Alternative Hamiltonian representation for gravity

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Abstract. By using a Hamiltonian formalism for fields wider than the canonical one, we write the Einstein vacuum field equations in terms of alternative variables. This variables emerge from the Ashtekar's formalism for gravity.

1. Introduction

In 3+1 dimensions the only successful attempt to obtain a canonical theory for gravity in terms of a connection that yields first class constraints is that due to Ashtekar [1, 2]. In this formulation one can use a (complex) $SO(3)$ spatial connection as coordinate for the gravitational phase space instead of the 3-metric introduced by Arnowitt, Deser and Misner (ADM) [3]. Ashtekar's canonical gravity has given rise to a promising quantization project.

In this work we show that the evolution equations for the gravitational field, given by the Einstein vacuum field equations in an alternative representation derived from the Ashtekar's canonical gravity, can be expressed in a Hamiltonian form with the canonical Hamiltonian structure and in terms of non canonical variables.

We will use a Hamiltonian formulation in which the time evolution of the field variables ϕ_α ($\alpha = 1, 2, ..., n$), which represent the state of the system, can be written in the form [4, 5]

$$\dot{\phi}_\alpha = D_{\alpha\beta}\frac{\delta H}{\delta \phi_\beta}, \tag{1}$$

where H (the Hamiltonian) is a suitable functional of the ϕ_α, and the $D_{\alpha\beta}$ are, in general, differential operators of an arbitrary finite order with the coefficients depending on the variables ϕ_α and their derivatives. The appropriate expression for the Poisson bracket between two functionals has the form

$$\{F, G\} = \int \frac{\delta F}{\delta \phi_\alpha} D_{\alpha\beta} \frac{\delta G}{\delta \phi_\beta} \mathrm{d}v \tag{2}$$

The Hamiltonian operator $D_{\alpha\beta}$ must satisfy certain restrictions in order for (2) to be a true Poisson bracket (antisymmetric and satisfying the Jacobi identity).

The outline of this paper is as follows. In the next section we review the ADM formalism. Then we analyze the change of variables leading to the Ashtekar formalism. In Sect. 4 the alternative form of the dynamical equations of vacuum general relativity is derived. We end the paper with some concluding remarks.

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2. ADM formalism

We can consider the spacetime as a 4-manifold M, arising as a result of the time evolution of a three-dimensional space-like hypersurface Σ. The manifold M is assumed to be orientable, and have the global topology $\Sigma \times \Re$, where $\Re$ is the real line. We assume that Σ is compact without boundary. The dynamical variables are the Riemannian 3-metric tensor field q_{ab}, and the tensor density field of the conjugate momenta p^{ab} [3], which are linearly related to the extrinsic curvature tensor K_{ab} of the hypersurface by $p^{ab} = -q^{1/2}(K^{ab} - q^{ab}K)$, where q^{ab} is the inverse matrix to q_{ab}, $K = q^{ab}K_{ab}$, $q = \det(q_{ab})$, and the Latin indices $a, b, \ldots$ label spatial coordinates and run over the values $1, 2, 3$. These indices are raised and lowered by means of q_{ab}. (See, *e.g.*, Ref. [6] for a nice treatment of this formulation.)

Dynamic equations are generated by the Hamiltonian

$$H = \int \left(N\mathcal{H} + N^b\mathcal{H}_b\right) \mathrm{d}^3x, \tag{3}$$

which is a linear combination of the (scalar and vectorial) constraints

$$\mathcal{H} \ = q^{1/2}\left(-{}^3R + q^{-1}p^{ab}p_{ab} - \frac{1}{2}q^{-1}p^2\right), \tag{4}$$

$$\mathcal{H}^a = \ - 2q^{1/2}D_b\left(q^{-1/2}p^{ab}\right), \tag{5}$$

and by the canonical Poisson bracket

$$\{q_{ab}(\mathbf{x}), p^{cd}(\mathbf{y})\} = \delta^c_{(a}\delta^d_{b)}\delta^3(\mathbf{x} - \mathbf{y}), \tag{6}$$

so that

$$\dot{q}_{ab} = \{q_{ab}, H\}, \quad \dot{p}^{ab} = \{p^{ab}, H\}. \tag{7}$$

In Eqs. (4) and (5) $p = p^a{}_a = p^{ab}q_{ab}$, and D_a is the torsion-free covariant derivative compatible with q_{ab}, with Riemann curvature tensor $2D_{[a}D_{b]}v_c \equiv {}^3R_{abc}{}^dv_d$, where v_a is an arbitrary covector on Σ. ${}^3R$ is the Ricci scalar of this curvature. The scalar N is known as the lapse and N^a is the shift vector on Σ; they appear in Eq. (3) as Lagrange multipliers.

