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Jost–Lehmann–Dyson representation in higher dimensional field theories

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ABSTRACT

The Jost–Lehmann–Dyson representation is derived for massive scalar field theories in higher spacetime dimensions, $D > 4$, for the four point scattering amplitude. The representation is very crucial to investigate the analyticity properties of the amplitude. The axiomatic approach of Lehmann–Symanzik–Zimmermann is adopted to show the existence of such a representation. Consequently, a host of interesting results will follow from derivation of JLD representation such as proof of analyticity properties and asymptotic behavior of the amplitude.

The purpose of this letter is to derive the Jost–Lehmann–Dyson (JLD) representation [1,2] for the four point scattering amplitude of higher dimensional, $D > 4$, scalar, neutral, massive field theories. It plays a central role in the investigation of the analyticity properties of the amplitude as is elaborated in sequel. We work in the Lehmann–Symanzik–Zimmermann (LSZ) [3] formalism of axiomatic field theory to derive the representation.

There exist rigorous bounds on experimentally measurable parameter in the context of four dimensional theories. These have been derived from the study of the analyticity properties of scattering amplitudes. The most important attribute of these rigorous results is that their derivations are based on the results proved in the axiomatic field theoretic formalisms without resorting to any specific model. Moreover, there are no observed experimental violation of exacts results on total cross sections, width of the diffraction peak in elastic scattering to mention a few. The Froissart–Martin [4,5] bound on the total cross section, σ_t, is the most celebrated of all

$$\sigma_t \le \frac{4\pi}{t_0} ln(\frac{s}{s_0})^2 \tag{1}$$

where $t_0 = 4m_\pi^2$ for most of the hadronic processes; s is the center of mass (c.m.) energy squared and s_0 is a scale introduced to make the argument of the logarithm dimensionless. The crucial ingredients leading to the derivation of the bound (1) are the unitarity of S-matrix, crossing symmetry and analyticity of the amplitude in s and t, the momentum transfer squared. The afore mentioned results follow from the axiomatic field theoretic formalism [6–14]. The analyticity of scattering amplitude in s and t facilitates writing of the fixed t dispersion relation in s; t lying within the Lehmann ellipse [15]. The partial wave expansion of the scattering amplitude converges inside this ellipse.

The role of the (JLD) representation is very fundamental in this context. The proof of the existence of Lehmann ellipse rests on the works of JLD. Moreover, the edge-of-the-wedge theorem [16] is proved by using this construction.

There is growing consensus that theories in higher spacetime dimensions, $D > 4$, are important to study fundamental interactions and geometric attributes of spacetime. There is increasing interest to investigate properties of the scattering amplitude afresh from various perspectives [18,19] in $D > 4$ theories. The supersymmetric theories and supergravity theories have opened up the possibilities to explore such theoretical constructions. Superstring theories hold the promise of unifying the fundamental forces of Nature. Moreover, the five perturbatively consistent string theories live in $D = 10$. Of course, the standard model is defined in $D = 4$ and its predictions have been experimentally tested with great accuracy. Thus the extra spatial dimensions of the D-dimensional theories must be compactified. In the context of string theories, most of the phenomenological analyses argue that the energy scale of compactification is very high and therefore, the present accessible accelerators cannot reveal the existence of the compact extra dimensions. However, in recent years elaborate models have been constructed where the compactification scale is quite low i.e. the presence of extra dimensions can be explored in LHC energy range. There have been considerable phenomenological activities to examine and propose the signatures for the spectra of new particles [20,21]. Moreover, the present data does not rule out existence of low compactification scales.

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In this investigation, we consider the idealized scenario where scattering of neutral massive scalers in a D-dimensional flat Minkowski space is envisaged. We are aware that scattering of physical scattering experiments is (mostly) carried out with hadrons in accelerators like LHC. In turn the hadrons are composite objects with quarks and gluons as constituents. The phenomenological analyses of the data are carried out taking into account that the targets and projectiles contain constituents alluded to above. It is pertinent to record that the rigorous total cross sections, σ_t, the celebrated Froissart–Martin bound (1) are derived in the framework of axiomatic field theories of four dimensional flat spacetime. The most general analyticity properties were derived for scattering of scalar fields in the axiomatic formulations. There are several other rigorous bounds on experimentally observable parameters in high energy collisions, also derived within the axiomatic frameworks. Although these bounds have been derived within the frameworks of axiomatic approach, they are tested against the experimental data. It is argued and generally accepted that violation of such rigorous constraints might lead to reexaminations of the validity of some of the basic axioms. In view of these remarks, this investigation is undertaken for an ideal axiomatic theory. Although we mention compactification of higher dimensional theory *en passant*; however, the issues pertaining to compactifications, even for a scalar field theory defined in D-dimensional flat spacetime, is not investigated at any length here.

