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A simple model of universe with a polytropic equation of state

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Abstract. We construct a simple model of universe with a generalized equation of state $p = (\alpha + k\rho^{1/n})\rho c^2$ having a linear component $p = \alpha\rho c^2$ and a polytropic component $p = k\rho^{1+1/n}c^2$. For $\alpha = 1/3$, $n = 1$ and $k = -4/(3\rho_P)$, where $\rho_P = 5.16 \times 10^{99} \text{ g/m}^3$ is the Planck density, the equation of state $p = \frac{1}{3}(1 - 4\rho/\rho_P)\rho c^2$ provides a model of early universe without singularity describing the transition between the vacuum energy era (Planck era) and the radiation era. The universe starts from $t = -\infty$ but, when $t < 0$, its size is less than the Planck length $l_P = 1.62 \times 10^{-35} \text{ m}$, requiring a quantum gravity treatment. For $t \geq 0$, the universe undergoes an accelerated expansion (early inflation) that brings it from the Planck size $l_P = 1.62 \times 10^{-35} \text{ m}$ to a size $a_1 = 2.61 \times 10^{-6} \text{ m}$ on a timescale of about 23.3 Planck times $t_P = 5.39 \times 10^{-44} \text{ s}$. When $t > t_1 = 23.3t_P = 1.25 \times 10^{-42} \text{ s}$, the universe decelerates and enters in the radiation era. For $\alpha = 0$, $n = -1$ and $k = -\rho_\Lambda$, where $\rho_\Lambda = 7.02 \times 10^{-24} \text{ g/m}^3$ is the cosmological density, the equation of state $p = -\rho_\Lambda c^2$ describes the transition from a decelerating universe dominated by dark matter to an accelerating universe (late inflation) dominated by dark energy (de Sitter era). This transition takes place at a size $a_2 = 0.204l_\Lambda = 8.95 \times 10^{25} \text{ m}$ corresponding to a time $t_2 = 0.203t_\Lambda = 2.97 \times 10^{17} \text{ s}$ where $l_\Lambda = 4.38 \times 10^{26} \text{ m}$ is the cosmological length and $t_\Lambda = 1.46 \times 10^{18} \text{ s}$ the cosmological time. The present universe turns out to be just at the transition ($t_0 \sim t_2$). Our model generalizes the standard Λ CDM model by incorporating naturally a phase of early inflation that avoids the primordial singularity. Furthermore, it reveals a nice “symmetry” between the early universe (vacuum energy + radiation) and the late universe (dark matter + dark energy). They are described by two polytropic equations of state with index $n = +1$ and $n = -1$ respectively. Furthermore, the cosmological constant Λ in the late universe plays a role similar to the Planck constant \hbar in the early universe. The mathematical formulae in the early and in the late universe are strikingly symmetric. We interpret the cosmological constant as a fundamental constant of nature describing the “cosmophysics” just like the Planck constant describes the “microphysics”. The Planck density and the cosmological density represent fundamental upper and lower bounds differing by 123 orders of magnitude. They are responsible for a phase of inflation in the early and late universe. The cosmological constant “problem” may be a false problem. Our model does not present any singularity and exists eternally in the past and in the future (aioniotic universe). On the other hand, it admits a scalar field interpretation based on an inflaton, quintessence, or tachyonic field. This correspondence is interesting because the early inflation and the late acceleration of the universe are usually described in terms of a scalar field. We determine the potential and the mass of this scalar field both in the early and late universe.



1. Introduction

According to contemporary cosmology, the present energy content of the universe is composed of approximately 5% baryonic matter, 20% dark matter, and 75% dark energy. The expansion of the universe began in a tremendous inflationary burst driven by vacuum energy (Planck era). Between 10^{-35} and 10^{-33} seconds after the Big Bang, the universe expanded by a factor 10^{30} [1]. The universe then entered in the radiation era and, when the temperature cooled down below approximately 10^3 K, in the matter era [2]. At present, it undergoes an accelerated expansion (de Sitter era) presumably due to the cosmological constant Λ or to some form of dark energy [3]. This corresponds to a second period of inflation. Despite the success of the standard cold dark matter model (Λ CDM), the nature of vacuum energy, dark matter and dark energy, remains very mysterious and leads to many speculations.

The phase of inflation in the early universe, which is necessary to solve notorious difficulties such as the singularity problem, the flatness problem (or the fine-tuning problem) and the horizon problem (or the causality problem) and to account for the observed spectrum of primordial density fluctuations, is usually described by some hypothetical scalar field ϕ with its origin in the quantum fluctuations of vacuum [1]. This leads to an equation of state $p = -\rho c^2$ with a negative pressure, implying a constant energy density, called the vacuum energy. This energy density is usually identified with the Planck density $\rho_P = c^5/G^2\hbar = 5.16 \times 10^{99}$ g/m³. As a result of the vacuum energy, the universe expands exponentially rapidly on a timescale of the order of the Planck time $t_P = 1/(G\rho_P)^{1/2} = (\hbar G/c^5)^{1/2} = 5.39 \times 10^{-44}$ s. This phase of early inflation (Planck era) is followed by the radiation era described by an equation of state $p = \rho c^2/3$ ($\rho \propto a^{-4}$, $a \propto t^{1/2}$ and $\rho = 3/32\pi Gt^2$) and by the matter era described by an equation of state $p = 0$ ($\rho \propto a^{-3}$, $a \propto t^{2/3}$ and $\rho = 1/6\pi Gt^2$).

The phase of acceleration in the late universe, which has been revealed by the observations of distant type Ia supernovae is usually ascribed to the cosmological constant Λ which is equivalent to a constant energy density $\rho_\Lambda = \Lambda/(8\pi G) = 7.02 \times 10^{-24}$ g/m³ called the dark energy [3]. This acceleration can be modeled by an equation of state $p = -\rho c^2$ with a negative pressure, implying a constant energy density identified with the cosmological density ρ_Λ . As a result of the dark energy, the universe expands exponentially rapidly on a timescale of the order of the cosmological time $t_\Lambda = 1/(G\rho_\Lambda)^{1/2} = (8\pi/\Lambda)^{1/2} = 1.46 \times 10^{18}$ s. This leads to a phase of late inflation (de Sitter era). Instead of introducing a cosmological constant, it has been proposed to explain the acceleration of the universe in terms of a dark energy with a time-varying density. Inspired by the analogy with the early inflation, some authors have represented the dark energy by a scalar field called quintessence [4]. As an alternative to quintessence, other authors have proposed to model the acceleration of the universe by an exotic fluid with an equation of state of the form $p/c^2 = -A/\rho$ called the Chaplygin gas [5]. At late times, this equation of state leads to a constant energy density implying an exponential growth of the scale factor that is similar to the effect of the cosmological constant. At earlier times, this equation of state returns the results of the CDM model ($p \simeq 0$) where the density decreases as $\rho \propto a^{-3}$ (Einstein-de Sitter solution). Therefore, it provides a unification of dark matter ($\rho \propto a^{-3}$) and dark energy ($\rho = \rho_\Lambda$) in the late universe. Furthermore, it gives a real speed of sound which is non-trivial for fluids with negative pressure. The Chaplygin gas has some connection with string theory and can be obtained from the Nambu-Goto action for d -branes moving in a $(d+2)$ -dimensional spacetime in the light-cone parametrization. However, the Chaplygin gas model does not give a good description of the evolution of the universe between the matter era and the dark energy era [6, 7].

In this paper, we construct a simple model of universe taking into account the effect of vacuum energy, radiation, dark matter and dark energy through a generalized equation of state of the form

$$p = (\alpha\rho + k\rho^{1+1/n})c^2. \quad (1)$$

This is the sum of a standard linear equation of state $p = \alpha\rho c^2$ and a polytropic equation of

state $p = k\rho^\gamma c^2$, where k is the polytropic constant and $\gamma = 1 + 1/n$ the polytropic index. An exhaustive study of this equation of state, considering all the possible cases for arbitrary values of α , k and n , is given in [8, 9]. We have found the following structure. Positive indices $n > 0$ describe the early universe where the polytropic component dominates the linear component because the density is high. Negative indices $n < 0$ describe the late universe where the polytropic component dominates the linear component because the density is low. On the other hand, a positive polytropic pressure ($k > 0$) leads to past or future singularities (or peculiarities) while a negative polytropic pressure ($k < 0$) leads to a phase of exponential expansion (inflation) in the past or in the future. The case of a phantom universe where the density increases with the scale factor has been studied in [10].

