constant. There are at least two physical theories where the role of $H_0$ is fundamental, Statistical Mechanics and Quantum Mechanics.

In Statistical Mechanics we associate $H_0$ to the theory of Brownian Motion and to the parabolic equation

$$\hbar\partial_t\eta = H_0\eta \tag{1}$$

where $\partial_t$ denotes the (partial) time derivative. Eq. (1) is known as the heat of diffusion equation, and describe many classical phenomena. In particular, its connection with the Brownian motion is known since the time of A. Einstein (for Physics) and N. Wiener (for Mathematics).

In the twenties of the last century, the second author constructed a mathematical model of Einstein's qualitative description, namely a probability measure $\mu_w^\hbar$, named after him, on a space of continuous paths ($\mathbb{R}$ valued, for simplicity) starting from the origin:

$$\Omega_0 = \{\omega \in C(\mathbb{R}^+, \mathbb{R}) \quad t.q. \quad \omega(0) = 0\} \tag{2}$$

The properties of the Wiener measure $\mu_w^\hbar$ are the following:

The distribution of $\omega(t)$ under $\mu_w^\hbar(d\omega)$ has the form:

$$\mu_w^\hbar(\omega(t) \in dq) = h_0(0,t,q)dq = (2\pi\hbar t)^{-\frac{1}{2}} \exp\left\{-\frac{1}{2\hbar t}q^2\right\} dq \tag{3}$$

i.e. $h_0$ is the fundamental solution of Eq. (1).

The increments $\omega(t_i) - \omega(t_{i-1}), i = 1, ..., n$, $0 = t_0 < t_1 < ... < t_n$ are independent under $\mu_w^\hbar$.

Then Wiener proved that there is a probability space $(\Omega_0, \sigma, \mu_w^\hbar)$, where $\sigma$ is the Borel $\sigma$-field of $\Omega_0$, with respect to which $\{\omega(t) \equiv w_t\}_{t\in\mathbb{R}^+}$ is, indeed, the Brownian motion. Let us stress that it is crucial, in the definition of any stochastic process, not only Wiener's one, to be able to construct its probability space.

The above definition of the Wiener measure is the constructive one of mathematicians, but there is a lack of intuition in it with respect to the following one inspired by R. Feynman:

$$\mu^{\hbar}(d\omega) = C \exp\left(-\frac{1}{\hbar} S_{L_0}[\omega]\right) \mathcal{D}\omega \tag{4}$$

where $C$ is a normalization constant, $S_{L_0}$ denotes the Action functional of the classical system underlying Eq. (1), i.e. the free one:

$$S_{L_0}[\omega] = \int_0^\infty L_0(\dot{\omega}, \omega) dt, \quad \text{for} \quad L_0 = \frac{1}{2} |\dot{\omega}(t)|^2, \tag{5}$$

and $\mathcal{D}\omega$ is used as a Lebesgue measure on $\Omega_0$.

In spite of the fact that there is no such Lebesgue measure on the (infinite dimensional!) space $\Omega_0$, the representation (4) is more intuitive than Wiener's original approach because it involves an explicit relation with a classical system, namely the Action (5). It suggests, moreover, that the dynamics founded on Eq. (1) is an $\hbar$-deformation of the classical dynamics.

Feynman's representation (4) was mostly designed not for Eq. (1) but for its quantum mechanical counterpart:

$$i\hbar \partial_t \psi = H_0 \psi. \tag{6}$$

Although, for Eq. (1), we may choose various function spaces where to look for the solution, this is not the case for Schrödinger's equation (6); its solution $\psi_t$ must be in $L^2(\mathbb{R}, dq)$ because of Born probabilistic interpretation of quantum states. On the other hand, Brownian motion is simpler than the one of a quantum free particle. In fact, denoting the expectation by $E$

$$E[w_t] = 0, \quad \forall t \in \mathbb{R}^+. \tag{7}$$

In contrast, if $\langle \ \rangle_{\psi_t}$ denotes the quantum expectation in the state $\psi_t$, and $Q, P$ the position and momentum observables respectively, the quantum particle satisfies:

$$\begin{aligned} \langle Q \rangle_{\psi_t} &= \int q |\psi_t(q)|^2 dq = \alpha + \beta t \\ \langle P \rangle_{\psi_t} &= \int (-i\hbar \nabla \log \psi_t(q)) |\psi_t(q)|^2 dq = \beta \end{aligned} \tag{8}$$

where $\alpha$ and $\beta$ are constant. In other words the free quantum particle has, in general, a constant drift $\beta$. So it is insufficient to claim (as in most textbooks) that the Wiener process is the probabilistic counterpart of the free quantum evolution under Eq. (6).

For a good analogy we need diffusion processes $z_t : I \to \mathbb{R}$ (where $I \subset \mathbb{R}^+$ is a time interval) built from the Wiener process $w_t$, more precisely whose distribution is absolutely continuous with respect to the one of $w_t$, for any $t \in I$. This construction can only involves the solutions $\eta_t$ of Eq. (1). The relations (8) suggest how to produce the probabilistic version of a quantum drift:

$$dz_t = \hbar^{\frac{1}{2}} dw_t + \hbar \nabla \log \eta_t(z_t) dt. \tag{9}$$

The first term of this equation represents a quantum noise, whose scale is given by $\hbar$. We cannot write $\frac{dw_t}{dt}$ in the strong sense, this is only defined as a distribution, the "white noise". The second term of (9), defined for $\eta_t > 0$, is a regular one, in the sense of ODE's theory. Eq. (9) is a (Itô's) stochastic differential equation (SDE), for each given positive solution of Eq. (1). Its solution $z_t$ is a time indexed random variable, whose path $t \mapsto z_t$ are not differentiable. In particular, only to the trivial solution $\eta_t = 1$ is associated the (Planck's scaled) Brownian motion itself.

The differential and integral calculus associated with SDE like Eq. (9) is due to K. Itô (cf. [2]). We shall interpret it as an $\hbar$-deformation of Leibniz-Newton calculus along smooth ("classical") paths $t \mapsto q(t)$, built on properties of the heat equation (1).

The fundamental kinematical characteristic of the irregular paths $t \mapsto z_t$ is that

$$E_t[(z_{t+\Delta t} - z_t)^2] = \hbar \Delta t + o(\Delta t) \tag{10}$$

where $E_t$ denotes a conditional expectation given $z_t$. This is a sum only over those continuous paths starting from a fixed point $q = z_t$. Notice that $\Delta t$ appears on the r.h.s. of (10), not $(\Delta t)^2$. This is why $t \mapsto z_t$ must be very irregular. Itô's differential calculus is built on relation (10). In Itô's terms, one write that $(dz_t)^2 = (dw_t)^2 = \hbar dt$, i.e. $z_t$ inherits the irregularities of the Wiener's paths. The key consequence is that any Taylor expansion of a smooth function $F$ of $z_t$ must be computed up to the second order: this is the mathematical origin of the $\hbar$-deformation of classical calculus.

The above framework extends without major difficulties when the paths $t \mapsto z_t$ stay on a $n$-dimensional Riemannian manifold $M$. The $H_0$ operator in (1) must be replaced by the Laplace-Beltrami operator

$$H_0 = -\frac{\hbar^2}{2} \nabla^j \nabla_j \tag{11}$$

where $\nabla_j$ denotes the covariant derivative with respect to the Riemannian connection. The Christoffel symbols associated with the metric $g^{ij}$ are $\Gamma^i_{jk}$.

The SDE (9) becomes

$$dz_t^j = dw_t^j + \left(\hbar\nabla^j \log \eta_t(z_t) - \frac{\hbar}{2}\Gamma^j_{ik}g^{ik}\right) dt \tag{12}$$

where $w_t^j$ is defined by $dw_t^j = \sigma_k^j(w_t)d\xi_t^j$, with $\sigma_k^i\sigma_k^j = g^{ij}$ and the $\xi_t^j$ denote $n$ independent copies of Euclidean Brownian motions as before. The trivial solution of Eq. (1) is still associated with the Wiener process $z_t^j = w_t^j$ on the manifold, whose drift is now non zero, except in normal (or harmonic) coordinates. The kinematical characteristic of the process solving (12) provides us with the metric:

$$dz_t^i dz_t^j = dw_t^i dw_t^j = g^{ij}dt. \tag{13}$$

With respect to the Euclidean case there are interesting new features, notably the construction of the stochastic notion of parallel transport (cf. [3], for instance) but no qualitatively new ideas. This is why the theory of SDE on Riemannian manifolds is almost as old as the Euclidean theory.

Let us notice, however, that it is also possible to construct a geometry directly on the path space $\Omega_0$ of such processes (cf. [7], [8]).

## 2. Some aspects of the theory of stochastic differential equations and of its geometry

Let us come back to the historical development of the theory of SDE. The general version of Eq. (9), for a $n$ dimensional diffusion process $z_t$, is

$$dz_t = \widetilde{\sigma}_i(z_t)dw_t^i + \widetilde{B}(z_t)dt \tag{14}$$

when $\widetilde{B}$ is a given $\mathbb{R}^n$-valued measurable function and $\widetilde{\sigma}$ a $n \times n$ matrix valued measurable function. After integration $\int_{t_0}^{t_1}$ the second integral of the r.h.s. is a regular (Lebesgue or possibly Riemann) integral. But the first one (an Itô's "stochastic integral") is not: it is too singular to be regarded as a Riemann-Stieltjes integral defined path by path. This is due to the fact that each part of almost every path of $w_t$ is of unbounded variation in a finite time interval. In short, its lenght is infinitive. On the other hand when $\widetilde{\sigma} = 0$ , we know that the ODE (14) has a unique solution if $\widetilde{B}$ is Lipschitz. Under this kind of restrictions on $\widetilde{B}$ and $\widetilde{\sigma}$ a differential and integral calculus on path space $\Omega_0$ has been designed, based on Wiener measure $\mu_w^{\hbar}$. Some of the mathematicians associated with this construction are Wiener, Cameron, Martin, Girsanov, Itô, Malliavin...

It appeared quite early, in this context, that there is a privileged subspace (named after Cameron and Martin) of $\Omega_0$:

$$\mathfrak{H} = \left\{ q \in \Omega_0, \text{ abs. cont. and s.t. } ||q||_1^2 = \int_0^\infty |\dot{q}(\tau)|^2 d\tau < \infty \right\}. \tag{15}$$

This is an Hilbert space. Now, to take Feynman's "Lebesgue measure" seriously would require to be able to do at least any translation

$$\begin{array}{rcccc} T_q & : & \Omega_0 & \rightarrow & \Omega_0 \\ & & \omega & \mapsto & \omega + q \end{array}$$

without problem in the paths space. In probability theory "without problem" means that the resulting measure $P_q$ is absolutely continuous with respect to $\mu_w^\hbar$ (otherwise we cannot compare them). But this is the case, precisely, iff $q \in \mathfrak{H}$. In fact, the Wiener measure does not "see" such "classical" paths:

$$\mu_w^\hbar(\mathfrak{H}) = 0.$$

Nevertheless, we have an explicit (Cameron, Martin, Girsanov) formula for the Radon-Nikodym density $\frac{dP_q}{d\mu_w^\hbar}$ involving Itô's stochastic integral; this one is so defined as an extension from ODE to SDE (cf. [8]). On this basis, a differential geometry can be constructed, where $\mathfrak{H}$ is regarded as a tangent space of $\Omega_0$. But, by the same token, the intuitive definition (4) stresses what will be the key technical difficulty of the framework. If we pick a typical $\omega \in \Omega_0$, $S_{L_0}[\omega]$ will be divergent. Only $S_{L_0}[q]$, for $q \in \mathfrak{H}$ would make sense but those "classical" paths are not representative in any way.

This situation is very reminiscent of Feynman's path integral method of quantization (cf. [4]). According to him, the fundamental (kinematical) property of (one dimensional) quantum paths $t \rightarrow \omega(t)$ is

$$\left\langle (\omega(t + \Delta t) - \omega(t))^2 \right\rangle_{S_L} = i\hbar\Delta t \tag{16}$$

where $S_L[q] = \int L(\dot{q}, q)dt$ is the Action functional of the classical system to be quantized. We shall prefer interpret geometrically $S_L$ as integral of the Poincaré-Cartan differential form:

$$S_L = \int \omega_{pc}, \quad \text{where} \quad \omega_{pc} = pdq - hdt, \tag{17}$$

for $p$ and $h = h(q, p)$ respectively, the momentum and Hamiltonian observables and (17) is computed along smooth solutions of the classical equations of motion.

In relation (16), $\langle\cdot\rangle_{S_L}$ denotes Feynman's path integral reinterpretation of quantum expectations in term of spectral measures of self-adjoint operators in $L^2$ (as in (8), for example). Nobody has ever been able to construct a probability space where Feynman's "process" underlying Eq. (16) could live. His path integral method is, from the probabilistic viewpoint, without content. But, in this symbolic method as in the rigorous discussion above, we observe the same coexistence of smooth (classical) and divergent expressions. Clearly, Feynman was discovering, in his historical 1948 paper, a kind of informal Itô's calculus.

Probability theory, when we can use it, provides tools to (partly) eliminate some divergences. Let us come back to Eq. (14), for $n = 1$. As said before, $\frac{dz_t}{dt}$ does not make sense. But, using the conditional expectation involved in Eq. (10) we can define a random variable which does:

$$\lim_{\Delta t \downarrow 0} E_t \left[ \frac{z_{t+\Delta t} - z_t}{\Delta t} \right] \equiv Dz_t \tag{18}$$

To such a $z_t$ is associated, in fact, a differential operator (when $\widetilde{\sigma} = \hbar^{\frac{1}{2}}$, as needed for quantum physics (cf. (9))):

$$D = \frac{\partial}{\partial t} + \widetilde{B} \frac{\partial}{\partial q} + \frac{\hbar}{2} \frac{\partial^2}{\partial q^2} \tag{19}$$

acting on regular $F = F(q,t)$, with $q = z_t$. When $F = q$, we have $Dz_t = \widetilde{B}(z_t)$. Now under $E_t$, $D$ behaves indeed as a time derivative:

$$\forall F \in \mathcal{D}_D, \quad E_{q,t} \int_t^T DF(z_\tau, \tau) d\tau = E_{q,t}[F(z_T, T)] - F(q,t). \tag{20}$$

Notice that, on the left hand side we are dealing with an usual (Lebesgue or Riemann) integral. Still, as it is clear from definition (19), before the expectation we must expect $\hbar$-deformations of elementary calculus. For example, Leibniz rule becomes, for $F = F(q)$ and $G = G(q)$ regular enough,

$$D(F.G) = DF.G + F.DG + \hbar \nabla F \nabla G. \tag{21}$$

Clearly, the Lagrangian in $S_L$ (cf. Eq. (16)) is defined on a space (a contact bundle, in fact) where $q$ and $\dot{q}$ are regarded as independent variables. How could we do that with $z_t$ and $Dz_t$? For a given drift $\widetilde{B}$ in the SDE (14), of course $z_t$ and $Dz_t$ are dependent...

This puzzling question is closely related with the lack of any "Stochastic Symplectic Geometry".

## 3. Cartan's geometry of PDE and Stochastic Symplectic Geometry

How could we build a Liouville measure $dpdq(\equiv dp \wedge dq)$ when $q = z_t$, a diffusion solving Eq. (14)? In quantum theory, the status of position and momentum observables are sharply distinguished: when one is a multiplication operator, the other is a partial differential operator. Our probabilistic framework suggests to find a model where the process and its drift can be regarded as independent variables.

We shall use Cartan's ideas on the geometry of Eq. (1) (cf. [5]). Or, better, motivated by the fundamental role of $\omega_{pc}$ in (17) and Feynman's expression (4), the geometry of the PDE solved by $S = -\hbar \, ln \, \eta_t(q)$, $\eta_t > 0$ solving Eq. (1):

$$\frac{\partial S}{\partial t} = \frac{1}{2}\left(\frac{\partial S}{\partial q}\right)^2 - \frac{\hbar}{2}\frac{\partial^2 S}{\partial q^2} \tag{22}$$

Eq. (22) is called the (free) Hamilton-Jacobi-Bellman equation ($HJB$). It is clearly an $\hbar$-deformation of its classical counterpart. On this basis one defines two new variables,

$$E = -\frac{\partial S}{\partial t}, \; B = -\frac{\partial S}{\partial q} \tag{23}$$

respectively called energy and momentum. Any solution of $HJB$ annuls the set of classical differential forms on $M = \mathbb{R}^5$ (with coordinates $q, t, B, E, S$):

$$\begin{aligned} \omega &= dS + Bdq + Edt \\ \Omega &= d\omega = dBdq + dEdt \\ \beta &= \left(E + \frac{1}{2}B^2\right) dqdt + \frac{\hbar}{2} dBdt \end{aligned} \tag{24}$$

where $d$ denotes, here, the classical exterior derivative. $\omega$ is called a contact form; clearly $\omega - dS$ is the form playing the role of Poincaré-Cartan one. $\Omega$ is a Liouville form in extended phase space. $\beta$ seems artificial but results simply from the product of Eq. (22) with $dqdt$. The exterior derivative of $\beta$ is a product of $d\omega$ with another form so the set (24) is closed under exterior multiplication and differentiation. It is called the differential ideal $I_{HJB}$ of Eq. (22). In this sense, the geometrical structure of Hamilton-Jacobi-Bellman contains all what should be needed, in particular for a symplectic (or a contact) geometry.

One of the main purposes of the representation of Eq. (22) as the ideal (24) is to study, in a way independent on the coordinates, the symmetries of $HJB$. If

$$N = N^q \frac{\partial}{\partial q} + N^t \frac{\partial}{\partial t} + N^B \frac{\partial}{\partial B} + N^E \frac{\partial}{\partial E} + N^S \frac{\partial}{\partial S} \tag{25}$$

is an infinitesimal symmetry then, for each differential form $\alpha$ in $I_{HJB}$ the Lie derivative $\mathcal{L}_N \alpha$ must stay in $I_{HJB}$. In short

$$\mathcal{L}_N(I_{HJB}) \subset I_{HJB}. \tag{26}$$

Such vector fields $N$ are called isovectors. In (25), the coefficients $N^q, N^t$, ..., $N^S$ are unknown. The condition (26) provides a set of linear PDE for them. Their solution is as follow (cf. [6] for details)

$$N_1 = \frac{\partial}{\partial t}, \quad N_2 = \frac{\partial}{\partial q}, \quad N_3 = -\hbar \frac{\partial}{\partial S}$$

$$N_4 = 2t \frac{\partial}{\partial t} + q \frac{\partial}{\partial q} - 2E \frac{\partial}{\partial E} - B \frac{\partial}{\partial B}$$

$$N_5 = -t \frac{\partial}{\partial t} + q \frac{\partial}{\partial S} + B \frac{\partial}{\partial E} - \frac{\partial}{\partial B} \tag{27}$$

$$N_6 = 2t^2 \frac{\partial}{\partial t} + 2qt \frac{\partial}{\partial q} + (\hbar t - q^2) \frac{\partial}{\partial S} - (2qB + 4tE + \hbar) \frac{\partial}{\partial E} + 2(q - tB) \frac{\partial}{\partial B}$$

$$N_g = e^{\frac{1}{\hbar} S} \left\{ -\hbar g \frac{\partial}{\partial S} + (\hbar g_t - Eg) \frac{\partial}{\partial E} + (\hbar g_q - Bg) \frac{\partial}{\partial B} \right\},$$

for $g$ a positive solution of Eq. (1) (the isovector $N_g$ expresses the superposition principle for the underlying linear equation (1)). Of course any linear combination of these $N$ is also an isovector.

The second form of (24) suggests to define, for $\forall$ pair $\delta, \delta'$ of vector fields in $M$ the following Cartan's "bilinear covariant":

$$\Omega(\delta, \delta') = (\delta(B)\delta'(q) - \delta(q)\delta'(B)) + (\delta(E)\delta'(t) - \delta(t)\delta'(E)). \tag{28}$$

The basic property of isovectors, in term of this definition, is that, for any vector field $\delta$ in $M$

$$\Omega(N, \delta) = -\delta(n_N) \tag{29}$$

as long as $N$ is generated by $N_1, ..., N_6$. In Eq. (29), $n_N$ denotes the contraction of theisovector $N$ with the contact form $\omega$:

$$n_N \equiv \omega(N) = N^t E + N^q B + N^S. \tag{30}$$

Given $M$ and $I_{HJB}$ it is clear how we should define, now, our (free) Lagrangian and the Poincaré-Cartan form on $M$:

$$L(B,q) = \frac{1}{2}B^2 \tag{31}$$

and

$$\omega_{pc} = \omega - dS = Bdq + Edt \tag{32}$$

The set of forms (24) is in involution with respect to our starting variables $q$ and $t$, i.e. gives back $HJB$ if $(q,t)$ are independent. This means that the "sectioned" forms (traditionally denoted by a $\sim$) are annulled on the 2d submanifold of $M$ where $dq, dt$ are independent forms. Or, equivalently, that the two dimensional "integral submanifold" of the ideal $I_{HJB}$ (annuling all forms of $I_{HJB}$) represents geometrically the solutions of the $HJB$ equation.

The section map $\theta_\eta$ has been already used implicitly in our definitions of $S$, the drift $\widetilde{B}$ of Eq. (9) (cf. also (19)) etc...

$$\theta_\eta(q) = q, \quad \theta_\eta(t) = t, \quad \theta_\eta(S) = \hbar\, ln\, \eta_t \equiv \widetilde{S}(q,t)$$
$$\theta_\eta(B) = \hbar\nabla\, ln\, \eta_t \equiv \widetilde{B}(q,t), \quad \theta_\eta(E) = \hbar\, \partial_t\, ln\, \eta_t \equiv \widetilde{E}(q,t). \tag{33}$$

Then we can re-express our main tools on the space-time submanifold:

$$\theta_\eta(\omega_{pc}) = \widetilde{B}dq + \widetilde{E}dt$$
$$\theta_\eta(n_N) = \hbar\frac{\nu_N \eta_t}{\eta_t}, \text{ for } \nu_N = N^t\frac{\partial}{\partial t} + N^q\frac{\partial}{\partial q} + \frac{1}{\hbar}N^S \tag{34}$$
$$\theta_\eta \circ \Omega = \Omega_\eta.$$

The 2-form $\Omega$ takes, then, a simple form when reduced to isovectors $N, N'$ not transforming the $S$-variable:

$$\Omega_\eta(N,N') = \hbar\frac{\nu_{[N,N']}\eta}{\eta} \quad \text{and} \quad d\Omega_\eta = 0. \tag{35}$$

To get the probabilistic interpretation we just have to read our various geometrical objects on $q = z_t$, solution of the SDE (9):

$$\widetilde{S}(q,t) = E_{q,t}\int_t^T \theta_\eta(\omega_{pc})d\tau. \tag{36}$$

By (20) and (34), the integrand, i.e. our Lagrangian, can be identified with $-D\widetilde{S}(z_\tau,\tau)$. Using (19) and (33) this reduces, as expected (cf. (31)), to

$$L_0(Dz_\tau, z_\tau) = \frac{1}{2}(Dz_\tau)^2 \tag{37}$$

Let us stress that this Lagrangian is not anymore singular along the "quantum" paths $\tau \mapsto z_\tau$, and can be used for the calculus whose existence was suggested by Feynman. For example, the condition of invariance of $L_0$ under the symmetry generated by any isovector $N$ is

$$\mathcal{L}_N(L_0) + L_0 \frac{dN^t}{dt} = -DN^S \tag{38}$$

This is a key result of a stochastic calculus of variations for the action (36), not so easy to obtain directly via Itô's calculus. We shall conclude with two examples, showing that this geometrical framework contains new informations both in probabilistic terms and in physical ones.

## 1) Probability theory

In the table (27), consider $N = -\frac{1}{2}N_6$. On the 2d space-time submanifold (cf. (34)) we have

$$\widehat{N}_6 = -\frac{1}{2}\nu_{N_6} = t^2\frac{\partial}{\partial t} + qt\frac{\partial}{\partial q} - \frac{1}{2\hbar}(q^2 - \hbar t). \tag{39}$$

Then $U_\alpha^N = \exp(\alpha \widehat{N}_6)$ maps a positive solution $\eta$ of (1) into

$$\eta_\alpha(q,t) = \frac{1}{\sqrt{1-\alpha t}}\, e^{-\frac{\alpha q^2}{2\hbar(1-\alpha t)}}\, \eta\left(\frac{q}{1-\alpha t}, \frac{t}{1-\alpha t}\right), \alpha \in \mathbb{R} \tag{40}$$

Clearly this corresponds to the change of space time variables

$$(Q,T) \mapsto \left(\frac{Q}{1+\alpha T}, \frac{T}{1+\alpha T}\right)$$

If the diffusion $z_t$ is built from $\eta$, according to the SDE (9), a one parameter family $z_t^\alpha$ will, therefore, be defined via $\eta_\alpha$, $\forall \alpha \in \mathbb{R}$, by

$$z^\alpha(t) = (1-\alpha t) z\left(\frac{t}{1-\alpha t}\right). \tag{41}$$

In particular, for the Wiener itself (i.e. the trivial solution $\eta_t = 1$)

$$dz_t^\alpha = \hbar^{\frac{1}{2}} dw_t - \frac{\alpha z_t^\alpha}{1-\alpha t}\, dt. \tag{42}$$

This SDE makes sense for $t < \alpha^{-1}$. Its solution is the famous "Brownian bridge", coming back at the origin at $t = \alpha^{-1}$, namely the process needed, for example, to study Loop spaces.

All invariances of the Brownian motion (like the projective one: $w_t^\alpha = \alpha^{-\frac{1}{2}} w_{\alpha t}$, etc...) follow as well from this method.

### 2) Quantum Physics

Coming back to physics, we lose, sadly, the diffusion $z_t$. But Cartan's geometrical analysis of Eq. (6) is almost identical to the one of Eq. (1). The isovector $\widehat{N}_6$ should now act on $\psi_t$. Interpretating $q$ as Heisenberg position observable etc... this means that

$$N(t) = -t^2 H_0(t) + Q(t)\, t \circ P(t) - \frac{1}{2}(Q^2(t) + i\hbar t) \tag{43}$$

is an observable (we had, first, to symmetrize it using Jordan's product $\circ$). It is easy to check that $N(t)$ is, indeed, a quantum constant of motion:

$$i\hbar \frac{\partial}{\partial t} N(t) + [N(t), H_0] = 0,$$

something hard to guess without the help of the above probabilistic analysis.

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# Least-perimeter partitions of the disk

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**Abstract.**

In this work we study the isoperimetric problem of partitioning a planar disk into $n$ regions of prescribed areas using the least-possible perimeter. We obtain the regularity conditions that must be satisfied by the solutions, and solve completely the problem in the cases of two and three regions.

*Keywords:* Isoperimetric partition, stability.

*2000 Mathematics Subject Classification:* 49Q10, 51M25.

## 1. Introduction

In the last years, the study of isoperimetric problems has been of great interest not only for its own nature but also for its relation with multitude of physical phenomena. Among them, the ones relating to isoperimetric partitions can model several natural situations.

In this work we treat the following partitioning problem: consider a planar disk $D$, and $n$ positive numbers $a_1, \ldots, a_n$ whose sum is the total area of the disk. Then we want to find the way of dividing the disk into $n$ regions $R_i$, each one of area $a_i$, with the least possible perimeter.

This question explains properly many situations in Nature: the first steps in the process of division of a cell, the shape of the interfaces separating different fluids in a round dish, and other ones (see [4, Ch. VII and VIII]).

For this problem, the main trouble is that regions may have several components, since we do not assume them to be connected. Therefore there are many ways of dividing the disk into $n$ regions of the given areas.

From [3] we obtain the existence of a solution, for any number of regions we consider:

**Theorem 1.** (Existence Theorem) *Given a planar disk $D$ and $n$ positive numbers $a_1, \dots, a_n$ such that $\sum_{i=1}^{n} a_i = area\,(D)$, there exists a least-perimeter way of dividing the disk into $n$ regions of areas $a_1, \dots, a_n$, consisting of smooth curves meeting in threes in the interior of the disk, and meeting $\partial D$ only one curve at each time.*

## 2. Regularity Conditions

Let $D \subset \mathbb{R}^2$ be a closed disk, centered at the origin. Along this work, a *graph* $C$ will consist of a finite number of vertices and edges in $D$ such that at every interior vertex (a vertex in the interior of the disk) three edges meet, and at every vertex in $\partial D$, only one edge arrives. Note that, in view of Theorem 1, this is a natural definition.

We shall assume that a graph $C$ decomposes the open disk into $n$ regions $R_i$, possibly nonconnected as commented before, and we shall denote by $C_{ij} \subset C$ the curve separating two adjacent regions $R_i$ and $R_j$ (this curve may not be connected), by $N_{ij}$ the normal vector to $C_{ij}$ pointing into $R_i$, and by $h_{ij}$ the geodesic curvature of $C_{ij}$ with respect to $N_{ij}$ (as we are working in the plane, the geodesic curvature coincides with the usual curvature of a curve).

We will call *minimizing graph* to the least-perimeter graph dividing the disk into $n$ regions of the given areas.

Let $\varphi_t : C \to D$ be a smooth variation of a graph $C$, for t small, such that $\varphi_t(C \cap \partial D) \subset \partial D$. Denoting the associated vector field by $X = d\varphi_t/dt|_{t=0}$, with normal components $u_{ij} = X \cdot N_{ij}$ on $C_{ij}$, then the derivative of the area $A_i$ enclosed by a region $R_i$ at $t = 0$ is equal to

$$\left.\frac{dA_i}{dt}\right|_{t=0} = -\sum_{j \in I(i)} \int_{C_{ij}} u_{ij},$$

where $I(i) = \{j \neq i; R_j \text{ touches } R_i\}$.

**Proposition 2.** (First variation of length) *Given a graph $C$ and a smooth variation $\varphi_t : C \to D$, the first derivative of the length functional of $\varphi_t(C)$ at $t = 0$ is equal to*

$$\left.\frac{dL}{dt}\right|_{t=0} = -\frac{1}{2} \sum_{\substack{i \in \{1,\dots,n\} \\ j \in I(i)}} \left\{ \int_{C_{ij}} h_{ij} u_{ij} + \sum_{p \in \partial C_{ij}} X(p) \cdot \nu_{ij}(p) \right\}, \tag{1}$$

*where $\nu_{ij}(p)$ is the inner conormal to $C_{ij}$ in $p$.*

A graph will be said *stationary* if $\frac{dL}{dt}|_{t=0} = 0$, for any variation preserving the areas. Observe that stationary graphs are critical points for the length functional when the areas $A_i$ are fixed. Since we want to minimize such a functional, it follows that a minimizing graph must be stationary.

From Proposition 2 we obtain the regularity conditions that stationary graphs (and then minimizing graphs) must verify:

**Theorem 3.** (Regularity Conditions) *Given a stationary graph $C$, the following regularity conditions must be satisfied:*

*i) The curvature $h_{ij}$ is constant on $C_{ij}$, and accordingly the edges will be circular arcs or line segments.*

*ii) The edges of $C$ meet in threes at 120-degree angles in interior vertices.*

*iii) Three edges $C_{ij}$, $C_{jk}$, $C_{ki}$ meeting in an interior vertex satisfy*

$$h_{ij} + h_{jk} + h_{ki} = 0.$$

*iv) The edges of $C$ meet $\partial D$ orthogonally.*

*Proof.* By considering appropriate area-preserving variations in (1), the conditions above easily follow.

# 3. Minimizing graph for two regions

The problem for two regions can now be solved from Theorem 3. In this case no triple interior vertices can appear in the minimizing graph (any interior vertex would be surrounded by three different regions), and so any edge will be a circular arc or a line segment meeting $\partial D$ orthogonally.

Let us assume that the minimizing graph has more than one edge. Then, by rotating one of them about the origin until touching another one, we will obtain a new minimizing graph (since the perimeter and the areas enclosed are preserved) with a non-allowed vertex in $\partial D$, which is contradictory.

Hence we have the following

**Theorem 4.** *Let $C \subset D$ be a minimizing graph for two given areas. Then $C$ consists of a circular arc or a line segment meeting orthogonally $\partial D$.*

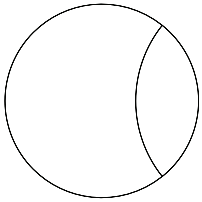

Figure 1: The least-perimeter partition of the disk into two given areas

## 4. Minimizing graph for three regions

Now we treat the problem for three regions. Straightforward calculations give the next proposition.

**Proposition 5.** (Second variation of length) *For a stationary graph $C$ and a variation $\{\varphi_t\}$ that preserves the areas, the second derivative of the length functional at $t=0$ is given by*

$$-\frac{1}{2}\sum_{\substack{i=1,\dots,n\\ j\in I(i)}}\left\{\int_{C_{ij}}(u_{ij}''+h_{ij}^2u_{ij})\,u_{ij}+\sum_{\substack{p\in\partial C_{ij}\\ p\in\mathrm{int}(D)}}\left(-q_{ij}u_{ij}^2+u_{ij}\frac{\partial u_{ij}}{\partial\nu_{ij}}\right)(p)\right. \tag{2}$$
$$\left.+\sum_{\substack{p\in\partial C_{ij}\\ p\in\partial D}}\left(u_{ij}^2+u_{ij}\frac{\partial u_{ij}}{\partial\nu_{ij}}\right)(p)\right\},$$

*where $q_{ij}(p)=(h_{ki}+h_{kj})(p)/\sqrt{3}$, and $R_k$ is the third region touching the vertex $p$.*

Let us introduce an important concept in this work: for a region $R_i$, it is possible to define its *pressure* $p_i$ as a real number such that, for any edge $C_{ij}\subset C$,

$$h_{ij}=p_i-p_j.$$

Then, for area-preserving variations *by stationary graphs* (that is, at each instant of the deformation we obtain stationary graphs, satisfying the conditions of Theorem 3), the second variation of length (2), expressed in terms of the pressures, turns

$$\frac{d^2L}{dt^2}=\sum_\alpha\frac{dp_\alpha}{dt}\,\frac{dA_\alpha}{dt}, \tag{3}$$

where $\alpha$ labels the *components* of the graph (recall that regions may have various components).

We will say that a stationary graph $C$ is *stable* if $\frac{d^2L}{dt^2}\big|_{t=0} \geqslant 0$, for any area-preserving variation. This means that stable graphs are second order local minima for the length when the areas $A_i$ are preserved. Then it is clear that a minimizing graph must be stable.

A *hexagonal component* of a region is a component bounded by six edges. Fix $R_1$ as the *region of largest pressure.* From Proposition 5 we get the following result:

**Proposition 6.** *Consider a stable graph dividing the disk into $n$ regions. Then, $R_1$ has at most $n-1$ nonhexagonal components.*

In the case $n = 3$, this result allows us to discuss the possible configurations for a minimizing graph, once we have checked, using Gauss-Bonnet Theorem, that hexagonal components cannot occur:

**Proposition 7.** *Let $C \subset D$ be a minimizing graph dividing $D$ into three regions. Then $C$ is one of the graphs in Figure 2.*

Configuration 2(**10**) will be called *standard graph.* Now we will show how to discard the non-standard possibilities, to conclude that the minimizing graph for three regions is the standard graph.

Let us check that configurations 2(**1**) and 2(**2**) are unstable, and hence not minimizing. The motive is that both of them have a region with two *triangles* (components of three edges) touching $\partial D$.

**Proposition 8.** *Given a stationary graph with a triangle $\Omega$ touching $\partial D$, there exists a variation by stationary graphs such that*

*(i) increases the area of $\Omega$,*

*(ii) decreases the pressure of $\Omega$, keeping the other pressures unchanged, and*

*(iii) leaves invariant the edges of the graph not placed in $\partial\Omega$.*

*Remark.* For this variation, we have that $\frac{dp}{dt}\frac{dA}{dt} < 0$ in $\Omega$.

**Proposition 9.** *Any stationary graph with a region with two triangles touching the boundary of the disk is unstable.*

*Sketch of the proof.* First, consider in each triangle $\Omega_i$ the variation of Proposition 8. By combining both variations, we can construct another variation by stationary graphs too, preserving the areas. Finally, by using Equation (3) to compute the second variation of length we have

$$\frac{d^2L}{dt^2}\bigg|_{t=0} = \sum_{\alpha} \frac{dp_\alpha}{dt}\frac{dA_\alpha}{dt} = \frac{dp_1}{dt}\frac{dA_1}{dt} + \frac{dp_2}{dt}\frac{dA_2}{dt} < 0,$$

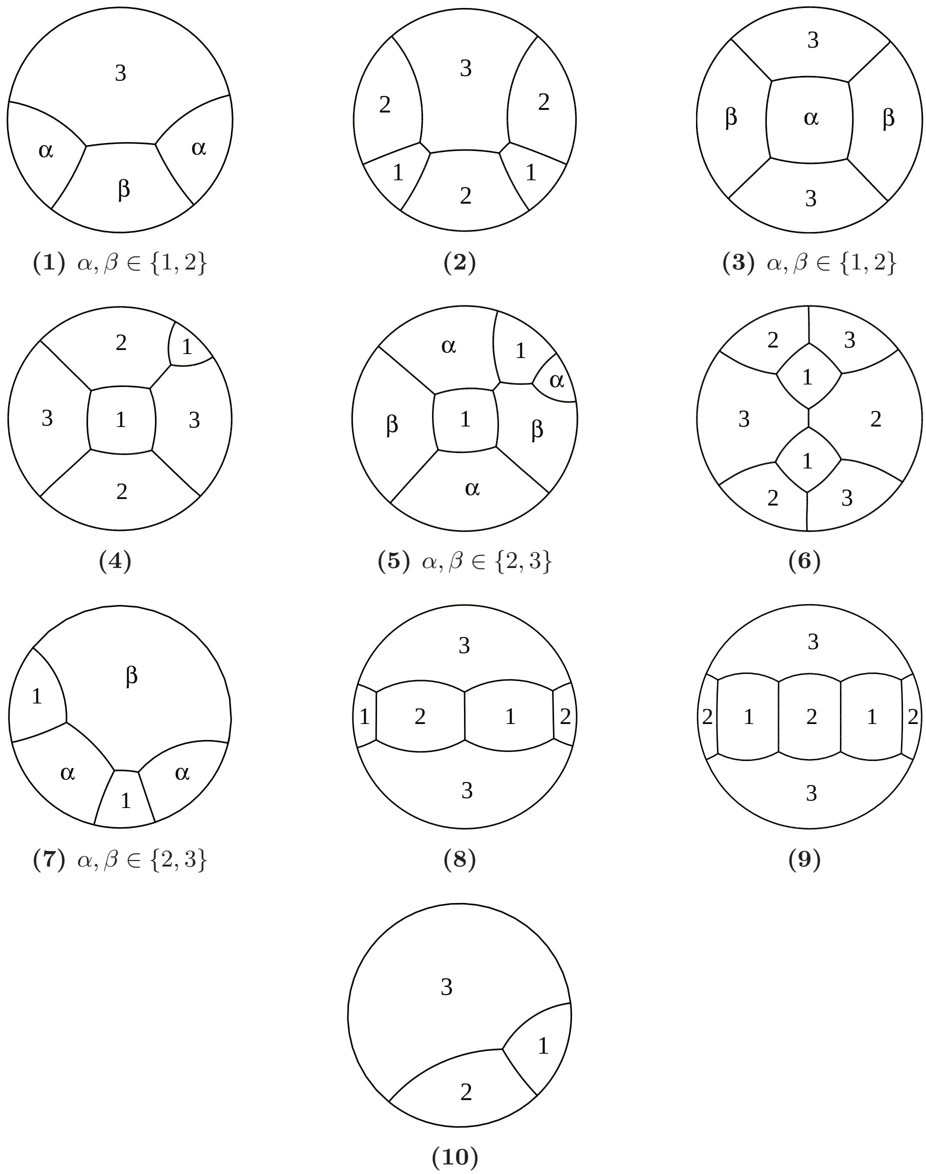


Figure 2: The ten possible configurations for a minimizing graph

since the only pressures that change along the deformation are the ones of the triangles $\Omega_1$ and $\Omega_2$.

