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Non-relativistic supergravity in three space–time dimensions

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Introduction

This chapter shall serve as a motivation for the analysis performed in this thesis. We aim to motivate the problem, put it into context and briefly describe it with as little technical detail as possible.

Gravitation is the only unquestionably ubiquitous, fundamental force in Nature. It is so common to us that we can hardly envision a world without it. Obviously, it was also the first one subject to investigation by physicists. Newton and Kepler were the first to successfully describe the motion of particles and planets that undergo gravitational interactions. Nowadays though, we believe that Newton's theory is but an effective description of another more fundamental theory, the theory of general relativity that was formally described first by Einstein exactly one hundred years ago, see e.g. [1].

Einstein's theory rests only on a few pillars, statements that give restrictions, certain laws that the theory should obey. Notably, one of them is that the theory must be independent of the coordinate system that is used to describe it. In contemporary (theoretical, but not exclusively) physics this is an obvious thing to do, i.e. one requires invariance of the theory and its predictions under general coordinate transformations. In hindsight though, we might wonder why no one addressed this question for the Newtonian theory. Even before the discovery of general relativity, the concept of diffeomorphism invariance could have been addressed for any theory that has so far only been formulated in such a way that it is invariant under Galilean coordinate transformations.

At the time, this might have been sufficient. Physical models were mostly considered in so-called free-falling, or inertial, frames. These are coordinate frames where an observer is not subject to any gravitational force. Different non-relativistic inertial frames are then related by Galilean coordinate transformations. (In the relativistic context these would be the Lorentz transformations.) Galilean transformations relate two different coordinate systems via *constant* rotations, boosts and shifts. In contrast, arbitrary general coordinate transformations are not subject to any limitation, such as constancy of the symmetry parameters. As mentioned before, in the context of non-relativistic theories, one oftentimes considers only Galilean transformations. However, in principle there should exist a description of any non-relativistic theory, and thus of Newtonian gravity, that is invariant under arbitrary transformations of the coordinates.

So, what is the coordinate independent description of Newtonian gravity? Only a few years after Einstein's discovery of general relativity this question was eventually answered. It was discovered by the French mathematician Cartan and the theory now goes by the name Newton–Cartan gravity [2, 3]. It is this coordinate independent description of Newtonian gravity that we are most concerned with in this thesis. Note that, while the Newton–Cartan theory obeys one of the statements that Einstein's theory is based on, it does differ from general relativity in another crucial aspect. Namely, in the Newton–Cartan theory there exists no maximal velocity. We come back to this issue in chapter 3 when we investigate how Einstein's theory gets deformed when we let its limiting speed, the speed of light, go to infinity. As we shall see this provides one way to derive the Newton–Cartan theory from general relativity.

1.1 Newton–Cartan (like) structures in physics

Most of the recent interest in describing theories that are invariant under non-relativistic general coordinate transformations has been stirred by developments in condensed matter theories, see e.g. [4–7] for some earlier works. This engendered more systematic studies recently, especially on how to couple non-relativistic field theories to Newton–Cartan backgrounds [8–17] (see also [18–24]). It has been noted that besides the "usual" fields, the time-like and spatial vielbein, one needs an additional vector field to consistently couple non-relativistic field theories to arbitrary curved backgrounds, see in particular [8–11] and [16]. These references together with [25–29] also point towards another important aspect of the background geometry: that one should not restrict the torsion of the non-relativistic gravity theory.

The necessity to add an extra vector gauge-field to the Newton–Cartan fields was also realized by Son [30]. The use of Newton–Cartan structures is obligatory in his effective action for the quantum Hall effect [8, 30–33]. This model naturally includes a so-called Wen–Zee term [34], which describes the coupling between the extra gauge-field and the curvature, thus encoding the Hall viscosity and giving rise to more universal features of the quantum Hall effect than only the quantized Hall conductivity. Through a similar coupling with a $U(1)$ gauge-field and a Wen–Zee term, one obtains an effective action for chiral superfluids as well [35, 36]. It was realized in [36] that Newton–Cartan structures offer a particularly neat way to write the action in a covariant form, and hence offer a convenient framework to calculate e.g. the energy current of the superfluid. Newton–Cartan structures are also essential in describing Newtonian fluids [37, 38]. In particular, those models were used to describe the effects of superfluidity in Neutron stars [38–40]. A study of non-relativistic fluids, or "Galilean fluids", i.e. hydrodynamics on Galilean backgrounds features in [41, 42]. Indeed, the applications of Newton–Cartan structures in describing effective actions for phenomena in condensed matter physics seem so vast that this short introduction can hardly do justice to all works in this fields.

Some works on condensed matter theory are motivated by the emergence of new "holographic" techniques in condensed matter physics, see [43–46]. These techniques lend their roots to the holographic principle [47, 48], which is a conjectured correspondence of gravitational theories in anti-de Sitter space-times and conformal field theories, the so-called AdS/CFT correspondence. Some simple, yet insightful, manifestations of this duality were found in three-dimensional space-times, see e.g. [49–51], building on the seminal work of Brown and Henneaux [52], which is indeed one of the precursors for this duality. While, as we mentioned before, such techniques are already being applied, much still needs to be investigated when it comes to test the generality of this correspondence. The work on finding non-relativistic realizations of this duality is one effort in this direction. More concretely, let us draw the attention to the recent works [15, 25–27] where Newton–Cartan structures play a very prominent role.

Newton–Cartan like structures, in fact dual structures that are more related to ultra-relativistic physics than non-relativistic physics, also feature in warped conformal field theories [53, 54]. They are also conjectured to play a role in the tensionless limit of string theory [55, 56]. The tensionless limit amounts to taking an ultra-relativistic limit [55, 57], that leads to "Carrollian" physics and space-times, which are in many ways dual to Newton–Cartan structures, see e.g. [54, 55, 57–61]. Other, in fact opposite, non-relativistic limits of string theory have also been considered. Non-relativistic string theory [62, 63] was studied as a possible soluble sector within string or M-theory [64–70]. These works are also important for the applications of the holographic principle in the context of condensed matter theory which we mentioned before.

A more systematic approach towards non-relativistic (and ultra-relativistic) geometry was taken in [71, 72], where Newton–Cartan structures are discussed with a particular emphasis on the metric structures.

In this thesis we will be mostly interested in supersymmetric extensions of Newton–Cartan structures. While we will discuss (non-relativistic) supersymmetry more at a later stage, let us give some motivations for our general interest in them here. For example, one of the papers on the quantum Hall effect does in fact make use a (non-relativistic) supersymmetric model [73]. Further motivations, and indeed one of the main motivations for this thesis in general, is related to the very successful application of localization techniques in the relativistic context, see e.g. [74, 75]. The applications of localization techniques have proved useful to obtain exact results for relativistic supersymmetric field theories. Perhaps we can expect similarly successful results in the non-relativistic case too. The method relies on the coupling of non-relativistic field theories to arbitrary supersymmetric backgrounds. To do so, one needs of course non-relativistic supergravity theories to provide such backgrounds. However, the construction that we have in mind [76], makes use of so-called off-shell formulations of supergravity, a feature that the only known theory of non-relativistic supergravity so far, see [77], does not possess. Suitable extensions of Newton–Cartan supergravity would be necessary to check whether those techniques can be extended to non-relativistic theories.

We discussed the main motivations for our interest in non-relativistic physics and that Newton–Cartan structures provide a convenient way of denoting the non-relativistic gravity background. Moreover, we gave one particular motivation to consider supersymmetric theories. Let us now return to the point where we started: symmetries.

1.2 Non-relativistic (super)symmetry

Requiring a theory, Newtonian gravity in this case, to be invariant under general coordinate transformations, means that it should be invariant under a special kind of symmetry transformation. Symmetries play a major role in contemporary theoretical physics. They impose strong restrictions on a theory, for example they restrict the kind of interaction terms that one can write down. This is of vital importance as particular symmetries of a given theory can often be found in experiments and thus the underlying "fundamental" theory can be much better described. All (quantum) theories that are used to describe the standard model of particle physics are gauge-theories, i.e. they are by construction invariant under a certain symmetry. The symmetry groups are $U(1)$ for electrodynamics, $SU(2)$ for the weak interactions and $SU(3)$ for the strong interactions. In a way, general relativity can be understood as a gauge-theory too. It is the gauge-theory of diffeomorphisms. As we will see later, this feature, the gauge-theory aspect of general relativity, is of particular importance to us.

In view of such vast applications, an interesting question is what is the most general symmetry (group) that we can allow? This has been extensively studied and one of the most celebrated theorems by Coleman and Mandula [78] states that a physical theory can at most be invariant under the conformal group. All other symmetries must be "internal" ones, i.e. physical observables must always be scalar representations of those symmetries. This theorem holds but for one exception: it does not account for fermionic symmetries. As was later shown by Haag, Lopuszanski and Sohnius [79] the addition of supersymmetry, or superconformal symmetries to be precise, really exhausts all possibilities. This, in a way, fundamental nature of supersymmetry serves as one more motivation for us to consider supersymmetric theories and to study supersymmetric Newton–Cartan structures in this thesis.

We must now put our focus more towards non-relativistic symmetries. To understand how a theory can be invariant under non-relativistic diffeomorphisms one must of course know what non-relativistic diffeomorphisms are. Interestingly, this is part of current research too. One approach consists of mimicking some ideas that can be applied to general relativity as well. General relativity can be seen as gauge-theory of the Poincaré algebra and hence one could try to derive a non-relativistic version thereof by gauging instead a non-relativistic symmetry algebra. This approach was pursued in [77, 80]. In particular, in [77] it was shown how the gauge-theory of the Bargmann algebra can be reduced to Newtonian gravity by gauge-fixing. The Newton–Cartan theory was thus linked to Newtonian gravity and it was shown explicitly that the difference between those theories is precisely due to the amount of symmetry that we allow. We will apply similar ideas, i.e. gauging techniques, to derive some of the new (supergravity) theories that we put forward in this thesis.

Let us pause for a moment to fix nomenclature and to quickly define the different theories of non-relativistic gravity. Newton–Cartan gravity is invariant under general coordinate transformations, while in Newtonian gravity we allow only for Galilei transformations with constant symmetry parameters. There is one in-between step that we shall also consider in this thesis. We can allow for arbitrary time-dependent spatial translations, keeping all other symmetry parameters constant. These symmetries are sometimes referred to as "acceleration-extended" Galilean symmetries, or Milne symmetries [4]. The theory is then called Galilean gravity.

What kind of non-relativistic symmetries will we be interested in? We already mentioned Galilean symmetries and also the Bargmann algebra. Just like in the relativistic case, we can extend those with yet more symmetry generators. In the relativistic case this leads to conformal symmetry. In the non-relativistic case however, we have the choice between two different "conformal" extensions of the Galilei algebra. The Galilean conformal algebra [46, 81] is the non-relativistic analog of the conformal algebra. Another possibility is the Schrödinger algebra [82, 83]. We will choose to work with the latter one because it is the only possible extension of the Bargmann algebra.[1] This in particular implies that we can allow for non-zero mass.[2] Indeed, the additional symmetry of the Bargmann algebra (w.r.t. the Galilei algebra) is often interpreted as being related to the conservation of mass (and being non-relativistic this implies conservation of particle number).

The Schrödinger algebra comprises symmetry generators which give rise to the symmetries of the Schrödinger equation (hence the name). The rigid Schrödinger transformations also leave invariant the simple action of the non-relativistic point-particle:

$$S = \frac{m}{2} \int dt \, \dot{x}^i \dot{x}^i \, .$$

More extensions of this formula will also follow in this thesis.

So, in the spirit of Coleman–Mandula we add "conformal" extensions. But in this thesis in particular, we will also be interested in another extension of non-relativistic symmetries, namely the addition of supersymmetry.

Above we gave some "physical" reasons to study Newton–Cartan theories and also for studying supersymmetry in that context. Another motivation comes from the analysis of [78, 79] in the relativistic context. In the relativistic case, supersymmetry or superconformal symmetries are the most general symmetries that we can allow for a theory. In the non-relativistic case, there are no arguments similar to those of [78, 79], but it is still interesting to see if we are even able to find non-relativistic theories that are supersymmetric.

Non-relativistic supersymmetry algebras are not per se difficult to construct and some simple field theories that realize those symmetries have also been found, see e.g. [84–94]. However, the first local realization of non-relativistic supersymmetry dates back only to the year 2013 [77]. There are no (known) obstacles that would prevent anyone from finding such theories in principle. However, it turns out that this is not an easy task and one faces quite some difficulties. This leads us to first consent ourselves with constructing examples in a simpler setting. Such simplicity can result for example from considering theories in lower dimensions. Therefore, the non-relativistic supergravity theories that we will construct in this thesis, all of which are generalizations of [77], are theories in three space-time dimensions. We

[1] We will discuss the reasons for our particular interest in the Bargmann algebra instead of the Galilei algebra at a later stage, as they are mostly technical in nature.

[2] In the relativistic case, conformal symmetry is too strong to allow for massive representations. Therefore, since our "conformal" Schrödinger symmetry does allow $m \neq 0$, we will reserve the name (non-relativistic) conformal symmetry for the Galilean conformal ones.

hope that a more thorough understanding of the mechanics of three-dimensional case will provide useful insight for the construction of four-dimensional theories.

1.3 Outline of the thesis

So, this thesis deals with non-relativistic supergravity theories in three space-time dimensions. We motivated our interest in such theories, which stems from effective models in condensed matter theory to possible applications of localization techniques. The fact that we work in three dimensions is due to a more straightforward motto of ours: simplicity first. In the following we will work out many examples of non-relativistic supergravity theories and shall compare our findings with the relativistic case.

At first we must address the question of how we are going to construct those supergravities. Throughout this work we will use two different techniques. The first one lends its roots to relativistic supergravity and the fact that it can be obtained by gauging the Poincaré superalgebra [95]. The application of similar techniques will allow us to obtain non-relativistic supergravities by gauging non-relativistic superalgebras instead. This approach will feature in chapters 4 and 5. The other method to derive non-relativistic supergravities is a non-relativistic limiting procedure that we develop in chapter 3. This method is also used in chapter 6 to derive non-relativistic matter multiplets.

We can depict the two methods that we use to construct non-relativistic supergravities in the following diagram, see figure 1.1. (The adjective 'super' can be

Ways to obtain Newton–Cartan gravity

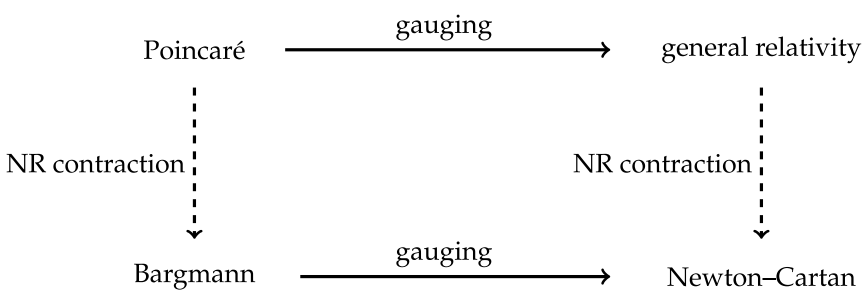

Figure 1.1: *A diagrammatic way to explain the relations between gauging procedures and non-relativistic (NR) contractions. Symmetry algebras feature on the right side and the respective gravitational theories on the left side, see main text for more explanation.*

added at each point/corner in figure 1.1 to make the diagram more suitable to our specific purpose.) On the left side we present the respective symmetry algebras that are gauged. The horizontal arrows stand for the gauging procedures, e.g. the upper line translates to "general relativity can be obtained by gauging the Poincaré algebra". The dashed, vertical arrow on the left corresponds to the Inönü–Wigner contraction [96], which is a way to obtain the Bargmann (super)algebra from the Poincaré (super)algebra. The dashed arrow on the right stands for the limiting procedure that we adopt in chapter 3. It is no coincidence that this graph seems to suggest the limiting procedure is motivated by the Inönü–Wigner contraction of the respective symmetry algebras.

There are particular reasons why we are using two rather than just one method to obtain non-relativistic supergravities. Gauging techniques, especially in three dimensions, lead to representations that contain only gauge-fields. This means that gauging techniques will not lead to representations with extra (auxiliary) fields that are typical for so-called off-shell formulations for supergravity. While performing the application of localization techniques goes beyond the scope of this thesis, we do aim to construct such off-shell formulations for their putative use. Namely, in a convenient way to couple supergravity to curved backgrounds, described in [76], we are forced us to choose particular (non-zero!) values for the auxiliary fields, hence the need for formulations that contain such fields.

In chapter 6 we will introduce means to obtain off-shell formulations also via gauging techniques. However, this calculation will rely on the existence of matter multiplets and we will use the limiting procedure to obtain those. We will use a non-relativistic version of the superconformal tensor calculus, to derive off-shell formulations of three-dimensional non-relativistic supergravity. It turns out that the structures are very similar to the relativistic case.

We have argued at length that we are interested in off-shell formulations of non-relativistic supergravity. But this is not the only extension that we will consider in the thesis. For example, one generalization of [77] consists of "cosmological" extensions, i.e. gauging a non-relativistic symmetry algebra with a "cosmological constant", which is done in chapter 4. As mentioned, we will look at theories that possess a "maximal" amount of symmetry. To this end, we will also construct a Schrödinger extension of three-dimensional non-relativistic supergravity in chapter 5.

One important aspect in the coupling of non-relativistic field theories to arbitrary backgrounds is that those backgrounds should allow for non-zero torsion, i.e. the curvature of the gauge-field of time-translations should be un-constrained. We shall therefore be particularly interested in extensions with torsion.

In conclusion, we will extend the theory that was presented in [77] in the directions depicted in figure 1.2. While this figure gives an idea of what are the contents of each chapter of the thesis, we shall briefly describe them in a little more detail now.

The main goal of the thesis is to derive theories of non-relativistic supergravity in three space-time dimensions. Naturally, we have to introduce all of the background material needed for such an endeavor. This is done in concise form in chapter 2. The reader who is familiar with those concepts can skip that chapter and

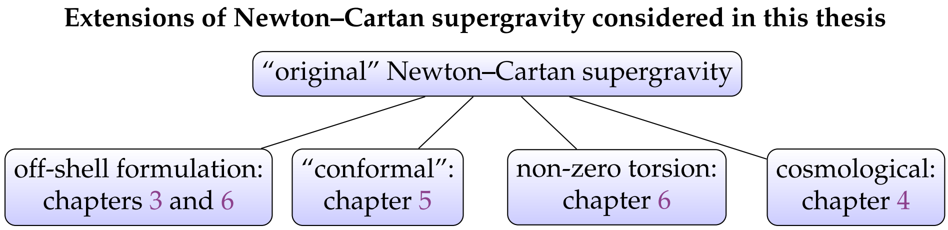

Figure 1.2: *Extensions of the "original", i.e. three-dimensional, $\mathcal{N} = 2$, torsion-less, on-shell, Newton–Cartan supergravity that we consider in this thesis.*

immediately proceed to chapters 3–6 where we present the new materials. A quick glance at the summary of the conventions used in this work, presented in section 2.1, might be useful.

The chapter on background material, chapter 2, focuses only on the most important points. We introduce the concept of symmetries, gauge symmetries and gauge theories and also present an introduction to general relativity from this point of view. More importantly, especially for this work, we introduce non-relativistic gravity, supersymmetry and supergravity. The combination of those is the main theme of our work and makes up the bulk part of the thesis, chapters 3–6. The chapter is quite technical and we proceed rather quickly to cover all but the necessary concepts that we use later on.

The presentation of new material starts in chapter 3, where we develop a non-relativistic contraction/limiting procedure that can be used to derive non-relativistic supergravity theories from relativistic ones. After describing the procedure in quite abstract terms, we apply it to some examples. We re-derive Newton–Cartan gravity in the formulation in which it appeared earlier in the literature. We will also comment shortly on why we are mostly concerned with theories with more than one supersymmetry charge, which are called extended supersymmetries in the relativistic context. However, we shall argue that those are indeed "minimal" in the non-relativistic setting. Then we re-derive three-dimensional Newton–Cartan supergravity and we end with a novel off-shell formulation of the latter. Finally, we apply the procedure to derive a non-relativistic version of the superparticle in a curved background. We also comment on the relation of our limiting procedure to other non-relativistic limits that have been put forward in the literature.

A novel non-relativistic supergravity theory is derived in chapter 4. By gauging a particular non-relativistic superalgebra we find a cosmological extension of Newton–Cartan supergravity, i.e. a generalization of the theory put forward in [77]. We will obtain a $1/R$-modification of that theory, where R is related to the cosmological constant $\Lambda_{CC} = -1/R^2$. We keep the same name for the supergravity theory that we use for the algebra, Newton–Hooke in this case, hence the title Newton–Hooke supergravity. We proceed with a particular choice of gauge-fixing, similar to the one performed in [77] to derive $1/R$-modifications of Galilean supergravity.[3] At the end, we use the procedure of the previous chapter 3, and the new off-shell supergravity found there, to show that we could have obtained the same results by use of the limiting procedure.

In chapter 5 we expand on our study of three-dimensional non-relativistic supergravities by deriving a "conformal" extension of the non-relativistic supergravities of chapters 3 and 4. This is done, again, by using gauging techniques. The non-relativistic algebra that we use in this chapter is a Schrödinger superalgebra. This theory will serve as a base for the "conformal" tensor calculus of the next chapter.

A non-relativistic version of the superconformal tensor calculus, a "Schrödinger tensor calculus", is introduced in chapter 6. To this end, we first introduce matter multiplets that are coupled to the non-relativistic supergravity background of chapter 5. These will serve as so-called compensator multiplets in the Schrödinger tensor calculus. By gauge-fixing all extra (Schrödinger) symmetries, we derive an off-shell version of Newton–Cartan supergravity. Because the Schrödinger theory allows for non-vanishing torsion we are thus able to derive a non-relativistic supergravity theory with non-zero torsion. We discuss how truncating to zero torsion leads to the theory of chapter 3.

We conclude in chapter 7 and give an outlook on open problems that are not addressed in this thesis and, perhaps more interestingly, those that can be addressed in the future based on the new findings of our work.

[3] Galilean supergravity is the supersymmetric extension of Galilean gravity, which by definition is the non-relativistic theory that is invariant under 'acceleration-extended' Galilean symmetries.

2

Background material

This chapter aims to set the basis for the later chapters. Here we review all physical theories and principles that will be used later on.

In this chapter we quickly review the very basics of symmetries, gauge-symmetries, general relativity and supergravity. The purpose is to give a brief introduction to all concepts that will be needed in later parts of the thesis in an effort to make this work self-contained. Naturally, since there are many textbooks devoted to teach exactly those subjects, there is too much to say about them. Here, we will proceed at a pace that cannot do justice to the details and we shall restrict ourselves to features that we use in later parts of the thesis.

The expert is likely to know all the material covered in this chapter and might proceed to the next chapter 3 immediately, possibly after a quick glance over our conventions in the next section 2.1.

2.1 Conventions

In order to get this technical part out of the way let us devote this first section to a brief overview of the conventions used in this work.

We work mostly in three space-time dimensions, i.e. one time and two spatial dimensions. For the flat Minkowski background we choose the "mostly plus" signature

$$\eta_{\mu\nu} = \mathrm{diag}(-1, 1, 1)\,. \tag{2.1}$$

The first entry is the time-like direction. When we are in more than three dimensions a suitable number of (plus) ones must be added.

Greek indices, as in (2.1) are used for curved space-time indices. They run over all coordinates, including time. The spatial subset of such indices is denoted by Latin letters $i, j, k, \ldots$ Flat, Lorentz, indices are denoted by Latin indices from the beginning of the alphabet. In particular, in this thesis we must also differentiate between relativistic and non-relativistic indices because in the non-relativistic case we need to split space and time explicitly. Capital letters usually refer to relativistic indices while lowercase letters are again reserved for the spatial part only and usually appear in the non-relativistic context only. An example is given by writing $\mu = (0, i)$ and $A = (0, a)$. Using zero for both time-like components of curved and flat indices might cause confusion. However, oftentimes we will simply not write the zero for flat components, e.g. $\lambda^a{}_0 = \lambda^a$. Then it is important to keep track of where the zero was because a zero up and a zero down differ by a minus sign due to the metric (2.1).

Symmetrization and anti-symmetrization of indices are denoted with round and square brackets respectively and we include a normalization factor in the definition, e.g. $2\, a_{[\mu} b_{\nu]} = a_\mu\, b_\nu - a_\nu b_\mu$.

We will come back to spinor and gamma-matrix conventions in section 2.6. Here, let us briefly mention that we will use Majorana spinors and real gamma-matrices given by

$$\gamma^A = (i\sigma_2, \sigma_1, \sigma_3) \tag{2.2}$$

where σ_i are the usual Pauli matrices. The charge conjugation matrix C is purely imaginary

$$C = i\gamma^0 \,. \tag{2.3}$$

Note that numerically $\gamma^{\mu\nu\rho} = \varepsilon^{\mu\nu\rho}$. This identity holds numerically as $\gamma^{\mu\nu\rho}$ is of course a matrix while the completely anti-symmetric epsilon symbol is a number. The two-dimensional, purely spatial epsilon symbol is related to the three-dimensional by

$$\varepsilon^{0ij} = \varepsilon^{ij}\,, \qquad \varepsilon^{012} = 1\,. \tag{2.4}$$

We reverse the order of spinors when we do a complex conjugation. Spinor indices are Greek indices from the beginning of the alphabet and lowercase Latin indices on spinors are used if there are multiple supercharges Q^i.

This ends our short excursion on conventions. Now are ready to dive into the realm of "physics".

2.2 Gauge symmetries

Symmetries play an immensely important role in this work. Let us introduce them using a simple example. It is not one that we will need later on in this thesis, it just serves the purpose of introducing gauge-symmetries. Consider the action of a complex scalar field $\Phi(x)$:

$$S = -\frac{m}{2} \int d^3x\, \eta^{\mu\nu}\, \partial_\mu \Phi(x) \partial_\nu \Phi^*(x)\,. \tag{2.5}$$

Never mind the prefactor $m/2$ or the fact that we chose three-dimensional space-time. This is not important. One more piece of information is that the (inverse) Minkowski metric $\eta^{\mu\nu}$ which appears in (2.5) is given by the diagonal matrix

$$\eta^{\mu\nu} = \text{diag}(-1, 1, 1)\,. \tag{2.6}$$

This just means that we are in flat space here.

For now the only observation we are interested in is that (2.5) is left invariant if we rotate the complex field $\Phi(x)$ by an arbitrary phase $\exp(i\,\alpha)$, i.e. we can replace the field $\Phi(x)$ by $\exp(i\,\alpha)\,\Phi(x)$ and this does not alter the form of the action (2.5).

The infinitesimal version of this transformation is

$$\delta\Phi(x) = i\,\alpha\,\Phi(x)\,, \qquad \delta\Phi^*(x) = -i\,\alpha\,\Phi^*(x)\,, \tag{2.7}$$

and those contributions will cancel exactly when we calculate the variation of (2.5). The action is invariant under global $U(1)$ transformations because $\exp(i\,\alpha)$ span the group $U(1)$. The transformations are *global* because the parameter α is constant. If it was not constant the variation of (2.5) would yield additional terms that arise when the derivative acts on α.

We can nevertheless modify the action in such a way that the parameter α becomes a local function of x while still leaving the action invariant. To do so, we need to introduce a new field $A_\mu(x)$ that compensates for the additional contributions that we get from the derivatives acting on $\alpha(x)$. We define a new so-called covariant derivative

$$D_\mu\Phi(x) = \partial_\mu\Phi(x) - i\,A_\mu(x)\,, \tag{2.8}$$

together with a corresponding $D_\mu\Phi^*(x)$ for the complex conjugate. Then, if we use

$$\delta A_\mu(x) = \partial_\mu\alpha(x)\,, \tag{2.9}$$

we note that the quantity (2.8) transforms just like the field $\Phi(x)$, i.e. its transformation is given by $\delta D_\mu\Phi(x) = i\alpha\,D_\mu\Phi(x)$. The gauge-field $A_\mu(x)$ is added precisely for that purpose, to cancel the derivative of the parameter $\alpha(x)$. This way we gauged our first symmetry. The field $A_\mu(x)$ is the gauge-field for the (local $U(1)$ gauge) symmetry $\Phi(x) \to \exp[i\,\alpha(x)]\,\Phi(x)$.

Note that there is no covariant derivative of $A_\mu(x)$ because it already transforms with the derivative of the parameter. Taking this thought a little bit further we realize that gauge-fields must always be part of some covariant expression, like the covariant derivative that we just introduced, or a curvature which we will introduce below. A gauge-field cannot "stand alone" as it would not lead to an invariant or covariant expression.

The symmetry here is $U(1)$ because $\alpha(x)$ is just a number. If we think of this as an identity times the parameter $\alpha(x)$ we can also envision generalizing this to other groups where we have several $\alpha(x)$s labeled by $\alpha(x)^a$ multiplying some matrices T_a. This is, in very short terms, how other transformations, such as for example rotations, work.

For any field such as $\Phi(x)$ or $A_\mu(x)$ and their symmetry transformations we can define quantities that are invariant or covariant under those symmetry transformations. The first such quantity was the covariant derivative introduced in (2.8). Another very important quantity is the curvature of the gauge-field. In our present case, it is given by

$$F_{\mu\nu}(A) = \partial_\mu A_\nu(x) - \partial_\nu A_\mu(x) = 2\,\partial_{[\mu}A_{\nu]}(x)\,. \tag{2.10}$$

Curvatures are always invariant under the gauge-transformation of the respective gauge-fields but they transform covariantly under all other transformations of the field. (For example $F_{\mu\nu}(A)$ is invariant under transformations with parameter $\alpha(x)$ but it does transform like a covariant two tensor under diffeomorphisms.)

We can write down a dynamical theory for the gauge-field $A_\mu(x)$ by making use of the curvature (2.10). The action

$$S = -\frac{1}{4} \int d^3x \, \eta^{\mu\nu} \eta^{\rho\sigma} \, F_{\mu\rho}(A) \, F_{\nu\sigma}(A) \tag{2.11}$$

can be used for that purpose. Much like (2.5) (using covariant derivatives) it is second order in derivatives and also invariant under the $U(1)$ symmetry transformations.

The actions (2.5) and (2.11) are examples of the few actions that we will introduce in this chapter. The main reason is that, while actions exist for "relativistic" gravity, no such action exists for the non-relativistic gravitational theories that we will use and hence we mostly work without actions. We will come back to this point in chapter 3. Indeed, we would like to stress that there is no need for an action to describe a theory, usually transformation rules and possibly equations of motion suffice. Having an action however, simply reduces the amount of information that is needed as transformation rules and equations of motion can always be deduced from the action.

In the following we will always use the infinitesimal version of the symmetry transformations, for example (2.7) instead of $\Phi(x) \to \exp[i\,\alpha(x)]\,\Phi(x)$ and we will in general not denote the dependence of gauge-fields and parameters on the coordinates. Unless stated otherwise, we will assume that they are all functions of the space-time coordinates x^μ.

In summary, all (gauge) theories that we will deal with later on will be given in terms of their field content, e.g. A_μ, the transformations of the fields under all symmetries, e.g. (2.9), and possibly some additional constraints on the fields, which are usually constraints on the curvatures. For example, using the definition of $F_{\mu\nu}(A)$ we see that

$$\partial_{[\mu} F_{\nu\rho]}(A) = 0 \,. \tag{2.12}$$

We noted earlier that there is no covariant derivative for A_μ, so we are inclined to generalize (2.12) to the—indeed correct and very powerful—statement that

$$D_{[\mu} F_{\nu\rho]}(A) = 0 \,. \tag{2.13}$$

This is know as the Bianchi identity. It says that the totally anti-symmetrized covariant derivative of any curvature must vanish.

Another constraint on $F_{\mu\nu}(A)$, and thus on A_μ, can be derived from the action

(2.11). It is the equation of motion for A_μ given by

$$\eta^{\mu\nu}\partial_\mu F_{\nu\rho}(A) = 0\,. \tag{2.14}$$

As we will not always have the luxury of an action we do not immediately know which constraints are really equations of motion and which ones follow for example from Bianchi identities or are imposed by hand. There might be a way to find out, however. The constraints transform under symmetry transformations like the curvatures themselves. Equations of motion always transform to equations of motion and likewise for other constraints. If we know one equation of motion we can always derive other ones by symmetry variations.

The main aim of this chapter is to introduce yet another gauge-theory that will turn out to be nothing else but the theory of general relativity. Having done so, we will also be able to couple the scalar and vector theories of this section to a curved background. Before doing so, we shortly digress and talk about the symmetry algebras that underly our gauge-theories.

2.3 Symmetry algebras

We should shortly comment on how algebras link to everything we heard so far. Remember the theory of the last section was a $U(1)$ theory because α was just a number. We thought about replacing it with several αs that multiply matrices T_a. Now, those matrices T_a will (in general) not commute, but their commutation relations will form an algebra.

A well-known theorem by Emmy Nöther dictates that for every symmetry which leaves a given theory invariant there exists a conserved current. This current gives rise to a quantity called the generator of the symmetry. Oftentimes these operators are “scalar quantities”, e.g. differential operators, sometimes, as we alluded to before, they can be written as matrices. If the generators of the symmetries of a given theory do not commute, we say that the non-vanishing commutation relations of those symmetry generators form the symmetry algebra of the theory.

We will always denote symmetry algebras using commutation relations. However, this is only for notational purposes, we do not imply any quantization of the theory. Equivalently, we could use Poisson brackets everywhere.

An example of such an algebra, that we will also use in the next section, is the Poincaré algebra. Its non-vanishing commutation relations are

$$\left[\hat{P}_A, M_{BC}\right] = 2\,\eta_{A[B}\,\hat{P}_{C]}\,, \qquad \left[M_{AB}, M_{CD}\right] = 4\,\eta_{[A[C}\,M_{D]B]}\,. \tag{2.15}$$

Note, that in contrast to the standard literature we prefer representations that do not have too many i's. All our generators, $\hat{P}_A$ for translations and M_{AB} for rotations, are anti-hermitian operators. In particular, this means that an infinitesimal transformation is given by, e.g.

$$\delta \bullet = \zeta^A \hat{P}_A \bullet \, . \tag{2.16}$$

The bullet shall denote an arbitrary field in our theory. The finite transformation is given by the exponential of this expression. Compare e.g. to $exp(i\,\mathbb{I}\,\alpha)\Phi$, where the "generator" $\mathbb{I}$ (the identity) is real and we have an explicit factor of i such that the hermitian conjugate of $exp(i\,\mathbb{I}\,\alpha)$ gives the inverse transform. We will not have those i's in the algebra, nor in the finite transformation.

2.4 Gauging the Poincaré algebra

In this section we will show how the vielbein formulation of general relativity can be seen as a gauge-theory of the Poincaré algebra. In the following we will "gauge" the algebra (2.15). This exercise is done in many textbooks on general relativity and/or supergravity. It will serve as an important basis for us, first to compare with results that we will derive later on, and secondly for the gauging of the non-relativistic Newton–Hooke superalgebra which we will perform in chapter 4, as well as the Schrödinger algebra in chapter 5.