In this paper, we show that the early and late evolution of the universe can be described “symmetrically” by two polytropic equations of state of the form of Eq. (1) with index $n = 1$ and $n = -1$, respectively. Our model [see Eq. (57)] generalizes the standard Λ CDM model by removing the primordial singularity and accounting for a phase of early inflation. We study the dynamical and thermodynamical evolution of the universe and determine a generalized Stefan-Boltzmann law [see Eq. (13)] valid in the very early universe, before the radiation era. On the other hand, we show that our model admits a scalar field interpretation based on an inflaton, quintessence, or tachyonic field. This correspondence is interesting because the early inflation and the late acceleration of the universe are usually described in terms of a scalar field. We determine the potential and the mass of this scalar field both in the early [see Eqs. (19) and (25)] and late [see Eqs. (38), (42) and (49)] universe.

2. Basic equations of cosmology

We assume that the universe is isotropic and homogeneous at large scales and contains a uniform perfect fluid of energy density $\epsilon(t) = \rho(t)c^2$ and pressure $p(t)$. We also assume that the universe is flat in agreement with observations of the cosmic microwave background (CMB). In that case, the Einstein equations reduce to

$$\frac{d\rho}{dt} + 3\frac{\dot{a}}{a}\left(\rho + \frac{p}{c^2}\right) = 0, \quad (2)$$

$$H^2 = \left(\frac{\dot{a}}{a}\right)^2 = \frac{8\pi G}{3}\rho, \quad (3)$$

where $a(t)$ is the scale factor (by an abuse of language, we shall sometimes call it the “radius” of the universe) and $H = \dot{a}/a$ is the Hubble parameter. These are the well-known Friedmann equations describing an expanding universe [2]. The first equation is the energy equation (it can also be viewed as an equation of continuity). For a given barotropic equation of state $p = p(\rho)$, it determines the relation between the energy density and the scale factor. Then, the temporal evolution of the scale factor is given by Eq. (3). We have not written the cosmological constant Λ because its effect will be taken into account in the equation of state.

The equation of state parameter w , the deceleration parameter q and the speed of sound c_s are defined by

$$w = \frac{p}{\rho c^2}, \quad q(t) = -\frac{\ddot{a}a}{\dot{a}^2} = \frac{1 + 3w(t)}{2}, \quad c_s^2 = p'(\rho), \quad (4)$$

where the second equality in the expression of q , obtained from the Friedmann equations, is valid for a flat universe. The universe is decelerating when $q > 0$ (or $w > -1/3$) (strong energy condition) and accelerating when $q < 0$ (or $w < -1/3$). On the other hand, according to Eq. (2), the density decreases with the scale factor if $w > -1$ (null dominant energy condition) and increases with the scale factor if $w < -1$. The latter case corresponds to a phantom universe [11].

We will also need the thermodynamic equation

$$\frac{dp}{dT} = \frac{1}{T}(\rho c^2 + p), \quad (5)$$

which can be derived from the first principle of thermodynamics [2]. For a given barotropic equation of state $p = p(\rho)$, this equation can be integrated to obtain the relation $T = T(\rho)$ between the temperature and the density. Combining Eqs. (2) and (5) we get

$$\frac{dS}{dt} = 0, \quad \text{where} \quad S = \frac{a^3}{T}(p + \rho c^2) \quad (6)$$

is the entropy of the universe in a volume a^3 . This shows that the Friedmann equations imply the conservation of the entropy.

To close the system of equations, we have to specify an equation of state. We shall consider a generalized equation of state of the form of Eq. (1) in the early and late universe [8, 9, 12].

3. Early universe: vacuum energy + radiation

3.1. Equation of state

We first consider the equation of state (1) with $\alpha = 1/3$, $n = 1$ and $k = -4/(3\rho_P)$ which provides a model of early universe dominated by vacuum energy and radiation. This equation of state can be rewritten as [8, 12]:

$$p = \frac{1}{3} \left(1 - \frac{4\rho}{\rho_P} \right) \rho c^2. \quad (7)$$

The continuity equation (2) can be integrated into

$$\rho = \frac{\rho_P}{(a/a_1)^4 + 1}, \quad (8)$$

where a_1 is a constant of integration.

The equation of state parameter, the deceleration parameter, and the speed of sound are given by

$$w = \frac{1}{3} - \frac{4}{3} \frac{\rho}{\rho_P}, \quad q = 1 - 2 \frac{\rho}{\rho_P}, \quad \frac{c_s^2}{c^2} = \frac{1}{3} - \frac{8}{3} \frac{\rho}{\rho_P}. \quad (9)$$

As the universe expands from $a = 0$ to $a = +\infty$, the density decreases from ρ_P to 0, the equation of state parameter w increases from -1 to $1/3$, the deceleration parameter q increases from -1 to 1 , and the ratio $(c_s/c)^2$ increases from $-7/3$ to $1/3$.

3.2. Vacuum energy era: early inflation

When $a \ll a_1$, the density is approximately constant ($\rho \simeq \rho_P$) and the pressure tends to the value $p = -\rho_P c^2$. Since this solution is expected to describe the very early universe, it is natural to identify the constant ρ_P with the Planck density $\rho_P = c^5/G^2\hbar = 5.16 \times 10^{99} \text{ g/m}^3$ (vacuum energy) which may represent a fundamental upper bound for the density. This is how quantum mechanics is taken into account in our model. A constant value of the density gives rise to a phase of early inflation. From the Friedmann equation (3), we find that the scale factor increases exponentially rapidly with time as

$$a(t) \sim l_P e^{(8\pi/3)^{1/2} t/t_P}, \quad (10)$$

where $t_P = 1/(G\rho_P)^{1/2} = (\hbar G/c^5)^{1/2} = 5.39 \times 10^{-44} \text{ s}$ is the Planck time which gives the timescale of the exponential growth. We have defined the time $t = 0$ such that $a(0)$ is equal

to the Planck length $l_P = ct_P = (G\hbar/c^3)^{1/2} = 1.62 \times 10^{-35}$ m. Mathematically speaking, this universe exists at any time in the past ($a \rightarrow 0$ and $\rho \rightarrow \rho_P$ for $t \rightarrow -\infty$), so there is no primordial singularity. For $t < 0$, the radius of the universe is smaller than the Planck length. However, when $a \rightarrow 0$, we cannot ignore the quantum fluctuations associated with the spacetime. In that case, we cannot use the classical Einstein equations anymore and a theory of quantum gravity is required. It is not known whether quantum gravity will remove, or not, the primordial singularity. Therefore, we cannot extrapolate the solution (10) to the infinite past. However, this solution may provide a *semi-classical* description of the phase of early inflation when $a \geq l_P$.

3.3. Radiation era

When $a \gg a_1$, we get $\rho/\rho_P \sim (a_1/a)^4$. When the density is “low”, the equation of state (7) reduces to the linear equation of state $p = \rho c^2/3$ which describes the radiation era. The conservation of $\rho_{\text{rad}} a^4$ implies that $\rho_P a_1^4 = \rho_{\text{rad},0} a_0^4$ where $\rho_{\text{rad},0}$ is the present density of radiation and $a_0 = c/H_0 = 1.32 \times 10^{26}$ m the present distance of cosmological horizon determined by the Hubble constant $H_0 = 2.27 \times 10^{-18} \text{ s}^{-1}$ (the Hubble time is $H_0^{-1} = 4.41 \times 10^{17}$ s). Writing $\rho_{\text{rad},0} = \Omega_{\text{rad},0} \rho_0$ where $\rho_0 = 3H_0^2/8\pi G = 9.20 \times 10^{-24} \text{ g/m}^3$ is the present density of the universe (the present mass of the universe is $M_0 = \rho_0 a_0^3 = 3c^3/8\pi G H_0 = 2.11 \times 10^{55} \text{ g}$) and $\Omega_{\text{rad},0} = 8.48 \times 10^{-5}$ is the present fraction of radiation in the universe, we obtain $a_1/a_0 = (\Omega_{\text{rad},0} \rho_0/\rho_P)^{1/4}$, hence $a_1 = 2.61 \times 10^{-6}$ m. This scale is intermediate between the Planck length l_P and the present size of the universe a_0 ($a_1/l_P = 1.61 \times 10^{29}$ and $a_1/a_0 = 1.97 \times 10^{-32}$). It marks the transition between the vacuum energy era and the radiation era. When $a \gg a_1$, the Friedmann equation (3) yields $a/a_1 \sim (32\pi/3)^{1/4} (t/t_P)^{1/2}$ and $\rho/\rho_P \sim (3/32\pi)(t_P/t)^2$.