Therefore, Proposition 9 yields instability of configurations $2(\mathbf{1})$ and $2(\mathbf{2})$.

A function $u : C = \bigcup_{i,j} C_{ij} \to \mathbb{R}$ is said a *Jacobi function* if the restrictions to $C_{ij}$ satisfy

$$u''_{ij} + h^2_{ij}\, u_{ij} = 0.$$

The following result will show that configurations $2(\mathbf{3})$ to $2(\mathbf{7})$ of Figure 2 are unstable:

**Proposition 10.** ([2]) *Let $C$ be a stationary graph separating the disk into three regions. If there exists a Jacobi function defined on $C$ with at least four nodal domains, then $C$ is unstable.*

In this case, the normal components of the rotations vector field about the origin constitute a suitable Jacobi function (recall that a nodal domain is a domain in $C$ where that function does not vanish). Then, because of some symmetries we find in these configurations, we can apply Proposition 10 and obtain the instability of all of them.

Finally let us see that configuration $2(\mathbf{8})$ cannot be a minimizing graph. It is possible to construct a new configuration with the same perimeter and enclosing the same areas by this geometric transformation:

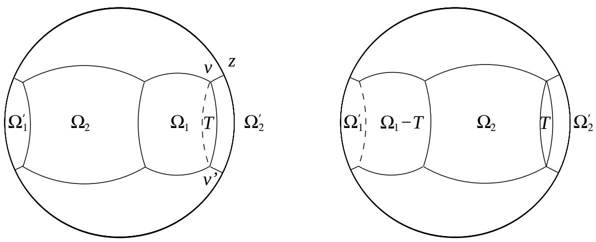


Figure 3: Geometric transformation creating two non-allowed vertices

As the new configuration has two non-allowed vertices, it cannot be minimizing, and so, neither configuration $2(\mathbf{8})$. The same argument can be applied for configuration $2(\mathbf{9})$.

Then, the nine non-standard possibilities have been discarded, so we obtain the following theorem:

**Theorem 11.** ([1]) *Let $C \subset D$ be a minimizing graph for three given areas. Then $C$ is a standard graph.*

*Remark.* For two and three regions, the solutions are *unique* up to rigid motions of the disk.

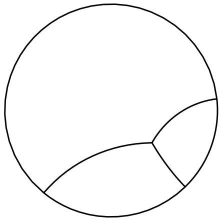

Figure 4: The least-perimeter partition of the disk into three given areas

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# Periodic maximal graphs in the Lorentz-Minkowki space $\mathbb{L}^3$

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**Abstract.** We study maximal graphs in the Lorentz-Minkowski space $\mathbb{L}^3$ invariant under a discrete group of isometries and having a finite number of singularities in its fundamental piece. We also give a method to construct them, based on the Weierstrass representation for maximal surfaces.

*Keywords:* Maximal surfaces, conelike singularities, periodic surfaces.

*2000 Mathematics Subject Classification:* 53C50, 53C42, 53A10

## 1. Introduction

Maximal surfaces in a Lorentzian manifold are spacelike surfaces with zero mean curvature. In the Lorentz-Minkowski space $\mathbb{L}^3$ these surfaces arise as local maxima for the area functional associated to variations of the surface by spacelike surfaces. Also, maximal graphs in $\mathbb{L}^3$ are the solutions for a quasi-linear elliptic differential equation, and therefore a maximum principle for them is satisfied. As in the case of minimal surfaces in the Euclidean space, maximal surfaces have a conformal representation (*Weierstrass representation*) in terms of meromorphic data on a Riemann surface.

A classical result by Calabi [1] asserts that the unique complete maximal surfaces in $\mathbb{L}^3$ are the spacelike planes. However, if we allow the existence of singularities, there is a vast theory of complete maximal surfaces, see for example [10], [2], [4], [5], [6]. In this paper we focus our attention on isolated embedded singularities of maximal surfaces, also called *conelike singularities* (see [7]). If in adittion the surface is complete or proper, it turns out that it is a graph over any spacelike plane of $\mathbb{L}^3$.

We say that a surface is *periodic* if it is invariant under a group of isometries $G$ of $\mathbb{L}^3$ acting properly and freely on $\mathbb{L}^3$. This paper develops the main results obtained by the authors in [3] for periodic maximal surfaces in the embedded case. In concrete, we show that the group $G$ contains a finite index subgroup $G_0$ which is a group of translations of rank 0 (that is, $G_0 = \{Id\}$), 1 (singly periodic surfaces), or 2 (doubly periodic surfaces). We also use the Weierstrass representation of maximal surfaces to give a recipe recovering these surfaces.

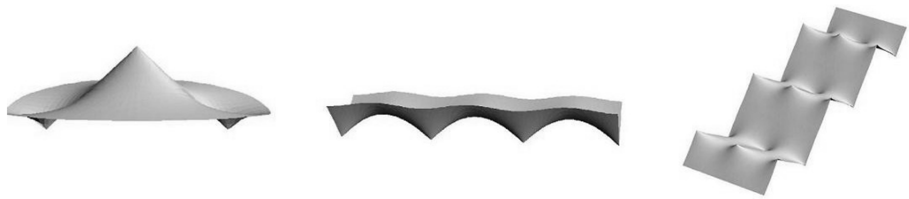

Figure 1: Examples of maximal graphs with isolated singularities

## 2. Preliminaries

### 2.1. Spacelike immersion with isolated singularities

Through this paper $\mathbb{L}^3$ will denote the 3-dimensional Lorentz-Minkowski space, that is $\mathbb{L}^3 = (\mathbb{R}^3, dx^2 + dy^2 - dz^2)$, and $\mathcal{M}$ a differentiable surface.

An immersion $X : \mathcal{M} \longrightarrow \mathbb{L}^3$ is said to be *spacelike* if for any $p \in \mathcal{M}$, the tangent plane $T_p\mathcal{M}$ with the induced metric is spacelike, that is to say, the induced metric on $\mathcal{M}$ is Riemannian. This metric induces a conformal structure on $\mathcal{M}$, and so it becomes in a Riemann surface.

Let $F \subset \mathcal{M}$ be a *discrete closed* subset of a differentiable surface $\mathcal{M}$ and $ds^2$ a Riemannian metric in $\mathcal{M} - F$. Take a point $q \in F$, an open disk $\mathcal{D}(q)$ in $\mathcal{M}$ such that $\mathcal{D}(q) \cap F = \{q\}$ and an isothermal parameter $z$ for $ds^2$ on $\mathcal{D}(q) - \{q\}$. Then write $ds^2 = h|dz|^2$, where $h(w) > 0$ for any $w \in z(\mathcal{D}(q) - \{q\})$. By definition, the Riemannian metric $ds^2$ is *singular* at $q$ if for any disk $\mathcal{D}(q)$ and any parameter $z$ as above, the limit $\lim_{p \to q} h(z(p))$ vanishes (as a matter of fact, it suffices to check this condition just for one disc and conformal parameter). The metric $ds^2$ is said to be singular at $F$ if it is singular at any point of $F$. In this case, $(\mathcal{M}, ds^2)$ is said to be a Riemannian surface with isolated singularities and $F$ is the singular set of $(\mathcal{M}, ds^2)$.

**Definition 1.** Let $X : \mathcal{M} \to \mathbb{L}^3$ be a continuous map. Suppose there is a discrete closed $F \subset \mathcal{M}$ subset such that $X|_{\mathcal{M}-F}$ is a spacelike immersion and $(\mathcal{M}, ds^2)$ is a Riemannian surface with isolated singularities in $F$, where $ds^2$ is the metric induced by $X$.

Then, $X$ is said to be a spacelike immersion with (isolated) singularities at $F$, and $X(\mathcal{M})$ a spacelike surface with (isolated) singularities at $X(F)$.

The following lemma describes the behavior of a spacelike immersion around an isolated singularity.

**Lemma 1** ([3]). *Let $X : \mathcal{M} \to \mathbb{L}^3$ be a spacelike immersion with isolated singularities and $\Pi$ a spacelike plane. Label $\pi : \mathbb{L}^3 \to \Pi$ as the Lorentzian orthogonal projection.*

*Then, $h := \pi \circ X$ is a branched local homeomorphism and its branch points correspond to the locally non embedded singularities of $X$.*

*As a consequence, if $X$ is an embedding locally around the singular points and is proper, then $X(\mathcal{M})$ is a graph over any spacelike plane (in particular $X$ is an embedding). The same conclusion holds if we replace proper by complete.*

## 2.2. Maximal surfaces

A maximal immersion $X : \mathcal{M} \to \mathbb{L}^3$ is a spacelike immersion with vanishing mean curvature. The notion of maximal immersion with (isolated) singularities is defined analogously.

If $X : \mathcal{M} \to \mathbb{L}^3$ is a (everywhere regular) maximal immersion, it is known that there exist a meromorphic map $g$ with $|g| \neq 1$ and a holomorphic 1-form $\phi_3$ defined on the Riemann surface $\mathcal{M}$ satisfying that the vectorial 1-form $\Phi = (\phi_1, \phi_2, \phi_3) := (\frac{i}{2}(\frac{1}{g} - g)\phi_3, \frac{-1}{2}(\frac{1}{g} + g)\phi_3, \phi_3)$ is holomorphic, non vanishing and without real periods in $\mathcal{M}$. Moreover, up to a translation $X$ is given by $X(p) = \operatorname{Re} \int_{p_0}^{p} \Phi$, where $p_0$ is an arbitrary point.

Either the pair $(g, \phi_3)$ or the vectorial 1-form $\Phi$ is called *the Weierstrass representation* of the maximal immersion $X$.

As mentioned before, we will focus our attention in embedded surfaces, and therefore, we will consider only embedded singularities. For a more general treatment of non-embedded maximal surfaces with isolated singularities see [3]. As a consequence of Lemma 1, any proper (or complete) maximal surface with isolated embedded singularities is a graph over any spacelike plane.

The behaviour of a maximal immersion around embedded isolated singularities is well known (see for example [4], [7]). As a matter of fact, if $\mathcal{D}$ is a disc around such a singularity $p$, then $\mathcal{D} \setminus \{p\}$ is conformally equivalent to an annulus $\mathcal{A}$, and the Gauss map of the immersion becomes lightlike at the boundary component of $\mathcal{A}$ corresponding to the singularity. Moreover, around $X(p)$ the surface $X(\mathcal{M})$ is asymptotic to a component of the light cone at $X(p)$. For this reason, isolated embedded singularities of maximal surfaces are also called *conelike singularities*.

**Definition 2.** We say that a maximal graph with isolated singularities $X : \mathcal{M} \to \mathbb{L}^3$ is $G$-periodic if $X(\mathcal{M})$ is invariant under a discrete subgroup $G$ of isometries acting freely and properly of $\mathbb{L}^3$. We say that $X$ is singly (resp. doubly) periodic if $G$ is a group of translations of rank one (resp. two).

If in addition the quotient of the singular set of $X$ under the relation induced by $G$ is finite we say that $X$ is *of finite type.*

Let $X : \mathcal{M} \to \mathbb{L}^3$ be a maximal graph of finite type and label $F = \{p_\alpha : \alpha \in \Lambda\} \subset \mathcal{M}$ as its singular set. Taking into account the local behavior around the singularities described above and the results about the Koebe uniformization given in [8], we can deduce that the Riemann surface $\mathcal{M} \setminus F$ is biholomorphic to a circular domain $\mathbb{C} \setminus \cup_{\alpha\in\Lambda} D_\alpha$, where $D_\alpha$ are pairwise disjoint closed discs in $\overline{\mathbb{C}}$ whose boundaries $\gamma_\alpha := \partial(D_\alpha)$, $\alpha \in \Lambda$, correspond to the singularities. In this setting, we label

$$\mathcal{M}_0 := \mathbb{C} \setminus \cup_{\alpha\in\Lambda}\mathrm{Int}(D_\alpha) \tag{1}$$

and we refer to it as the *conformal support* of $X$. The conformal reparameterization $X_0 : \mathbb{C} \setminus \cup_{\alpha\in\Lambda} D_\alpha \to \mathbb{L}^3$ extends to $\mathcal{M}_0$ by putting $X_0(\gamma_\alpha) = X(p_\alpha)$.

## 3. Main Results

**Theorem 2** ([3]). *Let $X : \mathcal{M} \to \mathbb{L}^3$ be a $G$-periodic maximal graph of finite type. Then the subgroup $G_0$ of $G$ consisting of the positive and orthochronous (that is, preserving $\mathbb{H}^2_+$) isometries of $G$, which is a finite index subgroup of $G$, is either the identity or a group of spacelike translations of rank 1 or 2.*

Our aim now is to describe the global behavior of the $G$-periodic maximal graphs of finite type when $G$ is one of the three groups given in the above theorem in terms of its Weierstrass data.

Thus, let $X : \mathcal{M} \to \mathbb{L}^3$ be as in the statement of the theorem, and consider its conformal support $\mathcal{M}_0$ and the conformal reparameterization $X_0 : \mathcal{M}_0 \to \mathbb{L}^3$ (see Equation (1)). Since the isometries in $G$ preserves the singular set we can regard $G$ as group of transformations in $\mathcal{M}_0$. So, we can consider the induced immersion $\hat{X}_0 : \hat{\mathcal{M}}_0 = \mathcal{M}_0/G \to \mathbb{L}^3/G$. It follows that $\hat{X}_0$ is a complete maximal immersion with a finite number of singularities. Moreover, since $\mathcal{M}$ is simply connected, $\hat{X}_0(\hat{\mathcal{M}}_0)$ is an embedded surface in $\mathbb{L}^3/G$. If in addition $G$ is a translational group, the Weierstrass data of $X_0$ can be also pushed out to $\hat{\mathcal{M}}_0$.

Observe that the Riemann surface with boundary $\hat{\mathcal{M}}_0 = \mathcal{M}_0/G$ is biholomorphic to $\Sigma \setminus \cup_{j=0}^k \mathrm{Int}(D_j)$, where $D_j$ are pairwise disjoint closed discs and

$\Sigma = \mathbb{C}$ if $G = \{Id\}$, $\Sigma = \mathbb{C}^*$ in the singly periodic case, and $\Sigma$ is a torus in the doubly periodic case.

In order to use the tools we need for our purposes is useful to work with boundaryless surfaces, for this reason we introduce the notion of the *double surface* of the conformal support. This surface is nothing but the quotient of $\hat{\mathcal{M}}_0 \cup \hat{\mathcal{M}}_0^*$ by identifying their boundary components, $\partial(\hat{\mathcal{M}}_0) \equiv \partial(\hat{\mathcal{M}}_0^*)$, where $\hat{\mathcal{M}}_0^*$ is the mirror surface associated to $\hat{\mathcal{M}}_0$ (see [9] for more details). It follows that the Weierstrass data $\Phi$ can be extended holomorphically to the double surface $\mathfrak{S}$ and satisfy $J^*(\Phi) = -\overline{\Phi}$, where $J : \mathfrak{S} \to \mathfrak{S}$ is the mirror involution, that maps each point of $\hat{\mathcal{M}}_0$ into its mirror image and vice versa (observe that the fixed point set of $J$ coincides with $\partial(\hat{\mathcal{M}}_0)$).

**Theorem 3** ([3]). *Let $\overline{\mathfrak{S}}$ be a compact Riemann surface of genus $k \geq 0$ and $J : \overline{\mathfrak{S}} \to \overline{\mathfrak{S}}$ be an antiholomorphic involution having $k+1$ pairwise disjoint Jordan curves of fixed points $\gamma_0, \ldots, \gamma_k$. Suppose also that $\overline{\mathfrak{S}} \setminus \cup_{j=0}^k \gamma_j$ has two connected components, namely $\Omega$ and $J(\Omega)$, any one of them homeomorphic (and so biholomorphic) to a circular domain[1] in the extended complex plane $\overline{\mathbb{C}}$ or in a torus $\mathbb{T}$.*

*Consider a meromorphic vectorial 1-form $\Phi = (\phi_1, \phi_2, \phi_3)$ defined on $\overline{\mathfrak{S}}$, non vanishing, with $J^*(\Phi) = -\overline{\Phi}$ and having poles at $F_\infty \cup J(F_\infty)$, where $F_\infty \subset \Omega$ consists of one (and in this case the poles are double) or two points (and in this case the poles are simple) if $\Omega \subset \overline{\mathbb{C}}$ and $F_\infty = \emptyset$ if $\Omega \subset \mathbb{T}$.*

*Finally, label $G$ as the group of translations of vectors $\{Re \int_\gamma \Phi \ : \ \gamma \in H_1(\Omega_0, \mathbb{Z})\}$, where $\Omega_0$ is the quotient surface obtained from $\overline{\Omega} \setminus F_\infty$ by identifying each component $\gamma_j$ of $\partial(\Omega)$ to a point $q_j \notin \Omega$, $j = 0, \ldots, k$ ($q_j \neq q_h$ for $j \neq h$).*

*Then, $G$ has rank 0 (if $F_\infty$ has 1 point), 1 (if $F_\infty$ contains 2 points) or 2 ($F_\infty = \emptyset$), and the map*

$$\hat{X}_0 : \overline{\Omega} \setminus F_\infty \to \mathbb{L}^3/G, \quad \hat{X}_0 = Re(\int \Phi),$$

*is well defined and provides a complete maximal surface with $k+1$ singular points (namely $\hat{X}_0(\gamma_j)$, $j = 0, \ldots, k$) whose lifting to $\mathbb{L}^3$ is a $G$-periodic maximal graph with $k+1$ singular points in its fundamental piece.*

*Conversely, any such surface can be obtained in this way.*

From the behaviour of the Weierstrass data described above we can deduce the asymptotic behaviour of the lifted periodic surface in $\mathbb{L}^3$. It turns out that if $G = \{Id\}$ the surface is asymptotic at infinity to either half catenoid or a spacelike plane, and in the singly periodic case the surface is asymptotic to

[1] that is, an open domain bounded by analytical circles

two spacelike half planes. In the doubly periodic case the resulting surface is contained in a slab (see [4], [3] for a detailed proof).

Examples of surfaces constructed using the above representation (for example, the surfaces in Figure 1) can be found in [3].

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# Hamiltonization and the study of Jacobi equations: some perspectives and results

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**Abstract.** Identifying and recasting a given set of ordinary differential equations as a finite-dimensional Poisson system is an open problem of Hamiltonian dynamics. Such problem is revisited in this work with special emphasis on the point of view of the analysis of the partial differential equation formulation of the Jacobi identities in local coordinates. The characterization of two different solution families of the Jacobi equations (one of them in dimension three, the other in dimension $n$) and the global and explicit study of their main properties (including their symplectic structure, Casimir invariants and the reduction to the Darboux canonical form) will be reviewed in what follows. Such analysis underlines how the study of skew-symmetric solutions of the Jacobi equations is not only unavoidable in order to solve the Hamiltonization problem, but also important for classification purposes and for the establishment of a common and more economic description of different Poisson systems previously unrelated, that now appear as particular instances of more general Poisson structures and can be jointly analyzed in a unified way.

*Keywords:* Hamiltonian systems, Poisson structures, Jacobi equations, Darboux theorem, Casimir invariants.

*2000 Mathematics Subject Classification:* 35Q72, 37J05, 53D17, 70H05.

## 1. Introduction

Finite-dimensional Poisson structures [1] have an important presence in all fields of mathematical physics, such as dynamical systems theory, mechanics, optics, electromagnetism, etc. Describing a given physical system in terms of such a structure opens the possibility of obtaining a wide range of information which may be in the form of perturbative solutions, invariants, nonlinear stability analysis and determination of Liapunov functionals, study of bifurcation properties, characterization of chaotic behaviour, or integrability results, just to cite a few.

Mathematically, a finite-dimensional dynamical system is said to have a Poisson structure if it can be written in terms of a set of ODEs of the form $\dot{x}_i = \sum_{j=1}^{n} J_{ij}\partial_j H$, $i = 1, \ldots, n$, or $\dot{x} = \mathcal{J} \cdot \nabla H$ in short, where $H(\mathbf{x})$, which is usually taken to be a time-independent first integral, plays the role of Hamiltonian function. The $J_{ij}(\mathbf{x})$ are the entries of a $n \times n$ matrix $\mathcal{J}$ which may be degenerate in rank —known as the structure matrix— and they have the property of being solutions of the Jacobi identities:

$$\sum_{l=1}^{n}(J_{li}\partial_l J_{jk} + J_{lj}\partial_l J_{ki} + J_{lk}\partial_l J_{ij}) = 0 \ , \quad i, j, k = 1, \ldots, n \tag{1}$$

The $J_{ij}$ must also verify the additional condition of being skew-symmetric:

$$J_{ij} = -J_{ji} \quad \text{for all } \ i, j \tag{2}$$

The possibility of describing a given finite-dimensional dynamical system in terms of a Poisson structure is still an open problem [2]-[5]. The source of the difficulty arises not only from the need of a known first integral playing the role of the Hamiltonian, but mainly due to the necessity of associating a suitable structure matrix to the problem. In other words, finding an appropriate solution of the Jacobi identities (1), complying also with the additional conditions (2), is unavoidable. This explains, together with the intrinsic mathematical interest of the problem, the permanent attention deserved in the literature by the obtainment and classification of skew-symmetric solutions of the Jacobi identities. In the simplest case of three-dimensional flows it has been possible to rewrite equations (1–2) in more manageable forms allowing the determination of some families of solutions, as well as some general results on classification of Poisson structures [2]. However, most of these strategies are not applicable when analyzing the general $n$-dimensional problem (1–2). In such a case [3]–[5], the evolution of results leading to the present-day classification of solutions of (1–2) can be summarized, roughly speaking, as a sequence of solution families of increasing nonlinearity: from the constant structures, to the linear (i.e. Lie-Poisson) structures [6], affine-linear structures [7] and finally quadratic structures [4, 8].

As an illustration of the previous trends, in these pages the characterization and main properties of two different families of solutions are reviewed [9, 10]. One of them is of dimension three [9], and will be considered in Section 2. The other [10] is of dimension $n$, and will be the subject of Section 3. As a result of the generality of such families —both consist of solutions containing functions not limited to a given degree of nonlinearity— many known Poisson structures appear embraced as particular cases, thus unifying many different systems seemingly unrelated. As we shall see, this unification has relevant consequences for the analysis of such systems, since it leads to the development of a common framework for the explicit determination of key features such as the symplectic structure or the Darboux canonical form. Such properties can now be characterized globally in a unified and economic way. The work is concluded in Section 4 with some final remarks.

## 2. A three dimensional family of solutions

In the 3d case the joint problem (1-2) can be reduced to the single equation: $J_{12}\partial_1 J_{31} - J_{31}\partial_1 J_{12} + J_{23}\partial_2 J_{12} - J_{12}\partial_2 J_{23} + J_{31}\partial_3 J_{23} - J_{23}\partial_3 J_{31} = 0$. Now for an open domain $\Omega \subset I\!\!R^3$, consider the family of functions of the form

$$J_{ij}(x) = \eta(x)\psi_i(x_i)\psi_j(x_j)\sum_{k=1}^{3}\epsilon_{ijk}\phi_k(x_k) \tag{3}$$

where indexes $i, j$ run from 1 to 3, $\{\eta, \psi_i, \phi_i\}$ are arbitrary $C^1(\Omega)$ functions of their respective arguments which do not vanish in $\Omega$ and $\epsilon$ is the Levi-Civita symbol. This family of functions is a skew-symmetric solution family of the 3d Jacobi equations. Actually such family is very general, therefore containing numerous previously known structure matrices of very diverse three-dimensional systems as particular cases. Instances of this are found in Poisson structures reported for the Euler top, the Kermack-McKendrick model, the Lorenz system, the Lotka-Volterra and generalized Lotka-Volterra systems, the Maxwell-Bloch equations, the Ravinovich system, the RTW interaction equations, etc. which are enumerated in detail in [9].

However, the generality of this solution family is not an obstacle for the characterization of the main properties. We begin by the symplectic structure and the Casimir invariant. It can be shown that the rank of such Poisson structures is constant in $\Omega$ and equal to 2, and a Casimir invariant is:

$$C(x) = \sum_{i=1}^{3}\int \frac{\phi_i(x_i)}{\psi_i(x_i)}\mathrm{d}x_i$$

Moreover, this Casimir invariant is globally defined in $\Omega$ and $C^2(\Omega)$.

To conclude, we proceed now to construct globally the Darboux canonical form. For every three-dimensional Poisson system $\dot{x} = \mathcal{J}\cdot\nabla H$ defined in an open domain $\Omega \subset I\!\!R^3$ and such that $\mathcal{J}$ is of the form (3), the Darboux canonical form is accomplished globally in $\Omega$ in the new coordinate system $\{z_1, z_2, z_3\}$ and the new time $\tau$, where $\{z_1, z_2, z_3\}$ is related to $\{x_1, x_2, x_3\}$ by the diffeomorphism globally defined in $\Omega$

$$z_1(x_1) = \int \frac{\phi_1(x_1)}{\psi_1(x_1)} \mathrm{d}x_1 \ , \ \ z_2(x_2) = \int \frac{\phi_2(x_2)}{\psi_2(x_2)} \mathrm{d}x_2 \ , \ \ z_3(x) = \sum_{i=1}^{3} \int \frac{\phi_i(x_i)}{\psi_i(x_i)} \mathrm{d}x_i$$

and the new time $\tau$ is given by a time reparametrization of the form $\mathrm{d}\tau = \eta(x(z))\phi_1(x_1(z))\phi_2(x_2(z))\phi_3(x_3(z))\mathrm{d}t$. The result of these transformations is a new Poisson system with structure matrix

$$\mathcal{J}_{\mathcal{D}}(z) = \begin{pmatrix} 0 & 1 & 0 \\ -1 & 0 & 0 \\ 0 & 0 & 0 \end{pmatrix}$$

and time $\tau$. The reduction is thus explicitly and globally completed. In this way it can be seen how the different Poisson structures embraced by family (3) become unified in a common framework.

## 3. A $n$-dimensional family of solutions

Let $\{\varphi_1(x_1), \varphi_2(x_2), \ldots, \varphi_n(x_n)\}$ be a set of nonvanishing $C^1$ functions defined in a subset $\Omega \subset I\!\!R^n$. A separable matrix is a $n \times n$ matrix defined in $\Omega$ of the form

$$J_{ij} = a_{ij}\varphi_i(x_i)\varphi_j(x_j)$$

with $a_{ij} \in I\!\!R$, $a_{ij} = -a_{ji}$ for all $i, j$. Obviously, every separable matrix thus defined is skew-symmetric because $A = (a_{ij})$ is. Moreover, every separable matrix is also a solution of the Jacobi identities (1). Therefore, every separable matrix is a structure matrix [10].

Determining the Casimir invariants is not difficult [11]: in fact, if

$$C_{\mathbf{k}} = \sum_{j=1}^{n} k_j \int \frac{\mathrm{d}x_j}{\varphi_j(x_j)} \tag{4}$$

where $\mathbf{k} = (k_1, k_2, \ldots, k_n)^T \in \mathrm{Ker}(A)$, then $C_{\mathbf{k}}$ is a Casimir function of $\mathcal{J}$. In addition, we have that $\dim\{\mathrm{Ker}(A)\} = n - \mathrm{Rank}(A) \equiv m$, and thus there are $m$ linearly independent vectors that span $\mathrm{Ker}(A)$. In fact, $m$ is also the number of independent Casimir invariants, and actually a basis of $\mathrm{Ker}(A)$ provides a complete set of independent Casimir invariants of $\mathcal{J}$ of the form (4).

We now examine the reduction to the Darboux canonical form. For this we first introduce the following diffeomorphic transformation globally defined in $\Omega$: $y_i = \int (\varphi_i(x_i))^{-1} \mathrm{d}x_i$, $i = 1, \ldots, n$. The transformed Poisson structure is given by $\tilde{J}_{ij}(y) = a_{ij}$, for all $i, j$. In other words, we have transformed the matrix in such a way that now $\tilde{\mathcal{J}} = A$ is a structure matrix of constant entries. In addition, we now perform a second transformation, which is also globally defined: $z_i = \sum_{j=1}^{n} P_{ij} y_j$, $i = 1, \ldots, n$, where $P$ is a constant, $n \times n$ invertible matrix. Then the structure matrix $\tilde{J}$ now becomes $\hat{\mathcal{J}} = P \cdot \tilde{\mathcal{J}} \cdot P^T = P \cdot A \cdot P^T$. It is well known that matrix $P$ can be chosen to have $\hat{\mathcal{J}} = \mathrm{diag}(D_1, D_2, \ldots, D_{r/2}, 0, \overset{(n-r)}{\ldots}, 0)$ where $r = \mathrm{Rank}(A)$ is even, and

$$D_1 = D_2 = \ldots = D_{r/2} = \begin{pmatrix} 0 & 1 \\ -1 & 0 \end{pmatrix}$$

Thus the structure matrix has been reduced to the Darboux form. It is worth emphasizing that the reduction has been completed explicitly and globally in the domain of interest $\Omega$. This is remarkable, since the number of Poisson structures for which this can be done is very limited.

These results have allowed a unified treatment of many different Poisson structures, including Lotka-Volterra and Generalized Lotka-Volterra systems (of importance in contexts such as plasma physics or population dynamics), mechanical systems such as the Toda and relativistic Toda models, and other Poisson formulations such as the ones used in the Kermack-McKendric model, in circle maps, or in bimatrix games (see [10] for the full details).

## 4. Concluding remarks

It can be appreciated how the analysis and classification of skew-symmetric solutions of the Jacobi equations is not only necessary for the Hamiltonization of dynamical systems, but also useful for classification purposes of Poisson structures. Moreover, this perspective leads to a common description of diferent Poisson systems previously unrelated that now can be regarded and analyzed in a more general framework. Given that Jacobi equations are a set of nonlinear coupled p.d.e.s, we are still far from a complete knowledge of their solutions. Such investigation will be the subject of future research.

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# Semi-Riemannian metrics as deformations of a constant curvature metric

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**Abstract.** A semi-Riemannian metric in a $n$-manifold has $n(n-1)/2$ degrees of freedom, as many as the number of components of a differential 2-form. We prove that any analytic semi-Riemannian metric can be obtained as a deformation of a constant curvature metric, this deformation being parametrized by an analytic 2-form.

*Keywords:* Riemannian geometry, Cauchy-Kowalevski theorem

*2000 Mathematics Subject Classification:* 53.B20,53.C21,53.C50

## 1. Introduction

It is known, since an old result by Riemann [1], that an $n$-dimensional metric is locally equivalent to the giving of $f = n(n-1)/2$ functions. Since this feature is related to a particular choice of either a local chart or a local base, it seems to be a non-covariant property.

For $n = 2$ it is known that (Gauss theorem)[2] *any two-dimensional metric* $g$ *is locally conformally flat,* $g = \phi\eta$, *where* $\eta$ *is the flat metric and* $\phi$ *a* conformal *deformation scalar factor*. This result is *covariant*, because the sole degree of freedom is represented by the *scalar* $\phi$, which only depends on the metric $g$.

The question thus arises of, whether or not, for $n > 2$ they exist similar intrinsic and covariant local relations between an arbitrary metric $g$, on the one hand, and the corresponding flat metric $\eta$ together with a set of $f$ covariant quantities on the other.

There is a number of results concerning the diagonalization of any three-dimensional metric (see ref.[3], to quote some few) which are not covariant because, in addition to the metric, an orthogonal triad or a specific coordinate system is involved. To our knowledge, the first published result of this kind for $n = 3$ is [4], where the following theorem was proved:

**Theorem 1.** *Any three-dimensional analytic Riemannian metric $g$ may be locally obtained from a constant curvature metric $\eta$ by a deformation of the form*

$$g = a\eta + \epsilon\, \boldsymbol{s} \otimes \boldsymbol{s} \ , \tag{1}$$

*where $a$ and $\boldsymbol{s}$ are respectively a scalar function and a differential 1-form. The sign $\epsilon = \pm 1$, the curvature of $\eta$ and an analytic constraint $\Psi(a, \boldsymbol{s})$ between the scalar $a$ and the Riemannian norm $|\boldsymbol{s}|$ may be arbitrarily prescribed.*

After realizing that $n(n-1)/2$ is precisely the number of independent components of a $n$-dimensional 2-form, in the context of the general theory of relativity B. Coll [5] has conjectured that any $n$-dimensional metric $g$ can be locally obtained as a *deformation* of a constant curvature metric $\eta$, parametrized by a 2-form $F$, according to:

$$g = \lambda(F)\, \eta + \mu(F) F^2 \tag{2}$$

where $\lambda$ and $\mu$ are scalar functions of $F$ and $F^2 := F\eta^{-1}F$.

In particular, it can be easily proved that the Kerr-Schild class of metrics in general relativity [6] satisfy this relation.

We here prove the following variant of Coll's conjecture:

**Theorem 2** (**Deformation theorem**). *Let $(\mathcal{V}_n, g)$ be an analytic semi-Riemannian manifold. Locally there always exist an analytic 2-form $F$ and an analytic scalar function $a$ such that:*
(a) *they satisfy a previously chosen arbitrary analytic scalar constraint:*
$\Psi(a, F) = 0$ *and*
(b) *the semi-Riemannian deformed metric:*

$$\overline{g}_{\alpha\beta} = a g_{\alpha\beta} - \epsilon F^2_{\alpha\beta} \ , \tag{3}$$

*with $F^2_{\alpha\beta} := g^{\mu\nu} F_{\alpha\mu} F_{\nu\beta}$ and $|\epsilon| = 1$, has constant curvature.*

Since the constraint $\Psi(a, F)$ is a scalar, it will only depend on the invariants of $F^{\alpha}_{\ \beta} = g^{\alpha\nu} F_{\nu\beta}$.

Our proof relies on the Cauchy-Kowalevski theorem for partial differential systems. Therefore, analyticity of $g$, $F$, $a$ and $\Psi$ is implied hereon (even though, we will not always mention it explicitly).

What we shall actually prove is the following extension of theorem 2

**Theorem 3.** *Let $(\mathcal{V}_d, g)$ be a semi-Riemannian manifold and $\lambda$ a 2-covariant symmetric tensor. Locally there always exist a 2-form $F$ and a scalar function $a$ such that: (a) they meet a previously chosen constraint: $\Psi(a, F, x) = 0$ and (b) the semi-Riemannian deformed metric:*

$$\overline{g}_{\alpha\beta} = a g_{\alpha\beta} + \lambda_{\alpha\beta} - \epsilon F^2_{\alpha\beta} \,, \qquad \text{with} \qquad F^2_{\alpha\beta} := g^{\mu\nu} F_{\alpha\mu} F_{\nu\beta} \tag{4}$$

*has constant curvature ($|\epsilon| = 1$).*

We shall prove theorem 3 instead of theorem 2 because it can be proved by iteration on the number of dimensions. The proof that it is true for $n = 2$ is similar to the proof of the above mentioned Gauss theorem. We shall now see that: *if theorem 3 holds for $d = n - 1$, then it also holds for $d = n$.*

## 2. Sketch of the proof

Let $(\mathcal{V}_n, g)$ be a semi-Riemannian manifold and let $\{e_\alpha\}_{\alpha=1\ldots n}$ be a base of vectors. The metrics $g_{\alpha\beta}$ and $\overline{g}_{\alpha\beta}$ coexist through the proof. Indices are always lowered with $g_{\alpha\beta}$ and raised with its inverse $g^{\alpha\beta}$. The inverse metric for $\overline{g}_{\alpha\beta}$ is denoted $\overline{h}^{\alpha\beta}$.

The Riemannian connections $\nabla$ and $\overline{\nabla}$ for $g$ and $\overline{g}$ define the difference tensor

$$B_{\rho\mu\nu} := \overline{g}_{\alpha\rho} \left(\overline{\gamma}^\alpha_{\mu\nu} - \gamma^\alpha_{\mu\nu}\right) = \frac{1}{2} \left(\nabla_\mu \overline{g}_{\rho\nu} + \nabla_\nu \overline{g}_{\rho\mu} - \nabla_\rho \overline{g}_{\mu\nu}\right) \,, \tag{5}$$

and the respective Riemann tensors are related to each other by

$$\overline{R}_{\mu\nu\alpha\beta} := \overline{g}_{\mu\rho} \overline{R}^\rho{}_{\nu\alpha\beta} = \overline{g}_{\mu\rho} R^\rho{}_{\nu\alpha\beta} + 2\, \nabla_{[\alpha} \overline{B}_{\mu\beta]\nu} + 2\, B^\rho{}_{\nu[\alpha} \overline{B}_{\rho|\beta]\mu} \tag{6}$$

The condition that $\overline{g}_{\alpha\beta}$ has constant curvature is expressed as

$$\mathcal{E}_{\mu\nu\alpha\beta} := \overline{R}_{\mu\nu\alpha\beta} + k \left(\overline{g}_{\mu\alpha} \overline{g}_{\nu\beta} - \overline{g}_{\mu\beta} \overline{g}_{\nu\alpha}\right) = 0 \,. \tag{7}$$

Part of these equations yield a partial differential system (PDS) for the unknowns $F_{\alpha\beta}$ and $a$.