The first step in the gauging procedure is to assign a gauge-field to every generator of the algebra. From (2.16) one can derive very general rules that determine how those gauge-fields must transform in terms of the structure constants of the symmetry algebra, see e.g. [97]. In particular, we define a connection $\mathcal{A}_\mu$ that takes values in the adjoint of the gauge group. For the Poincaré algebra we choose

$$\mathcal{A}_\mu = E_\mu{}^A \, \hat{P}_A + \frac{1}{2} \, \Omega_\mu{}^{AB} \, M_{AB} \, , \tag{2.17}$$

where $E_\mu{}^A$ and $\Omega_\mu{}^{AB}$ will eventually take over the roles of the relativistic vielbein and spin-connection. We can then realize the algebra on $\mathcal{A}_\mu$ using the transformation rule

$$\bar{\delta} \mathcal{A}_\mu = \partial_\mu \zeta + \left[\mathcal{A}_\mu , \zeta \right] , \tag{2.18}$$

where the gauge-parameter ζ is given by

$$\zeta = \zeta^A \, \hat{P}_A + \frac{1}{2} \, \lambda^{AB} \, M_{AB} \, . \tag{2.19}$$

This leads to the following transformations of the relativistic vielbein $E_\mu{}^A$ and the spin-connection $\Omega_\mu{}^{AB}$:

$$\delta E_\mu{}^A = \partial_\mu \zeta^A - \Omega_\mu{}^{AB} \zeta^B + \lambda^A{}_B \, E_\mu{}^B \, , \tag{2.20}$$

$$\delta \Omega_\mu{}^{AB} = \partial_\mu \lambda^{AB} + 2 \, \lambda^{[A}{}_C \, \Omega_\mu{}^{CB]} \, . \tag{2.21}$$

In a similar manner we define curvatures

$$\mathcal{R}_{\mu\nu}(\mathcal{A}) = 2\,\partial_{[\mu}\mathcal{A}_{\nu]} + \left[\mathcal{A}_\mu, \mathcal{A}_\nu\right] = R_{\mu\nu}{}^A(E)\,\hat{P}_A + \frac{1}{2}\,R_{\mu\nu}{}^{AB}(\Omega)\,M_{AB}\,, \tag{2.22}$$

and find

$$R_{\mu\nu}{}^A(E) = 2\,\partial_{[\mu}E_{\nu]}{}^A - 2\,\Omega_{[\mu}{}^A{}_B\,E_{\nu]}{}^B\,, \tag{2.23}$$

$$R_{\mu\nu}{}^{AB}(\Omega) = 2\,\partial_{[\mu}\Omega_{\nu]}{}^{AB} - 2\,\Omega_{[\mu}{}^A{}_C\,\Omega_{\nu]}{}^{CB}\,. \tag{2.24}$$

The connection $\mathcal{A}_\mu$ is a vector and also transforms in the usual way under general coordinate transformations:

$$\delta\mathcal{A}_\mu = \xi^\rho\partial_\rho\,\mathcal{A}_\mu + \mathcal{A}_\rho\partial_\mu\xi^\rho = \mathcal{L}_\xi\mathcal{A}_\mu\,. \tag{2.25}$$

From (2.18) and (2.25) we can define a new transformation, using the symmetry parameter

$$\Sigma = \Lambda - \xi^\mu\mathcal{A}_\mu\,. \tag{2.26}$$

Unlike Λ though, Σ only takes values in the internal/homogeneous part of the symmetry group. In the present case, this means

$$\Sigma = \frac{1}{2}\,\lambda^{AB}\,M_{AB}\,, \tag{2.27}$$

and it leads to the following general transformation

$$\delta A_\mu = \bar{\delta}\mathcal{A}_\mu - \xi^\nu\,\mathcal{R}_{\mu\nu}(\mathcal{A}) = \mathcal{L}_\xi + \partial_\mu\Sigma + \left[\mathcal{A}_\mu, \Sigma\right]. \tag{2.28}$$

Those are exactly the transformations that we are interested in. All fields will transform as vectors under general coordinate transformations and keep their original symmetry transformations under the internal symmetries.

This is slightly different for supersymmetry. There, we would like to replace only local translations by diffeomorphisms, but not the supersymmetry transformations. Thus Σ will consist of the homogeneous part plus supersymmetry. This is not straightforward though because the anti-commutator of two supersymmetries leads to a local translation, hence two "ordinary" symmetries lead to one that we replaced by (2.28). The result is that the anti-commutators of supercharges will take very peculiar forms and for most theories that we consider in this thesis we will be particularly interested in the closure of the algebra of those anti-commutators. For the time being let us focus on the bosonic problem.

While it is not strictly necessary to do so, one would oftentimes like to identify the transformations (2.18) and (2.28). For example, we will make this identification for all bosonic theories. Then it is useful to set the curvature of the independent gauge-fields, in this case only $E_\mu{}^A$, to zero:

$$R_{\mu\nu}{}^A(E) = 0\,. \tag{2.29}$$

On the one hand, this removes the curvature contributions to the formula (2.28), thus enabling us to identify local translations with general coordinate transformations. On the other hand, it allows us to solve for the spin-connection $\Omega_\mu{}^{AB}(E)$ in terms of $E_\mu{}^A$:

$$\Omega_\mu{}^{AB}(E) = -2\,E^{\rho[A}\partial_{[\mu}E_{\rho]}{}^{B]} + E_{\mu C}E^{\rho A}E^{\nu B}\partial_{[\rho}E_{\nu]}{}^C\,. \tag{2.30}$$

From now on we shall always assume that the spin-connection is a dependent field that is given in terms of the independent gauge-fields through the solution of a (curvature) constraint such as (2.29). This also brings us closer to the theory of general relativity as there is no independent spin-connection in that theory either. Now, the transformation of the vielbein follows from (2.28) and it is given by

$$\delta E_\mu{}^A = \xi^\rho\partial_\rho E_\mu{}^A + E_\rho{}^A\partial_\mu\xi^\rho + \lambda^A{}_B E_\mu{}^B\,. \tag{2.31}$$

The transformation of $\Omega_\mu{}^{AB}(E)$ follows from its expression in term of $E_\mu{}^A$ and using (2.31). In the case at hand, it is still given by (2.21) for local Lorentz rotations. It also transforms under diffeomorphisms in the usual way, i.e.

$$\delta\Omega_\mu{}^{AB}(E) = \xi^\rho\partial_\rho\Omega_\mu{}^{AB} + \Omega_\rho{}^{AB}\partial_\mu\xi^\rho + \partial_\mu\lambda^{AB} + 2\,\lambda^{[A}{}_C\,\Omega_\mu{}^{CB]}\,. \tag{2.32}$$

This finishes the gauging procedure. We have obtained the kinematics of general relativity. In a next step, we will impose equations of motion on the only independent field left, the vielbein $E_\mu{}^A$. Before doing so we digress shortly for a remark on the nature of the transformations under general coordinate transformations. Indeed, the first two terms in (2.31) and (2.32) are the transformation of a covariant tensor under general coordinate transformations. It is the infinitesimal version, using the transformation $\delta x^\alpha = \xi^\alpha$, of the defining law

$$\bar{T}_\mu\left(\bar{x}^\nu\right) = \frac{\partial x^\alpha}{\partial\bar{x}^\mu}\,T_\alpha\left(x^\beta\right), \tag{2.33}$$

for a covariant tensor. Two-tensors such as the metric (2.36) transform with two factors $\partial x/\partial\bar{x}$ and any contravariant (upper) index transforms with an inverse factor $\partial\bar{x}/\partial x$.

Putting the torsion to zero also implies that the relativistic curvature (2.24) identically satisfies the Bianchi identity

$$R_{[\mu\nu\rho]}{}^B(\Omega) = R_{[\mu\nu}{}^{AB}(\Omega)\,E_{\rho]A} = 0\,, \tag{2.34}$$

which follows from (2.13) and the transformation of $E_\mu{}^A$ given in (2.20).

Finally, we can put the theory on-shell by imposing yet another constraint on the curvature, the Einstein equation. In the current formulation, it reads

$$E^{\mu}{}_{A}\, R_{\mu\nu}{}^{AB}(\Omega) = 0\,, \tag{2.35}$$

where $\Omega_{\mu}{}^{AB}(E)$ is expressed in terms of $E_{\mu}{}^{A}$ using (2.30).

In chapter 3 we will consider the non-relativistic limit of the formulas (2.20)–(2.35) that we collected here.

To put field theories such as the scalar or vector of section 2.2 in a curved background we have to introduce one more connection. In the same way that we introduced A_μ to cancel the derivative of the parameter α we need a new (in this case dependent) field that deals with derivatives of the parameter of diffeomorphisms, such that we have a derivative ∇_μ that is covariant with respect to general coordinate transformations. This Christoffel connection can be defined by requiring that the covariant derivative of the vielbein, and by extension of the metric, given by

$$g_{\mu\nu} = \eta_{AB} E_{\mu}{}^{A} E_{\nu}{}^{B}\,, \tag{2.36}$$

is zero. Eta is again the Minkowski metric (2.1), hence all Latin indices are flat indices while Greek indices are curved space-time indices. If the vielbein is just a delta function then the space-time is flat.

The equivalence principle of general relativity states that we are always able to go to a flat frame locally. The vielbeins are doing exactly this. We will see later that the introduction of vielbeins is an immense simplification when working with spinors, as it is much easier to work with spinors and gamma-matrices in flat space.

Coming back to the connection we define it by

$$\nabla_\mu E_{\nu}{}^{A} = 0 = D_\mu E_{\nu}{}^{A} - \Gamma^{\rho}_{\mu\nu} E_{\rho}{}^{A}\,. \tag{2.37}$$

Here D_μ is the covariant derivative with respect to the transformations given by the algebra and we solve (2.8) by

$$\Gamma^{\rho}_{\mu\nu}(E) = E^{\rho}{}_{A} D_\mu E_{\nu}{}^{A}\,. \tag{2.38}$$

Now the curved space analogues of (2.5) and (2.11) are given by writing an invariant measure for the integral

$$d^3x \to d^3x\ \det(E_{\mu}{}^{A})\,, \tag{2.39}$$

replacing the (inverse) Minkowski metric by the curved (inverse) metric

$$\eta^{\mu\nu} \to g^{\mu\nu}\,, \tag{2.40}$$

and replacing all partial derivatives with covariant ones

$$\partial_\mu \to \nabla_\mu \,. \tag{2.41}$$

These substitutions suffice to put any theory in a curved background.

Finally, let us connect our theory of general relativity, as we presented it here, to the form that one usually finds in textbooks. In fact, we have derived gravity in what is referred to as the second order formulation of general relativity. Normally, one introduces the metric (2.36), the covariant derivative ∇_μ with the Christoffel connection (2.38), and the Riemann tensor (2.24). Further contractions of the Riemann tensor are the Ricci tensor given in (2.35), which still has to vanish on-shell, and the Ricci scalar

$$R(\Omega) = E^\mu{}_A E^\nu{}_B \, R_{\mu\nu}{}^{AB}(\Omega) \,. \tag{2.42}$$

The Einstein–Hilbert action of general relativity in D space-time dimensions is then given by

$$S = -\frac{1}{\kappa^2} \int d^D x \; \det(E_\mu{}^A) \, R(\Omega) \,. \tag{2.43}$$

The equation of motion for the only independent field $E_\mu{}^A$, or the metric (2.36), is the Einstein equation (2.35). The term “second order formulation” refers to the fact that the action (2.43) is second order in derivatives and that $E_\mu{}^A$ is the only independent field.

This finishes our discussion of “relativistic” gravity. In the next section we briefly describe some notions of non-relativistic gravity with different amount of symmetries.

2.5 Non-relativistic gravity

As we briefly pointed out in the introduction, there are different theories of non-relativistic gravity. The difference lies only in the amount of symmetry that we allow. In this thesis, we refer to Newtonian gravity for a system that is in fact not subject to any gravitational force, i.e. a free-falling frame. Obviously, the symmetry transformations that related different such frames are the Galilei transformations. Galilean gravity, see 2.5.1, is the first generalizations of such a system. There, we allow for arbitrary time-dependent spatial translations, and the gravitational force is determined by a Newton potential Φ. The fully gauged version, where we allow for arbitrary coordinate transformations is Newton–Cartan gravity, see 2.5.2, and the gravitational fields are $(\tau_\mu, e_\mu{}^a, m_\mu)$. In the subsections 2.5.3 and 2.5.4 we comment on some issues that are related to the dynamics of non-relativistic gravity theories and the non-relativistic limit of gravity in three space-time dimensions, respectively.

2.5.1 Galilean gravity

Here any gravitational attraction is governed by the Newton potential $\Phi(x)$ only. This Newton potential is subject to the Laplace equation,

$$\partial^i \partial_i \Phi(x) = 0 \,. \tag{2.44}$$

We will see later on, in chapter 4, that we should interpret (2.44) as the equation of motion for a "flat space" Newton potential, as opposed to the Galilean version of Newton–Hooke gravity.

A theory is non-relativistic if it is invariant under Galilei transformations, i.e. its symmetry algebra is the Galilei algebra. Its non-vanishing commutators are given by

$$\begin{aligned} \left[P_a, J_{bc}\right] &= 2\,\delta_{a[b}\,P_{c]}\,, & \left[J_{ab}, J_{cd}\right] &= 4\,\delta_{[a[c}\,J_{d]b]}\,, \\ \left[G_a, J_{bc}\right] &= 2\,\delta_{a[b}\,G_{c]}\,, & \left[H, G_a\right] &= P_a\,, \end{aligned} \tag{2.45}$$

where H, P_a, G_a and J_{ab} are the generators of time- and space-translations, boosts and rotations, respectively. The Galilei algebra admits a central extension [98], i.e. we can add one more generator Z that commutes with all other generators. It would appear in the following commutator

$$\left[P_a, G_b\right] = \delta_{ab}\, Z\,. \tag{2.46}$$

In the following subsection, we will give some more reasons for our use of this central extension. The centrally extended Galilei algebra is usually called the Bargmann algebra. We will also adopt this nomenclature and from now on we are mostly concerned with this centrally extended version of the Galilei algebra. Note, the three-dimensional Galilei or Bargmann algebra also admits another central extension, see e.g. [99]. With generalizations to higher dimensions in mind we will not consider that extension in this work.

Using the argument that a non-relativistic theory must be in variant under the Galilei or Bargmann algebra, and with the gauge-theory description of general relativity in mind, we will come to the conclusion that non-relativistic gravity could also be a gauge-theory of non-relativistic diffeomorphisms, i.e. a gauged version of the Galilei or Bargmann algebra. This is exactly the way Newton–Cartan gravity was derived in [80]. We shall come back to this point later.

Let us now consider un-gauged representations of non-relativistic algebras. We can write down a representation of the Bargmann algebra using a single scalar ϕ (note that this ϕ is not the Newton potential) in the same way as (2.20) and (2.21) are a representation of the Poincaré algebra (2.15). We would use

$$\delta\phi = \zeta\partial_t\phi + \xi^i\partial_i\phi - \lambda^i{}_j x^j\partial_i\phi + t\lambda^i\partial_i\phi + m\,\lambda^i x^i\phi + m\,\sigma\,\phi\,. \tag{2.47}$$

The constant parameters ζ, ξ^i, λ^i, $\lambda^i{}_j$ and σ are for time- and space-translations, boosts, rotations and central charge transformations. The parameter m here is a mass parameter and for $m \to 0$ (2.47) realizes the Galilei algebra. This is one way to see that the Bargmann algebra is connected to massive representations.

Note, the transformation rules of the Newton potential are not given by (2.47), see (2.48) below.

This far we are in what we will later refer to as (non-relativistic) "flat space". It is characterized by the fact that the underlying symmetry algebra is given by the Galilei algebra (2.45). In particular, only constant translations ξ^i are allowed. We could consider a partial gauging of the Galilei or Bargmann algebra—a full gauging would of course lead to Newton–Cartan gravity [80]—such that we allow for arbitrary time-dependent translations $\xi^i(t)$. In this case the symmetry algebra is not the Galilei algebra anymore but we will speak of "acceleration-extended Galilei", or Milne [4], symmetries and we are in a "curved" background that is characterized by the Newton potential Φ [100].

The transformation of this Newton potential is given by [77,101]

$$\delta\Phi = \zeta\partial_t\Phi + \xi^i(t)\,\partial_i\Phi - \lambda^i{}_j x^j\partial_i\Phi + \partial_t\partial_t\xi^i(t)\,x^i + m\,\sigma(t)\,\Phi\,. \tag{2.48}$$

Note the difference to the transformation of an arbitrary scalar field, the gauged version of (2.47),

$$\delta\phi = \zeta\partial_t\phi + \xi^i(t)\,\partial_i\phi - \lambda^i{}_j x^j\partial_i\phi + m\,\partial_t\xi^i(t)\,x^i\phi + m\,\sigma(t)\,\phi\,. \tag{2.49}$$

It is obvious that the Newton potential is not a normal scalar field. The $\partial_t\partial_t\xi^i(t)x^i$ term reminds us of its (gravitational) spin-two origin. This term survives the $m \to 0$ limit, which eliminates the central charge transformations $\sigma(t)$ (which are also gauged with respect to the Bargmann transformation (2.47) where σ is a constant).

2.5.2 Newton–Cartan gravity

Another, more general formulation of non-relativistic gravity is due to Cartan [2,3]. This so-called Newton–Cartan theory is a reformulation of Newtonian gravity that is invariant under general coordinate transformations, i.e. local transformations, not only constant ones like in (2.47). We shall not describe that theory in too much detail here as we are essentially going to re-derive it in chapter 3 by taking a non-relativistic limit of general relativity.

Let us mention that one can obtain this Newton–Cartan theory in the same way as we obtained general relativity in the previous section, by gauging the underlying symmetry algebra. In the case of Newton–Cartan gravity one needs to gauge the Bargmann algebra. This was done in [80]. Similar techniques were used to derive a non-relativistic supergravity theory in [77] and we will also use such techniques to derive Newton–Hooke supergravity in chapter 4 and Schrödinger supergravity in chapter 5.

The reasons for using the Bargmann algebra are the following. On a technical level, the addition of the central charge gauge-field m_μ and its related curvature

$R_{\mu\nu}(Z)$ enable us to derive the dependent spin- and boost-connections $\omega_\mu{}^{ab}$ and $\omega_\mu{}^a$ in terms of the independent fields τ_μ, $e_\mu{}^a$ and m_μ. This would not be possible without m_μ. Secondly, the action of the non-relativistic point-particle, the gauged version of the action that we presented in the introduction, see (3.93), is not invariant under Galilean boosts. If we would not add the m_μ terms its transformation would be a total derivative instead.

A third reason is that without m_μ we would not be able to define an invariant (again, Galilean boosts are the problem) spatial metric $\bar{h}_{\mu\nu}$. In Newton–Cartan gravity the spatial vielbein $e_\mu{}^a$ and the time-like vielbein τ_μ take over the role of the metric in general relativity. A degenerate, rank $D-1$ metric can be defined as $h_{\mu\nu} = e_\mu{}^a e_\nu{}^b \delta_{ab}$. This metric has one null eigenvector τ^μ, which is the inverse of the temporal vielbein. In turn, the time-like vielbein is a null vector of the spatial inverse metric $h^{\mu\nu}$ which consists of two inverse spatial vielbeins $e^\mu{}_a$. (We come back to the definition of the projective inverses τ^μ and $e^\mu{}_a$, which are used in (2.53), in chapter 3.) Note that the metric $h_{\mu\nu}$ is not invariant under Galilean boosts, see the transformations (2.50). However, the object $\bar{h}_{\mu\nu} = h_{\mu\nu} - 2\,\tau_{(\mu} m_{\nu)}$ is indeed invariant under boost transformations.

Lastly, a fourth reason to add the generator Z is the respective symmetry, conservation of mass, or particle number. For example, if we think about the Schrödinger equation this generator is related to the symmetry of the equation under constant phase shifts of the wave function.

Let us return to describing the theory of Newton–Cartan gravity. For the purpose of a brief introduction it should suffice to say that the transformation rules of the independent fields are given by

$$\begin{aligned} \delta\tau_\mu &= 0\,, \\ \delta e_\mu{}^a &= \lambda^a{}_b\, e_\mu{}^b + \lambda^a \tau_\mu\,, \\ \delta m_\mu &= \partial_\mu \sigma + \lambda^a e_\mu{}^a\,, \end{aligned} \tag{2.50}$$

and all fields transform under diffeomorphisms in the usual way. There are two dependent spin-connections $\omega_\mu{}^{ab}(e,\tau,m)$ and $\omega_\mu{}^a(e,\tau,m)$ that are solutions to the constraints

$$\begin{aligned} R_{\mu\nu}{}^a(P) &= 2\,\partial_{[\mu} e_{\nu]}{}^a - 2\,\omega_{[\mu}{}^{ab} e_{\nu]}{}^b - 2\,\omega_{[\mu}{}^a \tau_{\nu]} = 0\,, \\ R_{\mu\nu}(Z) &= 2\,\partial_{[\mu} m_{\nu]} - 2\,\omega_{[\mu}{}^a e_{\nu]}{}^a = 0\,. \end{aligned} \tag{2.51}$$

Furthermore we set

$$R_{\mu\nu}(H) = 2\,\partial_{[\mu} \tau_{\nu]} = 0\,, \tag{2.52}$$

by convention, and an equation of motion for those non-relativistic background fields is given by the trace of the curvature of Lorentz boosts

$$\tau^{\mu} e^{\nu}{}_{a} R_{\mu\nu}{}^{a}(G) = R_{0a}{}^{a}(G) = 0\,. \tag{2.53}$$

Furthermore, there are more constraints on the curvatures $R_{\mu\nu}{}^{ab}(J)$ and $R_{\mu\nu}{}^{a}(G)$ that follow from the Bianchi identities.

Note that the constraint (2.52) implies that the theory has no torsion. Generalizations of Newton–Cartan gravity with torsion are given in [25–29].

As a final remark for this very short summary on Newton–Cartan gravity let us mention that one can obtain the formulas for the acceleration-extended Galilei theory through a particular gauge-fixing of the Newton–Cartan theory. This was shown in [77] and we will use the same gauge-fixing to get a Galilean version of Newton–Hooke supergravity in chapter 4. This gauge-fixing offers a clear way to see how the Newton potential is related to the background fields of the Newton–Cartan theory.

2.5.3 Non-trivial dynamics

Another, not yet fully understood point is related to the dynamics of Newton–Cartan gravity. On the one side, [77, 80] gives equations of motion for the Newton–Cartan fields for "flat" space-times, i.e. space-times for which the curvature of spatial rotations vanishes $\hat{R}_{\mu\nu}{}^{ab}(J) = 0$. It is, at the time of the writing of this thesis, currently under investigation how this must be generalized for curved space-times, i.e. space-times were $\hat{R}_{\mu\nu}{}^{ab}(J) \neq 0$, see [102]. On the other side there are proposals for dynamics determined by an action [103], see also [20]. In particular it is argued that the action for Newton–Cartan gravity is given by (extended) Hořava–Lifshitz gravity [104, 105]. However, it remains to check if these actions indeed give rise to the same equations of motion that are given in [102] or [77, 80].

For more details we must refer the reader to [102]. However, let us make some intriguing remarks about why it is so difficult to describe dynamics for the Newton–Cartan background fields. It was shown in [77, 80] that Newton–Cartan gravity is ultimately related to the Bargmann algebra, rather than the Galilei algebra. The difference is that the Bargmann algebra allows for one more symmetry generator Z, i.e. one more symmetry that the equations of motion have to obey. Once $\hat{R}_{\mu\nu}{}^{ab}(J) \neq 0$ the equations of motion that were proposed in [77, 80] are not anymore invariant under Galilean boosts. This problem can be solved by adding extra terms, proportional to $\hat{R}_{\mu\nu}{}^{ab}(J)$ times m_{μ}, the gauge-field of the central charge symmetry Z. The drawback is that these new equations of motion are not invariant under central charge transformations anymore. Moreover, without introducing new fields one cannot overcome this problem. So, either one adds a Stückelberg field [102], see also [29], or the equations of motion are not invariant under central charge symmetry. One is always free to opt for the first option, but we would like to point out an interesting fact about the second one.

The central charge symmetry is related to the conservation of the particle number. Thus, if the equations of motion are not invariant under Z it would mean that in non-relativistic curved space-times the total number of particles is not conserved. But this should not come as a big surprise given that we know that this is certainly the case for some relativistic curved space-times. Indeed, in de Sitter space-times, which are space-times with a non-vanishing positive cosmological constant, we cannot define a unique vacuum, see e.g. [106], and particle number is certainly not conserved when we are able to create particles simply by choosing a new vacuum state.

2.5.4 Taking a non-relativistic limit in three dimensions

For technical reasons we aim to construct non-relativistic theories of supergravity in three space-time dimensions only. This approach proved useful in previous works and instead of generalizing this particular result to higher dimensions, we rather aim to find more examples of non-relativistic supergravity in three dimensions. However, this leads to an apparent problem when we want to take, or even define, limits that are supposed to take us from a relativistic theory to a non-relativistic one. The problem is the following. It is well known that three-dimensional gravity is trivial in the sense that there are no local degrees of freedom and hence there are no dynamics [107–109]. (There are global effects though, but these are not important for our discussion here.) This implies, in particular, that there are no forces between test particles in ordinary three-dimensional gravity.

The curious thing now is that we want to describe non-relativistic gravitational theories that are generalizations of Newtonian gravity. Newtonian gravity is defined in the same manner in any space-time dimension, namely the gravitational interactions are determined by a scalar field, the Newton potential, and test particles are subject to a force that is proportional to the (spatial) gradient of this scalar field. The number of space-time dimensions does not play a role in the Newtonian theory and test particles always feel a force.

The question is how is it possible, or is it possible at all, to go from general relativity in three dimensions, a theory with no force between test particles, to the Newtonian theory with a force?

The answer to this apparent discrepancy that was given given in [107–109] is that we must acknowledge that there is no Newtonian limit in three dimensions. Note that this does not imply that there is no Newtonian gravity, rather that it cannot be obtained from general relativity.

A wide-held believe is that there is no Newton potential in three dimensions. This is not true. The Newton potential is determined by the Poisson equation. If there is no source term we only find trivial solutions, in which case we can indeed say there is no Newton potential. If there is a source term then we can have a non-trivial Newton potential in four or in three or in any space-time dimension.

The crucial point that sets three dimensions apart is that when we take a non-relativistic limit all source terms in the Poisson equation will vanish. (They come with a prefactor $D-3$.) So we cannot consider sources that have a relativistic origin. But if we forget, or do not require a relativistic origin of the source, we can simply add it by hand and we can find non-trivial solutions to the Poisson equation also in three space-time dimensions.

This does not pose any problems to us if we try do construct non-relativistic supergravity theories from scratch because in this scenario we don't think about connections to the relativistic theory. But it is confusing in view of the limiting procedure that we adopt in chapter 3. This procedure does seem to yield Newton–Cartan gravity as some sort of limit of general relativity, for any space-time dimension. Nevertheless, also in this case we obtain the correct equation of motion in three dimensions (mainly because we derive a source free equation).

This completes our discussion of symmetries and relativistic and non-relativistic gravitational backgrounds, for the bosonic case. We already saw in the transformation rule (2.20) that symmetries might link different fields of the theory, albeit this is not true anymore in the equations that we will use (2.31). In the following, we will introduce a new symmetry that must, due to its very nature, mix fields of different spin, hence different fields. But first we shall take a small detour and talk more about spin and what spinors are.

2.6 Spinors

We encountered fields of different spin, the scalar and the vector in 2.2. Now we will introduce one more, the spinor. Spin, the defining quantity for those fields, is determined by how those fields transform under the rotations of the (homogeneous) Poincaré group. The scalar is invariant, the vector rotates in the "normal" way, i.e. a rotation by an angle of 360 degrees leaves it invariant. Spinors, weirdly, go to minus themselves if you rotate them by 360 degrees. The existence of these objects is due to the fact that the little groups of massive and massless particles, $SO(3)$ and $SU(2)$, mathematically speaking have a double cover. Thus, for any normal representation there is one with half its spin. Therefore, we find objects with half-integer spin and curious transformation properties.

Scalars and vectors, and other integer spin fields like the spin-two graviton, transform under the vector representation of the Lorentz group. Spinors transform under the fermionic, spinor representation. Elements of this representation are Grassmann valued, (oftentimes, but not necessarily) anti-commuting, objects. A spinor is an array of Grassmann variables λ_α. The spinor index α labels the different components. How many components there are depends on the number of space-time dimensions. All our supergravities are in three dimensions so our spinors all have two components $\alpha = 1, 2$.

Spinors are representations of the Lorentz group with half integer spin. That means by combining two of them we should obtain an object with integer spin. This is indeed correct, for example a combination of two spinors ψ and χ is a scalar for $\bar{\psi}\chi$ and a vector for $\bar{\psi}\gamma^A\chi$. Barred spinors are introduced for exactly that purpose.

The index A on the gamma-matrix is the vector index of the (Lorentz) vector $V^A = \bar{\psi}\gamma^A\chi$.

So how do we define a barred spinor and what are gamma-matrices? The answer to the second part of the question is simple. Gamma-matrices are a representation of the Clifford algebra

$$\{\gamma^A, \gamma^B\} = \gamma^A\gamma^B + \gamma^B\gamma^A = 2\,\eta^{AB}\,\mathbb{I}\,. \tag{2.54}$$

Eta is again the flat (inverse) Minkowski metric (2.6). We added an identity matrix on the right-hand-side. This is a matrix in spinor space because the left-hand-side is a matrix too. As you (do not) see we omitted the spinor indices. Explicit spinor indices will not be necessary for any calculation that we perform in this thesis.

This is were the vielbeins that connect the space-time dependent metric (2.36) to a local, constant Minkowski metric are so useful. This way, we only ever have to use a representation of (2.54) with constant gamma-matrices.

The defining equation (2.54) in fact determines the dimension of spinors too. In three space-time dimensions we can realize (2.54) using objects $\gamma^A = (i\sigma_2, \sigma_1, \sigma_3)$ with σ_i the Pauli matrices

$$\sigma_1 = \begin{pmatrix} 0 & 1 \\ 1 & 0 \end{pmatrix}\,, \qquad \sigma_2 = \begin{pmatrix} 0 & i \\ -i & 0 \end{pmatrix}\,, \qquad \sigma_3 = \begin{pmatrix} 1 & 0 \\ 0 & -1 \end{pmatrix}\,. \tag{2.55}$$

So spinors are two-dimensional objects, e.g.

$$\lambda = \begin{pmatrix} \lambda_1 \\ \lambda_2 \end{pmatrix}\,. \tag{2.56}$$

In order to obtain a scalar quantity, one that does not have any free spinors indices, we need to multiply objects like (2.56) with a suitable transpose. Those transposes are the barred spinors that we define by

$$\bar{\lambda} = \lambda^T C = \left(\bar{\lambda}_1, \bar{\lambda}_2\right)\,, \tag{2.57}$$

where the matrix C is called the charge conjugation matrix. It is given by $C = i\gamma^0$.

The reason why we put an i here is that we (choose to) reverse the order of spinors when we complex conjugate. Since they are anti-commuting variables we get a minus sign when we put them in the original order. This sign is compensated by the i in C. We conclude that, due to our conventions, all spinor bilinears are real, no matter what gamma-matrices are wedged between the spinors.

About complex conjugation. We should mention that we use "real" spinors, so-called Majorana spinors that are subject to the identity

$$(\lambda)^* = \lambda\,. \tag{2.58}$$

Majorana spinors do not exists in every dimension, but they do in three space-time dimensions so we make use of that. If we would use for example Dirac spinors, then bilinears would be complex numbers and our previous statement about them being real would not hold. It follows that Majorana spinors have half as many degrees of freedom as Dirac spinors.

Because we use Majorana spinors it makes sense to consider the transpose of a spinor bilinear. The transpose of the gamma-matrices and of C follows from their representation through the Pauli matrices. They can be written as

$$C^T = -C\,, \qquad \left(\gamma^A\right)^T = -C\gamma^A C^{-1}\,, \tag{2.59}$$

which holds for any representation, not just for ours. Using (2.59) we can show for example that

$$\bar{\psi}\chi = \bar{\chi}\psi\,, \qquad \bar{\psi}\gamma^A\chi = -\bar{\chi}\gamma^A\psi\,. \tag{2.60}$$

What if we have two spinors, a barred and a normal one that are not contracted? Can we write them in terms of bilinears? We certainly can. Two spinors that are not contracted essentially give rise to a quantity with two free indices in spinor space, i.e. a matrix in spinor space. Such a matrix consists of four independent entries, hence we need four independent basis elements to determine it. Those four independent elements are readily found by the identity plus the three Pauli matrices or the three gamma-matrices γ^A. The completeness relation, also called Fierz identity, in three dimensions is given by

$$\chi\bar{\psi} = \frac{1}{2}\,\bar{\psi}\chi\,\mathbb{I} - \frac{1}{2}\,\bar{\psi}\gamma^A\chi\,\gamma_A\,. \tag{2.61}$$

This ends the rather technical discussion on properties of spinors and how to manipulate when doing calculations.

Finally, let us come back to our starting point, the transformation of spinors under Lorentz rotations. This is given by

$$\delta\psi = \frac{1}{4}\,\lambda^{AB}\gamma_{AB}\psi\,. \tag{2.62}$$

We use $\gamma_{AB} = (\gamma_A\gamma_B - \gamma_B\gamma_A)/2$. Any object that transforms under a rotation in this manner is a spinor and we can use (2.62) to show that any spinor-bilinear is a boson, a scalar, a vector or a spin-two field.

This concludes our short excursion on relativistic spinors and some of their characteristics. As this thesis deals with non-relativistic supergravity, we will of course encounter non-relativistic spinors. Let us anticipate here what the non-relativistic analog of (2.62) will be. In (2.62) the indices A and B run over all space- and time-components. In the non-relativistic case, we will differentiate between them. Hence we will find

$$\delta\psi = \frac{1}{4}\,\lambda^{ab}\gamma_{ab}\psi\,, \tag{2.63}$$

if the representation contains only one spinor variable. Since we will mostly deal with so-called extended supersymmetry, we will in general have at least two spinorial objects, which will generically transform as

$$\begin{aligned} \delta\psi_1 &= \frac{1}{4}\,\lambda^{ab}\gamma_{ab}\psi_1\,, \\ \delta\psi_2 &= \frac{1}{4}\,\lambda^{ab}\gamma_{ab}\psi_2 - \frac{1}{2}\,\lambda^{a}\gamma_{a0}\psi_1\,. \end{aligned} \tag{2.64}$$

We will come across many representations of this form in chapters 3–6 whenever there are at least two spinors in the multiplet. Moreover, from the point of view of the limiting procedure which we present in chapter 3 it will become clear how the relativistic (2.62) transforms into (2.64).

2.7 Global supersymmetry

So far our symmetry parameters, $\lambda^{A}{}_{B}$, ζ, ξ^i, etc. were bosonic parameters. In particular, this allowed for representations of bosonic symmetry algebras that use only a single field, e.g. (2.47). Now we will take a look at a symmetry whose parameter is a spinor, ϵ or η. We will use η as the parameter of relativistic supersymmetry and ϵ as parameter of non-relativistic supersymmetry.

This new symmetry transforms bosons into fermions and vice versa. Obviously, the representations of this new symmetry must consist of more that one field. At least we need one boson and one fermion. For this transformation to be a good symmetry it should also be bijective. If we can assign a fermion to each boson we should be able to assign a boson to each fermion. It follows that in any given representation of a supersymmetry algebra the number of degrees of freedom of bosonic fields must match the number of degrees of freedom of fermionic fields.

The generators of this new symmetry Q must be fermions too. An algebra with fermionic generators is referred to as a superalgebra. For example, the (simplest) supersymmetric extension of the Poincaré algebra (2.15) is given by

$$[M_{AB}, Q] = -\frac{1}{2}\,\gamma_{AB}Q\,, \qquad \{Q, Q\} = -\gamma^{A}C^{-1}\,P_{A}\,. \tag{2.65}$$

We say the simplest, because there might be more than one fermionic supercharge Q. The number of supersymmetry charges is usually denoted by $\mathcal{N}$. In this work, we shall deal only with $\mathcal{N} = 1$ and $\mathcal{N} = 2$. If $\mathcal{N}$ is more than one we speak of extended supersymmetry and we will dress the different supercharges, parameters and gauge-fields with a label i, where $i = 1, \ldots, \mathcal{N}$.

There are physical reasons why there should not be more than four supersymmetries in four space-time dimensions. Since we only deal with toy models in three dimensions we shall not be concerned with this any further.