We mention in passing that the issues concerned with ultra high energy scatterings have been of considerable interests over a period of two decades. We refer the reader to the review lectures of Giddings [22] where role of gravitational interaction is dealt with. It is to be mentioned that the conceptual frame works of axiomatic field theory such as locality and microcausality are to be critically address for such theories as have been alluded to in [22]. Our goal here is not to address those issues. We consider scattering process in energy domains lying much below the Planckian energy regime.

The bound on total cross section have been derived for $D > 4$ theories with certain reasonable assumptions [23,24]; however these assumptions were not proved from the field theoretic basis. Our endeavors are the first step in a direction to derive important results for the amplitudes which will lead to rigorous study of the analyticity of four point amplitude in D-dimensional theories in the LSZ formalism. The axioms of LSZ [8] generalized to a D-dimensional field theory are:

(A1). The states of the system are represented by vectors in the Hilbert space, $\mathcal{H}$. All the physical observables are self-adjoint operators in $\mathcal{H}$. **(A2)**. The theory is invariant under inhomogeneous Lorentz transformations. **(A3)**. There exist the energy, momentum operators, P_μ, which commute and there are complete set of states $P_\mu|p,\alpha> = p_\mu|p,\alpha>$ which belong to $\mathcal{H}$. Existence of a unique invariant vacuum state i.e. $U(a,\Lambda)|0> = |0>$, where $U(a,\Lambda)$ are the representations of the Lorentz group. Consequently, $P_\mu|0> = 0$, and $M_{\mu\nu}|0> = 0$; $M_{\mu\nu}$ being generators of the Lorentz transformation. Moreover, if $\mathcal{F}(x)$ is a Heisenberg operator, $[P_\mu, \mathcal{F}(x)] = i\partial_\mu \mathcal{F}(x)$ when the operator does not depend on x^μ, $\mu = 0, 1..D-1$ explicitly. **(A4)**. Microcausality: a (bosonic) local operator at the space time point x^μ commutes with another (bosonic) local operator at x'^μ when their separation is spacelike i.e. if $(x-x')^2 < 0$; note $x.y = x^0y^0 - x^1y^1 - ...x^{D-1}y^{D-1}$ in our convention.

In order to study analyticity properties of the scattering amplitude we are to deal with the retarded, advanced and the causal functions which are vacuum expectation values of product of local operators to be defined in sequel. The expression for the $2 \to 2$ scattering amplitude for D-dimensional scalar theory assume the following form in the LSZ formalism with suitable generalization

$$
\begin{aligned}
&< -p_d \; -p_c \; out|p_a \; p_b \; in> - < -p_d \; -p_c \; in|p_a \; p_b \; in> \\
&= -\frac{i}{(2\pi)^{D-1}} \int dx^D dy^D e^{-ip_c.x - ip_b.y} (\Box_x - m_c^2)(\Box_y - m_b^2) \\
&\quad \times < -p_d|R\phi_c(x)\phi_b^\dagger(y)|p_a> \\
&= -\frac{i}{(2\pi)^{D-1}} \int d^Dx d^Dy e^{-ip_c.x - ip_d.y} (\Box_x - m_c^2)(\Box_y - m_d^2) \\
&\quad \times <0|R\phi_c(x)\phi_d(y)|p_a \; p_b \; in>
\end{aligned} \tag{2}
$$

Our conventions are as follows: the incoming initial and outgoing final two particle states are $|p_a \; p_b \; in>$ and $< -p_d \; -p_c \; out|$. The incoming D-momenta are p_a and p_b whereas the outgoing momenta are $-p_c$ and $-p_d$ so that energy momentum conservation rule is $p_a + p_b + p_c + p_d = 0$. Correspondingly, the Mandelstam variables are: $s = (p_a + p_b)^2$, $t = (p_a + p_d)^2$, $u = (p_a + p_c)^2$ and $s + t + u = m_a^2 + m_b^2 + m_c^2 + m_d^2 = 4m^2$. Although we deal with identical particles of mass m we continue to label the particles in order to keep tag of each one. We define the R-product [10,12] as $R \; \phi(x)\phi_1(x_1)...\phi_n(x_n) = (-1)^n \sum_{\mathcal{P}} \theta(x_0 - x_{10})\theta(x_{10} - x_{20})...\theta(x_{n-10} - x_{n0})[[...[\phi(x), \phi_{i_1}(x_{i_1})], \phi_{i_2}(x_{i_2})]..], \phi_{i_n}(x_{i_n})]$ and $\mathcal{P}$ stands for all permutations $(i_1, ...i_n)$ of $(1, 2, ...n)$. Note that in the first expression of (2) we have reduced fields b and c whereas in the latter one c and d have been reduced. The expressions involve interacting fields and their equations of motion is related to a source current: $(\Box_x - m_l^2)\phi_l(x) = j_l(x)$, $l = a, b, c, d$. The amplitude is expressed as