To pose the Cauchy problem, consider a hypersurface $\mathcal{S}$ with a non-null unit normal vector $n^\alpha$ and denote $\Pi^\alpha_\mu := \delta^\alpha_\mu - \sigma n^\alpha n_\mu$ the $g$-orthogonal projector and $\sigma = n_\alpha n^\alpha$. Not all equations (7) contain second order normal derivatives of the unknowns. In particular, the only equations contributing to the PDS's principal part [7] are the $n(n-1)/2$ independent combinations:

$$\mathcal{E}_{\mu\alpha} := \mathcal{E}_{\mu\nu\alpha\beta}\, n^\nu n^\beta = 0 \tag{8}$$

As it will be seen below, a well posed Cauchy problem with data on $\mathcal{S}$ can be set up for these equations. The remaining equations (7) are equivalent to

$$\mathcal{L}_{\mu\nu\alpha\beta} := \mathcal{E}_{\lambda\rho\sigma\gamma}\,\Pi^\lambda_\mu\,\Pi^\rho_\nu\,\Pi^\sigma_\alpha\,\Pi^\gamma_\beta = 0 \qquad \mathcal{L}_{\mu\nu\alpha} := \mathcal{E}_{\lambda\rho\sigma\gamma}\,\Pi^\lambda_\mu\,\Pi^\rho_\nu\,\Pi^\sigma_\alpha\, n^\gamma = 0 \tag{9}$$

and can be taken as subsidiary conditions to be fulfilled by the Cauchy data on $\mathcal{S}$. Indeed, from the second Bianchi identity for $\overline{R}_{\mu\nu\alpha\beta}$ it easily follows that:

**Proposition.** *Any analytical solution of (8) satisfying the constraints (9) on $\mathcal{S}$, also satisfies them in a neighborhood of $\mathcal{S}$.*

## 2.1. Non-characteristic hypersurfaces

Let us now see that Cauchy data can be given on $\mathcal{S}$ so that it is non-characteristic for the PDS (8). We first write $F_{\alpha\beta} := n_\alpha E_\beta - E_\alpha n_\beta + \tilde{F}_{\alpha\beta}$, i.e., splitting $F_{\alpha\beta}$ into its parallel and transversal parts relatively to $n_\alpha$ and, on substituting it into (8), we obtain

$$\nabla^2_{\mathbf{n}} a\,\hat{g}_{\alpha\beta} + \sigma\epsilon\,\left(E_\beta\nabla^2_{\mathbf{n}}E_\alpha + E_\alpha\nabla^2_{\mathbf{n}}E_\beta\right) - \epsilon\,\left(\tilde{F}_{\rho\beta}\,\nabla^2_{\mathbf{n}}\tilde{F}_\alpha^{\;\rho} + \tilde{F}_\alpha^{\;\rho}\,\nabla^2_{\mathbf{n}}\tilde{F}_{\rho\beta}\right) = \mathcal{H}_{\alpha\beta} \tag{10}$$

where $\mathcal{H}_{\alpha\beta}$ includes all non-principal terms. This equation must be supplemented with the second normal derivatives of the constraint

$$\nabla^2_{\mathbf{n}}\Psi(a, F_{\rho\beta}, x) := \Psi_1\nabla^2_{\mathbf{n}}a + \Psi_2^\alpha\nabla^2_{\mathbf{n}}E_\alpha + \Psi_3^{\alpha\beta}\nabla^2_{\mathbf{n}}\tilde{F}_{\alpha\beta} - \Psi_0 = 0\,. \tag{11}$$

Now $\mathcal{S}$ is non-characteristic for the prescribed Cauchy data if, and only if, the linear system (10-11) can be solved for the second order normal derivatives $\nabla^2_{\mathbf{n}}E_\alpha$, $\nabla^2_{\mathbf{n}}\tilde{F}_{\alpha\beta}$ and $\nabla^2_{\mathbf{n}}a$. A sufficient condition on the Cauchy data is

$$\Delta(d) \neq 0 \qquad \text{and} \qquad \Gamma(d) \neq 0 \tag{12}$$

where $\Gamma(d)$ is the determinant of $G_{ij} := (-1)^{i-1}E_\alpha E_\beta(\tilde{F}^{i+j-2})^{\alpha\beta}$, $i, j = 1\ldots d$ and $\Delta(d)$ is a function of $E_\beta$ and $\tilde{F}_{\alpha\beta}$ (see [9] for details).

## 2.2. Geometrical meaning of the subsidiary conditions

Let us now assume that $\mathcal{S}$ and the Cauchy data have been chosen so that $\overline{h}^{\alpha\beta}n_\alpha n_\beta \neq 0$. Consider the $\overline{g}$-orthogonal unit vector

$$\overline{n}^\alpha := \overline{h}^{\alpha\beta}\zeta_\beta \qquad \text{with} \qquad \zeta_\beta := |\overline{h}^{\alpha\rho}n_\alpha n_\rho|^{-1/2}n_\beta\,,$$

and denote $\overline{\Pi}^\alpha_\beta := \delta^\alpha_\beta - \bar\sigma \overline{n}^\alpha \zeta_\beta$ the $\overline{g}$-orthogonal projector, with $\overline{g}_{\alpha\beta}\overline{n}^\alpha\overline{n}^\beta = \bar\sigma = \pm 1$.

It follows that $\Pi^\alpha_\beta \overline{\Pi}^\beta_\mu = \overline{\Pi}^\alpha_\mu$ and $\overline{\Pi}^\alpha_\beta \Pi^\beta_\mu = \Pi^\alpha_\mu$. As a consequence it is obtained that conditions (9) are equivalent to:

$$\overline{\mathcal{L}}_{\mu\nu\alpha\beta} := \mathcal{E}_{\lambda\rho\sigma\gamma}\,\overline{\Pi}^\lambda_\mu\,\overline{\Pi}^\rho_\nu\,\overline{\Pi}^\sigma_\alpha\,\overline{\Pi}^\gamma_\beta = 0 \quad \text{and} \quad \overline{\mathcal{L}}_{\mu\nu\alpha} := \mathcal{E}_{\lambda\rho\sigma\gamma}\,\overline{\Pi}^\lambda_\mu\,\overline{\Pi}^\rho_\nu\,\overline{\Pi}^\sigma_\alpha\,\overline{n}^\gamma = 0 \tag{13}$$

The theory of Riemannian submanifolds [8] applied to $(\mathcal{V}_4, \overline{g})$ and the hypersurface $\mathcal{S}$ provides a clear geometrical meaning to these subsidiary conditions. Consider the isometrical embedding of $\mathcal{S}$ into $(\mathcal{V}_4, \overline{g})$ and respectively denote by $\tilde g$, $\tilde\nabla$ and $\overline{\phi}$, the first fundamental form, the induced connection and the second fundamental form. Including the Codazzi-Mainardi equation, the second subsidiary condition (13) amounts to

$$\tilde\nabla_v \overline{\phi}(w,z) - \tilde\nabla_w \overline{\phi}(v,z) = 0 \qquad \forall v,\, w,\, z \text{ tangential to } \mathcal{S} \tag{14}$$

which has $\overline{\phi} = 0$ as a particular solution. For this particular choice, including Gauss equation, the first of conditions (13) yields

$$\tilde R(v,w,t,z) + k\,[\overline{g}(v,t)\overline{g}(w,z) - \overline{g}(v,z)\overline{g}(w,t)] = 0\,. \tag{15}$$

$\forall v,\, w,\, z,\, t$ tangential to $\mathcal{S}$.

Take now a base of vectors adapted to $\mathcal{S}$, i.e. $\{t^\alpha_1 \dots t^\alpha_d, t^\alpha_n = n^\alpha\}$, with $t^\alpha_j n_\alpha = t^\alpha_j \zeta_\alpha = 0$, $j = 1 \dots d$. It is obvious that $\Pi^\alpha_\mu t^\mu_j = t^\alpha_j$ and, since by definition $E_\alpha n^\alpha = \tilde F_{\alpha\beta} n^\alpha = 0$, the induced metric $\tilde g$ in this base is:

$$\tilde g_{ij} = a\,\hat g_{ij} + \tilde\lambda_{ij} - \epsilon \tilde F^2_{ij}\,, \qquad i,j = 1 \dots d \tag{16}$$

with $\tilde\lambda_{ij} := \lambda_{ij} + \sigma\epsilon E_i E_j$. Also in this base, equation (15) reads:

$$\tilde R_{ijkl} + k\,[\tilde g_{ik}\tilde g_{jl} - \tilde g_{il}\tilde g_{jk}] = 0\,, \qquad i,j,k,l = 1 \dots d$$

i. e., the Cauchy data on $\mathcal{S}$ must be chosen so that the induced metric $\tilde g_{ij}$ has constant curvature.

Therefore, if we choose one half of the Cauchy data $\nabla_{\mathbf{n}} a$, $\nabla_{\mathbf{n}} E_\alpha$, $\nabla_{\mathbf{n}} \tilde F_{\alpha\beta}$ so that $\overline{\phi} = 0$ (which can be achieved provided that the inequalities (12) are satisfied [9]) and choose the remaining data $a$, $E_\alpha$, $\tilde F_{\alpha\beta}$ so that the induced metric $\tilde g$ has constant curvature (which is possible by the hypothesis), then these Cauchy data on $\mathcal{S}$ satisty the subsidiary conditions (9). □

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# Cohomology of Lie groups associated to compatible linear cocycles and Lie group structures on $T^*G$

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**Abstract.** In this paper we introduce the cohomology associated to a pair of compatible linear cocycles for the left and right actions on a Lie group $G$. We show that every Lie group structure on $T^*G$, compatible with the Lie group $G$, can be constructed in terms of this cohomology. In general, these new Lie group structures on $T^*G$ are not isomorphic to the standard one and they are (non-central) Lie group extensions of $G$ by $\mathfrak{g}^*$. These new products on $T^*G$ can be used to construct invariant symplectic forms or invariant complex structures on $T^*G$.

*Keywords:* invariant symplectic form, cotangent extensions, linear cocycles, cohomology of Lie groups

*2000 Mathematics Subject Classification:* 53C15, 53D05, 57R25, 57T10, 22E15

## 1. Introduction

The cotangent bundle $T^*G$ of a Lie group $G$ has a Lie group structure with product $\omega_g \cdot \omega'_{g'} = \mathbf{L}_g^{ctg}[\omega'_{g'}] + \mathbf{R}_{g'}^{ctg}[w_g]$, where $\mathbf{L}_g^{ctg}$ and $\mathbf{R}_{g'}^{ctg}$ are the natural cotangent lifts of the left and right actions of $G$. Therefore, invariant geometric objects on $T^*G$, with respect to this Lie group structure, are determined by algebraic conditions on the Lie algebra $\mathfrak{g} \oplus \mathfrak{g}^*$ of this group.

It is well known that the canonical symplectic form on $T^*G$, in general, is not invariant under this product. On the other hand,, if one considers the left invariant 2-form ${\Omega_2}^0$ on $T^*G$ determined by the linear 2-form ${\omega_2}^0$ on $T_{(\epsilon,0)}(T^*G) \simeq \mathfrak{g} \oplus \mathfrak{g}^*$ defined by ${\omega_2}^0((A,\alpha),(A',\alpha')) = \alpha(A') - \alpha'(A)$, it is easy to check that ${\Omega_2}^0$ is closed if and only if $G$ is abelian.

However it is possible [1] to deform the Lie group structure of $T^*G$ in such a way that the left invariant 2-form ${\Omega_2}^{inv}$ defined by ${\omega_2}^0$ becomes closed, giving thus an invariant symplectic structure on $T^*G$. N. Boyom proved in [2] that if a simply connected Lie group $G$ admits an invariant flat torsion free linear connection (i.e. $G$ has a left invariant affine structure), it is possible to endow $T^*G$ with a Lie group structure such that $d{\Omega_2}^{inv} = 0$. Similar ideas have been used to construct other invariant geometric structures on the Lie group $T^*G$ ([3], [4], [5]).

In order to have a better geometric understanding of Boyom's result, we analyze all the inequivalent Lie group extensions of $G$ by $\mathfrak{g}^*$ compatible with the differentiable structure of $T^*G$; that is, we classify all non-equivalent Lie group structures on the manifold $T^*G$ such that

$$0 \longrightarrow \mathfrak{g}^* \xrightarrow{\iota} T^*G \xrightarrow{\pi} G \longrightarrow 1$$

is an exact sequence of Lie groups. One can proof ([6], [7], [8]) that in this case, there exists a trivialization of $T^*G$ as a trivial $\mathfrak{g}^*$-principal bundle over $G$ such that the product can be written

$$(g, \alpha) \cdot (g', \alpha') = (gg', \alpha + \rho_{g'^{-1}}(\alpha') + \xi(g, g'))$$

where $\xi : G \times G \to \mathfrak{g}^*$ is a smooth map fulfilling some normalized 2-cocycle condition and $\rho : G \to Aut\,\mathfrak{g}^*$ is a linear representation.

In this paper we present a new geometric and constructive approach to this problem in terms of a pair of compatible linear cocycles for the left and right actions of $G$. These techniques are closely related to the geometric way of lifting an action of $G$ on a manifold $X$ to an action of $G$ on a trivial vector bundle over $X$ by means of automorphic factors. In order to solve the equivalence problem for these group structures on $T^*G$, we introduce a cohomology associated to a pair of compatible $Aut\,\mathfrak{g}^*$-cocycles for the left and right actions on $G$. In our understanding, this cohomology is new in the literature and, for some particular compatible linear constant cocycles, it coincides with the standard cohomology of the Lie group $G$ with values on the $G$-module $\mathfrak{g}^*$.

We would like to point out that all these constructions can be generalized, in a straightforward way, in order to define Lie group structures on any trivial vector bundle over $G$; for instance, in all the tensorial bundles $T^r_s(G)$.

## 2. Compatible linear cocycles for a Lie group

Given a Lie group action $\phi : G \times X \to X$, any lift $\mathbf{\Phi}$ of this action to an action on a trivial vector bundle $\pi : E \simeq X \times \mathcal{E} \to X$ can be described as

$$\begin{array}{rcl} \mathbf{\Phi} : G \times (X \times \mathcal{E}) & \longrightarrow & (X \times \mathcal{E}) \\ (h, (x, e)) & \longrightarrow & h \cdot (x, e) = (\phi_h(x), \mathcal{J}_{(h,x)}(e)) \end{array}$$

where $\mathcal{J} : G \times X \longrightarrow Aut\, \mathcal{E}$ is a linear $\phi$-cocycle; that is, a differentiable map fulfilling the cocycle condition $\mathcal{J}_{(h_1 h_2, x)} = \mathcal{J}_{(h_1, \phi_{h_2}(x))} \circ \mathcal{J}_{(h_2,x)}$.

**Definition 1.** A set of compatible $Aut\, \mathfrak{g}^*$-valued cocycles for a Lie group $G$ is a pair of differentiable maps $\mathcal{L} : G \times G \longrightarrow Aut\, \mathfrak{g}^*$ and $\mathcal{R} : G \times G \longrightarrow Aut\, \mathfrak{g}^*$ fulfilling : (a) cocycle conditions

$$\begin{array}{rcl} \mathcal{L}_{(h_1 h_2, g)} & = & \mathcal{L}_{(h_1, h_2 g)} \circ \mathcal{L}_{(h_2, g)} \\ \mathcal{R}_{(h_1 h_2, g)} & = & \mathcal{R}_{(h_1, g h_2^{-1})} \circ \mathcal{R}_{(h_2, g)} \end{array}$$

for the left and right actions, respectively, and (b) the compatibility condition

$$\mathcal{L}_{(g_1, g_2 g_3)} \circ \mathcal{R}_{(g_3{}^{-1}, g_2)} = \mathcal{R}_{(g_3^{-1}, g_1 g_2)} \circ \mathcal{L}_{(g_1, g_2)}$$

*Remark* 1. Given a trivialization of the bundle $T^*G \simeq G \times \mathfrak{g}^*$, we see that $\mathcal{L}$ and $\mathcal{R}$ allow us to define non standard cotangent lifts of the left and right actions of $G$. The compatibility condition implies that the lifted actions commute.

**Proposition 1.** *If $\{\mathcal{L}, \mathcal{R}\}$ is a set of compatible $Aut\, \mathfrak{g}^*$-valued cocycles for $G$, the map $\rho^{(\mathcal{L},\mathcal{R})} : G \longrightarrow Aut\, \mathfrak{g}^*$ defined by*

$$\rho_g^{(\mathcal{L},\mathcal{R})} \equiv \mathcal{R}_{(g,g)} \circ \mathcal{L}_{(g,\epsilon)} = \mathcal{L}_{(g,g^{-1})} \circ \mathcal{R}_{(g,\epsilon)}$$

*is a linear representation of $G$.*

**Example.** If $T^*G$ is trivialized by left invariant forms, the natural cotangent lift of the left action $L$ (resp. the right action $R$) of the Lie group $G$ are defined by the $L$-cocycle $\mathcal{L}_{(h,g)} = Id_{\mathfrak{g}^*}$ (resp. by the $R$-cocycle $\mathcal{R}_{(h,g)} = CoAd_h$). It is easy to check that these linear cocycles are compatible and, in this case, $\rho^{(\mathcal{L},\mathcal{R})}$ is the coadjoint representation.

**Definition 2.** We say that two compatible $Aut\, \mathcal{G}^*$-valued cocycles $\{\mathcal{L}, \mathcal{R}\}$ and $\{\mathcal{L}', \mathcal{R}'\}$ for a Lie group $G$ are gauge equivalent if there exists a gauge transformation on $T^*G \simeq G \times \mathfrak{g}^*$, defined by a differentiable map $\varrho : G \to Aut\, \mathfrak{g}^*$, such that

$$\begin{array}{rcl} \mathcal{L}'_{(h,g)} & = & \varrho_{hg} \circ \mathcal{L}_{(h,g)} \circ \varrho_g^{-1} \\ \mathcal{R}'_{(h,g)} & = & \varrho_{gh^{-1}} \circ \mathcal{R}_{(h,g)} \circ \varrho_g^{-1} \end{array}$$

**Theorem 2.** (Rigidity of compatible linear cocycles) *Every pair $\{\mathcal{L},\mathcal{R}\}$ of compatible $Aut\,\mathcal{G}^*$-valued cocycles for a Lie group $G$ is gauge equivalent to the compatible linear cocycles $\{\mathcal{L}^{(Id)},\mathcal{R}^{(\rho^{(\mathcal{L},\mathcal{R})})}\}$ defined by $\mathcal{L}^{(Id)}_{(h,g)} = Id_{\mathfrak{g}^*}$ and $\mathcal{R}^{(\rho^{(\mathcal{L},\mathcal{R})})}_{(h,g)} = \rho^{(\mathcal{L},\mathcal{R})}_h$.*

**Proof.** The gauge transformation relating both sets of compatible linear cocycles is defined by $\varrho_g = \mathcal{L}_{(g^{-1},g)}$.

**Corollary 3.** *Any set of compatible $Aut\,\mathcal{G}^*$-valued cocycles $\{\mathcal{L},\mathcal{R}\}$ for $G$ can be constructed as*

$$\begin{aligned}\mathcal{L}_{(h,g)} &= \phi_{hg}\circ\phi_g^{-1}\\ \mathcal{R}_{(h,g)} &= \phi_{gh^{-1}}\circ\rho_h\circ\phi_g^{-1}\end{aligned}$$

*where $\rho : G \to Aut\,\mathfrak{g}^*$ is a linear representation and $\phi : G \to Aut\,\mathfrak{g}^*$ is a differentiable map satisfying $\phi_\epsilon = Id$, where $\epsilon$ is the unit element of $G$. Moreover, the gauge equivalence class of any set of linear cocycles $\{\mathcal{L},\mathcal{R}\} \equiv \{\rho,\phi\}$ is determined by the linear equivalence class of the representation $\rho$.*

# 3. Cohomology associated to a set of compatible $Aut\,\mathfrak{g}^*$-valued cocycles of a Lie group

**Definition 3.** A differentiable p-cochain of $G$ with values in $\mathfrak{g}^*$ is a differentiable map $\xi : G\times \overset{(p}{\ldots} \times G \longrightarrow \mathfrak{g}^*$. We say that a differentiable p-cochain $\xi$ is normalized if $\xi(g_1,\ldots,g_{i-1},\epsilon,g_{i+1}\ldots g_p) = 0$, $\forall g_1,\ldots,g_{i-1},g_{i+1}\ldots g_p \in G$, where $\epsilon$ is the unit element of $G$.

**Definition 4.** For every $p$, let us denote by $C^p_{norm}(G,\mathfrak{g}^*)$ the vector space of all normalized differentiable p-cochains of $G$ with values in $\mathfrak{g}^*$. For $p = 0$, we define $C^0_{norm}(G,\mathfrak{g}^*) = \mathfrak{g}^*$.

**Definition 5.** Given a set of compatible linear cocycles $\{\mathcal{L},\mathcal{R}\}$ for the Lie group $G$, we define the following complex of vector spaces $(C^\bullet_{norm}(G,\mathfrak{g}^*),\delta^{(\mathcal{L},\mathcal{R})}_\bullet)$ where the coboundary operators are :

- $\delta^{(\mathcal{L},\mathcal{R})}_0 : C^0_{norm}(G,\mathfrak{g}^*) = \mathfrak{g}^* \longrightarrow C^1_{norm}(G,\mathfrak{g}^*)$ with

$$[\delta^{(\mathcal{L},\mathcal{R})}_0(\alpha)](g) \equiv \mathcal{L}_{(g,\epsilon)}(\alpha) - \mathcal{R}_{(g^{-1},\epsilon)}(\alpha)$$

- $\delta_p^{(\mathcal{L},\mathcal{R})} : C^p_{norm}(G,\mathfrak{g}^*) \longrightarrow C^{p+1}_{norm}(G,\mathfrak{g}^*)$, if $p > 0$, with

$$\begin{aligned}[][\delta_p^{(\mathcal{L},\mathcal{R})}\xi](g_1,g_2,\dots,g_{p+1}) &= \mathcal{L}_{(g_1,g_2\cdots g_{p+1})}\,\xi(g_2,\dots,g_{p+1}) + \\ &+ \sum_{i=1}^{p}(-1)^i\,\xi(g_1,\dots,g_ig_{i+1},\dots,g_{p+1}) + \\ &+ (-1)^{p+1}\,\mathcal{R}_{(g_{p+1}^{-1},g_1\cdots g_p)}\,\xi(g_1,\dots,g_p)\end{aligned}$$

These maps $\delta_p^{(\mathcal{L},\mathcal{R})}$ are linear and $\delta_p^{(\mathcal{L},\mathcal{R})}\xi$ is a normalized differentiable $(p+1)$-cocycle , $\forall \xi \in C^p_{norm}(G,\mathfrak{g}^*)$.

**Proposition 4.** *One has* $\delta_{p+1}^{(\mathcal{L},\mathcal{R})} \circ \delta_p^{(\mathcal{L},\mathcal{R})} = 0$ .

**Definition 6.** Let us denote by $Z^p_{(\mathcal{L},\mathcal{R})}(G,\mathfrak{g}^*) = ker\,\delta_p^{(\mathcal{L},\mathcal{R})}$ the subspace of normalized differentiable p-cocycles and by $B^p_{(\mathcal{L},\mathcal{R})}(G,\mathfrak{g}^*) = Im\,\delta_{p-1}^{(\mathcal{L},\mathcal{R})}$ the subspace of normalized differentiable p-coboundaries. The p-cohomology space of $G$ with values in $\mathfrak{g}^*$ associated to the compatible linear cocycles $\{\mathcal{L},\mathcal{R}\}$ is

$$H^p_{(\mathcal{L},\mathcal{R})}(G,\mathfrak{g}^*) = \frac{Z^p_{(\mathcal{L},\mathcal{R})}(G,\mathfrak{g}^*)}{B^p_{(\mathcal{L},\mathcal{R})}(G,\mathfrak{g}^*)}$$

**Lemma 5.** *If we denote by* $[\mathfrak{g}^*]^G$ *the subspace of invariant elements under the linear representation* $\rho^{(\mathcal{L},\mathcal{R})} : G \to Aut\,\mathfrak{g}^*$ *associated to the compatible cocycles* $\{\mathcal{L},\mathcal{R}\}$*, then* $Z^0_{(\mathcal{L},\mathcal{R})}(G,\mathfrak{g}^*) = ker\,\delta_0^{(\mathcal{L},\mathcal{R})} = [\mathfrak{g}^*]^G$.

Some particular cases :

- $\theta \in Z^1_{(\mathcal{L},\mathcal{R})}(G,\mathfrak{g}^*) \Longleftrightarrow \theta(g_1g_2) = \mathcal{L}_{(g_1,g_2)}\,\theta(g_2) + \mathcal{R}_{(g_2^{-1},g_1)}\,\theta(g_1)$.

- $\xi \in Z^2_{(\mathcal{L},\mathcal{R})}(G,\mathfrak{g}^*)$ if and only if

$$\mathcal{L}_{(g_1,g_2g_3)}\,\xi(g_2,g_3) + \xi(g_1,g_2g_3) = \xi(g_1g_2,g_3) + \mathcal{R}_{(g_3^{-1},g_1g_2)}\,\xi(g_1,g_2)$$

**Proposition 6.** *If* $\{\mathcal{L},\mathcal{R}\}$ *and* $\{\mathcal{L}',\mathcal{R}'\}$ *are gauge equivalent under a gauge transformation* $\varrho : G \to Aut\,\mathfrak{g}^*$*, then there exists* $\Phi^p_\varrho \in Aut\,C^p_{norm}(G,\mathfrak{g}^*)$ *fulfilling* $\Phi^{p+1}_\varrho \circ \delta_p^{(\mathcal{L}',\mathcal{R}')} = \delta_p^{(\mathcal{L},\mathcal{R})} \circ \Phi^p_\varrho$ *and inducing linear isomorphisms* $\Phi^p_\varrho : H^p_{(\mathcal{L}',\mathcal{R}')}(G,\mathfrak{g}^*) \to H^p_{(\mathcal{L},\mathcal{R})}(G,\mathfrak{g}^*)$.

As a consequence of the rigidity theorem for compatible linear cocycles, we have

**Corollary 7.** *Given a pair $\{\mathcal{L},\mathcal{R}\}$ of compatible $Aut\,\mathcal{G}^*$-valued cocycles for a Lie group $G$, the cohomology spaces $H^\bullet_{(\mathcal{L},\mathcal{R})}(G,\mathfrak{g}^*)$ are isomorphic to the usual cohomology spaces $H^\bullet(G,\mathfrak{g}^*;\rho^{(\mathcal{L},\mathcal{R})})$ of the Lie group $G$ with values on $\mathfrak{g}^*$ via the representation $\rho^{(\mathcal{L},\mathcal{R})}: G \to Aut\,\mathfrak{g}^*$.*

**Corollary 8.** *Given two sets $\{\mathcal{L},\mathcal{R}\}$, $\{\mathcal{L}',\mathcal{R}'\}$ of compatible $Aut\,\mathfrak{g}^*$-valued cocycles for a Lie group $G$, every equivariant linear isomorphism $\phi:\mathfrak{g}^*\to\mathfrak{g}^*$ with respect to the actions $\rho^{(\mathcal{L},\mathcal{R})}$ and $\rho^{(\mathcal{L}',\mathcal{R}')}$, induces linear isomorphisms $H^p_{(\mathcal{L},\mathcal{R})}(G,\mathfrak{g}^*)\to H^p_{(\mathcal{L}',\mathcal{R}')}(G,\mathfrak{g}^*)$.*

# 4. Lie group structures on $T^*G$ associated to a pair of compatible linear cocycles

**Theorem 9.** *Let $\{\mathcal{L},\mathcal{R}\}$ be a set of compatible $Aut\,\mathfrak{g}^*$-valued cocycles for a Lie group $G$ and let $\xi\in Z^2_{(\mathcal{L},\mathcal{R})}(G,\mathfrak{g}^*)$ be a normalized 2-cocycle. Fixing a trivialization of the cotangent bundle $T^*G\simeq G\times\mathfrak{g}^*$, we define the product*

$$(g_1,\alpha_1)\cdot(g_2,\alpha_2)=(g_1g_2,\mathcal{L}_{(g_1,g_2)}(\alpha_2)+\mathcal{R}_{(g_2^{-1},g_1)}(\alpha_1)+\xi(g_1,g_2))$$

*Then, $(T^*G,\cdot)$ is a Lie group that we will denote by $(G\times\mathfrak{g}^*)_{(\mathcal{L},\mathcal{R},\xi)}$.*

- Associativity follows since $(\mathcal{L},\mathcal{R})$ are compatible and $\delta_2^{(\mathcal{L},\tilde{\mathcal{R}})}\xi=0$.
- The unit element is $(\epsilon,0)$.
- The inverse is $(g,\alpha)^{-1}=(g^{-1},-\mathcal{L}_{(g^{-1},\epsilon)}[\mathcal{R}_{(g,g)}(\alpha)+\xi(g,g^{-1})])$.
- The differentiability of the multiplication law and the inverse are trivial since all the cocycles are assumed to be linear maps of $\mathfrak{g}^*$.

**Theorem 10.** *Fixed a trivialization of the vector bundle $T^*G\simeq G\times\mathfrak{g}^*$, all Lie group extensions of $G$ by $\mathfrak{g}^*$ compatible with the differentiable structure of $T^*G$ can be written as in the previous theorem.*

**Proof.** Fixing a trivialization of $T^*G$ as a trivial $\mathfrak{g}^*$-principal bundle over $G$, it is easy to see that the product can be written

$$(g,\beta)\cdot(g',\beta')=(gg',\beta+\rho_{g'^{-1}}(\beta')+\eta(g,g'))$$

where $\eta: G\times G\to\mathfrak{g}^*$ is a smooth map fulfilling a normalized 2-cocycle condition and $\rho: G\to Aut\,\mathfrak{g}^*$ is a linear representation. By means of a gauge transformation $\phi : G \to Aut\,\mathfrak{g}^*$, we can express this product, in the initially fixed trivialization of $T^*G$ as a vector bundle, in the following way

$$(g,\alpha)\cdot(g',\alpha') = (\,gg', \phi_{gg'}\phi_g^{-1}(\alpha) + \phi_{gg'}\rho_{g'^{-1}}\phi_{g'}^{-1}(\alpha') + \phi_{gg'}(\eta(g,g'))\,)$$

where $\alpha = \phi_g(\beta)$. Defining $\mathcal{L}_{(g,g')} \equiv \phi_{gg'} \circ \phi_g^{-1}$, $\mathcal{R}_{(g'^{-1},g)} \equiv \phi_{gg'} \circ \rho_{g'^{-1}} \circ \phi_{g'}^{-1}$ and $\xi(g,g') \equiv \phi_{gg'}(\eta(g,g'))$, one can see that $\{\mathcal{L},\mathcal{R},\xi\}$ are the set of compatible $Aut\,\mathfrak{g}^*$-valued cocycles and the normalized 2-cocycle stated on the theorem.

**Corollary 11.** *We have the following exact sequence of Lie groups*

$$0 \to \mathfrak{g}^* \longrightarrow T^*G \simeq (G\times\mathfrak{g}^*)_{(\mathcal{L},\mathcal{R},\xi)} \xrightarrow{\pi} G \to 1$$

*where $\mathfrak{g}^*$ is endowed with its natural abelian additive Lie group structure as a vector space.*

In general, $T^*G \simeq (G\times\mathfrak{g}^*)_{(\mathcal{L},\mathcal{R},\xi)}$ is not a central extension of $G$ because, in general, $(\epsilon,\alpha)\cdot(g,\beta) \neq (g,\beta)\cdot(\epsilon,\alpha)$.

**Proposition 12.** *The group extensions $(G\times\mathfrak{g}^*)_{(\mathcal{L},\mathcal{R},\xi)}$ and $(G\times\mathfrak{g}^*)_{(\mathcal{L}',\mathcal{R}',\xi')}$ are equivalent if, and only if, the following conditions are fulfilled :*

*a) There exists a linear isomorphism $\phi : \mathfrak{g}^* \to \mathfrak{g}^*$ such that*

$$\rho_g^{(\mathcal{L}',\mathcal{R}')} \circ \phi = \phi \circ \rho_g^{(\mathcal{L},\mathcal{R})} \qquad \forall g \in G$$

*b) There exists $\theta \in C^1_{norm}(G,\mathfrak{g}^*)$ such that*

$$[\delta_1^{(\mathcal{L},\mathcal{R})}\theta] = [\phi\cdot\xi] - \xi'$$

*In that case, the isomorphism $\Phi : (G\times\mathfrak{g}^*)_{(\mathcal{L},\mathcal{R},\xi)} \longrightarrow (G\times\mathfrak{g}^*)_{(\mathcal{L}',\mathcal{R}',\xi')}$ is given by*

$$\Phi(g,\alpha) = (g, \mathcal{L}'_{(g,\epsilon)}\phi\mathcal{L}_{(g^{-1},g)}(\alpha) + \theta(g)) = (g, \mathcal{R}'_{(g^{-1},\epsilon)}\,\phi\,\mathcal{R}_{(g,g)}(\alpha) + \theta(g))$$

**Corollary 13.** *If $\xi = \delta_1^{(\mathcal{L},\tilde{\mathcal{R}})}\theta \in B^2_{(\mathcal{L},\mathcal{R})}(G,\mathfrak{g}^*)$ then $(G\times\mathfrak{g}^*)_{(\mathcal{L},\mathcal{R},\xi)}$ is isomorphic, as a Lie group, to the semidirect product $G\times_{\rho(\mathcal{L},\mathcal{R})}\mathfrak{g}^*$.*

*Remark* 2. With these Lie group structures on $T^*G$, it is possible to give a better geometric approach to Boyom's result on the existence of invariant symplectic structures on $T^*G$. Also by means of the results of this paper, [9] we are searching for invariant complex structures on $T^*G$.

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# Trapped submanifolds in Lorentzian geometry

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**Abstract.** In Lorentzian geometry, the concept of trapped submanifold is introduced by means of the mean curvature vector properties. Trapped submanifolds are generalizations of the standard maximal hypersurfaces and minimal surfaces, of geodesics, and also of the trapped surfaces introduced by Penrose. Selected applications to gravitational theories are mentioned.

*Keywords:* Trapped surfaces, horizons, maximal/minimal surfaces.

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## 1. Introduction

The concept of closed trapped surface, first introduced by Penrose [1], is extremely useful in many physical problems and mathematical developments, with truly versatile applications. It was a cornerstone for the achievement of the singularity theorems, the analysis of gravitational collapse, the study of the cosmic censorship hypothesis, or the numerical evolution of initial data, just to mention a few, see e.g. [1, 2, 3] (a more complete list of references can be found in [4].) Trapped surfaces are usually introduced as co-dimension 2 imbedded spatial surfaces such that all its local portions have, at least initially, a decreasing (increasing) area along *any* future evolution direction. However, it has been seldom recognized that the concept of trapped surface is genuinely and purely geometric, closely related to the traditional concepts of geodesics, minimal surfaces and variations of submanifolds. The purpose of this short note is to present this novel view, which may be clarifying for, and perhaps arouse interest of, the mathematical community.

## 2. Basics on semi-Riemannian submanifolds

Let $(V, g)$ be any $D$-dimensional semi-Riemannian manifold with metric tensor $g$ of any signature. An imbedded submanifold is a pair $(S, \Phi)$ where $S$ is a $d$-dimensional manifold on its own and $\Phi : S \longrightarrow V$ is an imbedding [5]. As is customary in mathematical physics, for the sake of brevity $S$ will be identified with its image $\Phi(S)$ in $V$. $D - d$ is called the co-dimension of $S$ in $V$.

At any $p \in \Phi(S)$ one has the decomposition of the tangent space

$$T_pV = T_pS \oplus T_pS^\perp$$

*if and only if* the inherited metric (or first fundamental form) $\Phi^* g \equiv \gamma$ is non-degenerate at $p$. Henceforth, I shall assume that $\gamma$ is non-degenerate everywhere. Let us note in passing that $\Phi(S)$ is called *spacelike* if $\gamma$ is also positive definite. Thus, $\forall p \in S$, $\forall \vec{v} \in T_pV$ we have $\vec{v} = \vec{v}^T + \vec{v}^\perp$ which are called the *tangent* and *normal* parts of $\vec{v}$ relative to $S$.

Obviously, $(S, \gamma)$ is a semi-Riemannian manifold on its own, and its intrinsic structure as such is inherited from $(V, g)$. However, $(S, \gamma)$ inherits also *extrinsic* properties. Important inherited intrinsic objects are (i) the canonical volume element $d$-form $\boldsymbol{\eta}_S$ associated to $\gamma$; (ii) a Levi-Civita connection $\overline{\nabla}$ such that $\overline{\nabla}\gamma = 0$. An equivalent interesting characterization is

$$\forall \vec{x}, \vec{y} \in TS, \qquad \overline{\nabla}_{\vec{x}}\, \vec{y} = (\nabla_{\vec{x}}\, \vec{y})^T \tag{1}$$

(where $\nabla$ is the connection on $(V, g)$); and (iii) of course, the curvature of $\overline{\nabla}$ and all derived objects thereof.

Concerning the extrinsic structure, the basic object is the *shape tensor* $K : TS \times TS \longrightarrow TS^\perp$, also called extrinsic curvature of $S$ in $V$, defined by

$$\forall \vec{x}, \vec{y} \in TS, \qquad K(\vec{x}, \vec{y}) = -\left(\nabla_{\vec{x}}\, \vec{y}\right)^\perp . \tag{2}$$

The combination of (1) and (2) provides

$$\forall \vec{x}, \vec{y} \in TS, \qquad \nabla_{\vec{x}}\, \vec{y} = \overline{\nabla}_{\vec{x}}\, \vec{y} - K(\vec{x}, \vec{y}) .$$

An equivalent way of expressing the same is

$$\forall \boldsymbol{\omega} \in T^*S, \qquad \Phi^*(\boldsymbol{\nabla}\boldsymbol{\omega}) = \overline{\nabla}(\Phi^*\boldsymbol{\omega}) + \boldsymbol{\omega}(K)$$

where by definition $\boldsymbol{\omega}(K)(\vec{x}, \vec{y}) = \boldsymbol{\omega}\left(K(\vec{x}, \vec{y})\right)$ for all $\vec{x}, \vec{y} \in TS$.

The shape tensor contains the information concerning the "shape" of $\Phi(S)$ within $V$ along *all* directions normal to $\Phi(S)$. Observe that $K(\vec{x}, \vec{y}) \in TS^\perp$. If one chooses a particular normal direction $\vec{n} \in TS^\perp$, then one defines a 2-covariant symmetric tensor field $K_{\vec{n}} \in T_{(0,2)}S$ by means of

$$K_{\vec{n}}(\vec{x}, \vec{y}) = \boldsymbol{n}(K)(\vec{x}, \vec{y}) = g\left(\vec{n}, K(\vec{x}, \vec{y})\right), \quad \forall \vec{x}, \vec{y} \in TS$$

which is called the *second fundamental form* of $S$ in $(V, g)$ relative to $\vec{n}$.

## 3. The mean curvature vector

The main object to be used in this contribution is the *mean curvature vector* $\vec{H}$ of $S$ in $(V, g)$. This is an averaged version of the shape tensor defined by

$$\vec{H} = \operatorname{tr} K, \qquad \vec{H} \in TS^{\perp}$$

where the trace tr is taken with respect to $\gamma$, of course. Each component of $\vec{H}$ along a particular normal direction, that is to say, $g(\vec{H}, \vec{n})$ ($=$ tr $K_{\vec{n}}$) is termed "expansion along $\vec{n}$" in some physical applications.

The classical interpretation of $\vec{H}$ can be understood as follows. Let us start with the simplest case $d = 1$, so that $S$ is a curve in $V$. Then there is only one independent tangent vector, say $\vec{x}$, and $(\nabla_{\vec{x}}\,\vec{x})^{\perp} = -K = -\vec{H}$ is simply (minus) the proper acceleration vector of the curve. In other words, $S$ is a geodesic if and only if $K = 0$ (equivalently in this case, $\vec{H} = \vec{0}$). Hence, an immediate and standard generalization of a geodesic to arbitrary codimension $d$ is: "$S$ is *totally geodesic* if and only if $K = 0$". Totally geodesic submanifolds are those such that all geodesics within $(S, \gamma)$ are geodesics on $(V, g)$.