3. Ashtekar formalism

Originally Ashtekar's variables were $SU(2)$ spinors. However, we will use $SO(3)$-valued variables. The translation from $SO(3)$-valued variables to $SU(2)$ spinors can be see in Ref. [7]. Instead of the metric tensor q_{ab} we introduce the triad $e_a{}^i$, such that the spatial metric is given by

$$q_{ab} = e_a{}^ie_b{}^j\delta_{ij}. \tag{8}$$

Latin indices $i, j, ... = 1, 2, 3$ are $SO(3)$ indices. They are raised and lowered with the Kronecker delta δ^{ij}. The inverse matrices to the triad are denoted by $e^a{}_i$. It is not difficult to verify that $q^{ab} = e^{ai}e^{bi}$ and that $q = \det(e_a{}^ie_{bi}) = (\det(e_a{}^i))^2 \equiv e^2$.

Let us introduce the momenta $p^a{}_i$ conjugate to the triad. They satisfy the equations

$$\left\{e_a{}^i(\mathbf{x}), p^b{}_j(\mathbf{y})\right\} = \delta^b_a\delta^i_j\delta^3(\mathbf{x} - \mathbf{y}), \tag{9}$$

and can be easily related to the momenta p^{ab} by means of $p^a{}_i = 2p^{ab}e_{bi}$. It now turns out that part of the Poisson brackets for the ADM variables has been modified, one has also (6), but now $\{p^{ab}(\mathbf{x}), p^{cd}(\mathbf{y})\} \neq 0$ [2]. To preserve the correspondence between Poisson structures, one has to impose three additional constraints which generate $SO(3)$ rotations and can be represented in the form $\mathcal{J}^i = \epsilon^{ijk} p^a{}_j e_{ak} = 0$ [8]. This constraint also ensures the conservation of the number of degrees of freedom.

In terms of $(e_a{}^i, p^a{}_i)$, the Hamiltonian becomes

$$H = \int \left(N\mathcal{H} + N^a \mathcal{H}_a + N_i \mathcal{J}^i \right) \mathrm{d}^3x, \tag{10}$$

where $\mathcal{H}$, $\mathcal{H}_a$ are given by (5), with q_{ab} and p^{ab} considered here as derived quantities, and we have annexed the additional constraint with the Lagrange multiplier N_i.

For the transition to the Ashtekar variables it is more convenient to use the variables $(E^{ai}, K_a{}^i)$ defined by $E^a{}_i \equiv e e^a{}_i$, $K_a{}^i \equiv K_{ab} e^b{}_i + J_{ab} e^b{}_i$, where $K_{ab} = K_{(ab)}$ is the extrinsic curvature, and $J^{ab} = \frac{1}{4}(e^{ai} p^b{}_i - e^{bi} p^a{}_i) = J^{[ab]}$. Then one has

$$\{E^a{}_i(\mathbf{x}), K_b{}^j(\mathbf{y})\} = \frac{1}{2} \delta^a_b \delta^j_i \delta^3(\mathbf{x} - \mathbf{y}). \tag{11}$$

In [1] Ashtekar proposed a transformation that allows one to represent the density of the gravitational Hamiltonian as a polynomial in canonical variables. Ashtekar also introduced a complex parametrization in which the new variables are represented as $A_a{}^i = \frac{1}{2} \epsilon^{ijk} e^b{}_k D_a e_b{}^j + i K_a{}^i$. In this parametrization, we have

$$\{E^a{}_i(\mathbf{x}), A_b{}^j(\mathbf{y})\} = i \delta^a_b \delta^j_i \delta^3(\mathbf{x} - \mathbf{y}), \tag{12}$$

$$\{E^a{}_i(\mathbf{x}), E^b{}_j(\mathbf{y})\} = 0, \quad \{A_a{}^i(\mathbf{x}), A_b{}^j(\mathbf{y})\} = 0. \tag{13}$$

For any two functionals in phase space $F(E, A)$, $G(E, A)$, the Poisson brackets are thus given by

$$\{F, G\} \equiv i \int \left(\frac{\delta F}{\delta E^a{}_i} \frac{\delta G}{\delta A_a{}^i} - \frac{\delta F}{\delta A_a{}^i} \frac{\delta G}{\delta E^a{}_i} \right) \mathrm{d}^3x. \tag{14}$$