Of particular interest to us are $\mathcal{N} = 2$ extensions. For example, the $\mathcal{N} = 2$ Poincaré superalgebra is given by (2.15) and

$$[M_{AB}, Q^i] = -\frac{1}{2}\gamma_{AB}Q^i , \qquad \{Q^i, Q^j\} = -\gamma^A C^{-1} P_A \delta^{ij} + C^{-1} \mathcal{Z} \epsilon^{ij} . \quad (2.66)$$

Note that the addition of more supercharges opens up the possibility to have central charges, $\mathcal{Z}$ in the anti-commutator of two supercharges Q^i.

Like in the bosonic case we are looking for representations of such symmetry algebras. Here, we shall start off with the simpler $\mathcal{N} = 1$ Poincaré superalgebra, (2.15) and (2.65). The simplest representation is given by the so-called scalar multiplet

$$\begin{aligned} \delta\phi &= \frac{1}{4}\bar{\eta}\chi , \\ \delta\chi &= \gamma^\mu \eta\, \partial_\mu \phi - \frac{1}{4} D\, \eta , \\ \delta D &= -\bar{\eta}\gamma^\mu \partial_\mu \chi . \end{aligned} \quad (2.67)$$

This is a representation of constant, or rigid, supersymmetry. We come back to space-time dependent supersymmetry parameters in the next section. The algebra (2.65) is realized in the following way. The anti-commutator of two supersymmetries, e.g. on ϕ,

$$[\delta_Q(\eta_1), \delta_Q(\eta_2)]\phi = \frac{1}{2}\bar{\eta}_2\gamma^\mu\eta_1\, \partial_\mu\phi , \quad (2.68)$$

leads to a translation with parameter

$$\zeta^\mu = \frac{1}{2}\bar{\eta}_2\gamma^\mu\eta_1 . \quad (2.69)$$

Here, we do not need to care whether we label our indices by μ or A as we are in flat space anyways. The coupling to (super)gravity is also part of the next section.

The multiplet (2.67) realizes the $\mathcal{N} = 1$ Poincaré superalgebra without further ado, i.e. without any additional constraints. However, there is a somewhat stripped down version as well, where we consider only the transformation rules

$$\delta\phi = \frac{1}{4}\bar{\eta}\chi , \qquad \delta\chi = \gamma^\mu \eta\, \partial_\mu \phi . \quad (2.70)$$

Now the algebra is realized as well, if we impose the constraints

$$\partial^\mu \partial_\mu \phi = 0\,, \qquad \gamma^\mu \partial_\mu \chi = 0\,. \tag{2.71}$$

These are, in fact, the equations of motion of the fields ϕ and χ and we refer to (2.70) as the "on-shell" multiplet, meaning that the supersymmetry algebra is realized upon using the field's equations of motion.

We note two things. First, the off-shell multiplet contains more fields, because the equations of motion effectively remove degrees of freedom. Hence, if we want the same number of degrees of freedom for bosonic and fermionic fields then we must add fields that simply vanish when the equations of motion are enforced. The other point is that one can easily check that the equations of motion (2.71) transform into each other under supersymmetry transformations of ϕ and χ. We pointed towards this at the end of section 2.2. Constraints always transform to constraints under symmetry transformations.

Next, we are going to investigate the implications of making the supersymmetry parameter a local function of the space-time coordinates.

2.8 Local supersymmetry: supergravity

Having gone through the analysis of section 2.2, we know what will happen when we consider local supersymmetry parameters $\eta(x)$. We will get derivatives of this parameter in the algebra and to fore-come this we have to introduce a gauge-field that cancels those derivatives. In the case of fermionic symmetry parameters, the gauge-fields will be fermions, more precisely vector-spinors. The gauge-fields for supersymmetry are called gravitinis and we use the symbol Ψ_μ for them.

For extended supersymmetry, i.e. if there is more than one supersymmetry parameter, we will find one gravitini $\Psi_{\mu i}$ for each parameter $\eta_i(x)$.

The easiest representations with local supersymmetry are the graviton multiplets. Otherwise, because of the local nature of $\eta(x)$ and the necessity for a gauge-field for this local symmetry, we would need to consider for example the scalar multiplet coupled to supergravity. We will not consider any such "matter" multiplets in this chapter. In chapter 6 we will perform such a coupling to the supergravity background, albeit in the non-relativistic case.

The simplest off-shell representation of the $\mathcal{N} = 1$ Poincaré superalgebra is given by the multiplet

$$\begin{aligned} \delta E_\mu{}^A &= \frac{1}{2}\,\bar\eta\,\gamma^A \Psi_\mu\,, \\ \delta \Psi_\mu &= D_\mu \eta + \frac{1}{2}\,S\,\gamma_\mu \eta\,, \\ \delta S &= \frac{1}{8}\,\bar\eta \gamma^{\mu\nu} \hat\Psi_{\mu\nu}\,, \end{aligned} \tag{2.72}$$

where S is the auxiliary field that we need to add to close the supersymmetry algebra off-shell. Here we defined the Lorentz covariant derivative of a spinor D_μ, which is given by

$$D_\mu \eta = \partial_\mu \eta - \frac{1}{4}\, \Omega_\mu{}^{AB}(E, \Psi) \gamma_{AB} \eta \,, \tag{2.73}$$

and the supercovariant curvature of the gravitino is

$$\hat{\Psi}_{\mu\nu} = 2\, \partial_{[\mu} \Psi_{\nu]} - \frac{1}{2}\, \Omega_{[\mu}{}^{AB} \gamma_{AB} \Psi_{\nu]} + S\, \gamma_{[\mu} \Psi_{\nu]} \,. \tag{2.74}$$

To verify that (2.72) realizes the $\mathcal{N} = 1$ Poincaré superalgebra one needs to know the transformation of $\Omega_\mu{}^{AB}(E, \Psi)$ under supersymmetry (and the bosonic transformations). These can be deduced from the expression of $\Omega_\mu{}^{AB}(E, \Psi)$ in terms of the independent fields $E_\mu{}^A$ and Ψ_μ:

$$\begin{aligned} \Omega_\mu{}^{AB}(E, \Psi) = & -2\, E^{\rho[A} \Big(\partial_{[\mu} E_{\rho]}{}^{B]} - \frac{1}{4}\, \Psi_{[\mu} \gamma^{B]} \Psi_{\nu]} \Big) \\ & + E_{\mu C} E^{\rho A} E^{\nu B} \Big(\partial_{[\rho} E_{\nu]}{}^C - \frac{1}{4}\, \Psi_{[\rho} \gamma^C \Psi_{\nu]} \Big) \,. \end{aligned} \tag{2.75}$$

Much like its bosonic counterpart (2.30), this formula follows from setting to zero the supercovariant curvature of $E_\mu{}^A$, see (2.79), the torsion constraint. We find the following supersymmetry transformation for the spin-connection:

$$\delta \Omega_\mu{}^{AB}(E, \Psi) = \frac{1}{2}\, E^{\rho[A}\, \bar{\eta}\, \gamma^{B]} \hat{\Psi}_{\mu\rho} + \frac{1}{4}\, E_{\mu C} E^{\rho A} E^{\nu B}\, \bar{\eta}\, \gamma^C\, \hat{\Psi}_{\rho\nu} + \frac{1}{2}\, S\, \bar{\epsilon} \gamma^{AB} \Psi_\mu \,. \tag{2.76}$$

The closure of the supersymmetry algebra on the multiplet (2.72) takes a slightly different form than the algebra of e.g. the multiplet (2.67). The anti-commutator of two supercharges leads to a so-called soft-algebra,

$$\begin{aligned} \big[\delta_Q(\eta_1), \delta_Q(\eta_2) \big] = & \, \delta_{g.c.t.}(\tilde{\xi}^\rho) + \delta_Q(-\tilde{\xi}^\rho \Psi_\rho) \\ & + \delta_M\big(-\tilde{\xi}^\rho \Omega_\rho{}^{AB} + \frac{1}{2}\, S\, \bar{\eta}_2 \gamma^{AB} \eta_1 \big) \,, \end{aligned} \tag{2.77}$$

and the parameter of diffeomorphisms, and contributor to other symmetries, $\tilde{\xi}^\mu$ is

$$\tilde{\xi}^\mu = \frac{1}{2}\, \bar{\eta}_2 \gamma^A \eta_1\, E^\mu{}_A \,. \tag{2.78}$$

Note that the subscripts 1 and 2 refer to the two independent supersymmetry transformations that are performed in (2.77). They do not indicate the existence of multiple supercharges. Also, since we use that $\gamma^\mu = E^\mu{}_A\, \gamma^A$ this coincides with (2.69) when we take the rigid limit. The appearance of this soft-algebra stems from the fact that we introduced diffeomorphisms instead of local translations using (2.28). This is responsible for all the terms of the form $\tilde{\xi}^\mu$ times a gauge-field.

Of course all the curvatures shown in section 2.4 receive fermionic corrections to account for the additional supersymmetry transformations of the gauge-fields. For example, we find

$$\hat{R}_{\mu\nu}{}^A(E) = 2\,\partial_{[\mu}E_{\nu]}{}^A - 2\,\Omega_{[\mu}{}^A{}_B\,E_{\nu]}{}^B - \frac{1}{2}\,\Psi_{[\mu}\gamma^A\Psi_{\nu]}\,. \tag{2.79}$$

Supercovariant quantities such as (2.74) and (2.79) will be denoted with a hat to distinguish them from their bosonic counterparts.

In chapter 3, in fact in this whole thesis, we will focus mostly on extended supergravities with four supercharges, or $\mathcal{N}=2$ supersymmetry. A more detailed argument against $\mathcal{N}=1$ theories will be given in section 3.3. In short, non-relativistic superalgebras with only two supercharges do not lead to diffeomorphisms.

For later reference, we need on- and off-shell formulations of $\mathcal{N}=2$ Poincaré supergravity. We give the on-shell one here and we shall defer the presentation of the off-shell multiplet to the next chapter.

In general, superalgebras with more than one supercharge allow for central extensions in the anti-commutator of two such charges. We will also include such a central element, denoted by $\mathcal{Z}$. Then, the $\mathcal{N}=2$ Poincaré superalgebra is given by (2.15) and (2.66). Here we will focus on the on-shell version, where the gauge-field that is related to the central charge transformation is not needed to close the algebra. Nevertheless, we will have to add it to take the non-relativistic limit, as we will show in chapter 3, see section 3.4.

The on-shell $\mathcal{N}=2$ supergravity multiplet is given by

$$\delta E_\mu{}^A = \frac{1}{2}\,\delta^{ij}\,\bar{\eta}_i\,\gamma^A\Psi_{\mu j}\,, \tag{2.80}$$

$$\delta\Psi_{\mu i} = D_\mu\eta_i = \partial_\mu\eta_i - \frac{1}{4}\,\Omega_\mu{}^{AB}\gamma_{AB}\eta_i\,. \tag{2.81}$$

The supersymmetry transformation of the dependent spin-connection,

$$\delta\Omega_\mu{}^{AB}(E,\Psi_i) = -\frac{1}{2}\,\delta^{ij}\,E^{\rho[A}\,\bar{\eta}_i\,\gamma^{B]}\hat{\Psi}_{\mu\rho j} + \frac{1}{4}\,\delta^{ij}\,E_{\mu C}E^{\rho A}E^{\nu B}\,\bar{\eta}_i\,\gamma^C\,\hat{\Psi}_{\rho\nu j}\,, \tag{2.82}$$

is zero on-shell, i.e. when we use the fermionic equation of motion

$$\hat{\Psi}_{\mu\nu i} = 0\,. \tag{2.83}$$

The variation of $\Omega_\mu{}^{AB}(E,\Psi_i)$ follows from

$$\begin{aligned}\Omega_\mu{}^{AB}(E,\Psi) = -2\,E^{\rho[A}\Big(\partial_{[\mu}E_{\rho]}{}^{B]} - \frac{1}{2}\,\delta^{ij}\,\Psi_{[\mu i}\gamma^{B]}\Psi_{\nu]j}\Big) \\ + E_{\mu C}E^{\rho A}E^{\nu B}\Big(\partial_{[\rho}E_{\nu]}{}^{C} - \frac{1}{4}\,\delta^{ij}\,\Psi_{[\rho i}\gamma^{C}\Psi_{\nu]j}\Big)\,,\end{aligned} \tag{2.84}$$

which in turn is the solution to the supercovariant torsion constraint

$$\hat{R}_{\mu\nu}{}^{A}(E) = 2\,\partial_{[\mu}E_{\nu]}{}^{A} - 2\,\Omega_{[\mu}{}^{A}{}_{B}\,E_{\nu]}{}^{B} - \delta^{ij}\,\Psi_{[\mu i}\gamma^{A}\Psi_{\nu]j}\,. \tag{2.85}$$

This finishes our quick review of $\mathcal{N}=2$ on-shell Poincaré supergravity.

We note that in essence the only difference between the $\mathcal{N}=1$ and $\mathcal{N}=2$ theories is that all of the spinor variables take extra indices. This is somewhat special to three dimensions though, because all supergravity fields are pure gauge and do not constitute physical degrees of freedom. Thus, by adding more gravitini we do not add fermionic degrees of freedom and the counting between bosonic and fermionic degrees of freedom is not upset. In higher dimensions, adding more supercharges usually enforces also the addition of extra physical and auxiliary fields.

3

A non-relativistic limiting procedure

This chapter contains the first main part of the thesis. Its aim is to introduce a procedure that can be used to derive non-relativistic supergravity theories. We start by stating the main ideas that underly this limiting procedure. Since this is quite abstract we go on to discuss several examples. As a test we re-derive Newton–Cartan gravity in arbitrary dimensions and on-shell Newton–Cartan supergravity in three space-time dimensions. We end by deriving a novel off-shell version of Newton–Cartan supergravity and, by extension of those results, an on-shell version of Newton–Hooke supergravity.

This chapter begins the main part of this thesis. As laid out in the introduction, we want to construct non-relativistic theories of supergravity. This chapter serves to describe one way of doing so, based on our work in [110]. It is mainly organized along the same lines as that paper including one additional section on "less than minimal" $\mathcal{N} = 1$ Newton–Cartan supergravity.

In the following we develop a non-relativistic limiting procedure that enables us to derive non-relativistic geometries and generally covariant (super)gravity theories from relativistic ones. In section 3.1 we explain the procedure itself in a rather abstract way. To emphasize possible pitfalls, we use the procedure to re-derive known examples, i.e. theories that featured in the literature already. We begin by deriving with Newton–Cartan gravity in the formulation of [80] in section 3.2. Then, before moving on to "normal" $\mathcal{N} = 2$ supergravity theories, we review the arguments for why we are mostly interested in theories with extended supersymmetry in section 3.3. We simply apply the limiting procedure on the $\mathcal{N} = 1$ Poincaré supergravity theory which, for obvious reasons, does not lead to our preferred choice of a non-relativistic theory of supergravity. In section 3.4 we re-derive the known on-shell version of three-dimensional Newton–Cartan supergravity of [77] and in section 3.5 we obtain a new off-shell formulation of that theory. The limiting procedure is then applied to the relativistic point-particle with the aim of re-deriving the results of [111] in section 3.6. A short conclusion is presented in section 3.7.

3.1 The general procedure

In this section we will describe the limiting procedure that we adopt in this chapter in general terms. This procedure can be viewed as an extension of the contraction of a relativistic space-time symmetry algebra to a non-relativistic one. Such contractions are known as so-called Inönü–Wigner contractions [96]. In particular, we try to implement a similar contraction on an irreducible multiplet of fields, that forms a representation of the relativistic algebra, to obtain an irreducible multiplet of fields that represents the contracted non-relativistic algebra.

Recall that when performing a standard Inönü–Wigner contraction of a symmetry algebra, one first redefines the generators of the algebra by taking linear combinations of the original generators with coefficients that involve a contraction parameter ω. The contracted algebra is then obtained by calculating commutators of the redefined generators, re-expressing the result in terms of the redefined generators and taking $\omega \to \infty$ in the end. Note that this procedure does not change the number of generators and that for finite ω the algebra of redefined generators is equivalent to the original one.

We now wish to extend this contraction to an irreducible multiplet of fields that forms a representation of a relativistic algebra. Such a multiplet will in general contain a number of independent fields that can be associated to certain generators of the algebra. For instance, the vielbein of general relativity can be viewed as a gauge-field of local translations [95]. The other generators of the algebra are associated to gauge-fields, that are however not independent, but that in general depend on all other independent fields in the multiplet. This is the way in which the spin-connection of general relativity can be viewed as a gauge-field of local Lorentz transformations. Finally, the multiplet can also contain independent fields that can not be interpreted as gauge-fields of the underlying space-time symmetry algebra. This is for instance the case when considering off-shell supergravity multiplets, where typically auxiliary fields are necessary to ensure that the number of bosonic and fermionic degrees of freedom match.

In a first step, one can extend the algebra contraction to all fields in the multiplet. This can be done first by extending the contraction from the generators to the parameters of symmetry transformations and to the gauge-fields that are associated to the generators. Let us denote the original algebra generators collectively by T_A, the original symmetry parameters by ζ^A and the original fields, that are associated to the generators by A_μ^A. The redefinition of the generators T_A to generators $\tilde{T}_A$, that involves ω and defines the contraction, can then be extended to redefinitions of ζ^A to $\tilde{\zeta}^A$ and of A_μ^A to $\tilde{A}_\mu^A$ such that

$$\zeta^A T_A = \tilde{\zeta}^A \tilde{T}_A \,, \qquad\qquad A_\mu^A T_A \simeq \tilde{A}_\mu^A \tilde{T}_A \,. \tag{3.1}$$

Strictly speaking, as we will clarify later in the specific examples, the second equation only holds up to terms that are subleading in ω. For that reason we do not put an equality sign there.

The equation (3.1) defines the tilded parameters and fields in terms of the original ones and the contraction parameter ω. These definitions also guarantee that, for finite ω, the redefined multiplet is equivalent to the original one.

The defining equation (3.1) involves independent and dependent fields that are associated to algebra generators. As far as the independent fields are concerned, this is the end of the story. Regarding the dependent fields, one should take into account that the redefinition obtained from (3.1) should be consistent with the one induced by performing the redefinitions on the independent fields in the expressions that define dependent fields in terms of independent ones. This amounts to a non-trivial consistency check, as we will explain in the next paragraph. By examining the transformation rules in terms of redefined parameters and fields, one can then define redefinitions of the fields that are not associated to gauge transformations, by requiring that no term in the transformation rules diverges when taking the limit $\omega \to \infty$. As we will see later in a specific example, this step usually amounts to rescaling these fields with the contraction parameter ω.

After having determined the necessary redefinitions of all fields, one can send $\omega \to \infty$, at which point the consistency of the procedure needs to be checked. In a first step, checking consistency involves ensuring that no divergences appear in the procedure. This means for instance that one needs to examine whether the transformation rules in terms of redefined quantities are finite in the limit $\omega \to \infty$. Moreover, one needs to examine the expressions of the dependent fields in terms of the independent ones and check whether one obtains a finite result, consistent with the redefinition implied by (3.1), when writing these expressions in terms of redefined independent fields and taking $\omega \to \infty$. As we will see in specific examples, this step involves imposing additional constraints on the fields.

Typically, these constraints involve putting certain covariant quantities to zero. These covariant quantities might for instance correspond to gauge covariant curvatures of fields that are interpreted as gauge-fields of the algebra. Constraining the fields in such a covariant way will ensure that the final transformation rules are the proper ones.

Once a set of constraints is obtained by requiring that transformation rules and other quantities do not diverge when $\omega \to \infty$, one needs to check whether these constraints form a consistent set. This involves varying all non-trivial constraints found so far under all symmetry transformations and checking that they form a closed set. In this way, one ensures that one is performing a consistent truncation.

Finally, we mention that the limiting procedure can lead to the elimination of a number of auxiliary fields. This is due to the fact that we are interested in obtaining an irreducible multiplet. The non-relativistic theory can have less equations of motion than the relativistic one hence the number of auxiliary fields that are needed to realize the non-relativistic algebra can differ from the number that is needed to realize the relativistic algebra. This is responsible for the fact that some auxiliary fields have to be eliminated in the limiting process.

We can summarize the procedure in the following way:

I. We first write the relativistic gauge-fields in terms of new 'non-relativistic' ones, using a contraction parameter ω. For finite ω this is a field redefinition that is dictated by the contraction of the generators of the algebra. At this point the scaling of the auxiliary fields is still arbitrary.

II. Using the above redefinitions and taking the limit $\omega \to \infty$ we can derive a first set of non-relativistic constraints by taking the limit $\omega \to \infty$ of the relativistic un-conventional constraints.

III. In a next step, we derive the transformation rules of all fields. Requiring that no terms diverge in the limit $\omega \to \infty$ fixes the scalings of the auxiliary fields. At this point, we can check the limit of dependent gauge-fields, such as e.g. the spin-connection. Requiring that they have a well-defined limit can involve the use of un-conventional constraints, written in terms of redefined fields, to replace divergent terms by terms with a proper limit.

IV. In this step, we check whether the constraints found in step two are a closed set under the different symmetry transformations or whether we are forced to introduce additional constraints. An example were many new constraints are found by continuous variation under supersymmetry is given by the chain of constraints in eqs. (3.65)–(3.67).

V. The number of auxiliary fields that are needed in the non-relativistic case can be less then the number that is needed in the relativistic case. In such cases, in order to obtain an irreducible multiplet, we eliminate the redundant auxiliary fields. This occurs, for instance, in the example of section 3.5.

In the next sections, we will illustrate this procedure by using it to re-derive various results on Newton–Cartan geometry and (super)gravity that have been obtained by other methods.

3.2 Newton–Cartan gravity

In the following, we illustrate the limiting procedure in one of its simplest manifestations, by deriving the Newton–Cartan theory of gravity of [80] from the vielbein formulation of general relativity. The starting point was conveniently summarized in the last chapter in section 2.4 where we discussed general relativity from precisely the point of view that we need here. We will pay attention to how we derive the transformation rules of the non-relativistic gauge-fields and how we derive constraints that those background fields have to obey.

As should be clear from the general discussion in the last section, the limiting procedure is based on the Inönü–Wigner contraction of symmetry algebras. In particular, now we need to look at the non-relativistic contraction of the Poincaré algebra that yields the Bargmann algebra [96,98]. To obtain the Galilei algebra with a central extension we need to add an extra generator to the Poincaré algebra. The number of generators stays the same in the contraction and the Poincaré and Galilei algebras already have the same number of generators. Hence, to get an extra generator we need to add it already to the Poincaré algebra. In the bosonic case, the only way to do so is to trivially extend the Poincaré algebra with an extra $U(1)$ symmetry that is generated by the new operator $\mathcal{Z}$. The contraction then consists of rescaling the generators of the Poincaré algebra and $\mathcal{Z}$ in the following way:

$$\hat{P}_0 \to \frac{1}{2\omega}\,\tilde{H} + \omega\,\tilde{Z}\,, \qquad \mathcal{Z} \to \frac{1}{2\omega}\,\tilde{H} - \omega\,\tilde{Z}\,, \qquad M_{a0} \to \omega\,\tilde{G}_a\,. \tag{3.2}$$

The spatial translations $\tilde{P}_a$ and rotations, which we will denote by $M_{ab} = \tilde{J}_{ab}$, are not rescaled. Calculating the commutators of $\tilde{H}$, $\tilde{P}_a$, $\tilde{G}_a$, $\tilde{J}_{ab}$ and $\tilde{Z}$, re-expressing everything in terms of the relativistic generators and taking $\omega \to \infty$ leads to the non-vanishing commutators of the Bargmann algebra given in (2.45) and (2.46).

In [80] Newton–Cartan gravity was obtained by gauging the Bargmann algebra. Our aim is to derive the same formulation of Newton–Cartan gravity via a contraction of the relativistic background fields $E_\mu{}^A$ and $\Omega_\mu{}^{AB}(E)$ that follow from gauging the Poincaré algebra as we explained in chapter 2.

Let us mention at this point that the contraction (3.2) is not unique. Rather, it corresponds a particular non-relativistic "point-particle limit" where time is singled out as a special direction. One can define more general p-brane limits where one time and p spatial directions are singled out, see e.g. [62,63,65,101]. For the purpose of this thesis we will only concentrate on the "particle limit".

In this first instance of the limiting procedure we shall be very careful about which quantities are non-relativistic ones, i.e. which are obtained in the limit $\omega \to \infty$, and which ones are merely redefined relativistic ones, i.e. with finite contraction parameter ω. We will denote the latter ones with tildes and drop those tildes as we let $\omega \to \infty$.

The main idea of the limiting procedure is to extend the contraction (3.2) to the gauge-fields. Hence, we will take the first two formulas in (3.2) as a motivation for the expansion of our relativistic background fields in terms of the Newton–Cartan ones. The result is the following ansatz for the relativistic vielbein:

$$E_\mu{}^A = \delta_0^A \left(\omega\,\tilde\tau_\mu + \frac{1}{2\omega}\,\tilde m_\mu\right) + \delta_a^A\,\tilde e_\mu{}^a\,. \tag{3.3}$$

Ultimately, $\tilde\tau_\mu$, $\tilde e_\mu{}^a$ and $\tilde m_\mu$ will become the background fields of Newton–Cartan gravity. Then we shall also use projective inverses of $\tilde\tau_\mu$ and $\tilde e_\mu{}^a$, which we denote by $\tilde\tau^\mu$ and $\tilde e^\mu{}_a$, that are defined as follows:

$$\begin{aligned} &\tilde e^\mu{}_a \tilde e_\mu{}^b = \delta_a^b\,, && \tilde\tau^\mu\tilde\tau_\mu = 1\,, && \tilde\tau^\mu \tilde e_\mu{}^a = 0\,, \\ &\tilde\tau_\mu \tilde e^\mu{}_a = 0\,, && \tilde e^\rho{}_a \tilde e_\mu{}^a = \delta_\mu^\rho - \tilde\tau_\mu\tilde\tau^\rho\,. \end{aligned} \tag{3.4}$$

From here we can immediately deduce the following expansion of the relativistic inverse vielbein in terms of the non-relativistic components:

$$E^\mu{}_A = \delta_A^a\left[\tilde e^\mu{}_a + \mathcal{O}\Big(\frac{1}{\omega^2}\Big)\right] + \frac{1}{\omega}\,\delta_A^0\left[\tilde\tau^\mu + \mathcal{O}\Big(\frac{1}{\omega^2}\Big)\right]. \tag{3.5}$$

Note that we have only explicitely given the terms of leading order in ω. There are in principle an infinite number of corrections which, however, will not be needed in the following as they do not contribute in the limit $\omega \to \infty$.

Extending the contraction (3.2) to the gauge-field M_μ that is related to the $U(1)$ generator $\mathcal{Z}$, we find the expansion

$$M_\mu = \omega\,\tilde\tau_\mu - \frac{1}{2\omega}\,\tilde m_\mu\,. \tag{3.6}$$

Since this gauge-field did not take part in the discussion on the kinematics of general relativity in chapter 2 we shall collect some more important formulas concerning it here. The transformation rule of M_μ is given by

$$\delta M_\mu = \xi^\rho\partial_\rho M_\mu + M_\rho\partial_\mu\xi^\rho + \partial_\mu\Lambda\,, \tag{3.7}$$

where Λ is the parameter of the spurious $U(1)$ symmetry. The covariant curvature of M_μ reads

$$F_{\mu\nu}(M) = 2\,\partial_{[\mu}M_{\nu]}\,. \tag{3.8}$$

In the following we will need to impose constraints on this curvature. In the bosonic case that we discuss in this section we have two options:

$$\text{dynamical}: \quad F_{\mu\nu}(M) = 0\,, \tag{3.9}$$

$$\text{kinematical}: \quad \tilde{e}^\mu{}_a \tilde{e}^\nu{}_b\, F_{\mu\nu}(M) = \tilde{F}_{ab}(M) = 0\,. \tag{3.10}$$

If we are interested in the dynamical theory, we set the full curvature $F_{\mu\nu}(M)$ to zero. Then M_μ is a pure gauge-field and does not propagate, so we do not add any degrees of freedom to general relativity. However, we will see that, in order to take the non-relativistic limit in a consistent manner, we need not impose such a strict constraint. It turns out that if we are only interested in deriving the correct kinematics, it is sufficient to require that only the spatial projection of the curvature vanishes, i.e. we can opt for the "kinematical" constraint (3.10).

In principle we can extend the contraction (3.2) to find the rescaling of the dependent gauge-fields. Their dependence on other independent gauge-fields is a consequence of constraints. We might consider taking the non-relativistic limit of those un-conventional constraints too. However, this should not lead to any new information, provided we are able to take the limit in the expression for the dependent gauge-fields in a consistent way. Having used the relativistic torsion constraint to solve for the relativistic spin-connection, we cannot obtain new information by taking its limit. It still remains an identity that we use to solve for $\Omega_\mu{}^{AB}(E)$.

It turns out that in order to take the limit of the explicit expression (2.30) we need an extra constraint to replace a term that diverges in the limit $\omega \to \infty$ by a finite one. To this end, we have to make use of the extra gauge-field M_μ and impose one of the two constraints (3.9) and (3.10) on its curvature (3.8). Now we can define the non-relativistic spin- and boost-connection by

$$\Omega_\mu{}^{ab}(E) + \frac{\omega}{2}\,\tilde{\tau}_\mu \tilde{e}^{\rho a}\tilde{e}^{\nu b}\, F_{\rho\nu}(M) = \tilde{\omega}_\mu{}^{ab}(\tilde{e},\tilde{\tau},\tilde{m}) + \mathcal{O}\Big(\frac{1}{\omega^2}\Big)\,, \tag{3.11}$$

$$\Omega_\mu{}^{0a}(E) - \frac{1}{2}\,\tilde{e}_\mu{}^b \tilde{e}^{\rho b}\tilde{e}^{\nu a}\, F_{\rho\nu}(M) = \frac{1}{\omega}\,\tilde{\omega}_\mu{}^{a}(\tilde{e},\tilde{\tau},\tilde{m}) + \mathcal{O}\Big(\frac{1}{\omega^3}\Big)\,. \tag{3.12}$$

When inserting the redefinitions (3.3) and (3.6) in (2.30) we find

$$\tilde{\omega}_\mu{}^{ab}(\tilde{e},\tilde{\tau},\tilde{m}) = -2\,\tilde{e}^{\nu[a}\partial_{[\mu}\tilde{e}_{\nu]}{}^{b]} + \tilde{e}_\mu{}^c \tilde{e}^{\rho a}\tilde{e}^{\nu b}\partial_{[\rho}\tilde{e}_{\nu]}{}^c - \tilde{\tau}_\mu \tilde{e}^{\rho a}\tilde{e}^{\nu b}\partial_{[\rho}\tilde{m}_{\nu]}\,, \tag{3.13}$$

$$\tilde{\omega}_\mu{}^{a}(\tilde{e},\tilde{\tau},\tilde{m}) = \tilde{\tau}^\nu\partial_{[\mu}\tilde{e}_{\nu]}{}^a + \tilde{e}_\mu{}^b \tilde{e}^{\rho a}\tilde{\tau}^\nu\partial_{[\rho}\tilde{e}_{\nu]}{}^b + \tilde{e}^{\nu a}\partial_{[\mu}\tilde{m}_{\nu]} - \tilde{\tau}_\mu \tilde{e}^{\rho a}\tilde{\tau}^\nu\partial_{[\rho}\tilde{m}_{\nu]}\,. \tag{3.14}$$

A few comments are in order. First, the subleading contributions in the definitions (3.11) and (3.12) are due to the subleading terms in the expansion of the relativistic inverse vielbein (3.5). Secondly, note that while the constraint $F_{\mu\nu}(M)$ is trivial in the relativistic case, by adding it to the spin-connection we are effectively splitting it up into different parts. The leading part will cancel the leading (divergent) contribution of $\Omega_\mu{}^{AB}(E)$, while the subleading part remains as a finite contribution to the non-relativistic connections. Hence, we need to be very careful and check if the definitions (3.11) and (3.12) are indeed consistent when we take the limit $\omega \to \infty$.

Moreover, we have to ensure that the constraints (3.9) and (3.10) remain valid in the limit, i.e. we shall investigate their limits too.

Note that the scaling of all gauge-fields (3.3), (3.6), (3.11) and (3.12) are such that the sum of the products of the gauge-fields with their respective symmetry generator remains invariant, up to subleading terms in ω. One thus has

$$\begin{aligned}&\hat{P}_A\, E_\mu{}^A + \mathcal{Z}\, M_\mu + M_{AB}\, \Omega_\mu{}^{AB}(E) = \\ &\qquad \tilde{P}_a\, \tilde{e}_\mu{}^a + \tilde{H}\, \tilde{\tau}_\mu + \tilde{Z}\, \tilde{m}_\mu + \tilde{J}_{ab}\, \tilde{\omega}_\mu{}^{ab}(\tilde{e},\tilde{\tau},\tilde{m}) - 2\, \tilde{G}_a\, \tilde{\omega}_\mu{}^a(\tilde{e},\tilde{\tau},\tilde{m}) + \mathcal{O}\Big(\frac{1}{\omega^2}\Big)\,.\end{aligned} \tag{3.15}$$

The factor -2 in the last term is due to our definition of $\tilde{\omega}_\mu{}^a(\tilde{e},\tilde{\tau},\tilde{m})$, see (3.12). Note also the appearance of the subleading terms in ω that we referred to after (3.1). They are also due to the definitions (3.11) and (3.12).

Now we proceed to take the limit $\omega \to \infty$ and drop the tildes on all fields. We start by calculating the transformation rules of the Newton–Cartan background fields τ_μ, $e_\mu{}^a$ and m_μ, by applying (3.3) and (3.6) to (2.20). To do so, we express the new fields in terms of the old ones, i.e.

$$\tilde{\tau}_\mu = \frac{1}{2\omega}\left(E_\mu{}^0 + M_\mu\right), \qquad \tilde{m}_\mu = \omega\left(E_\mu{}^0 - M_\mu\right). \tag{3.16}$$

Then it is straightforward to get

$$\begin{aligned}\delta\tau_\mu &= 0\,,\\ \delta e_\mu{}^a &= \lambda^a{}_b\, e_\mu{}^b + \lambda^a \tau_\mu\,,\\ \delta m_\mu &= \partial_\mu\sigma + \lambda_a\, e_\mu{}^a\,,\end{aligned} \tag{3.17}$$

where we defined

$$\lambda^a = \omega\, \lambda^a{}_0\,, \qquad \Lambda = -\frac{\sigma}{\omega}\,. \tag{3.18}$$

All fields transform also under diffeomorphisms in the usual way. The transformations of the spin-connections can be found as well:

$$\delta\omega_\mu{}^{ab}(e,\tau,m) = \partial_\mu\lambda^{ab} + 2\,\lambda^{[a}{}_c\,\omega_\mu{}^{cb]}\,, \tag{3.19}$$

$$\delta\omega_\mu{}^a(e,\tau,m) = \partial_\mu\lambda^a + \lambda^a{}_b\,\omega_\mu{}^b - \omega_\mu{}^a{}_c\,\lambda^c\,. \tag{3.20}$$

The transformation rules (3.17)–(3.20) agree with those found in [80] by gauging the Bargmann algebra.

Next, we take a look at the constraints of the non-relativistic theory. The expressions (3.13) and (3.14) (without the tildes) agree with those of [80]. In their analysis these expressions were obtained by solving the constraints

$$\begin{aligned} R_{\mu\nu}{}^a(P) &= 2\,\partial_{[\mu}e_{\nu]}{}^a - 2\,\omega_{[\mu}{}^{ab}e_{\nu]}{}^b - 2\,\omega_{[\mu}{}^a\tau_{\nu]} = 0\,, \\ R_{\mu\nu}(Z) &= 2\,\partial_{[\mu}m_{\nu]} - 2\,\omega_{[\mu}{}^a e_{\nu]}{}^a = 0\,. \end{aligned} \tag{3.21}$$

Hence, these are also satisfied in our case. Next, we look at the implications of (3.9) and (3.10). Here, we have two possibilities. The "dynamical" constraint (3.9) leads to

$$R_{\mu\nu}(H) = 2\,\partial_{[\mu}\tau_{\nu]} = 0\,, \tag{3.22}$$

a constraint that was also used in [80]. The other option is to set only the spatial projection to zero, i.e.

$$R_{ab}(H) = 2\,e^\mu{}_a e^\nu{}_b\,\partial_{[\mu}\tau_{\nu]} = 0\,. \tag{3.23}$$

Note that this could be solved by

$$2\,\partial_{[\mu}\tau_{\nu]} = b_{[\mu}\tau_{\nu]}\,, \tag{3.24}$$

where b_μ is arbitrary. This equation resembles the curvature constraint of the twistless torsional Newton–Cartan theory [29]. In both cases, (3.22) or (3.23), the variation of this constraint does not lead to any further restriction as τ_μ is invariant under gauge transformations.