Nevertheless, one can also generalize the concept of geodesic to arbitrary $d$ by assuming just that $\vec{H} = \vec{0}$. To grasp the meaning of this condition, let us first consider the opposite extreme case: $d = D - 1$ or codimension 1. Then, $S$ is a hypersurface and there exists only one independent normal direction, say $\vec{n}$, so that necessarily $\vec{H} = \theta\vec{n}$ where $\theta$ is the (only) expansion, or divergence. Classical results imply that the vanishing of $\vec{H}$ (ergo $\theta = 0$) defines the situation where there is no local variation of volume along the normal direction. Actually, this interpretation remains valid for arbitrary $d$. Indeed, let $\vec{\xi}$ be an arbitrary $C^1$ vector field on $V$ defined on a neighbourhood of $S$, and let $\{\varphi_\tau\}_{\tau\in I}$ be its flow, that is its local one-parameter group of local transformations, where $\tau$ is the canonical parameter and $I \ni 0$ is a real interval. This defines a one-parameter family of surfaces $S_\tau \equiv \varphi_\tau(S)$ in $V$, with corresponding imbeddings $\Phi_\tau : S \to V$ given by $\Phi_\tau = \varphi_\tau \circ \Phi$. Observe that $S_0 = S$. Denoting by $\boldsymbol{\eta}_{S_\tau}$ their associated canonical volume element $d$-forms, it is a matter of simple calculation to get

$$\left.\frac{d\boldsymbol{\eta}_{S_\tau}}{d\tau}\right|_{\tau=0} = \frac{1}{2}\operatorname{tr}\left[\Phi^*(\pounds_{\vec{\xi}}\, g)\right]\boldsymbol{\eta}_S$$

where $\pounds_{\vec{\xi}}$ is the Lie derivative with respect to $\vec{\xi}$. Another straightforward computation using the standard formulae relating the connections on $\nabla$ and $\overline{\nabla}$ leads to

$$\frac{1}{2}\operatorname{tr}\left[\Phi^*(\pounds_{\vec{\xi}}\, g)\right] = \operatorname{div}(\varphi^*\boldsymbol{\xi}) + g(\vec{\xi}, \vec{H}) \tag{3}$$

where div is the divergence operator on $S$. Combining the two previous formulas one readily gets the expression for the variation of $d$-volume:

$$\left.\frac{dV_{S_\tau}}{d\tau}\right|_{\tau=0} = \int_S \left(\mathrm{div}\vec{\bar{\xi}} + g(\vec{\xi}, \vec{H})\right) \boldsymbol{\eta}_S$$

where $V_{S_\tau} = \int_{S_\tau} \boldsymbol{\eta}_{S_\tau}$ is the volume of $S_\tau$. In summary:

*Among the set of all submanifolds without boundary (or with a fixed boundary under appropriate restrictions) those of extremal volume must have* $\vec{H} = \vec{0}$.

## 4. Lorentzian case. Future-trapped submanifolds

If $(V, g)$ is a proper Riemannian manifold, then $g(\vec{H}, \vec{H}) \geq 0$ and the only distinguished case is $g(\vec{H}, \vec{H}) = 0$ which is equivalent to $\vec{H} = \vec{0}$: a extremal submanifold. However, in general semi-Riemannian manifolds $g(\vec{H}, \vec{H})$ can be also negative, as well as zero with non-vanishing $\vec{H}$. Thus, new possibilities and distinguished cases arise.

To fix ideas, let us concentrate in the physically relevant case of a Lorentzian manifold $(V, g)$ with signature $(-,+,\ldots,+)$. Let $(S, \gamma)$ be spacelike. Then, $\vec{H}$ can be classified according to its causal character:

$$g(\vec{H}, \vec{H}) = \begin{cases} > 0 & \vec{H} \text{ is spacelike} \\ = 0 & \vec{H} \text{ is null (or zero)} \\ < 0 & \vec{H} \text{ is timelike} \end{cases}$$

Of course, this sign can change from point to point of $S$. Recall that non-spacelike vectors can be subdivided into future- and past-pointing. Hence, $S$ can be classified as (omitting past duals) [4, 6]:

1. *future trapped* if $\vec{H}$ is timelike and future-pointing all over $S$.
2. *nearly future trapped* if $\vec{H}$ is non-spacelike and future-pointing all over $S$, and timelike at least at a point of $S$.
3. *marginally future trapped* if $\vec{H}$ is null and future-pointing all over $S$, and non-zero at least at a point of $S$.
4. *extremal* or *symmetric* if $\vec{H} = \vec{0}$ all over $S$.
5. *absolutely non-trapped* if $\vec{H}$ is spacelike all over $S$.

The original definition of "closed trapped surface", which is of paramount importance in General Relativity ($D = 4$), is due to Penrose [1, 2, 3] and was for codimension two, in which case points 1, 4 and 5 coincide with the standard nomenclature; point 2 was coined in [4], while 3 is more general than the standard concept in GR (e.g. [2, 3]) —still, all standard marginally trapped $(D-2)$-surfaces are included in 3—. On the other hand, the above terminology is unusual for the cases $d = D-1$ or $d = 1$, see [4, 6] for explanations.

## 5. Applications

One of the advantages of having defined trapped submanifolds via $\vec{H}$ is —apart from being generalizable to arbitrary codimension and thereby comparable with well-known cases such as maximal hypersurfaces and geodesics— that many simple results and applications can be derived. As an example, let us consider the case in which $\vec{\xi}$ is a conformal Killing vector $\pounds_{\vec{\xi}}\, g = 2\Psi g$ (including the particular cases of homotheties ($\Psi$ =const.) and proper Killing vectors ($\Psi = 0$)). Then formula (3) specializes to $\Psi\, d = \mathrm{div}(\varphi^* \boldsymbol{\xi}) + g(\vec{\xi}, \vec{H})$ so that, integrating over any *closed* $S$ (i.e. compact without boundary) we get

$$\int_S \Psi \boldsymbol{\eta}_S = \frac{1}{d} \int_S g(\vec{\xi}, \vec{H}) \boldsymbol{\eta}_S \, .$$

Therefore, if $\Psi|_S$ has a sign, then $g(\vec{\xi}, \vec{H})$ must have the same sign, clearly restricting the possibility of $\vec{H}$ being non-spacelike. For instance, if $\vec{\xi}$ is timelike, then $\vec{H}$ (if non-spacelike) must be oppositely directed to $\mathrm{sign}(\Psi|_S)\vec{\xi}$; in particular, if $\Psi = 0$, then there cannot be closed (nearly, marginally) trapped submanifolds at all [4, 6]. Analogously, if $\vec{\xi}$ is null on $S$ and $\Psi|_S = 0$, then the only possibility for a non-spacelike $\vec{H}$ is that the mean curvature vector be null and proportional to $\vec{\xi}$.

Specific consequences of the above are, for example, [4, 6, 7]

- that in Robertson-Walker spacetimes (where there is a conformal Killing vector), closed spacelike geodesics are forbiden (!), and closed submanifolds can only be *past*-trapped if the model is expanding [4, 6]; furthermore, there cannot be maximal closed hypersurfaces, nor minimal surfaces [4, 6].
- in stationary regions of $(V, g)$, any marginally trapped, nearly trapped, or trapped submanifold is necessarily non-closed and non-orthogonal to the timelike Killing vector [4, 6].
- in regions with a null Killing vector $\vec{\xi}$, all trapped or nearly trapped submanifolds must be non-closed and non-orthogonal to $\vec{\xi}$, and any marginally trapped submanifold must have a mean curvature vector parallel (and orthogonal!) to the null Killing vector.

- the impossibility of existence of closed trapped surfaces (co-dimension 2) in spacetimes (arbitrary dimension) with vanishing curvature invariants [7]. This includes, in particular, the case of pp-waves [4, 6, 7]. This has applications to modern string theories, implying that the spacetimes with vanishing curvature invariants, which are in particular exact solutions of the full non-linear theory, do not posses any horizons.

More details and applications can be found in [4, 6, 7, 8].

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# Posters

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# Riemannian geometry of the twistor space of a symplectic manifold

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## 1. The metric

In this short communication we show some computations about the curvature of a metric defined on the twistor space of a symplectic manifold.

Let $(M,\omega,\nabla)$ be a symplectic manifold endowed with a symplectic connection (that is $\nabla\omega=0,\ T^\nabla=0$). Recall that the twistor space

$$\mathcal{Z}=\big\{j\in\operatorname{End}T_xM:\ x\in M,\ j^2=-1,\ \omega\text{ type (1,1) for } j \text{ and } \omega(\ ,j\ )>0\big\}$$

is a bundle $\pi:\mathcal{Z}\rightarrow M$, with obvious projection, together with an almost complex structure $\mathcal{J}^\nabla$ defined as follows. First, notice the connection induces a splitting

$$0\longrightarrow\mathcal{V}\longrightarrow T\mathcal{Z}=\mathcal{H}^\nabla\oplus\mathcal{V}\stackrel{\mathrm{d}\pi}{\longrightarrow}\pi^*TM\longrightarrow 0$$

into horizontal and vertical vectors, which is to be preserved by $\mathcal{J}^\nabla$. Since the fibres of $\mathcal{Z}$ are hermitian symmetric spaces $Sp(2n,\mathbb{R})/U(n)$ — the Siegel domain —, we may identify

$$\mathcal{V}_j=\{A\in\mathfrak{sp}(\pi^*TM,\pi^*\omega):\ Aj=-jA\}$$

and hence $\mathcal{J}^\nabla_j$ acts like left multiplication by $j$: $\mathcal{J}^\nabla_j(A)=jA$.

On the horizontal part, the twistor almost complex structure is defined in a tautological fashion as $j$ itself, up to the bundle isomorphism $\mathrm{d}\pi_|:\mathcal{H}^\nabla\rightarrow\pi^*TM$ which occurs pointwise: thus $\mathcal{J}^\nabla_j(X)=(\mathrm{d}\pi)^{-1}j\mathrm{d}\pi(X),\ \forall X$ horizontal.

Notice that we understand that $j \in \mathcal{Z}$ also belongs to $\mathrm{End}\,(\pi^*TM)_j$, so there exists a canonical section $\Phi$ of the endomorphisms bundle defined by $\Phi_j = j$.

In [1,2] a few properties and examples of this twistor theory are explored. Between them, the integrability equation is recalled (cf. [3]), dependent on the curvature of $\nabla$ only. A natural hermitian metric on $\mathcal{Z}$ was also considered in [1,2] and our aim now is to find the sectional curvature in a special case. First we define its associated non-degenerate 2-form $\Omega^\nabla$. By analogy with the Killing form in Lie algebra theory and a Cartan's decomposition of $\mathfrak{sp}(2n,\mathbb{R}) = \mathfrak{u}(n) \oplus \mathfrak{m}$, the subspace $\mathfrak{m}$ playing the role of $\mathcal{V}_j$, one defines a symplectic form on $\mathcal{Z}$ by $\Omega^\nabla = t\,\pi^*\omega - \tau$, where $t \in ]0,+\infty[$ is a fixed parameter and

$$\tau(X,Y) \;=\; \frac{1}{2}\mathrm{Tr}\,(PX)\Phi(PY).$$

$P$ is the projection $T\mathcal{Z}$ onto $\mathcal{V}$ with kernel $\mathcal{H}^\nabla$, thus a $\mathcal{V}$-valued 1-form on $\mathcal{Z}$. It is easy to see that $\mathcal{J}^\nabla$ is compatible with $\Omega^\nabla$ and that the induced metric is positive definite. The following results are proved in the cited thesis.

**Theorem 1.** *$\Omega^\nabla$ is closed iff $\nabla$ is flat. In such case, $\mathcal{Z}$ is a Kähler manifold.*

Let $\langle\ ,\ \rangle$ be the induced metric, so that

$$\langle X,Y\rangle \;=\; t\,\pi^*\omega(X,\mathcal{J}^\nabla Y) + \frac{1}{2}\mathrm{Tr}\,(PXPY)$$

and thus $\mathcal{H}^\nabla \perp \mathcal{V}$.

**Lemma 2.** *$P$ is a $\mathcal{V} \subset \mathrm{End}\,(\pi^*TM)$-valued 1-form on $\mathcal{Z}$. The connection $D = \pi^*\nabla - P$ on $\pi^*TM$ preserves $\mathcal{V}$ and hence induces a new linear connection $D$ over the twistor space such that $D\mathcal{J}^\nabla = 0$ and $D$ preserves the splitting of $T\mathcal{Z}$. Moreover, the torsion $T^D = P(\pi^*R^\nabla) - P \wedge \mathrm{d}\pi$.*

Let $\cdot^h$ denote the horizontal part of any tangent-valued tensor.

**Theorem 3.** *(i) The Levi-Civita connection of $\langle\ ,\ \rangle$ is given by*

$$\mathfrak{D}_X Y \;=\; D_X Y - PY(\pi_* X) - \frac{1}{2}P(\pi^* R^\nabla_{X,Y}) + S(X,Y)$$

*where $S$ is symmetric and defined both by*

$$\langle P(S(X,Y)),A\rangle \;=\; \langle A\pi_*X,\pi_*Y\rangle, \qquad \forall A \in \mathcal{V},$$

*and*

$$\langle S^h(X,B),Y\rangle \;=\; \frac{1}{2}\langle P(\pi^* R^\nabla_{X,Y}),B\rangle, \qquad \forall Y \in \mathcal{H}^\nabla.$$

*Hence for* $X, Y \in \mathcal{H}^\nabla$ *and* $A, B \in \mathcal{V}$ *we have*

$$P(S(X,A)) = P(S(A,B)) = 0,$$
$$S^h(X,Y) = S^h(A,B) = 0.$$

*(ii) The fibres* $\pi^{-1}(x),\ x \in M$, *are totally geodesic in* $\mathcal{Z}_M$.
*(iii) If* $\nabla$ *is flat, then* $\mathfrak{D}\mathcal{J}^\nabla = 0$.

One may write $P(S(X,Y))$ explicitly and construct a symplectic-orthonormal basis of $\mathcal{V}$ induced by a given such basis on $\mathcal{H}^\nabla$. We show the first of these assertions.

**Proposition 4.** *For* $X, Y$ *horizontal*

$$S_j(X,Y) \ = \ -\frac{t}{2}\Big\{\omega(X,\ \ )jY + \omega(jY,\ \ )X + \omega(jX,\ \ )Y + \omega(Y,\ \ )jX\Big\}.$$

*In particular,* $\langle S_j(X,Y)X,Y\rangle = \frac{1}{2}\big(\langle X,Y\rangle^2 + \|X\|^2|Y\|^2 + t^2\omega(X,Y)^2\big)$ *and* $\langle S_j(X,X)Y,Y\rangle = \langle X,Y\rangle^2 - t^2\omega(X,Y)^2$.

The proof of the last result is accomplished by simple verifications. The following is the relevant linear algebra used in its discovery, explained to us by J. Rawnsley. Since $\mathfrak{sp}(2n,\mathbb{R}) \simeq S^2\mathbb{R}^{2n}$, the symmetric representation space, which is irreducible under $Sp(2n,\mathbb{R})$, and since

$$\omega^2(XY,ZT) = \omega(X,Z)\omega(Y,T) + \omega(X,T)\omega(Y,Z)$$

is a non-degenerate symmetric bilinear form, it follows that $\omega^2$ must be a multiple of the Killing form of $\mathfrak{sp}$, ie. the trace form!

The twistor space is not compact, nor does the metric extend to any compact space that we know. Indeed, we have not yet found a proof for the following **conjecture**: if $\nabla$ is complete, the same is true for $D$ and $\mathfrak{D}$.

## 2. Kählerian twistor spaces

The next result appeared in [1] without a proof. Until the end of the subsection assume $R^\nabla = 0$, ie. that the metric $\langle\ ,\ \rangle$ is Kählerian.

**Theorem 5.** *Let* $\Pi$ *be a 2-plane in* $T_j\mathcal{Z}$ *spanned by the orthonormal basis* $\{X+A, Y+B\},\ X, Y \in \mathcal{H}^\nabla,\ A, B \in \mathcal{V}$. *Then the sectional curvature of* $\Pi$ *is*

$$\begin{aligned} k_j(\Pi) \ &= \ -\langle R^{\mathfrak{D}}(X+A,Y+B)(X+A), Y+B\rangle \\ &= \ \tfrac{1}{2}\Big(\|X\|^2\|Y\|^2 + 3t^2\omega(X,Y)^2 - \langle X,Y\rangle^2\Big) + \\ &\qquad + \|BX - AY\|^2 - 2\langle [B,A]X,Y\rangle - \|[B,A]\|^2 \end{aligned}$$

*where* $[\,,\,]$ *is the commutator bracket. Thus*

$$k_j(\Pi)\begin{cases} >0 & \textit{for } \Pi\subset\mathcal{H}^\nabla\\ <0 & \textit{for } \Pi\subset\mathcal{V}\ . \end{cases}$$

*Proof.* Following the previous theorem, notice that $S$ is vertical only. Let $U,V$ be any two tangent vector fields over $\mathcal{Z}$. Then

$$\begin{aligned} \mathrm{d}^{\pi^*\nabla}P(U,V) &= \pi^*\nabla_U(PV)-\pi^*\nabla_V(PU)-P[U,V]\\ &= D_UPV+[PU,PV]-D_VPU-[PV,PU]-P[U,V]\\ &= PT^D(U,V)+2[PU,PV]\ =\ 2[PU,PV]. \end{aligned}$$

Hence, from well known connection theory,

$$R^D=R^{\pi^*\nabla}-\mathrm{d}^{\pi^*\nabla}P+P\wedge P\ =\ -P\wedge P.$$

Now let us use the notation $\mathcal{R}_{uvwz}=\langle R^{\mathfrak{D}}(U,V)W,Z\rangle$. Recall the symmetries $\mathcal{R}_{uvwz}=\mathcal{R}_{wzuv}=-\mathcal{R}_{uvzw}$ and Bianchi identity $\mathcal{R}_{uvwz}+\mathcal{R}_{vwuz}+\mathcal{R}_{wuvz}=0$. Now we want to find

$$\begin{aligned} -k_j(\Pi) &= \langle R^{\mathfrak{D}}(X+A,Y+B)(X+A),Y+B\rangle\\ &= \mathcal{R}_{xyxy}+\mathcal{R}_{xyxb}+\mathcal{R}_{xyay}+\mathcal{R}_{xyab}\\ &\quad+\mathcal{R}_{xbxy}+\mathcal{R}_{xbxb}+\mathcal{R}_{xbay}+\mathcal{R}_{xbab}\\ &\quad+\mathcal{R}_{ayxy}+\mathcal{R}_{ayxb}+\mathcal{R}_{ayay}+\mathcal{R}_{ayab}\\ &\quad+\mathcal{R}_{abxy}+\mathcal{R}_{abxb}+\mathcal{R}_{abay}+\mathcal{R}_{abab} \end{aligned}$$

and, if we see this sum as a matrix, then we deduce that it is symmetric.

Notice that $R^{\mathfrak{D}}(X,Y)Z$, with $X,Y,Z$ horizontal, and $R^{\mathfrak{D}}(A,B)C$, with $A,B,C$ vertical, can be obtained immediately from Gauss-Codazzi equations. First, notice that the horizontal distribution is integrable when $\nabla$ is flat. Then the horizontal leaves are immediately seen to have $D$, or simply $\pi^*\nabla$, for Levi-Civita connection with the induced metric; hence they are flat. Finally, $S$ is the $2^{\mathrm{nd}}$ fundamental form, so a formula of Gauss says $R^{\mathfrak{D}}_{X,Y}Z=R^{\pi^*\nabla}_{X,Y}Z+S(X,Z)Y-S(Y,Z)X$. Therefore

$$\begin{aligned} -\mathcal{R}_{xyxy} &= \langle S(X,Y)X,Y\rangle-\langle S(X,X)Y,Y\rangle\\ &= \tfrac12\big(\langle X,Y\rangle^2+\|X\|^2|Y\|^2+t^2\omega(X,Y)^2\big)-2\langle X,Y\rangle^2+2t^2\omega(X,Y)^2\big)\\ &= \tfrac12\big(\|X\|^2|Y\|^2+3t^2\omega(X,Y)^2-\langle X,Y\rangle^2\big). \end{aligned}$$

which is positive, as we have deduced following proposition 4.

By the same principles, $R^{\mathfrak{D}}_{A,B}C = R^{D}_{A,B}C = [-P\wedge P(A,B),C] = -[[A,B],C]$. For the (totally geodesic) vertical fibres of $\mathcal{Z}$, we recall that

$$\mathcal{R}_{abab} = -\langle[[A,B],A],B\rangle = \|[B,A]\|^2$$

is minus the sectional curvature of the hyperbolic space $Sp(2n,\mathbb{R})/U(n)$. We also note that the previous curvatures return, respectively, to the horizontal and vertical subspaces. Hence we get

$$\mathcal{R}_{xyxb} = \mathcal{R}_{xyay} = \mathcal{R}_{xbab} = \mathcal{R}_{ayab} = 0.$$

Now we want to find $\mathcal{R}_{xbay}$. First we deduce via theorem 3 the formulae $\mathfrak{D}_AX = D_AX$, $\mathfrak{D}_XA = D_XA - AX$, $\mathfrak{D}_AB = D_AB$. Also, the Lie bracket $[X,B] = D_XB - D_BX - T^D(X,B) = D_XB - D_BX - BX$ by lemma 2. Thus

$$\begin{aligned} R^{\mathfrak{D}}_{X,B}A &= \mathfrak{D}_X\mathfrak{D}_BA - \mathfrak{D}_B\mathfrak{D}_XA - \mathfrak{D}_{[X,B]}A \\ &= \mathfrak{D}_XD_BA - \mathfrak{D}_BD_XA + \mathfrak{D}_B(AX) - \mathfrak{D}_{D_XB-D_BX-BX}A \\ &= D_XD_BA - (D_BA)X - D_BD_XA + D_B(AX) \\ &\quad - D_{[X,B]}A - A(D_BX) - ABX \\ &= R^D_{X,B}A - ABX \;=\; -ABX. \end{aligned}$$

Hence $\mathcal{R}_{xbay} = -\langle ABX,Y\rangle$, $\mathcal{R}_{xbxb} = \langle B^2X,X\rangle = -\|BX\|^2$ and

$$\mathcal{R}_{xyab} = \mathcal{R}_{abxy} = -\mathcal{R}_{xaby} - \mathcal{R}_{bxay} = \langle BAX,Y\rangle - \langle ABX,Y\rangle = \langle[B,A]X,Y\rangle.$$

Finally

$$\begin{aligned} k_j(\Pi) &= -\mathcal{R}_{xyxy} - 2\mathcal{R}_{xyab} - \mathcal{R}_{xbxb} - 2\mathcal{R}_{xbay} - \mathcal{R}_{ayay} - \mathcal{R}_{abab} \\ &= -\mathcal{R}_{xyxy} + 2\langle[A,B]X,Y\rangle + \|BX\|^2 + 2\langle ABX,Y\rangle + \|AY\|^2 - \mathcal{R}_{abab} \\ &= -\mathcal{R}_{xyxy} + 2\langle[A,B]X,Y\rangle + \|BX-AY\|^2 - \mathcal{R}_{abab} \end{aligned}$$

as we wished. The second part of the result follows by Cauchy inequality. ■

It is possible to prove that the sectional curvature attains the value $-4$ in vertical planes and a the maximum value 2 in horizontal planes. The following problem is closely related to this.

## 3. A problem in variational calculus

Let $T$ be a real vector space. Let $R$ be a Riemannian curvature-type tensor, ie. an element of $\bigwedge^2 T^*\otimes\bigwedge^2 T^*$ satisfying Bianchi identity and $R(u,v,z,w) = R(z,w,u,v)$. Let

$$k : Gr(2,T)\to\mathbb{R}$$

be the induced sectional curvature function on the real Grassmannian of 2-planes of $T$. Let $H \oplus V$ be a direct sum decomposition of $T$ and suppose $k$ is positive in $H$ and negative in $V$. Then, are the maximum and minimum of $k$, respectively, in $H$ and $V$?

We do not know a reference for this result — which we believe to be true. We thank any comments or guidance to the related literature.

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# $q$-Poisson coalgebras and integrable dynamics on spaces with variable curvature

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**Abstract.**

A direct connection between $q$-deformation and integrability on spaces with variable curvature is presented. The non-standard deformation of a $sl(2)$ Poisson coalgebra is used to introduce an integrable Hamiltonian that describes a geodesic motion on two-dimensional spaces with non-constant curvature. Another superintegrable Hamiltonian defined on the same deformed coalgebra is also shown to generate integrable motions on the two-dimensional Cayley–Klein spaces. In this case, the constant curvature of the space is just the deformation parameter $z = \ln q$.

*Keywords:* Quantum groups, deformation, curvature, hyperbolic spaces, de Sitter spacetime.

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## 1. Poisson coalgebra symmetry and integrability

A systematic approach to the construction of integrable and superintegrable Hamiltonian systems with $N$ degrees of freedom from Poisson coalgebras has been recently introduced (see [1, 2, 3] and references therein). From this perspective, a specific class of Poisson coalgebras associated to quantum groups [4, 5] can be understood as the dynamical symmetries that generate integrable deformations of well-known dynamical systems with an arbitrary number of degrees of freedom. In this contribution we show that one of such deformations can be interpreted in the 2D case as the algebraic structure that underlies the integrability of the geodesic dynamics on certain spaces of variable curvature. Moreover, the formalism allows for the definition of a superintegrable geodesic motion, which is shown to live on the so-called Cayley–Klein (CK) spaces of constant curvature. A more detailed exposition of these results can be found in [6], and their application to the definition of certain integrable potentials on the spaces here studied will be presented elsewhere [7].

Let us briefly recall the basics of this construction by using the non-deformed Poisson coalgebra $(sl(2), \Delta)$, which is defined by the following Poisson brackets and coproduct map $\Delta$:

$$\{J_3, J_+\} = 2J_+, \qquad \{J_3, J_-\} = -2J_-, \qquad \{J_-, J_+\} = 4J_3, \tag{1}$$

$$\Delta(J_i) = J_i \otimes 1 + 1 \otimes J_i, \qquad i = +, -, 3. \tag{2}$$

The Casimir function is $\mathcal{C} = J_-J_+ - J_3^2$ and a one-particle symplectic realization of (1), labeled by $\mathcal{C}^{(1)} = b_1$, is given by

$$J_-^{(1)} = q_1^2, \qquad J_+^{(1)} = p_1^2 + \frac{b_1}{q_1^2}, \qquad J_3^{(1)} = q_1 p_1. \tag{3}$$

A two-particle symplectic realization is obtained through the coproduct (2):

$$J_-^{(2)} = q_1^2 + q_2^2, \quad J_+^{(2)} = p_1^2 + p_2^2 + \frac{b_1}{q_1^2} + \frac{b_2}{q_2^2}, \quad J_3^{(2)} = q_1 p_1 + q_2 p_2. \tag{4}$$

Thus, given any function $\mathcal{H}$ defined on the generators of $(sl(2), \Delta)$, the coalgebra symmetry ensures that the associated two-body Hamiltonian $\mathcal{H}^{(2)} := \Delta(\mathcal{H}) = \mathcal{H}(J_-^{(2)}, J_+^{(2)}, J_3^{(2)})$ is integrable, since the two-particle Casimir

$$\mathcal{C}^{(2)} = \Delta(\mathcal{C}) = (q_1 p_2 - q_2 p_1)^2 + \left(b_1 \frac{q_2^2}{q_1^2} + b_2 \frac{q_1^2}{q_2^2}\right) + b_1 + b_2, \tag{5}$$

Poisson commutes with $\mathcal{H}^{(2)}$. Some well-known (super)integrable Hamiltonian systems can be recovered as specific choices for $\mathcal{H}^{(2)}$. In particular, if we set $\mathcal{H} = \frac{1}{2}J_+ + \mathcal{F}(J_-)$, where $\mathcal{F}$ is an arbitrary smooth function, we find the following family of integrable systems defined on the 2D Euclidean space

$$\mathcal{H}^{(2)} = \frac{1}{2}\left(p_1^2 + p_2^2\right) + \frac{b_1}{2q_1^2} + \frac{b_2}{2q_2^2} + \mathcal{F}\left(q_1^2 + q_2^2\right). \tag{6}$$

The case $\mathcal{F}(J_-) = \omega^2 J_- = \omega^2(q_1^2 + q_2^2)$ is just the 2D Smorodinsky–Winternitz system [8, 9, 10]. Obviously, the free motion on the 2D Euclidean space is described by $\mathcal{H} = \frac{1}{2}J_+$, and the $N$-body generalization of this construction [2] follows from the $N$-body coassociative iteration of the coproduct map (2).

## 2. $q$-Poisson coalgebras

By considering coalgebra deformations of the previous construction we are able to find integrable deformations of (6). For instance, the non-standard quantum deformation of $sl(2)$ [11] is associated to the $q$-Poisson coalgebra:

$$\{J_3, J_+\} = 2J_+ \cosh zJ_-, \quad \{J_3, J_-\} = -2\,\frac{\sinh zJ_-}{z}, \quad \{J_-, J_+\} = 4J_3, \tag{7}$$

$$\Delta_z(J_-) = J_- \otimes 1 + 1 \otimes J_-, \quad \Delta_z(J_{+,3}) = J_{+,3} \otimes \mathrm{e}^{zJ_-} + \mathrm{e}^{-zJ_-} \otimes J_{+,3}, \tag{8}$$

where $z = \ln q$ is a real deformation parameter. The deformed Casimir is

$$\mathcal{C}_z = \frac{\sinh zJ_-}{z}\, J_+ - J_3^2, \tag{9}$$

and we consider the $b_i = 0$ deformed symplectic realization [2] with $\mathcal{C}_z^{(1)} = 0$:

$$J_-^{(1)} = q_1^2, \qquad J_+^{(1)} = \frac{\sinh zq_1^2}{zq_1^2}\, p_1^2, \qquad J_3^{(1)} = \frac{\sinh zq_1^2}{zq_1^2}\, q_1 p_1. \tag{10}$$

The coproduct $\Delta_z$ (8) provides the two-particle realization of (7):

$$\begin{aligned} J_-^{(2)} &= q_1^2 + q_2^2, \qquad J_+^{(2)} = \frac{\sinh zq_1^2}{zq_1^2}\, \mathrm{e}^{zq_2^2} p_1^2 + \frac{\sinh zq_2^2}{zq_2^2}\, \mathrm{e}^{-zq_1^2} p_2^2, \\ J_3^{(2)} &= \frac{\sinh zq_1^2}{zq_1^2}\, \mathrm{e}^{zq_2^2} q_1 p_1 + \frac{\sinh zq_2^2}{zq_2^2}\, \mathrm{e}^{-zq_1^2} q_2 p_2. \end{aligned} \tag{11}$$

Consequently, the two-particle Casimir given by

$$\mathcal{C}_z^{(2)} = \Delta_z(\mathcal{C}_z) = \frac{\sinh zq_1^2}{zq_1^2}\,\frac{\sinh zq_2^2}{zq_2^2}\, \mathrm{e}^{-zq_1^2}\mathrm{e}^{zq_2^2}\,(q_1 p_2 - q_2 p_1)^2, \tag{12}$$

is, by construction, a constant of the motion for any Hamiltonian $\mathcal{H}_z^{(2)} = \Delta_z(\mathcal{H}) = \mathcal{H}(J_-^{(2)}, J_+^{(2)}, J_3^{(2)})$.

Therefore, if we consider $\mathcal{H} = \frac{1}{2}J_+$, we find the Hamiltonian for an *integrable* deformation of the free motion on the 2D Euclidean space, namely

$$\mathcal{H}_z^{\mathrm{I}} = \frac{1}{2}\left(\frac{\sinh zq_1^2}{zq_1^2}\, \mathrm{e}^{zq_2^2} p_1^2 + \frac{\sinh zq_2^2}{zq_2^2}\, \mathrm{e}^{-zq_1^2} p_2^2\right). \tag{13}$$

It can also be shown that a *maximally superintegrable* deformation (an additional constant of the motion does exist [2]) of the free motion is achieved by considering $\mathcal{H} = \frac{1}{2}J_+\mathrm{e}^{zJ_-}$, which gives rise to the slightly different Hamiltonian

$$\mathcal{H}_z^{\mathrm{S}} = \frac{1}{2}\left(\frac{\sinh zq_1^2}{zq_1^2}\, \mathrm{e}^{zq_1^2}\mathrm{e}^{2zq_2^2} p_1^2 + \frac{\sinh zq_2^2}{zq_2^2}\, \mathrm{e}^{zq_2^2} p_2^2\right). \tag{14}$$

This contribution focuses on the geometric interpretation of the manifolds on which the Hamiltonians (13) and (14) describe geodesic motions.

## 3. Spaces with non-constant curvature

The kinetic energy $\mathcal{T}_z^{\rm I}(q_i,p_i)$ coming from (13) can be rewritten as the Lagrangian

$$\mathcal{T}_z^{\rm I}(q_i,\dot q_i)=\frac{1}{2}\left(\frac{zq_1^2}{\sinh zq_1^2}\,{\rm e}^{-zq_2^2}\dot q_1^2+\frac{zq_2^2}{\sinh zq_2^2}\,{\rm e}^{zq_1^2}\dot q_2^2\right), \tag{15}$$

that defines a geodesic flow on a 2D Riemannian space with a definite positive metric with signature diag$(+,+)$ given by

$${\rm d}s^2=\frac{2zq_1^2}{\sinh zq_1^2}\,{\rm e}^{-zq_2^2}\,{\rm d}q_1^2+\frac{2zq_2^2}{\sinh zq_2^2}\,{\rm e}^{zq_1^2}\,{\rm d}q_2^2. \tag{16}$$

The corresponding Gaussian curvature $K$ reads

$$K(q_1,q_2;z)=-z\sinh\left(z(q_1^2+q_2^2)\right), \tag{17}$$

so that the space is of hyperbolic type, with a variable negative curvature, and endowed with a radial symmetry. Now we introduce a pair of new coordinates $(\rho,\theta)$ defined through

$$\cosh(\lambda_1\rho)=\exp\left\{z(q_1^2+q_2^2)\right\},\quad \sin^2(\lambda_2\theta)=\frac{\exp\left\{2zq_1^2\right\}-1}{\exp\left\{2z(q_1^2+q_2^2)\right\}-1}, \tag{18}$$

where both $\lambda_1=\sqrt{z}$ and $\lambda_2\neq 0$ can be either real or pure imaginary numbers.

In this way, we will be able to rewrite the initial metric (16) as a family of six metrics on spaces with different signature and curvature. In fact, under the transformation (18), the metric (16) takes a simpler form:

$${\rm d}s^2=\frac{1}{\cosh(\lambda_1\rho)}\left({\rm d}\rho^2+\lambda_2^2\,\frac{\sinh^2(\lambda_1\rho)}{\lambda_1^2}\,{\rm d}\theta^2\right)=\frac{1}{\cosh(\lambda_1\rho)}\,{\rm d}s_0^2. \tag{19}$$

In these new coordinates the Gaussian curvature (17) turns out to be

$$K(\rho)=-\frac{1}{2}\lambda_1^2\,\frac{\sinh^2(\lambda_1\rho)}{\cosh(\lambda_1\rho)}. \tag{20}$$

Now, by recalling the description of the 2D CK spaces in terms of geodesic polar coordinates [12, 13], we find that ${\rm d}s_0^2$ is just the CK metric provided that we identify $z=\lambda_1^2\equiv-\kappa_1$ and $\lambda_2^2\equiv\kappa_2$. Therefore, by taking into account the possible values of $z$ and $\lambda_2$ six (deformed) spaces arise:

- When $\lambda_2$ is real, we get a 2D deformed sphere $\mathbf{S}_z^2$ $(z<0)$, and a deformed hyperbolic or Lobachewski space $\mathbf{H}_z^2$ $(z>0)$.

- When $\lambda_2$ is imaginary, we obtain a deformation of the (1+1)D anti-de Sitter spacetime $\mathbf{AdS}_z^{1+1}$ ($z < 0$) and of the de Sitter one $\mathbf{dS}_z^{1+1}$ ($z > 0$).
- In the non-deformed case $z \to 0$, the Euclidean space $\mathbf{E}^2$ ($\lambda_2$ real) and Minkowskian spacetime $\mathbf{M}^{1+1}$ ($\lambda_2$ imaginary) are recovered.

Thus the variable curvature of the space is directly related to the deformation parameter $z$ and the "additional" parameter $\lambda_2$ governs the signature of the metric. The integration of the geodesic motion on all these spaces can be explicitly performed in terms of elliptic integrals [7].

## 4. Spaces with constant curvature

Let us consider now the superintegrable Hamiltonian (14). The associated metric is given by

$$\mathrm{d}s^2 = \frac{2zq_1^2}{\sinh zq_1^2}\,\mathrm{e}^{-zq_1^2}\mathrm{e}^{-2zq_2^2}\,\mathrm{d}q_1^2 + \frac{2zq_2^2}{\sinh zq_2^2}\,\mathrm{e}^{-zq_2^2}\,\mathrm{d}q_2^2. \tag{21}$$

Surprisingly, in this case the Gaussian curvature turns out to be *constant* and coincides with the deformation parameter $K = z$.

Under the change of coordinates (18), the metric (21) becomes

$$\mathrm{d}s^2 = \frac{1}{\cosh^2(\lambda_1\rho)}\left(\mathrm{d}\rho^2 + \lambda_2^2\,\frac{\sinh^2(\lambda_1\rho)}{\lambda_1^2}\,\mathrm{d}\theta^2\right) = \frac{1}{\cosh^2(\lambda_1\rho)}\,\mathrm{d}s_0^2, \tag{22}$$

where $\mathrm{d}s_0^2$ is again the metric of the 2D CK spaces. As these spaces are also of constant curvature, a further change of coordinates should allow us to reproduce exactly the CK metric. By introducing a new radial coordinate as

$$r = \int_0^\rho \frac{\mathrm{d}x}{\cosh(\lambda_1 x)}, \tag{23}$$

we finally obtain

$$\mathrm{d}s^2 = \mathrm{d}r^2 + \lambda_2^2\,\frac{\sin^2(\lambda_1 r)}{\lambda_1^2}\,\mathrm{d}\theta^2, \tag{24}$$

which is just the CK metric written in geodesic polar coordinates $(r, \theta)$ and provided that $z = \lambda_1^2 \equiv \kappa_1$ and $\lambda_2^2 \equiv \kappa_2$ [13]. Note that in the limiting case $z \to 0$ the coordinate $\rho \to r$.

Therefore, we conclude that (at least in 2D) the existence of an additional integral of the motion leads to a geodesic dynamics on a set of spaces with constant curvature. Finally, we stress that, as a consequence of the underlying coalgebra structure, the generalization of the approach here presented to $N$-dimensions can be performed and seems worthy to be investigated.

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# Perturbative and non-perturbative quantum Einstein-Rosen waves

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**Abstract.**

We discuss the connection between the Fock space introduced by Ashtekar and Pierri for the quantization of the cylindrically symmetric Einstein-Rosen gravitational waves and its perturbative counterpart based on the concept of particle that arises in linearized gravity with a de Donder gauge. We show that the vacua of these two Fock spaces cannot be related by means of a power series in the gravitational constant. This result is interpreted as indicating that the two Fock quantizations are unitarily inequivalent.