Changing the variables in the Hamiltonian leads to the expression

$$H = i \int \left(-\frac{i}{2} \mathcal{N} \mathcal{S} + \frac{1}{2} N^a \mathcal{V}_a + N^i \mathcal{G}_i \right) \mathrm{d}^3x, \tag{15}$$

where

$$\mathcal{G}_i(A, E) \equiv \mathcal{D}_a E^a{}_i \equiv i \epsilon^{abc} J_{ab} e_c{}^i = 0, \tag{16}$$

$$\mathcal{V}_a(A, E) \equiv E^b{}_i F_{ab}{}^i = 0, \tag{17}$$

$$\mathcal{S}(A, E) \equiv E^a{}_i E^b{}_j F_{abk} \epsilon^{ijk} = 0. \tag{18}$$

are the (Gauss, vectorial and scalar) constraints, $\mathcal{N} = e^{-1} N$ and ϵ^{abc} is the totally antisymmetric Levi-Civita symbol ($\epsilon^{123} = 1$). The new covariant derivative $\mathcal{D}_a$ is defined by $\mathcal{D}_a v_i \equiv \partial_a v_i + \frac{1}{2} \epsilon_{ijk} A_a{}^j v^k$. The curvature of the connection $A_a{}^i$ can be found from $2\mathcal{D}_{[a} \mathcal{D}_{b]} v_i = \frac{1}{2} \epsilon_{ijk} F_{ab}{}^j v^k$, hence

$$F_{ab}{}^i = \partial_a A_b{}^i - \partial_b A_a{}^i + \frac{1}{2} \epsilon^i{}_{jk} A_a{}^j A_b{}^k. \tag{19}$$

The evolution equations for the canonical variables are obtained taking the Poisson bracket of the variables with the Hamiltonian (15), and, neglecting boundary terms, they are given by

$$\dot{A}_a{}^i(x) = \{A_a{}^i, H\} = -i \mathcal{N} \epsilon_{ijk} E^{bj} F_{ab}{}^k + \frac{1}{2} N^b F_{ba}{}^i, \tag{20}$$

$$\dot{E}^a{}_i(x) = \{E^a{}_i, H\} = i \mathcal{D}_b(\mathcal{N} \epsilon_{ijk} E^{[a|j|} E^{b]k}) + \mathcal{D}_b(N^{[b} E^{a]}{}_i). \tag{21}$$

4. Alternative representation

In this section we will describe the phase space of gravity using the "magnetic" field $B^a{}_i \equiv \epsilon^{abc} F_{bci} = \epsilon^{abc}\left(2\partial_b A_{ci} + \frac{1}{2}\epsilon^{ijk} A_{bj} A_{ck}\right)$ rather than A_{ai}. For a non-Abelian theory the Bianchi identity,

$$\mathcal{D}_a B^{ai} = \partial_a B^{ai} + \frac{1}{2}\epsilon^{ijk} A_{aj} B^a{}_k = 0, \tag{22}$$

is a relation between $B^a{}_i$ and A_{ai} which can be used as a linear relation to be solved for A_{ai}, *i.e.*, it is generically possible to obtain $A_{ai} = A_{ai}(B)$ (see [9] for some examples). Note that $\dot{B}^a{}_i = \{B^a{}_i, H\} = 2\epsilon^{abc}\mathcal{D}_b \dot{A}_{ci}$, thus, we have a new set of equations of evolution for the gravitational field, equivalent to Eqs. (21) and (20), given by

$$\dot{E}^a{}_i = \mathcal{D}_b(i\mathcal{N}\epsilon_{ijk} E^{[a|j|} E^{b]k} + N^{[b} E^{a]}{}_i), \tag{23}$$

$$\dot{B}^a{}_i = -\mathcal{D}_b(2i\mathcal{N}\epsilon_{ijk} E^{[a|j|} B^{b]k} + N^{[a} B^{b]}{}_i). \tag{24}$$

In order to express the alternative set of evolution equations in the Hamiltonian form (1), we introduce the Hamiltonian

$$H = i\int \mathrm{d}^3x \left(-\frac{i}{2}\mathcal{N}\mathcal{S}(E,B) + \frac{1}{2}N^a \mathcal{V}_a(E,B) + N^i \mathcal{G}_i(E,B)\right) \tag{25}$$

where, now,

$$\mathcal{V}_a(E,B) \equiv \frac{1}{2}\epsilon_{abc} E^b{}_i B^{ci} = 0, \tag{26}$$

$$\mathcal{S}(E,B) \equiv \frac{1}{2}\epsilon_{abc} E^a{}_i E^b{}_j B^c{}_k \epsilon^{ijk} = 0, \tag{27}$$

$$\mathcal{G}_i(E,B) \equiv \mathcal{D}_a E^a{}_i, \tag{28}$$

are the constraints. The Hamiltonian (25) is the same of Ashtekar [*cf.* Eq. (15)], with $A = A(B)$.