Further constraints stem from the relativistic Bianchi identity. Using the inverse vielbein (3.5) and the expansion

$$R_{\mu\nu}{}^{AB}(\Omega) = \delta^A_a\delta^B_b\,\tilde{R}_{\mu\nu}{}^{ab}(\tilde{J}) - \frac{1}{\omega}\,\delta^A_a\delta^B_0\,\tilde{R}_{\mu\nu}{}^a(\tilde{G}) + \frac{1}{\omega}\,\delta^A_0\delta^B_b\,\tilde{R}_{\mu\nu}{}^b(\tilde{G})\,, \tag{3.25}$$

in (2.34) we obtain the non-relativistic Bianchi identities

$$R_{[\mu\nu}{}^a(G)\,e_{\rho]}{}^a = 0\,, \qquad\qquad R_{[\mu\nu}{}^{ab}(J)\,e_{\rho]}{}^a + R_{[\mu\nu}{}^b(G)\,\tau_{\rho]} = 0\,, \tag{3.26}$$

for the following curvatures of spatial rotations and Galilean boosts:

$$\begin{aligned} R_{\mu\nu}{}^{ab}(J) &= 2\,\partial_{[\mu}\omega_{\nu]}{}^{ab} - 2\,\omega_{[\mu}{}^a{}_c\omega_{\nu]}{}^{cb}\,, \\ R_{\mu\nu}{}^a(G) &= 2\,\partial_{[\mu}\omega_{\nu]}{}^a - 2\,\omega_{[\mu}{}^{ab}\omega_{\nu]}{}^b\,. \end{aligned} \tag{3.27}$$

This concludes the search for constraints of the kinematical theory. As we are interested primarily in the kinematics we did not derive equations of motion yet. We may impose equations of motion on the Newton–Cartan background fields, for example by taking the non-relativistic limit of the Einstein equation. This will lead to the exact theory presented in [80]. The limit yields the equations

$$R_{\mu\nu}{}^{ab}(J)\, e^{\mu}{}_{a} = 0\,, \qquad\qquad R_{\mu\nu}{}^{a}(G)\, e^{\mu}{}_{a} = 0\,. \tag{3.28}$$

If we choose to solve the first equation by setting $R_{\mu\nu}{}^{ab}(J) = 0$, which will be needed in the supersymmetric on-shell case, it follows from the Bianchi identities that the only real equation of motion left is

$$R_{0a}{}^{a}(G) = 0\,, \tag{3.29}$$

all other (components of) curvatures being zero due to conventional constraints (3.21), one of the foliation constraints (3.22) or (3.23), or the choice $R_{\mu\nu}{}^{ab}(J) = 0$.

The equation (3.29), when considering observers in a Galilean frame with only time-dependent acceleration, reduces to the Poisson equation for the Newton potential, see [77,80]. In general, the right-hand-side is non-zero, but here it is because we took the limit of the Einstein equation without a source term. However, as we mentioned in the introduction, in the special case of three space-time dimensions, we cannot write down a relativistic source term that would give rise to non-trivial sources after taking the non-relativistic limit. Therefore, in three dimensions the limit of the Einstein equation will always be (3.29) unless we add a source by hand. Hence, there is no non-trivial, normalizable solution for the Newton potential in three dimensions. In this sense, we do not find any contradiction with the fact there should be no force between point-particles in "ordinary" three-dimensional non-relativistic gravity.

Finally, let us try to understand, from the point of this limiting procedure, why there is no action for Newton–Cartan gravity. What happens if we simply take the limit of the Einstein–Hilbert action (2.43)? The determinant of the relativistic vielbein will scale with one power of ω and the Ricci scalar expands to

$$R(\Omega) = R_{ab}{}^{ab}(J) - \frac{2}{\omega^2}\, R_{0a}{}^{a}(G) + \mathcal{O}\Big(\frac{1}{\omega^3}\Big)\,. \tag{3.30}$$

Obviously the equations of motion (3.28) set this to zero. On the other hand, it is not obvious how an action consisting of only the leading term $R_{ab}{}^{ab}(J)$ would lead to both equations of (3.28). Moreover, due to the extra factor of ω from the determinant this term would diverge.

Previously we dealt with such divergences by imposing curvature constraints. However, none of those constraints was a limitation on $R_{\mu\nu}{}^{ab}(J)$. It seems quite unnatural to impose yet another, additional, constraint just to be able to get a finite action. An action that does not even lead to the correct equations of motion. While this is not tight enough to serve as a no-go argument, it certainly is yet another indication that there is no action for Newton–Cartan gravity, at least not at the two-derivative level. Actions which consist of higher-derivative invariants have been proposed in [20,103]. However, those papers do not derive the equations of motion that constrain the background fields. Therefore, their connection to the theory of

Newton–Cartan gravity as we discussed it here is not completely clear to us.

This concludes our first check of the non-relativistic limiting procedure. In the following, we are going to investigate the limits of several supergravity theories. Therefore, from now on we shall restrict ourselves to theories in three space-time dimensions. We give a short exposition of non-relativistic $\mathcal{N} = 1$ "supergravity" next, before we (re-)derive the on- and off-shell formulations of three-dimensional Newton–Cartan supergravity theory.

3.3 Non-relativistic $\mathcal{N} = 1$ supergravity

The discussion here and in the following sections parallels the one of the previous section. We will therefore skip most intermediate steps, where the contraction parameter ω is finite, and we will only focus on the results obtained in the $\omega \to \infty$ limit. Here and in the following, we will therefore no longer resort to the notation using tildes, to denote quantities at finite ω.

In this section we briefly review the argument of why we consider non-relativistic $\mathcal{N} = 2$ supergravities as the minimal case, in place of the technically simpler $\mathcal{N} = 1$ versions. To this end, we look at the non-relativistic contraction of the $\mathcal{N} = 1$ Poincaré superalgebra, (2.15) and (2.65). Again, we add an extra $U(1)$ to get the central charge of the Bargmann superalgebra. The scaling of the bosonic generators is given in (3.2) and the fermionic supercharge Q must scale as

$$Q = \sqrt{\omega}\, Q_- \,. \tag{3.31}$$

In addition to the bosonic commutation relations of the Bargmann algebra we get

$$[J_{ab}, Q_-] = -\frac{1}{2}\,\gamma_{ab} Q_- \,, \qquad\qquad \{Q_-, Q_-\} = \gamma^0 C^{-1}\, Z \,. \tag{3.32}$$

Notice that the anti-commutator of two supercharges leads to a central charge transformation but no time- or space-translation. Normally, the anti-commutator of two supercharges leads to a diffeomorphism, see e.g. (2.65) or (2.66). So while the $\mathcal{N} = 1$ extended Bargmann algebra is perfectly well-defined, we rather not call it a superalgebra in the usual sense, simply because the anti-commutator of two supercharges does not lead to diffeomorphisms.

We can demonstrate this too by looking at the explicit transformation rules of the background fields τ_μ, ${e_\mu}^a$, m_μ and $\psi_{\mu-}$, which we derive from (2.72) using the limiting procedure. This shall also serve as a first easy example of the limiting procedure in the supersymmetric case. It also comprises a new result that has not been derived in the literature before, unlike the two $\mathcal{N} = 2$ formulations that follow in section 3.4 and 3.5.

We must now specify the transformation of M_μ under supersymmetry. We use

$$\delta M_\mu = 0 \,, \tag{3.33}$$

and in order not to upset the counting of degrees of freedom we set the curvature of M_μ to zero

$$\hat{F}_{\mu\nu}(M) = 2\,\partial_{[\mu}M_{\nu]} = 0\,. \tag{3.34}$$

This coincides of course with the bosonic constraint on M_μ that we imposed in the last section 3.2, see eq. (3.8). Then M_μ is pure gauge and does not propagate any degrees of freedom. Otherwise we would, due to the addition of the extra field M_μ, have more bosonic degrees of freedom than fermionic ones in the multiplet (2.72).

Furthermore, the transformation rule (3.33) makes sure that the constraint (3.34) does not lead to any additional conditions on the gauge-fields as its supersymmetry transformation does not lead to more constraints. Also, the supersymmetry algebra (2.77) is realized on M_μ through (3.34):

$$\left[\delta_1, \delta_2\right] M_\mu = 0 = \tilde{\xi}^\rho \partial_\rho M_\mu + M_\rho \partial_\mu \tilde{\xi}^\rho - \partial_\mu\left(\tilde{\xi}^\rho M_\rho\right) = \tilde{\xi}^\rho \hat{F}_{\rho\mu}(M)\,. \tag{3.35}$$

The last term in the soft-algebra on M_μ is not present in (2.77) because we did not consider the additional $U(1)$ generator then. It should be clear, by comparing various soft-algebras in this work, that we naturally expect it to take this form.

This concludes our discussion of the relativistic starting point. Next, we will take the non-relativistic limit to derive transformation rules and constraints. First, we need to define how to scale the spinors. Following our general rules on how to take the limit we use the contraction (3.3) and (3.6), plus the new rule

$$\Psi_\mu = \frac{1}{\sqrt{\omega}}\,\psi_{\mu-}\,, \tag{3.36}$$

which follows promptly from the formula (3.31). Furthermore, we need to use

$$S \to \frac{1}{\omega}\,S\,, \tag{3.37}$$

when taking the limit of the off-shell multiplet (2.72). Then, we find the transformation rules of the non-relativistic $\mathcal{N} = 1$ off-shell multiplet

$$\begin{aligned}
\delta\tau_\mu &= 0\,,\\
\delta e_\mu{}^a &= 0\,,\\
\delta m_\mu &= \frac{1}{2}\,\bar{\epsilon}_-\gamma^0\psi_{\mu-}\,,\\
\delta\psi_{\mu-} &= D_\mu\epsilon_- + \frac{1}{2}\,S\,\tau_\mu\,\gamma_0\epsilon_-\,,\\
\delta S &= \frac{1}{8}\,\bar{\epsilon}_-\gamma^{ab}\hat{\psi}_{ab-}\,.
\end{aligned} \tag{3.38}$$

The transformation rules of the dependent spin-connections, which are obtained from (3.11) and (3.12) and take the expressions

$$\begin{aligned}\omega_\mu{}^{ab}(e,\tau,m,\psi) &= -2\,e^{\nu[a}\partial_{[\mu}e_{\nu]}{}^{b]} + e_\mu{}^c e^{\rho a}e^{\nu b}\partial_{[\rho}e_{\nu]}{}^c \\ &\quad - \tau_\mu e^{\rho a}e^{\nu b}\left(\partial_{[\rho}m_{\nu]} - \frac{1}{4}\,\bar\psi_{[\rho-}\gamma^0\psi_{\nu]-}\right), \end{aligned} \tag{3.39}$$

$$\begin{aligned}\omega_\mu{}^a(e,\tau,m,\psi) &= \tau^\nu\partial_{[\mu}e_{\nu]}{}^a + e_{\mu b}e^{\rho a}\tau^\nu\partial_{[\rho}e_{\nu]}{}^b + e^{\nu a}\left(\partial_{[\mu}m_{\nu]} - \frac{1}{4}\,\bar\psi_{[\mu-}\gamma^0\psi_{\nu]-}\right) \\ &\quad - \tau_\mu e^{\rho a}\tau^\nu\left(\partial_{[\rho}m_{\nu]} - \frac{1}{4}\,\bar\psi_{[\rho-}\gamma^0\psi_{\nu]-}\right), \end{aligned} \tag{3.40}$$

are given by

$$\delta_Q\omega_\mu{}^{ab}(e,\tau,m,\psi) = -\frac{1}{4}\,\tau_\mu\,\bar\epsilon_-\gamma^0\hat\psi^{ab}{}_-\,, \tag{3.41}$$

$$\delta_Q\omega_\mu{}^a(e,\tau,m,\psi) = \frac{1}{4}\,\bar\epsilon_-\gamma^0\hat\psi_\mu{}^a{}_- - \frac{1}{4}\,\tau_\mu\,\bar\epsilon_-\gamma^0\hat\psi^a{}_{0-}\,. \tag{3.42}$$

Only the first one is needed when calculating the commutator algebra. The curvature of the gravitino is given by

$$\hat\psi_{\mu\nu-} = 2\,\partial_{[\mu}\psi_{\nu]-} - \frac{1}{2}\,\omega_{[\mu}{}^{ab}\gamma_{ab}\psi_{\nu]-} - \gamma_0\psi_{[\mu-}\,\tau_{\nu]}\,S\,. \tag{3.43}$$

One can check explicitely that in this case the algebra closes using only central charge transformations,

$$\left[\delta(Q_1),\delta(Q_2)\right] = \delta_Z\left(\tilde\sigma\right), \tag{3.44}$$

with the parameter

$$\tilde\sigma = \frac{1}{2}\,\epsilon_{2-}\gamma^0\epsilon_{1-}\,. \tag{3.45}$$

In contrast to all other realizations of superalgebras on supergravity multiplets, it is not given by a algebra with diffeomorphisms on the right-hand-side.

One can go on-shell by setting

$$S = 0\,, \qquad\qquad \hat\psi_{ab-} = 0\,. \tag{3.46}$$

This can also be inferred by taking the limit of the on-shell multiplet. From (3.46) we can deduce

$$R_{ab}{}^{cd}(J) = 0\,. \tag{3.47}$$

There are no further constraints from varying this one because the spin-connection $\omega_\mu{}^{ab}(e,\tau,m,\psi)$ now is invariant under supersymmetry transformations.

The short analysis of this section served two purposes. It gives a more detailed argument for why we use only $\mathcal{N} = 2$ algebras in the following. On the other hand, it hints towards the kind of supergravity theories that we will be dealing with. The formulas (3.38)–(3.42) will not only receive additional contributions due to the extra supersymmetry, but their $\epsilon_-/\psi_{\mu-}$ terms will also change slightly, mainly because the transformation rule of M_μ, (3.33) will become non-trivial.

The minus index for the non-relativistic gravitini and the parameter ϵ_- was chosen with the results of the next sections and chapters in mind. There we will use two spinors ϵ_+ and ϵ_-. The results of this section can be retrieved by setting the parameter ϵ_+ to zero in the later sections. (One also has to rescale the remaining spinor by a factor of $\sqrt{2}$.) Thereby, as we will see explicitely, one also eliminates the diffeomorphism parameter $\tilde{\xi}^\mu$ in the algebra because this parameter always depends on ϵ_+. Note also that $\hat{\psi}_{ab-} = 0$ becomes an un-conventional constraint in the later theories. It follows from the foliation constraint $\hat{R}(H) = 0$ by a Q_+ transformation. Since there is no Q_+ symmetry in the current case we need not impose this constraint here though.

3.4 On-shell Newton–Cartan supergravity

In this section we re-derive the Newton–Cartan supergravity theory put forward in [77]. This is an on-shell theory, hence we shall start from the three-dimensional relativistic on-shell multiplet. In order to determine how we are going to take the limit, i.e. impose the scalings on the (independent) gauge-fields of the relativistic supergravity multiplet, we will review how the $\mathcal{N} = 2$ Bargmann superalgebra is obtained by contracting the $\mathcal{N} = 2$ Poincaré superalgebra. This will tell us in particular how to rescale the gravitini $\psi_{\mu\pm}$.

We start from the three-dimensional $\mathcal{N} = 2$ Poincaré superalgebra with central extension $\mathcal{Z}$, (2.15) and (2.66). Then we define

$$Q_\pm = \frac{1}{\sqrt{2}} \left(Q_1 \pm \gamma_0 Q_2 \right), \tag{3.48}$$

and split the three-dimensional flat indices A, B into time-like and space-like indices $\{0, a\}$. Furthermore, we set $M_{ab} = J_{ab}$ to describe purely spatial rotations. The motivation for choosing these combinations of the relativistic spinors, eq. (3.48), can be seen in the non-relativistic algebra (and later on in the transformation rules of the gravitini). It leads to particularly simple transformations of the spinors under boosts. The contraction consists of (3.2) and the following rescaling of the fermionic generators:

$$Q_- \to \sqrt{\omega}\, Q_-, \qquad Q_+ \to \frac{1}{\sqrt{\omega}}\, Q_+ . \tag{3.49}$$

Taking the limit $\omega \to \infty$ leads to the supersymmetric extension of the Bargmann algebra. We get the following non-vanishing commutation relations:

$$\begin{aligned} [J_{ab}, P_c] &= -2\,\delta_{c[a}P_{b]}\,, & [J_{ab}, G_c] &= -2\,\delta_{c[a}G_{b]}\,, \\ [G_a, H] &= -P_a\,, & [G_a, P_b] &= -\delta_{ab}\,Z\,, \\ [J_{ab}, Q_\pm] &= -\frac{1}{2}\,\gamma_{ab}Q_\pm\,, & [G_a, Q_+] &= -\frac{1}{2}\,\gamma_{a0}Q_-\,, \\ \{Q_+, Q_+\} &= -\gamma^0 C^{-1}\,H\,, & \{Q_+, Q_-\} &= -\gamma^a C^{-1}\,P_a\,, \\ \{Q_-, Q_-\} &= -2\,\gamma^0 C^{-1}\,Z\,. \end{aligned} \tag{3.50}$$

The bosonic part of the algebra is the usual Bargmann algebra, see (2.45) and (2.46). Note also that the spatial rotations are abelian because we have only two space dimensions.

The difference of a factor two in the anti-commutator of two Q_- supercharges compared to the $\mathcal{N} = 1$ algebra (3.32) is due to the extra central extension $\mathcal{Z}$ in the relativistic algebra (2.66). Moreover, in the $\mathcal{N} = 1$ case there is no Jacobi identity that would fix this factor, i.e. relate it to the commutator of translations P_a with boosts G_a. In this sense the extension in the P_a–G_b commutator is somewhat "independent" of the Z in the Q_-–Q_- anti-commutator. This is not true anymore for the $\mathcal{N} = 2$ algebra (3.50) because of the non-vanishing commutator between boosts G_a and Q_+ supersymmetry.

Lets move on to the contraction of the supergravity theory. We first need to collect the transformation rules and constraints of our relativistic starting point. Again, we add the central charge gauge-field M_μ. Here is where this field makes its first physical appearance: it is the gauge field of the central charge operator of the $\mathcal{N} = 2$ algebra. Its transformation rule under supersymmetry is now given by

$$\delta M_\mu = \frac{1}{2}\,\varepsilon^{ij}\,\bar{\eta}_i \Psi_{\mu j}\,, \tag{3.51}$$

and the anti-commutator of two supersymmetries ($[\delta^1_i, \delta^2_j]$) on M_μ leads to a central charge transformation with parameter $\Lambda = \frac{1}{2}\,\varepsilon^{ij}\,\bar{\eta}^1_i \eta^2_j$. In order not to add any bosonic degrees of freedom to our theory we need to set the curvature of M_μ to zero:

$$\hat{F}_{\mu\nu}(M) = 2\,\partial_{[\mu}M_{\nu]} - \varepsilon^{ij}\,\bar{\Psi}_{[\mu i}\Psi_{\nu]j} = 0\,. \tag{3.52}$$

This agrees with our equation of motion as the supersymmetry variation of (3.52) leads to $\hat{\Psi}_{\mu\nu i} = 0$. Like in the bosonic case, this constraint, which is the supercovariant version of (3.9), will allow us to obtain finite expressions for the non-relativistic spin-connections from the relativistic one. Moreover, we can show that the full set of equations of motion is given by

$$\hat{F}_{\mu\nu}(M) = 0 \quad \rightarrow \quad \hat{\Psi}_{\mu\nu i} = 0 \quad \rightarrow \quad \hat{R}_{\mu\nu}{}^{AB}(\Omega) = 0\,, \tag{3.53}$$

using supersymmetry variations.

So much for the relativistic supergravity background. In the following we shall see what are the non-relativistic limits of the transformations (2.80), (2.81) and the constraints (3.51) and (3.53).

Now we extend the algebra contraction to the fields of the on-shell multiplet. For the bosonic fields this entails the redefinitions involving ω that were given in the previous section. Next, we need to define how we are going to redefine the spinors. This follows from the way we contract the generators of the three-dimensional $\mathcal{N} = 2$ Poincaré superalgebra to get the Bargmann superalgebra, see (3.48). Hence, we define new spinors

$$\Psi_\pm = \frac{1}{\sqrt{2}}\Big(\Psi_1 \pm \gamma_0 \Psi_2\Big)\,, \tag{3.54}$$

and we will scale the two projections $\Psi_\pm$ differently:

$$\Psi_{\mu+} = \sqrt{\omega}\,\psi_{\mu+}\,, \qquad \eta_+ = \sqrt{\omega}\,\epsilon_+\,, \tag{3.55}$$

$$\Psi_{\mu-} = \frac{1}{\sqrt{\omega}}\,\psi_{\mu-}\,, \qquad \eta_- = \frac{1}{\sqrt{\omega}}\,\epsilon_-\,. \tag{3.56}$$

Then the next, usually trivial, step it to get the transformation rules. Since there are no additional auxiliary fields in this formulation (whose scaling we should take care of at this point) it is straightforward to get

$$\begin{aligned} \delta\tau_\mu &= \frac{1}{2}\,\bar{\epsilon}_+\gamma^0\psi_{\mu+}\,, \\ \delta e_\mu{}^a &= \frac{1}{2}\,\bar{\epsilon}_+\gamma^a\psi_{\mu-} + \frac{1}{2}\,\bar{\epsilon}_-\gamma^a\psi_{\mu+}\,, \\ \delta m_\mu &= \bar{\epsilon}_-\gamma^0\psi_{\mu-}\,, \end{aligned} \tag{3.57}$$

and

$$\begin{aligned} \delta\psi_{\mu+} &= \partial_\mu\epsilon_+ - \frac{1}{4}\,\omega_\mu{}^{ab}\gamma_{ab}\epsilon_+\,, \\ \delta\psi_{\mu-} &= \partial_\mu\epsilon_- - \frac{1}{4}\,\omega_\mu{}^{ab}\gamma_{ab}\epsilon_- + \frac{1}{2}\,\omega_\mu{}^a\gamma_{a0}\epsilon_+\,. \end{aligned} \tag{3.58}$$

It is understood that the spin-connections $\omega_\mu{}^{ab}$ and $\omega_\mu{}^a$ in (3.58) are dependent, i.e. $\omega_\mu{}^{ab} = \omega_\mu{}^{ab}(e, \tau, m, \psi_\pm)$ and $\omega_\mu{}^a = \omega_\mu{}^a(e, \tau, m, \psi_\pm)$. The bosonic symmetries act on the spinors in the following way:

$$\begin{aligned}\delta\psi_{\mu+} &= \frac{1}{4}\,\lambda^{ab}\gamma_{ab}\psi_{\mu+}\,,\\ \delta\psi_{\mu-} &= \frac{1}{4}\,\lambda^{ab}\gamma_{ab}\psi_{\mu-} - \frac{1}{2}\,\lambda^{a}\gamma_{a0}\psi_{\mu+}\,.\end{aligned} \tag{3.59}$$

This stems from the relativistic $\delta\Psi_\mu = \frac{1}{4}\,\lambda^{AB}\gamma_{AB}\Psi_\mu$.

The expressions for the spin-connections follow from (2.84) and (3.52). Like in the bosonic case discussed in section 3.2, we use (3.52) to replace terms that diverge in the limit $\omega \to \infty$, by terms with the expected ω-order. We get

$$\begin{aligned}\omega_\mu{}^{ab} &= -2e^{\nu[a}\big(\partial_{[\mu}e_{\nu]}{}^{b]} - \frac{1}{2}\,\bar\psi_{[\mu+}\gamma^{b]}\psi_{\nu]-}\big) + e_\mu{}^c e^{\rho a}e^{\nu b}\big(\partial_{[\rho}e_{\nu]}{}^c - \frac{1}{2}\,\bar\psi_{[\rho+}\gamma^c\psi_{\nu]-}\big)\\ &\quad - \tau_\mu e^{\rho a}e^{\nu b}\big(\partial_{[\rho}m_{\nu]} - \frac{1}{2}\,\bar\psi_{[\rho-}\gamma^0\psi_{\nu]-}\big)\,,\end{aligned} \tag{3.60}$$

and

$$\begin{aligned}\omega_\mu{}^{a} &= \tau^\nu\big(\partial_{[\mu}e_{\nu]}{}^{a} - \frac{1}{2}\,\bar\psi_{[\mu+}\gamma^{a}\psi_{\nu]-}\big) + e_{\mu b}e^{\rho a}\tau^\nu\big(\partial_{[\rho}e_{\nu]}{}^b - \frac{1}{2}\,\bar\psi_{[\rho+}\gamma^b\psi_{\nu]-}\big)\\ &\quad + e^{\nu a}\big(\partial_{[\mu}m_{\nu]} - \frac{1}{2}\,\bar\psi_{[\mu-}\gamma^0\psi_{\nu]-}\big) - \tau_\mu e^{\rho a}\tau^\nu\big(\partial_{[\rho}m_{\nu]} - \frac{1}{2}\,\bar\psi_{[\rho-}\gamma^0\psi_{\nu]-}\big)\,.\end{aligned} \tag{3.61}$$

These expressions for the spin- and boost-connection also solve the supercovariant curvature constraints

$$\begin{aligned}\hat R_{\mu\nu}{}^a(P) &= R_{\mu\nu}{}^a(P) - \bar\psi_{[\mu+}\gamma^a\psi_{\nu]-} = 0\,,\\ \hat R_{\mu\nu}(Z) &= R_{\mu\nu}(Z) - \bar\psi_{[\mu-}\gamma^0\psi_{\nu]-} = 0\,,\end{aligned} \tag{3.62}$$

identically. To check the consistency of the set of constraints, i.e. to derive the implications of (3.52) we need the transformations of the spin-connections under supersymmetry. The explicit expressions (3.60) and (3.61), or the constraints (3.62), can be used for that purpose. We find

$$\begin{aligned}\delta_Q\omega_\mu{}^{ab} &= \frac{1}{2}\,\bar\epsilon_+\gamma^{[b}\hat\psi^{a]}{}_{\mu-} + \frac{1}{4}\,e_{\mu c}\,\bar\epsilon_+\gamma^c\hat\psi^{ab}{}_- - \frac{1}{2}\,\tau_\mu\,\bar\epsilon_-\gamma^0\hat\psi^{ab}{}_-\\ &\quad + \frac{1}{2}\,\bar\epsilon_-\gamma^{[b}\hat\psi^{a]}{}_{\mu+} + \frac{1}{4}\,e_{\mu c}\,\bar\epsilon_-\gamma^c\hat\psi^{ab}{}_+\,,\end{aligned} \tag{3.63}$$

$$\begin{aligned}\delta_Q\omega_\mu{}^{a} &= \frac{1}{2}\,\bar\epsilon_-\gamma^0\hat\psi_\mu{}^a{}_- + \frac{1}{2}\,\tau_\mu\,\bar\epsilon_-\gamma^0\hat\psi_0{}^a{}_- + \frac{1}{4}\,e_{\mu b}\,\bar\epsilon_+\gamma^b\hat\psi^a{}_{0-} + \frac{1}{4}\,\bar\epsilon_+\gamma^a\hat\psi_{\mu0-}\\ &\quad + \frac{1}{4}\,e_{\mu b}\,\bar\epsilon_-\gamma^b\hat\psi^a{}_{0+} + \frac{1}{4}\,\bar\epsilon_-\gamma^a\hat\psi_{\mu0+}\,.\end{aligned} \tag{3.64}$$

Now we readily derive that the following set of constraints is generated under supersymmetry transformations:

$$\xrightarrow{Q_-} \quad \hat{\psi}_{ab-} = 0 \tag{3.65}$$

$$\hat{R}_{\mu\nu}(H) = 0 \quad \xrightarrow{Q_+} \quad \hat{\psi}_{\mu\nu+} = 0 \quad \xrightarrow{Q_+} \quad R_{\mu\nu}{}^{ab}(J) = 0 \tag{3.66}$$

$$\xrightarrow{Q_+} \quad \gamma^a \hat{\psi}_{a0-} = 0 \quad \xrightarrow{Q_+} \quad \hat{R}_{0a}{}^a(G) = 0\,. \tag{3.67}$$

Using the last equation of (3.66), the non-relativistic Bianchi identities reduce to

$$\hat{R}_{ab}{}^c(G) = 0\,, \qquad \hat{R}_{0[a}{}^{b]}(G) = 0\,, \tag{3.68}$$

which ensures e.g. that (3.65) does not lead to further constraints. The supercovariant curvature of Galilean boosts, which is used in the formulas above, is given by

$$\begin{aligned} \hat{R}_{\mu\nu}{}^a(G) = R_{\mu\nu}{}^a(G) - 2\,\bar{\psi}_{[\mu-}\gamma^0\hat{\psi}_{\nu]}{}^a{}_- \\ - \frac{1}{2}\,e_{[\nu}{}^b\,\bar{\psi}_{\mu]+}\gamma^b\hat{\psi}^a{}_{0-} - \frac{1}{2}\,\bar{\psi}_{[\mu+}\gamma^a\hat{\psi}_{\nu]0-}\,. \end{aligned} \tag{3.69}$$

The variation of the last constraint in (3.67) is quite involved and has also not been carried out in [77]. We will, however, show in section 3.5 that the full set of constraints (3.65)–(3.67) can be derived from an off-shell version of this multiplet. There it is easier to check the consistency of the whole set of constraints.

At this point we have finished the derivation of the three-dimensional on-shell Newton–Cartan supergravity constructed in [77], i.e. we obtained all constraints and transformation rules. The terminology "on-shell" stems from the fact that the constraints given in eq. (3.67) both can be interpreted as equations of motion for Newton–Cartan supergravity: the first constraint is necessary to obtain closure of the supersymmetry algebra, while the bosonic part of the second constraint is precisely the equation of motion of the bosonic Newton–Cartan gravity theory. Note, however, that to call some constraints "equation of motion" and others not is slightly ambiguous when talking about Newton–Cartan (super)gravity, due to the absence of an action principle that can be used to derive these equations of motion. In section 3.5, we will construct a different, "off-shell" version of three-dimensional Newton–Cartan supergravity, that includes an auxiliary scalar field in the supermultiplet. The terminology "off-shell" will be justified in the sense that the first constraint given in eq. (3.67) will no longer be needed for closure of the supersymmetry algebra. Both constraints given in eq. (3.67) will in fact not appear at all. Equations of motion can thus be identified in a pragmatic way as those constraints that can be removed by adding auxiliary degrees of freedom to a non-relativistic supermultiplet.

Let us stress/repeat some important aspects. In the case at hand, we can draw the following diagram, see figure 3.1. It shows that we can always derive a non-

$$
\begin{array}{ccc}
\hat{F}_{\mu\nu}(M) = 0 & \xrightarrow{\omega \to \infty} & \hat{R}_{\mu\nu}(H) = 0 \\
\big\downarrow Q_i & & \big\downarrow Q_+ \\
\hat{\Psi}_{\mu\nu i} = 0 & \xrightarrow{\omega \to \infty} & \hat{\psi}_{\mu\nu +} = 0 \\
\big\downarrow Q_i & & \big\downarrow Q_+ \\
R_{\mu\nu}{}^{AB}(\Omega) = 0 & \xrightarrow{\omega \to \infty} & R_{\mu\nu}{}^{ab}(J) = 0
\end{array}
$$

Figure 3.1: *The chains of constraints in the relativistic and non-relativistic case. In the non-relativistic case we do not denote the complete chain, see* (3.65)–(3.67).

relativistic constraint from a relativistic one. However, we cannot obtain the full set of constraints this way. This is due to the fact that in the relativistic case we always vary under two supersymmetries, while the chain on the right hand side only consists of Q_+ transformations. Further constraints follow from variation under Q_- transformations, but those are not limits of relativistic constraints.

We note that in the limit $\omega \to \infty$ the two relativistic constraints given in the second row of the left column, namely those containing the two gravitino curvatures, lead to just the single non-relativistic constraint given in the second row of the right column. This is in line with the fact that the constraint $\hat{R}_{\mu\nu}(H) = 0$ only varies under one of the two non-relativistic supersymmetries and hence eliminates only one of the non-relativistic gravitino curvatures. This observation is of vital importance to understand the off-shell case treated in the next section. There we are also going to impose the constraint $\hat{F}_{\mu\nu}(M) = 0$, but since its non-relativistic limit does not necessarily lead to the non-relativistic equations of motion, imposing this constraint does not force us to immediately go to the non-relativistic on-shell multiplet of the current section. Of course, in that case figure 3.1 changes. For example, the constraint in the bottom right corner gets extra contributions from the auxiliary field.

3.5 3D non-relativistic off-shell supergravity

In this section we derive an off-shell version of the three-dimensional Newton–Cartan supergravity theory that we revisited in the last section. Such an off-shell formulation will necessarily contain auxiliary fields, that cannot be interpreted as gauge-fields of an underlying symmetry algebra. Therefore, one cannot use gauging techniques to find off-shell multiplets. The limiting procedure that we adopt in this chapter provides an efficient tool to derive such off-shell formulation, as we will show in this section.

There is another approach that also allows one to systematically derive off-shell formulations, the superconformal tensor calculus. This though is the theme of chapter 6, were will derive two off-shell formulations, one of them being the new minimal Newton–Cartan supergravity theory that we also derive in this section. We proceed now with taking the limit.

We shall take the non-relativistic limit of a three-dimensional $\mathcal{N} = 2$ off-shell theory of supergravity. There exist two different off-shell formulations of $\mathcal{N} = 2$ supergravity in three dimensions [112–114]. To determine which one we should focus on, we take a look at the fields and transformation rules that we used to derive the on-shell theory. We will use the so-called new minimal $\mathcal{N} = 2$ Poincaré multiplet because this multiplet contains the abelian central charge gauge-field M_μ that was already part of the on-shell analysis. It is not obvious to us how to take the non-relativistic limit of the old minimal $\mathcal{N} = 2$ Poincaré multiplet due to the lack of this abelian gauge-field with the transformation rule given in (3.51). Therefore, we shall not try to do so here but we focus only on the new minimal formulation.