3.4. The general solution

The general solution of the Friedmann equation (3) with Eq. (8) is [8, 12]:

$$\sqrt{(a/a_1)^4 + 1} - \ln \left(\frac{1 + \sqrt{(a/a_1)^4 + 1}}{(a/a_1)^2} \right) = 2 \left(\frac{8\pi}{3} \right)^{1/2} \frac{t}{t_P} + C, \quad (11)$$

where $C \simeq -134$ is a constant of integration determined such that $a = l_P$ at $t = 0$. For $t \rightarrow -\infty$, Eq. (11) returns Eq. (10) with an excellent approximation. The transition between the vacuum energy era and the radiation era ($a = a_1$) corresponds to a density $\rho_1 = \rho_P/2$ and a time $t_1 = (3/32\pi)^{1/2} [\sqrt{2} - \ln(1 + \sqrt{2}) - C] t_P$. The universe is accelerating when $a < a_c$ (i.e. $\rho > \rho_c$) and decelerating when $a > a_c$ (i.e. $\rho < \rho_c$) where $\rho_c = \rho_P/2$ and $a_c = a_1$. The time t_c at which the universe starts decelerating corresponds to the time at which the curve $a(t)$ presents a first inflexion point. It turns out that this inflexion point coincides with a_1 so it also marks the end of the inflation ($t_c = t_1$).

3.5. The pressure

The pressure is given by Eq. (7). Using Eq. (8), we get

$$p = \frac{\frac{1}{3}(a/a_1)^4 - 1}{[(a/a_1)^4 + 1]^2} \rho_P c^2. \quad (12)$$

The pressure starts from $p = -\rho_P c^2$ at $t = -\infty$, remains approximately constant during the inflation, increases at the end of the inflation, becomes positive, reaches a maximum value p_e , and decreases algebraically during the radiation era as $p \sim (1/3)\rho c^2 \sim (1/3)\rho_P c^2 (a_1/a)^4 \sim (1/32\pi)\rho_P c^2 (t_P/t)^2$. At the transition/deceleration point $t = t_1 = t_c$, we have $p_1/(\rho_P c^2) = -1/6$. The pressure vanishes ($w = 0$) at $\rho_w/\rho_P = 1/4$, $a_w/a_1 = 3^{1/4}$. This corresponds to

a time $t_w = (3/32\pi)^{1/2} [2 - (1/2) \ln(3) - C] t_P$. On the other hand, the pressure reaches its maximum value $p_e/(\rho_P c^2) = 1/48$ when $\rho_e/\rho_P = 1/8$, $a_e/a_1 = 7^{1/4}$. This corresponds to a time $t_e = (3/32\pi)^{1/2} [\sqrt{8} - \ln(1 + \sqrt{8}) + \ln 7/2 - C] t_P$. The speed of sound is imaginary ($c_s^2 < 0$) when $a < a_e$, vanishes ($c_s = 0$) when $a = a_e$, and is real ($c_s^2 > 0$) when $a > a_e$. It is always less than the speed of light.

3.6. Evolution of the temperature

The thermodynamical equation (5) with the equation of state Eq. (7) can be integrated into

$$T = T_P \left(\frac{15}{\pi^2} \right)^{1/4} \left(1 - \frac{\rho}{\rho_P} \right)^{7/4} \left(\frac{\rho}{\rho_P} \right)^{1/4} = T_P \left(\frac{15}{\pi^2} \right)^{1/4} \frac{(a/a_1)^7}{[(a/a_1)^4 + 1]^2}, \quad (13)$$

where T_P is a constant of integration and we have used Eq. (8) to obtain the second equality. In the radiation era $\rho \ll \rho_P$, Eq. (13) reduces to $\rho/\rho_P \sim (\pi^2/15)(T/T_P)^4$ which is the Stefan-Boltzmann law $\rho c^2 = (\pi^2/15)(k_B T)^4/\hbar^3 c^3$ provided that T_P is identified with the Planck temperature $T_P = (\hbar c^5/Gk_B^2)^{1/2} = 1.42 \times 10^{32}$ K. Accordingly, Eq. (13) may be viewed as a generalized Stefan-Boltzmann law [8, 12].

Using Eqs. (7), (8) and (13), we find that the entropy (6) is given by

$$S = \frac{4}{3} \left(\frac{\pi^2}{15} \right)^{1/4} \frac{\rho_P c^2}{T_P} a_1^3. \quad (14)$$

We check that it is a constant. Numerically, $S/k_B = 5.04 \times 10^{87}$ [8, 12].

The temperature starts from $T = 0$ at $t = -\infty$. In the vacuum energy era, it increases exponentially rapidly with time as

$$\frac{T}{T_P} \sim \left(\frac{15}{\pi^2} \right)^{1/4} \left(1 - \frac{\rho}{\rho_P} \right)^{7/4} \sim \left(\frac{15}{\pi^2} \right)^{1/4} \left(\frac{a}{a_1} \right)^7 \sim \left(\frac{15}{\pi^2} \right)^{1/4} \left(\frac{l_P}{a_1} \right)^7 e^{7(8\pi/3)^{1/2} t/t_P}. \quad (15)$$

At $t = t_i = 0$, $T_i/T_P \simeq (15/\pi^2)^{1/4} (l_P/a_1)^7$. At the transition/deceleration point $t = t_1 = t_c$, we have $T_1/T_P = (1/4)(15/\pi^2)^{1/4}$. At the point where the pressure vanishes, we have $T_w = (9/16)(5/\pi^2)^{1/4} T_P$. The temperature reaches its maximum value $T_e = (7/8)^{7/4} (15/8\pi^2)^{1/4} T_P$ at the point where the pressure reaches its maximum value. In the radiation era, the temperature decreases algebraically rapidly as

$$\frac{T}{T_P} \sim \left(\frac{15}{\pi^2} \right)^{1/4} \left(\frac{\rho}{\rho_P} \right)^{1/4} \sim \left(\frac{15}{\pi^2} \right)^{1/4} \frac{a_1}{a} \sim \left(\frac{45}{32\pi^3} \right)^{1/4} \left(\frac{t_P}{t} \right)^{1/2}. \quad (16)$$

Remark: In our model, the temperature is initially very low, increases exponentially rapidly during the inflation up to a fraction (~ 0.523) of the Planck temperature $T_P = 1.42 \times 10^{32}$ K, then decreases algebraically during the radiation era. On the other hand, our model generates a value of the entropy as large as $S/k_B = 5.04 \times 10^{87}$. This is very different from the standard inflationary scenario [1]. In that scenario, the universe is radiation dominated up to $t_i = 10^{-35}$ s and expands exponentially rapidly by a factor 10^{30} in the interval $t_i < t < t_f$ with $t_f = 10^{-33}$ s. For $t > t_f$, the evolution is again radiation dominated. At $t = t_i$, the temperature is about 10^{27} K. This corresponds to the epoch at which most grand unified theories (GUTs) have a significant influence on the evolution of the universe. During the exponential inflation, the temperature drops drastically and one must advocate a phase of re-heating by various high energy processes (not very well understood) to restore the initial temperature.