On the other hand, Eqs. (23) and (24) can be written in the Hamiltonian form

$$\dot{E}^a{}_i = \mathcal{D}^{ab}{}_{ij}\frac{\delta H}{\delta B^b{}_j}, \quad \dot{B}^a{}_i = -\mathcal{D}^{ab}{}_{ij}\frac{\delta H}{\delta E^b{}_j} \tag{29}$$

where

$$\mathcal{D}^{ab}{}_{ij} \equiv -2i\epsilon^{abc}\mathcal{D}_c\delta_{ij} \equiv -2i\epsilon^{abc}\left(\partial_c\delta_{ij} + \frac{1}{2}\epsilon_{ikl} A_c{}^k \delta^l_j\right) \tag{30}$$

and H is given by (25).

Making use of the $\mathcal{D}^{ab}{}_{ij}$ given by Eq. (30), a Poisson bracket between any pair of functionals of the field $F(E,B)$ and $G(E,B)$ can be defined as

$$\{F,G\}_n \equiv \int \left(\frac{\delta F}{\delta E^a{}_i}\mathcal{D}^{ab}{}_{ij}\frac{\delta G}{\delta B^b{}_j} - \frac{\delta F}{\delta B^a{}_i}\mathcal{D}^{ab}{}_{ij}\frac{\delta G}{\delta E^b{}_j}\right)\mathrm{d}^3x, \tag{31}$$

where the subscript n (*non* canonical variables) is introduced to distinguish it from the canonical Poisson bracket. However, one can see that the antisymmetry and the Jacobi identity of the Poisson bracket (31) follows from the fact that the Hamiltonian structure is the canonical one, *i.e*, from the canonical Poisson bracket (14), by using the fact of that $\frac{\delta}{\delta A_a{}^i} = 2\epsilon^{abc}\mathcal{D}_b\frac{\delta}{\delta B^c{}_i}$, which follows from the chain rule, one can see that (integrating by parts)

$$\{F,G\} = \{F,G\}_n. \tag{32}$$

Therefore, the Hamiltonian structure is the canonical one, only the variables are new. Thus, in what follows, we will use the subscript n in order to point out that we are using E and B as variables of the phase space.

The new variables satisfy the Poisson brackets relations

$$\{E^a{}_i(\mathbf{x}), E^b{}_j(\mathbf{y})\}_n = 0, \quad \{B^a{}_i(\mathbf{x}), B^b{}_j(\mathbf{y})\}_n = 0 \tag{33}$$

and

$$\begin{aligned}\{E^a{}_i(\mathbf{x}), B^b{}_j(\mathbf{y})\}_n &= -2i\epsilon^{abc}\mathcal{D}_c\delta_{ij}\delta^3(\mathbf{x}-\mathbf{y}) \\ &= -2i\epsilon^{abc}\left(\partial_c\delta_{ij} + \frac{1}{2}\epsilon_{ilj}A_c{}^l\right)\delta^3(\mathbf{x}-\mathbf{y}).\end{aligned} \tag{34}$$

The Poisson bracket (31) yields the expected relations between the Hamiltonian and any functional of the field. If $F(E,B)$ is any functional of the field that does not depend explicitly on the time then Eqs. (31) and (29) give

$$\{F, H\}_n = \int \left(\frac{\delta F}{\delta E^a{}_i}\dot{E}^a{}_i + \frac{\delta F}{\delta B^a{}_i}\dot{B}^a{}_i\right) \mathrm{d}^3x = \dot{F}, \tag{35}$$

i.e., H generates time translations.

5. Concluding remarks

As one result we have shown that it is possible to write the dynamic equations of general relativity in terms of new variables, which are not canonical. We obtained a Poisson bracket (associated with the canonical Hamiltonian structure) and it was shown that it yields the expected relations between the Hamiltonian and any functional of the field.

The Poisson algebra of the constraints in terms of the new variables is first class, *i.e.*, it closes. Furthermore the number of degrees of freedom is two. (See [10] for a review of this points and for a sketch of quantization in this formalism.)

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