$$
F(p_a, ...p_d) = -\int d^D z e^{iP.z} < -p_d|Rj_c(\frac{z}{2})j_b^\dagger(-\frac{z}{2})|p_a> \tag{3}
$$

where $P = \frac{(p_b - p_c)}{2}$. Here b and c are reduced and a similar expression will appear when c and d are reduced. The generic expression for the retarded function

$$
F_R(q) = \int d^D z e^{iq.z}\theta(z_0) < Q_f|[j_l(\frac{z}{2}), j_m(-\frac{z}{2})]|Q_i> \tag{4}
$$

with two states $|Q_f>$ and $|Q_i>$ carrying D-dimensional fixed momenta Q_f and Q_i respectively is of importance. We also define the advanced and causal functions F_A and F_C respectively as [9,10]

$$
F_A = -\int d^D z e^{iq.z}\theta(-z_0) < Q_f|[j_l(\frac{z}{2}), j_m(-\frac{z}{2})]|Q_i> \tag{5}
$$

and

$$
F_C(q) = \int d^D z e^{iq.z} < Q_f|[j_l(\frac{z}{2}), j_m(-\frac{z}{2})]|Q_i> \tag{6}
$$

The functions defined above (4)–(6) play a crucial role in the study of analyticity properties of the amplitude. Moreover, these are taken to be tempered distributions and consequently, their Fourier transforms are polynomially bounded in appropriate Lorentz invariant momentum variables. Notice that $F_C(q) = F_R(q) - F_A(q)$ and (6) is a commutator of currents and therefore, $\tilde{F}_C(z) = 0$ for $z^2 < 0$ due to micro causality. Moreover, in arriving at the expressions on the *r.h.s* of (4)–(6), we used the relation $(\Box_x - m_c^2)(\Box_y - m_d^2)(R\phi_c(x)\phi_d(y)) = R(j_c(x)j_d(y))$. It begs a qualifying remark: in this operation, a finite number of derivatives of δ-function will appear in general since we deal with a local quantum field theory [25]. Therefore, $F_C(q), F_R(q)$ and $F_A(q)$ are defined up to a finite polynomial in q. The kinematical region where $F_C(q) = 0$ i.e. $F_R(q) = F_A(q)$, is the coincidence region. It is proved that $F_R(q)$ and $F_A(q)$ are analytic continuations of each other from the edge-of-the-wedge theorem [16]. We have argued elsewhere [17] that this theorem is valid for the four point amplitude of D-dimensional theories.

Jost and Lehmann [1], in their pioneering work derived the representation for $F_C(q)$ and $F_R(q)$ in the LSZ formalism for equal mass particles. Dyson [2] introduced an elegant and indigenous mathematical formalism to obtain a necessary and sufficient condition for the representation for the unequal mass case in four dimensional theories in a general setting. He considered a six dimensional wave equation in the momentum space. He introduced two extra spatial coordinates and correspondingly an extra pair of momenta. As is well known, the solution to such an equation is uniquely determined once the initial value of the function and its normal derivative on a spacelike surface are specified. Thus the representation for the causal function is expressed as boundary value of the solution to the six dimensional wave equation with specific boundary conditions.

We generalize Dyson's theorem for massive neutral scalar field theories in D-dimensions. The crucial ingredient is to envisage a $D+2$ dimensional space and correspondingly introduce the same number of momentum variables. The coordinates are : $\tilde{z}=\{\tilde{z}_0=x_0, \tilde{z}_1=x_1...\tilde{z}_{D-1}=x_{D-1}, \tilde{z}_D=y_1, \tilde{z}_{D+1}=y_2\}$ and the momenta are: $\tilde{r}=\{\tilde{r}_0=q_0, \tilde{r}_1=q_1, ...\tilde{r}_{D-1}=q_{D-1}, \tilde{r}_D=p_1, \tilde{r}_{D+1}=p_2\}$. It is a flat space with Lorentzian signature metric $(+,-,-...-)$ and $\tilde{z}^2=\tilde{x}^2-y^2=x_0^2-x_1^2-...-x_{D-1}^2-y_1^2-y_2^2$. Recall $\tilde{F}_C(x)$ is the Fourier transform of $F_C(q)$ and $\tilde{F}_C(x)$ is the Fourier transform of $\tilde{F}_C(q)$ and thus

$$\tilde{F}_C(\tilde{z})=4\pi\tilde{F}_C(x)\delta(x^2-y^2)=4\pi\tilde{F}_C(x)\delta(\tilde{z}^2) \tag{7}$$