*Keywords:* Einstein-Rosen waves, quantum field theory.

*2000 Mathematics Subject Classification:* 83C45, 83C47

## 1. Introduction

The ability to provide a model with the field complexity of general relativity, but with known exact solutions which describe gravitational waves, has endowed the family of Einstein-Rosen (ER) spacetimes with a prominent role in the analysis of the quantization of gravitational systems [1, 2]. A key remark for the canonical quantization of these waves is that the dynamics of the ER spacetimes is equivalent to that of a cylindrically symmetric, massless scalar field propagating on an auxiliary Minkowski background [1]. Thanks to this fact, one can recast the system as three-dimensional gravity coupled to a scalar field with rotational symmetry. Employing this three-dimensional formulation, a consistent and essentially complete quantization of the ER waves was obtained by Ashtekar and Pierri (AP). This quantization was achieved after a careful treatment of the regularity conditions at the symmetry axis, and of the boundary conditions at spatial infinity that ensure asymptotic flatness in cylindrical gravity [2].

In the AP quantization, the Hilbert space is the Fock space corresponding to the rotationally symmetric scalar field that propagates in the three-dimensional, auxiliary Minkowski spacetime. An important issue that has only recently been considered [4] is whether this space is the kind of Fock space that one would introduce in a standard perturbative treatment of the model and, if they differ, what relation exists between them. The main aim of this contribution is to discuss this point. This is a fundamental question in order to answer whether one can or cannot attain the correct non-perturbative results by adopting a perturbative approach. In our analysis, we will adopt units such that $c = \hbar = 1$, with $c$ being the speed of light and $\hbar$ the Planck constant. We also call $G$ the Newton constant per unit length in the direction of the symmetry axis.

## 2. ER waves and AP quantization

The ER waves are linearly polarized, cylindrical waves in vacuum general relativity. They can be described by the metric

$$ds^2 = e^{-\psi}\left[-N^2 dt^2 + e^{\gamma}(dR + N^R dt)^2 + (8Gr)^2 d\theta^2\right] + e^{\psi} dZ^2. \qquad (1)$$

All these metric functions depend only on the radial and time coordinates, $R$ and $t$. $Z$ is the coordinate of the symmetry axis and $\theta \in S^1$ is the axial coordinate. The lapse $N$ and the radial shift $N^R$ can be fixed by removing the gauge-freedom associated with the Hamiltonian and diffeomorphism constraints [4]. An admissible gauge is provided by the AP gauge conditions $r = R/(8G)$ and $p_\gamma = 0$, where $p_\gamma$ is the canonical momentum of $\gamma$. In this manner, and imposing regularity on the axis $R = 0$, one attains a reduced system whose metric can be written in the form [2, 4]

$$ds^2 = e^{-\psi}\left[e^{\gamma}(dT^2 + dR^2) + R^2 d\theta^2\right] + e^{\psi} dZ^2, \gamma = \int_0^R d\tilde{R}\,\frac{\tilde{R}}{2}\left[(\psi')^2 + \frac{(8Gp_\psi)^2}{\tilde{R}^2}\right].$$

The field $\psi$ (with momentum $p_\psi$) provides the only physical degree of freedom, and $\gamma$ (up to a constant factor) can be interpreted as the energy in a circle of radius $R$ of a free, rotationally symmetric scalar field $\psi$ in three dimensions. Actually, the reduced dynamics of $\psi$ is generated by the free-field Hamiltonian $H_0$ that would provide the total energy according to this interpretation, namely $8GH_0 = \lim_{R\to\infty}\gamma := \gamma_\infty$. Thus, the field $\psi$ satisfies a cylindrical wave equation, whose regular solutions can be obtained in terms of the zeroth-order Bessel function $J_0$: $\psi(R,T) = \sqrt{4G}\int_0^\infty dk\, J_0(Rk)\left[A(k)e^{-ikT} + A^\dagger(k)e^{ikT}\right]$. The function $A(k)$ and its complex conjugate $A^\dagger(k)$ are determined by the initial conditions and play the role of annihilation and creation variables.

The AP quantization is essentially based on the introduction of a Fock space in which $A^\dagger(k)$ and $A(k)$ are promoted to creation and annihilation operators [2] (with $k \in \mathbb{R}^+$). The field $\psi$ (e.g. at $T = 0$) becomes then an operator valued-distribution $\hat{\psi}(R)$. In order to obtain a truly well-defined operator, it suffices to insert a regulator $g(k)$ in the $k$-decomposition of the field, where $g$ is any square-integrable function of $k \in \mathbb{R}^+$. In the following, we will concentrate our attention on the case in which $g(k)$ corresponds to a cut-off $k_c$, i.e., $g(k) = \Theta(k_c - k)$ with $\Theta$ being the Heaviside function.

## 3. Field redefinition and linearized theory

In standard perturbative treatments of gravity, the basic metric fields are linear in the excess of the metric around the Minkowski background. Expanding the gravitational action in powers of these basic fields, the quadratic term provides the action of linearized gravity, while the higher-order terms can be regarded as describing interactions. At this stage, it is convenient to adopt a gauge that simplifies the linearized equations. A frequently used gauge is the de Donder or Lorentz gauge, in which the linearized gravitational equations reduce to wave equations, so that one easily arrives at a notion of particle. When considering the full gravitational system, rather than its linearization, one would correct the de Donder gauge with higher-order terms in the fields to ensure that it continues to be well posed.

We will now show that the gauge fixing introduced by AP is nothing but a valid generalization of the de Donder gauge from linearized to full ER gravity. Therefore, adopting it as a suitable gauge, the relation between the AP and the perturbative treatments follows straightforwardly from the transformation in configuration space that connects the metric variables used in each of the two descriptions. Let us start by introducing a different field parametrization for the ER metric (1) such that the new (over-barred) fields are linear in the metric excess around Minkowski,

$$\begin{aligned} ds^2 &= -(1 - 2\bar{N} - \bar{\psi})dt^2 + 2\bar{N}^R dt dR + (1 + \bar{\gamma} - \bar{\psi})dR^2 \\ &\quad + (R^2 - R^2\bar{\psi} + 16GR\bar{\rho})d\theta^2 + (1 + \bar{\psi})dZ^2. \end{aligned}$$

Note that this new parametrization can also be regarded as the linearization of the metric (1) around the background solution. In addition, our field redefinition can be easily completed into a canonical transformation. One may then fix the gauge freedom by simply imposing the AP gauge conditions translated to the new canonical variables [4]. In this way, one arrives again at the reduced model, but described in terms of the new scalar field $\bar{\psi}$,

$$ds^2 = \left(1 + \bar{\gamma}[\bar{\psi}] - \bar{\psi}\right)\left(-dT^2 + dR^2\right) + \frac{R^2 d\theta^2}{1+\bar{\psi}} + (1+\bar{\psi})dZ^2,$$

with $\bar{\gamma}[\bar{\psi}]$ a functional of $\bar{\psi}$ (and its momentum) [4]. This basic field and its canonical momentum are related to those of the AP formulation by means of the canonical transformation $\psi = \ln(1+\bar{\psi})$, $p_\psi = \bar{p}_{\bar{\psi}}(1+\bar{\psi})$. It is worth emphasizing that, whereas $\psi$ provides a free-field realization of the reduced ER model, the field $\bar{\psi}$ displays a much more complicated evolution [4]. On the contrary, while $\psi$ is non-linear in the metric excess with respect to Minkowski, the field $\bar{\psi}$ can be identified as the excess of the norm of the translational Killing vector $\partial_Z$.

On the other hand, the linearization of the AP conditions around the Minkowski solution (where, in particular, $r = R/8G$) adopts the expressions $\bar{\rho} = 0$; $\bar{p}_{\bar{\gamma}} = 0$. It has been recently proved [4] that these linearized conditions fix a valid gauge for the ER system in linearized gravity. With such a gauge, the reduced model that one attains in the linearized theory has the metric

$$ds_l^2 = (1-\bar{\psi})(-dT^2 + dR^2 + R^2 d\theta^2) + (1+\bar{\psi})dZ^2 \tag{2}$$

and, in this linearized theory, the cylindrically symmetric field $\bar{\psi}$ is free [4].

We are now in an adequate position to prove that the linearization of the AP conditions is in fact a de Donder gauge. Let $h_{\mu\nu} = g_{\mu\nu} - \eta_{\mu\nu}$ ($\mu, \nu = 1, ...4$) be the excess of the spacetime metric $g_{\mu\nu}$ around the Minkowski metric $\eta_{\mu\nu}$ and call $\bar{h}_{\mu\nu} = h_{\mu\nu} - \eta_{\mu\nu}h/2$, where $h = h_{\mu\nu}\eta^{\mu\nu}$. The de Donder (or Lorentz) gauge conditions are $\bar{h}_{\mu\nu,}{}^{\nu} = 0$ (in Cartesian coordinates), and provide an acceptable gauge fixing in linearized gravity, although they still leave some freedom in the choice of coordinates. The most straightforward way to see that conditions $\bar{\rho} = 0$ and $\bar{p}_{\bar{\gamma}} = 0$ determine a de Donder gauge is to compute $\bar{h}_{\mu\nu}$ for the reduced metric (2); one gets that the only non-vanishing component is $\bar{h}_{ZZ} = 2\bar{\psi}$. Since the field $\bar{\psi}$ is independent of $Z$, the considered gauge is then of de Donder type. Furthermore, a detailed study of the ER system in linearized gravity shows that the linearization of the AP gauge is indeed the only de Donder gauge which is compatible with the requirement of regularity of the metric at the symmetry axis, assumed to be located at the origin of the radial coordinate [4]. We therefore conclude that the AP gauge is a non-linear generalization to full cylindrical gravity of the de Donder gauge compatible with regularity .

## 4. Annihilation and creation variables

Once we have found the field redefinition that relates the AP and the perturbative descriptions of the ER model (with the same gauge fixation), we can proceed to discuss the connection between the annihilation and creation variables that are associated with each of these descriptions. For this, we restrict our attention to a section of constant time, e.g. $T = 0$, where we evaluate from now on our fields and momenta. Recalling that the considered descriptions adopt as basic canonical pairs, respectively, $(\psi, p_\psi)$ and $(\bar{\psi}, \bar{p}_{\bar{\psi}})$, a simple exercise shows that, in each of these cases, the following variables have the Poisson-bracket algebra of a set of particle-like variables:

$$A(k) := \int_0^\infty dR \, \frac{J_0(Rk)}{2\sqrt{4G}} \left[Rk\psi(R) + i8Gp_\psi(R)\right]; \quad A^\dagger(k) := A^*(k); \quad (3)$$

$$a(k) := \int_0^\infty dR \, \frac{J_0(Rk)}{2\sqrt{4G}} \left[Rk\bar{\psi}(R) + i8G\bar{p}_{\bar{\psi}}(R)\right]; \quad a^\dagger(k) := a^*(k). \quad (4)$$

Here, the star denotes complex conjugation. These definitions reproduce the annihilation and creation variables of the AP formulation and of the linearized ER model with (regular) de Donder gauge. In order to determine the relation between these two sets of variables, one can substitute in Eq. (4) the inverse of transformation $\psi = \ln(1+\bar{\psi})$, $p_\psi = \bar{p}_{\bar{\psi}}(1+\bar{\psi})$, and employ the expression of the AP canonical pair in terms of their corresponding particle-like variables:

$$\bar{\psi}(R) = e^{\psi(R)} - 1, \qquad \bar{p}_{\bar{\psi}}(R) = p_\psi(R)e^{-\psi(R)}; \quad (5)$$

$$\psi(R) = \sqrt{4G}\int_0^\infty dk J_0(Rk)[A(k) + A^\dagger(k)], \quad (6)$$

$$p_\psi(R) = i\int_0^\infty dk \frac{J_0(Rk)}{\sqrt{16G}} Rk[-A(k) + A^\dagger(k)]. \quad (7)$$

In this way, we get the expression of the variables $(a, a^\dagger)$ in terms of their AP counterpart, $(A, A^\dagger)$. The next natural step is to try to promote the particle-like variables of the perturbative description to operators in the AP quantization by replacing the variables $(A, A^\dagger)$ with annihilation and creation operators. We focus the analysis on the AP quantization because it is precisely this approach to the quantum theory which is known to be well-defined and under control for the ER model.

However, there exist some difficulties in promoting $(a, a^\dagger)$ to operators. In particular, to cope with the quantum analog of expressions (5), one needs to regularize the operator-valued distributions $\hat{\psi}(R)$ and $\hat{p}_\psi(R)$. We do this by introducing a regulator $g(k)$ corresponding to a generic cut-off $k_c$, as explained in Sec. 2, thus arriving to well-defined operators $\hat{\psi}(R|g)$ and $\hat{p}_\psi(R|g)$. Functions of the field $\psi$ such as its exponential can then be defined quantum mechanically by means of the spectral theorem. In addition, factor ordering ambiguities arise in the definition of the particle-like operators, although the conclusions of our discussion can be shown to be rather insensitive to them [4]. The annihilation and creation-like operators obtained in this manner will be called $\hat{a}(k|g)$ and $\hat{a}^\dagger(k|g)$. By construction, they are adjoint to each other. It is not difficult to calculate their commutators; although relatively complicated, they have the following noticeable properties [4]. Firstly, one can compute their limit when the gravitational constant $G$ vanishes and check that this reproduces the algebra of a set of annihilation and creation operators restricted to the region of wave-numbers under the cut-off. Secondly and most important, when the cut-off is removed (so that $g \to 1$), the commutators formally reproduce the algebra of a set of annihilation and creation operators (regardless of the value of $G$).

On the other hand, notice that, from the above expressions, the particle-like operators that we have introduced have a non-linear dependence in $\sqrt{G}$. At least formally, it is possible to expand them as power series in this constant, $\hat{a}(k|g) = \sum_{n=0}^{\infty}(G)^{n/2}\hat{a}_{(n)}(k|g)$. Increasing powers of $\sqrt{G}$ reflect contributions from a larger number of AP particles (with a certain factor ordering) [4]. In this sense, we can regard $\sqrt{G}$ as playing the role of interaction constant for the system. In the above expansion, the leading contribution coincides with the AP operator in the sector of particles with wave-numbers smaller than the cut-off, $\hat{a}_{(0)}(k|g) = g(k)\hat{A}(k)$. This is due to the fact that the canonical pair of the perturbative approach $(\bar{\psi}, \bar{p}_{\bar{\psi}})$ can be considered the linearization of the AP pair $(\psi, p_\psi)$ [see Eq. (5)], whose expressions are homogeneous in $\sqrt{G}$.

## 5. The perturbative vacuum

In principle, the perturbative vacuum may be reached by considering a state $|\bar{0}_g\rangle$ annihilated by all the introduced particle-like operators $\hat{a}(k|g)$ and taking the limit when the cut-off is removed. The state $|\bar{0}_g\rangle$ is totally determined from the above annihilation conditions and the requirement that it contains no contribution from AP particles with energies above that corresponding to the cut-off. In fact, taking into account the power series for $\hat{a}(k,g)$, $|\bar{0}_g\rangle$ can (at least formally) be written as the AP vacuum corrected by a perturbative series in $\sqrt{G}$, $|\bar{0}_g\rangle = |0\rangle + \sum_{n=1}^{\infty}(G)^{n/2}\,|\Phi_{n,g}\rangle$, where $|\Phi_{n,g}\rangle$ is a superposition of non-vacuum m-particle states with wave-numbers smaller than the cut-off.

The explicit form of $|\Phi_{n,g}\rangle$ can be determined by expanding the condition $\hat{a}(k|g)\,|\bar{0}_g\rangle = 0$ in powers of $\sqrt{G}$ and imposing each order independently. In this way, one can in particular show that $|\Phi_{1,g}\rangle$ is a sum of a three-particle state and a one-particle state [4]. Furthermore, although the one-particle state is partially affected by the factor ordering ambiguities that arise in the definition of $\hat{a}(k|g)$, the form of the three-particle contribution $|\Phi^{(3)}_{1,g}\rangle$ can be found unambiguously and exactly. The norm of such a state turns out to be given by [4]

$$\langle\Phi^{(3)}_{1,g}|\Phi^{(3)}_{1,g}\rangle = \int_0^{k_c}\frac{dk_1}{6\pi^2}\int_0^{k_c}dk_2\int_{|k_1-k_2|}^{\min\{k_c,k_1+k_2\}}\frac{(k_1+k_2+k_3)\;dk_3}{(k_1+k_2-k_3)(k_3-|k_1-k_2|)(k_3+|k_1-k_2|)}.$$

This integral diverges regardless of the value of the cut-off $k_c$ since the integrand is positive and there exists a simple pole at the boundary $k_3 = |k_1-k_2|$ of the integration region for $k_3$. Therefore, we see that the first-order correction to the AP vacuum is in fact a state of infinite norm. So, the vacuum of the perturbative approach is not accessible as a power series in $\sqrt{G}$ in the Fock space of the AP quantization.

It is worth remarking that the divergence of the norm of $|\Phi^{(3)}_{1,g}\rangle$ does not arise as the result of taking the limit in which the cut-off disappears. If that were the case, one could proceed to normalize first the perturbative vacuum $|\bar{0}_g\rangle$ for fixed regulator, and remove the cut-off afterwards. However, the divergence exists for all positive values of the cut-off; hence renormalization does not solve the problem.

This result is a clear indication of the inequivalence of the Fock quantizations of the two approaches and, in any case, proves that the perturbative vacuum is not analytic in $\sqrt{G}$. Hence, a standard perturbative quantum analysis fails.

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# Generalized $\mathcal{W}_\infty$ higher-spin algebras and quantization

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**Abstract.** The aim of this paper is to discuss the group theoretical structure underlying the $\mathcal{W}_\infty$ algebra, viewed as a tensor operator algebra of the group $SU(1,1)$. From this perspective, generalizations to arbitrary pseudo-unitary groups $U(N_+, N_-)$ are straightforward. The classical limit of these tensor operator algebras turns out to be Poisson and symplectic diffeomorphism algebras on flag/Kähler manifolds. The Poisson bracket can be obtained from the star-commutator of operator covariant symbols [the mean value of a tensor operator in a coherent state of $U(N_+, N_-)$] in the limit of large quantum numbers.

*Keywords:* Infinite-dimensional Lie algebras, Virasoro and $\mathcal{W}_\infty$ symmetries, Berezin and geometric quantization, coherent states, tensor operator symbols, classical limit, Poisson bracket, coadjoint orbit.

*2000 Mathematics Subject Classification:* 81R10, 17B65, 16S30, 81S10, 20C35, 81R15, 81R30

## 1. Introduction

In the last decade, a large body of literature has been devoted to the study of $\mathcal{W}$-algebras, and the subject still continues to be fruitful. These algebras were first introduced as higher-conformal-spin $s > 2$ extensions of the Virasoro algebra ($s = 2$) through the operator product expansion of the stress-energy tensor and primary fields in two-dimensional conformal field theory. $\mathcal{W}$-algebras have been widely used as a useful tool in two-dimensional physical systems. Only when all ($s \to \infty$) conformal spins $s \geq 2$ are considered, the algebra (denoted by $\mathcal{W}_\infty$) is proven to be of Lie type. The process of elucidating the mathematical structure underlying $\mathcal{W}$ algebras has led to various directions. On the one hand, geometric approaches identify the classical ($\hbar \to 0$) limit $w_\infty$ of $\mathcal{W}_\infty$ algebras with area-preserving (symplectic) diffeomorphism algebras of two dimensional surfaces (cone and hyperboloid). On the other hand, group-theoretic approaches identify $\mathcal{W}_\infty$ algebras as particular members of a one-parameter family $\mathcal{W}_\infty(c)$ of non-isomorphic infinite-dimensional Lie-algebras of $SU(1,1)$ tensor operators. Going from three-dimensional algebras $su(2)$ and $su(1,1)$ to higher-dimensional pseudo-unitary algebras $su(N_+,N_-)$ entails non-trivial problems. Some steps in this direction have been already done in Ref. [1]. Here we just want to discuss the basic mathematical structures underlying all these constructions without entering into concrete details (we address the interested reader to Ref. [1])

## 2. Underlying group-theoretic structures

Let us start by fixing notation and reminding some definitions and results on group, tensor operator, Poisson-Lie algebras, coherent states and symbols of a Lie group $G$; in particular, we shall focus on pseudo-unitary groups:

$$G = U(N_+,N_-) = \{g \in M_{N\times N}(\mathbb{C}) \ / \ g\Lambda g^\dagger = \Lambda\}, \ \ N = N_+ + N_-, \qquad (1)$$

that is, groups of complex $N \times N$ matrices $g$ that leave invariant the indefinite metric $\Lambda = \mathrm{diag}(1, \overset{N_+}{\ldots}, 1, -1, \overset{N_-}{\ldots}, -1)$, which is used to raise and lower indices. The Lie-algebra $\mathcal{G} = u(N_+,N_-)$ is generated by the step operators $\hat{X}_\alpha^\beta$ with commutation relations $\left[\hat{X}_{\alpha_1}^{\beta_1}, \hat{X}_{\alpha_2}^{\beta_2}\right] = \hbar(\delta_{\alpha_2}^{\beta_1}\hat{X}_{\alpha_1}^{\beta_2} - \delta_{\alpha_1}^{\beta_2}\hat{X}_{\alpha_2}^{\beta_1})$ (we introduce the Planck constant $\hbar$ for convenience to discuss the classical limit). There is a standard *oscillator realization* of these step operators in terms of $N$ boson operator variables $(\hat{a}_\alpha^\dagger, \hat{a}^\beta)$, given by: $\hat{X}_\alpha^\beta = \hat{a}_\alpha^\dagger \hat{a}^\beta$, with $[\hat{a}^\beta, \hat{a}_\alpha^\dagger] = \hbar\delta_\alpha^\beta \mathbb{I}, \ \ \alpha, \beta = 1, \ldots N$. Thus, for unitary irreducible representations of $U(N_+,N_-)$ we have the conjugation relation: $(\hat{X}_\alpha^\beta)^\dagger = \Lambda^{\beta\mu}\hat{X}_\mu^\nu\Lambda_{\nu\alpha}$ (sum over doubly occurring indices is understood unless otherwise stated).

**Definition 1.** Let $\mathcal{G}^\otimes$ be the tensor algebra over $\mathcal{G}$, and $\mathcal{I}$ the ideal of $\mathcal{G}^\otimes$ generated by $[\hat{X}, \hat{Y}] - (\hat{X} \otimes \hat{Y} - \hat{Y} \otimes \hat{X})$ where $\hat{X}, \hat{Y} \in \mathcal{G}$. The universal enveloping algebra $\mathcal{U}(\mathcal{G})$ is the quotient $\mathcal{G}^\otimes/\mathcal{I}$. [From now on we shall drop the $\otimes$ symbol in writing tensor products]

**Theorem 1** (Poincaré-Birkhoff-Witt)**.** *The monomials* $\hat{X}_{\alpha_1\beta_1}^{k_1} \ldots \hat{X}_{\alpha_n\beta_n}^{k_n}$, *with* $k_i \geq 0$, *form a basis of* $\mathcal{U}(\mathcal{G})$.

Casimir operators are especial elements of $\mathcal{U}(\mathcal{G})$, which commute with everything. There are $N$ Casimir operators for $U(N_+,N_-)$, which are written as polynomials of degree $1, 2, \ldots, N$ of step operators as follows:

$$\hat{C}_1 = \hat{X}^\alpha_\alpha, \;\; \hat{C}_2 = \hat{X}^\beta_\alpha \hat{X}^\alpha_\beta, \;\; \hat{C}_3 = \hat{X}^\beta_\alpha \hat{X}^\gamma_\beta \hat{X}^\alpha_\gamma, \ldots \tag{2}$$

The universal enveloping algebra $\mathcal{U}(\mathcal{G})$ decomposes into *factor or quotient Lie algebras* $\mathcal{W}_c(\mathcal{G})$ as follows:

**Theorem 2.** *Let* $\mathcal{I}_c = \prod_{\alpha=1}^{N}(\hat{C}_\alpha - \hbar^\alpha c_\alpha)\mathcal{U}(\mathcal{G})$ *be the ideal generated by the Casimir operators* $\hat{C}_\alpha$*. The quotient* $\mathcal{W}_c(\mathcal{G}) = \mathcal{U}(\mathcal{G})/\mathcal{I}_c$ *is a Lie algebra (roughly speaking, this quotient means that we replace* $\hat{C}_\alpha$ *by the complex c-number* $C_\alpha \equiv \hbar^\alpha c_\alpha$ *whenever it appears in the commutators of elements of* $\mathcal{U}(\mathcal{G})$*). We shall refer to* $\mathcal{W}_c(\mathcal{G})$ *as a c-tensor operator algebra.*

According to Burnside's theorem, for some critical values $c_\alpha = c_\alpha^{(0)}$, the infinite-dimensional Lie algebra $\mathcal{W}_c(\mathcal{G})$ "collapses" to a finite-dimensional one. In a more formal language:

**Theorem 3** (Burnside)**.** *When* $c_\alpha, \alpha = 1, \ldots, N$ *coincide with the eigenvalues of* $\hat{C}_\alpha$ *in a* $d_c$*-dimensional irrep* $D_c$ *of* $G$*, there exists an ideal* $\chi \subset \mathcal{W}_c(\mathcal{G})$ *such that* $\mathcal{W}_c(\mathcal{G})/\chi = sl(d_c, \mathbb{C})$*, or* $su(d_c)$*, by taking a compact real form of the complex Lie algebra.*

For other values of $c$ the algebra $\mathcal{W}_c(\mathcal{G})$ is, in general, infinite-dimensional. For example, the standard $\mathcal{W}_\infty$ and $\mathcal{W}_{1+\infty}$ algebras appear to be distinct members ($c = 0$ and $c = -1/4$ cases, respectively) of the one-parameter family $\mathcal{W}_c(su(1,1))$ of non-isomorphic infinite-dimensional factor Lie-algebras of $SU(1,1)$ tensor operators. It is also precisely for the specific value of $c = 0$ (resp. $c = -\frac{1}{4}$) that $\mathcal{W}_c(su(1,1))$ can be consistently truncated to a closed algebra containing only those generators $\hat{L}^s_m$ with positive "conformal-spins" $s = 2, 3, 4, \ldots$ (resp. $s = 1, 2, 3, \ldots$) and "conformal-dimension" $m \in \mathbb{Z}$.

Another interesting structure related to the previous one is the *group* $C^*$*-algebra* $\mathbb{C}^*(G)$ [in order to avoid some technical difficulties, let us restrict ourselves to the compact $G$ case in the next discussion]:

**Definition 2.** Let $C^\infty(G)$ be the set of analytic complex functions $\Psi$ on $G$ ,

$$C^\infty(G) = \{\Psi : G \to \mathbb{C}, \;\; g \mapsto \Psi(g)\}\,. \tag{3}$$

The group algebra $\mathbb{C}^*(G)$ is a $C^*$-algebra with an invariant associative *-product (convolution product):

$$(\Psi * \Psi')(g') \equiv \int_G d^L g\, \Psi(g)\Psi'(g^{-1} \bullet g'), \tag{4}$$

($g \bullet g'$ denotes the composition group law and $d^L g$ stands for the left Haar measure) and an involution $\Psi^*(g) \equiv \bar{\Psi}(g^{-1})$.

The dual space $R(G)$ of $C^\infty(G)$ consists of all generalized functions with compact supports. The space $M_0(G)$ of all regular Borel measures with compact support is a subspace of $R(G)$. The set $R(G,H)$ of all generalized functions on $G$ with compact supports contained in a subgroup $H$ also forms a subspace of $R(G)$. The following theorem reveals a connection between $R(G,\{e\})$ [$e\in G$ denotes the identity element] and the enveloping algebra:

**Theorem 4** (L. Schwartz)**.** *The algebra $R(G,\{e\})$ is isomorphic to the enveloping algebra $\mathcal{U}(\mathcal{G})$.*

This isomorphism is apparent when we realize the Lie algebra $\mathcal{G}$ by left invariant vector fields $\hat{X}^L$ on $G$ and consider the mapping $\Phi:\mathcal{G}\to R(G), \hat{X}\mapsto \Phi_{\hat{X}}$, defined by the formula $\langle\Phi_{\hat{X}}|\Psi\rangle \equiv (\hat{X}^L\Psi)(e), \quad \forall\Psi\in C^\infty(G)$, where $\langle\Phi|\Psi\rangle\equiv\int_G d^Lg\,\bar{\Phi}(g)\Psi(g)$ denotes a scalar product and $(\hat{X}^L\Psi)(e)$ means the action of $\hat{X}^L$ on $\Psi$ restricted to the identity element $e\in G$. One can also verify the relation $\langle\Phi_{\hat{X}_1}*\cdots*\Phi_{\hat{X}_n}|\Psi\rangle = (\hat{X}^L_1\ldots\hat{X}^L_n\Psi)(e)$, $\forall\Psi\in C^\infty(G)$, between star products in $R(G)$ and tensor products in $\mathcal{U}(\mathcal{G})$:

## 3. Underlying geometric structures. Classical Limit

Let us comment now on the geometric counterpart of the previous algebraic structures, by using the language of Geometric Quantization and recalling the Kostant-Kirillov-Souriau construction of the symplectic structure on the orbits of the coadjoint (ir-)representations of $G$.

The classical limit $\lim_{\hbar\to 0}(i/\hbar^2)[\Psi,\Psi'](g)$ of the convolution commutator $[\Psi,\Psi']=\Psi*\Psi'-\Psi'*\Psi$ corresponds to the Poisson-Lie bracket

$$\{\psi,\psi'\}_{PL}(g)=i(\Lambda_{\alpha_2\beta_1}x_{\alpha_1\beta_2}-\Lambda_{\alpha_1\beta_2}x_{\alpha_2\beta_1})\frac{\partial\psi}{\partial x_{\alpha_1\beta_1}}\frac{\partial\psi'}{\partial x_{\alpha_2\beta_2}} \tag{5}$$

between smooth functions $\psi\in C^\infty(\mathcal{G}^*)$ on the coalgebra $\mathcal{G}^*$, where $x_{\alpha\beta},\alpha,\beta=1,\ldots,N$ denote a coordinate system in the coalgebra $\mathcal{G}^*=u(N_+,N_-)^*\simeq\mathbb{R}^{N^2}$, seen as a $N^2$-dimensional vector space. The "quantization map" relating $\Psi$ and $\psi$ is symbolically given by the expression:

$$\Psi(g)=\int_{\mathcal{G}^*}\frac{d^{N^2}\Theta}{(2\pi\hbar)^{N^2}}e^{\frac{i}{\hbar}\Theta(\hat{X})}\psi(\Theta), \tag{6}$$

where $g=\exp(\hat{X})=\exp(x^{\alpha\beta}\hat{X}_{\alpha\beta})$ is an element of $G$ and $\Theta=\theta_{\alpha\beta}\Theta^{\alpha\beta}$ is an element of $\mathcal{G}^*$. The constraints $\hat{C}_\alpha(x)=C_\alpha=\hbar^\alpha c_\alpha$ defined by the Casimir operators (2) (written in terms of the coordinates $x_{\alpha\beta}$ instead of $\hat{X}_{\alpha\beta}$) induce a foliation $\mathcal{G}^*\simeq\bigcup_C\mathbb{O}_C$ of the coalgebra $\mathcal{G}^*$ into leaves $\mathbb{O}_C$: coadjoint orbits, algebraic (flag) manifolds (see e.g. [1]). This foliation is the (classical) analogue of the (quantum) standard Peter-Weyl decomposition of the group algebra $\mathbb{C}^*(G)$:

**Theorem 5** (Peter-Weyl)**.** *Let $G$ be a compact Lie group. The group algebra $\mathbb{C}^*(G)$ decomposes $\mathbb{C}^*(G) \simeq \bigoplus_{c\in\hat{G}} \mathcal{W}_c(\mathcal{G})$ into factor algebras $\mathcal{W}_c(\mathcal{G})$, where $\hat{G}$ denotes the space of all (equivalence classes of) irreducible representations of $G$ of dimension $d_c$.*

The leaves $\mathbb{O}_C$ admit a symplectic structure $(\mathbb{O}_C, \Omega_C)$, where $\Omega_C$ denotes a closed 2-form (a Kähler form), which can be obtained from a Kähler potential $K_C$ as:

$$\Omega_C(z,\bar{z}) = \frac{\partial^2 K_C(z,\bar{z})}{\partial z_{\alpha\beta}\partial \bar{z}_{\sigma\nu}} dz_{\alpha\beta}\wedge d\bar{z}_{\sigma\nu} = \Omega_C^{\alpha\beta;\sigma\nu}(z,\bar{z}) dz_{\alpha\beta}\wedge d\bar{z}_{\sigma\nu}, \tag{7}$$

where $z_{\alpha\beta}$, $\alpha > \beta$ denotes a system of complex coordinates in $\mathbb{O}_C$ (see [1]). After the foliation of $C^\infty(\mathcal{G}^*)$ into Poisson algebras $C^\infty(\mathbb{O}_C)$, the Poisson bracket induced on the leaves $\mathbb{O}_C$ becomes:

$$\begin{aligned} \{\psi_l^c, \psi_m^c\}_P(z,\bar{z}) &= \sum_{\alpha_j>\beta_j} \Omega^C_{\alpha_1\beta_1;\alpha_2\beta_2}(z,\bar{z}) \frac{\partial \psi_l^c(z,\bar{z})}{\partial z_{\alpha_1\beta_1}} \frac{\partial \psi_m^c(z,\bar{z})}{\partial \bar{z}_{\alpha_2\beta_2}} = \\ &= \sum_n f_{lm}^n(c)\psi_n^c(z,\bar{z}). \end{aligned} \tag{8}$$

The structure constants $f_{lm}^n(c)$ can be obtained through the scalar product $f_{lm}^n(c) = \langle \psi_n^c | \{\psi_l^c, \psi_m^c\}_P\rangle$, with integration measure $d\mu_C(z,\bar{z}) \propto \Omega_C^n(z,\bar{z}), 2n = \dim(\mathbb{O}_C)$, when the set $\{\psi_n^c\}$ is chosen to be orthonormal.

To each function $\psi \in C^\infty(\mathbb{O}_C)$, one can assign its Hamiltonian vector field $H_\psi \equiv \{\psi, \cdot\}_P$, which is divergence-free and preserves de natural volume form $d\mu_C(z,\bar{z})$. In general, any vector field $H$ obeying $L_H\Omega = 0$ (with $L_H \equiv i_H \circ d + d \circ i_H$ the Lie derivative) is called locally Hamiltonian. The space $\mathrm{LHam}(\mathbb{O})$ of locally Hamiltonian vector fields is a subalgebra of the algebra $\mathrm{sdiff}(\mathbb{O})$ of symplectic (volume-preserving) diffeomorphisms of $\mathbb{O}$, and the space $\mathrm{Ham}(\mathbb{O})$ of Hamiltonian vector fields is an ideal of $\mathrm{LHam}(\mathbb{O})$. The two-dimensional case $\dim(\mathbb{O}) = 2$ is special because $\mathrm{sdiff}(\mathbb{O}) = \mathrm{LHam}(\mathbb{O})$, and the quotient $\mathrm{LHam}(\mathbb{O})/\mathrm{Ham}(\mathbb{O})$ can be identified with the first de-Rham cohomology class $H^1(\mathbb{O},\mathbb{R})$ of $\mathbb{O}$ via $H \mapsto i_H\Omega$.

Poisson and symplectic diffeomorphism algebras of the orbits $\mathbb{O}_{C_+} \equiv S^2$ and $\mathbb{O}_{C_-} \equiv H^2$ (the sphere and the hyperboloid of $SU(2)$ and $SU(1,1)$, respectively) appear as the classical limit of factor algebras $\mathcal{W}_{c_+}(su(2))$ and $\mathcal{W}_{c_-}(su(1,1))$, respectively (see [1]). The classical limit means small $\hbar$ and large $su(2)$ (resp. $su(1,1)$) Casimir eigenvalues $c_\pm = s(s\pm 1)$ [$s$ denotes the (conformal) spin], so that the curvature radius $C_\pm = \hbar^2 c_\pm$ remains finite. Let us clarify these classical limits by making use of the *operator (covariant) symbols*: $L^c(z,\bar z) \equiv \langle cz|\hat L|cz\rangle$, $\hat L \in \mathcal{W}_c(\mathcal{G})$, constructed as the mean value of an operator $\hat L \in \mathcal{W}_c(\mathcal{G})$ in the coherent state $|cz\rangle$ (see Ref. [1] for a thorough discussion on coherent states of $U(N_+,N_-)$). Using the resolution of unity:

$$\int_{\mathbb{O}_C} |cu\rangle\langle cu| d\mu_C(u,\bar u) = 1 \tag{9}$$

for coherent states, one can define the so called *star multiplication of symbols* $L_1^c \star L_2^c$ as the symbol of the product $\hat L_1 \hat L_2$ of two operators $\hat L_1$ and $\hat L_2$:

$$(L_1^c \star L_2^c)(z,\bar z) \equiv \langle cz|\hat L_1 \hat L_2|cz\rangle = \int_{\mathbb{O}_C} L_1^c(z,\bar u) L_2^c(u,\bar z) e^{-s_c^2(z,u)} d\mu_c(u,\bar u), \tag{10}$$

where we introduce the non-diagonal symbols $L^c(z,\bar u) = \langle cz|\hat L|cu\rangle/\langle cz|cu\rangle$ and the overlap $s_c^2(z,u) \equiv -\ln|\langle cz|cu\rangle|^2$ can be interpreted as the square of the distance between the points $z,u$ on the coadjoint orbit $\mathbb{O}_C$. Using general properties of coherent states, it can be easily seen that $s_c^2(z,u) \geq 0$ tends to infinity with $c \to \infty$, if $z \neq u$, and equals zero if $z = u$. Thus, one can conclude that, in that limit, the domain $u \approx z$ gives only a contribution to the integral (10). Decomposing the integrand near the point $u = z$ and going to the integration over $w = u - z$, it can be seen that the Poisson bracket (8) provides the first order approximation to the star commutator for large quantum numbers $c$ (small $\hbar$); that is:

$$L_1^c \star L_2^c - L_2^c \star L_1^c = i\,\{L_1^c, L_2^c\}_P + \mathrm{O}(1/c_\alpha), \tag{11}$$

i.e. the quantities $1/c_\alpha \sim \hbar^\alpha$ (inverse Casimir eigenvalues) play the role of the Planck constant $\hbar$ to the $\alpha$, and one uses that $ds_c^2 = \Omega_C^{\alpha\beta;\sigma\nu} dz_{\alpha\beta} d\bar z_{\sigma\nu}$ (Hermitian Riemannian metric on $\mathbb{O}_C$). We address the reader to Ref. [1] for more details.

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# Triple structures on a manifold

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**Abstract.** Many geometric concepts can be defined by a suitable algebraic formalism. This point of view has interest because one can compare different geometric structures having similar algebraic expressions. In the present paper we study manifolds endowed with three (1,1)-tensor fields $F$, $P$ and $J$ satisfying

$$F^2 = \pm I,\ P^2 = \pm I,\ J = P \circ F,\ P \circ F \pm F \circ P = 0.$$

We analyze the geometries arising from the above algebraic conditions. According to the chosen signs there exist eight different geometries. We show that, in fact, there are only four. In particular, hypercomplex manifolds and manifolds endowed with a 3-web fit in this construction.We study geometric objects associated to these manifolds (such $G$-structures, connections, etc.), restrictions on dimensions, etc., and we show significative examples.