The new minimal $\mathcal{N} = 2$ Poincaré multiplet consists of the dreibein $E_\mu{}^A$, two gravitini $\Psi_{\mu i}$ $(i = 1, 2)$, two vector gauge-fields M_μ and V_μ and an auxiliary scalar D, see e.g. [114]. The supersymmetries, central charge transformations and R-symmetry transformations, with parameters η_i, Λ and ρ respectively, are given by

$$\begin{aligned} \delta E_\mu{}^A &= \frac{1}{2}\,\delta^{ij}\,\bar{\eta}_i\,\gamma^A\,\Psi_{\mu j}\,, \\ \delta\Psi_{\mu i} &= D_\mu\eta_i - \gamma_\mu\eta_i\,D + \varepsilon^{ij}\,\eta_j\,V_\mu + \frac{1}{4}\,\gamma_\mu\,\gamma\cdot\hat{F}(M)\,\varepsilon^{ij}\,\eta_j - \varepsilon^{ij}\,\Psi_{\mu j}\,\rho\,, \\ \delta M_\mu &= \frac{1}{2}\,\varepsilon^{ij}\,\bar{\eta}_i\,\Psi_{\mu j} + \partial_\mu\Lambda\,, \\ \delta V_\mu &= \frac{1}{2}\,\varepsilon^{ij}\,\bar{\eta}_i\,\gamma^\nu\hat{\Psi}_{\mu\nu j} - \frac{1}{8}\,\varepsilon^{ij}\,\bar{\eta}_i\,\gamma_\mu\gamma\cdot\hat{\Psi}_j - \frac{1}{4}\,\delta^{ij}\,\bar{\eta}_i\,\gamma\cdot\hat{F}(M)\,\Psi_{\mu j} \\ &\quad - \varepsilon^{ij}\,\bar{\eta}_i\,\Psi_{\mu j}\,D + \partial_\mu\rho\,, \\ \delta D &= -\frac{1}{16}\,\delta^{ij}\,\bar{\eta}_i\,\gamma\cdot\hat{\Psi}_j\,. \end{aligned} \tag{3.70}$$

The field strengths are given by (3.52) and

$$\hat{\Psi}_{\mu\nu i} = 2\,D_{[\mu}\Psi_{\nu]i} - 2\,\gamma_{[\mu}\Psi_{\nu]i}\,D - 2\,\varepsilon^{ij}\,\Psi_{[\mu j}\,V_{\nu]} + \frac{1}{2}\,\varepsilon^{ij}\,\gamma_{[\mu}\gamma\cdot\hat{F}(M)\,\Psi_{\nu]j}\,, \tag{3.71}$$

and dots refer to gamma traces as in $\gamma\cdot\hat{F}(M) = \gamma^{\mu\nu}\hat{F}_{\mu\nu}(M)$. The spin-connection is determined by requiring the torsion $\hat{R}_{\mu\nu}{}^A(E)$ be zero. Its supersymmetry variation follows from the expression in terms of the independent fields $E_\mu{}^A$ and $\Psi_{\mu i}$.

In order to take the non-relativistic limit of (3.70) we use the same rescalings as in the previous sections, supplemented with the rule

$$D = \frac{1}{\omega}\,S\,. \tag{3.72}$$

In the action of the $\mathcal{N} = 2$ new minimal supergravity the D^2 term plays the role of the cosmological constant Λ_{CC}. It is thus not surprising that in the non-relativistic limit, D scales like the square root of Λ_{CC}. For the non-relativistic contraction of the anti-de Sitter algebra see e.g. [111]. We do not rescale the field V_μ. Moreover, below we will argue that in the non-relativistic limit one must to eliminate V_μ.[1]

Going through similar arguments as in sections 3.2 and 3.4 we determine the non-relativistic dependent spin-connections $\omega_\mu{}^{ab}(e, \tau, m, \psi_\pm)$ and $\omega_\mu{}^a(e, \tau, m, \psi_\pm)$ to be given by (3.60) and (3.61). They also satisfy (3.62). As in the on-shell case we need to impose (3.52) as an extra constraint and we also need to eliminate the curvature $\hat{\psi}_{\mu\nu+}$ in order to take the limit in a consistent manner. Since, in the relativistic theory the constraint $\hat{F}_{\mu\nu}(M) = 0$ transforms under supersymmetry to the fermionic equation of motion, see (3.53), we thus effectively put the theory on-shell. In the following, we will show that, upon elimination of some auxiliary fields, the limiting procedure leads to an irreducible non-relativistic multiplet on which the Bargmann superalgebra is realized off-shell, in a sense that we will clarify below.

Let us now present a short argument for why we can eliminate the auxiliary field V_μ, when taking the limit. In a first approach the procedure leads to the constraints

$$\hat{R}_{\mu\nu}(H) = 0\,, \qquad \hat{\psi}_{\mu\nu+} = 0\,, \qquad \hat{\psi}_{ab-} = 0\,. \tag{3.73}$$

At this point we also find the following transformation rules for τ_μ and the auxiliary fields V_μ and S:

$$\begin{aligned}
\delta\tau_\mu &= \frac{1}{2}\,\bar{\epsilon}_+\gamma^0\psi_{\mu+}\,,\\
\delta V_\mu &= -\frac{1}{4}\,\bar{\epsilon}_+\gamma^{a0}\hat{\psi}_{\mu a-} - \bar{\epsilon}_+\gamma^0\psi_{\mu+}\,S\,,\\
\delta S &= -\frac{1}{8}\,\bar{\epsilon}_+\gamma^{a0}\hat{\psi}_{a0-}\,.
\end{aligned} \tag{3.74}$$

The supersymmetry variations of the last two constraints in (3.73) imply

$$e^\mu{}_a e^\nu{}_b \hat{V}_{\mu\nu} = \hat{V}_{ab} = 0\,, \tag{3.75}$$

which is the spatial part of the supercovariant curvature of V_μ:

$$\hat{V}_{\mu\nu} = 2\,\partial_{[\mu}V_{\nu]} + \frac{1}{2}\,\bar{\psi}_{[\mu}\gamma^{a0}\hat{\psi}_{\nu]a-} + \bar{\psi}_{[\mu+}\gamma^0\psi_{\nu]+}\,S\,. \tag{3.76}$$

Now we observe that the constraint (3.75) is always satisfied if we impose

$$V_\mu = -2\,\tau_\mu\,S\,. \tag{3.77}$$

[1] In chapter 6 we will find another way to understand the elimination of the auxiliary field V_μ.

Indeed, the inverse vielbeins in (3.75) eliminate any term with a free τ_μ and thus the derivative in (3.76) must hit the τ_μ when inserting (3.77) into (3.75). The remaining terms can then be canceled using the first constraint of (3.73). The identification (3.77) is preserved under all symmetry transformations using (3.73). In particular, the combination $V_\mu + 2\,\tau_\mu\,S$ does not transform under supersymmetry. It is thus not needed to close the algebra on any of the other fields. With the aim of deriving an irreducible multiplet we shall therefore eliminate V_μ, using (3.77). This sets the R-symmetry parameter $\rho = \text{const}$ in (3.70).

Performing the above manipulations, i.e. after taking the limit and eliminating the auxiliary field V_μ with (3.77), we end up with the following transformation rules for the complete non-relativistic new minimal off-shell multiplet. The bosonic fields transform as

$$\begin{aligned}
\delta\tau_\mu &= \frac{1}{2}\,\bar{\epsilon}_+\gamma^0\psi_{\mu+}\,,\\
\delta e_\mu{}^a &= \frac{1}{2}\,\bar{\epsilon}_+\gamma^a\psi_{\mu-} + \frac{1}{2}\,\bar{\epsilon}_-\gamma^a\psi_{\mu+}\,,\\
\delta m_\mu &= \bar{\epsilon}_-\gamma^0\psi_{\mu-}\,,\\
\delta S &= -\frac{1}{8}\,\bar{\epsilon}_+\gamma^{a0}\hat{\psi}_{a0-}\,,
\end{aligned} \tag{3.78}$$

while the transformations of the gravitini are given by

$$\begin{aligned}
\delta\psi_{\mu+} &= D_\mu\epsilon_+ + \gamma_0\epsilon_+\,S\,\tau_\mu\,,\\
\delta\psi_{\mu-} &= D_\mu\epsilon_- - 3\,\gamma_0\epsilon_-\,S\,\tau_\mu + \frac{1}{2}\,\omega_\mu{}^a\gamma_{a0}\epsilon_+ - \gamma_a\epsilon_+\,e_\mu{}^a\,S\,.
\end{aligned} \tag{3.79}$$

Given that there is only a single (fermionic) equation of motion in the on-shell theory, see eq. (3.67), it is not surprising that the number of auxiliary fields needed to close the algebra off-shell is reduced with respect to the relativistic multiplet we started with.

We have explicitely checked that the non-relativistic supersymmetry transformations given in (3.78) and (3.79) above close off-shell, i.e. upon use of the constraints (3.86)–(3.88) given below. Note that in this calculation we do not need to make use of the equations of motion (3.67). For explicit checks of the closure we use the transformations of the spin-connections, which are

$$\delta_Q\omega_\mu{}^{ab} = \frac{1}{2}\,\bar{\epsilon}_+\,\gamma^{[b}\hat{\psi}^{a]}{}_{\mu-} - S\,\bar{\epsilon}_+\gamma^{ab}\psi_{\mu+}\,, \tag{3.80}$$

$$\begin{aligned}
\delta_Q\omega_\mu{}^a = &\ \frac{1}{4}\,\bar{\epsilon}_+\gamma^a\hat{\psi}_{\mu0-} + \frac{1}{4}\,e_{\mu b}\,\bar{\epsilon}_+\gamma^b\hat{\psi}^a{}_{0-} + \bar{\epsilon}_-\gamma^0\hat{\psi}_\mu{}^a{}_-\\
&+ S\,\bar{\epsilon}_+\gamma^{a0}\psi_{\mu-} + S\,\bar{\epsilon}_-\gamma^{a0}\psi_{\mu+}\,,
\end{aligned} \tag{3.81}$$

and the gravitini curvatures take the form

$$\hat{\psi}_{\mu\nu+} = 2\,\partial_{[\mu}\psi_{\nu]+} - \frac{1}{2}\,\omega_{[\mu}{}^{ab}\gamma_{ab}\psi_{\nu]+} - 2\,\gamma_0\psi_{[\mu+}\,\tau_{\nu]}\,S\,, \tag{3.82}$$

$$\begin{aligned}\hat{\psi}_{\mu\nu-} &= 2\,\partial_{[\mu}\psi_{\nu]-} - \frac{1}{2}\,\omega_{[\mu}{}^{ab}\gamma_{ab}\psi_{\nu]-} + 6\,\gamma_0\psi_{[\mu-}\,\tau_{\nu]}\,S \\ &\quad + \omega_{[\mu}{}^{a}\gamma_{a0}\psi_{\nu]+} + 2\,\gamma_a\psi_{[\mu+}\,e_{\nu]}{}^{a}\,S\,.\end{aligned} \tag{3.83}$$

The anti-commutator of two supersymmetry transformations can be summarized in the following formula

$$\begin{aligned}\left[\delta_Q(\epsilon_1),\delta_Q(\epsilon_2)\right] = \delta_{g.c.t.}\left(\Xi^\rho\right) + \delta_J\left(\Lambda_a{}^b\right) + \delta_G\left(\Lambda^a\right) + \delta_Z\left(\Sigma\right) \\ + \delta_+\left(\Upsilon_+\right) + \delta_-\left(\Upsilon_-\right),\end{aligned} \tag{3.84}$$

where the parameters of the transformations on the right-hand-side are given by

$$\begin{aligned}\Xi^\mu &= \frac{1}{2}\,\bar{\epsilon}_{2+}\gamma^0\epsilon_{1+}\,\tau^\mu + \frac{1}{2}(\bar{\epsilon}_{2+}\gamma^a\epsilon_{1-} + \bar{\epsilon}_{2-}\gamma^a\epsilon_{1+})e^\mu{}_a\,, \\ \Lambda^{ab} &= -\Xi^\mu\omega_\mu{}^{ab} - S\,\bar{\epsilon}_{2+}\gamma^{ab}\epsilon_{1+}\,, \\ \Lambda^a &= -\Xi^\mu\omega_\mu{}^a + S\left(\bar{\epsilon}_{2+}\gamma^{a0}\epsilon_{1-} + \bar{\epsilon}_{2-}\gamma^{a0}\epsilon_{1+}\right), \\ \Upsilon_\pm &= -\Xi^\mu\psi_{\mu\pm}\,, \\ \Sigma &= -\Xi^\mu m_\mu + \bar{\epsilon}_{2-}\gamma^0\epsilon_{1-}\,.\end{aligned} \tag{3.85}$$

Using the transformations (3.78) and (3.79) supersymmetry variations lead to the following chain of constraints:

$$\xrightarrow{Q_-} \qquad \hat{\psi}_{ab-} = 0 \tag{3.86}$$

$$\hat{R}_{\mu\nu}(H) = 0 \quad \xrightarrow{Q_+} \quad \hat{\psi}_{\mu\nu+} = 0 \quad \xrightarrow{Q_+} \quad \hat{R}_{\mu\nu}{}^{ab}(J) = -4\,\varepsilon^{ab}\,\tau_{[\mu}\hat{D}_{\nu]}\,S\,. \tag{3.87}$$

Again, the starting point here follows from taking the limit of the constraint (3.52). The Bianchi identities, upon use of the last constraint in (3.87), get the following contributions from auxiliary fields:

$$\hat{R}_{0[a}{}^{b]}(G) = 0\,, \qquad\qquad \hat{R}_{ab}{}^{c}(G) = 2\,\varepsilon_{ab}\,e^{\mu c}\,\hat{D}_\mu\,S\,. \tag{3.88}$$

The supercovariant curvatures of spatial rotations and Galilean boosts are given by

$$\begin{aligned}\hat{R}_{\mu\nu}{}^{ab}(J) &= 2\,\partial_{[\mu}\omega_{\nu]}{}^{ab} + \bar{\psi}_{[\mu+}\gamma^{[a}\hat{\psi}^{b]}{}_{\nu]-} - \bar{\psi}_{[\mu+}\gamma^{ab}\psi_{\nu]+}\,S\,, \\ \hat{R}_{\mu\nu}{}^{a}(G) &= 2\,\partial_{[\mu}\omega_{\nu]}{}^{a} - 2\,\omega_{[\mu}{}^{ab}\omega_{\nu]}{}^{b} - 2\,\bar{\psi}_{[\mu-}\gamma^0\hat{\psi}_{\nu]}{}^{a}{}_{-} - \frac{1}{2}\,e_{[\nu}{}^{b}\bar{\psi}_{\mu]+}\gamma^b\hat{\psi}^a{}_{0-} \\ &\quad - \frac{1}{2}\,\bar{\psi}_{[\mu+}\gamma^a\hat{\psi}_{\nu]0-} + 2\,\bar{\psi}_{[\mu+}\gamma^{a0}\psi_{\nu]-}\,S\,.\end{aligned} \tag{3.89}$$

We can use these identities to show that (3.86) does not imply any extra constraints. The contribution of the auxiliary field in the last equation of (3.87) ensures that its variation does not lead to additional constraints either. Hence, the set of constraints given in (3.86)–(3.88) is complete because we varied *all* constraints under supersymmetry. The check here is more rigorous than in the on-shell case where we did not vary the bosonic equation of motion anymore. Moreover, because the on-shell case can be derived from the off-shell formulation, see below, we have thus also proven consistency of the on-shell formulation.

As a final consistency check we can reduce our new result to two on-shell formulations that were presented in the literature earlier, namely on-shell Newton–Cartan supergravity [77] and Newton–Hooke supergravity [111]. The latter one will also be the subject of the next chapter where we shall derive Newton–Hooke supergravity using gauging techniques.

By simply eliminating the auxiliary field S, i.e. setting

$$S = 0\,, \tag{3.90}$$

we arrive at the result of section 3.4. To obtain Newton–Hooke supergravity we need to introduce a cosmological constant. This can be done by choosing a constant value for the auxiliary field

$$S = \frac{1}{2R}\,, \tag{3.91}$$

with R constant and related to the cosmological constant by $\Lambda_{CC} = -1/R^2$. This reproduces the on-shell Newton–Hooke supergravity theory of [111]. The order $1/R$ corrections with respect to the flat case are hidden in the curvatures, e.g. the bosonic equation of motion for Newton–Hooke gravity is still given by (3.67), but $\hat{R}_{\mu\nu}{}^a(G)$ now contains additional order $1/R$ terms.

This concludes our derivation of an off-shell formulation of three-dimensional Newton–Cartan supergravity. In the next section we will apply the limiting procedure to derive a different result, the non-relativistic superparticle in a curved background.

3.6 The point-particle limit

In this last example we apply the limiting procedure to a superparticle moving in a curved background. To be concrete, we use it to derive the action and transformation rules of the non-relativistic superparticle in a curved background, put forward in [111]. The non-relativistic superparticle in a flat background was already discussed in [115–117]. We note that the limit that was taken in [65] to derive the non-relativistic superparticle in a flat background can be understood as a special case of the analysis in this section.

It is illustrative to first discuss the bosonic particle. To derive the action of a non-relativistic bosonic point-particle in an arbitrary Newton–Cartan background we start from the relativistic action

$$S_{\rm rel} = -M \int d\lambda \left[\sqrt{-\eta_{AB}(\dot{x}^\mu E_\mu{}^A)(\dot{x}^\nu E_\nu{}^B)} - \dot{x}^\mu M_\mu \right] . \tag{3.92}$$

All dots refer to derivatives with respect to the worldline parameter λ, i.e. $\dot{x}^\mu = dx^\mu/d\lambda$. We use mostly plus signature and we also added a "charge" term $\dot{x}^\mu M_\mu$. Here, we impose that the curvature of the abelian gauge-field M_μ vanishes, implying that it can locally be written as $M_\mu = \partial_\mu \Gamma$ and the second term in (3.92) corresponds to a total derivative. Using the expressions (3.3) and (3.6) in the relativistic action (3.92) and taking $M = \omega\, m$, we obtain, in the limit $\omega \to \infty$, the following non-relativistic action:

$$S_{\rm nr} = m \int d\lambda \left[\frac{\delta_{ab}(\dot{x}^\mu e_\mu{}^a)(\dot{x}^\nu e_\nu{}^b)}{2\tau_\rho \dot{x}^\rho} - m_\mu \dot{x}^\mu \right] . \tag{3.93}$$

This action agrees with the action, calculated by other means, in e.g. [101, 118, 119]. Note that one of the reasons to add the term $\dot{x}^\mu M_\mu$ is to cancel a divergent (total derivative) term that otherwise would arise in the limit $\omega \to \infty$, see also [62]. In contrast, the combination $\dot{x}^\mu m_\mu$ in the non-relativistic action is not a total derivative term. This non-relativistic term does not only follow from the relativistic $\dot{x}^\mu M_\mu$ term alone, but it also receives contributions from the kinematic term $\sqrt{-\dot{x}^2}$.

We now generalize the discussion of the non-relativistic bosonic particle to the non-relativistic superparticle. The relativistic superparticle in a curved background is most conveniently written using superspace techniques, see [120]. Since so far a non-relativistic superspace description is lacking, we will refrain from using superspace notation and simplify the discussion and notation by considering only the terms in the action that are at most quadratic in the fermions. Thus, the supersymmetric analog of (3.92) takes the form

$$S_{\rm rel} = -M \int d\lambda \left[\sqrt{-\eta_{AB}\,\Pi^A \Pi^B} - \frac{1}{4}\,\varepsilon^{ij}\,\bar{\theta}_i D_\lambda \theta_j - \dot{x}^\mu \left(M_\mu - \frac{1}{2}\,\varepsilon^{ij}\,\bar{\theta}_i\,\Psi_{\mu j} \right) \right] . \tag{3.94}$$

The background fields $E_\mu{}^A$, M_μ and $\Psi_{\mu i}$ are those of the relativistic on-shell theory discussed in section 3.4 and 2.8. The embedding coordinates are x^μ and θ_i. The supersymmetric line-element Π^A is defined as

$$\Pi^A = \dot{x}^\mu \left(E_\mu{}^A - \frac{1}{2}\,\delta^{ij}\,\bar{\theta}_i\,\gamma^A \Psi_{\mu j} \right) + \frac{1}{4}\,\delta^{ij}\,\bar{\theta}_i\,\gamma^A D_\lambda \theta_j\,, \tag{3.95}$$

where the derivative D_λ is covariantized with respect to Lorentz transformations, i.e.

$$D_\lambda \theta = \dot{\theta} - \frac{1}{4}\,\dot{x}^\mu\,\Omega_\mu{}^{AB}(E,\Psi_i)\,\gamma_{AB}\theta\,. \tag{3.96}$$

As we are only interested in terms up to second order in fermions, expressions like $\eta_{AB}\Pi^A\Pi^B$ are understood to contain only such terms and all terms quartic in fermions are discarded.

The action (3.94) is invariant under the following supersymmetry transformations of the embedding coordinates

$$\delta x^\mu = -\frac{1}{4}\,\delta^{ij}\,\bar{\eta}_i\,\gamma^A\theta_j\,E^\mu{}_A\,, \qquad \delta\theta_i = \eta_i\,. \tag{3.97}$$

These transformations should be accompanied by the following σ-model transformations [121, 122] of the background fields, as explained e.g. in [101, 111]:

$$\begin{aligned} \delta E_\mu{}^A &= \frac{1}{2}\,\delta^{ij}\,\bar{\eta}_i\,\gamma^A\Psi_{\mu j} - \frac{1}{4}\,\delta^{ij}\,\bar{\eta}_i\,\gamma^B\theta_j\,E^\rho{}_B\partial_\rho E_\mu{}^A\,, \qquad \delta\Psi_{\mu i} = D_\mu\eta_i\,, \\ \delta M_\mu &= \frac{1}{2}\,\varepsilon^{ij}\,\bar{\eta}_i\,\Psi_{\mu j} - \frac{1}{4}\,\delta^{ij}\,\bar{\eta}_i\,\gamma^B\theta_j\,E^\rho{}_B\partial_\rho M_\mu\,. \end{aligned} \tag{3.98}$$

The action (3.94) is also left invariant by the κ-transformations

$$\delta_\kappa x^\mu = -\frac{1}{4}\,\delta^{ij}\,\bar{\theta}_i\,\gamma^A\delta_\kappa\theta_j\,E^\mu{}_A\,, \qquad \delta_\kappa\theta_1 = \kappa\,, \qquad \delta_\kappa\theta_2 = -\frac{\Pi^A\gamma_A}{\sqrt{-\Pi^2}}\,\kappa\,. \tag{3.99}$$

In this case all background fields transform under κ-symmetry only through their dependence on the embedding coordinates. To show invariance under supersymmetry and κ-symmetry, one needs to use the equations of motion of the background fields.

With all these preliminaries at hand, it is now straightforward to apply the limiting procedure to the relativistic superparticle action (3.94). This yields the following result:

$$\begin{aligned} S_{\rm nr} = \frac{m}{2}\int d\lambda\,\Big[\frac{\hat{\pi}^a\hat{\pi}^b\delta_{ab}}{\hat{\pi}^0} - 2\,\dot{x}^\mu\Big(m_\mu - \bar{\theta}_-\gamma^0\psi_{\mu-}\Big) - \bar{\theta}_-\gamma^0\hat{D}\theta_- \\ - \frac{1}{2}\,\dot{x}^\mu\omega_\mu{}^a\,\bar{\theta}_+\gamma_a\theta_-\Big]\,, \end{aligned} \tag{3.100}$$

where we have defined the following supersymmetric line-elements

$$\hat{\pi}^0 = \dot{x}^\mu \Big(\tau_\mu - \frac{1}{2}\,\bar{\theta}_+\gamma^0\psi_{\mu+}\Big) + \frac{1}{4}\,\bar{\theta}_+\gamma^0\hat{D}\theta_+\,, \tag{3.101}$$

$$\hat{\pi}^a = \dot{x}^\mu \Big({e_\mu}^a - \frac{1}{2}\,\bar{\theta}_+\gamma^a\psi_{\mu-} - \frac{1}{2}\,\bar{\theta}_-\gamma^a\psi_{\mu+}\Big) + \frac{1}{4}\,\bar{\theta}_+\gamma^a\hat{D}\theta_- + \frac{1}{4}\,\bar{\theta}_-\gamma^a\hat{D}\theta_+ + \frac{1}{8}\,\bar{\theta}_+\gamma^a\gamma_{b0}\theta_+\,\dot{x}^\mu{\omega_\mu}^b\,. \tag{3.102}$$

Note that the supercovariant derivative $\hat{D}$ is covariant with respect to spatial rotations, not boosts. The boost-connection ${\omega_\mu}^a$ that appears in eqs. (3.100) and (3.102) is the dependent boost-connection (3.61). For notational simplicity we do not denote below its dependence on the other fields. The transformations of the embedding coordinates under κ-symmetry are given by

$$\begin{aligned} \delta t &= -\frac{1}{4}\,\bar{\theta}_+\gamma^0\kappa\,, & \delta\theta_+ &= \kappa\,, \\ \delta x^i &= -\frac{1}{4}\,\bar{\theta}_-\gamma^i\kappa - \frac{1}{8}\,\frac{\hat{\pi}^j}{\hat{\pi}^0}\,\bar{\theta}_+\gamma^{0i}\gamma_j\kappa\,, & \delta\theta_- &= -\frac{\hat{\pi}^i}{2\hat{\pi}^0}\,\gamma_{i0}\kappa\,. \end{aligned} \tag{3.103}$$

This reproduces precisely, to second order in fermions, the κ-symmetric non-relativistic superparticle in a curved background as presented in [111]. Fixing κ-symmetry by setting $\theta_+ = 0$ leads to the result of [111]. When we gauge-fix the Newton–Cartan background to a Galilean background with a Newton potential Φ, the one described in [77], the action (3.100) reduces to

$$S_{\rm nr} = \frac{m}{2}\int d\lambda\,\Big[\frac{\pi^i_\Phi\pi^i_\Phi}{\pi^0} - 2\,\dot{t}\big(\Phi - \bar{\theta}_-\gamma^0\Psi\big) - \bar{\theta}_-\gamma^0\dot{\theta}_- + \frac{\dot{t}}{2}\,\partial_i\Phi\,\bar{\theta}_+\gamma^i\theta_-\Big]\,, \tag{3.104}$$

with the “super-Galilean” line-elements given by

$$\pi^0 = \dot{t} + \frac{1}{4}\,\bar{\theta}_+\gamma^0\dot{\theta}_+\,, \tag{3.105}$$

$$\pi^i_\Phi = \dot{x}^i - \frac{1}{2}\,\dot{t}\,\bar{\theta}_+\gamma^i\Psi + \frac{1}{4}\,\bar{\theta}_+\gamma^i\dot{\theta}_- + \frac{1}{4}\,\bar{\theta}_-\gamma^i\dot{\theta}_+ - \frac{1}{8}\,\dot{t}\,\partial_j\Phi\,\bar{\theta}_+\gamma^i\gamma_{j0}\theta_+\,. \tag{3.106}$$

This finishes our discussion of the superparticle in a non-relativistic curved background. We have shown that the limiting procedure can be applied to also obtain results different from gravitational backgrounds. In the next section we present a short recap of this chapter and give an outlook on possible further applications of the limiting procedure that we have developed.

3.7 Conclusion

In this chapter we have established a procedure to derive a non-relativistic theory of supergravity from a relativistic one. Our ansatz is strongly motivated by the contraction of relativistic symmetry algebras to non-relativistic ones. This contraction determines the scaling of all gauge-fields. Usually, the easiest part then is to find the transformation rules of the independent non-relativistic gauge-fields. These can be compared to the transformations rules that we would find had we started out by gauging the non-relativistic symmetry algebra directly.

One non-trivial aspect of the procedure consists of finding the correct set of constraints. For example, in order to avoid divergences we need to set to zero some of the curvatures of the gauge-fields like e.g. $\hat{F}_{\mu\nu}(M) = 0$. Here, the difficulty lies in finding a complete set of constraints, such that we eliminate not only the divergent terms, but also all their putative, finite contributions in the commutator algebra. The chains of constraints (3.65)–(3.67) and (3.86)–(3.87) comprise such complete sets of constraints.

Also, as we established in the explicit exercise in section 3.4, it is not always straightforward to find all the equations of motion by taking the non-relativistic limit of the relativistic equation of motion. For example, the equations (3.67) do not follow from a non-relativistic limit of the relativistic equations of motion. We were able though to find the non-relativistic equation of motion by varying all constraints under supersymmetry.

In our last example we took the non-relativistic limit of a point-particle that moves in an arbitrary dynamical supergravity background. The aim of this analysis was to perform yet another check of the procedure, by deriving the results of [111], and also to show that the limit works on theories other than pure (super)gravity.

As hopefully made clear by the examples in this chapter, our procedure does not just consist of taking a limit. We also have to impose constraints and sometimes eliminate fields. In this sense it is a truncation as much as a limit. Therefore we prefer to refer to it as the "non-relativistic limiting procedure" rather than the "non-relativistic limit". Simply calling it a non-relativistic limit certainly also causes confusion with previous works in the literature.

Many different such non-relativistic limits have been put forward in the literature, including supersymmetric theories, see e.g. [13, 16, 90, 91, 118, 123, 124]. However, none of them quite compares to our limit because they do not make this systematic use of the contraction of symmetry algebras.

Our hope is that this new way of thinking about the non-relativistic limit enables us not only to re-derive known results, as we mainly did in this chapter, but also to use the limit to get novel non-relativistic (supergravity) theories, like in section 3.5. For example, in chapter 6 we make use of this procedure to derive novel non-relativistic matter multiplets.

In chapter 2, see section 2.5.4, we already mentioned the issue of taking the limit in three space-time dimensions. Very similar structures appear in the supersymmetric case as well, i.e. in the limit a relativistic source cannot yield any (source) contribution on the right-hand-side of the equation of motion (3.67).

Note that we did not look at the non-relativistic limit of electrodynamics in this chapter. This is an interesting extension as there exists more than one limit in this case [125]. This has to do with the fact that we only consider $c \to \infty$ here, where c is the speed of light. In electrodynamics c is related to two "coupling constants" ε_0 and μ_0 via $\varepsilon_0 \mu_0 = 1/c^2$. Sending $c \to \infty$ yields $\varepsilon_0 \mu_0 \to 0$, but we can still choose which of the two vanishes. This leads to an "electric" and a "magnetic" limit which lead to different non-relativistic theories.

A worthwhile exercise might be to apply this procedure to the four-dimensional supergravity multiplet. Ideally, this leads to a four-dimensional Newton–Cartan supergravity theory, which has not been found so far. However, our procedure is not bound to lead to a result, unless one chooses the correct starting point. In particular, without the correct constraints that allow one to get rid of putative divergent terms, such as e.g. $\hat{F}_{\mu\nu}(M) = 0$ in our examples, the procedure will not yield a good result. That is not to say that four-dimensional Newton–Cartan supergravity does not exist, but a bit of trial and error might be needed to find the correct starting point.

There also exist other, for example ultra-relativistic contractions of symmetry algebras. Having a limiting procedure that depends only on the contraction itself, but not the fact that it is a non-relativistic contraction, means that we should be able to generalize the procedure and apply it to derive an ultra-relativistic Carroll gravity [61] as well and to derive Carroll (super)particle actions such as those given in [59, 60].

We will come back to this in the last chapter of this thesis where we also aim to give a more detailed outlook on possible applications of our (not necessarily non-relativistic) limiting procedure.

4

Newton–Hooke supergravity

In this chapter we will derive a cosmological extension of non-relativistic supergravity. To obtain this theory, Newton–Hooke supergravity, we will use gauging techniques, like we discussed in chapter 2. To do so, we need the Newton–Hooke superalgebra, which we derive in section 4.1. Then, we derive non-relativistic supergravity on cosmological space-time manifolds, i.e. with a non-vanishing cosmological constant, in section 4.2. We proceed to discuss a particular gauge-fixing to obtain a Galilean version of Newton–Hooke supergravity in section 4.3. In section 4.4, we discuss how the results of this chapter are related to their relativistic counterpart, anti-de Sitter supergravity, through the non-relativistic limiting procedure introduced in chapter 3.

This chapter is largely based on the construction of Newton–Cartan supergravity in [77]. Here, we will perform a similar analysis for a slightly more general case by including a negative cosmological constant $\Lambda_{CC} = -1/R^2$. Of course, all formulas that we obtain in this chapter must reduce to those of Newton–Cartan supergravity in the limit $R \to \infty$. This always serves as a quick check on the results that we are about to derive.

Given that there are no actions for non-relativistic (super)gravity theories, gauging procedures so far were the only way to construct, and understand, such theories. This is why, in this chapter, we start out by constructing this new non-relativistic supergravity theory in precisely that way. The analysis proceeds along similar lines as we saw in chapter 2, when we showed how general relativity can be constructed by gauging the Poincaré algebra. In this chapter, we will gauge a non-relativistic superalgebra, the Newton–Hooke superalgebra. The bosonic version of this algebra can be derived from the (anti-)de Sitter algebra by an Inönü–Wigner contraction [126]. For more works on Newton–Hooke algebras and space-times see e.g. [127–132] and supersymmetric extensions can be found e.g. in [133–135]. Here, we will derive the superalgebra ourselves by a non-relativistic contraction of an anti-de Sitter superalgebra. To show how this contraction works on the algebra is important when we want to take the limit of the supergravity fields later.

The gauging of the Newton–Hooke superalgebra will lead to a theory with many, in some sense too many, symmetries. Like in the relativistic case we will impose curvature constraints, see also [77]. It will then be possible to solve for some of the gauge-fields explicitely. We will also consider a partial gauge-fixing, where we use some of the curvature constraints to fix particular values of the gauge-fields, i.e. to eliminate some of their components. We will make maximal use of this to derive a Galilean version of Newton–Hooke supergravity, i.e. a supergravity theory of a Galilean observer. This Galilean observer is living on a space-time manifold which is curved due to the presence of a cosmological constant. In this special scenario space-time is curved, but space is flat, i.e. $R_{\mu\nu}{}^{ab}(J) = 0$. In the end, we will relate the theory that we construct here to a relativistic one via the non-relativistic limiting procedure introduced in the previous chapter.

We have no prove that the limit must always lead to the same theory that we would get from gauging. Hence, this chapter also serves as a check on the non-relativistic contraction itself. In particular, we will derive the same novel supergravity theory in two different ways, via gauging and via the limiting procedure. For that purpose, let us recall the figure 1.1 that we used in the introduction. A similar diagram may serve as the motivation for the current chapter, see figure 4.1. The aim of this chapter is thus to show that the gravitational theories obtained by gauging the Newton–Hooke (super)algebra coincide with the theories that follow from taking the non-relativistic limit of anti-de Sitter (super)gravity.

This chapter is organized as follows. In section 4.1 we derive the Newton–Hooke (super)algebra. Then we proceed to gauge it to get Newton–Hooke supergravity in section 4.2. Section 4.3 deals with a particular gauge-fixing of the Newton–Hooke supergravity background. In section 4.4 we use the contraction of

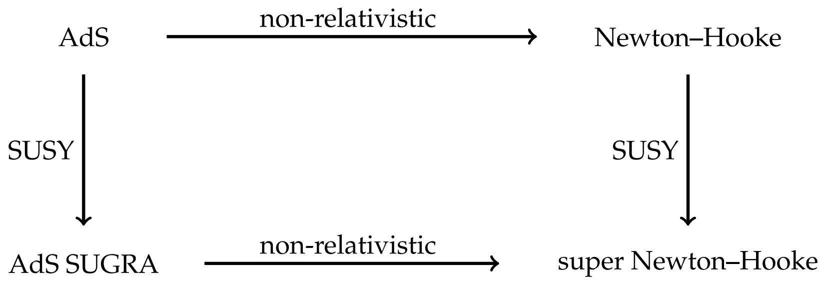

Figure 4.1: *Non-relativistic limits and supersymmetric extensions of cosmological gravity (AdS) and anti-de Sitter supergravity (AdS SUGRA).*

the algebra presented in section 4.1 and the non-relativistic limiting procedure of chapter 3 to re-derive the supergravity theory of section 4.2.

4.1 The Newton–Hooke superalgebra

Newton–Hooke (super)gravity is the cosmological extension of Newton–Cartan (super)gravity. While the latter one is related to the Bargmann (super)algebra the former one follows from gauging the cosmological extension of the Bargmann algebra, the Newton–Hooke (super)algebra. There are two versions of the Newton–Hooke algebra, one for a positive and one for a negative cosmological constant. They follow from an Inönü–Wigner contraction of the de Sitter and anti-de Sitter algebra, respectively. In the supersymmetric case, however, we will focus only on a negative cosmological constant.[1]

There are two different anti-de Sitter algebras with two supersymmetries, the so-called $\mathcal{N} = (1,1)$ and the $\mathcal{N} = (2,0)$ algebra. In this chapter, we shall use a Newton–Hooke superalgebra that stems from the latter one and the reason for doing so is as follows. In chapter 2 and again in section 3.3, we argued why we are interested in non-relativistic superalgebras with two supercharges: we want that the commutator of two supercharges yields time- and space-translations. This is the case for the non-relativistic analog of the $\mathcal{N} = (2,0)$ algebra but not for the other one. The $\mathcal{N} = (1,1)$ algebra is a direct product $\mathrm{OSp}(1|2)\otimes\mathrm{OSp}(1|2)$, i.e. it is essentially the product of two $\mathcal{N} = 1$ algebras. Taking a contraction thereof will only lead to a product of two non-relativistic $\mathcal{N} = 1$ algebras which, as we argued before, is not of the desired form. We shall show this in more detail in the following subsection.