Therefore, $\tilde{F}_C(\tilde{z})$ is defined on the light cone of the $\tilde{z}$-space.

$$\int_{-\infty}^{+\infty}\int_{-\infty}^{+\infty} dy_1dy_2\tilde{F}_C(\tilde{z})=4\pi^2\tilde{F}_C(x)\ , \text{for } x^2\geq 0$$
$$=0,\ \text{for } x^2<0 \tag{8}$$

Thus, by construction, $\tilde{F}(\tilde{z})$ and $\tilde{F}_C(x)$ are ($\tilde{F}_C(x)=0$, for $x^2<0$) equivalent since latter is recovered from the former after integrating over $\int d^2y$ (8). Let us choose a special $(D+2)$-dimensional vector; $\hat{q}=(q_0,q_1,...q_{D-1},0,0)$. The Fourier transform of $\tilde{F}_C(\tilde{z})$ is given by

$$\bar{F}_C(\tilde{r})=\frac{1}{(2\pi)^{D+2}}\int e^{i\tilde{r}.\tilde{z}}\tilde{F}_C(\tilde{z})d^{D+2}\tilde{z} \tag{9}$$

If we insert (7) into (9), we get

$$\bar{F}_C(\tilde{r})=\frac{4\pi}{(2\pi)^{D+2}}\int d^{D+2}\tilde{z}d^Dqe^{i\hat{q}.\tilde{z}}\bar{F}_C(q)$$
$$=\int D^{(1)}(\tilde{r}-\hat{q})\bar{F}_C(q)d^Dq \tag{10}$$

where

$$D^{(1)}(\tilde{r})=\frac{2}{(2\pi)^{D+1}}\int e^{-i\tilde{r}.\tilde{z}}\delta(\tilde{z}^2)d^{D+2}\tilde{z}$$
$$=\frac{2}{(2\pi)^{D+1}}P\frac{1}{(\tilde{r})^{D/2}} \tag{11}$$

P stands for the principal value. From now on we drop factors like $\frac{1}{(2\pi)^{D+2}}$, $\frac{1}{(2\pi)^{D+1}}$ etc. coming from Fourier transforms and inverse transforms. We now derive an expression for $\bar{F}_C(\tilde{r})$ to display the singularity structure in q and their locations. Insert expression for $D^{(1)}(\tilde{r})$, (11) into (10):

$$\bar{F}_C(\tilde{r})=\int d^Dq\frac{F_C(q)}{\left((\tilde{r}-\hat{q})^2\right)^{D/2}}=\int d^Dq\frac{F_C(q)}{[(u-q)^2-\bar{s}]^{D/2}} \tag{12}$$

with $\bar{s}=p_1^2+p_2^2$. The important point to note is $\tilde{F}_C(\tilde{z})=F_C(x)\delta(\tilde{z}^2)$ whose support is on the light cone of the enlarged spacetime. Furthermore, the Fourier transformed $\tilde{F}_C(\tilde{r})$, is rotationally invariant in p_1-p_2 plane due its dependence on $\bar{s}$. A crucial observation is that $D^{(1)}(\tilde{r})$ satisfies $(D+2)$-dimensional wave equation $\tilde{r}$-space

$$\Box_{D+2}D^{(1)}(\tilde{r})=0,\quad \text{where}\quad \Box_{D+2}=\frac{\partial^2}{\partial\tilde{r}_0^2}-\sum_{k=1}^{D+1}\frac{\partial^2}{\partial\tilde{r}_k^2} \tag{13}$$

Furthermore, $\tilde{F}_C(\tilde{r})$ also satisfies the same wave equation: $\Box_{D+2}\bar{F}_C(\tilde{r})=0$. We may argue if $\tilde{F}_C(x)$ vanishes for $x^2<0$ then $F_C(q)$ is the boundary value of $\bar{F}_C(q)$ on $\bar{s}=0$; therefore, $\bar{F}_C(\hat{q})=F_C(q)$, $\hat{q}=(q_0,q_1,..q_{D-1},0,0)$. Moreover,

$$\bar{F}_C(\hat{q})=\int d^{D+2}\tilde{z}e^{i\hat{q}.\tilde{z}}4\pi\delta(x^2-y^2)\tilde{F}_C(x)$$
$$=\int d^Dxe^{iq.x}4\pi\theta(x^2)\tilde{F}_C(x) \tag{14}$$