3.7. History of the early universe

In our model, the universe “starts” at $t = -\infty$ with a vanishing radius $a = 0$, a finite density $\rho = \rho_P = 5.16 \times 10^{99} \text{ g/m}^3$, a finite pressure $p = -\rho_P c^2 = -4.64 \times 10^{116} \text{ g/ms}^2$, and a vanishing temperature $T = 0$. The universe exists at any time in the past and does not present any singularity. For $t < 0$, the radius of the universe is less than the Planck length $l_P = 1.62 \times 10^{-35} \text{ m}$. In the Planck era, quantum gravity should be taken into account so our semi-classical approach is probably not valid in the infinite past. We define the “original” time $t = t_i = 0$ as the time at which the radius of the universe is equal to the Planck length: $a_i = l_P = 1.62 \times 10^{-35} \text{ m}$. The corresponding density, pressure and temperature are $\rho_i \simeq \rho_P = 5.16 \times 10^{99} \text{ g/m}^3$, $p_i \simeq -\rho_P c^2 = -4.64 \times 10^{116} \text{ g/ms}^2$ and $T_i = 3.91 \times 10^{-205} T_P = 5.54 \times 10^{-173} \text{ K}$. We note that quantum mechanics regularizes the finite time singularity present in the standard Big Bang theory. This is similar to finite size effects in second order phase transitions (the standard Big Bang theory is recovered for $\hbar \rightarrow 0$) [8, 12]. We also note that the universe is very cold at $t = t_i = 0$, unlike what is predicted by the Big Bang theory (a naive extrapolation of the law $T \propto t^{-1/2}$ leads to $T(0) = +\infty$). The universe first undergoes a phase of inflation driven by the vacuum energy during which its radius and temperature increase exponentially rapidly while its density and pressure remain approximately constant. The inflation “starts” at $t_i = 0$ and “ends” at $t_1 = 23.3 t_P = 1.25 \times 10^{-42} \text{ s}$. During this very short lapse of time, the radius of the universe grows from $a_i = l_P = 1.62 \times 10^{-35} \text{ m}$ to $a_1 = 2.61 \times 10^{-6} \text{ m}$, and the temperature grows from $T_i = 3.91 \times 10^{-205} T_P = 5.54 \times 10^{-173} \text{ K}$ to $T_1 = 0.278 T_P = 3.93 \times 10^{31} \text{ K}$. By contrast, the density and the pressure do not change significantly: they go from $\rho_i \simeq \rho_P = 5.16 \times 10^{99} \text{ g/m}^3$ and $p_i \simeq -\rho_P c^2 = -4.64 \times 10^{116} \text{ g/ms}^2$ to $\rho_1 = \rho_P/2 = 2.58 \times 10^{99} \text{ g/m}^3$ and $p_1 = -(1/6)\rho_P c^2 = -7.73 \times 10^{115} \text{ g/ms}^2$. The pressure passes from negative values to positive values at $t_w = 23.4 t_P = 1.26 \times 10^{-42} \text{ s}$ (at that moment $\rho_w = \rho_P/4 = 1.29 \times 10^{99} \text{ g/m}^3$, $a_w = 1.32 a_1 = 3.43 \times 10^{-6} \text{ m}$ and $T_w = 0.475 T_P = 6.74 \times 10^{31} \text{ K}$). After the inflation, the universe enters in the radiation era and, from that point, we recover the standard model [2]. The radius increases algebraically as $a \propto t^{1/2}$ while the density and the temperature decrease algebraically as $\rho \propto t^{-2}$ and $T \propto t^{-1/2}$. The pressure and the temperature achieve their maximum values $p_e = 2.08 \times 10^{-2} \rho_P c^2 = 9.66 \times 10^{114} \text{ g/ms}^2$ and $T_e = 0.523 T_P = 7.40 \times 10^{31} \text{ K}$ at $t = t_e = 23.6 t_P = 1.27 \times 10^{-42} \text{ s}$. At that moment, the density is $\rho_e = \rho_P/8 = 6.44 \times 10^{98} \text{ g/m}^3$ and the radius $a_e = 1.63 a_1 = 4.24 \times 10^{-6} \text{ m}$. During the inflation, the universe is accelerating and during the radiation era it is decelerating. The transition (marked by an inflexion point) takes place at a time $t_c = t_1 = 23.3 t_P = 1.25 \times 10^{-42} \text{ s}$ coinciding with the end of the inflation ($a_c = a_1$). The evolution of the scale factor, density and temperature of the universe as a function of time are represented in Figs. 1-3 below.

3.8. Scalar field: inflaton

The phase of inflation in the very early universe is usually described by some hypothetical scalar field ϕ , called inflaton, with its origin in the quantum fluctuations of the vacuum [1]. A canonical scalar field minimally coupled to gravity evolves according to the Klein-Gordon equation

$$\ddot{\phi} + 3H\dot{\phi} + \frac{dV}{d\phi} = 0, \quad (17)$$

where $V(\phi)$ is the potential of the scalar field. The scalar field tends to run down the potential towards lower energies. The density and the pressure of the universe are related to the scalar field by

$$\rho c^2 = \frac{1}{2} \dot{\phi}^2 + V(\phi), \quad p = \frac{1}{2} \dot{\phi}^2 - V(\phi). \quad (18)$$

Using standard techniques [9, 12], we find that the inflaton potential corresponding to the

equation of state (7) is

$$V(\psi) = \frac{1}{3}\rho_P c^2 \frac{\cosh^2 \psi + 2}{\cosh^4 \psi} \quad \psi = \left(\frac{8\pi G}{c^2}\right)^{1/2} \phi, \quad (\phi \geq 0). \quad (19)$$

We can also show [9, 12] that the scale factor, the energy density and the pressure of the universe are related to the scalar field by

$$\left(\frac{a}{a_1}\right)^2 = \sinh \psi, \quad \rho = \frac{\rho_P}{\cosh^2 \psi}, \quad p = \frac{\rho_P c^2}{3 \cosh^2 \psi} \left(1 + \frac{2}{\cosh^2 \psi}\right). \quad (20)$$

Using Eq. (20) and the results of the previous sections we can obtain the temporal evolution of the inflaton in the early universe [9, 12].

For $\psi \rightarrow 0$, the scalar field potential (19) can be expanded up to fourth order giving

$$\frac{V(\psi)}{\rho_P c^2} \simeq 1 - \frac{5}{3}\psi^2 + \frac{16}{9}\psi^4 + \dots \quad (21)$$

This expression can be compared with the standard form of a quartic potential

$$V = V_0 + \frac{m^2 c^4}{2\hbar^2} \phi^2 + \frac{\lambda c^3}{4\hbar} \phi^4, \quad (22)$$

where V_0 is the energy of point zero, m is the mass of the SF and λ is the dimensionless self-interaction constant. When the SF describes the wave function of a Bose-Einstein condensate (BEC), we have

$$\frac{\lambda}{8\pi} = \frac{a_s}{\lambda_C} = \frac{a_s m c}{\hbar}, \quad (23)$$

where a_s is the scattering length of the bosons and λ_C is their Compton length. Comparing Eq. (21) with Eq. (22), we obtain

$$V_0 = \rho_P c^2, \quad (24)$$

$$m^2 = -\frac{80\pi}{3} \frac{\rho_P G \hbar^2}{c^4} = -\frac{80\pi}{3} M_P^2, \quad (25)$$

$$\lambda = \frac{4096\pi^2}{9}, \quad a_s^2 = -\frac{16384\pi}{135} l_P^2, \quad (26)$$

where $M_P = (\hbar c/G)^{1/2} = 2.17 \times 10^{-5} \text{ g}$ is the Planck mass. From Eqs. (25) and (26) we get

$$a_s = \frac{16}{15} \frac{2Gm}{c^2} = \frac{16}{15} r_S, \quad (27)$$

where $r_S = 2Gm/c^2$ is the effective Schwarzschild radius of the particle. We note that the mass of the inflaton is imaginary, but its modulus is of the order of the Planck mass (similarly $\lambda \sim 1$ and $|a_s| \sim l_P$). In theories of extended supergravity, the mass is quantized according to the rule [13]:

$$m^2 = n_* \frac{H^2 \hbar^2}{c^4}, \quad (28)$$

where n_* is an integer and H is the Hubble factor in the de Sitter era. Since $H^2 = 8\pi G \rho_P / 3$ during the early inflation, we can rewrite Eq. (28) as

$$m^2 = n_* \frac{8\pi G \rho_P \hbar^2}{3c^4}. \quad (29)$$

We see that this quantization condition is consistent with Eq. (25) for the integer value $n_* = -10$ [14].

4. Late universe: dark matter + dark energy

4.1. Equation of state

We now consider the equation of state (1) with $\alpha = 0$, $n = -1$ and $k = -\rho_\Lambda$ which provides a model of late universe dominated by dark matter and dark energy. This equation of state can be rewritten as [9, 12]:

$$p = -\rho_\Lambda c^2. \quad (30)$$

The pressure has a constant negative value. The continuity equation (2) can be integrated into

$$\rho = \rho_\Lambda \left[\left(\frac{a_2}{a} \right)^3 + 1 \right], \quad (31)$$

where a_2 is a constant of integration. The relation (31) between the energy density and the scale factor is formally equivalent to that of the Λ CDM model.

The equation of state parameter, the deceleration parameter, and the speed of sound are given by

$$w = -\frac{\rho_\Lambda}{\rho}, \quad q = \frac{1}{2} - \frac{3}{2} \frac{\rho_\Lambda}{\rho}, \quad \frac{c_s^2}{c^2} = 0. \quad (32)$$

As the universe expands from $a = 0$ to $a = +\infty$, the density decreases from $+\infty$ to ρ_Λ , the equation of state parameter w decreases from 0 to -1 , the deceleration parameter q decreases from $1/2$ to -1 , and the speed of sound c_s remains equal to zero.