[1] Formally, one can also define a Newton–Hooke superalgebra with positive cosmological constant. However, as in the relativistic case, one has problems defining a strictly positive Hamiltonian, see [136].

4.1.1 The $\mathcal{N} = (1,1)$ anti-de Sitter superalgebra

The $\mathcal{N} = (1,1)$ anti-de Sitter superalgebra is given by

$$\begin{aligned}
\big[M_{AB}, P_C\big] &= -2\,\eta_{C[A}\,P_{B]}\,, \qquad & \big[M_{AB}, Q^\pm\big] &= -\frac{1}{2}\gamma_{AB}Q^\pm\,,\\
\big[M_{AB}, M_{CD}\big] &= 4\,\eta_{[A[C}\,M_{D]B]}\,, \qquad & \big[P_A, Q^\pm\big] &= \pm x\,\gamma_A Q^\pm\,,\\
\big[P_A, P_B\big] &= 4\,x^2 M_{AB}\,,\\
\big\{Q^\pm_\alpha, Q^\pm_\beta\big\} &= 4\,[\gamma^A C^{-1}]_{\alpha\beta}P_A \pm 4\,x\,[\gamma^{AB}C^{-1}]_{\alpha\beta}M_{AB}\,.
\end{aligned} \tag{4.1}$$

If you compare it to the bosonic Poincaré algebra (2.15) you note that the difference is that translations do not commute. They lead to a rotation and scale with the cosmological parameter x. It is sometimes given in terms of the anti-de Sitter radius R by $x = 1/2R$. The last formula in (4.1), the anti-commutator of two supercharges, should be read using either only upper or only lower signs. The anti-commutator of Q^+ with Q^- is zero in general. We could opt to add a central charge in this commutator but since we will not use the algebra we refrain from adding yet more detail.

It is easy to show that this is the $\mathcal{N} = (1,1)$ algebra by defining new ’rotation’ generators by

$$M_C = \epsilon_{CAB}\,M^{AB}\,, \qquad J^\pm_A = P_A \pm x\,M_A\,. \tag{4.2}$$

Then we can show that the supercharges rotate in the following way:

$$\big[J^\pm_A, Q^\pm\big] = \pm 2\,x\,\gamma_A Q^\pm\,, \qquad \big\{Q^\pm_\alpha, Q^\pm_\beta\big\} = 4\,[\gamma^A C^{-1}]_{\alpha\beta}\,J^\pm_A\,. \tag{4.3}$$

Again, it is understood that any (anti-)commutator of a plus generator with a minus generator vanishes, hence the $\mathcal{N} = (1,1)$ character is manifest.

The Inönü–Wigner contraction should proceed similar to the contraction of the Poincaré algebra. The cosmological parameter x scales as

$$x \to \frac{1}{2\omega\,R}\,, \tag{4.4}$$

but we cannot find suitable combinations of the spinor charges Q^+ and Q^- such that we could scale them differently. It turns out that the only way not to run into problems with divergences is to scale both generators in the same way:

$$Q^\pm \to \frac{1}{\sqrt{\omega}}\,Q^\pm\,. \tag{4.5}$$

As a result the anti-commutator of two supercharges is identically zero (because we did not add a central charge).

4.1.2 The $\mathcal{N} = (2,0)$ anti-de Sitter superalgebra

Now let us proceed to discuss the $\mathcal{N} = (2,0)$ algebra. It is given by

$$\begin{aligned}
\left[M_{AB}, M_{CD}\right] &= 2\,\eta_{A[C}\,M_{D]B} - 2\,\eta_{B[C}\,M_{D]A}\,, & \left[M_{AB}, Q^i\right] &= -\frac{1}{2}\gamma_{AB}Q^i\,,\\
\left[M_{AB}, P_C\right] &= -2\,\eta_{C[A}\,P_{B]}\,, & \left[P_A, Q^i\right] &= x\,\gamma_A Q^i\,,\\
\left[P_A, P_B\right] &= 4\,x^2\,M_{AB}\,, & \left[\mathcal{R}, Q^i\right] &= 2\,x\,\varepsilon^{ij}Q^j\,,\\
\{Q^i_\alpha, Q^j_\beta\} &= 2\,[\gamma^A C^{-1}]_{\alpha\beta}\,P_A\,\delta^{ij} + 2\,x\,[\gamma^{AB}C^{-1}]_{\alpha\beta}\,M_{AB}\,\delta^{ij} + 2\,C^{-1}_{\alpha\beta}\,\varepsilon^{ij}\,\mathcal{R}\,.
\end{aligned} \tag{4.6}$$

In the flat limit $x \to 0$ the SO(2) R-symmetry generator $\mathcal{R}$ becomes the central element of the Poincaré superalgebra, see (2.66).

From here, the contraction proceeds similar to the one for the Poincaré algebra, i.e. we use (3.2) and (3.49). The only difference is that we replace the central element of the Poincaré algebra $\mathcal{Z}$ with the R-symmetry generator $\mathcal{R}$,

$$\mathcal{R} \to \frac{1}{2\omega}\,H - \omega\,Z\,, \tag{4.7}$$

and we add the aforementioned scaling of the cosmological parameter (4.4). After the contraction we find the non-vanishing commutators of the $\mathcal{N} = (2,0)$ Newton–Hooke superalgebra:

$$\begin{aligned}
\left[J_{ab}, (P/G)_c\right] &= -2\,\delta_{c[a}(P/G)_{b]}\,, & \left[H, G_a\right] &= P_a\,,\\
\left[H, P_a\right] &= -\frac{1}{R^2}\,G_a\,, & \left[J_{ab}, Q^\pm\right] &= -\frac{1}{2}\,\gamma_{ab}Q^\pm\,,\\
\left[H, Q^+\right] &= -\frac{1}{2R}\,\gamma_0 Q^+\,, & \left[H, Q^-\right] &= \frac{3}{2R}\,\gamma_0 Q^-\,,\\
\left[G_a, Q^+\right] &= -\frac{1}{2}\,\gamma_{a0}Q^-\,, & \left[P_a, Q^+\right] &= \frac{1}{2R}\,\gamma_a Q^-\,,\\
\{Q^+_\alpha, Q^+_\beta\} &= [\gamma^0 C^{-1}]_{\alpha\beta}\,H + \frac{1}{2R}\,[\gamma^{ab}C^{-1}]_{\alpha\beta}J_{ab}\,,\\
\{Q^+_\alpha, Q^-_\beta\} &= [\gamma^a C^{-1}]_{\alpha\beta}\,P_a + \frac{1}{R}\,[\gamma^{a0}C^{-1}]_{\alpha\beta}G_a\,.
\end{aligned} \tag{4.8}$$

The above formulas give the Newton–Hooke superalgebra without the central extension Z. The generator Z appears in the following (anti-)commutation relations:

$$\left[P_a, G_b\right] = \delta_{ab}\,Z\,, \qquad \{Q^-_\alpha, Q^-_\beta\} = 2\,[\gamma^0 C^{-1}]_{\alpha\beta}\,Z\,. \tag{4.9}$$

Of course, as is indeed the case here, we must obtain the Bargmann superalgebra when we send $R \to \infty$. In the following we try to realize this algebra on a set of independent gauge-fields.

4.2 Gauging the Newton–Hooke superalgebra

It is not difficult to gauge the Newton–Hooke superalgebra and to realize it on independent gauge-fields τ_μ, $e_\mu{}^a$, m_μ, $\omega_\mu{}^{ab}$, $\omega_\mu{}^a$, $\psi_{\mu+}$ and $\psi_{\mu-}$. These are the gauge-fields for the generators $(H, P_a, Z, J_{ab}, G_a, Q^+, Q^-)$ of time- and space-translations, central charge transformations, rotations, boosts and the two supersymmetries, respectively. The non-trivial part, in principle, is to find a set of constraints that simplifies replacing local translations by diffeomorphisms and at the same time enables us to solve for the spin- and boost-connection such that the only independent fields left are τ_μ, $e_\mu{}^a$, m_μ, $\psi_{\mu+}$ and $\psi_{\mu-}$. As we will see, we can use the same constraints that were imposed in [77], hence leading to a torsionless version of Newton–Hooke gravity.

4.2.1 Transformation rules

We find that the bosonic symmetries of the independent fields are given by

$$\begin{aligned} \delta\tau_\mu &= 0\,, \\ \delta e_\mu{}^a &= \lambda^a{}_b\, e_\mu{}^b + \lambda^a\, \tau_\mu\,, \\ \delta m_\mu &= \partial_\mu\sigma + \lambda^a\, e_\mu{}^a\,, \end{aligned} \tag{4.10}$$

and

$$\begin{aligned} \delta\psi_{\mu+} &= \frac{1}{4}\,\lambda^{ab}\gamma_{ab}\psi_{\mu+}\,, \\ \delta\psi_{\mu-} &= \frac{1}{4}\,\lambda^{ab}\gamma_{ab}\psi_{\mu-} - \frac{1}{2}\,\lambda^a\gamma_{a0}\psi_{\mu+}\,. \end{aligned} \tag{4.11}$$

After we solve for the spin-connections in terms of the fields above their new transformation rules will be determined through their dependence on other fields. As it turns out, the bosonic transformation rules will be un-altered but we need to be very careful about their transformation under supersymmetry. Therefore, we shall give those transformations only after solving for the spin-connections in terms of the independent fields.

The transformation rules of the independent fields under supersymmetry follow from the same principle as the bosonic ones and we find

$$\begin{aligned} \delta\tau_\mu &= \frac{1}{2}\,\bar\epsilon_+\gamma^0\psi_{\mu+}\,, \\ \delta e_\mu{}^a &= \frac{1}{2}\,\bar\epsilon_+\gamma^a\psi_{\mu-} + \frac{1}{2}\,\bar\epsilon_-\gamma^a\psi_{\mu+}\,, \\ \delta m_\mu &= \bar\epsilon_-\gamma^0\psi_{\mu-}\,, \end{aligned} \tag{4.12}$$

as well as

$$\begin{aligned} \delta\psi_{\mu+} &= D_\mu\epsilon_+ + \frac{1}{2R}\,\tau_\mu\,\gamma_0\epsilon_+\,, \\ \delta\psi_{\mu-} &= D_\mu\epsilon_- - \frac{3}{2R}\,\tau_\mu\,\gamma_0\epsilon_- + \frac{1}{2}\,\omega_\mu{}^a\gamma_{a0}\epsilon_+ - \frac{1}{2R}\,e_\mu{}^a\,\gamma_a\epsilon_+\,. \end{aligned} \tag{4.13}$$

The supercovariant curvatures associated to those transformation rules are found in much the same way as the transformation rules themselves, they are dictated by the structure constants of the algebra:

$$\begin{aligned} \hat{R}_{\mu\nu}(H) &= 2\,\partial_{[\mu}\tau_{\nu]} - \frac{1}{2}\,\bar{\psi}_{+[\mu}\gamma^0\psi_{\nu]+}\,, \\ \hat{R}_{\mu\nu}{}^a(P) &= 2\,\partial_{[\mu}e_{\nu]}{}^a - 2\,\omega_{[\mu}{}^{ab}e_{\nu]b} - 2\,\omega_{[\mu}{}^a\tau_{\nu]} - \bar{\psi}_{+[\mu}\gamma^a\psi_{\nu]-}\,, \\ \hat{R}_{\mu\nu}(Z) &= 2\,\partial_{[\mu}m_{\nu]} - 2\,\omega_{[\mu}{}^a e_{\nu]a} - \bar{\psi}_{\mu-}\gamma^0\psi_{\nu-}\,, \\ \hat{\psi}_{\mu\nu+} &= 2\,\partial_{[\mu}\psi_{\nu]+} - \frac{1}{2}\,\omega_{[\mu}{}^{ab}\gamma_{ab}\psi_{\nu]+} + \frac{1}{R}\,\tau_{[\mu}\,\gamma_0\psi_{\nu]+}\,, \\ \hat{\psi}_{\mu\nu-} &= 2\,\partial_{[\mu}\psi_{\nu]-} - \frac{1}{2}\,\omega_{[\mu}{}^{ab}\gamma_{ab}\psi_{\nu]-} - \frac{3}{R}\,\tau_{[\mu}\gamma_0\psi_{\nu]-} \\ &\quad + \omega_{[\mu}{}^a\gamma_{a0}\psi_{\nu]+} - \frac{1}{R}\,e_{[\mu}{}^a\gamma_a\psi_{\nu]+}\,. \end{aligned} \tag{4.14}$$

and

$$\begin{aligned} \tilde{R}_{\mu\nu}{}^a(G) &= 2\,\partial_{[\mu}\omega_{\nu]}{}^a - 2\,\omega_{[\mu}{}^{ab}\omega_{\nu]b} + \frac{2}{R^2}\,e_{[\mu}{}^a\tau_{\nu]} - \frac{1}{R}\,\bar{\psi}_{[\mu-}\gamma^{a0}\psi_{\nu]+}\,, \\ \hat{R}_{\mu\nu}{}^{ab}(J) &= 2\,\partial_{[\mu}\omega_{\nu]}{}^{ab} + \frac{1}{2R}\,\bar{\psi}_{\mu+}\gamma^{ab}\psi_{\nu+}\,. \end{aligned} \tag{4.15}$$

We should point out that $\tilde{R}_{\mu\nu}{}^a(G)$ as it is defined here, will not be covariant anymore when the spin- and boost-connections become dependent fields. This is due to the fact that its supersymmetry transformation changes, hence the curvature that is given in (4.15) will get additional contributions to accommodate those changes in the transformation rules. We will denote the covariant curvature $\hat{R}_{\mu\nu}{}^a(G)$ later in this chapter.

The Bianchi identities, that we will refer to later on, are defined for the curvatures (4.15). However, taking into account the full set of constraints we are able to show that for the on-shell case it does not matter which curvature we write in the Bianchi identities. This is different in the off-shell formulation that we discussed in chapter (3). There we saw terms that depend on (derivatives of the) auxiliary field appear in the Bianchi identities.

4.2.2 Curvature constraints and solving for the spin-connections

We want to replace the local translations, H and P transformations, by diffeomorphisms. In connection to this we are going to impose constraints on the curvatures (4.14) for two purposes. First, we would like to simplify our system as much as possible and remove gauge degrees of freedom. Therefore, we set

$$\hat{R}_{\mu\nu}(H) = 0\,, \qquad \hat{\psi}_{\mu\nu+} = 0\,, \qquad \hat{R}_{\mu\nu}{}^{ab}(J) = 0\,. \tag{4.16}$$

We will see in the next section how these constraints can be used to choose a particular gauge-fixing that will lead to the Galilean version of Newton–Hooke supergravity.

The last constraint tells us that space, not space-time (!), is flat. The first two are related to the choice of a preferred time-frame and its supersymmetric analog. In the bosonic case, the constraint $R_{\mu\nu}(H) = 2\,\partial_{[\mu}\tau_{\nu]} = 0$ implies that the gauge-field of time-translations is pure gauge, $\tau_\mu = \partial_\mu t$. Hence, τ_μ is a Stückelberg field and so is its fermionic partner $\psi_{\mu+}$. All constraints in (4.16) are related to each other via supersymmetry variation, in particular Q_+ transformations.

On a different note, the first constraint $\hat{R}_{\mu\nu}(H) = 0$ implies that there is no torsion. In the bosonic case, one can consider imposing slightly weaker conditions, for example requiring that only the spatial projection of $\hat{R}_{\mu\nu}(H)$ vanishes. One would then speak of twistless torsional Newton–Cartan geometry, see [29], and also chapter 5.

There is a second set of constraints, in addition to (4.16), which is motivated by the fact that we want the spin-connections $\omega_\mu{}^{ab}$ and $\omega_\mu{}^a$ to be dependent fields. Thus, we impose the so-called conventional constraints

$$\hat{R}_{\mu\nu}{}^{a}(P) = 0\,, \qquad \hat{R}_{\mu\nu}(Z) = 0\,. \tag{4.17}$$

Below we will explicitely show how these constraints can be used to solve for the non-relativistic spin-connections.

In a next step, we have to check whether or not the set (4.16) and (4.17) is indeed a consistent truncation of our theory. This means we need to vary the constraints under all symmetries to see if they imply further restrictions on the gauge-fields. In fact, we need to do this only for the constraints (4.16). The second set (4.17) is used to determine the dependent fields in terms of the independent ones and thus they are fulfilled identically.

Before we can look to the variations of constraints we need to solve for the now dependent $\omega_\mu{}^{ab}$ and $\omega_\mu{}^a$, to calculate their transformation rules. Equivalently, we could also derive the transformation rules by varying the curvatures $\hat{R}_{\mu\nu}{}^a(P)$ and $\hat{R}_{\mu\nu}(Z)$. Requiring that these variations must vanish also leads to the expressions for $\delta\omega_\mu{}^{ab}$ and $\delta\omega_\mu{}^a$. Once we have those transformation rules, we can proceed to investigate the consistency of our constraints.

We solve for the spin-connections in the following way. First, we use

$$e_\rho{}^a\,\hat{R}_{\mu\nu}{}^a(P) + e_\nu{}^a\,\hat{R}_{\rho\mu}{}^a(P) - e_\mu{}^a\,\hat{R}_{\nu\rho}{}^a(P) = 0\,, \tag{4.18}$$

to obtain

$$\begin{aligned}\omega_\mu{}^{ab}(e,\tau,m,\psi_\pm) = -2\,e^{\nu[a}\big(\partial_{[\mu}e_{\nu]}{}^{b]} - \tfrac{1}{2}\,\bar{\psi}_{[\mu+}\gamma^{b]}\psi_{\nu]-}\big) \\ + e_{\mu c}\,e^{\rho a}e^{\nu b}\big(\partial_{[\mu}e_{\nu]}{}^{c} - \tfrac{1}{2}\,\bar{\psi}_{[\mu+}\gamma^{c}\psi_{\nu]-}\big) \\ - \tau_\mu\,e^{\rho[a}\omega_\rho{}^{b]}(e,\tau,m,\psi_\pm)\,.\end{aligned} \tag{4.19}$$

To get an expression for $\omega_\mu{}^a(e,\tau,m,\psi_\pm)$ we contract $\hat{R}_{\mu\nu}{}^a(P)$ and $\hat{R}_{\mu\nu}(Z)$ with $e^\mu{}_a$ and τ^μ. We find

$$\begin{aligned}e^{\mu(a}\omega_\mu{}^{b)}(e,\tau,m,\psi_\pm) &= -2\,\tau^\mu e^{\nu(a}\big(\partial_{[\mu}e_{\nu]}{}^{b)} - \tfrac{1}{2}\,\bar{\psi}_{[\mu+}\gamma^{b)}\psi_{\nu]-}\big)\,, \\ e^{\mu[a}\omega_\mu{}^{b]}(e,\tau,m,\psi_\pm) &= e^{\mu a}e^{\nu b}\big(\partial_{[\mu}m_{\nu]} - \tfrac{1}{2}\,\bar{\psi}_{[\mu-}\gamma^{0}\psi_{\nu]-}\big)\,, \\ \tau^\mu\omega_\mu{}^{a}(e,\tau,m,\psi_\pm) &= 2\,\tau^\mu e^{\nu a}\big(\partial_{[\mu}m_{\nu]} - \tfrac{1}{2}\,\bar{\psi}_{[\mu-}\gamma^{0}\psi_{\nu]-}\big)\,.\end{aligned} \tag{4.20}$$

Now we can use (4.19) and (4.20) to derive the explicit forms

$$\begin{aligned}\omega_\mu{}^{ab}(e,\tau,m,\psi_\pm) = -2\,e^{\nu[a}\big(\partial_{[\mu}e_{\nu]}{}^{b]} - \tfrac{1}{2}\,\bar{\psi}_{[\mu+}\gamma^{b]}\psi_{\nu]-}\big) \\ + e_\mu{}^{c}\,e^{\rho a}e^{\nu b}\big(\partial_{[\rho}e_{\nu]}{}^{c} - \tfrac{1}{2}\,\bar{\psi}_{[\rho+}\gamma^{c}\psi_{\nu]-}\big) \\ - \tau_\mu\,e^{\rho a}e^{\nu b}\big(\partial_{[\rho}m_{\nu]} - \tfrac{1}{2}\,\bar{\psi}_{[\rho-}\gamma^{0}\psi_{\nu]-}\big)\,,\end{aligned} \tag{4.21}$$

$$\begin{aligned}\omega_\mu{}^{a}(e,\tau,m,\psi_\pm) = \tau^\nu\big(\partial_{[\mu}e_{\nu]}{}^{a} - \tfrac{1}{2}\,\bar{\psi}_{[\mu+}\gamma^{a}\psi_{\nu]-}\big) \\ + e_{\mu b}\,e^{\rho a}\tau^\nu\big(\partial_{[\rho}e_{\nu]}{}^{b} - \tfrac{1}{2}\,\bar{\psi}_{[\rho+}\gamma^{b}\psi_{\nu]-}\big) \\ + e^{\nu a}\big(\partial_{[\mu}m_{\nu]} - \tfrac{1}{2}\,\bar{\psi}_{[\mu-}\gamma^{0}\psi_{\nu]-}\big) \\ - \tau_\mu\,e^{\rho a}\tau^\nu\big(\partial_{[\rho}m_{\nu]} - \tfrac{1}{2}\,\bar{\psi}_{[\rho-}\gamma^{0}\psi_{\nu]-}\big)\,.\end{aligned} \tag{4.22}$$

We can use these expressions to check that the bosonic transformations are

$$\begin{aligned}\delta\omega_\mu{}^{ab} &= \partial_\mu\lambda^{ab}\,, \\ \delta\omega_\mu{}^{a} &= \partial_\mu\lambda^a + \lambda^a{}_b\,\omega_\mu{}^b - \omega_\mu{}^{ab}\lambda^b\,,\end{aligned} \tag{4.23}$$

while the supersymmetry transformations result in

$$\delta\omega_\mu{}^{ab}(e,\tau,m,\psi_\pm) = -\frac{1}{2R}\,\bar\epsilon_+\gamma^{ab}\psi_{\mu+}\,, \tag{4.24}$$

$$\begin{aligned}\delta\omega_\mu{}^{a}(e,\tau,m,\psi_\pm) &= \bar\epsilon_-\gamma^0\hat\psi_\mu{}^a{}_- + \frac{1}{2R}\,\bar\epsilon_-\gamma^{a0}\psi_{\mu+}\\ &\quad + \frac{1}{4}\,e_\mu{}^b\,\bar\epsilon_+\gamma^b\hat\psi^a{}_{0-} + \frac{1}{4}\,\bar\epsilon_+\gamma^a\hat\psi_{\mu0-} + \frac{1}{2R}\,\bar\epsilon_+\gamma^{a0}\psi_{\mu-}\,.\end{aligned} \tag{4.25}$$

Now we turn to checking consistency of the constraints (4.16). In fact, in writing (4.24) and (4.25) we already used the constraint

$$\hat\psi_{ab-} = 0\,, \tag{4.26}$$

which follows from a Q_- transformation of (4.16). As we mentioned already the constraints transform to each other under Q_+ transformations. In fact, this is true only if we also require

$$\gamma^{[a}\hat\psi^{b]}{}_\mu = 0\,. \tag{4.27}$$

This constraint can also be written in a slightly different, simpler form, see (4.31) below. Additional constraints can be found by looking at Bianchi identities. They imply that

$$\hat R_{ab}{}^c(G) = 0\,, \qquad\qquad \hat R_{0[a}{}^{b]}(G) = 0\,, \tag{4.28}$$

where the supercovariant boost curvature is given by

$$\hat R_{\mu\nu}{}^a(G) = \tilde R_{\mu\nu}{}^a(G) - 2\,\bar\psi_{[\mu-}\gamma^0\hat\psi_{\nu]}{}^a{}_- - \frac{1}{2}\,e_{[\nu}{}^b\,\bar\psi_{\mu]+}\gamma^b\hat\psi^a{}_{0-} - \frac{1}{2}\,\bar\psi_{[\mu+}\gamma^a\hat\psi_{\nu]0-}\,, \tag{4.29}$$

If we add the bosonic equation of motion,

$$\hat R_{0a}{}^a(G) = 0\,, \tag{4.30}$$

we have collected all constraints in eqs. (4.16) and (4.26)–(4.30). To close the commutator algebra of the transformation rules (4.10)–(4.13), we can in principle opt for an on-shell or an off-shell version. Here, we choose the on-shell version where the equations of motion are given by

$$\hat R_{0a}{}^a(G) = 0\,, \qquad\qquad \gamma^a\hat\psi_{a0-} = 0\,. \tag{4.31}$$

To show that the supersymmetry algebra closes, only the latter constraint is needed. Under supersymmetry it transforms into the bosonic equation of motion, i.e. the first one.

At last, we find that Newton–Hooke supergravity is given by a multiplet of five independent gauge-fields τ_μ, $e_\mu{}^a$, m_μ, $\psi_{\mu+}$ and $\psi_{\mu-}$ with transformation rules given in eqs. (4.10)–(4.13). The spin-connections transform according to (4.23)–(4.25). The full set of constraints is given by (4.16), (4.26)–(4.30). The commutator algebra of all transformations realizes the $\mathcal{N} = 2$ Newton–Hooke superalgebra (4.8), where the anti-commutator of two supercharges is given by

$$\begin{aligned}\left[\delta_Q(\epsilon_1), \delta_Q(\epsilon_2)\right] = \delta_{g.c.t.}\left(\Xi^\rho\right) + \delta_J\left(\Lambda_a{}^b\right) + \delta_G\left(\Lambda^a\right) + \delta_Z\left(\Sigma\right) \\ + \delta_+\left(\Upsilon_+\right) + \delta_-\left(\Upsilon_-\right).\end{aligned} \tag{4.32}$$

The parameters of the transformations on the right-hand-side are given by

$$\begin{aligned}
\Xi^\mu &= \frac{1}{2}\,\bar{\epsilon}_{2+}\gamma^0\epsilon_{1+}\,\tau^\mu + \frac{1}{2}\left(\bar{\epsilon}_{2+}\gamma^a\epsilon_{1-} + \bar{\epsilon}_{2-}\gamma^a\epsilon_{1+}\right)e^\mu{}_a\,, \\
\Lambda^{ab} &= -\Xi^\mu\omega_\mu{}^{ab} - \frac{1}{2R}\,\bar{\epsilon}_{2+}\gamma^{ab}\epsilon_{1+}\,, \\
\Lambda^a &= -\Xi^\mu\omega_\mu{}^a + \frac{1}{2R}\left(\bar{\epsilon}_{2+}\gamma^{a0}\epsilon_{1-} + \bar{\epsilon}_{2-}\gamma^{a0}\epsilon_{1+}\right), \\
\Upsilon_\pm &= -\Xi^\mu\psi_{\mu\pm}\,, \\
\Sigma &= -\Xi^\mu m_\mu + \bar{\epsilon}_{2-}\gamma^0\epsilon_{1-}\,.
\end{aligned} \tag{4.33}$$

This completes the construction of Newton–Hooke supergravity through the gauging procedure. Note that the equations of motion (4.31) are introduced "by hand". There is no action for Newton–Hooke supergravity whose maximization would give rise to those equations.

In the last part of this chapter, we will re-derive the theory of Newton–Hooke supergravity that we constructed here, using the non-relativistic limiting procedure developed in chapter 3. Before doing so, we will proceed with solving some of the constraints (4.16), by choosing a particular gauge for the background fields.

4.3 Galilean Newton–Hooke supergravity

In this section we will make use of the constraints (4.16) and (4.26) to gauge-fix most of the fields. Starting out with three independent bosonic vector fields τ_μ, $e_\mu{}^a$ and m_μ, we shall at the end be left with a single scalar field Φ, which we will identify as Newton's potential on curved space. Similarly, the two fermionic vectors $\psi_{\mu+}$ and $\psi_{\mu-}$ will be reduced to a single spinor Ψ.

In the previous section, it was implied that all gauge-parameters are local functions of space and time. In this section some—but, crucially, not all of them—will become constants, which is why we shall find it useful to write all coordinate dependence explicitly. Wherever this does not lead to formulas that are too long we shall do the same also for the gauge-fields.

The starting point of the gauge-fixing performed in this section is given by the constraints (4.16). Since it essentially amounts to $\tau_\mu = \partial_\mu t$ with t being some gauge-parameter we can choose a foliation of space-time such that

$$\tau_\mu(x^\alpha) = \delta^0_\mu \,. \tag{4.34}$$

This needs to be supplemented with the supersymmetrically analog restriction

$$\psi_{\mu+}(x^\alpha) = 0 \,. \tag{4.35}$$

Setting $\hat{R}_{\mu\nu}{}^{ab}(J) = 0$ also enables us to choose a spatially flat space, such that we may also use

$$\omega_\mu{}^{ab}(x^\alpha) = 0 \,. \tag{4.36}$$

It follows from the transformations (4.10), (4.11), (4.23) and (4.24), that the gauge-fixings (4.34)–(4.36) imply that the following parameters become constant:

$$\xi^0(x^\alpha) = \zeta \,, \qquad \lambda^a{}_b(x^\alpha) = \lambda^a{}_b \,, \qquad \epsilon_+(x^\alpha) = \epsilon_{+,t} = e^{-\frac{t}{2R}\gamma_0}\,\epsilon_+ \,. \tag{4.37}$$

The fact that the manifold is flat also implies that the spatial components of the vielbein must obey

$$e_i{}^a(x^\alpha) = \delta^a_i \,. \tag{4.38}$$

Using all the gauge-choices above, and the fact that the spatial components of $\hat{R}_{ab}{}^c(G)$ vanish, we observe that

$$\partial_{[i}\omega_{j]}{}^a(x^\alpha) = 0 \,. \tag{4.39}$$

This means that locally we can always write

$$\omega_i{}^a(x^\alpha) = \partial_i\,\omega^a(x^\alpha) \,, \tag{4.40}$$

with ω^a a dependent field, since $\omega_\mu{}^a$ was dependent. Using (4.34) and (4.38), we can deduce from the definition of the projective inverses $e^\mu{}_a$ and τ^μ that

$$\tau^\mu(x^\alpha) = \big(1, \tau^i(x^\alpha)\big) \,, \qquad e_0{}^a(x^\alpha) = -\tau^a(x^\alpha) \,, \qquad e^\mu{}_a(x^\alpha) = \big(0, \delta^i_a\big) \,. \tag{4.41}$$

Returning to our gauge-fixing conditions, we note that in addition to (4.38) we should require

$$\psi_{i-}(x^\alpha) = 0 \,. \tag{4.42}$$

Certainly, this solves for example the constraint (4.26). The compensating transformations that we get from imposing (4.38) and (4.42) are

$$\begin{aligned}\xi^i(x^\alpha) &= \xi^i(t) - \lambda^i{}_j x^j\,,\\ \epsilon_-(x^\alpha) &= \epsilon_-(t) - \frac{1}{2}\,\omega^a(x^\alpha)\gamma_{a0}\epsilon_{+,t} + \frac{x^i}{2R}\,\gamma_i\epsilon_{+,t}\,.\end{aligned} \tag{4.43}$$

So far the motivations for our gauge-choices where of a somewhat physical nature. Since we want to derive a Galilean theory we required absolute time and that space is flat. In order to figure out our next step, we may pause for a moment and write down the transformations of all independent fields that are left still. These are, from (4.41), $\tau^i(x^\alpha)$, the complete $m_\mu(x^\alpha)$ and the time component of one of the gravitinis, $\psi_{0-}(x^\alpha)$. We shall find it useful to consider separately the time- and space-components of $m_\mu(x^\alpha)$. Also, to make the transformation rules more reader friendly we do not denote the dependence of the background fields on the space-time coordinates x^α. We have

$$\begin{aligned}\delta\tau^i(x^\alpha) &= \zeta\,\partial_t\tau^i + \xi^j(t)\,\partial_j\tau^i - \lambda^j{}_k\,x^k\partial_j\tau^i + \lambda^i{}_j\,\tau^j - \lambda^i(x^\alpha)\\ &\quad - \partial_t\xi^i(t) - \frac{1}{2}\,\bar\epsilon_{+,t}\gamma^i\psi_{0-}\,,\end{aligned} \tag{4.44}$$

$$\delta m_i(x^\alpha) = \zeta\,\partial_t m_i + \xi^j(t)\,\partial_j m_i - \lambda^j{}_k\,x^k\partial_j m_i + \lambda_i{}^j\,m_j + \lambda^i(x^\alpha) + \partial_i\sigma(x^\alpha)\,, \tag{4.45}$$

$$\begin{aligned}\delta m_0(x^\alpha) &= \zeta\,\partial_t m_0 + \xi^i(t)\,\partial_i m_0 - \lambda^i{}_j\,x^j\partial_i m_0 + m_i\xi^i(t) - \lambda^i(x^\alpha)\,\tau^i + \partial_t\sigma(x^\alpha)\\ &\quad + \bar\epsilon_-(t)\gamma^0\psi_{0-} + \frac{1}{2}\,\omega^a\,\bar\epsilon_{+,t}\gamma_a\psi_{0-} - \frac{x^i}{2R}\,\bar\epsilon_{+,t}\gamma^{i0}\psi_{0-}\,,\end{aligned} \tag{4.46}$$

$$\begin{aligned}\delta\psi_{0-}(x^\alpha) &= \zeta\,\partial_t\psi_{0-} + \xi^i(t)\,\partial_i\psi_{0-} - \lambda^i{}_j\,x^j\partial_i\psi_{0-} + \frac{1}{4}\,\lambda^{ab}\gamma_{ab}\psi_{0-}\\ &\quad + \partial_t\epsilon_-(t) - \frac{1}{2}\,\partial_t\big(\omega^a\gamma_{a0}\epsilon_{+,t}\big) - \frac{x^i}{4R^2}\,\gamma_{i0}\epsilon_{+,t} - \frac{3}{2R}\,\gamma_0\epsilon_-(t)\\ &\quad - \frac{3}{4R}\,\omega^a\gamma_a\epsilon_{+,t} + \frac{3x^i}{4R^2}\,\gamma_{i0}\epsilon_{+,t} + \frac{1}{2}\,\omega_0{}^a\gamma_{a0}\epsilon_{+,t} + \frac{1}{2R}\,\tau^i\gamma_i\epsilon_{+,t}\,.\end{aligned} \tag{4.47}$$

At this point we could already remove either $\tau^i(x^\alpha)$ or $m_i(x^\alpha)$, since they both are Stückelberg fields of λ^i. As it turns out, we can better use a constraint coming from (4.36) to first remove $m_i(\alpha)$ completely and only then we shall deal with $\tau^i(x^\alpha)$. This works as follows.

The zero component $\omega_0{}^{ab}(x^\alpha)$—using (4.21) and all other gauge-choices imposed in this section—reads

$$\omega_0{}^{ab}(x^\alpha) = -\partial_{[a}\big(\tau_{b]} + m_{b]}\big) = 0\,. \tag{4.48}$$

It follows that we can define

$$\tau_i(x^\alpha) + m_i(x^\alpha) = \partial_i m(x^\alpha)\,, \tag{4.49}$$

and the transformation rule of the new field $m(x^\alpha)$ is

$$\begin{aligned}\delta(\partial_i m) = \partial_i\Big(\zeta\,\partial_t m + \xi^j(t)\,\partial_j m - \lambda^j{}_k\,x^k\partial_j m - x^j\,\partial_t\xi^j(t) + \sigma(x^\alpha)\Big)\\ - \frac{1}{2}\,\bar\epsilon_{+,t}\gamma^i\psi_{0-}\,.\end{aligned} \tag{4.50}$$

Note that we can write each but the last term as a partial derivative ∂_i. If it was not for the $\bar\epsilon_{+,t}$ term, $m(x^\alpha)$ would be a Stückelberg field too and we could easily eliminate it by gauge-fixing $\sigma(x^\alpha)$.