after integration $\int d^2y$ and setting $\hat{q}.\tilde{z}=q.x$. Thus there is a class of solutions of $F_C(q)$, the class denoted by $\mathcal{C}$, whose Fourier transform $\tilde{F}_C(x)=0$ for $x^2<0$. The D-dimensional version of Dyson's necessary condition is $\tilde{F}_C(x)=0$ for $x^2<0$ and $\tilde{F}_C(q)$ should be the boundary value on the plane $\bar{s}=0$ of a solution $\bar{F}_C(q,\bar{s})$ satisfying the momentum-space wave equation. Note that this class of solutions is ordained to be rotationally symmetric in the p_1-p_2 plane and $\bar{s}=0$ defines a timelike surface. More importantly, the boundary value of the solution to hyperbolic wave equation $\Box_{D+2}\bar{F}_C(\tilde{r})=0$ is not arbitrary on this surface. In a more general setting one envisages a function satisfying above wave equation with rotational symmetry on p_1-p_2 plane. The Fourier transformed $\tilde{z}$-space function is

$$\tilde{F}(\tilde{z})=\int d^{D+2}\tilde{r}e^{-i\tilde{r}.\tilde{z}}\bar{F}(\tilde{r}) \tag{15}$$

$\tilde{F}(\tilde{z})$ is endowed with the following attribute: $\Box_{D+2}\bar{F}(\tilde{r})=0$. Thus $\tilde{F}(\tilde{z})=\delta(\tilde{z}^2)G(\tilde{z})$ and the support of $\tilde{F}(\tilde{z})$ lies on the light cone of $\tilde{z}$-spacetime. Consequently,

$$\tilde{F}(\tilde{z})=\int d^{D+2}\tilde{r}e^{-i\tilde{r}.\tilde{z}}\bar{F}(u,|p|)$$
$$=\int d^Due^{-iu.x}\int_0^\infty pdp\int_0^{2\pi}e^{p|y|cos\theta}\bar{F}(u,|p|) \tag{16}$$

in the polar decomposition of (p_1,p_2) with $\bar{s}=p_1^2+p_2^2$. Therefore,

$$\tilde{F}(\tilde{z})=2\pi\int d^Due^{-iu.x}\int_0^\infty d\bar{s}J_0(\sqrt{\bar{s}}|y|)\bar{F}(u,\bar{s}) \tag{17}$$

The Bessel function $J_0(\sqrt{\bar{s}}y)=\sum_0^\infty\frac{(\bar{s}|y|)}{n!}$; its rotational invariance in the y-plane is displayed from the y^2 dependence. We argue that $\tilde{F}(\tilde{z})=\delta(\tilde{z}^2)G(x,y^2)=\delta(x^2-y^2)G(x,y^2)$ is an admissible structure and write

$$\tilde{F}(\tilde{z})=\delta(x^2-y^2)\tilde{f}(x) \tag{18}$$

Thus $\tilde{f}(x)$ needs not vanish for $x^2<0$ from the construction so far. Therefore, $\tilde{f}(x)=0$ for $x^2<0$ is to be imposed from outside as an extra constraint to relate it to the causal function. Under this constraint we could identify $\tilde{F}(\tilde{z})=\tilde{F}_C(\tilde{z})$. The generalized Dyson's condition is: the necessary and sufficient condition for a function to vanish outside the light cone of the D-dimensional spacetime, i.e. $x^2<0$, is that $F_C(q)$ be boundary value on the surface $\bar{s}=0$ of the solution to the $\tilde{r}$-space wave equation $\Box_{D+2}\bar{F}=0$ with requisite symmetries. Note that a solution to this differential equation can be obtained in terms of its value and its normal derivative on an arbitrary spacelike surface. Now introduce a singular function, $\bar{D}(\tilde{r})$, satisfying the same wave equation: $\Box_{D+2}\bar{D}(\tilde{r})=0$. The initial conditions are

$$\bar{D}(\tilde{r}_0=0,\tilde{r}_1,...,\tilde{r}_{D+1})=0, \text{ and } \frac{\partial\bar{D}(\tilde{r})}{\partial\tilde{r}_0}\Big|_{\tilde{r}_0=0}=\Pi_{i=1}^{D+1}\delta(\tilde{r}_i) \quad (19)$$

We can write (19) explicitly as

$$\bar{D}(\tilde{r})=\int d^{D+2}\tilde{z}e^{-i\tilde{r}.\tilde{z}}\epsilon(\tilde{z})\delta(\tilde{z}^2) \quad (20)$$