4.2. Dark energy era: late inflation

When $a \gg a_2$, the density is approximately constant: $\rho \simeq \rho_\Lambda$. Since this solution is expected to describe the very late universe, it is natural to identify the constant ρ_Λ with the cosmological density $\rho_\Lambda = \Lambda/8\pi G = 7.02 \times 10^{-24} \text{ g/m}^3$ (dark energy) which may represent a fundamental lower bound for the density.¹ A constant value of the density gives rise to a phase of late inflation (accelerated expansion of the universe). It is convenient to define a cosmological time $t_\Lambda = 1/(G\rho_\Lambda)^{1/2} = (8\pi/\Lambda)^{1/2} = 1.46 \times 10^{18} \text{ s}$, a cosmological length $l_\Lambda = ct_\Lambda = (8\pi c^2/\Lambda)^{1/2} = 4.38 \times 10^{26} \text{ m}$ and a cosmological mass $M_\Lambda = \rho_\Lambda l_\Lambda^3 = (8\pi c^6/\Lambda G^2)^{1/2} = 5.90 \times 10^{56} \text{ g}$. These are the counterparts of the Planck scales for the late universe [9]. Since $\Omega_{\Lambda,0} \sim 1$, these cosmological scales are of the same order as the Hubble scales defined in terms of $\rho_0 = 3H_0^2/8\pi G$. From the Friedmann equation (3), we find that the scale factor increases exponentially rapidly with time as

$$a(t) \sim l_\Lambda e^{(8\pi/3)^{1/2}(t-t_f)/t_\Lambda}, \quad (33)$$

where t_f (determined below) is the time at which $a_f = l_\Lambda$ and $\rho_f = [(a_2/l_\Lambda)^3 + 1]\rho_\Lambda$. This exponential growth corresponds to the de Sitter (dS) solution. The timescale of the exponential growth is the cosmological time $t_\Lambda = 1.46 \times 10^{18} \text{ s}$. This solution exists at any time in the future ($a \rightarrow +\infty$ and $\rho \rightarrow \rho_\Lambda$ for $t \rightarrow +\infty$), so there is no future singularity.²

¹ In our model, ρ_Λ and Λ appear as fundamental constants that characterize the late universe (in the same sense that ρ_P and \hbar characterize the early universe). To determine the value of ρ_Λ , we use the results of observations and the fact that our model of late universe is formally equivalent to the Λ CDM model. Thus, $\rho_\Lambda = \Omega_{\Lambda,0}\rho_0$ with $\Omega_{\Lambda,0} = 0.763$ and $\rho_0 = 3H_0^2/8\pi G = 9.20 \times 10^{-24} \text{ g/m}^3$ giving $\rho_\Lambda = 7.02 \times 10^{-24} \text{ g/m}^3$.

² This is not the case of all cosmological models. In a phantom universe [11], violating the null dominant energy condition ($w < -1$), the density increases as the universe expands. The models based on phantom dark energy usually predict a future singularity, called Big Rip [15], in which the scale factor and the energy density become infinite in a finite time. There can also be a Little Rip [16] in which the scale factor and the energy density becomes infinite in infinite time. Contrary to the Big Crunch, the universe is destroyed not by excessive contraction but rather by excessive expansion. The possibility that we live in a phantom universe is not ruled out by observations.

4.3. Matter era

When $a \ll a_2$, we get $\rho/\rho_\Lambda \sim (a_2/a)^3$. Because of the smallness of ρ_Λ , the equation of state (30) can be approximated by $p \simeq 0$ which describes the pressureless matter era.³ The conservation of $\rho_m a^3$ implies that $\rho_\Lambda a_2^3 = \rho_{m,0} a_0^3$. Using $\rho_\Lambda = \Omega_{\Lambda,0} \rho_0$ and $\rho_{m,0} = \Omega_{m,0} \rho_0$ with $\Omega_{\Lambda,0} = 0.763$ (dark energy) and $\Omega_{m,0} = \Omega_{b,0} + \Omega_{dm,0} = 0.237$ (baryonic and dark matter), we obtain $a_2/a_0 = (\Omega_{m,0}/\Omega_{\Lambda,0})^{1/3} = 0.677$, hence $a_2 = 8.95 \times 10^{25}$ m. This can be rewritten as $a_2 = 0.204 l_\Lambda$. The characteristic scale a_2 marks the transition between the matter era and the dark energy era. When $a \ll a_2$, the Friedmann equation (3) yields $a/a_2 \sim (6\pi)^{1/3} (t/t_\Lambda)^{2/3}$ and $\rho/\rho_\Lambda \sim (1/6\pi)(t_\Lambda/t)^2$. This is the Einstein-de Sitter (EdS) solution of a pressureless universe without cosmological constant.

4.4. The general solution

The general solution of the Friedmann equation (3) with Eq. (31) is [9, 12]:

$$\frac{a}{a_2} = \sinh^{2/3} \left(\sqrt{6\pi} \frac{t}{t_\Lambda} \right), \quad \frac{\rho}{\rho_\Lambda} = \frac{1}{\tanh^2 \left(\sqrt{6\pi} \frac{t}{t_\Lambda} \right)}. \quad (34)$$

Its asymptotic behavior for $t \rightarrow +\infty$ returns Eq. (33) with $t_f = (3/8\pi)^{1/2} [(2/3) \ln 2 + \ln(l_\Lambda/a_2)] t_\Lambda$. The universe is decelerating when $a < a'_c$ (i.e. $\rho > \rho'_c$) and accelerating when $a > a'_c$ (i.e. $\rho < \rho'_c$) where $a'_c = (1/2)^{1/3} a_2$ and $\rho'_c = 3\rho_\Lambda$. The time t'_c at which the universe starts accelerating is given by $t'_c = (1/6\pi)^{1/2} \operatorname{argsinh}(1/\sqrt{2}) t_\Lambda$. This corresponds to the time at which the curve $a(t)$ presents a second inflexion point. The time t_2 corresponding to the transition between the matter era and the dark energy era ($a = a_2$) is $t_2 = (1/\sqrt{6\pi}) \operatorname{argsinh}(1) t_\Lambda$. The corresponding density is $\rho_2 = 2\rho_\Lambda$.

4.5. The temperature of radiation

In the late universe, the temperature of radiation [see Eq. (16)] is given by

$$\frac{T}{T_P} = \left(\frac{15}{\pi^2} \right)^{1/4} \frac{a_1}{a}, \quad (35)$$

where the scale factor is given by Eq. (34). In the matter era

$$\frac{T}{T_P} = \left(\frac{15}{\pi^2} \right)^{1/4} \frac{a_1}{a_2} \frac{1}{(6\pi)^{1/3}} \left(\frac{t_\Lambda}{t} \right)^{2/3}. \quad (36)$$

In the dark energy era

$$\frac{T}{T_P} = \left(\frac{15}{\pi^2} \right)^{1/4} \frac{a_1}{l_\Lambda} e^{-(8\pi/3)^{1/2} (t-t_f)/t_\Lambda}. \quad (37)$$

4.6. History of the late universe

When $a \ll a_2$, the universe is in the matter era. Its radius increases algebraically as $a \propto t^{2/3}$ while its density decreases algebraically as $\rho \propto t^{-2}$ (EdS). The temperature of radiation decreases algebraically as $T \propto t^{-2/3}$. The expansion of the universe is decelerating. When $a \gg a_2$, the universe is in the dark energy era. It undergoes a phase of accelerating expansion (late inflation) driven by the dark energy during which its radius increases exponentially rapidly (dS) while

³ The equation of state (30) applies only to dark matter and dark energy. In principle, baryonic matter should be treated as a different fluid with $p_b = 0$ (see Sec. 5). However, for commodity, we shall include the contribution of baryonic matter in the density ρ because the final equations are the same.

its density remains constant and equal to the cosmological density $\rho_\Lambda = 7.02 \times 10^{-24} \text{ g/m}^3$. The temperature of radiation decreases exponentially rapidly. The time at which the universe starts accelerating is $t'_c = 0.152t_\Lambda = 2.22 \times 10^{17} \text{ s}$, corresponding to a radius $a'_c = 0.794a_2 = 0.162l_\Lambda = 7.11 \times 10^{25} \text{ m}$ and a density $\rho'_c = 3\rho_\Lambda = 2.11 \times 10^{-23} \text{ g/m}^3$. The transition between the matter era and the dark energy era takes place at $a_2 = 0.204l_\Lambda = 8.95 \times 10^{25} \text{ m}$, $\rho_2 = 2\rho_\Lambda = 1.40 \times 10^{-23} \text{ g/m}^3$, $t_2 = 0.203t_\Lambda = 2.97 \times 10^{17} \text{ s}$. At $t = t_f = 0.708t_\Lambda = 1.03 \times 10^{18} \text{ s}$, the radius of the universe is equal to the cosmological length ($a_f = l_\Lambda = 4.38 \times 10^{26} \text{ m}$) and its density $\rho_f = 1.008\rho_\Lambda = 7.08 \times 10^{-24} \text{ g/m}^3$. The evolution of the scale factor, density and temperature of the universe as a function of time are represented in Figs. 1-3 below. The pressure has a constant negative value $p = -\rho_\Lambda c^2 = -6.31 \times 10^{-7} \text{ g/ms}^2$.