*Keywords:* biparacomplex structure, hyperproduct structure, bicomplex structure, hypercomplex structure, canonical connection, Obata connection

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## 1. Introduction

A *triple structure* on a manifold $M$ is given by three tensor fields of type (1,1), $F$, $P$ and $J$ satisfying the relations:

*i*) $F^2 = \epsilon_1 I$, $P^2 = \epsilon_2 I$, $J^2 = \epsilon_3 I$, where $\epsilon_1, \epsilon_2, \epsilon_3 \in \{-1, +1\}$ and $I$ stands for the identity map, *i.e.*, each one of them is an almost product or almost complex structure.

*ii*) $P \circ F \pm F \circ P = 0$, *i.e.*, they commute or anti-commute.

*iii)* $J = P \circ F$.

These relations allow us to define four types of triple structures on a manifold, which provide four different geometries:

1. *Almost biparacomplex structure*: $F^2 = I$, $P^2 = I, P \circ F + F \circ P = 0$.
2. *Almost hyperproduct structure*: $F^2 = I$, $P^2 = I, P \circ F - F \circ P = 0$.
3. *Almost bicomplex structure*: $F^2 = -I$, $P^2 = -I, P \circ F - F \circ P = 0$.
4. *Almost hypercomplex structure*: $F^2 = -I$, $P^2 = -I, P \circ F + F \circ P = 0$.

One can expect to define four other triple structures:

5. $F^2 = I$, $P^2 = -I, P \circ F + F \circ P = 0$; in this case, $J^2 = I$.
6. $F^2 = I$, $P^2 = -I, P \circ F - F \circ P = 0$; in this case, $J^2 = -I$.
7. $F^2 = -I$, $P^2 = I, P \circ F + F \circ P = 0$; in this case, $J^2 = I$.
8. $F^2 = -I$, $P^2 = I, P \circ F - F \circ P = 0$; in this case, $J^2 = -I$,

but one can prove that the conditions 1, 5 and 7, and 3, 6 and 8 define the same triple structures. Our main aim is to compare the four geometries defined by the four previous triple structures, which have similar algebraic definitions.

Three of these triple structures have been studied in other works. For example, D. V. Alekseevsky and S. Marchiafava study almost hypercomplex and almost quaternionic structures in [1]. V. Crucenanu introduces the almost hyperproduct structures on a manifold in [3]. In this paper, he also studies metrics and connections attached to these structures. Almost biparacomplex structures have been deeply studied in [9]. Examples of almost biparacomplex structures on manifolds and adapted metrics can be found in [8]. Recently, in [4], the authors of the present work have established links among these structures, bi-Lagrangian manifolds and symplectic ones.

We will dedicate the next sections to show the principal objects attached to every type of triple structure. We focus our attention on the $G$-structure defined by the triple structure and the existence of functorial connections associated to them. The work [6] is a survey about this topic. At the end of every section we show several examples of the triple structures studied.

Let $(F, P, J)$ be a triple structure. We say that $(F, P, J)$ is *integrable* if the Nijenhuis tensor of the three tensor fields vanishes, $N_F = N_P = N_J = 0$.

**Notation.** If $H^2 = I$, we denote by $T_H^+(M)$ and $T_H^-(M)$ the distributions on $M$ defined by the eigenvectors of $H_p$ on $T_p(M)$ for every $p \in M$, by $H^+$ and $H^-$ the projections over these distributions. Also we denote by $\mathrm{Tor}_\nabla$ and $\mathrm{R}_\nabla$ the torsion and the curvature tensors of the connection $\nabla$, by $F(M)$ the principal bundle of linear frames. Definitions and basic results on functorial connections can be found in [6].

## 2. Almost biparacomplex structures

**Definition 1.** A manifold $M$ has an *almost biparacomplex structure* if it is endowed with two tensor fields $F$, $P$, satisfying $F^2 = P^2 = I, P \circ F + F \circ P = 0$.

In this case, $J^2 = -I$ and $M$ has got three equidimensional supplementary distributions defined by: $V_1 = T_F^+(M), V_2 = T_F^-(M), V_3 = T_P^+(M)$, with $J(V_1) = V_2$. Then $\dim M = 2n$, with $\dim V_i = n$, $i = 1, 2, 3$.

An almost biparacomplex structure $(F, P, J)$ on $M$ defines the following subbundle of $F(M)$:

$$\mathcal{B} = \bigcup_{p \in M} \left\{ \begin{array}{c} (X_1, \ldots, X_n, Y_1, \ldots, Y_n) \in F_p(M) \\ \{X_1, \ldots, X_n\} \text{ basis of } V_1(p), \{Y_1, \ldots, Y_n\} \text{ basis of } V_2(p), \\ \{X_1 + Y_1, \ldots, X_n + Y_n\} \text{ basis of } V_3(p) \end{array} \right\},$$

which has the following structural group

$$\Delta GL(n; \mathbb{R}) = \left\{ \begin{pmatrix} A & 0 \\ 0 & A \end{pmatrix} : A \in GL(n; \mathbb{R}) \right\}.$$

The Lie algebra of this group is invariant under matrix transpositions and its first prolongation vanishes; then $\Delta GL(n; \mathbb{R})$ admits functorial connections.

The connection characterized by the conditions: $\nabla F = 0$, $\nabla P = 0$, and $\mathrm{Tor}_\nabla(F^+X, F^-Y) = 0$, for every $X, Y$ vector fields on $M$, is a functorial connection. This connection is called the *canonical connection* of $(F, P, J)$.

The canonical conection characterizes the integrability of the almost biparacomplex structure $(F, P, J)$ and the $\Delta GL(n; \mathbb{R})$-structure: $(F, P, J)$ is integrable if and only if $\mathrm{Tor}_\nabla = 0$, and, the $\Delta GL(n; \mathbb{R})$-structure is integrable if and only if $\mathrm{Tor}_\nabla = 0, \mathrm{R}_\nabla = 0$ (see the details in [9]).

Manifolds endowed with a 3-web, the tangent bundle of a manifold, and Lie groups of even dimension are examples of manifolds which admit almost biparacomplex structures.These examples are carefully explained in [8] and [9].

## 3. Almost hyperproduct structures

**Definition 2.** A manifold $M$ has an *almost hyperproduct structure* if it is endowed with two tensor fields $F$, $P$, satisfying $F^2 = P^2 = I, P \circ F - F \circ P = 0$.

In this case, $J^2 = I$. The manifold has got six distributions defined by eigenvectors of $F$, $P$ and $J$: $V_1 = T_F^+(M)$, $V_2 = T_F^-(M)$, $V_3 = T_P^+(M)$, $V_4 = T_P^-(M)$, $V_5 = T_J^+(M)$, $V_6 = T_J^-(M)$. Then, one can define the following four distributions: $V_{13} = V_1 \cap V_3$, $V_{14} = V_1 \cap V_4$, $V_{23} = V_2 \cap V_3$, $V_{24} = V_2 \cap V_4$, with $\dim V_{ij} = n_{ij}, i \in \{1,2\}, j \in \{3,4\}$.

One almost hyperproduct structure $(F, P, J)$ on $M$ defines the following subbundle of $F(M)$:

$$\mathcal{H} = \bigcup_{p \in M} \left\{ \begin{array}{c} (X_1, \dots, X_{n_{13}}, Y_1, \dots, Y_{n_{14}}, U_1, \dots, U_{n_{23}}, W_1, \dots, W_{n_{24}}) \in F_p(M) \\ \{X_1, \dots, X_{n_{13}}\} \text{ basis of } V_{13}(p), \{Y_1, \dots, Y_{n_{14}}\} \text{ basis of } V_{14}(p), \\ \{U_1, \dots, U_{n_{23}}\} \text{ basis of } V_{23}(p), \{W_1, \dots, W_{n_{24}}\} \text{ basis of } V_{24}(p) \end{array} \right\}.$$

Therefore $\dim M = n_{13} + n_{14} + n_{23} + n_{24} = n$ and there are no restrictions over the dimension of the manifold $M$.

The structural group of $\mathcal{H}$, $\Delta(n_{13}, n_{14}, n_{23}, n_{24}; \mathbb{R})$, is the image of the product group $GL(n_{13}; \mathbb{R}) \times GL(n_{14}; \mathbb{R}) \times GL(n_{23}; \mathbb{R}) \times GL(n_{24}; \mathbb{R})$ by the diagonal inmersion in $GL(n; \mathbb{R})$. The Lie algebra of this group is

$$\Delta_*(n_{13}, n_{14}, n_{23}, n_{24}; \mathbb{R}) = \left\{ \begin{pmatrix} A & 0 & 0 & 0 \\ 0 & B & 0 & 0 \\ 0 & 0 & C & 0 \\ 0 & 0 & 0 & D \end{pmatrix} : \begin{array}{l} A \in \mathfrak{gl}(n_{13}; \mathbb{R}), \\ B \in \mathfrak{gl}(n_{14}; \mathbb{R}), \\ C \in \mathfrak{gl}(n_{23}; \mathbb{R}), \\ D \in \mathfrak{gl}(n_{24}; \mathbb{R}), \end{array} \right\}.$$

The first prolongation of $\Delta_*(n_{13}, n_{14}, n_{23}, n_{24}; \mathbb{R})$ does not vanish; therefore, the Lie group $\Delta(n_{13}, n_{14}, n_{23}, n_{24}; \mathbb{R})$ does not admit functorial connections.

The tensor fields of $\mathbb{R}^4$ defined in the standard basis by

$$F = \begin{pmatrix} 1 & 0 & 0 & 0 \\ 0 & 1 & 0 & 0 \\ 0 & 0 & -1 & 0 \\ 0 & 0 & 0 & -1 \end{pmatrix}, \quad P = \begin{pmatrix} 1 & 0 & 0 & 0 \\ 0 & -1 & 0 & 0 \\ 0 & 0 & 1 & 0 \\ 0 & 0 & 0 & -1 \end{pmatrix}$$

provide an elemental example of an almost hyperproduct structure in $\mathbb{R}^4$.

## 4. Almost bicomplex structures

**Definition 3.** A manifold $M$ has an *almost bicomplex structure* if it is endowed with two tensor fields $F$, $P$, satisfying $F^2 = P^2 = -I, P \circ F - F \circ P = 0$.

In this case, $J^2 = I$ and $M$ has got two distributions: $V_1 = T_J^+(M), V_2 = T_J^-(M)$. One can prove that $F_p(V_i(p)) = V_i(p)$, $\forall p \in M$, $i = 1, 2$, and then the two distributions are even-dimensional, but these dimensions do not coincide in general. Therefore $M$ is also even-dimensional: $\dim M = 2(r+s)$, with $\dim V_1 = 2r,\ \dim V_2 = 2s$.

An almost bicomplex structure $(F, P, J)$ on $M$ defines the following subbundle of $F(M)$:

$$\mathcal{BC} = \bigcup_{p\in M} \left\{ \begin{array}{c} (X_1, \dots, X_r, PX_1, \dots, PX_r, Y_1, \dots, Y_s, PY_1, \dots, PY_s) \in F_p(M) \\ \{X_1, \dots, X_r, PX_1, \dots PX_r\} \,\text{basis of}\, V_1(p), \\ \{Y_1, \dots, Y_s, PY_1, \dots, PY_s\} \,\text{basis of}\, V_2(p), \end{array} \right\},$$

The structural group is

$$\Delta(r, s; \mathbb{C}) = \left\{ \begin{pmatrix} A & 0 \\ 0 & B \end{pmatrix} : A \in GL(r; \mathbb{C}), B \in GL(s; \mathbb{C}) \right\}.$$

The Lie algebra of this group, $\Delta_*(r, s; \mathbb{C})$, is a subalgebra of $\mathfrak{gl}(r+s; \mathbb{C})$ which can be considered as a Lie subalgebra of $\mathfrak{gl}(2(r+s); \mathbb{R})$ by means of the real representation of $GL(r+s; \mathbb{C})$. The first prolongation of $\Delta_*(r, s; \mathbb{C})$ does not vanish, and then, the Lie group $\Delta(r, s; \mathbb{C})$ does not admit functorial connections.

This triple structure appears naturally on the product of manifolds $M$ and $N$, $M \times N$, endowed with an almost complex structure $J_M$ and $J_N$ respectively. One can easily prove that $M \times N$ is an almost bicomplex manifold when one considers the tensor fields $F$ and $P$ defined by:

$$\begin{aligned} F(X,0) &= (-J_M(X), 0), \quad & P(X,0) &= (J_M(X), 0), \quad & \forall X \in \mathfrak{X}(M), \\ F(0,Y) &= (0, J_N(Y)), \quad & P(0,Y) &= (0, J_N(Y)), \quad & \forall Y \in \mathfrak{X}(N), \end{aligned}$$

is an almost bicomplex structure on $M \times N$. Note that $J = P \circ F$ is the canonical almost product structure of $M \times N$. Recently, this structures have appeared in the study of bi-Hamiltonian systems (see [5]).

## 5. Almost hypercomplex structures

**Definition 4.** A manifold $M$ has an *almost hypercomplex structure* if it is endowed with two tensor fields $F$, $P$ satisfying $F^2 = P^2 = -I$, $P \circ F + F \circ P = 0$.

In this case, $J^2 = -I$ and neccesarily $\dim M = 4n$.

An almost hipercomplex structure $(F, P, J)$ on $M$ defines the following subbundle of $F(M)$:

$$\mathcal{HC} = \bigcup_{p\in M} \left\{ \begin{array}{c} (X_1, FX_1, PX_1, JX_1, \dots, X_n, FX_n, PX_n, JX_n) \in F_p(M) \\ \{X_1, \dots, X_n\} \,\text{linearly independent in}\, T_p(M) \end{array} \right\},$$

whose structural group is the linear general group over the field of quaternions, $GL(n;\mathbb{H})$.

Alekseevsky and Marchiafava have proved that the first prolongation of the Lie algebra of the group $GL(n;\mathbb{H})$ vanishes. They attached to each $GL(n;\mathbb{H})$-structure one $\mathcal{D}$-connection, which is a particular case of functorial connection with a condition over the torsion tensor (see the details in [1]). This connection is called the *Obata connection* of the almost hypercomplex structure.

As in the case of the canonical connection of an almost biparacomplex structure, this connection allows to characterize the integrability of the almost hypercomplex structure and the $GL(n;\mathbb{H})$-structure associated. One has: the triple structure $(F, P, J)$ is integrable if and only if the torsion tensor of the Obata connection vanishes, and the $GL(n;\mathbb{H})$-structure is integrable if and only if the Obata connection is locally flat.

A lot of examples, such as K3-surfaces and Kählerian complex 2-dimensional torus, can be found, *e.g.*, in [2] and [7].

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# Vector hulls and jet bundles

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**Abstract.** Every affine space $A$ can be canonically immersed as a hyperplane into a vector space $\hat{A}$, which is called the vector hull of $A$. This fact is greatly clarifying, both for affine geometry and for its applications. It can also be extended to affine bundles. We will study here some aspects of this construction and we will compute the vector hull of some jet bundles.

*Keywords:* vector hull, affine space, affine bundle, jet bundle.

*2000 Mathematics Subject Classification:* 55R25, 58A20, 58A32

## 1. Introduction

Any affine space has a canonical immersion, as a hyperplane, in a vector space; with it, affine maps can be understood as linear maps. This construction has been studied in several places, and has applications to different fields [1–9]. Among these applications, we can point out: the $(n+1)$-dimensional linear representation of the affine group in $n$ dimensions, barycentric calculus in affine spaces and the projective completion of an affine space, computer-aided geometric design, the study of solid mechanics, and the geometric description of time-dependent mechanics. In this paper we will study some aspects of this construction, both for affine spaces and affine bundles. We will also show that the vector hull of some interesting affine bundles can be identified with well-known vector bundles. More details will be given in a forthcoming paper [8].

## 2. A universal problem in affine geometry

First we establish the notation. If $A$ is an affine space (over a field $K$), we denote by $\vec{A}$ its associated vector space. The set of affine maps between the affine spaces $A$ and $B$ is denoted by $\mathcal{A}\!f\!f(A,B)$. Remember that an affine map $f \in \mathcal{A}\!f\!f(A,B)$ has an associated linear map, which we denote by $\vec{f} \in \mathcal{L}in(\vec{A},\vec{B})$.

Let $E$ be a vector space. Giving a proper hyperplane $H \subset E$ amounts to giving a non-vanishing linear form $w\colon E \to K$, and both are related by $H = w^{-1}(1)$. Now let $h\colon H \to F$ be an affine map with values in a vector space. Then there exists a unique linear map $\bar{h}\colon E \to F$ prolonging $h$. This map is defined by $\bar{h}(x) = \begin{cases} \vec{h}(x) & \text{if } w(x) = 0 \\ w(x)h\left(\frac{x}{w(x)}\right) & \text{if } w(x) \neq 0 \end{cases}$

Suppose we have an affine space $A$, a vector space $\widehat{A}$, and an affine map $j\colon A \to \widehat{A}$. Consider the following **universal property**: *for every vector space $F$ and affine map $h\colon A \to F$, there exists a unique linear map $\hat{h}\colon \widehat{A} \to F$ such that $h = \hat{h} \circ j$.*

$$\begin{tikzcd} A \arrow[r, "j"] \arrow[dr, "h"'] & \widehat{A} \arrow[d, dashed, "\hat{h}"] \\ & F \end{tikzcd}$$

**Proposition 1.** $(\widehat{A}, j)$ *satisfies the universal property iff $j$ is injective and $j(A) \subset \widehat{A}$ is a proper hyperplane.*

$(\widehat{A}, j)$ is called a *vector hull* of $A$. It is unique, up to isomorphism. We know that there is a unique linear form $w\colon \widehat{A} \to K$ such that $j(A) = w^{-1}(1)$; then $\vec{A}$ is identified with $w^{-1}(0)$. All this can be summarized in the following diagram:

$$\begin{tikzcd} 0 \arrow[r] & \vec{A} \arrow[r, "i"] & \widehat{A} \arrow[r, "w"] & K \arrow[r] & 0 \\ & & A \arrow[u, hook, "j"'] & & \end{tikzcd}$$

The assignment $h \mapsto \hat{h}$ is an isomorphism $\mathcal{A}\!f\!f(A,F) \simeq \mathcal{L}in(\widehat{A},F)$, and in particular $\mathcal{A}\!f\!f(A,K) \simeq \widehat{A}^*$.

Given an affine map $f\colon A \to B$, there is a unique linear map $\widehat{f}\colon \widehat{A} \to \widehat{B}$ such that $\widehat{f} \circ j_A = j_B \circ f$. We call it the *vector extension* of $f$.

$$\begin{tikzcd} A \arrow[r, "f"] \arrow[d, hook, "j_A"'] & B \arrow[d, hook, "j_B"] \\ \widehat{A} \arrow[r, dashed, "\widehat{f}"] & \widehat{B} \end{tikzcd}$$

The assignment $f \mapsto \widehat{f}$ is an affine inclusion $\mathcal{A}\!\mathit{ff}(A,B) \hookrightarrow \mathcal{L}\mathit{in}(\widehat{A},\widehat{B})$.

**Coordinate description** Consider a point $e_0 = a_0 \in A$ and a basis $(e_i)_{i\in I}$ of $\vec{A}$. Then, with the appropriate identifications, every point in $\widehat{A}$ can be uniquely written as $x = x^0 e_0 + x^i e_i$. A point $x \in \widehat{A}$ belongs to $A$ iff $x^0 = 1$, and belongs to $\vec{A}$ iff $x^0 = 0$.

With these coordinates, the linear extension of $y^j = c^j + A^j_i x^i$ is the linear map $y^\nu = A^\nu_\mu x^\mu$, with $A^j_0 = c^j$, $A^0_0 = 1$, $A^0_i = 0$.

## 3. A construction of the vector hull

Consider an affine map $X\colon A \to \vec{A}$, which can also be identified with a vector field on $A$. The map $X$ has an associated linear map $\vec{X} \in \mathcal{L}\mathit{in}(\vec{A},\vec{A})$; if this endomorphism is a homothety, let us call $X$ *homothetic*.

The set $\widehat{A}$ of homothetic vector fields is clearly a vector space, and the map $w\colon \widehat{A} \to K$ defined by $\vec{X} = -w(X)\mathrm{Id}$ is a nonzero linear form.

Consider the following special vector fields on $A$:

- Constant vector field $Y_{\mathbf{u}}(p) = \mathbf{u}$.
- Central vector field $Z_{a,\lambda}(p) = \lambda\vec{pa} = -\lambda\vec{ap}$.

**Proposition 2.** *Let $X \in \widehat{A}$ be a homothetic vector field. If $w(X) = 0$, then $X$ is a constant vector field $Y_{\mathbf{u}}$. If $w(X) = \lambda \neq 0$, then $X$ is a central vector field $Z_{a,\lambda}$.*

So the set $\widehat{A}$ is the disjoint union of constant vector fields and central vector fields.

**Proposition 3.** *The image of the linear map $i\colon \vec{A} \hookrightarrow \widehat{A}$ given by $i(\mathbf{u}) = Y_{\mathbf{u}}$ coincides with $w^{-1}(0)$. The image of the affine map $j\colon A \hookrightarrow \widehat{A}$ given by $j(a) = Z_{a,1}$ coincides with $w^{-1}(1)$.*

This shows that $(\widehat{A}, j)$ is a construction of the vector hull of $A$. Moreover, $A \rightsquigarrow \widehat{A}$, $f \rightsquigarrow \widehat{f}$, is a covariant functor from affine spaces to vector spaces.

## 4. Vector hull of an affine bundle

Let $\pi\colon A \to M$ be an affine bundle modelled on the vector bundle $\pi\colon \vec{A} \to M$. For each $m \in M$, $A_m$ is an affine space modelled on $\vec{A}_m$. Consider its vector hull $j_m\colon A_m \to \widehat{A}_m$ and define

$$\widehat{A} := \bigsqcup_{m\in M} \widehat{A}_m, \qquad \widehat{\pi}\colon \begin{array}{ccc} \widehat{A} & \longrightarrow & M \\ \widehat{a}_m & \longmapsto & m \end{array} \qquad j\colon \begin{array}{ccc} A & \longrightarrow & \widehat{A} \\ a_m & \longmapsto & j_m(a_m) \end{array}$$

**Proposition 4.** *$\widehat{\pi}$ is a vector bundle and $j$ is an injective affine morphism.*

The same properties as for affine spaces hold for affine bundles. Let us review some of them.

Let $E \to M$ be a vector bundle. If $f\colon A \to E$ is an affine bundle morphism there exists a unique vector bundle morphism $f\hat{}\colon \widehat{A} \to E$ such that $f\hat{} \circ j = f$.

Similarly, if $f\colon A \to B$ is an affine bundle morphism, there exists a unique vector bundle morphism $\widehat{f}\colon \widehat{A} \to \widehat{B}$ such that $\widehat{f} \circ j_A = j_B \circ f$.

Suppose we have an exact sequence of vector bundles over $M$

$$0 \longrightarrow \vec{A} \xrightarrow{\vec{\jmath}} F \xrightarrow{w} M \times \mathbf{R} \longrightarrow 0$$

where $w^{-1}(1)$ is isomorphic to the affine bundle $A$. Then we can identify $F$ with the vector hull $\widehat{A}$. In the next sections this property will be used to identify the vector hull of some jet bundles with some particular vector bundles.

# 5. Vector hull of jet bundles over R

### First order case

Let $\rho\colon M \to \mathbf{R}$ be a bundle. Consider its first order jet manifold $\mathrm{J}^1\rho$:

$$\begin{tikzcd} \mathrm{J}^1\rho \arrow[r, "\rho_{1,0}"] \arrow[dr, "\rho_1"'] & M \arrow[d, "\rho"] \\ & \mathbf{R} \end{tikzcd} \qquad \begin{tikzcd} (t, q^i, v^i) \arrow[r, "\rho_{1,0}"] \arrow[dr, "\rho_1"'] & (t, q^i) \arrow[d, "\rho"] \\ & t \end{tikzcd}$$

The bundle $\rho_{1,0}$ is affine, and it is modelled on the vertical vector bundle of $\rho$, which is $\mathrm{V}\rho = \operatorname{Ker} \mathrm{T}\rho \subset \mathrm{T}M$. The vertical bundle $\mathrm{V}\rho$ is locally generated by the vector fields $\{\frac{\partial}{\partial q^i}\}$.

There is a canonical immersion $j\colon \mathrm{J}^1\rho \to \mathrm{T}M$, defined by $\mathrm{j}^1_t\xi \mapsto \dot{\xi}(t)$; in coordinates, $(t, q^i, v^i) \mapsto (t, q^i; 1, v^i)$. Its image coincides with $\mathrm{d}t^{-1}(1)$ in the exact sequence $0 \to \mathrm{V}\rho \xrightarrow{\vec{\jmath}} \mathrm{T}M \xrightarrow{\mathrm{d}t} M \times \mathbf{R} \to 0$. Therefore

$$\widehat{\mathrm{J}^1\rho} = \mathrm{T}M.$$

### Higher order case

Consider now the $k$-th order jet bundle of $\rho$, $\mathrm{J}^k\rho \to \mathbf{R}$, with coordinates $(t, q^i_{(0)}, q^i_{(1)}, \ldots, q^i_{(k-1)}, q^i_{(k)})$. The bundle $\rho_{k,k-1}\colon \mathrm{J}^k\rho \to \mathrm{J}^{k-1}\rho$ is affine, and it is modelled on the vertical vector bundle of $\rho_{k-1,k-2}$, which is $\mathrm{V}\rho_{k-1,k-2} = \operatorname{Ker} \mathrm{T}\rho_{k-1,k-2} \subset \mathrm{TJ}^{k-1}\rho$. This vertical bundle is locally generated by the vector fields $\frac{\partial}{\partial q^i_{(k-1)}}$.

There is a canonical immersion of $\mathrm{J}^k\rho$ into $\mathrm{TJ}^{k-1}\rho$:

$$\begin{array}{rccc} j\colon & \mathrm{J}^k\rho & \longrightarrow & \mathrm{TJ}^{k-1}\rho \\ & \mathrm{j}^k_t\xi & \longmapsto & (\mathrm{j}^{k-1}\xi)^{\cdot}(t) \end{array}$$

The Cartan distribution on $\mathrm{J}^{k-1}\rho$ (denoted by $C\rho_{k-1,k-2}$) is the distribution generated by the vectors tangent to $(k-1)$-jet prolongations of sections of $\rho$. A local basis for $C\rho_{k-1,k-2}$ is given by the vector fields

$$\frac{\partial}{\partial t} + \sum_{l=0}^{k-2} q^i_{(l+1)} \frac{\partial}{\partial q^i_{(l)}}, \quad \frac{\partial}{\partial q^1_{(k-1)}}, \quad \ldots, \quad \frac{\partial}{\partial q^n_{(k-1)}}.$$

It is easily seen that $\operatorname{Im}(j) \subset C\rho_{k-1,k-2}$ coincides with $\mathrm{d}t^{-1}(1)$ in the exact sequence

$$0 \longrightarrow \mathrm{V}\rho_{k-1,k-2} \xrightarrow{\vec{\jmath}} C\rho_{k-1,k-2} \xrightarrow{\mathrm{d}t} \mathrm{J}^{k-1}\rho \times \mathbf{R} \longrightarrow 0$$

Therefore

$$\widehat{\mathrm{J}^k\rho} = C\rho_{k-1,k-2}.$$

# 6. Vector hull of $\mathrm{J}^1\pi$ over an arbitrary base

Let $\pi\colon M \to B$ be a bundle. We consider its first-order jet manifold $\mathrm{J}^1\pi$, with coordinates $(x^i, u^\alpha, u^\alpha_i)$. The projection $\pi_{1,0}\colon \mathrm{J}^1\pi \to M$ is an affine bundle modelled on the vector bundle $\mathrm{V}\pi \otimes \pi^*(\mathrm{T}^*B) \to M$.

We have an inclusion

$$\mathrm{V}\pi \otimes \pi^*(\mathrm{T}^*B) \simeq \operatorname{Hom}(\pi^*\mathrm{T}B, \mathrm{V}\pi) \subset \operatorname{Hom}(\pi^*\mathrm{T}B, \mathrm{T}M)$$

characterized by $\mathrm{V}\pi \otimes \pi^*(\mathrm{T}^*B) = \{A \in \operatorname{Hom}(\pi^*\mathrm{T}B, \mathrm{T}M) \mid \mathrm{T}\pi \circ A = 0\}$. On the other hand, each $\mathrm{j}^1_b\phi \in \mathrm{J}^1\pi$ induces a homomorphism

$$\begin{array}{ccc} (\pi^*\mathrm{T}B)_m & \longrightarrow & \mathrm{T}_mM \quad (m = \phi(b)) \\ (m, v_b) & \longmapsto & \mathrm{T}_b\phi(v_b) \end{array}$$

so there is also an inclusion

$$\mathrm{J}^1\pi \subset \operatorname{Hom}(\pi^*\mathrm{T}B, \mathrm{T}M)$$

characterized by $\mathrm{J}^1\pi = \{A \in \mathrm{Hom}(\pi^*\mathrm{T}B, \mathrm{T}M) \mid \mathrm{T}\pi \circ A = \mathrm{Id}_{\mathrm{T}B}\}$.

Then we can see that this sequence is exact:

$$0 \longrightarrow \mathrm{V}\pi \otimes \pi^*(\mathrm{T}^*B) \longrightarrow \widehat{\mathrm{J}^1\pi} \xrightarrow{w} M \times \mathbf{R} \longrightarrow 0$$

where the vector hull is identified with

$$\widehat{\mathrm{J}^1\pi} = \{A \in \mathrm{Hom}(\pi^*\mathrm{T}B, \mathrm{T}M) \mid \mathrm{T}\pi \circ A = \lambda\, \mathrm{Id}_{\mathrm{T}B}\}$$

and the 1-form is $w(A) = \lambda$.

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# Quantization of the regularized Kepler and related problems

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**Abstract.** The regularized Kepler problem is considered, as obtained from the Kustaanheimo-Stiefel (K-S) transformation, for negative energies. The symmetry group for the Kepler problem turns out to be $SU(2,2)$, and the Hamiltonian of the Kepler problem can be realized as the sum of the energies of four harmonic oscillator with the same frequency, with a certain constraint. Other related problems like the Stark effect are considered in this framework.

*Keywords:* Kepler problem. K-S regularization. Quantization

*2000 Mathematics Subject Classification:* 53D80 70F16 70G65 70H45

## 1. KS Regularization of the Kepler problem

In this work we affront the task of the quantization of the Kepler problem, given by the Hamiltonian defined on $\mathbb{R}^3_0 \times \mathbb{R}^3$, $\mathbb{R}_0 = \mathbb{R} - \{0\}$,

$$\mathcal{H} = \frac{\vec{\mathbb{Y}} \cdot \vec{\mathbb{Y}}}{2m} - \frac{\gamma}{r}\,, \tag{1}$$

where $r = \sqrt{\mathbb{X}^2}$, $(\mathbb{X}, \mathbb{Y}) \in \mathbb{R}^3_0 \times \mathbb{R}^3$.

For this purpose we shall use the linearization provided by the KS regularization, as introduced by P. Kustaanheimo and E. Stiefel, in the spinorial version due to Jost (see [1]). The KS transformation regularizes the Kepler problem and linearizes it, showing that the dynamical group of the Kepler problem is $SU(2,2)$. The regularization means that collision orbits, for which the potential is singular, are included in the state space on the same footing that the rest of the orbits. The linearization means that the Kepler problem, for the case of negative energy, can be seen as a system of **4 harmonic oscillators in resonance** subject to a constrain. For the case of positive energy, it turns to be a system of **4 repulsive harmonic oscillators in resonance** subject to constrains. Finally, the singular case of zero energy can be expressed as **4 free particles** subject to constrains. In this work we shall only consider the case of negative energies (bounded states), see [2] for a detailed study of the other cases.

The key point in the KS transformation is the commutativity of the diagram (see [1]):

$$\begin{array}{ccc} (z,\, w) \in (I^{-1}(0))' & \stackrel{\mathfrak{C}}{\to} & (\eta,\, \zeta) \in I^{-1}(0) \subset \mathbb{C}^4 \\ \pi \downarrow & \circlearrowleft & \downarrow \widehat{\pi} \\ (\vec{x},\, \vec{y}) \in \mathbb{R}^3_0 \times \mathbb{R}^3 & \stackrel{\nu^{-1}}{\to} & (q,\, p) \in T^+S^3 \end{array} \tag{2}$$

In this diagram $\nu$ is Moser transformation (see [1]), which allows us to see $T^+S^3$ as an embedded manifold in $\mathbb{R}^3_0 \times \mathbb{R}^3$, where $T^+S^3 = \{(q,p) \in \mathbb{R}^8,\ ||q|| = 1, <p,q>= 0,\ p \neq 0\}$, is named **Kepler manifold**.

The KS transformation is the map $\pi$, which can be seen as a symplectic lift of the Hopf fibration, $\pi_0 : \mathbb{C}^2_0 \to \mathbb{R}^3_0, \quad z = (z_1, z_2) \mapsto \pi_0(z) :=< z, \bar{\sigma} z >$, ($\vec{\sigma}$ are Pauli matrices):

$$\pi : T^*\mathbb{C}^2_0 \to T^*\mathbb{R}^3_0,\ (z,w) \mapsto (\vec{x} = \pi_0(z), \vec{y} = \frac{\mathrm{Im} < w, \vec{\sigma} z >}{< z, z >}), \tag{3}$$

such that, $\pi^*\theta_{\mathbb{R}^3_0} = \theta_{\mathbb{C}^2_0}|_{(I^{-1}(0))'} = 2\mathrm{Im} < w, dz >$ ( which equals $\theta_{\eta\zeta} = \mathrm{Im}(<\eta, d\eta> - <\zeta, d\zeta>)$ up to a total differential) and $\theta_{\mathbb{R}^3_0}$ is the canonical potential form restricted to $\mathbb{R}^3_0$. The map

$$\mathfrak{C} = \frac{1}{\sqrt{2}} \begin{pmatrix} \sigma_0 & \sigma_0 \\ \sigma_0 & -\sigma_0 \end{pmatrix}, \tag{4}$$

provides the injection of collision states. The function

$$I = \frac{1}{2}(<\eta, \eta> - <\zeta, \zeta>), \tag{5}$$

defines the regularized space $I^{-1}(0)$ ($(I^{-1}(0))'$ doesn't contain collision states), which is diffeomorphic to $\mathbb{C}^2_0 \times S^3$ while $I^{-1}(0)/U(1)$ is diffeomorphic to $\mathbb{R}^3_0 \times S^3$. The transformation, $\vec{\mathbb{X}} = \frac{1}{\sqrt{mk}}\vec{x}, \quad \vec{\mathbb{Y}} = k\sqrt{m}\vec{y}$, with $\rho = \sqrt{\vec{x}^2}$ and $k >$

0 is a scaling parameter to account for all (negative) energies, relates the variables in the Kepler problem to the variables used in $\nu$. The map $\widehat{\pi}$ is a symplectomorphism between $I^{-1}(0)/U(1)$ and $T^{+}S^{3}$, with the symplectic structures restricted to the corresponding spaces. The Kepler Hamiltonian for negative energy is associated with

$$\mathcal{J} = \frac{1}{2}(<\eta,\eta> + <\zeta,\zeta>)\,, \tag{6}$$

which corresponds to a system of 4 harmonic oscillators in resonance 1-1-1-1.

The potential 1-form $\theta_{\eta\,\zeta}$ is left invariant by the Lie group $U(2,2)$, which also leaves invariant the constrain $I$ when acting on $\mathbb{C}^4_0$. This is thus the dynamical group for the Kepler problem. A convenient basis for the Lie algebra $u(2,2)$ is given by the components of the momentum map associated with its action on $\mathbb{C}^4_0$ (here $I$ is central):

$$\begin{aligned} &I, \quad \mathcal{J}, \quad \vec{M} = -\tfrac{1}{2} <\eta,\vec{\sigma}\,\eta>, \quad \vec{N} = \tfrac{1}{2} <\zeta,\vec{\sigma}\,\zeta>, \\ &Q = (-\mathrm{Im} <\eta,\zeta>, \mathrm{Re} <\eta,\vec{\sigma}\zeta>), \quad P = (\mathrm{Re} <\eta,\zeta>, \mathrm{Im} <\eta,\vec{\sigma}\zeta>). \end{aligned} \tag{7}$$

| **Table I: KS regularization with physical constants** |
|---|
| $\vec{\mathbb{X}} = \frac{1}{\sqrt{m}\,k}\,(\vec{Q} - \vec{R}'), \qquad \vec{\mathbb{Y}} = k\,\sqrt{m}\,\frac{\vec{P}}{\lVert P\rVert + P_0},$ |
| $\mathcal{H} = \frac{\vec{\mathbb{Y}}^2}{2m} - \frac{\gamma}{\lVert\vec{\mathbb{X}}\rVert} = \frac{k}{2(\lVert P\rVert+P_0)}(k(\lVert P\rVert - P_0) - 2\gamma\sqrt{m}),$ <br> $\vec{\mathcal{AM}} = \vec{L} = \vec{\mathbb{X}} \times \vec{\mathbb{Y}} = \vec{M} + \vec{N}, \quad \vec{R}' = \vec{M} - \vec{N},$ <br> $\vec{\mathcal{RL}} = \frac{\vec{\mathbb{Y}} \times \vec{L}}{m} - \gamma\frac{\vec{\mathbb{X}}}{\lVert\vec{\mathbb{X}}\rVert} = \frac{\vec{R}'(kP_0 + \gamma\sqrt{m}) + \vec{Q}(k\lVert P\rVert - \gamma\sqrt{m})}{\sqrt{m}(\lVert P\rVert + P_0)}.$ |

## 2. Quantization of the Kepler problem for $E < 0$

The KS transformation reveals that the Kepler problem for negative energies can be seen as the Hamiltonian system $(\mathbb{C}^4, \theta_{(\eta,\zeta)}, \mathcal{J})$ restricted to $I^{-1}(0)$. Defining $\mathbf{C} = (\mathbf{C}_1, \mathbf{C}_2) = (\eta, \zeta^+)$, $\mathbf{C}_i \in \mathbb{C}^2$, the Hamiltonian $\mathcal{J}$ adopts the form $\mathcal{H}_{\rm ho} = \omega\mathbf{C}\cdot\mathbf{C}^+$ which corresponds to four harmonic oscillators. The quantization of this system can be obtained from the group law of the corresponding symmetry group (a central extension of it by $U(1)$, rather, see [3]):

$$\begin{aligned} &\lambda'' = \lambda' + \lambda, \qquad \mathbf{C}'' = \mathbf{C}'\,e^{-i\,\lambda} + \mathbf{C}, \quad \mathbf{C}''^{+} = \mathbf{C}'^{+}\,e^{i\,\lambda} + \mathbf{C}^{+}, \\ &\varsigma'' = \varsigma'\,\varsigma\,\exp[\tfrac{i}{2}\,(i\,\mathbf{C}'\cdot\mathbf{C}^{+}\,e^{-i\,\lambda} - i\,\mathbf{C}'^{+}\cdot\mathbf{C}\,e^{i\,\lambda})], \end{aligned} \tag{8}$$

where $\mathbf{C}, \mathbf{C}^+ \in \mathbb{C}^4$, $\varsigma \in U(1)$ and $\lambda = \omega\, t \in \mathbb{R}$. We can obtain the quantum version of this system using any geometrical (like Geometric Quantization, see [4]) or group-theoretical method, like Group Approach to Quantization (GAQ, see [3]), the one used here. The resulting wave functions (defined on the group) are $\psi = \varsigma e^{-\frac{1}{2}\boldsymbol{C}\cdot\boldsymbol{C}^+}\phi(\mathbf{C}^+, \lambda)$, and the Schrödinger equation for this system is $i\frac{\partial\phi}{\partial\lambda} = i\mathbf{C}^+ \cdot \frac{\partial\phi}{\partial\boldsymbol{C}^+}$.