Now choose a spacelike surface, Σ, and prescribe initial data on it. If $\bar{F}(\tilde{r})$ is solution to the wave equation it assume the value $\bar{F}(\tilde{r}')$ and $\left(\frac{\partial\bar{F}(\tilde{r}')}{\partial\tilde{r}'_\alpha}\right)n^\alpha(\tilde{r}')$ on Σ; n^α being the normal to the surface. The solution to $\bar{F}(\tilde{r})$ is

$$\bar{F}(\tilde{r})=\int_\Sigma d\Sigma_\alpha\left[\bar{F}(\tilde{r}'),\frac{\partial}{\partial\tilde{r}'_\alpha}\bar{D}(\tilde{r}'-\tilde{r})\right] \quad (21)$$

with the definition

$$\left[\bar{F}(\tilde{r}'),\frac{\partial}{\partial\tilde{r}'_\alpha}\bar{D}(\tilde{r}'-\tilde{r})\right]=\bar{F}(\tilde{r}')\frac{\partial}{\partial\tilde{r}'_\alpha}\bar{D}(\tilde{r}')-\frac{\partial\bar{F}(\tilde{r}')}{\partial\tilde{r}'_\alpha}\bar{D}(\tilde{r}') \quad (22)$$

$d\Sigma_\alpha$ is the surface element which is a $(D+2)$-dimensional vector normal to Σ. The purpose is to derive a representation for $F_C(q)$; and we set $\bar{F}_C(\hat{q})=F_C(q)$. The resulting integral equation is

$$F_C(q)=\int_\Sigma d\Sigma'\left[\bar{F}(\tilde{r}'),\frac{\partial}{\partial\tilde{r}'_\alpha}\bar{D}(\tilde{r}'-\hat{q})\right]$$
$$=\int_\Sigma d\Sigma'_\alpha\left[\bar{F}(\tilde{r}'),\frac{\partial}{\partial\tilde{r}'_\alpha}\left\{\epsilon(u_0-q_0)\delta'\left((u-q)^2-\bar{s}\right)\right\}\right] \quad (23)$$

and

$$\bar{F}(\tilde{r})=\int\bar{D}^{(1)}(\tilde{r},\hat{q})\bar{F}_C(\hat{q})d^Dq \quad (24)$$

This representation is unique as has been argued by Dyson [2]. Notice that (23) defines $F_C(q)$ with a given surface Σ with any function $\bar{F}(\tilde{r})=\bar{F}_C(u,\bar{s})$ so that the dependence is on the invariant $\bar{s}=p_1^2+p_2^2$. We argue that there is one-to-one correspondence between the class of functions, $F_C(q)$ (designated by the class $\mathcal{C}$) and the solutions, $\bar{F}(\tilde{r})$, to the $\tilde{r}$-space wave equation with desired rotational symmetry in p_1–p_2 plane and $\bar{F}(\tilde{r})$ is expressed in terms of $F_C(q)$. Our goal is to choose a suitable $F_C(q,\bar{s})$ and a Σ to obtain $F_C(q)$ with appropriate support in momentum space. Eventually, it is desirable to identify the coincidence region, $F_C(q)=0$. Let us define a region, **R**, in the q-space, bounded by two spacelike surfaces, σ_1 and σ_2

$$\mathbf{R}: \quad \bar{s}_1(\mathbf{q})<q_0<\bar{s}_2(\mathbf{q}) \quad (25)$$

$F_C(q)=0$ inside this domain. The two surfaces are chosen as

$$|\bar{s}_1(\mathbf{q})-\bar{s}_1(\mathbf{q}')|<|\mathbf{q}-\mathbf{q}'|, \quad |\bar{s}_2(\mathbf{q})-\bar{s}_2(\mathbf{q}')|<|\mathbf{q}-\mathbf{q}'| \quad (26)$$

$\mathbf{q}$ is the $(D-1)$ component vector along spatial directions of the D-vector q and the same definition holds for $\mathbf{q}'$. The two spacelike surfaces are $q^0=s_1(\mathbf{q})$ and $q^0=s_2(\mathbf{q})$. Let C_R be the class of functions such that $\tilde{F}_C(x)=0$ for $x^2<0$ and $F_C(q)=0$ for any $q\in\mathbf{R}$. The hyperboloid $(q-u)^2-\bar{s}=0$ is q-space admissible. This is valid as long as the upper sheet does not come below σ_2 and the lower sheet is above σ_1. For the enlarged space the hyperboloid in question corresponds to points $\tilde{r}=(u_0,u_1,...u_{D-1},p_1,p_2)$, $\bar{s}=p_1^2+p_2^2$ lying in a certain region **S** of $\tilde{r}$-space. The intent is to derive a representation for $F_C(q)$. For every $\tilde{r}\in\mathbf{S}$ and but $q\in\mathbf{R}$, $\bar{D}(\tilde{r}-\hat{q})$ vanishes. A perspective representation for $F_C(q)\in C_R$ is

$$F_C(q)=\int_\Sigma d\Sigma_\alpha\left[\bar{F}(\tilde{r}),\frac{\partial}{\partial\tilde{r}_\alpha}[\epsilon(u_0-q_0)\delta'\left((u-q)^2-\bar{s}\right)]\right] \quad (27)$$