4.7. The present universe

The present size of the visible universe $a_0 = 0.302l_\Lambda = 1.32 \times 10^{26} \text{ m}$ is precisely of the order of the scale $a_2 = 0.204l_\Lambda = 8.95 \times 10^{25} \text{ m}$ ($a_0 = 1.48a_2$). Therefore, we live just at the transition between the matter era and the dark energy era (see the bullets in Figs. 1-3). The present density of the universe is $\rho_0 = 1.31\rho_\Lambda = 9.20 \times 10^{-24} \text{ g/m}^3$. The present values of the deceleration parameter and of the equation of state parameter are $q_0 = (1 - 3\Omega_{\Lambda,0})/2 = -0.645$ and $w_0 = -\Omega_{\Lambda,0} = -0.763$. The age of the universe is $t_0 = (1/6\pi)^{1/2} \text{argsinh}[(a_0/a_2)^{3/2}]t_\Lambda$. Numerically, $t_0 = 0.310t_\Lambda = 4.54 \times 10^{17} \text{ s} \sim 14 \text{ Gyrs}$ ($t_0 = 13.7 \text{ Gyrs}$ if we use a more precise value of H_0). We have $t_0 = 1.03H_0^{-1}$. It is rather fortunate that the age of the universe almost coincides with the Hubble time H_0^{-1} . The present temperature of radiation is $T_0 = T_P(15/\pi^2)^{1/4}(a_1/a_0)$. Numerically, $T_0 = 3.11 \text{ K}$ ($T_0 = 2.7 \text{ K}$ if we take into account the radiation of neutrinos in thermal equilibrium).

4.8. Scalar field: quintessence

In alternative theories to the cosmological constant, the dark energy is usually described by a scalar field, defined by Eqs. (17) and (18), called quintessence [4, 3]. Using standard techniques [9, 12], we find that the quintessence potential corresponding to the equation of state (30) is

$$V(\psi) = \frac{1}{2}\rho_\Lambda c^2 (\cosh^2 \psi + 1), \quad \psi = \frac{3}{2} \left(\frac{8\pi G}{3c^2} \right)^{1/2} (\phi - \phi_{\max}) \quad (\psi \leq 0). \quad (38)$$

We can also show [9, 12] that the scale factor and the energy density of the universe are related to the scalar field by

$$\left(\frac{a_2}{a} \right)^{3/2} = -\sinh \psi, \quad \rho = \rho_\Lambda \cosh^2 \psi. \quad (39)$$

Using Eq. (39) and the results of the previous sections we can obtain the temporal evolution of the quintessence field in the late universe [9, 12].

For $\psi \rightarrow 0$, the scalar field potential (38) can be expanded up to fourth order giving

$$\frac{V(\psi)}{\rho_\Lambda c^2} \simeq 1 + \frac{1}{2}\psi^2 + \frac{1}{6}\psi^4 + \dots \quad (40)$$

Comparing this expression with the standard form (22) of a quartic potential, we obtain

$$V_0 = \rho_\Lambda c^2, \quad (41)$$

$$m^2 = \frac{6\pi G \rho_\Lambda \hbar^2}{c^4}, \quad (42)$$

$$\lambda = 24\pi^2 \frac{\rho_\Lambda}{\rho_P}, \quad a_s^2 = \frac{3\pi G^3 \rho_\Lambda \hbar^2}{2c^8}. \quad (43)$$

From Eqs. (42) and (43) we get

$$a_s = \frac{Gm}{2c^2} = \frac{1}{4}r_S. \quad (44)$$

We note that the mass of the quintessence field is real and is of order $m_\Lambda = \hbar H_0/c^2 = 1.43 \times 10^{-33} \text{ eV}/c^2$ (since $H_0^2 \sim G\rho_\Lambda \sim \Lambda$). This mass scale is often interpreted as the smallest mass of the bosons predicted by string theory [17] or as the upper bound on the mass of the graviton [18]. It is simply obtained by equating the Compton wavelength of the particle $\lambda_C = \hbar/mc$ with the Hubble radius $a_0 = c/H_0$ (the typical size of the visible universe). On the other hand, the self-interaction constant λ and the scattering length a_s are of the order of $\rho_\Lambda/\rho_P \sim 10^{-123}$ and $a_\Lambda = G\hbar H_0/c^4 = 1.90 \times 10^{-81} \text{ fm}$.

Since $H^2 = 8\pi G\rho_\Lambda/3$ during the late inflation, we can rewrite the quantization condition (28) as

$$m^2 = n_* \frac{8\pi G\rho_\Lambda \hbar^2}{3c^4}. \quad (45)$$

We see that Eq. (42) is *not* consistent with this quantization condition since it corresponds to $n_* = 9/4$ which is not an integer. We recall that our model of late universe is formally equivalent to the Λ CDM model. As a result, the quintessence field associated with the standard Λ CDM model does not satisfy the quantization condition (28) [14].

Remark: In this paper, we have described the late universe by a polytrope of index $n = -1$. We have shown that it is equivalent to the Λ CDM model. If we consider a polytropic equation of state with an arbitrary index $n < 0$, we find that the mass of the scalar field is given by [14]:

$$m^2 = -\frac{1+2n}{n^2} \frac{6\pi G\rho_\Lambda \hbar^2}{c^4}. \quad (46)$$

We note that this mass presents a maximum for $n = -1$. This maximum selects the Λ CDM model among all the polytropic models of the form of Eq. (1) with $n < 0$ [14].

4.9. Scalar field: tachyons

Some authors have proposed to represent the dark energy by a rolling tachyon condensate appearing in a class of string theories [19]. A tachyonic scalar field [20, 21, 22] has an equation of state $p = w(t)\rho c^2$ with $-1 \leq w(t) \leq 0$. This scalar field evolves according to the equation

$$\frac{\ddot{\phi}}{1-\dot{\phi}^2} + 3H\dot{\phi} + \frac{1}{V} \frac{dV}{d\phi} = 0. \quad (47)$$

The tachyonic field tends to run down the potential towards lower energies. The density and the pressure of the universe are related to the tachyonic field by

$$\rho c^2 = \frac{V(\phi)}{\sqrt{1-\dot{\phi}^2}}, \quad p = -V(\phi)\sqrt{1-\dot{\phi}^2}. \quad (48)$$

Since $w = p/\rho c^2 = -\rho_\Lambda/\rho$ is between -1 and 0 , the equation of state (30) can be associated with a tachyonic field. Using standard techniques [9, 12], we find that its potential is

$$V(\psi) = \frac{\rho_\Lambda c^2}{\cos \psi}, \quad \psi = \sqrt{\rho_\Lambda c^2} \left(\frac{6\pi G}{c^2} \right)^{1/2} (\phi - \phi_{\max}) \quad (-\pi/2 \leq \psi \leq 0). \quad (49)$$

We can also show [9, 12] that the scale factor and the density of the universe are related to the scalar field by

$$\left(\frac{a_2}{a} \right)^{3/2} = -\tan \psi, \quad \rho = \frac{\rho_\Lambda}{\cos^2 \psi}. \quad (50)$$

Using Eq. (50) and the results of the previous sections we can obtain the temporal evolution of the tachyonic scalar field in the late universe [9, 12].

5. The whole evolution of the universe

5.1. First point of view: Generalized radiation + dark fluid

In Sec. 3, we have described the transition between the vacuum energy era (Planck era) and the radiation era in the early universe by a single equation of state (7). This equation of state interpolates smoothly between the equation of state of vacuum energy ($p = -\rho c^2$) and the equation of state of radiation ($p = \rho c^2/3$). It provides therefore a unified description of vacuum energy and radiation. We shall consider that it describes a fluid of “generalized radiation”. The relation between the energy density of this fluid and the scale factor can be written as

$$\frac{\rho}{\rho_0} = \frac{\Omega_{\text{rad},0}}{\frac{\Omega_{\text{rad},0}\rho_0}{\rho_P} + (a/a_0)^4}, \quad (51)$$

where $\Omega_{\text{rad},0}\rho_0/\rho_P = 1.51 \times 10^{-127}$.

In Sec. 4, we have described the transition between the dark matter era and the dark energy era (de Sitter era) in the late universe by a single equation of state (30). This equation of state interpolates smoothly between the equation of state of pressureless matter ($p = 0$) and the equation of state of dark energy ($p = -\rho c^2$). It provides therefore a unified description of dark matter and dark energy. We shall consider that it describes a “dark fluid”. The relation between the energy density of this fluid and the scale factor can be written as

$$\frac{\rho}{\rho_0} = \frac{\Omega_{\text{m},0}}{(a/a_0)^3} + \Omega_{\Lambda,0}. \quad (52)$$

This is formally equivalent to the Λ CDM model.