One way around this issue would be to define a new spinor by

$$\gamma^i\psi_{0-} = \partial_i\chi\,, \qquad \psi_{0-} = \frac{1}{2}\,\gamma^i\partial_i\chi\,. \tag{4.51}$$

This also leads to the appearance of a new bosonic fields as we shall see later.

For now we will rather choose to keep the explicit partial derivatives and gauge-fix $\partial_i m(x^\alpha) = 0$ such that

$$\partial_i\sigma(x^\alpha) = \partial_t\xi^i(t) + \frac{1}{2}\,\bar\epsilon_{+,t}\gamma^i\psi_{0-}\,. \tag{4.52}$$

Independently of whether we proceed like this, or if we had used (4.51), we can next eliminate $\tau^i(x^\alpha)$ by use of the local Lorentz boosts $\lambda^i(x^\alpha)$. The compensating gauge-transformation that we obtain is

$$\lambda^i(x^\alpha) = -\partial_t\xi^i(t) - \frac{1}{2}\,\bar\epsilon_{+,t}\gamma^i\psi_{0-}\,. \tag{4.53}$$

Finally, we are left with only two fields, the bosonic $m_0(x^\alpha)$ and the fermionic $\psi_{0-}(x^\alpha)$. In the following we shall denote them by $\Phi(x^\alpha)$ and $\Psi(x^\alpha)$. Furthermore, we will also refrain from further denoting the dependence on the coordinates x^α. This was only useful to determine all compensating gauge-transformations and to solve for the new gauge-parameters. This task is now completed.

In (4.52) we only determined the partial space derivative of σ, but to write down the new transformation rule of Φ we need the expression for the full function σ. Given that (4.52) includes the unknown function Ψ we cannot simply integrate it as we did earlier. What we will do instead, is to add a partial derivative to (4.46) to obtain a transformation rule for the partial derivative of the Newton potential $\partial_i\Phi$.

We also have to find the expression of ω^a and $\omega_0{}^a$ in terms of Φ and Ψ. From

(4.22) we get

$$\omega_0{}^a = -\partial_a\Phi\,, \qquad \omega_i{}^a = \partial_i\omega^a = 0\,. \tag{4.54}$$

In summary, the transformation rules for the 'Newton force' $\partial_i\Phi$ and its supersymmetric partner Ψ are given by

$$\begin{aligned}\delta(\partial_i\Phi) &= \zeta\,\partial_t(\partial_i\Phi) + \xi^j(t)\,\partial_j(\partial_i\Phi) - \lambda^j{}_k\,x^k\partial_j(\partial_i\Phi) + \lambda_i{}^j\,(\partial_j\Phi) + \partial_t\partial_t\xi^i(t)\\ &\quad + \frac{1}{2}\,\partial_t\big(\bar\epsilon_{+,t}\gamma^i\Psi\big) + \bar\epsilon_-(t)\gamma^0\partial_i\Psi - \frac{x^j}{2R}\,\bar\epsilon_{+,t}\gamma^{j0}\partial_i\Psi - \frac{1}{2R}\,\bar\epsilon_{+,t}\gamma^{i0}\Psi\,,\end{aligned} \tag{4.55}$$

$$\begin{aligned}\delta\Psi &= \zeta\,\partial_t\Psi + \xi^i(t)\,\partial_i\Psi - \lambda^i{}_j\,x^j\partial_i\Psi + \frac{1}{4}\,\lambda^{ab}\gamma_{ab}\Psi\\ &\quad + \partial_t\epsilon_-(t) - \frac{3}{2R}\,\gamma_0\epsilon_-(t) + \frac{x^i}{2R^2}\,\gamma_{i0}\epsilon_{+,t} - \frac{1}{2}\,(\partial_i\Phi)\,\gamma_{i0}\epsilon_{+,t}\,.\end{aligned} \tag{4.56}$$

At this point we have successfully obtained a multiplet that realizes the "acceleration-extended" Newton–Hooke superalgebra. The non-vanishing commutators are given by

$$\begin{aligned}&\big[\delta_\zeta,\delta_{\xi^i}\big] = \delta_{\xi^i}\big(-\zeta\dot\xi^i(t)\big)\,, && \big[\delta_{\lambda^i{}_j},\delta_{\xi^i}\big] = \delta_{\xi^i}\big(-\lambda^i{}_j\xi^j(t)\big)\,,\\ &\big[\delta_\zeta,\delta_{\epsilon_-}\big] = \delta_{\epsilon_-}\big(-\zeta\dot\epsilon_-(t)\big)\,, && \big[\delta_{\lambda^i{}_j},\delta_{\epsilon_-}\big] = \delta_{\epsilon_-}\big(-\frac{1}{4}\,\lambda^{ab}\gamma_{ab}\epsilon_-\big)\,,\\ &\big[\delta_{\xi^i},\delta_{\epsilon_+}\big] = \delta_{\epsilon_-}\big(-\frac{1}{2}\,\dot\xi^i(t)\gamma_{i0}\epsilon_{+,t}\big)\,, && \big[\delta_{\lambda^i{}_j},\delta_{\epsilon_+}\big] = \delta_{\epsilon_+}\big(-\frac{1}{4}\,\lambda^{ab}\gamma_{ab}\epsilon_{+,t}\big)\,,\\ &\big[\delta_{\epsilon_{1+}},\delta_{\epsilon_{2+}}\big] = \delta_\zeta\big(\frac{1}{2}\,\bar\epsilon_{2+}\gamma^0\epsilon_{1+}\big) + \delta_{\lambda^i{}_j}\big(\frac{1}{2R}\,\bar\epsilon_{2+}\gamma^{ij}\epsilon_{1+}\big)\,,\\ &\big[\delta_{\epsilon_+},\delta_{\epsilon_-}\big] = \delta_{\xi^i}\big(\frac{1}{2}\,\bar\epsilon_-(t)\gamma^i\epsilon_{+,t}\big)\,.\end{aligned} \tag{4.57}$$

A few comments are in order. First, note that the parameters of the transformations on the right-hand-side of the anti-commutator of two $\epsilon_{+,t}$ supersymmetries are constant, even though $\epsilon_{+,t}$ itself is not, see (4.43). This is so because the two time-dependent exponential factors cancel each other in any bilinear of two $\epsilon_{+,t}$ with an even number of spatial gamma-matrices. For the same reason we omitted the subscript t.

Second, the only $1/R$ correction to the algebra appears in the anti-commutator of two $\epsilon_{+,t}$ supersymmetries, while there are many more in the algebra (4.8). This is because all G_a and P_a transformations are "combined" in the acceleration-extended translations $\xi^i(t)$. For example, there are $1/R$ terms in the commutator of ϵ_+ with ϵ_-, which leads to a translation ξ^i. There are no $1/R$ terms in the ξ^i transformations in (4.55) and (4.56), but the parameter does depend on R through $\epsilon_{+,t}$.

The multiplet (4.55) and (4.56) closes on-shell. The gauge-fixed versions of the equations of motion (4.31) are

$$\partial^i\partial_i\Phi = \frac{2}{R^2}\,, \qquad\qquad \gamma^i\partial_i\Psi = 0\,. \tag{4.58}$$

One can readily check that they transform into each other under a supersymmetry transformation.

Let us now come back to the idea put forward in (4.51). Our aim now is to derive the transformation rule, such as (4.55), for the Newton potential only, not its spatial derivative. We thus introduce the new fermionic potential χ by

$$\gamma^i\Psi = \partial_i\chi\,, \qquad\qquad \Psi = \frac{1}{2}\,\gamma^i\partial_i\chi\,. \tag{4.59}$$

Then we can immediately write down the transformation rule for Φ:

$$\begin{aligned}\delta\Phi = {} & \zeta\,\partial_t\Phi + \xi^j(t)\,\partial_j\Phi + x^i\partial_t\partial_t\xi^i(t) - \lambda^j{}_k\,x^k\partial_j\Phi + \sigma(t)\\ & + \frac{1}{2}\,\bar{\epsilon}_-(t)\gamma^{0i}\partial_i\chi + \frac{1}{2}\,\partial_t\big(\bar{\epsilon}_{+,t}\chi\big) + \frac{x^i}{2R}\,\bar{\epsilon}_{+,t}\gamma^0\partial_i\chi\,.\end{aligned} \tag{4.60}$$

We have to allow for an arbitrary time-dependent shift because we only know the transformation of $\partial_i\Phi$. Equivalently, we see from (4.52) that such a shift is allowed in the compensating transformation.

To obtain the same result for χ is more challenging as we have to write (4.56), or actually $\gamma_i\delta\Psi$, as a partial derivative acting on $\delta\chi$. The problem here lies with the $\epsilon_{+,t}$ terms. We can write them as

$$\Big(\frac{x^j}{2R^2} - \frac{1}{2}\,\partial_j\Phi\Big)\gamma_i\gamma_{j0}\epsilon_{+,t} = \Big(\frac{x^i}{2R^2} - \frac{1}{2}\,\partial_i\Phi\Big)\gamma_0\epsilon_{+,t} - \Big(\frac{x^j}{2R^2} - \frac{1}{2}\,\partial_j\Phi\Big)\varepsilon_{ij}\,\epsilon_{+,t}\,, \tag{4.61}$$

where we use $\varepsilon_{ij} = \gamma^{ij0}$. Now we define a "dual Newton potential" Ξ by

$$\partial_i\Phi - \frac{x^i}{R^2} = -\varepsilon_{ij}\partial_j\Xi\,, \qquad\qquad \partial_i\Xi = \varepsilon_{ij}\Big(\partial_j\Phi - \frac{x^j}{R^2}\Big)\,. \tag{4.62}$$

We thus get

$$\begin{aligned}\delta\chi = {} & \zeta\,\partial_t\chi + \xi^i(t)\,\partial_i\chi - \lambda^i{}_j\,x^j\partial_i\chi + \frac{1}{4}\,\lambda^{ab}\gamma_{ab}\chi + x^i\gamma_i\dot{\epsilon}_-(t)\\ & - \frac{3x^i}{2R}\,\gamma_{i0}\epsilon_-(t) - \frac{1}{2}\,\Big(\Phi - \frac{x^ix^i}{2R^2}\Big)\,\gamma_0\epsilon_{+,t} + \frac{1}{2}\,\Xi\,\epsilon_{+,t} + \eta(t)\,.\end{aligned} \tag{4.63}$$

It remains to derive the transformation rule for Ξ. From the variation of (4.62) we note that

$$\varepsilon_{ij}\partial_j(\delta\Phi) = \partial_i(\delta\Xi)\,. \tag{4.64}$$

This leads to

$$\begin{aligned}\delta\Xi &= \zeta\,\partial_t\Xi + \xi^j(t)\,\partial_j\Xi + \varepsilon_{ij}\,x^i\partial_t\partial_t\xi^j(t) + \varepsilon_{ij}\,\frac{x^i\xi^j(t)}{R^2} - \lambda^j{}_k\,x^k\partial_j\Xi + \tilde{\sigma}(t)\\ &\quad + \frac{1}{2}\,\bar{\epsilon}_-(t)\gamma^i\partial_i\chi + \frac{1}{2}\,\partial_t\big(\bar{\epsilon}_{+,t}\gamma^0\chi\big) - \frac{x^i}{2R}\,\bar{\epsilon}_{+,t}\partial_i\chi\,.\end{aligned} \tag{4.65}$$

Using the defining equations (4.62), we can show that the multiplet given by (4.60), (4.63) and (4.65) also satisfies (4.57). In addition, we find more commutators given by the time-dependent shift functions $\sigma(t)$ and $\eta(t)$ (and $\tilde{\sigma}(t)$). They lead to the relations

$$\begin{aligned}\big[\delta_\zeta,\delta_\sigma\big] &= \delta_\sigma\big(-\zeta\dot{\sigma}(t)\big)\,, & \big[\delta_{\xi_1^i},\delta_{\xi_2^i}\big] &= \delta_\sigma\big(\dot{\xi}_1^i(t)\xi_2^i(t) - \dot{\xi}_2^i(t)\xi_1^i(t)\big)\,,\\ \big[\delta_\zeta,\delta_\eta\big] &= \delta_\eta\big(-\zeta\dot{\eta}(t)\big)\,, & \big[\delta_{\xi^i},\delta_{\epsilon_-}\big] &= \delta_\eta\big(-\xi^i(t)\gamma_i\dot{\epsilon}_-(t)\big)\,,\\ \big[\delta_\sigma,\delta_{\epsilon_+}\big] &= \delta_\eta\big(\frac{1}{2}\,\sigma(t)\gamma^0\epsilon_{+,t}\big)\,, & \big[\delta_{\lambda^i{}_j},\delta_\eta\big] &= \delta_\eta\big(-\frac{1}{4}\,\lambda^{ab}\gamma_{ab}\eta(t)\big)\,,\end{aligned} \tag{4.66}$$

which form an ideal of the acceleration-extended Newton–Hooke superalgebra.

In this section we have derived a gauge-fixed version of Newton–Hooke supergravity that is very much like the Galilean supergravity theory found in [77]. Indeed, in the limit $R \to \infty$ we recover the same results. Slight modifications are needed in the gauge-fixing due to the additional $1/R$ contributions, such as the new fixed time-dependence of $\epsilon_{+,t}$.

Let us remark here that it would be possible to define new spinors $\tilde{\Psi}$ and $\tilde{\chi}$ that differ from to the old ones only by an exponential factor $\exp(-t\gamma_0/2R)$. This way we could also make ϵ_+ time-independent.

Note also that the Φ that we use here differs from the one used in [111]. There, we preferred to define the Newton potential such that its Laplacian is zero, rather than proportional to the cosmological constant, see (4.58). We will comment further on this in the conclusion section.

4.4 Connection to the non-relativistic limit

There are different ways to understand the results of section 4.2 as a non-relativistic limit of anti-de Sitter supergravity. In particular, we can start from an on-shell anti-de Sitter supergravity multiplet to derive the transformation rules given in 4.2, which is what we are going to do in this section. The other option would be to use the off-shell Newton–Cartan multiplet of chapter 3. In the latter case, we would simply get the on-shell multiplet derived here by choosing the correct values for the auxiliary field S. While we set $S = 0$ to obtain on-shell Newton–Cartan supergravity we get the formulas of on-shell Newton–Hooke supergravity by using

$$S = \frac{1}{2R}\,, \tag{4.67}$$

in the off-shell Newton–Cartan supergravity multiplet (3.78) and (3.79).

In the following we shall briefly go through the first option, i.e. we take the limit of on-shell $\mathcal{N} = (2,0)$ anti-de Sitter supergravity [112]. The reason why we use $\mathcal{N} = (2,0)$ instead of $\mathcal{N} = (1,1)$ was mentioned at several points previously. The corresponding multiplet is given by the cosmological extension of the $\mathcal{N} = 2$ theory of section 2.8, which consists of the transformation rules

$$\begin{aligned} \delta E_\mu{}^A &= \frac{1}{2}\,\delta^{ij}\,\bar{\eta}_i\,\gamma^A\Psi_{\mu j}\,, \\ \delta\Psi_{\mu i} &= D_\mu\eta_i + \varepsilon^{ij}\,\eta_j\,V_\mu - \frac{1}{2R}\,\gamma_\mu\,\eta_i\,, \\ \delta V_\mu &= -\frac{1}{2R}\,\varepsilon^{ij}\,\bar{\eta}_i\Psi_{\mu j}\,, \end{aligned} \tag{4.68}$$

and the equations of motion for the supergravity fields $E_\mu{}^A$, $\Psi_{\mu i}$ and V_μ are given by

$$\hat{\Psi}_{\mu\nu i} = 0\,, \qquad \hat{V}_{\mu\nu} = 0\,. \tag{4.69}$$

The spin-connection $\Omega_\mu{}^{AB}(E,\Psi_i)$ is determined by the torsion constraint (2.85) and it is given by (2.84). The multiplet (4.68) realizes the $\mathcal{N} = (2,0)$ anti-de Sitter superalgebra (4.6).

Recall that in order to take the non-relativistic limit in section 3.4, we had to add the gauge-field M_μ of the central charge generator $\mathcal{Z}$ of the $\mathcal{N} = 2$ Poincaré superalgebra. Its transformation rule was given in (3.51) and in order not to upset the counting of degrees of freedom we had to set its curvature to zero, eq. (3.52). Here, we need not add any such field as the gauge-field of R-symmetry V_μ will take over that role. This can be inferred from looking at the contractions (3.2), (3.49) and (4.7) that lead to the $\mathcal{N} = (2,0)$ Newton–Hooke superalgebra. Indeed, this is not too surprising as in the limit where the cosmological constant goes to zero, the R-symmetry generator of the anti-de Sitter algebra becomes the central element of the Poincaré superalgebra.

Before we apply the limiting procedure by using the contractions (3.2), (3.49) and (4.7), to decompose the relativistic gauge-fields in terms of non-relativistic ones, we shall find it convenient (and necessary) to make one more redefinition. When we compare the transformation rule of the gauge-field V_μ, and the V_μ terms in the transformation rule of the gravitini, to the structure constants of the $\mathcal{N} = (2,0)$ anti-de Sitter superalgebra (4.6), we note that there are additional factors of

R. In fact, V_μ is not exactly the gauge-field related to the generator $\mathcal{R}$, but R times V_μ is. Therefore, we define

$$\bar{M}_\mu = -R\, V_\mu\,. \tag{4.70}$$

In the limit $R \to \infty$ we go to Poincaré supergravity and $\bar{M}_\mu$ becomes M_μ. The equation of motion $\hat{V}_{\mu\nu} = 0$ then becomes equivalent to (3.52).

After the redefinition (4.70) the multiplet (4.68) goes to

$$\begin{aligned} \delta E_\mu{}^A &= \frac{1}{2}\,\delta^{ij}\,\bar{\eta}_i\,\gamma^A\Psi_{\mu j}\,,\\ \delta\Psi_{\mu i} &= D_\mu\eta_i - \frac{1}{R}\,\varepsilon^{ij}\,\eta_j\,\bar{M}_\mu - \frac{1}{2R}\,\gamma_\mu\,\eta_i\,,\\ \delta\bar{M}_\mu &= \frac{1}{2}\,\varepsilon^{ij}\,\bar{\eta}_i\Psi_{\mu j}\,. \end{aligned} \tag{4.71}$$

We take the non-relativistic limit in these transformation rules by using (3.3), (3.54) with the rescalings (3.55), along with

$$\bar{M}_\mu = \omega\,\tau_\mu - \frac{1}{2\omega}\,m_\mu\,, \tag{4.72}$$

which follows from (4.7), and $R \to \omega R$. Since the transformation rules of the bosonic fields are identical to those of on-shell Poincaré supergravity, see eqs. (2.80) and (3.51), we immediately derive the correct transformation rules for τ_μ, $e_\mu{}^a$ and m_μ given in (4.12). The correct transformation rules for the fermions, eq. (4.13), also follow from directly from (4.71). The expressions for the supercovariant curvatures (4.14) are derived from the torsion constraint (2.85) and the relativistic gravitino curvature

$$\hat{\Psi}_{\mu\nu i} = 2\,\partial_{[\mu}\Psi_{\nu]i} - \frac{1}{2}\,\Omega_{[\mu}{}^{AB}\gamma_{AB}\Psi_{\nu]i} - \frac{2}{R}\,\varepsilon^{ij}\,\Psi_{[\mu}\,\bar{M}_{\nu]} + \frac{1}{R}\,\gamma_{[\mu}\Psi_{\nu]i}\,. \tag{4.73}$$

The curvature of the relativistic spin-connection $\hat{R}_{\mu\nu}{}^{AB}(M)$ splits into the curvature of the boost-connection $\hat{R}_{\mu\nu}{}^a(G)$ and the non-relativistic spin-connection $\hat{R}_{\mu\nu}{}^{ab}(J)$. The derivation of the constraints also follows the analysis of section 3.4. We find (3.66), i.e. (4.16), and from consistency of the commutator algebra we deduce (4.26) and (4.27).

At this point we have derived all transformation rules, constraints and equations of motion of the on-shell Newton–Hooke supergravity theory of section 4.2. There, we used gauging techniques to obtain the result, while here we started from the relativistic one. Therefore, we have shown that the non-relativistic limiting procedure and gauging techniques indeed lead to the same result.

4.5 Summary

In this chapter we derived a new non-relativistic three-dimensional supergravity theory. In particular, we considered a cosmological extension of "flat" Newton–Cartan supergravity. Our analysis also served as a check for the non-relativistic limiting procedure that was developed in the last chapter.

Since this cosmological extension must reduce to the "flat" case, in the limit where the cosmological constant vanishes, i.e. for $R \to \infty$, we expected to find results that are similar to Newton–Cartan supergravity, that are only modified with order $1/R$ corrections.

As it turns out the gauging procedure can indeed be done in much the same way as Newton–Cartan supergravity was found in [77]. We also find the same constraint structure. The order $1/R$ correction with respect to the result of [77] can be seen in the transformation rules only, but not explicitely in the constraints.

The Newton–Hooke supergravity background of section 4.2, as well as the Galilean version of section 4.3, were used in [111] to consider a non-relativistic superparticle in such curved backgrounds. As noted earlier we use a slightly different definition of the central charge gauge-field m_μ and the Newton potential Φ in the Galilean version. If we denote the central charge gauge-field that was used in [111] by $\bar{m}_\mu$ we have

$$m_\mu = \bar{m}_\mu + \tau_\mu \frac{(x^\rho e_\rho{}^a)^2}{2\,R^2}\,. \tag{4.74}$$

The reason for this choice was that the equation of motion for the Newton potential $\bar{\phi}$ is given by $\partial^i \partial_i \bar{\phi} = 0$ rather than $\partial^i \partial_i \Phi = 1/R^2$.

There are several possible extensions to the analysis of this chapter. First, inspired by our work in the previous chapter one could try to find an off-shell formulation of three-dimensional Newton–Hooke supergravity. Since, $\hat{R}_{\mu\nu}{}^{ab}(J) = 0$ for $S = 0$, see e.g. the off-shell formulations of chapter 3, this might be equivalent to seeking an extension with $\hat{R}_{\mu\nu}{}^{ab}(J) \neq 0$. Using the limiting procedure, this should be a fairly straightforward exercise. One could also try to extend the theory by adding more supercharges and consider non-relativistic extended supersymmetry. Another interesting possibility would be to look for a formulation of Newton–Hooke (super)gravity where we do not set the curvature $\hat{R}_{\mu\nu}(H)$ to zero. This would be equivalent to asking if there exists a torsion-full version of this theory. For this purpose it might of course be better to first consider only the bosonic case.

5

Schrödinger supergravity

This chapter discusses a "conformal" extension of Newton–Cartan supergravity. The Schrödinger supergravity theory is obtained by gauging a Schrödinger superalgebra. This construction of a Schrödinger supergravity theory is interesting in its own right as a non-relativistic version of "conformal" supergravity, but it is also very useful as the basic building block for the non-relativistic version of the conformal tensor calculus of the next chapter.

In this chapter we take a step towards "completing" the realm of three-dimensional non-relativistic supergravity theories. So far, we managed to find examples of Poincaré, i.e. Newton–Cartan, and anti-de Sitter, i.e. Newton–Hooke, supergravity. One missing piece is a theory of supergravity with more symmetries such as e.g. dilatations. To derive such a theory is the main objective of this chapter. Another motivation is that we are going to need this theory as a basis for the analysis that will follow in the next chapter 6.

We will derive this non-relativistic "superconformal" theory by gauging a superalgebra that is a "conformal" extension of the supergravity theories that we saw in chapters 3 and 4 [137]. We refer to this supergravity theory as Schrödinger supergravity rather than conformal supergravity because we prefer to reserve the name non-relativistic "conformal" supergravity multiplet for the multiplet that realizes the gauging of the Galilean Conformal superalgebra, see [138–140]. The reason for this is that the Schrödinger superalgebra, with only a single special conformal generator, allows a mass parameter while the Galilean Conformal superalgebra does not. A related issue is that the Schrödinger algebra allows for the same central extension with the symmetry generator Z as the Bargmann algebra. This is important for gauging too, as it allows to solve for the dependent gauge-fields in the same way as when gauging the Bargmann algebra.

Note also that the algebra that we will use here, unlike the Galilean Conformal superalgebra mentioned before, is not obtained by contracting a relativistic one, hence we cannot employ the limiting procedure. But there is no need for that as most of what we need was already found in [29]. Indeed, this reference treats the bosonic case of the analysis that we are about to perform here, i.e. a gauging of the Schrödinger algebra along the lines of chapters 2 and 4. For us, it only remains to discuss the differences when we add fermionic generators.

In this section we thus discuss the gauging of a Schrödinger superalgebra. This is done in several steps. First, in section 5.1 we give the algebra. Then, in section 5.2 we go through the gauging procedure. The transformation rules of all independent fields, the constraint structure and the expressions of the dependent fields in terms of the independent ones is found in subsections 5.2.1–5.2.3. The full set of curvature constraints is discussed in detail in subsection 5.2.2. After calculating the expressions of the dependent gauge fields in terms of independent ones in subsection 5.2.3 we re-evaluate the transformation rules of the dependent fields to check whether or not they still coincide with those given by the algebra. Finally, having determined the transformations of all fields we need to check if the set of curvature constraints that we imposed is a consistent one. In order to shorten our presentation we moved this last point to subsection 5.2.2 too, even though it is in fact one of the last steps of the analysis and can be performed only after having determined the transformations of the fields in 5.2.3. Finally, we conclude in section 5.3.

5.1 A Schrödinger superalgebra

The non-relativistic algebra that we are interested in is a supersymmetric extension of the Schrödinger algebra [82, 83]. The Schrödinger algebra is the symmetry algebra of the Schrödinger equation and its rigid version also leaves the point-particle action invariant. Supersymmetric extensions were first found in [89] as the symmetry group of a spinning particle. However, this leads to an algebra with a Grassmann valued vector charge, instead of a spinor (Q_- in our notation). Because we are mainly interested in extensions of the Bargmann superalgebra with two spinorial supercharges, see (3.50), we prefer if our Schrödinger algebra also contains such operators. Only then it is guaranteed that omitting the extra conformal symmetries allows us to retrieve the Newton–Cartan supergravity theories that we dealt with previously. This is of particular importance for our analysis in the following chapter.

For the purpose of this work we restrict ourselves to using the $z = 2$ Schrödinger algebra. This algebra, as well as its supersymmetric extension, is similar to the Bargmann algebra in that it allows for the same central extension in the commutator of spatial translations and Galilean boosts. This is important because it enables us to solve for the non-relativistic spin- and Galilean boost-connections and thus the gauging works in the same way as e.g. in [29, 77, 80].

As we choose to work in three space-time dimensions our preferred choice shall be the algebra of [91]. For completeness let us mention that a more systematic study to find supersymmetric extensions of the Schrödinger algebra was carried out in [92], where the same Schrödinger superalgebra was found. Furthermore, the algebra that we are going to use also appeared in [141].

The most prominent difference with respect to the purely bosonic analysis in [29] is the existence (and necessity) of an extra bosonic symmetry generator that we denote by R. This leads to the addition of an extra bosonic gauge-field r_μ, a $U(1)$ gauge-field in our case, with respect to their analysis. As we shall show in chapter 6 this vector is related to the relativistic gauge-field of R-symmetry. Indeed, we will show later an exact relation between the field r_μ introduced here and the vector that results from taking the non-relativistic limit of the gauge-field of R-symmetry, V_μ, in the new minimal Poincaré multiplet, see section 3.5.

To be concrete, we use the following set of commutators. The bosonic commutation relations of the Bargmann algebra, which we repeat here for convenience,

$$
\begin{aligned}
\left[P_a, J_{bc}\right] &= 2\,\delta_{a[b}\,P_{c]}\,, & \left[H, G_a\right] &= P_a\,,\\
\left[G_a, J_{bc}\right] &= 2\,\delta_{a[b}\,G_{c]}\,, & \left[P_a, G_b\right] &= \delta_{ab}\,Z\,,
\end{aligned}
\tag{5.1}
$$

are supplemented by the action of the dilatation operator D and special conformal transformations K as follows:

$$\begin{aligned} [D,H] &= -2H\,, & [H,K] &= D\,, & [D,K] &= 2K\,, \\ [D,P_a] &= -P_a\,, & [D,G_a] &= G_a\,, & [K,P_a] &= -G_a\,. \end{aligned} \tag{5.2}$$

The extension to supersymmetry is done by adding two fermionic supersymmetry generators Q_+, Q_- and one so-called special conformal supersymmetry generator S. We also have to add one more bosonic so-called R-symmetry generator R which, however, does not contribute to the commutation relations (5.1) and (5.2). This leads to the superalgebra that was found in [91], see also [92, 141]. In this way, the commutators of the Bargmann superalgebra,

$$\begin{aligned} [J_{ab}, Q_\pm] &= -\frac{1}{2}\gamma_{ab}Q_\pm\,, & [G_a, Q_+] &= -\frac{1}{2}\gamma_{a0}Q_-\,, \\ \{Q_+,Q_+\} &= -\gamma^0 C^{-1} H\,, & \{Q_+,Q_-\} &= -\gamma^a C^{-1} P_a\,, \\ \{Q_-,Q_-\} &= -2\gamma^0 C^{-1} Z\,, \end{aligned} \tag{5.3}$$

are augmented by commutators with the extra bosonic and fermionic operators of the Schrödinger superalgebra. The new commutation relations read

$$\begin{aligned} [D,Q_+] &= -Q_+\,, & [D,S] &= S\,, & [K,Q_+] &= S\,, \\ [J_{ab},S] &= -\frac{1}{2}\gamma_{ab}S\,, & [S,H] &= Q_+\,, & [S,P_a] &= \frac{1}{2}\gamma_{a0}Q_-\,, \\ [R,Q_\pm] &= \pm\gamma_0 Q_\pm\,, & [R,S] &= \gamma_0 S\,, \end{aligned} \tag{5.4}$$

for the extra boson-fermion combinations and

$$\begin{aligned} \{S,S\} &= -\gamma^0 C^{-1} K\,, \qquad \{S,Q_-\} = \gamma^a C^{-1} G_a\,, \\ \{S,Q_+\} &= \frac{1}{2}\gamma^0 C^{-1} D + \frac{1}{4}\gamma^{0ab} C^{-1} J_{ab} + \frac{3}{4} C^{-1} R\,, \end{aligned} \tag{5.5}$$

for the anti-commutators of all additional fermionic supercharges.

5.2 Gauging

In this section we are going to perform the gauging procedure in the same way as we did e.g. in chapter 4 for the Newton–Hooke algebra. The transformation rules are given in subsection 5.2.1, the constraints in 5.2.2 and the dependent gauge-fields are treated in subsection 5.2.3.

After imposing the conventional constraints we will find that the gauge-fields $\omega_\mu{}^{ab}$, $\omega_\mu{}^a$, f_μ and ϕ_μ of spatial rotations, Galilean boosts, special conformal transformations and S-supersymmetry transformations, respectively, together with the spatial components $b_a = e^\mu{}_a b_\mu$ of the dilatation gauge-field b_μ are dependent. The time-component $b = \tau^\mu b_\mu$ of b_μ will turn out to be a Stückelberg field for special conformal transformations, just like in the bosonic case [29]. Eventually, we will use this to set b to zero, gauge-fixing special conformal transformations. For notational purposes though, and because the dependent field does not differ from the independent ones in e.g. transformation rules or the expression of the covariant curvature, it is easier to keep the full b_μ. For notational purposes we will also refrain from denoting the dependence of the dependent fields on the independent ones. From here on, the fields ${\omega_\mu}^{ab}$, ${\omega_\mu}^a$, f_μ, ϕ_μ and b_a are always understood to be dependent.

5.2.1 Transformation rules of the independent fields

We start with the transformations of the independent bosonic fields under the bosonic symmetries. They are

$$\begin{aligned}
\delta\tau_\mu &= 2\,\Lambda_D\,\tau_\mu\,,\\
\delta{e_\mu}^a &= \lambda^a{}_b\,{e_\mu}^b + \lambda^a\,\tau_\mu + \Lambda_D\,{e_\mu}^a\,,\\
\delta m_\mu &= \partial_\mu\sigma + \lambda^a\,{e_\mu}^a\,,\\
\delta b_\mu &= \partial_\mu\Lambda_D + \Lambda_K\,\tau_\mu\,,\\
\delta r_\mu &= \partial_\mu\rho\,.
\end{aligned} \tag{5.6}$$

For the fermionic fields we find

$$\begin{aligned}
\delta\psi_{\mu+} &= \frac{1}{4}\,\lambda^{ab}\gamma_{ab}\psi_{\mu+} + \Lambda_D\,\psi_{\mu+} - \gamma_0\psi_{\mu+}\,\rho\,,\\
\delta\psi_{\mu-} &= \frac{1}{4}\,\lambda^{ab}\gamma_{ab}\psi_{\mu-} - \frac{1}{2}\,\lambda^a\gamma_{a0}\psi_{\mu+} + \gamma_0\psi_{\mu-}\,\rho\,.
\end{aligned} \tag{5.7}$$

Here Λ_D and ρ are the parameters of dilatations and R-symmetry transformations, respectively. The parameters $\lambda^a{}_b$ and λ^a of spatial rotation and Galilean boosts are the same as in the previous chapters.

The fermionic symmetries act on the bosonic fields as follows:

$$\begin{aligned}
\delta\tau_\mu &= \frac{1}{2}\,\bar\epsilon_+\gamma^0\psi_{\mu+}\,,\\
\delta{e_\mu}^a &= \frac{1}{2}\,\bar\epsilon_+\gamma^a\psi_{\mu-} + \frac{1}{2}\,\bar\epsilon_-\gamma^a\psi_{\mu+}\,,\\
\delta m_\mu &= \bar\epsilon_-\gamma^0\psi_{\mu-}\,,\\
\delta b_\mu &= -\frac{1}{4}\,\bar\epsilon_+\gamma^0\phi_\mu - \frac{1}{4}\,\bar\eta\,\gamma^0\psi_{\mu+}\,,\\
\delta r_\mu &= -\frac{3}{8}\,\bar\epsilon_+\phi_\mu + \frac{3}{8}\,\bar\eta\,\psi_{\mu+}\,,
\end{aligned} \tag{5.8}$$

where $\epsilon_\pm$ are the two Q-supersymmetry parameters while η is the single S-supersymmetry parameter. Under these fermionic symmetries the fermionic fields transform as follows:

$$\begin{aligned}\delta\psi_{\mu+} &= D_\mu\epsilon_+ - b_\mu\,\epsilon_+ + r_\mu\,\gamma_0\epsilon_+ - \tau_\mu\,\eta\,,\\ \delta\psi_{\mu-} &= D_\mu\epsilon_- - r_\mu\,\gamma_0\epsilon_- + \frac{1}{2}\,\omega_\mu{}^a\gamma_{a0}\epsilon_+ + \frac{1}{2}\,e_\mu{}^a\gamma_{a0}\eta\,.\end{aligned} \tag{5.9}$$

Since we expect the transformation rules of the dependent gauge-fields to change when we solve for them we will not denote them here. Rather, we will first solve for the gauge-fields $\omega_\mu{}^{ab}$, $\omega_\mu{}^a$, b_a, f_μ and ϕ_μ. To do so, we need to impose curvature constraints. A full discussion of the complete set of constraints that we impose on our theory is given in the following section.