In this formalism, quantum operators are constructed from the right-invariant vector fields on the group (8), and in this case **creation** and **annihilation** operators are given by $\widehat{\mathbf{C}}^+ = X^R_{\mathbf{C}}$ and $\widehat{\mathbf{C}} = X^R_{\mathbf{C}^+}$, respectively, where $X^R$ stands for right-invariant vector fields. Since the momentum map (7) is expressed as quadratic functions on $\mathbf{C}$ and $\mathbf{C}^+$, we can resort to Weyl prescription to obtain the quantization of these functions on the (right) enveloping algebra of the group (8). In this way we obtain a Lie algebra of quantum operators isomorphic to the one satisfied by the momentum map (7) with the Poisson bracket associated with $\theta_{\eta\,\zeta}$. The Hamiltonian operator and the quantum version of the constrain, when acting on wave functions are given by:

$$\widehat{\mathcal{J}}\psi = -\frac{1}{2}\mathcal{W}(2+\mathbf{C}^+\cdot\frac{\partial}{\partial\boldsymbol{C}^+})\phi\,, \qquad \widehat{I}\psi = -\frac{1}{2}\mathcal{W}(\mathbf{C}^+_1\cdot\frac{\partial}{\partial\boldsymbol{C}^+_1} - \mathbf{C}^+_2\cdot\frac{\partial}{\partial\boldsymbol{C}^+_2})\phi\,, \quad (9)$$

where $\mathcal{W} = \varsigma e^{-\frac{1}{2}\mathbf{C}\cdot\mathbf{C}^+}$ is the "vacuum" state. To obtain the quantum version of the Kepler manifold (that is, the Hilbert space of states of the Hydrogen atom for $E < 0$), we must impose the constrain $\widehat{I}\psi = 0$. This means that the energy of the first two oscillators must equal the energy of the other two. It is easy to check that the operators in the (right) enveloping algebra of the group (8) preserving the constrain (see [5, 6] for a characterization of these operators) is the algebra $su(2, 2)$ of the quantum version of the momentum map (7). These operators act irreducibly on the constrained Hilbert space, as can be checked computing the Casimirs of $su(2, 2)$, which are constant.

The quantum operators commuting with the Hamiltonian (and providing the degeneracy of the spectrum) are $\widehat{\vec{M}}$ and $\widehat{\vec{N}}$. They define two commuting $su(2)$ algebras in the same representation $((\widehat{\vec{M}})^2 = (\widehat{\vec{N}})^2 = \frac{1}{4}\,(\widehat{\mathcal{J}})^2 - \frac{1}{4})$, and linear combinations of them provide us with the angular momentum and the Runge-Lenz vector (see Table I).

The relation between the Kepler Hamiltonian $\mathcal{H}$ and the Hamiltonian $\mathcal{J}$ is

$$\mathcal{H} = -\frac{m\,\gamma^2}{2\,\mathcal{J}^2}\,, \qquad (10)$$

and the same relation holds for their quantum counterparts. If we act on eigenstates of the number operator for each oscillator,

$$\psi_{n_1,n_2,n_3,n_4} = (\hat{C}^+_{11})^{n_1}(\hat{C}^+_{12})^{n_2}(\hat{C}^+_{21})^{n_3}(\hat{C}^+_{22})^{n_4}\mathcal{W}\,, \tag{11}$$

and taking into account that $\widehat{\mathcal{J}}\psi_{n_1,n_2,n_3,n_4} = \frac{1}{2}(2+\sum n_i)\psi_{n_1,n_2,n_3,n_4}$, we recover the **spectrum of the Hydrogen atom**,

$$E_n = -\frac{m\,\gamma^2}{2\,n^2}\,, \qquad n = 1 + n_1 + n_2\,, \tag{12}$$

where we have made use of the fact that $n_1 + n_2 = n_3 + n_4$ by the constrain $\widehat{I}\psi = 0$. The degeneracy is provided by the dimension of the representations of the algebra $su(2) \times su(2)$, which turn to be $n^2$ (if spin 1/2 is considered, the degeneracy is doubled).

An interesting application of these results is the fact that the linearization is preserved in some perturbed problems, such as the lunar problem or the Stark effect (see next section).

A similar study can be found in [7], where the quantization of the Kepler problem for $E \neq 0$ is considered in the Weyl-Wigner-Moyal formalism using the K-S transformation. A discussion of the hyperbolic version of the K-S transformation is given in [8]. A detailed study of the conformal symmetry (the group $SU(2,2)$ is locally isomorphic to the conformal group $SO(4,2)$), in the two-dimensional Kepler problem can be found in [9].

## 3. Stark Effect

Let us consider the Hamiltonian for the Stark Effect,

$$\mathcal{H}_S = \mathcal{H} + \sqrt{\gamma}\,\mathcal{E}\,\mathbb{Z}\,, \tag{13}$$

where $\mathcal{H}$ is the Kepler Hamiltonian and $\mathcal{E}$ is the electric field intensity, which is supposed to be constant and in the direction of the $\mathbb{Z}$ component. We need to express $\mathcal{H}_S$ in the K-S variables, and for this we use that $\mathbb{Z} = \frac{1}{\sqrt{m}\,k}(Q_3 - R'_3)$, where $\vec{R}' = \vec{M} - \vec{N}$. Thus we have:

$$\begin{aligned}\sqrt{\gamma}\,\mathcal{E}\,\mathbb{Z} &= \frac{\sqrt{\gamma}\,\mathcal{E}}{2\sqrt{m}\,k} < \eta + \zeta, \sigma_3\,(\eta + \zeta) > \\ &= \frac{\sqrt{\gamma}\,\mathcal{E}}{2\sqrt{m}\,k}\left(\bar{\eta}_1\eta_1 - \bar{\eta}_2\eta_2 + \bar{\zeta}_1\zeta_1 - \bar{\zeta}_2\zeta_2 + 2\mathrm{Re}(\bar{\eta}_1\zeta_1 - \bar{\eta}_2\zeta_2)\right)\,.\end{aligned} \tag{14}$$

Since the perturbation term commutes with $I$, the space of states is still defined by $I^{-1}(0)$, and therefore the dynamical group of this system is again

$SU(2,2)$. However, the complete Hamiltonian $\mathcal{H}_S$ does not commute with the angular momentum and Runge-Lenz vectors.

Following the scheme of Sec. 2, the quantization of this system is straightforward. The quantum version of the new Hamiltonian is

$$\begin{aligned}\hat{\mathcal{H}}_S &= \hat{\mathcal{H}} + \kappa\Big(\hat{C}_{11}^+\hat{C}_{11} - \hat{C}_{12}^+\hat{C}_{12} + \hat{C}_{21}^+\hat{C}_{21} - \hat{C}_{22}^+\hat{C}_{22} \\ &\qquad + \hat{C}_{11}^+\hat{C}_{21}^+ + \hat{C}_{11}\hat{C}_{21} - \hat{C}_{12}^+\hat{C}_{22}^+ - \hat{C}_{12}\hat{C}_{22}\Big) \qquad (15)\end{aligned}$$

where $\hat{\mathcal{H}}$ is the quantum version of the Kepler Hamiltonian given in (10), and $\kappa = \frac{\sqrt{\gamma}\,\mathcal{E}}{2\sqrt{m\,k}}$. From the form $\hat{\mathcal{H}}_S$ it is clear that preserves the constrain $\hat{I}\Psi = 0$, i.e., $n_1 + n_2 = n_3 + n_4$. Since it does not commute with the angular momentum and Runge-Lenz operators, the $n^2$ degeneracy of the eigenstates will be partially broken (the third component of the angular momentum operator still commutes with $\hat{\mathcal{H}}_S$).

The determination of the energy levels of this system is a standard problem in perturbation theory in quantum mechanics. A detailed account of the Stark effect can be found, for example, in [10].

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# Compatibility between Lie affgebroid and affine Jacobi structures on an affine bundle

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**Abstract.** We introduce the notion of compatibility between Lie affgebroids and affine Jacobi structures on an affine bundle. Some examples are presented and several properties are studied.

*Keywords:* Jacobi manifolds, Lie affgebroids, affine bundles.

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## 1. Introduction

Recently (see [5]), a correspondence has been obtained between affine Jacobi brackets on an affine bundle $A \to M$ and Lie algebroid structures on the dual bundle $A^+$, extending the correspondence between linear Poisson structures on a vector bundle $V \to M$ and Lie algebroid structures on the dual vector bundle $V^*$ (see [1]).

On the other hand, in [7] (see also [2]) a possible generalization of the notion of Lie algebroid to affine bundles is introduced in order to build a geometrical model for a time-dependent version of Lagrange equations on Lie algebroids. The new structures are called Lie affgebroid structures (in the terminology of [2]). In [2, 7], the authors show that a Lie affgebroid structure on an affine bundle is equivalent to a Lie algebroid structure on the bidual $\widetilde{A} = (A^+)^*$ with a non-vanishing 1-cocycle. The canonical example of a Lie affgebroid is the first-jet bundle $J^1M \to M$, where $M$ is a fibred manifold over the real line. In addition, if $M$ is a Poisson manifold then we can define an affine Jacobi structure on $J^1M$ which is compatible with the Lie affgebroid structure in a certain sense. This fact suggests us to introduce the compatibility notion between these types of structures defined over an arbitrary affine bundle. In this note we will study several aspects related with this concept of compatibility and we will present some examples which illustrate this notion.

## 2. Lie affgebroid structures on affine bundles

Let $\tau : A \to M$ be an affine bundle with associated vector bundle $\tau_V : V \to M$. Denote by $\tau^+ : A^+ \to M$ the dual bundle whose fibre over $m \in M$ consists of affine functions on the fibre $A_m$. Note that this bundle has a distinguished section $e^0 \in \Gamma(A^+)$ corresponding to the constant function 1 on $A$. We also consider the bidual bundle $\widetilde{\tau} : \widetilde{A} \to M$ whose fibre at $m \in M$ is the vector space $\widetilde{A}_m = (A^+_m)^*$. Then, $A$ can be identified with an affine subbundle of $\widetilde{A}$ via the inclusion $i : A \to \widetilde{A}$ given by $i(a)(\varphi) = \varphi(a)$, which is an injective affine map whose associated vector map is denoted by $i_V : V \to \widetilde{A}$. Thus, $V$ may be identified with a vector subbundle of $\widetilde{A}$. Using these facts, one can prove that there is a one-to-one correspondence between affine functions on $A$ and linear functions on $\widetilde{A}$. On the other hand, there is an obvious one-to-one correspondence between affine functions on $A$ and sections of $A^+$.

A *Lie affgebroid structure* on $A$ (see [2, 7]) consists of a Lie algebra structure $[\![\cdot,\cdot]\!]_V$ on the space $\Gamma(V)$ of sections of $\tau_V : V \to M$, a $\mathbb{R}$-linear action $D : \Gamma(A) \times \Gamma(V) \to \Gamma(V)$ of the sections of $A$ on $\Gamma(V)$ and an affine map $\rho : A \to TM$, the *anchor map*, satisfying the following conditions:

- $D_s[\![\bar{s}_1, \bar{s}_2]\!]_V = [\![D_s\bar{s}_1, \bar{s}_2]\!]_V + [\![\bar{s}_1, D_s\bar{s}_2]\!]_V, \;\; \forall \bar{s}_1, \bar{s}_2 \in \Gamma(V), \forall s \in \Gamma(A)$,
- $D_{s+\bar{s}_1}\bar{s}_2 = D_s\bar{s}_2 + [\![\bar{s}_1, \bar{s}_2]\!]_V, \;\; \forall \bar{s}_1, \bar{s}_2 \in \Gamma(V), \forall s \in \Gamma(A)$,
- $D_s(f\bar{s}) = \rho(s)(f)\bar{s} + fD_s\bar{s}, \;\; \forall \bar{s} \in \Gamma(V), \forall s \in \Gamma(A), \forall f \in C^\infty(M)$.

We recover the notion of *Lie algebroid* on a manifold $M$ when $A = V$ is a vector bundle, $D$ is the action defined by the Lie bracket $[\![\cdot,\cdot]\!]_V$ and $\rho : A = V \to TM$ is a morphism of vector bundles. In fact, for a Lie affgebroid $(A, [\![\cdot,\cdot]\!]_V, D, \rho)$ we have that $(V, [\![\cdot,\cdot]\!]_V, \bar{\rho})$ is a Lie algebroid, where $\bar{\rho} : V \to TM$ is the linear part of $\rho$. Moreover, one can induce a Lie algebroid structure on the bidual bundle $\widetilde{A}$ such that $e^0 \in \Gamma(A^+)$ is an 1-cocycle in the corresponding Lie algebroid cohomology. Conversely, a Lie algebroid structure on $\widetilde{A}$ such that $e^0$ is an 1-cocycle restricts to a Lie affgebroid structure on $A$ (see [2, 7]).

The canonical example of a Lie affgebroid is the first-jet bundle $J^1M \to M$ to a manifold $M$ fibred over the real line $\pi : M \to \mathbb{R}$. In this case, the vector bundle associated with $J^1M$, $Ver(\pi)$, is the set of vectors tangent to $M$ which are vertical with respect to $\pi$. We can identify the space of sections of $J^1M$ as the affine space $\Gamma(J^1M) = \{X \in \mathfrak{X}(M)/X(t \circ \pi) = 1\}$, where $t$ is the usual coordinate on $\mathbb{R}$. Under this identification, $(J^1M)^+ = T^*M$ and $\widetilde{J^1M} = TM$. The Lie affgebroid structure on $J^1M$ is given by the usual Lie bracket on $\Gamma(Ver(\pi)) \equiv \mathfrak{X}^V(M)$, the action $D : \Gamma(J^1M) \times \mathfrak{X}^V(M) \to \mathfrak{X}^V(M)$ is also the Lie bracket and the anchor map $\rho : \Gamma(J^1M) \to \mathfrak{X}(M)$ is the natural inclusion (see [7]).

## 3. Lie algebroids and affine Jacobi structures

A *Jacobi manifold* is a differentiable manifold $M$ endowed with a pair (*Jacobi structure*) $(\Lambda, E)$, where $\Lambda$ is a 2-vector and $E$ is a vector field on $M$ satisfying $[\Lambda, \Lambda] = -2E \wedge \Lambda$ and $[E, \Lambda] = 0$. For this type of manifolds, one can define the *Jacobi bracket* as $\{f, g\}_{(\Lambda,E)} = \Lambda(df, dg) + fE(g) - gE(f)$, for $f, g \in C^\infty(M)$. If $E = 0$ we recover the notion of *Poisson manifold.* A Jacobi structure on an affine bundle (respectively, on a vector bundle) $\tau : A \to M$ is called *affine* (respectively, *linear*) if the corresponding Jacobi bracket of affine functions is again an affine function (respectively, the Jacobi bracket of linear functions is again a linear function).

Recently, in [5] it is proved that there is a one-to-one correspondence between affine Jacobi structures on an affine bundle $\tau : A \to M$ and Lie algebroid structures on the dual bundle $\tau^+ : A^+ \to M$. More precisely, the Lie algebroid structure $([\![\cdot,\cdot]\!]^+, \rho^+)$ on $A^+$ induced by an affine Jacobi structure $(\Lambda, E)$ on $\tau : A \to M$ is defined by

$$\left.\begin{array}{rcl} [\![\widetilde{a}, \widetilde{b}]\!]^+ &=& \widetilde{\{a, b\}}_{(\Lambda,E)}, \\ \rho^+(\widetilde{a})(f) \circ \tau &=& \{a, f \circ \tau\}_{(\Lambda,E)} - (f \circ \tau)\{a, 1\}_{(\Lambda,E)}, \end{array}\right\} \tag{1}$$

for all $\widetilde{a}, \widetilde{b} \in \Gamma(A^+)$ and $f \in C^\infty(M)$, where $a$ and $b$ are the affine functions associated to the sections $\widetilde{a}$ and $\widetilde{b}$, respectively, and $\widetilde{\{a, b\}}_{(\Lambda,E)}$ is the section of $A^+$ associated to the affine function $\{a, b\}_{(\Lambda,E)}$. Conversely, if $([\![\cdot,\cdot]\!]^+, \rho^+)$ is a Lie algebroid structure on $\tau^+ : A^+ \to M$, then we have an affine Jacobi structure on A which is the restriction to $A$ of the following Jacobi structure on $\widetilde{A}$

$$\Lambda_{\widetilde{A}} = \Pi_{\widetilde{A}} - \Delta_{\widetilde{A}} \wedge E_{\widetilde{A}}, \quad E_{\widetilde{A}} = i_{d\widehat{e^0}} \Pi_{\widetilde{A}}, \tag{2}$$

where $\Pi_{\widetilde{A}}$ is the linear Poisson structure on $\widetilde{A}$ induced by $([\![\cdot,\cdot]\!]^+,\rho^+)$, $\Delta_{\widetilde{A}}$ is the Liouville vector field on $\widetilde{A}$ and $\widehat{e^0}$ is the linear function on $\widetilde{A}$ induced by $e^0 \in \Gamma(A^+)$.

## 4. Compatibility condition between an affine Jacobi structure and a Lie affgebroid structure on an affine bundle

Let $\Pi$ be a Poisson structure on a manifold $M$. Then, the cotangent bundle $T^*M$ admits a Lie algebroid structure $([\![\cdot,\cdot]\!]_\Pi, \#_\Pi)$ defined by $\#_\Pi(\alpha) = i_\alpha \Pi$ and $[\![\alpha,\beta]\!]_\Pi = \mathcal{L}_{\#_\Pi(\alpha)}\beta - \mathcal{L}_{\#_\Pi(\beta)}\alpha - d(\Pi(\alpha,\beta))$, for $\alpha,\beta \in \Omega^1(M)$.

**Proposition 1.** $(i)$ *Let $\pi : M \to \mathbb{R}$ be a fibration, $\Pi$ be a Poisson structure on $M$ and $\tau : J^1M \to M$ be the canonical projection. Suppose that $(\Lambda, E)$ is an affine Jacobi structure on $J^1M$ induced by the cotangent Lie algebroid $((J^1M)^+ = T^*M, [\![\cdot,\cdot]\!]_\Pi, \#_\Pi)$ and that $j^1 : C^\infty(M) \to C^\infty(J^1M)$ is the map given by $j^1 f(v) = v(f)$, for all $f \in C^\infty(M)$ and $v \in J^1M$. Then,*

$$\{f,g\}_\Pi \circ \tau = \{f\circ\tau, j^1 g\}_{(\Lambda,E)} - (f\circ\tau)\{1, j^1 g\}_{(\Lambda,E)} \tag{3}$$

*and $j^1$ is a Jacobi morphism, i.e., $j^1\{f,g\}_\Pi = \{j^1 f, j^1 g\}_{(\Lambda,E)}$, for all $f,g \in C^\infty(M)$.*

$(ii)$ *If $(\Lambda, E)$ is an affine Jacobi structure on $J^1M$ then there exists a bracket $\{\cdot,\cdot\}_\Pi$ on $C^\infty(M)$ such that (3) holds. Furthermore, if*

$$j^1\{f,g\}_\Pi = \{j^1 f, j^1 g\}_{(\Lambda,E)}, \quad \textit{for } f,g \in C^\infty(M), \tag{4}$$

*then $\{\cdot,\cdot\}_\Pi : C^\infty(M) \times C^\infty(M) \to C^\infty(M)$ defines a Poisson bracket on $M$.*

*Proof.* $(i)$ The linear Poisson structure on $TM$ induced by the Lie algebroid on $T^*M$ is the complete lift $\Pi^c$ of $\Pi$, which satisfies $\{f^c, g^c\}_{\Pi^c} = (\{f,g\}_\Pi)^c$, for $f,g \in C^\infty(M)$ (see [4]). Moreover, $(f^c)_{|J^1M} = j^1 f$ and $\Delta(f^c) = f^c$, $\Delta$ being the Liouville vector field on $TM$. Using these facts and (2), we get *(i)*.

$(ii)$ Since $(\Lambda, E)$ is an affine Jacobi structure on $J^1M$, we have (3) and that

$$\{f\circ\tau, g\circ\tau\}_{(\Lambda,E)} = (f\circ\tau)\{1, g\circ\tau\}_{(\Lambda,E)} + (g\circ\tau)\{f\circ\tau, 1\}_{(\Lambda,E)}, \tag{5}$$

for all $f \in C^\infty(M)$ (see [5]). Assuming that (4) holds, from (3) and (5), we obtain that $\{\cdot,\cdot\}_\Pi$ satisfies the Jacobi identity and it acts as a derivation on each of its arguments. Thus, $\{\cdot,\cdot\}_\Pi$ is skew-symmetric (see [3]) and, therefore, is a Poisson bracket on $M$. $\square$

Let $\pi : M \to \mathbb{R}$ be a fibration and $(\Lambda, E)$ be an affine Jacobi structure on the first-jet bundle $J^1M$ satisfying (3). Then, from Proposition 1, $\widetilde{(J^1M)} = TM$ and $(J^1M)^+ = T^*M$ are Lie algebroids over $M$ and the pair $(\widetilde{J^1M}, J^1M^+)$ is a Lie bialgebroid over $M$ (see, for instance, [8] for the definition of a Lie bialgebroid).

Motivated by the above example, we introduce the following definition.

**Definition 1.** Let $\tau : A \to M$ be an affine bundle, $\tau_V : V \to M$ the vector bundle associated with $A$, $(\Lambda, E)$ be an affine Jacobi structure on $A$ and $([\![\cdot,\cdot]\!]_V, D, \rho)$ be a Lie affgebroid structure on $A$. Consider the Lie algebroid structures on $\widetilde{A}$ and $A^+$ induced by $([\![\cdot,\cdot]\!]_V, D, \rho)$ and $(\Lambda, E)$, respectively. Then, $([\![\cdot,\cdot]\!]_V, D, \rho)$ and $(\Lambda, E)$ are compatible if $(\widetilde{A}, A^+)$ is a Lie bialgebroid over $M$.

**Others examples .** *(i)* Let $(V, [\![\cdot,\cdot]\!]_V, \rho_V)$ be a Lie algebroid structure on $V \to M$ such that $(V, V^*)$ is a Lie bialgebroid. Denote by $\Pi_V$ the linear Poisson structure on $V$ induced by the Lie algebroid structure on $V^*$ and by $([\![\cdot,\cdot]\!]_V, D, \rho_V)$ the Lie affgebroid structure on $V$ induced by $([\![\cdot,\cdot]\!]_V, \rho_V)$. Then, $\Pi_V$ and $([\![\cdot,\cdot]\!]_V, D, \rho_V)$ are compatible.

*(ii)* Let $p : Q \to M$ be a principal $G$-bundle over a manifold $M$ and $\pi : M \to \mathbb{R}$ be a fibration. Denote by $\tau : J^1Q \to Q$ and by $\mu = \pi \circ p : Q \to \mathbb{R}$ the corresponding fibrations. Then, $G$ acts on $J^1Q$ and on the vertical bundle to $\mu$, Ver $\mu$, so that $J^1Q/G$ is an affine bundle over $M = Q/G$ with associated vector bundle Ver $(\mu)/G \to M$.

Consider the Lie affgebroid on $J^1Q/G$ (first introduced in [7]) which associated bidual is the *Atiyah Lie algebroid* [6].

Now, suppose that $\Pi \in \mathcal{V}^2(Q)$ is a $G$-invariant Poisson structure on $Q$. Then, the Lie bialgebroid $(TQ, T^*Q)$ descends to a Lie bialgebroid structure $(TQ/G, T^*Q/G)$ on $M$, where the Lie bracket on $T^*Q/G$ comes from the fact that the Lie bracket of two $G$-invariant 1-forms on $Q$ is $G$-invariant. Therefore, the Lie affgebroid structure on $J^1Q/G$ and the affine Jacobi structure induced by the Lie algebroid $T^*Q/G$ are compatible.

In the following $\tau : A \to M$ is an affine bundle on $M$, $(\Lambda_A, E_A)$ is an affine Jacobi structure on $A$ and $([\![\cdot,\cdot]\!]_V, D, \rho)$ is a Lie affgebroid structure on $A$ which are compatible. Then, the Lie bialgebroid $(\widetilde{A}, A^+)$ induces a Poisson structure $\Pi_M$ on $M$ defined by $\{f, g\}_{\Pi_M} = \widetilde{d}f(d^+g)$, where $d^+$ and $\widetilde{d}$ are the differentials of $A^+$ and $\widetilde{A}$, respectively (see [8]).

Define the map $j^1 : C^\infty(M) \to C^\infty(A)$ such that $j^1f$ is the affine function on $A$ associated with $\widetilde{d}f \in \Gamma(A^+)$. Then, using (1), we deduce

**Proposition 2.** *Suppose that* $(\Lambda_A, E_A)$ *and* $(\llbracket\cdot,\cdot\rrbracket_V, D, \rho)$ *are compatible. Then, the Poisson structure* $\Pi_M$ *induced on* $M$ *is related with* $(\Lambda_A, E_A)$ *as follows*

$$\{f,g\}_{\Pi_M} \circ \tau = \{f\circ\tau, j^1 g\}_{(\Lambda_A,E_A)} - (f\circ\tau)\{1, j^1 g\}_{(\Lambda_A,E_A)}.$$

*In addition, if* $e^0$ *generates an ideal of* $\Gamma(A^+)$*, the dual bundle to* $V$*,* $V^*$*, admits a Lie algebroid structure such that* $(V,V^*)$ *is a Lie bialgebroid.*

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# Constraint algorithm for singular field theories

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**Abstract.** We present a geometric constraint algorithm for multisymplectic field theories in a covariant way, which is suitable both for the Lagrangian and Hamiltonian formalisms.

*Keywords:* Fibre bundles, connections, multisymplectic manifolds, constraints, field theories.

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## 1. Introduction

It is well known that for systems of ODE's describing singular dynamical systems, the existence and uniqueness of solutions are not assured. In many cases there are geometrical constraint algorithms that, in the most favourable situations, give a maximal submanifold of the phase space of the system, where consistent solutions exist [6], [7].

The same problems arise when considering system of PDE's associated with field theories described by singular Lagrangians, as well as in some other applications related with optimal control theories. Working in the framework of the multisymplectic description for this theories, we present a geometric algorithm for finding the maximal submanifold where there are consistent solutions of the system, reducing the problem to another one in the realm of linear algebra. (For the proofs of the results and other details see [9]).

Manifolds are real, paracompact, connected and $C^\infty$. Maps are $C^\infty$. Sum over crossed repeated indices is understood.

## 2. Linear theory

Let $\mathcal{W}$ and $\mathcal{E}$ be $\mathbb{R}$-vector spaces, such that $\dim \mathcal{E} = m$, $\dim \mathcal{W} = m+n$, and $\sigma\colon \mathcal{W} \to \mathcal{E}$ a surjective morphism. Denote $\mathrm{V}(\sigma) = \ker \sigma$, and $\jmath\colon \mathrm{V}(\sigma) \hookrightarrow \mathcal{W}$ the natural injection. Let $\eta \in \Lambda^m\mathcal{E}^*$ be a volume element, and $\omega = \sigma^*(\eta)$. Finally, consider $\Omega \in \Lambda^{m+1}\mathcal{W}^*$, and a subspace $\mathcal{C} \subseteq \mathcal{W}$. We consider the following problems in $(\sigma; \eta, \Omega, \mathcal{C})$:

*Statement* 1. To find an $m$-vector $\mathcal{X} \in \Lambda^m\mathcal{C}$ such that:
1. $\mathcal{X}$ is decomposable. 2. $i(\mathcal{X})\omega = 1$. 3. $i(\mathcal{X})\Omega = 0$.

*Statement* 2. To find a linear map $\mathbf{h}\colon \mathcal{E} \to \mathcal{C} \subseteq \mathcal{W}$ such that:
1. $\sigma \circ \mathbf{h} = \mathrm{Id}_{\mathcal{E}}$. 2. $[i(w)\Omega]|_{\mathrm{Im}\,\mathbf{h}} = 0$, $\forall w \in \mathcal{W}$.

**Proposition 1.** *The statements 1 and 2 are equivalent; that is, every solution to some of these problems is also a solution to the other.*

Let $\nabla\colon \mathcal{E} \to \mathcal{W}$ be a section of $\sigma$, and denote $\mathrm{H}(\nabla) := \mathrm{Im}\,\nabla$. We have the splitting $\mathcal{W} = \mathrm{H}(\nabla) \oplus \mathrm{V}(\sigma)$, where $\mathrm{H}(\nabla)$ is the *horizontal subspace* of $\nabla$, and $\mathrm{V}(\sigma)$ is the *vertical subspace* of $\sigma$. We also have the induced projections $\sigma^H_\nabla\colon \mathcal{W} \to \mathcal{W}$ and $\sigma^V_\nabla\colon \mathcal{W} \to \mathcal{W}$, with $\sigma^H_\nabla + \sigma^V_\nabla = \mathrm{Id}_{\mathcal{W}}$, and $w = w^H_\nabla + w^V_\nabla$, $\forall w \in \mathcal{W}$. $w^H_\nabla \in \mathrm{H}(\nabla)$, and $w^V_\nabla \in \mathrm{V}(\sigma)$ are called the *horizontal* and *vertical components* of $w$ induced by $\nabla$. Furthermore we have the splitting $\mathcal{W}^* = \mathrm{H}^*(\nabla) \oplus \mathrm{V}^*(\sigma)$, which induces the bigradation $\Lambda^k\mathcal{W}^* = \bigoplus\limits_{p,q=0,\ldots,k;\ p+q=k} (\Lambda^p\mathrm{H}^*(\nabla) \oplus \Lambda^q\mathrm{V}^*(\sigma))$. As a consequence we can write $\Omega = \Omega^{(m,1)} + \Omega^\nabla$; $\Omega^{(m,1)}$ being a $(m+1)$-form of bidegree $(m,1)$ and $\Omega^\nabla$ a $(m+1)$-form that includes the rest of components.

**Definition 1.** Let $Z \in \Lambda^m\mathcal{E} \mid \eta(Z) = 1$; then it is unique and decomposable. We define $\mathcal{Y}^\nabla_\eta = \Lambda^m\nabla(Z) \in \Lambda^m\mathcal{W}$, which is the $m$-vector associated to $\nabla$ and $\eta$. (It generates $\Lambda^m\mathrm{H}(\nabla)$).

$\mathcal{Y}^\nabla_\eta$ is decomposable and $\omega(\mathcal{Y}^\nabla_\eta) = 1$.

**Proposition 2.** $\Omega^{(m,1)} = \omega\wedge\gamma^\nabla_\eta$, *with* $\gamma^\nabla_\eta := i(\mathcal{Y}^\nabla_\eta)\Omega$. *Thus* $\Omega = \Omega^\nabla + \omega\wedge\gamma^\nabla_\eta$.

Finally, if $\mathbf{h}\colon \mathcal{E}\to\mathcal{C}$ is a linear map, $\nabla$ induces a splitting $\mathbf{h} = \mathbf{h}^H_\nabla + \mathbf{h}^V_\nabla = \sigma^H_\nabla\circ\mathbf{h} + \sigma^V_\nabla\circ\mathbf{h}$, and the endomorphism $\widetilde{\mathbf{h}^V_\nabla} = \mathbf{h}^V_\nabla\circ\sigma = \sigma^V_\nabla\circ\mathbf{h}\circ\sigma\colon \mathcal{W}\to\mathcal{W}$.
*Assumption* 1. $\Omega^\nabla$ is of bidegree $(m-1,2)$. Then $\Omega = \Omega^{(m,1)} + \Omega^{(m-1,2)}$.

$\nabla$ induces also the $\mathbb{R}$-bilinear map

$$\begin{array}{rcccc}\flat^\nabla_{\Omega\mathcal{C}} & : & \mathcal{E}^*\otimes\mathcal{C} & \to & \mathcal{E}^*\otimes \mathrm{H}(\nabla)\times \mathrm{V}^*(\sigma) \\ & & \mathbf{h} & \mapsto & (\mathbf{h}^H_\nabla, i(i([\widetilde{\mathbf{h}^V_\nabla}]^t)\mathcal{Y}^\nabla_\eta)(\Omega|_{\mathrm{V}(\sigma)})\ ,\end{array}$$

where $(i([\widetilde{\mathbf{h}^V_\nabla}]^t)\mathcal{Y}^\nabla_\eta)(\beta^1,\dots,\beta^m) := \sum_{\alpha=1}^m \mathcal{Y}^\nabla_\eta(\beta^1,\dots,[\widetilde{\mathbf{h}^V_\nabla}]^t(\beta^\alpha),\dots,\beta^m)$, for every $\beta^1,\dots,\beta^m\in\mathcal{W}^*$. If $\mathcal{Y}^\nabla_\eta = w_1\wedge\ldots\wedge w_m$, for $w_\alpha\in\mathcal{W}$, then $i([\widetilde{\mathbf{h}^V_\nabla}]^t)\mathcal{Y}^\nabla_\eta = \sum_{\alpha=1}^m w_1\wedge\ldots\wedge\widetilde{\mathbf{h}^V_\nabla}(w_\alpha)\wedge\ldots\wedge w_m$.

**Theorem 3.** *The necessary and sufficient condition for a linear map* $\mathbf{h}\colon\mathcal{E}\to\mathcal{C}$ *to be a solution to the problem posed in Statement 2 is that*

$$\flat^\nabla_{\Omega\mathcal{C}}(\mathbf{h}) = (\jmath_{\mathrm{H}(\nabla)}\circ(\sigma|_{\mathrm{H}(\nabla)})^{-1}, -\gamma^\nabla_\eta|_{\mathrm{V}(\sigma)})\ ,$$

*where* $\jmath_{\mathrm{H}(\nabla)}\colon \mathrm{H}(\nabla)\to\mathcal{W}$ *denotes the natural injection.*

**Corollary 4.** *A linear map* $\mathbf{h}\colon\mathcal{E}\to\mathcal{C}$ *is a solution to the problem posed in Statement 2 iff* $\mathbf{h}^H_\nabla = \jmath_{\mathrm{H}(\nabla)}\circ(\sigma|_{\mathrm{H}(\nabla)})^{-1}$ *and* $[i(i([\widetilde{\mathbf{h}^V_\nabla}]^t)\mathcal{Y}^\nabla_\eta)\Omega]|_{\mathrm{V}(\sigma)} = -\gamma^\nabla_\eta|_{\mathrm{V}(\sigma)}$.

**Definition 2.** The orthogonal complement of $\mathcal{C}$ with respect to $\Omega$ and $\nabla$ is

$$(\mathcal{C}^\perp)^\nabla_\Omega := (\mathrm{Im}\,\flat^\nabla_{\Omega\mathcal{C}})^0 \subset (\mathcal{E}\otimes\mathrm{V}(\sigma)^0)\times\mathrm{V}(\sigma)\ .$$

**Theorem 5.** *There exists a solution* $\mathbf{h}\colon\mathcal{E}\to\mathcal{W}$ *to the problem posed in Statement 2 if, and only if,*

$$\mathbf{h}^t(\jmath_{\mathrm{H}(\nabla)}\circ(\sigma|_{\mathrm{H}(\nabla)})^{-1}) - \gamma^\nabla_\eta(Z) = 0 \quad , \quad \text{for every } (\mathbf{h}^t, Z)\in(\mathcal{C}^\perp)^\nabla_\Omega\ . \tag{1}$$

## 3. The general multisymplectic case

Let $\kappa\colon F\to M$ be a fibre bundle with $\dim M = m > 1$, $\dim F = n+m$, and $\eta\in\Omega^m(M)$ a volume form on $M$. Denote $\omega = \kappa^*\eta$.

**Definition 3.** $\Omega\in\Omega^{m+1}(F)$ is a multisymplectic form if it is closed and 1-nondegenerate (the map $\flat_\Omega : \mathrm{T}M \longrightarrow \Lambda^m\mathrm{T}^*M$ is injective). Then $(F,\Omega,\omega)$ is a multisymplectic (regular) system. Otherwise, $\Omega$ is a pre-multisymplectic form, and the system is pre-multisymplectic (singular).

Now we state the following problem:

*Statement* 3. Given a pre-multisymplectic system $(F,\Omega,\omega)$. To find a submanifold $\jmath_C\colon C\hookrightarrow F$, and a $\kappa$-transverse, locally decomposable, integrable $m$-vector field $\mathcal{X}_C$ along $C$, such that $i(\mathcal{X}_C(y))\Omega(y)=0$, $\forall y\in C$.

First we obviate the *integrability condition*, and the problem consists in finding $C\hookrightarrow F$ and a locally decomposable $m$-vector field $\mathcal{X}_C\in\mathfrak{X}^m(F)$ along $C$ such that $i(\mathcal{X}_C(y))\omega(y)=1$, and $i(\mathcal{X}_C(y))\Omega(y)=0$, $\forall y\in C$.

Then, from proposition 1, we have:

**Proposition 6.** *In $C\hookrightarrow F$, there is a solution of the problem stated in Statement 3 if, and only if,, $\forall y\in C$, $\exists\mathbf{h}_y\in \mathrm{T}^*_{\kappa(y)}M\otimes\mathrm{T}_yC$ such that*

1. *$\mathbf{h}_y$ is $\kappa$-transverse (it is a connection along $C$): $\mathrm{T}_y\kappa|_{\mathrm{T}_yC}\circ\mathbf{h}_y=Id$.*
2. *$\forall(X'_1)_{\kappa(y)},\dots,(X'_m)_{\kappa(y)}\in\mathrm{T}_{\kappa(y)}M$, $\forall Y_y\in\mathrm{T}_yF$,*

$$\Omega(y)(\mathbf{h}_y((X'_1)_{\kappa(y)}),\dots,\mathbf{h}_y((X'_m)_{\kappa(y)}),Y_y)=0$$

Let $\nabla$ be a connection in $\kappa\colon F\to M$, and $\mathcal{Y}^\nabla_\eta\in\mathfrak{X}^m(F)$ the corresponding locally decomposable $m$-vector field in $F$ such that $i(\mathcal{Y}^\nabla_\eta)\omega=1$. $\nabla$ induces the splitting $\Lambda^k\mathrm{T}^*F=\bigoplus\limits_{p,q=0,\dots,k;\ p+q=k}(\Lambda^p\mathrm{H}^*(\nabla)\oplus\Lambda^q\mathrm{V}^*(\kappa))$; where $\mathrm{H}(\nabla)\to M$ and $\mathrm{V}(\kappa)\to M$ are the *horizontal* and *vertical subbundles* associated with $\nabla$. Thus we have $\Omega=\Omega^{(m,1)}+\Omega^\nabla$.