We can combine these two models by assuming the the universe is filled with a fluid of “generalized radiation” (vacuum energy + radiation) and a “dark fluid” (dark matter + dark energy). Summing the densities of these two fluids, considered as two different species, we get

$$\frac{\rho}{\rho_0} = \frac{\Omega_{\text{rad},0}}{\frac{\Omega_{\text{rad},0}\rho_0}{\rho_P} + (a/a_0)^4} + \frac{\Omega_{\text{m},0}}{(a/a_0)^3} + \Omega_{\Lambda,0}. \quad (53)$$

5.2. Second point of view: Unitary fluid + dark matter

Alternatively, we can unify vacuum energy, radiation and dark energy by a single (quadratic) equation of state of the form [23, 24]:⁴

$$p = -\frac{4\rho^2}{3\rho_P}c^2 + \frac{1}{3}\rho c^2 - \frac{4}{3}\rho_\Lambda c^2. \quad (54)$$

It is interesting to note that the Planck density ρ_P , the radiation parameter $\alpha = 1/3$ and the cosmological density ρ_Λ appear as the coefficients of this quadratic equation of state. We shall consider that this equation of state describes a “unitary fluid”. We can integrate analytically the energy equation (2) with the equation of state (54) without approximation [24]. However, using the fact that $\rho_P \gg \rho_\Lambda$, we obtain in very good approximation [23, 24]:

$$\frac{\rho}{\rho_0} = \frac{\Omega_{\text{rad},0}}{\frac{\Omega_{\text{rad},0}\rho_0}{\rho_P} + (a/a_0)^4} + \Omega_{\Lambda,0}. \quad (55)$$

⁴ Many cosmological models, such as the Chaplygin gas model [5], attempt to unify two fluids: dark matter and dark energy. The quadratic equation of state (54) attempts to unify three fluids: vacuum energy, radiation and dark energy. This model is discussed in detail in [23, 24]. In particular, we determine the scalar field potential corresponding to the equation of state (54) that unifies the inflaton and quintessence fields.

On the other hand, we consider dark matter as an independent fluid with a pressureless equation of state $p = 0$. In that case, the energy equation (2) gives

$$\frac{\rho}{\rho_0} = \frac{\Omega_{m,0}}{(a/a_0)^3}. \quad (56)$$

Assuming that the universe is filled with a “unitary fluid” (vacuum energy + radiation + dark energy) and “dark matter”, and summing the densities of these two fluids, we obtain again Eq. (53).

5.3. A cosmic equation that generalizes the Λ CDM model

Substituting Eq. (53) into the Friedmann equation (3) which can be rewritten as $(H/H_0)^2 = \rho/\rho_0$, we obtain a cosmic equation of the form [9, 12, 23, 24]:

$$\left(\frac{H}{H_0}\right)^2 = \frac{\Omega_{\text{rad},0}}{\frac{\Omega_{\text{rad},0}\rho_0}{\rho_P} + (a/a_0)^4} + \frac{\Omega_{m,0}}{(a/a_0)^3} + \Omega_{\Lambda,0}. \quad (57)$$

This equation describes the whole evolution of the universe, from its early inflation to its late acceleration (see Figs. 1-3). For $\hbar \rightarrow 0$ (i.e. $\rho_P \rightarrow +\infty$), we recover the standard Λ CDM model which presents a singularity at $t = 0$ (Big Bang) and does not describe the phase of early inflation. For $\hbar \neq 0$, we obtain a model of universe in which the initial singularity is removed and the universe displays a phase of early inflation. After the early inflation, i.e. for $t > t_1 = 23.3t_P = 1.25 \times 10^{-42}$ s, this model gives the same results as the Λ CDM model: the universe undergoes a decelerating power-law expansion in the radiation and matter eras, then an accelerated exponential expansion (late inflation) in the dark energy era. A nice feature of this model is its simplicity since it incorporates a phase of early inflation in a very simple manner. We just have to add a cut-off $(a_1/a_0)^4 = \Omega_{\text{rad},0}\rho_0/\rho_P$ in the cosmic equation of the standard Λ CDM model. Therefore, the modification implied by Eq. (57) to describe the early inflation is very natural. On the other hand, the equation of state (30) in the late universe gives the same results as the Λ CDM model where we add the contribution of dark matter and dark energy individually, so this does not bring any modification to the standard cosmic equation. This is an interest of our description since the Λ CDM model gives results that agree with observations at late times. In our model, the universe exists eternally in the past⁵ and in the future without any singularity (aioniotic universe). As explained in Ref. [9], this model is the most natural non-singular symmetric solution of the cosmological Einstein equations.

5.4. Cosmological constant problem

It is oftentimes argued that the dark energy ρ_Λ (cosmological constant) corresponds to the vacuum energy. Since the vacuum energy density is expected to be of the order of the Planck density ρ_P this leads to the so-called cosmological constant problem [26, 27] because the cosmological density $\rho_\Lambda = 7.02 \times 10^{-24}$ g/m³ and the Planck density $\rho_P = 5.16 \times 10^{99}$ g/m³ differ by about 123 orders of magnitude ($\rho_P/\rho_\Lambda \sim 10^{123}$). We think it is a mistake to identify the dark energy with the vacuum energy. As illustrated in Fig. 2, the Planck density and the cosmological density represent fundamental upper and lower density bounds acting in the early

⁵ As already stated in Sec. 3, our model certainly breaks down in the Planck era ($t \leq 0$) where $a \leq l_P$. The Planck era may not be described in terms of an equation of state $p(\rho)$, or even in terms of the Einstein equations, as we have assumed. It probably requires the development of a theory of quantum gravity. An interesting description of the early inflation has been given in Ref. [25] in terms of a quantized model based on a simplified Wheeler-DeWitt equation. In that model, a quantum tunneling process explains the birth of the universe with a well defined size after tunneling.

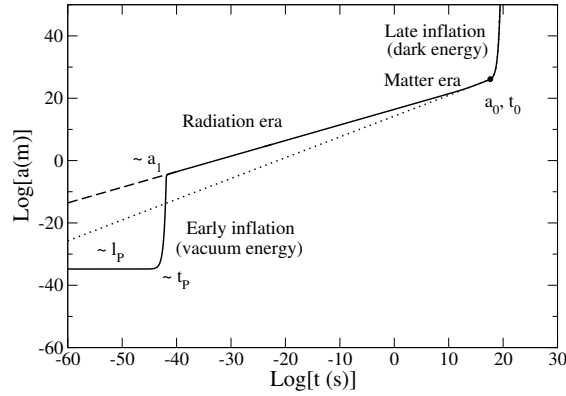


Figure 1. Evolution of the scale factor a as a function of time obtained by integrating Eq. (57). The universe undergoes an early inflation driven by the Planck density ρ_P connecting the vacuum energy era (Planck era) to the radiation era. During the early inflation, the scale factor increases by 29 orders of magnitude in less than 10^{-42} s. In the radiation era, the scale factor increases algebraically as $t^{1/2}$ and in the matter era it increases algebraically as $t^{2/3}$. The universe finally undergoes a late inflation (accelerated expansion) driven by the cosmological density ρ_Λ connecting the matter era to the dark energy era (de Sitter era). At present, we live just at the transition (bullet). The universe is accelerating during the vacuum energy era, decelerating during the radiation and matter eras, and accelerating again during the dark energy era. The evolution of the early and late universe is remarkably symmetric. In our model, it is described by two polytropic equations of state with index $n = 1$ and $n = -1$ respectively. The dashed and dotted lines corresponds to the Λ CDM model (with or without radiation) leading to a primordial singularity at $t = 0$ (Big Bang).

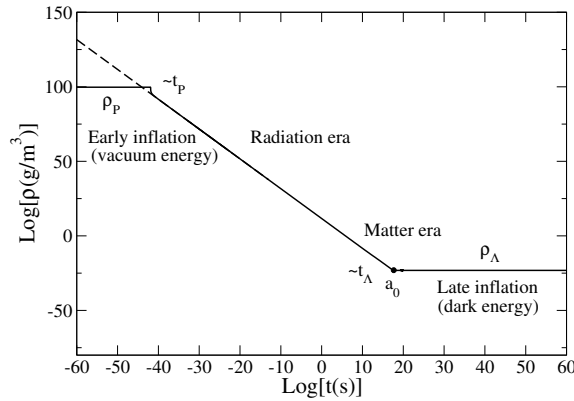


Figure 2. Evolution of the energy density ρ as a function of time from Eq. (57). The density goes from a maximum value $\rho_{\max} = \rho_P$ determined by the Planck constant (quantum mechanics) to a minimum value $\rho_{\min} = \rho_\Lambda$ determined by the cosmological constant (general relativity). These two bounds, which are fixed by fundamental constants of physics, are responsible for the early and late inflation (acceleration) of the universe. They differ by a factor of the order 10^{123} . In the radiation and matter eras, the density decreases algebraically as t^{-2} .

and late universe, respectively. Therefore, it is not surprising that they differ by 123 orders of magnitude. In this paper, we regard the vacuum energy and the dark energy as two distinct