the points of $\tilde{r}$ are constrained to be in **S**. Note that every point of $\tilde{r}$ and Σ in **S** are required to belong to C_R from the conditions stated above. The important point is that $F_C(q)$ has a representation using the admissible hyperboloid. Thus the intended $F_C(q)$ to be constructed must depend on variables belonging to the above domain: $(q-u)^2-\bar{s}=0$. The second constraint is – this must not cross the surface defined by $q_0=\bar{s}_1(\mathbf{q})$ and $q_0=\bar{s}_2(\mathbf{q})$ (see (26)). Let us focus attention on the upper sheet of the hyperboloid corresponding to the branch

$$q_0=u_0+\sqrt{(\mathbf{q}-\mathbf{u})^2+\bar{s}} \quad (28)$$

This will cross σ_2 if

$$u_0+\sqrt{(\mathbf{q}-\mathbf{u})^2+\bar{s}}\geq\bar{s}_2(\mathbf{q}) \quad (29)$$

for $\mathbf{q}$ held fixed. The above constraint and a corresponding one for the lower sheet are respectively rephrased as following two equations

$$u_0\geq\mathrm{Max}_\mathbf{q}\ \{\bar{s}_2(\mathbf{q})-\sqrt{(\mathbf{q}-\mathbf{u})^2+\bar{s}}\}=m(\mathbf{u},\bar{s}) \quad (30)$$

$$u_0\leq\mathrm{Min}_\mathbf{q}\ \{\bar{s}_1(\mathbf{q})+\sqrt{(\mathbf{q}-\mathbf{u})^2+\bar{s}}\}=M(\mathbf{u},\bar{s}) \quad (31)$$

In this generalized version of Dyson's formalism [2], the results are valid for the case of scattering of unequal mass particles unlike the approach of Jost and Lehmann [1] which is only applicable for equal mass scatterings. Moreover, as evident, Dyson formulation is quite general and mathematically elegant. We identify region **S** in the $\tilde{r}$-space for the case at hand to be

$$m(\mathbf{u},\bar{s})\leq u_0\leq M(\mathbf{u},\bar{s}) \quad (32)$$

which is bounded by two surfaces Σ_1 and Σ_2 in the $\tilde{r}$-space. Define **T**: complement of **S** i.e. it contains the set of points in the $\tilde{r}$-space such that

$$M(\mathbf{u},\bar{s})\leq u_0\leq m(\mathbf{u},\bar{s}) \quad (33)$$

The purpose is to derive a representation for $F_C(q)$ by imposing desired constraints on $\bar{F}(\tilde{r})$ so that the Fourier transform of $F_C(q)$ belongs to the class C_R. In order to fulfill this demand $\bar{F}(\tilde{r})$ must vanish for every $\tilde{r}\in\mathbf{T}$. We choose a spacelike surface, Σ, lying between the two spacelike surfaces, Σ_1 and Σ_2 and identify it to be

$$u_0=\frac{1}{2}[m(\mathbf{u},\bar{s})+M(\mathbf{u},\bar{s})] \quad (34)$$

Notice that u_0 is constrained to lie in the domain defined by (30) and (31) and is chosen to be (34). Therefore, every point of the chosen spacelike surface, Σ is either in **S** or lies in the complement **T**. We stipulated that $\bar{F}(\tilde{r})$ vanish for every $\tilde{r}\in\mathbf{T}$. A function $F_C(q)$ belongs to C_R (its Fourier transform is meant to be in C_R) if and only if it admits a unique representation

$$F_C(q)=\int_\Sigma d\Sigma_\alpha\left[\bar{F}(\tilde{r}),\frac{\partial}{\partial\tilde{r}_\alpha}D(\tilde{r}-\hat{q})\right] \quad (35)$$

where $\Sigma \in \mathbf{S}$. Thus the integral extends over only those points of $\tilde{r}$ of Σ which belong to $\mathbf{S}$. Recall that the set of points in domain $\mathbf{S}$ are determined by (30) and (31) and $\mathbf{S}: \; m(\mathbf{u}, \bar{s}) \le q_0 \le M(\mathbf{u}, \bar{s})$. Thus the generalized Dyson's theorem for D-dimensional theory is

Theorem. *For a function $F_C(q)$ to vanish in the region $\bar{s}_1(\mathbf{q}) < q_0 < \bar{s}_2(\mathbf{q})$ and to have a Fourier transform, $\tilde{f}(x)$ such that $\tilde{f}(x) = 0$ for $x^2 < 0$, it is necessary and sufficient to have a representation*

$$F_C(q) = \int d^D u \int_0^\infty \epsilon(q_0 - u_0)\delta[(\mathbf{q}-\mathbf{u})^2 - \bar{s}]\Phi(u, \bar{s}) \tag{36}$$