**Proposition 7.** $\Omega^{(m,1)}=\omega\wedge\gamma^\nabla_\eta$*, with* $\gamma^\nabla_\eta:=i(\mathcal{Y}^\nabla_\eta)\Omega$*. Thus* $\Omega=\Omega^\nabla+\omega\wedge\gamma^\nabla_\eta$*.*

*Assumption* 2. The $(m+1)$-form $\Omega^\nabla$ is of bidegree $(m-1,2)$.

(This is the situation in the Lagrangian and Hamiltonian field theories).

In order to solve the problem, we work at every point of the manifolds involved in this problem, and we apply the results given in section 2. Thus, if $y$ is a point of $C$, we make the following identifications:

$$\mathcal{E}\equiv\mathrm{T}_{\kappa(y)}M\quad,\quad\mathcal{W}\equiv\mathrm{T}_yF\quad,\quad\mathcal{C}\equiv\mathrm{T}_yC\quad,\quad\mathrm{V}(\sigma)\equiv\mathrm{V}_y(\kappa)\ .$$

*Assumption* 3. In the sequel, every subset $C_i$ is a regular submanifold of $F$, and its natural injection is an embedding.

Thus, we consider the submanifold $C_1\hookrightarrow F$ where a solution exists:

$$\begin{aligned}C_1\ =\ \{y\in F\ \mid\ &\exists\mathbf{h}_y\in Lin(\mathrm{T}_{\kappa(y)}M,\mathrm{T}_yF)\text{ such that }(\mathbf{h}_y)^H_\nabla=\mathrm{T}_y\kappa_{\mathrm{H}(\nabla)},\\ &[i(i([\widetilde{(\mathbf{h}_y)^V_\nabla}]^t)(\mathcal{Y}^\nabla_\eta(y)))(\Omega(y))]|_{\mathrm{V}_y(\kappa)}=-(\gamma^\nabla_\eta(y))|_{\mathrm{V}_y(\kappa)}\}.\end{aligned}$$

Then, there exists a locally decomposable section $\mathcal{X}_1$ of $\Lambda^m \mathrm{T}_{C_1} F \to C_1$ such that $(i(\mathcal{X}_1)\omega)|_{C_1} = 1$ and $(i(\mathcal{X}_1)\Omega)|_{C_1} = 0$. But,in general, $\mathbf{h}_y(\mathrm{T}_{\kappa(y)}M)$ is not a subspace of $\mathrm{T}_y C_1$, therefore $\mathcal{X}_1$ is not tangent to $C_1$. Thus we define

$$\begin{aligned} C_2 \;=\; &\{y_1 \in C_1 \mid \exists \mathbf{h}_{y_1} \in Lin(\mathrm{T}_{\kappa(y_1)}M, \mathrm{T}_{y_1}C_1) \text{ such that } (\mathbf{h}_{y_1})^H_\nabla = \mathrm{T}_{y_1}\kappa_{\mathrm{H}(\nabla)}, \\ &[i(i([\widetilde{(\mathbf{h}_y)^V_\nabla}]^t)(\mathcal{Y}^\nabla_\eta(y)))(\Omega(y_1))]|_{\mathrm{V}_{y_1}(\kappa)} = -(\gamma^\nabla_\eta(y_1))|_{\mathrm{V}_{y_1}(\kappa)}\}, \end{aligned}$$

and there exists a locally decomposable section $\mathcal{X}_2$ of $\Lambda^m \mathrm{T}_{C_2} C_1 \to C_2$ such that $(i(\mathcal{X}_2)\omega)|_{C_2} = 1$ and $(i(\mathcal{X}_2)\Omega)|_{C_2} = 0$. But $\mathcal{X}_2$ is not tangent to $C_2$, and the procedure follows giving a sequence of submanifolds $\cdots \overset{j^i_{i+1}}{\hookrightarrow} C_i \overset{j^{i-1}_i}{\hookrightarrow} \cdots \overset{j^1_2}{\hookrightarrow} C_1 \overset{j_1}{\hookrightarrow} C_0 \equiv F$. For every $i \geq 1$, $C_i$ is called the *ith constraint submanifold*. We have two possibilities: $\exists k$ such that $\dim C_k < m$, and then the problem has no solution, or $\exists k$ such that $C_{k+1} = C_k \equiv C_f$; then there exists a connection $\mathcal{X}_f$ in $\kappa\colon F \to M$ along $C_f$ such that $i(\mathcal{X}_f(y_f))(\Omega(y_f)) = 0$, $\forall y_f \in C_f$, and $C_f$ is the *final constraint submanifold*, and $\dim C_f \geq m$.

**Theorem 8.** *Every constraint submanifold can be defined as*

$$C_i = \{y_{i-1} \in C_{i-1} \mid \langle((\mathrm{T}\kappa_{\mathrm{H}(\nabla)})^{-1}, -(\gamma^\nabla_\eta)|_{\mathrm{V}(\kappa)})(y_{i-1}), (\mathrm{T}^\perp_{y_{i-1}} C_{i-1})^\nabla_\Omega\rangle = 0\}.$$

*Therefore, considering the vector bundle over $F$, $W(\kappa, \nabla) = (\kappa^*(\mathrm{T}^*M) \otimes \mathrm{H}(\nabla)) \oplus_F \mathrm{V}^*(\kappa)$, if $(\mathrm{T}^\perp C_{i-1})^\nabla_\Omega$ is a vector subbundle of rank $r$ of $W^*_{C_{i-1}}(\kappa, \nabla)$, and $\{(\mathbf{h}^*_1, Z_1)^{(i-1)}, \dots, (\mathbf{h}^*_r, Z_r)^{(i-1)}\}$ is a set of sections of $W^*(\kappa, \nabla) \to F$ spanning locally the space $\Gamma((\mathrm{T}^\perp C_{i-1})^\nabla_\Omega)$, then $C_i$, as a submanifold of $C_{i-1}$, is defined locally as the zero set of the functions $\xi^{(i)}_j \in C^\infty(F)$ given by*

$$\xi^{(i)}_j = ((\mathrm{T}_y\kappa_{\mathrm{H}(\nabla)})^{-1}, -(\gamma^\nabla_\eta)|_{\mathrm{V}(\kappa)})((\mathbf{h}^*_j, Z_j)^{(i-1)}).$$

*These functions are called* $i$th-generation constraints.

In general, $\mathcal{X}_f$ is not a flat connection. Nevertheless, in many cases, after applying an *integrability algorithm*, one may find a submanifold $\mathcal{I}$ of $C_f$ such that $(\mathcal{X}_f)|_{\mathcal{I}}$ is a flat connection in the fibration $\kappa\colon F \to M$ along $\mathcal{I}$.

## 4. Application to field theories

**Lagrangian field theory** (See [1], [3], [4], [10]): $M$ is an oriented manifold with volume form $\eta \in \Omega^m(M)$. $\pi\colon E \to M$ is the *configuration fiber bundle* ($\dim M = m$, $\dim E = N + m$). $\pi^1\colon J^1E \to E$ is the associated first-order jet bundle (*multivelocity phase bundle*), with $\bar{\pi}^1 = \pi \circ \pi^1\colon J^1E \to M$.

A *Lagrangian density* is a $\bar{\pi}^1$-semibasic $m$-form on $J^1E$, $\mathcal{L} = L\bar{\pi}^{1*}\eta$, where $L \in \mathrm{C}^\infty(J^1E)$ is the *Lagrangian function*. Then, we can define the associated *Poincaré-Cartan* $m$ *and* $(m+1)$*-forms* $\Theta_\mathcal{L} \in \mathit{\Omega}^m(J^1E)$ and $\Omega_\mathcal{L} = -\mathrm{d}\Theta_\mathcal{L} \in \mathit{\Omega}^{m+1}(J^1E)$. $\mathcal{L}$ is *regular* (resp. *singular*) if $\Omega_\mathcal{L}$ is a multisymplectic (resp. *pre-multisymplectic*) $(m+1)$-form.

The *Lagrangian problem* consists in finding integral sections $\phi \in \Gamma(M,E)$ of a class of *holonomic* $m$-vector fields $\{\mathcal{X}_\mathcal{L}\} \subset \mathfrak{X}^m(J^1E)$, such that $i(\mathcal{X}_\mathcal{L})\Omega_\mathcal{L} = 0$, for every $\mathcal{X}_\mathcal{L} \in \{\mathcal{X}_\mathcal{L}\}$ (Euler-Lagrange equations). Then, if the Lagrangian is singular, this is a particular case of the general problem stated in the statement 3, taking the pre-multisymplectic system $(J^1E, \Omega_\mathcal{L}, \bar{\pi}^{1*}\eta)$.

**Hamiltonian field theory** (See [2], [5], [8]): The *extended* and *restricted multimomentum bundles* are $\mathcal{M}\pi \equiv \Lambda_1^m\mathrm{T}^*E$ (the bundle of $m$-forms on $E$ vanishing by contraction with two $\pi$-vertical vector fields), and $J^{1*}E \equiv \mathcal{M}\pi/\Lambda_0^m\mathrm{T}^*E$. The natural projections are denoted by $\mu\colon \mathcal{M}\pi \to J^{1*}E$; $\tau^1\colon J^{1*}E \to E$; $\bar{\tau}^1 = \pi \circ \tau^1\colon J^{1*}E \to M$.

Introducing the *extended Legendre map* associated with $\mathcal{L}$, $\widetilde{\mathcal{FL}}\colon J^1E \to \mathcal{M}\pi$, we can define the *restricted Legendre map* associated with $\mathcal{L}$ as $\mathcal{FL} := \mu \circ \widetilde{\mathcal{FL}}\colon J^1E \to J^{1*}E$. Then $\mathcal{L}$ is *regular* if $\mathcal{FL}$ is a local diffeomorphism. Elsewhere it is called *singular*. $\mathcal{L}$ is *hyper-regular* if $\mathcal{FL}$ is a global diffeomorphism. $\mathcal{L}$ is *almost-regular* if $\mathcal{P} := \mathcal{FL}(J^1E) \hookrightarrow J^{1*}E$ is a closed submanifold of $J^{1*}E$, $\mathcal{FL}$ is a submersion onto its image, and the fibres $\mathcal{FL}^{-1}(\mathcal{FL}(\bar{y}))$ are connected submanifolds of $J^1E$. We denote by $\jmath_0\colon \mathcal{P} \hookrightarrow J^{1*}E$ the natural embedding, and by $\bar{\tau}_0^1 = \bar{\tau}^1 \circ \jmath_0\colon \mathcal{P} \to M$ the corresponding submersion.

$\mathcal{M}\pi$ is endowed with canonical forms $\Theta \in \mathit{\Omega}^m(\mathcal{M}\pi)$ and $\Omega := -\mathrm{d}\Theta \in \mathit{\Omega}^{m+1}(\mathcal{M}\pi)$. In the almost-regular case, there is a diffeomorphism $\tilde{\mu}\colon \mathrm{Im}\,\widetilde{\mathcal{FL}} \to \mathcal{P}$. Then, if $\tilde{\jmath}_0\colon \widetilde{\mathcal{FL}} \hookrightarrow \mathcal{M}\pi$ is the natural embedding, we can define the *Hamilton-Cartan forms* $\Theta_0 = (\tilde{\jmath}_0 \circ \tilde{\mu}^{-1})^*\Theta$, and $\Omega_0 = (\tilde{\jmath}_0 \circ \tilde{\mu}^{-1})^*\Omega$. Therefore, the *Hamiltonian problem* consists in finding integral sections $\psi \in \Gamma(M,\mathcal{P})$ of a class of integrable and $\bar{\tau}_0^1$-transverse $m$-vector fields $\{\mathcal{X}_0\} \subset \mathfrak{X}^m(\mathcal{P})$ satisfying that $i(\mathcal{X}_0)\Omega_0 = 0$, for every $\mathcal{X}_0 \in \{\mathcal{X}_0\}$ *(Hamilton-De Donder-Weyl equations)*. Thus, this is a particular case of the general problem stated in the statement 3, taking the pre-multisymplectic system $(\mathcal{P}, \Omega_0, \bar{\tau}_0^{1*}\eta)$.

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# A normal form theorem for integrable systems on contact manifolds

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**Abstract.** We present a normal form theorem for singular integrable systems on contact manifolds.

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## 1. Introduction

The theorem of Darboux is probably the first normal form theorem in symplectic geometry. This theorem has its analogue in contact geometry. Normal forms let us reduce the study of our system to model-like situations in which the computations are simplified. However the theorem of Darboux is local and does not take into account additional geometrical structures on the manifold. In this paper we review some normal forms results for integrable systems on symplectic manifolds and find an application to study normal forms for the contact analogous situation.

## 2. Completely integrable Hamiltonian systems on a symplectic manifold

We consider a completely integrable Hamiltonian system on a symplectic manifold $(M, \omega)$. It is given by a moment map $\mathbf{F} = (f_1, \dots, f_n)$. The condition $\{f_i, f_j\} = 0, \forall i, j$ implies that the distribution generated by the Hamiltonian vector fields $X_{f_i}$ is involutive. We denote this foliation by $\mathcal{F}$. This foliation has Lagrangian regular orbits and isotropic singular ones. A natural question in this situation arises: Can we find a classification theorem for completely integrable Hamiltonian systems in a neighbourhood of an orbit $L$?

In the case $L$ is a regular compact orbit, the theorem of Liouville-Mineur-Arnold for integrable systems gives a positive answer to this question. The existence of action-angle coordinates in a neighbourhood of $L$ implies that the completely integrable system is equivalent in a neighbourhood of $L$ to the completely integrable system determined by the action functions and the Darboux symplectic form.

If $L$ is singular the problem remains unsolved in general. In the case the orbits of the integrable system are compact and $L$ is a singular nondegenerate orbit, the answer is given by the following theorem due to the author and Nguyen Tien Zung:

**Theorem 1** (Miranda, Nguyen Tien Zung [11]). *Under the hypotheses mentioned above, the completely integrable Hamiltonian system is symplectically equivalent to the linearized integrable Hamiltonian system with the Darboux symplectic form. In the case there exists a symplectic action of a compact Lie group $G$ preserving the system, this equivalence can be established in a $G$-equivariant way.*

**Remarks.**

1) In the case $\dim L = 0$ and $L$ is nondegenerate Eliasson ([4], [5]) established local linear models for the singularity and provided a complete proof for the symplectic equivalence with the linear model in any dimension in the completely elliptic case.

2) Details of Eliasson's proof have been recently clarified by the author and Vu Ngoc San in [12]. In [10] the author provided a complete proof of Eliasson's result in cases other than elliptic. This proof uses a generalization of the Morse Isochore lemma and Moser's path method for foliations to achieve a symplectically orthogonal decomposition into 2 and 4-dimensional cells depending on the Williamson type of the singularity.

3) This result generalizes previous partial results for nondegenerate compact singular orbits of rank greater than 0. In particular it generalizes the result of Eliasson ([5]) in the case the orbits are of completely elliptic type. It also generalizes the results of Colin de Verdière and San Vu Ngoc ([2]) and Currás-Bosch and Miranda ([3]) in the case $\dim M = 4$ and $\dim L = 1$.

# 3. Integrable systems on contact manifolds

The aim of this section is to present an analogue to the linearization result for singular integrable Hamiltonian systems stated but in the case of singular integrable system in contact manifolds.

Consider a contact manifold $M^{2n+1}$ together with a contact form. We assume that the Reeb vector field associated to $\alpha$ coincides with the infinitesimal generator of an $S^1$ action. We assume further than there exists $n$-first integrals of the Reeb vector field which commute with respect to the Jacobi bracket. Then there are two foliations naturally attached to the situation. On the one hand, we can consider the foliation associated to the distribution generated by the contact vector fields. We call this foliation $\mathcal{F}'$. On the other hand we can consider a foliation $\mathcal{F}$ given by the horizontal parts of the contact vector fields. The functions determining the contact vector fields may have singularities. We will always assume that those singularities are of non-degenerate type. Observe that $\mathcal{F}'$ is nothing but the enlarged foliation determined by the foliation $\mathcal{F}$ and the Reeb vector field.

Let $\alpha'$ be another contact form in a neighbourhood of a compact orbit $\mathcal{O}$ of $\mathcal{F}'$ for which $\mathcal{F}$ is generically Legendrian and such that the Reeb vector field with respect to $\alpha'$ coincides with the Reeb vector field associated to $\alpha$. A natural question is to know if $\alpha$ is equivalent to $\alpha'$. This entails naturally the study of the existence of normal forms for $\alpha$ in a neighbourhood of $\mathcal{O}$ preserving the foliation $\mathcal{F}$.

The condition that the Reeb vector field is an infinitesimal generator of an $S^1$-action is fulfilled in many examples present in contact geometry. For instance, model contact structures for a transverse knot can be obtained by considering contact forms satisfying this condition (see for example [6]). Furthermore, as proved in [7], a contact form whose Reeb flow generates a torus action is "stable" in the sense that the Reeb flow of any $C^2$-close contact form has at least one periodic orbit.

The problem of determining normal forms for foliations related to Legendrian foliations has its own story. P. Libermann in [8] established a local equivalence theorem for $\alpha$-regular foliations. Loosely speaking, those foliations are regular foliations containing the Reeb vector field and a Legendrian foliation. The problem of classifying contact structures which are invariant under a Lie group was considered by Lutz in [9]. The foliations studied by Libermann and Lutz are regular. The singular counterpart to the result of Lutz was proved by Banyaga and Molino in [1] but for contact forms. Namely, Banyaga and Molino study the problem of finding normal forms under the additional assumption of transversal ellipticity. The assumption of transversal ellipticity allows to relate the foliation $\mathcal{F}'$ of generic dimension $(n+1)$ with the foliation given by the orbits of a torus action.

The results that we present here and whose proof is contained in [10] pretend to extend these results for foliations which are related in the same sense to foliations with generical $(n+1)$-dimensional leaves but which are not necessarily identified with the orbits of a torus action. All our study of the problem is done in a neighbourhood of a compact singular orbit.

## The linear model for the contact setting

Let $(M^{2n+1}, \alpha)$ be a contact pair and let $Z$ be its Reeb vector field. We assume that $Z$ coincides with the infinitesimal generator of an $S^1$ action. We also assume that there are $n$ first integrals $f_1, \dots, f_n$ of $Z$ which are generically independent and which are pairwise in involution with respect to the Jacobi bracket associated to $\alpha$. Let $\mathcal{O}$ be the orbit of the foliation $\mathcal{F}'$ through a point $p$ in $M^{2n+1}$. We will assume that $\mathcal{O}$ is diffeomorphic to a torus of dimension $k+1$ and that the singularity is non-degenerate in the Morse-Bott sense along $\mathcal{O}$.

In [10] it is proven that there exists a diffeomorphism from a neighbourhood of $\mathcal{O}$ to a model manifold $M_0^{2n+1}$ taking the foliation $\mathcal{F}'$ to a linear foliation in the model manifold with a finite group attached to it and taking the initial contact form to the Darboux contact form.

**Theorem 2** ((Miranda [10])).
*There exist coordinates $(\theta_0, ..., \theta_k, p_1, ..., p_k, x_1, y_1, ..., x_{n-k}, y_{n-k})$ in a finite covering of a tubular neighbourhood of $\mathcal{O}$ such that,*

- *The Reeb vector field is $Z = \frac{\partial}{\partial \theta_0}$.*
- *There exists a triple of natural numbers $(k_e, k_h, k_f)$ with $k_e + k_h + 2k_f = n-k$ and such that the first integrals $f_i$ are of the following type, $f_i = p_i, \quad 1 \leq i \leq k$, and*

  $$\begin{aligned} &f_{i+k} = x_i^2 + y_i^2 \quad \text{for} \quad 1 \leq i \leq k_e \ , \\ &f_{i+k} = x_i y_i \quad \text{for} \quad k_e + 1 \leq i \leq k_e + k_h \ , \\ &f_{i+k} = x_i y_{i+1} - x_{i+1} y_i \quad \text{and} \\ &f_{i+k+1} = x_i y_i + x_{i+1} y_{i+1} \quad \text{for} \quad i = k_e + k_h + 2j - 1, \ 1 \leq j \leq k_f \end{aligned}$$

- *The foliation $\mathcal{F}$ is given by the orbits of $\mathcal{D} =< Y_1, \dots Y_n >$ where $Y_i = X_i - f_i Z$ being $X_i$ the contact vector field of $f_i$ with respect to the contact form $\alpha_0 = d\theta_0 + \sum_{i=1}^{n-k} \frac{1}{2}(x_i dy_i - y_i dx_i) + \sum_{i=1}^{k} p_i d\theta_i$.*

The model manifold is the manifold $M_0^{2n+1} = \mathbb{T}^{k+1} \times U^k \times V^{2(n-k)}$, where $U^k$ and $V^{2(n-k)}$ are $k$-dimensional and $2(n-k)$ dimensional disks respectively endowed with coordinates $(\theta_0, \ldots, \theta_k)$ on $\mathbb{T}^{k+1}$, $(p_1, \ldots, p_k)$ on $U^k$ and $(x_1, \ldots, x_{n-k}, y_1, \ldots y_{n-k})$ on $V^{2(n-k)}$. The linear model for the foliation $\mathcal{F}'$ is the foliation expressed in the coordinates provided by the theorem together with a finite group attached to the finite covering. The different smooth submodels corresponding to the model manifold $M_0^{2n+1}$ are labeled by a finite group which acts in a contact fashion and preserves the foliation in the model manifold. This is the only differentiable invariant. In fact, this finite group comes from the isotropy group of an associated Hamiltonian action. In the symplectic case this finite group was already introduced in [13].

### Contact equivalence in the model manifold

**Theorem 3** (Miranda [10]). *Let $\alpha$ be a contact form on the model manifold $M_0^{2n+1}$ for which $\mathcal{F}$ is a generically Legendrian foliation and such that the Reeb vector field is $\frac{\partial}{\partial \theta_0}$. Then there exists a diffeomorphism $\phi$ defined in a neighbourhood of the singular orbit $\mathcal{O} = (\theta_0, \ldots, \theta_k, 0, \ldots, 0)$ preserving $\mathcal{F}'$ and taking $\alpha$ to $\alpha_0$.*

### The $G$-equivariant result

Consider a compact Lie group $G$ acting on contact model manifold in such a way that preserves the $n$ first integrals of the Reeb vector field and preserves the contact form as well. In [10] we prove that there exists a diffeomorphism in a neighbourhood of $\mathcal{O}$ preserving the $n$ first integrals, preserving the contact form and linearizing the action of the group. Namely we prove,

**Theorem 4** (Miranda [10]). *There exists a diffeomorphism $\phi$, preserving $F = (f_1, \ldots, f_n)$ defined in a tubular neighbourhood of $\mathcal{O}$ such that $\phi^*(\alpha_0) = \alpha_0$ and such that $\phi \circ \rho_g = \rho_g^{(1)} \circ \phi$, being $\rho_g^{(1)}$ the linearization of $\rho_g$ .*

### Contact linearization

Applying this $G$-equivariant version to the particular case of the finite group attached to the finite covering, we obtain as a consequence the following contact linearization result:

**Theorem 5** (Miranda [10]). *Let $\alpha$ be a contact form for which $\mathcal{F}$ is generically Legendrian and such that $Z$ is the Reeb vector field then there exists a diffeomorphism defined in a neighbourhood of $\mathcal{O}$ taking $\mathcal{F}'$ to the linear foliation, the orbit $\mathcal{O}$ to the torus $\{x_i = 0, y_i = 0, p_i = 0\}$ and taking the contact form to the Darboux contact form $\alpha_0$.*

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# Isoperimetric problems inside Euclidean convex domains

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**Abstract.** In these notes we review some facts about sets minimizing a perimeter functional under a volume constraint inside a Euclidean domain. First, we summarize what is known about these sets when the functional is the *Euclidean perimeter* and the ambient domain is a *convex body*. Second, we give a brief description of our results in [**RRo**], where we have studied some properties of sets minimizing the *relative perimeter* within a *solid cone* for fixed volume.

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## 1. Introduction

An *isoperimetric problem* is one in which we try to minimize a perimeter functional under one of more volume constraints. In this paper we study two of the simplest formulations of the isoperimetric problem inside a Euclidean domain. Let us precise the situation. We denote by $\Omega$ a smooth *domain* (connected, open set) of $\mathbb{R}^{n+1}$. For any $E \subseteq \Omega$, we consider

- $\mathrm{vol}(E) = (n+1)$-Lebesgue measure of $E$,
- $\mathcal{P}(E) =$ Euclidean perimeter of $E$,
- $\mathcal{P}(E, \Omega) =$ perimeter of $E$ relative to $\Omega$.

These notions of perimeter are defined in the sense of De Giorgi [**M**]. If, for instance, $E$ has $C^2$ boundary, then $\mathcal{P}(E) = \mathcal{H}^n(\partial E)$ and $\mathcal{P}(E,\Omega) = \mathcal{H}^n(\partial E \cap \Omega)$, where $\mathcal{H}^n(\cdot)$ is the $n$-dimensional Hausdorff measure in $\mathbb{R}^{n+1}$. This means that only the boundary area of $E$ *inside* $\Omega$ contributes to the relative perimeter.

In these notes we show some properties of sets minimizing one of the perimeter functionals defined above for a fixed volume $V \leqslant \mathrm{vol}(\Omega)$. These sets are called *isoperimetric regions* -or simply *minimizers*- of volume $V$. When the perimeter to minimize is the relative one, the problem is usually referred to as a *free boundary problem*.

In the study of an isoperimetric problem the first questions taken into consideration are related to the existence and regularity of minimizers. Recently, geometric and topological properties have been treated by many authors. However, in spite of the last advances, the complete description of isoperimetric regions has been achieved only for some few domains of $\mathbb{R}^{n+1}$. A survey of most of these results which also includes some recent progress and open questions can be found in [**R**], see also [**RR**].

Our main purpose here is to review the results we have obtained in two different situations

(i) $\Omega$ is a *convex body* and the functional to minimize for fixed volume is the Euclidean perimeter.

(ii) $\Omega$ is a *solid cone* and the perimeter functional is the relative one.

## 2. Convexity of minimizers inside a convex body

Let $\Omega$ be a bounded, smooth, convex domain of $\mathbb{R}^{n+1}$. In this section we will study minimizers for the Euclidean perimeter restricted to the class of subsets of $\Omega$ enclosing a given volume.

First, as $\overline{\Omega}$ is compact, we can apply standard results in Geometric Measure Theory [**M**] to ensure that for any $V \leqslant \mathrm{vol}(\Omega)$ there exists an isoperimetric region $E$ of volume $V$. Moreover, we have that $\partial E \cap \Omega$ is a smooth hypersurface with constant mean curvature, except for a closed set of singularities whose Hausdorff dimension is less than or equal to $n-7$. As $\Omega$ has smooth boundary, it is also known ([**SZ**, Theorem 3.6]) that $\partial E$ is a $C^{1,1}$ hypersurface around any $p \in \partial E \cap \partial\Omega$, and that $\partial E$ meets $\partial\Omega$ *tangentially*.

The classical isoperimetric inequality in $\mathbb{R}^{n+1}$ implies that any minimizer $E$ in $\Omega$ such that $\mathrm{vol}(E)$ does not exceed the volume of a largest ball contained in $\Omega$ must be a round ball. For larger volumes, minimizers cannot be round balls and the regularity properties mentioned above are not enough, in general, to describe them. In fact, the classification of isoperimetric regions inside an arbitrary convex body is an interesting and difficult problem. In the study of this question we find the following plausible conjecture proposed by E. Stredulinsky and W. Ziemer

**Conjecture** ([**SZ**]): *Isoperimetric regions for the Euclidean perimeter inside a convex body must be convex.*

What is known at this moment? Some partial results but not the complete answer. For example, the conjecture is true for any planar convex body; this easily follows from the fact that the convex hull of a domain of $\mathbb{R}^2$ increases volume while decreasing boundary length. For higher dimension the conjecture has a positive answer when assuming additional conditions on $\Omega$. In [**SZ**, Theorem 3.32] it is shown that the conjecture holds if $\Omega$ satisfies a *great circle condition*. This means that there is a largest ball $B$ of $\Omega$, and a hyperplane $P$ passing through the center of $B$, such that $\partial B \cap P \subset \partial\Omega$. This property allows also to prove that isoperimetric regions in $\Omega$ of volumes $V \geqslant V_0$ are unique and nested. Here $V_0$ is the volume of the set in $\Omega$ given by the union of all largest balls of $\Omega$.

In [**Ro**] we provide a new condition under which the conjecture turns out to be true. In precise terms, by using symmetrization arguments and the classification of stable constant mean curvature hypersurfaces of revolution in $\mathbb{R}^{n+1}$, it is proved

**Theorem 1.** *Let $\Omega \subset \mathbb{R}^{n+1}$ be a smooth convex body which is rotationally symmetric about a straight line. Then, minimizers inside $\Omega$ are convex.*

In [**Ro**] we also give an example illustrating that isoperimetric regions of volumes $V \geqslant V_0$ inside a rotationally symmetric $\Omega$ need not be nested.

The last result to our knowledge about the convexity of minimizers in a convex body has been recently established by F. Alter, V. Caselles and A. Chambolle [**ACC**]. These authors have shown the existence of a value $V_1 \geqslant V_0$ such that minimizers in $\Omega$ of volumes $V \geqslant V_1$ are unique, convex and nested (recall that $V_0$ is the volume of the set given by the union of all largest balls inside $\Omega$). This result together with the example in [**Ro**] mentioned above suggests that in the interval of volumes $[V_0, V_1]$ we cannot control the convexity nor the nestedness of minimizers. In fact, the conjecture by E. Stredulinsky and W. Ziemer remains open for an arbitrary convex body in $\mathbb{R}^{n+1}$, $n \geqslant 2$.

## 3. The free boundary problem inside a solid cone

In this section, the domain $\Omega$ will be a *smooth solid cone* of $\mathbb{R}^{n+1}$, that is, a cone $0 \times\!\!\!\!\times C$ over a smooth domain $C \subset \mathbb{S}^n$. The isoperimetric *free boundary problem* in $\Omega$ consists of finding a minimum for the *relative perimeter* functional $\mathcal{P}(\cdot\,,\Omega)$ in the class of sets contained in $\Omega$ and enclosing a given volume $V > 0$. The *isoperimetric profile* of $\Omega$ is the function

$$I_\Omega(V) = \inf\,\{\mathcal{P}(E,\Omega) : E \subset \Omega,\ \mathrm{vol}(E) = V\}, \qquad V > 0. \tag{1}$$

As a particular case, if $C$ is a half-sphere of $\mathbb{S}^n$ then $\Omega$ coincides with a half-space of $\mathbb{R}^{n+1}$. It is well-known that the isoperimetric regions in a half-space are the half-balls centered at the boundary of the half-space.

In general, it is not easy to decide whether minimizers exist or not inside an unbounded domain of $\mathbb{R}^{n+1}$. In fact, we can find non convex cones in $\mathbb{R}^{n+1}$ $(n \geqslant 2)$ for which there are no minimizers of any given volume. In the next section we will study this important question in more detail.

## 3.1. Existence and regularity results. Planar cones

The direct method of the Calculus of Variations to minimize a functional consists of taking a *minimizing sequence* and trying to extract a convergent subsequence. The problem which appears when working in a noncompact setting is that part or all of a minimizing sequence could diverge. Let us precise this fact. Consider a fixed $V > 0$, and a sequence in $\Omega$ of sets of volume $V$ whose relative perimeter tends to $I_\Omega(V)$. In [**RRo**, Theorem 2.1] we have proved that such a sequence splits in two disjoint parts: one of them converges to an isoperimetric region $E$, and the other one diverges. At first, a fraction of $V$ could disappear at infinity and so, the volume of $E$ may not coincide with $V$. However, by using that a cone is invariant under *dilations* centered at its vertex, we can control the volume and perimeter of the diverging part. With this scheme we have established the following

**Theorem 2** ([**RRo**]). *Let $\Omega$ be a smooth solid cone in $\mathbb{R}^{n+1}$. Then, either there are isoperimetric regions in $\Omega$ of any given volume, or the isoperimetric profile $I_\Omega$ defined in* (1) *coincides with the one of a half-space in $\mathbb{R}^{n+1}$.*

The theorem above allows us to give some criteria ensuring existence of minimizers in a cone. Recall that $\mathcal{H}^k(\cdot)$ represents the $k$-dimensional Hausdorff measure in $\mathbb{R}^{n+1}$.

**Corollary 3** ([**RRo**]). *Suppose that the cone $\Omega = 0 \times\!\!\!\!\times C$ of $\mathbb{R}^{n+1}$ satisfies one of the following conditions*

(i) *$\Omega$ admits a local supporting hyperplane at a point $p \in \partial\Omega \setminus \{0\}$.*

(ii) *$\mathcal{H}^n(C) \leqslant \mathcal{H}^n(\mathbb{S}^n)/2$.*

*Then, there are bounded isoperimetric regions in $\Omega$ of any given volume. In particular, we have existence of minimizers in convex cones.*

Once we have assured existence under certain conditions, we must study the regularity of minimizers. By using standard results in Geometric Measure Theory [**M**] we obtain that the relative boundary $\overline{\partial E \cap \Omega}$ of a minimizer $E$

is, up to a closed singular set whose Hausdorff dimension does not exceed $n-7$, a smooth, embedded hypersurface $\Sigma$ with boundary and with constant mean curvature. The behaviour of the minimizer near the boundary $\partial\Omega$ is quite different from the one we indicated in Section 2 for minimizers of the Euclidean perimeter. In fact, the hypersurface $\Sigma$ meets $\partial\Omega$ *orthogonally.*

Now, we can prove the following result which does not appear in [**RRo**]

**Proposition 4.** *Let $\Omega \subset \mathbb{R}^2$ be a cone over an open arc $C \subset \mathbb{S}^1$. Then, isoperimetric regions in $\Omega$ exist and they are*

(i) *Circular sectors centered at the vertex if* $\operatorname{length}(C) < \pi$,

(ii) *Half-discs centered at $\partial\Omega \setminus \{0\}$ if* $\operatorname{length}(C) \geqslant \pi$.

*Proof.* The existence of bounded minimizers is guaranteed since assertion (i) in Corollary 3 holds. Let $E$ be a minimizer in $\Omega$. By the regularity results mentioned above, we have that $\Sigma = (\overline{\partial E \cap \Omega}) \setminus \{0\}$ is a smooth, embedded curve with constant geodesic curvature. As $E$ is bounded, we deduce that $\Sigma$ is the union of finitely many circular arcs with the same radius, and meeting $\partial\Omega \setminus \{0\}$ orthogonally. This clearly implies that the center of any of these arcs is a point in $\partial\Omega$. It follows that any component of $\Sigma$ is a half-circle centered at $\partial\Omega \setminus \{0\}$ or a circular arc homothetic to $C$. From here, it is not difficult to see that $\Sigma$ is connected. Finally, a direct comparison between the lengths of the different candidates indicates us that circular sectors centered at the vertex are isoperimetrically better than half-discs only if $\operatorname{length}(C) < \pi$. □

## 3.2. Classification of stable regions in convex cones

In this section we study sets that *locally* minimize the perimeter for fixed volume inside a cone $\Omega$. Let $E$ be a bounded set in $\Omega$ such that $\mathcal{P}(E, \Omega) < +\infty$. A *volume preserving variation* of $E$ in $\Omega$ is a smooth family of diffeomorphisms $\{\varphi_t\}_{t \in (-\varepsilon,\varepsilon)}$ of $\overline{\Omega}$, preserving $\partial\Omega$, and such that the volume of any $E_t = \varphi_t(E)$ coincides with the volume of $E_0 = E$. We say that $E$ is a *stable region* in $\Omega$ if the perimeter functional $\mathcal{P}(t) = \mathcal{P}(E_t, \Omega)$ associated to any volume preserving variation of $E$ is a $C^2$ function near the origin such that $\mathcal{P}'(0) = 0$ and $\mathcal{P}''(0) \geqslant 0$. By using the first and second variation for perimeter and volume together with an appropriate volume preserving variation, we have completely described stable regions in convex cones allowing the presence of a small singular set in the relative boundary

**Theorem 5** ([**RRo**]). *Let $E$ be a bounded stable region inside a convex cone $\Omega \subset \mathbb{R}^{n+1}$ $(n \geqslant 2)$. Suppose that the boundary $\overline{\partial E \cap \Omega}$ is the union of a smooth embedded hypersurface $\Sigma$ with boundary $\partial\Sigma \subset \partial\Omega$, and a closed singular set $\Sigma_0$ such that $\mathcal{H}^{n-2}(\Sigma_0) = 0$ or consists of isolated points. Then $E$ is either*

(i) *A round ball contained in* $\Omega$*, or*

(ii) *A half-ball centered and lying over a flat piece of* $\partial\Omega$*, or*

(iii) *The intersection with the cone of a round ball centered at the vertex.*

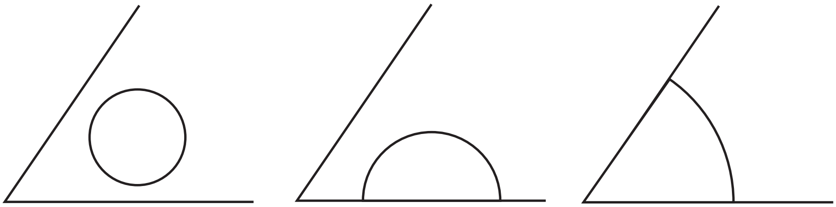

Figure 1: Stable bounded regions in a convex cone.

The idea of the proof of Theorem 5 is based on the method used by J. Barbosa and M. do Carmo [**BdC**] to characterize bounded stable regions without singularities in $\mathbb{R}^{n+1}$. Geometrically speaking we consider the volume preserving variation of $E$ which results when one considers first a contraction of $\Sigma$ by parallel hypersurfaces to its relative boundary, and then applies a dilation centered at the vertex of the cone to restore the enclosed volume. After some computations we obtain that the inequality $\mathcal{P}''(0) \geqslant 0$ associated to this variation implies that $\Sigma$ is a totally umbilical hypersurface. The proof finishes by using [**RRo**, Lemma 4.10] where we classify totally umbilical hypersurfaces inside a convex cone meeting orthogonally the boundary of the cone.

The existence of bounded minimizers in convex cones (Corollary 3) and the classification of stable regions in Theorem 5 allow us to describe, after an easy comparison among the perimeters of the different candidates, which are the isoperimetric regions in a smooth convex cone

**Theorem 6** ([**LP**], [**RRo**]). *Isoperimetric regions in a smooth convex cone are the intersection with the cone of round balls centered at the vertex.*

Theorem 6 was previously proved by P. Lions and F. Pacella [**LP**] by using the Brunn-Minkowski inequality in $\mathbb{R}^{n+1}$. The complete solution to the free boundary problem inside a convex cone over a *non-smooth* spherical domain of $\mathbb{S}^n$ is still an open question.

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