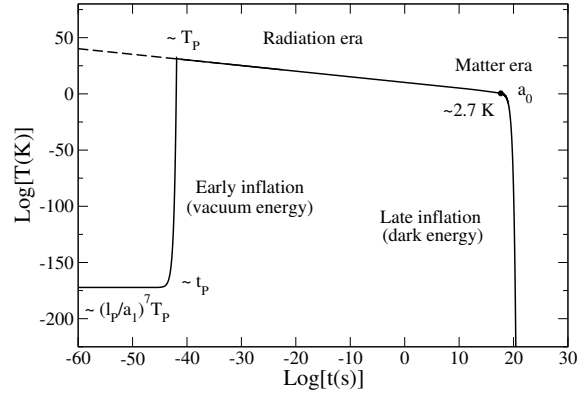


Figure 3. Evolution of the temperature of radiation as a function of time ($T/T_0 = (a/a_0)^7 / [(a/a_0)^4 + (a_1/a_0)^4]^2$). Before the early inflation, the universe is extremely cold ($T < 10^{-173}$ K). During the early inflation, the temperature increases by 204 orders of magnitude in less than 10^{-42} s. During the radiation and matter eras, the temperature decreases algebraically as $T \sim t^{-1/2}$ and $T \sim t^{-2/3}$, respectively. The present value of the temperature is $T_0 \simeq 2.7$ K (bullet). During the late inflation, the temperature decreases exponentially rapidly.

entities. We call vacuum energy the energy associated with the Planck density ρ_P and dark energy the energy associated with the cosmological density ρ_Λ . The vacuum energy is responsible for the inflation of the early universe and the dark energy for the inflation (acceleration) of the late universe. In this viewpoint, the vacuum energy is due to quantum mechanics and the dark energy is an effect of general relativity. The cosmological constant Λ may be interpreted as a fundamental constant of nature applying to the cosmophysics (late universe) in the same way the Planck constant \hbar applies to the microphysics (early universe). Accordingly, the origin of the dark energy density ρ_Λ should not be sought in quantum mechanics, but in pure general relativity. In this sense, the cosmological constant “problem” may be a false problem. Of course, the origin of the cosmological constant (or dark energy) still remains to be understood.

5.5. Further generalisations of the cosmic equation (57)

We have described the late universe by a constant equation of state $p/c^2 = -\rho_\Lambda$ interpolating between pressureless matter ($p = 0$) and dark energy ($p = -\rho c^2$). More generally, we could consider an equation of state of the form $p/c^2 = \alpha\rho - (\alpha + 1)\rho_\Lambda$ [8, 9] interpolating between isothermal matter ($p = \alpha\rho c^2$) and dark energy ($p = -\rho c^2$). In that case, Eq. (57) is replaced by

$$\left(\frac{H}{H_0}\right)^2 = \frac{\Omega_{\text{rad},0}}{\frac{\Omega_{\text{rad},0}\rho_0}{\rho_P} + (a/a_0)^4} + \frac{\Omega_{\text{m},0}}{(a/a_0)^{3(1+\alpha)}} + \Omega_{\Lambda,0}. \quad (58)$$

We have assumed that the universe is made of vacuum energy, radiation, dark matter and dark energy (some of them being unified in one fluid). We could also add the contribution of other species such as baryonic matter, stiff matter [28, 29, 30], or more generally an hypothetical exotic α -fluid [24]. Finally, we could add the contribution of a scalar field representing for example BEC dark matter [31, 32, 33, 34].

We have described the early universe by a polytrope of index $n = +1$ and the late universe by a polytrope of index $n = -1$. More generally, we could describe the early universe by a polytrope of index $n_e > 0$ and the late universe by a polytrope of index $n_l < 0$ [8, 9, 24].

A cosmic equation taking into account some of the possible generalizations proposed previously is given by [24]:

$$\left(\frac{H}{H_0}\right)^2 = \frac{\Omega_{\alpha,0}}{[(a/a_0)^{3(1+\alpha)/n_e} + (a_1/a_0)^{3(1+\alpha)/n_e}]^{n_e}} + \frac{\Omega_{s,0}}{(a/a_0)^6} + \frac{\Omega_{\text{rad},0}}{(a/a_0)^4} + \frac{\Omega_{m,0}}{(a/a_0)^3} + \Omega_{\Lambda,0}. \quad (59)$$

The model of late universe considered in this paper is formally equivalent to the Λ CDM model. We could account for a slight deviation to the Λ CDM model (actually the slightest one) by considering a logotropic equation of state [35, 36, 37]. In that case, we obtain a cosmic equation of the form

$$\left(\frac{H}{H_0}\right)^2 = \frac{\Omega_{\text{rad},0}}{\frac{\Omega_{\text{rad},0}\rho_0}{\rho_P} + (a/a_0)^4} + \frac{\Omega_{m,0}}{(a/a_0)^3} + \Omega_{\Lambda,0} [1 + 3B \ln(a/a_0)], \quad (60)$$

where $B = 3.53 \times 10^{-3}$ is the so-called dimensionless logotropic temperature [35, 36, 37]. It is given by $B \simeq 1/\ln(\rho_P/\rho_\Lambda) \simeq 1/[123 \ln(10)]$. We note that the nonzero value of B is due to quantum mechanics. For $\hbar \rightarrow 0$ (i.e. $\rho_P \rightarrow +\infty$), we find that $B \rightarrow 0$ and we recover the standard Λ CDM model. For $\hbar \neq 0$, the logotropic model is very close to the Λ CDM model (at large scales) up to the present, but will become phantom in about 25 Gyrs and will finally experience a little rip [35, 36, 37]. Owing to this remark and to the remark following Eq. (57), we note that quantum mechanics plays a role both in the early and late universe.

6. Conclusion

We have introduced a model of universe based on a (generalized) polytropic equation of state of the form of Eq. (1). Taking $n = +1$, $k = -4/(3\rho_P)$ and $\alpha = 1/3$, we obtain a quadratic equation of state $p/c^2 = -4\rho^2/3\rho_P + \rho/3$ that describes the transition between the vacuum energy era (Planck era) and the radiation era. Taking $n = -1$, $k = -\rho_\Lambda$ and $\alpha = 0$ we obtain a constant equation of state $p/c^2 = -\rho_\Lambda$ that describes the transition between the matter era and the dark energy era (de Sitter era). An interest of our model is that it describes the early universe and the late universe in a symmetric manner. The early universe is described by a polytropic equation of state of index $n = +1$ and the late universe by a polytropic equation of state of index $n = -1$. The first one unifies vacuum energy and radiation and the second one unifies dark matter and dark energy. The mathematical formulae are then strikingly symmetric. In the early universe, this amounts to summing the inverse Planck density and the inverse radiation density ($1/\rho = 1/\rho_P + 1/\rho_{\text{rad}}$). In the late universe, this amounts to summing the matter density and the cosmological density ($\rho = \rho_m + \rho_\Lambda$). The Planck density ρ_P in the early universe plays the same role as the cosmological density ρ_Λ in the late universe. They represent fundamental upper and lower density bounds differing by 123 orders of magnitude. They lead to phases of early and late inflation with very different timescales $t_P = 1/(G\rho_P)^{1/2} = 5.39 \times 10^{-44}$ s and $t_\Lambda = 1/(G\rho_\Lambda)^{1/2} = 1.46 \times 10^{18}$ s. The early universe is governed by quantum mechanics (\hbar) and the late universe by general relativity (Λ).

We have also mentioned an alternative model (developed in [23, 24]) where vacuum energy, radiation and dark energy are unified by a single quadratic equation of state $p/c^2 = -4\rho^2/3\rho_P + \rho/3 - 4\rho_\Lambda/3$. This equation of state displays both a phase of early inflation driven by the Planck density ρ_P and a phase of late acceleration driven by the cosmological density ρ_Λ . These two phases are bridged by a phase of deceleration. The pressure is successively negative (vacuum energy), positive (radiation), and negative again (dark energy). In this model, dark matter is treated as an independent species.

We have obtained a cosmic equation (57) that generalizes the Λ CDM model (the Λ CDM model is recovered for $\hbar \rightarrow 0$). Our model does not present any singularity and exists “eternally”

in the past and in the future, so that it has no origin nor end (we call it aioniotic universe). In particular, the phase of early inflation avoids the Big Bang singularity and replaces it by a sort of second order phase transition where the non-zero value of the Planck constant \hbar plays the same role as finite size effects in statistical mechanics [8]. Of course, it is probably incorrect to extrapolate the results in the infinite past since our model is purely classical, or semi-classical, and does not take quantum gravity into account. In our model, the universe starts from $t = -\infty$ but, for $t < 0$, its size is less than the Planck length $l_P = 1.62 \times 10^{-35}$ m. In the Planck era, the classical Einstein equations may be incorrect and should be replaced by a theory of quantum gravity that still has to be constructed.

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