$\Phi(u, \bar{s})$ vanishes outside the regions ($u_0 \ge \mathrm{Max}_{\mathbf{q}}$ $\{\bar{s}_2(\mathbf{q}) - \sqrt{(\mathbf{q}-\mathbf{u})^2 + \bar{s}}\}$ and $u_0 \le \mathrm{Min}_{\mathbf{q}}$ $\{\bar{s}_1(\mathbf{q}) + \sqrt{(\mathbf{q}-\mathbf{u})^2 + \bar{s}}\}$ as already noted earlier). The region $\mathbf{S}: \; m(\mathbf{u}, \bar{s}) \le u_0 \le M(\mathbf{u}, \bar{s})$. $\Phi(u, \bar{s})$ is arbitrary otherwise. Note that $\Phi(u, \bar{s})$, appearing in (36), depends on q's determined by $(u-q)^2 = \bar{s}$ which lie entirely in $\mathbf{R}$. It reproduces the function, $F_C(q)$ on the left hand side of (36) with the requisite support properties in q-space and the support properties of $\tilde{F}_C(x)$ are satisfied. Thus we can write

$$\tilde{F}_C(x) = \int_0^\infty d\bar{s}\Delta(x; \bar{s})\Phi(x, \bar{s}) \tag{37}$$

where $\Phi(x, \bar{s})$ is the Fourier transform of $\Phi(u, \bar{s})$ with respect to u and is the well known invariant function (now defined in D-dimensions) with mass $\sqrt{\bar{s}}$. Thus the causality properties of $\tilde{F}_C(x)$, as desired by us, is satisfied.

The spectral representation derived in this investigation has important consequences in the study of the analyticity properties of amplitudes in higher dimensional field theories. Several comments are in order in this context.

(i) The analyticity of scattering amplitude in s and t variables can be analyzed. Now the analog of the Lehmann ellipse can be derived in the sense that existence of the domain of analyticity in complex t-plane can proved. We have argued that the amplitude will be polynomially bounded in s invoking the arguments of Symanzik [25]. Thus a fixed t dispersion relation can be written down since the theory is crossing symmetric.

(ii) The next step is to show the analyticity in the product domain of $D_s \otimes D_t$. This is achieved through the generalization of Martin's theorem [26]. A consequence of this theorem is that the semimajor axis of Large Lehmann Ellipse (LLE) can be determined from first principles.

(iii) The importance of the above results (i) and (ii) is realized in that the analog of Froissart–Martin bound for total cross section in a D-dimensional theory can be proved. The number of subtractions, N, needed to write dispersion relation will be determined. Indeed the analog of the Jin–Martin [27] theorem is proved leading to the conclusion that $N = 2$.

(iv) Therefore, the ad hoc assumptions of [23,24] that the amplitude is polynomially bounded in s and that it converges inside an analog Lehmann ellipse now can be proved in the frame work of LSZ formalism.

(v) The present investigation is the first step to address issues related to the study of the analyticity properties of scattering amplitude in higher dimensional scalar massive neutral field theories in flat spacetime. We would like to remind the reader that the derivation of Froissart–Martin bound (1) in 4-dimensional field theories is also derived for the ideal scenario as envisaged in this work. As long as we consider D-dimensional flat Minkowski space with the scalar fields, as is the case here, theory fulfills the requirements of the axioms stated earlier. Thus the concepts such Lorentz invariance, uniqueness of vacuum and microcausality are defined. As alluded to earlier, the high energy scattering experiments involve hadrons which are composite and we would like to resort the arguments advanced in derivation of rigorous results in 4-dimensional theories in the axiomatic formulations of scalar, massive field theories. We would like to reiterate here that ours is the first effort in investigating analyticity properties of scattering amplitude in higher dimensional field theories under ideal conditions within the axiomatic framework. Of course, string theory is formulated in higher spacetime dimensions. It holds the prospect of unifying fundamental interactions since certain string theories admit graviton and nonabelian gauge bosons in their massless spectrum besides other massless states. Therefore, the scatterings in the stringy energy regime have to take into account presence of these states. There are well defined techniques to compute the amplitudes in string theory. It is important to recall that the scattering amplitudes derived in string theory are computed in the first quantized framework. Thus axiomatic formalism of string field theory is yet to be formulated which is at par with the axiomatic frameworks of point particle case. Therefore, the fundamental problems which have been addressed in the context of scattering in the frameworks of axiomatic point particle field theories are not discussed in the present formulation of string theories as far as scattering of stringy states are concerned.

In summary, we have proved the existence of Jost–Lehmann–Dyson representation for a massive scalar field theory in LSZ formalism. Our result paves way to investigate analyticity properties of scattering amplitude in D-dimensional field theories and their asymptotic growth properties. The elaborations of the results alluded to above will be published elsewhere [28].

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