5.2.2 Curvature constraints

While gauging the Schrödinger superalgebra we impose several curvature constraints. These follow mostly from requiring the correct transformation properties under diffeomorphisms. At the same time, they allow us to solve for some of the gauge-fields in terms of the remaining independent ones. According to the Schrödinger superalgebra the curvatures of the independent gauge-fields are given by

$$\begin{aligned}\mathcal{R}_{\mu\nu}(H) &= 2\,\partial_{[\mu}\tau_{\nu]} - 4\,b_{[\mu}\tau_{\nu]} - \frac{1}{2}\,\bar{\psi}_{[\mu+}\gamma^0\psi_{\nu]+}\,,\\ \mathcal{R}_{\mu\nu}{}^a(P) &= 2\,\partial_{[\mu}e_{\nu]}{}^a - 2\,\omega_{[\mu}{}^{ab}e_{\nu]}{}^b - 2\,\omega_{[\mu}{}^a\tau_{\nu]} - 2\,b_{[\mu}e_{\nu]}{}^a - \bar{\psi}_{[\mu+}\gamma^a\psi_{\nu]-}\,,\\ \mathcal{R}_{\mu\nu}(Z) &= 2\,\partial_{[\mu}m_{\nu]} - 2\,\omega_{[\mu}{}^a e_{\nu]}{}^a - \bar{\psi}_{[\mu-}\gamma^0\psi_{\nu]-}\,,\\ \mathcal{R}_{\mu\nu}(D) &= 2\,\partial_{[\mu}b_{\nu]} - 2\,f_{[\mu}\tau_{\nu]} + \frac{1}{2}\,\bar{\psi}_{[\mu+}\gamma^0\phi_{\nu]}\,,\\ \mathcal{R}_{\mu\nu}(R) &= 2\,\partial_{[\mu}r_{\nu]} + \frac{3}{4}\,\bar{\psi}_{[\mu+}\phi_{\nu]}\,,\end{aligned} \tag{5.10}$$

and

$$\begin{aligned}\hat{\Psi}_{\mu\nu+}(Q_+) &= 2\,\partial_{[\mu}\psi_{\nu]+} - \frac{1}{2}\,\omega_{[\mu}{}^{ab}\gamma_{ab}\psi_{\nu]+}\\ &\quad - 2\,b_{[\mu}\psi_{\nu]+} + 2\,r_{[\mu}\gamma_0\psi_{\nu]+} - 2\,\tau_{[\mu}\phi_{\nu]}\,,\\ \hat{\Psi}_{\mu\nu-}(Q_-) &= 2\,\partial_{[\mu}\psi_{\nu]-} - \frac{1}{2}\,\omega_{[\mu}{}^{ab}\gamma_{ab}\psi_{\nu]-}\\ &\quad - 2\,r_{[\mu}\gamma_0\psi_{\nu]-} + \omega_{[\mu}{}^a\gamma_{a0}\psi_{\nu]+} + e_{[\mu}{}^a\gamma_{a0}\phi_{\nu]}\,.\end{aligned} \tag{5.11}$$

Note that we slightly changed the notation with respect to previous chapter. The covariant curvatures of the dependent gauge-fields, which are denoted by $\mathcal{R}$ here, are not a priori given by the "curvatures" R that follow from the structure constants of the Schrödinger superalgebra. This is so because the transformation rules of the dependent gauge-fields are not necessarily equal to the ones that follow from the structure constants of the algebra, see e.g. the fermionic transformation rules given in eqs. (5.37). For the following discussion we will need the curvatures of spatial rotations, Galilean boosts and S-supersymmetry. In the case of spatial rotations the full curvature coincides with the expression that follows from the structure constants, i.e. $\mathcal{R}(J) = R(J)$, but in the other two cases there are additional terms in $\mathcal{R}$ since the fermionic transformation rules of those gauge-fields contain extra terms beyond those that are determined by the structure constants, see eq. (5.37). We therefore have that

$$\mathcal{R}_{\mu\nu}{}^{ab}(J) = 2\,\partial_{[\mu}\omega_{\nu]}{}^{ab} - \frac{1}{2}\,\bar{\phi}_{[\mu}\gamma^{0ab}\psi_{\nu]+}\,, \tag{5.12}$$

but that

$$\mathcal{R}_{\mu\nu}{}^{a}(G) = R_{\mu\nu}{}^{a}(G) + \text{additional terms}\,, \tag{5.13}$$

with the structure constant dependent part $R_{\mu\nu}{}^{a}(G)$ given by

$$R_{\mu\nu}{}^{a}(G) = 2\,\partial_{[\mu}\omega_{\nu]}{}^{a} - 2\,\omega_{[\mu}{}^{ab}\omega_{\nu]}{}^{b} - 2\,\omega_{[\mu}{}^{a}b_{\nu]} - 2\,f_{[\mu}e_{\nu]}{}^{a} + \bar{\phi}_{[\mu}\gamma^{a}\psi_{\nu]-}\,. \tag{5.14}$$

We will not need the "additional terms" in $\mathcal{R}(G)$ except for a special trace combination in which case the full expression for $\mathcal{R}(G)$ is given by

$$\mathcal{R}_{0a}{}^{a}(G) = R_{0a}{}^{a}(G) - e^{\mu}{}_{a}\,\bar{\psi}_{\mu-}\gamma^{0}\hat{\Psi}_{a0-}(Q_{-})\,. \tag{5.15}$$

The additional terms in (5.13) were given explicitly in (3.69), using a different notation though. Moving on with our new notation we find the curvature of the gauge-field of S-supersymmetry is given by

$$\begin{aligned}\mathcal{R}_{\mu\nu}(S) = 2\,\partial_{[\mu}\phi_{\nu]} - \frac{1}{2}\,\omega_{[\mu}{}^{ab}\gamma_{ab}\phi_{\nu]} + 2\,b_{[\mu}\,\phi_{\nu]} + 2\,r_{[\mu}\,\gamma_{0}\phi_{\nu]} + 2\,f_{[\mu}\,\psi_{\nu]+} \\ + 2\,\gamma^{0}\psi_{[\mu+}\left[\frac{1}{4}\,\varepsilon^{ab}\,\mathcal{R}_{\nu]0}{}^{ab}(J) - \mathcal{R}_{\nu]0}(R)\right] \\ - 2\,\gamma^{c}\psi_{[\mu-}\left[\frac{1}{4}\,\varepsilon^{ab}\,\mathcal{R}_{\nu]c}{}^{ab}(J) + \mathcal{R}_{\nu]c}(R)\right],\end{aligned} \tag{5.16}$$

where only the first line comprises terms that follow from the structure constants.

In the following subsection we will solve for the gauge-fields $\omega_\mu{}^{ab}$, $\omega_\mu{}^{a}$, b_a, f_μ and ϕ_μ in terms of the independent ones using the following set of conventional constraints:

$$\begin{aligned}&\mathcal{R}_{\mu\nu}{}^{a}(P) = 0\,, && \mathcal{R}_{\mu\nu}(Z) = 0\,, && \mathcal{R}_{a0}(H) = 0\,, \\ &\hat{\Psi}_{a0+}(Q_{+}) = 0\,, && \gamma^{a}\hat{\Psi}_{a0-}(Q_{-}) = 0\,, && \\ &\mathcal{R}_{a0}(D) = 0\,, && \mathcal{R}_{0a}{}^{a}(G) = 0\,. &&\end{aligned} \tag{5.17}$$

Note that the last constraint involves the curvature of the dependent Galilean boost gauge-field, whose definition in terms of the part of the curvature that is determined by the structure constants is given in eq. (5.15). Since the conventional constraints are used to solve for some of the gauge-fields their supersymmetry transformations do not lead to new constraints. We note that, imposing constraints on the curvatures, the Bianchi identities generically imply further constraints on the curvatures, which holds for the constraints in (5.17) and those to be discussed below.

Besides the conventional constraints we also impose the foliation constraint

$$\mathcal{R}_{\mu\nu}(H) = 0\,. \tag{5.18}$$

The time-space component of this constraint is conventional but the space-space part is not. Its Q_+-supersymmetry transformation leads to

$$\hat{\Psi}_{\mu\nu+}(Q_+) = 0\,, \tag{5.19}$$

where, again, only the space-space part is a new, un-conventional constraint. The latter two equations lead to

$$\mathcal{R}_{ab}(D) = 0\,, \tag{5.20}$$

as the consequence of a Bianchi identity. Now we consider supersymmetry transformations of the un-conventional constraint $\hat{\Psi}_{ab+} = 0$. We find that a Q_--variation enforces [1]

$$\hat{\Psi}_{ab-}(Q_-) = 0\,. \tag{5.21}$$

Upon use of all know constraints and Bianchi identities, we find that the only non-trivial variation of (5.21) is its Q_--variation which we combine with a Q_+-variation of (5.19) to get

$$\mathcal{R}_{ab}(R) = 0\,, \qquad\qquad \mathcal{R}_{ab}{}^{cd}(J) = 0\,. \tag{5.22}$$

At this point we have checked the symmetry variations of all constraints except the last two, i.e. (5.22). Before we go on determining the implications of their transformations we note that using all constraints so far we find the Bianchi identity

$$\mathcal{R}_{ab}(S) = 0\,. \tag{5.23}$$

[1] One might wonder how the supersymmetry transformation of a fermionic [bosonic] constraint can lead to another fermionic [bosonic] constraint. It is true that this is not possible when following generic transformation rules of covariant quantities. However, those rules only apply if we already know the full set of constraints and the commutator algebra closes precisely because some constraints are needed to eliminate apparently non-covariant terms. Hence, we can certainly take guidance from those covariant rules, but when we use them too naively we might miss some constraints.

This identity is useful in showing that the only non-trivial transformation of the first constraint in (5.22) leads to [1]

$$\frac{3}{4}\,\varepsilon^{ab}\,\mathcal{R}_{\mu\nu}{}^{ab}(J) = \mathcal{R}_{\mu\nu}(R)\,. \tag{5.24}$$

Since (5.24) essentially identifies $\mathcal{R}(J)$ with $\mathcal{R}(R)$ we have derived all consequences of (5.22). The constraint (5.24) itself is inert under all symmetries and hence we have derived the full set of un-conventional constraints that follow from (5.18).

In summary, the set of constraints comprises the following chain of un-conventional constraints:

$$\begin{array}{ccccc} \mathcal{R}_{ab}(H) = 0 & \xrightarrow{Q_+} & \hat{\Psi}_{ab+} = 0 & \xrightarrow{Q_-} & \hat{\Psi}_{ab-} = 0 \quad \longrightarrow \\ \left.\begin{array}{l}\hat{\Psi}_{ab+} = 0\\ \hat{\Psi}_{ab-} = 0\end{array}\right\} & \xrightarrow{Q_\pm} & \mathcal{R}_{ab}(R) = 0 & \xrightarrow{Q_+} & \frac{3}{4}\,\varepsilon^{ab}\,\mathcal{R}_{\mu\nu}{}^{ab}(J) = \mathcal{R}_{\mu\nu}(R)\,. \end{array} \tag{5.25}$$

The most important Bianchi identities that feature in the discussion above are given by

$$\begin{aligned} &\mathcal{R}_{ab}(D) = 0\,, && \mathcal{R}_{0[a}{}^{b]}(G) = 0\,, \\ &\mathcal{R}_{ab}(S) = 0\,, && \mathcal{R}_{ab}{}^{c}(G) = 2\,\mathcal{R}_{0[a}{}^{b]c}(J)\,. \end{aligned} \tag{5.26}$$

The equations (5.17), (5.25) and (5.26) comprise all constraints of the Schrödinger supergravity theory.

5.2.3 The dependent gauge-fields

Let us now determine the expressions of the dependent gauge-fields. We first determine the spatial component of b_μ. Using $\mathcal{R}_{a0}(H) = 0$ we find

$$b_a = e^\mu{}_a b_\mu = \frac{1}{2}\,e^\mu{}_a \tau^\nu \left(2\,\partial_{[\mu}\tau_{\nu]} - \frac{1}{2}\,\bar{\psi}_{[\mu+}\gamma^0\psi_{\nu]+}\right). \tag{5.27}$$

Like in the bosonic case [29], the (independent) scalar $b = \tau^\mu b_\mu$ is a Stückelberg field for special conformal transformations:

$$\begin{aligned} \delta b = {}& \Lambda_K + \tau^\mu\partial_\mu\Lambda_D - 2\,\Lambda_D\,b - \lambda^a\,b_a - \frac{1}{4}\,\tau^\mu\left(\bar{\epsilon}_+\gamma^0\phi_\mu + \bar{\eta}\,\gamma^0\psi_{\mu+}\right) \\ & - \frac{1}{2}\,b\,\bar{\epsilon}_+\gamma^0\psi_{\rho+}\tau^\rho - \frac{1}{2}\,b_a\,\tau^\rho\left(\bar{\epsilon}_+\gamma^a\psi_{\rho-} + \bar{\epsilon}_-\gamma^a\psi_{\rho+}\right). \end{aligned} \tag{5.28}$$

Therefore, we could choose to set $b = 0$ already at this point. Doing so would induce the compensating transformation

$$\begin{aligned}\Lambda_K &= -\tau^\mu \partial_\mu \Lambda_D + \lambda^a\, b_a + \frac{1}{4}\,\tau^\mu\,(\bar\epsilon_+ \gamma^0 \phi_\mu + \bar\eta\,\gamma^0 \psi_{\mu+}) \\ &\quad + \frac{1}{2}\, b_a\, \tau^\rho\,(\bar\epsilon_+ \gamma^a \psi_{\rho-} + \bar\epsilon_- \gamma^a \psi_{\rho+})\,.\end{aligned} \tag{5.29}$$

However, we will not set $b = 0$ just yet because we want to keep special conformal transformations in our supergravity background. Still, we would also like to point out that since no independent field transforms under special conformal transformations there would in essence be no effect from this gauge-fixing, when considering only independent fields.

We proceed with determining the spin- and boost-connection $\omega_\mu{}^{ab}$ and $\omega_\mu{}^a$. This calculation works in precisely the same way as we described in chapter 4, using the constraints $\mathcal{R}_{\mu\nu}{}^a(P) = 0$ and $\mathcal{R}_{\mu\nu}(Z) = 0$. It leads to the expressions

$$\begin{aligned}\omega_\mu{}^{ab} &= 2\, e^{\nu[a}\big(\partial_{[\nu} e_{\mu]}{}^{b]} - \frac{1}{2}\,\psi_{[\nu+}\gamma^{b]}\psi_{\mu]-} - b_{[\nu}\, e_{\mu]}{}^{b]}\big) \\ &\quad + e_\mu{}^c e^{\rho a} e^{\nu b}\big(\partial_{[\rho} e_{\nu]}{}^c - \frac{1}{2}\,\psi_{[\rho+}\gamma^c\psi_{\nu]-} - b_{[\rho}\, e_{\nu]}{}^c\big) \\ &\quad - \tau_\mu e^{\rho a} e^{\nu b}\big(\partial_{[\rho} m_{\nu]} - \frac{1}{2}\,\psi_{[\rho-}\gamma^0\psi_{\nu]-}\big)\,,\end{aligned} \tag{5.30}$$

$$\begin{aligned}\omega_\mu{}^a &= -\tau^\nu\big(\partial_{[\nu} e_{\mu]}{}^a - \frac{1}{2}\,\psi_{[\nu+}\gamma^a\psi_{\mu]-} - b_{[\nu}\, e_{\mu]}{}^a\big) \\ &\quad + e_\mu{}^c e^{\rho a}\tau^\nu\big(\partial_{[\rho} e_{\nu]}{}^c - \frac{1}{2}\,\psi_{[\rho+}\gamma^c\psi_{\nu]-} - b_{[\rho}\, e_{\nu]}{}^c\big) \\ &\quad + e^{\nu a}\big(\partial_{[\mu} m_{\nu]} - \frac{1}{2}\,\psi_{[\mu-}\gamma^0\psi_{\nu]-}\big) - \tau_\mu e^{\rho a}\tau^\nu\big(\partial_{[\rho} m_{\nu]} - \frac{1}{2}\,\psi_{[\rho-}\gamma^0\psi_{\nu]-}\big)\,,\end{aligned} \tag{5.31}$$

which of course differ from (4.21) and (4.22) because of the additional dilatation gauge-field b_μ. The field ϕ_μ is determined by the constraints in the third line of (5.17) giving

$$\begin{aligned}e^\mu{}_a\,\phi_\mu &= -e^\mu{}_a\tau^\nu\,\big(2\,\partial_{[\mu}\psi_{\nu]+} - \frac{1}{2}\,\omega_{[\mu}{}^{ab}\gamma_{ab}\psi_{\nu]+} - 2\, b_{[\mu}\psi_{\nu]+} + 2\, r_{[\mu}\gamma_0\psi_{\nu]+}\big)\,, \\ \tau^\mu\phi_\mu &= \tau^\mu e^\nu{}_c\,\gamma^{0c}\,\big(2\,\partial_{[\mu}\psi_{\nu]-} - \frac{1}{2}\,\omega_{[\mu}{}^{ab}\gamma_{ab}\psi_{\nu]-} + \omega_{[\mu}{}^a\gamma_{a0}\psi_{\nu]+} - 2\, r_{[\mu}\gamma_0\psi_{\nu]-}\big)\,.\end{aligned} \tag{5.32}$$

Thus, we obtain

$$\begin{aligned}\phi_\mu = & -\tau^\nu \left(2\,\partial_{[\mu}\psi_{\nu]+} - \frac{1}{2}\,\omega_{[\mu}{}^{ab}\gamma_{ab}\psi_{\nu]+} - 2\,b_{[\mu}\psi_{\nu]+} + 2\,r_{[\mu}\gamma_0\psi_{\nu]+}\right) \\ & + \tau_\mu\tau^\rho e^\nu{}_c\gamma^{0c}\left(2\,\partial_{[\rho}\psi_{\nu]-} - \frac{1}{2}\,\omega_{[\rho}{}^{ab}\gamma_{ab}\psi_{\nu]-} + \omega_{[\rho}{}^a\gamma_{a0}\psi_{\nu]+} - 2\,r_{[\mu}\gamma_0\psi_{\nu]-}\right).\end{aligned} \tag{5.33}$$

Finally, to solve for f_μ we use $\mathcal{R}_{a0}(D) = 0$ and $\mathcal{R}_{0a}{}^a(G) = 0$. We get, from the first and the second identity, respectively,

$$\begin{aligned} e^\mu{}_a\, f_\mu &= e^\mu{}_a\tau^\nu \left(2\,\partial_{[\mu}b_{\nu]} + \frac{1}{2}\,\bar{\psi}_{[\mu+}\gamma^0\phi_{\nu]}\right), \\ \tau^\mu f_\mu &= \frac{1}{2}\,\tau^\mu e^\nu{}_a\left(2\,\partial_{[\mu}\omega_{\nu]}{}^a - 2\,\omega_{[\mu}{}^{ab}\omega_{\nu]}{}^b - 2\,\omega_{[\mu}{}^a b_{\nu]} + \bar{\phi}_{[\mu}\gamma^a\psi_{\nu]-}\right) \\ &\quad - \frac{1}{2}\,e^\mu{}_a\,\bar{\psi}_{\mu-}\gamma^0\hat{\psi}_{a0-}\,. \end{aligned} \tag{5.34}$$

Hence, the full expression for f_μ is

$$\begin{aligned} f_\mu = & \,\tau^\nu\left(2\,\partial_{[\mu}b_{\nu]} + \frac{1}{2}\,\bar{\psi}_{[\mu+}\gamma^0\phi_{\nu]}\right) - \frac{1}{2}\,\tau_\mu e^\rho{}_a\,\bar{\psi}_{\rho-}\gamma^0\hat{\psi}_{a0-} \\ & + \frac{1}{2}\,\tau_\mu\tau^\rho e^\nu{}_a\left(2\,\partial_{[\rho}\omega_{\nu]}{}^a - 2\,\omega_{[\rho}{}^{ab}\omega_{\nu]}{}^b - 2\,\omega_{[\rho}{}^a b_{\nu]} + \bar{\phi}_{[\rho}\gamma^a\psi_{\nu]-}\right). \end{aligned} \tag{5.35}$$

At this point we have found all dependent gauge-fields in terms of the independent ones. In the following we derive the transformation rules of the dependent fields. We find

$$\begin{aligned} \delta\omega_\mu{}^{ab} &= \partial_\mu\lambda^{ab}\,, \\ \delta\omega_\mu{}^a &= \partial_\mu\lambda^a - \omega_\mu{}^a{}_b\lambda^b + b_\mu\lambda^a + \lambda^a{}_b\,\omega_\mu{}^b - \Lambda_D\,\omega_\mu{}^a + \Lambda_K\,e_\mu{}^a\,, \\ \delta f_\mu &= \partial_\mu\Lambda_K + 2\,\Lambda_K\,b_\mu - 2\,\Lambda_D\,f_\mu - \tau_\mu\,\lambda^b\,\mathcal{R}_{0a}{}^{ab}(J)\,, \\ \delta\phi_\mu &= \frac{1}{4}\,\lambda^{ab}\gamma_{ab}\phi_\mu - \Lambda_D\,\phi_\mu - \Lambda_K\,\psi_{\mu+} - \gamma_0\phi_\mu\,\rho\,, \end{aligned} \tag{5.36}$$

i.e. most transformation rules are un-altered. The only change appears for f_μ, which acquires a non-trivial transformation under Galilean boosts. In [29] this was circumvented by redefining f_μ by adding terms with m_μ and $\mathcal{R}_{\mu\nu}{}^{ab}(J)$ in one of the conventional constraints that is used to solve for f_μ. However, then the field acquired a non-trivial transformation under the central charge symmetry Z. We will not attempt any redefinition of that kind here.

Concerning the fermionic Q and S-transformations we calculate

$$
\begin{aligned}
\delta\omega_\mu{}^{ab} &= -\frac{1}{4}\,\bar{\epsilon}_+\gamma^{ab0}\phi_\mu + \frac{1}{4}\,\bar{\eta}\,\gamma^{ab0}\psi_{\mu+}\,,\\
\delta\omega_\mu{}^{a} &= \bar{\epsilon}_-\gamma^0\hat{\Psi}_\mu{}^a{}_- - \frac{1}{2}\,\bar{\epsilon}_-\gamma^a\phi_\mu + \frac{1}{4}\,e_{\mu b}\,\bar{\epsilon}_+\gamma^b\hat{\Psi}^a{}_{0-}\\
&\quad + \frac{1}{4}\,\bar{\epsilon}_+\gamma^a\hat{\Psi}_{\mu 0-} - \frac{1}{2}\,\bar{\eta}\,\gamma^a\psi_{\mu-}\,,\\
\delta\phi_\mu &= D_\mu\eta + b_\mu\,\eta + r_\mu\,\gamma_0\eta + f_\mu\,\epsilon_+ + \gamma_0\epsilon_+\left[\frac{1}{4}\,\varepsilon^{ab}\,\mathcal{R}_{\mu 0}{}^{ab}(J) - \mathcal{R}_{\mu 0}(R)\right]\\
&\quad + \gamma^c\epsilon_-\left[\frac{1}{4}\,\varepsilon^{ab}\,\mathcal{R}_{\mu c}{}^{ab}(J) + \mathcal{R}_{\mu c}(R)\right].
\end{aligned}
\tag{5.37}
$$

These transformations allow us to explicitly check that the commutator algebra (of the fermionic symmetries) is realized by the formula

$$
\begin{aligned}
\big[\delta(Q_1,S_1),\delta(Q_2,S_2)\big] &= \delta_{\text{g.c.t.}}\big(\Xi^\mu\big) + \delta_J\big(\Lambda^{ab}\big) + \delta_G\big(\Lambda^a\big) + \delta_Z\big(\Sigma\big) + \delta_R\big(\rho_R\big)\\
&\quad + \delta_{Q_+}\big(\Upsilon_+\big) + \delta_{Q_-}\big(\Upsilon_-\big) + \delta_D\big(\lambda_D\big) + \delta_K\big(\lambda_K\big)\,,
\end{aligned}
\tag{5.38}
$$

where the parameters are given by

$$
\begin{aligned}
\Xi^\mu &= \frac{1}{2}\,\bar{\epsilon}_{2+}\gamma^0\epsilon_{1+}\,\tau^\mu + \frac{1}{2}\big(\bar{\epsilon}_{2+}\gamma^a\epsilon_{1-} + \bar{\epsilon}_{2-}\gamma^a\epsilon_{1+}\big)e^\mu{}_a\,,\\
\Lambda^{ab} &= -\Xi^\mu\omega_\mu{}^{ab} + \frac{1}{4}\big(\bar{\epsilon}_{1+}\gamma^{0ab}\eta_2 - \bar{\eta}_1\,\gamma^{0ab}\epsilon_{2+}\big)\,,\\
\Lambda^a &= -\Xi^\mu\omega_\mu{}^a + \frac{1}{2}\big(\bar{\epsilon}_{1-}\gamma^a\eta_2 + \bar{\eta}_1\,\gamma^a\epsilon_{2-}\big)\,,\\
\Sigma &= -\Xi^\mu m_\mu + \bar{\epsilon}_{2-}\gamma^0\epsilon_{1-}\,,\\
\lambda_D &= -\Xi^\mu b_\mu + \frac{1}{4}\big(\bar{\epsilon}_{1+}\gamma^0\eta_2 + \bar{\eta}_1\,\gamma^0\epsilon_{2+}\big)\,,\\
\Upsilon_\pm &= -\Xi^\mu\psi_{\mu\pm}\,,\\
\lambda_K &= -\Xi^\mu f_\mu + \frac{1}{2}\,\bar{\eta}_2\,\gamma^0\eta_1\,,\\
\rho_R &= -\Xi^\mu r_\mu + \frac{3}{8}\big(\bar{\epsilon}_{1+}\eta_2 - \bar{\eta}_1\,\epsilon_{2+}\big)\,,\\
\eta &= -\Xi^\mu\phi_\mu\,.
\end{aligned}
\tag{5.39}
$$

This finishes the discussion of the "conformal" supergravity background.

Note that our analysis of the Schrödinger theory is not fully complete, since we did not derive the variation of the dependent field f_μ under fermionic symmetries. Even so, this was not needed to show that the set of constraints (5.25) is a consistent one and that the commutator algebra closes on all independent fields.

5.3 Concluding remarks

In previous works, see [77] and also chapter 4, the gauging was followed by a gauge-fixing to a Galilean observer. However, as shown in [29] this procedure does not lead to any new results with respect to the Bargmann case. The reason is that the Schrödinger theory is "gauge-equivalent" to the Newton–Cartan theory plus a scalar, see also [102]. Gauge-fixing the scalar, much in the same way as we are going to do in the next chapter, eliminates the dilatation symmetry and from there on the theory is equivalent to Newton–Cartan gravity. Therefore, we refrain from performing this analysis here.

An interesting problem is to define dynamics for this supergravity background. It is indeed highly non-trivial to find an equation of motion for the independent fields that is invariant under all symmetries. In the case of Newton–Cartan (super)gravity we would interpret the constraint $\hat{R}_{0a}{}^{a}(G) = 0$ as the equation of motion. Here, $\mathcal{R}_{0a}{}^{a}(G) = 0$ is a conventional constraint that we used to solve for f_μ. Because this constraint is not invariant under boost transformations we cannot reduce it to an invariant equation of motion. Not unless we require that $\mathcal{R}_{\mu\nu}{}^{ab}(J) = 0$ which, on the other hand, is not part of the constraints that we discussed in subsection 5.2.2. A systematic approach to find an equation of motion for Schrödinger (super)gravity can be found in [102].

While it is interesting to derive a non-relativistic theory of "conformal" supergravity, this was not the only motivation for us to do so. In the next chapter, we will use the Schrödinger supergravity theory as the basis for the Schrödinger tensor calculus. For example, we will derive a theory of Newton–Cartan supergravity with torsion. This is not surprising as e.g. the analysis in [29] proved that gauging the Schrödinger algebra can be used for precisely that purpose. In hindsight, the interesting and surprising fact is that there indeed does exist a truncation to zero torsion, i.e. to the Newton–Cartan supergravity theories that we have seen so far.

Last but not least, let us draw your attention to a connection of the theory discussed in this chapter, to condensed matter theory. The authors of [73] used the very same non-relativistic Chern–Simons theory of [91], whose algebra we just gauged in this section, and showed that it too can be used to describe features of the fractional Quantum Hall effect.

6

Schrödinger tensor calculus

In this chapter we present another generalization of three-dimensional Newton–Cartan supergravity, namely the theory with non-zero torsion. To derive it we use "superconformal", hereafter called Schrödinger, techniques where compensating matter multiplets are used to gauge-fix the extra symmetries (with respect to the Bargmann algebra) of the Schrödinger supergravity theory of the last chapter.

In this last chapter we derive yet another generalization of Newton–Cartan supergravity, the theory with non-zero torsion [137]. To do so, we will use a non-relativistic version of the superconformal tensor calculus, see e.g. [97] for an introduction and references. This chapter will also serve as an exposition of how these methods work and how they can be applied in the non-relativistic case.

Before doing this in the non-relativistic case we will briefly recall how it works in the relativistic case. The main idea is to gauge-fix the extra symmetries that the superconformal theory possesses with respect to the Poincaré symmetries. For this purpose, one adds (at least one) so-called compensator multiplet that also realizes the superconformal algebra. Then, one fixes dilatations and conformal S-supersymmetry by eliminating/gauge-fixing fields of the compensator multiplet.

Let us be more concrete and discuss a well-known example. The four-dimensional $\mathcal{N} = 1$ Weyl multiplet consists of the independent fields $(E_\mu{}^A, \Psi_\mu, A_\mu)$ which realize the superconformal algebra. (Special conformal symmetries were gauge-fixed by setting the gauge-field of dilatations B_μ to zero.) As a compensator, we might consider the chiral multiplet which comprises two complex scalars and a spinor (Φ, χ, F). To derive a Poincaré multiplet from the Weyl multiplet we need to gauge-fix dilatations D, R-symmetry and conformal S-supersymmetry. We may choose

$$\begin{aligned} \Phi = 1 \;: &\qquad \text{fixes dilatations } D \text{ and } R\text{-symmetry} \;, \\ \chi = 0 \;: &\qquad \text{fixes conformal } S\text{-supersymmetry} \,. \end{aligned} \tag{6.1}$$

With this analysis we derive the old minimal Poincaré multiplet which consists of $(E_\mu{}^A, \Psi_\mu, A_\mu, F)$ [142–145]. Alternatively, one can use a different compensator multiplet to obtain the new minimal formulation with the fields $(E_\mu{}^A, \Psi_\mu, V_\mu, A_\mu, D)$ [146–148]. Here D is a real scalar and the theory still enjoys a global $U(1)$-symmetry.

We will find that the non-relativistic case works quite similarly. Our equivalent of the Weyl multiplet is given by the Schrödinger multiplet which we derived in chapter 5 and it consists of the independent fields $(\tau_\mu, e_\mu{}^a, m_\mu, r_\mu, \psi_{\mu\pm})$. In analogy to the relativistic case we eliminate the scalar b to get rid of special conformal transformations.[1] Furthermore, we need to gauge-fix dilatations and S-supersymmetry using a compensating matter multiplet.

In this chapter we will use two different multiplets for that purpose, which are related to the scalar and the vector multiplet, respectively, in the relativistic case. This will lead to the non-relativistic analogs of the old and the new minimal formulation of Poincaré supergravity, which we discussed above. In particular, we find the "old" minimal multiplet with the independent field content (see subsection 6.2.1)

$$\text{non-relativistic old minimal} \;: \qquad (\tau_\mu, e_\mu{}^a, m_\mu, r_\mu, \psi_{\mu\pm}, \chi_-, F_1, F_2) \,, \tag{6.2}$$

[1] As shown in the (bosonic) analysis in [102] there is in fact no need to eliminate b as it simply drops out of all expressions. We did not investigate if this is also true in the supersymmetric case.

and the "new" minimal formulation with (see subsection 6.2.2)

$$\text{non-relativistic new minimal}: \quad \left(\tau_\mu, e_\mu{}^a, m_\mu, r_\mu, \psi_{\mu\pm}, S\right). \tag{6.3}$$

It was shown in [29], and chapter 5, that the gauging of the Schrödinger algebra naturally leads to Newton–Cartan geometry with torsion. In the previous chapter we found that the torsion is provided by the spatial components of the dilatation gauge-field b_a, that are dependent on the other fields. This feature remained in the construction of the Schrödinger supergravity multiplet and our non-relativistic Schrödinger tensor calculus therefore naturally leads to *torsionfull* Newton–Cartan supergravity theories. In this way, we are thus able to extend the constructions of [77] and chapter 3 to the torsionfull case. The torsionless case can be retrieved by putting the torsion to zero. As the torsion is provided by gauge-field components that depend on the other fields in the supergravity multiplet, this truncation is non-trivial and its consistency has to be examined. We will study this truncation only in the case of non-relativistic new minimal supergravity and we will show that this truncation leads to the off-shell three-dimensional $\mathcal{N} = 2$ theory of chapter 3.

We have organized this chapter as follows. The next section 6.1 is entirely devoted to the derivation of two non-relativistic matter multiplets. These will be used in section 6.2 to derive Newton–Cartan supergravity with non-zero torsion. The truncation to a theory with zero torsion is investigated in section 6.3 and in section 6.4 we end with a discussion.

6.1 Matter couplings

In this section we present matter multiplets that realize the same commutators corresponding to the Schrödinger superalgebra as we derived for the Schrödinger supergravity multiplet, see eq. (5.39). These multiplets will be used in the following section as compensator multiplets to derive off-shell formulations of Newton–Cartan supergravity.

One of those off-shell formulations was discussed in chapter 3. We obtained it by taking a non-relativistic limit of the three-dimensional $\mathcal{N} = 2$ new minimal Poincaré multiplet [114]. Recall that the new minimal Poincaré multiplet follows from superconformal techniques using a compensating (relativistic) vector multiplet. Hence, in order to derive its non-relativistic analog, we should use as a compensator a non-relativistic vector multiplet. This is one of the two non-relativistic matter multiplets which we derive in this section. The other one is the scalar multiplet, which we shall later use to derive a new off-shell formulation of Newton–Cartan supergravity.

This section also focuses on the method that we use to derive those novel matter multiplets. It would be very efficient if we could use the non-relativistic limiting procedure of chapter 3. However, we cannot do so because the Schrödinger superalgebra does not follow from the contraction of any relativistic algebra and the same applies to the Schrödinger supergravity theory. Instead, we shall proceed in the following way. We start from the rigid version of a relativistic matter multiplet that realizes the Poincaré superalgebra and take the non-relativistic limit thereof.[2] The important thing is that at this point we have derived the field content of the non-relativistic multiplet. It turns out that the same multiplet also provides a representation of the rigid Schrödinger superalgebra. Therefore, once we have obtained this non-relativistic matter multiplet, we can couple it to the fields of the Schrödinger supergravity theory, thereby realizing the commutator algebra derived in the previous section, in the standard way.

6.1.1 The scalar multiplet

In this subsection we construct the non-relativistic scalar multiplet. We start with the three-dimensional rigid relativistic $\mathcal{N} = 2$ scalar multiplet which comprises two complex scalar and two spinors. In real notation we are thus left with the fields $(\varphi_1, \varphi_2, \chi_1, \chi_2; F_1, F_2)$:

$$\begin{aligned}
\delta\varphi_1 &= \bar{\eta}_1\chi_1 + \bar{\eta}_2\chi_2\,,\\
\delta\varphi_2 &= \bar{\eta}_1\chi_2 - \bar{\eta}_2\chi_1\,,\\
\delta\chi_1 &= \frac{1}{4}\,\gamma^\mu\partial_\mu\varphi_1\,\eta_1 - \frac{1}{4}\,\gamma^\mu\partial_\mu\varphi_2\,\eta_2 - \frac{1}{4}\,F_1\,\eta_1 - \frac{1}{4}\,F_2\,\eta_2\,,\\
\delta\chi_2 &= \frac{1}{4}\,\gamma^\mu\partial_\mu\varphi_2\,\eta_1 + \frac{1}{4}\,\gamma^\mu\partial_\mu\varphi_1\,\eta_2 - \frac{1}{4}\,F_2\,\eta_1 + \frac{1}{4}\,F_1\,\eta_2\,,\\
\delta F_1 &= -\bar{\eta}_1\gamma^\mu\partial_\mu\chi_1 + \bar{\eta}_2\gamma^\mu\partial_\mu\chi_2\,,\\
\delta F_2 &= -\eta_1\gamma^\mu\partial_\mu\chi_2 - \eta_2\gamma^\mu\partial_\mu\chi_1\,.
\end{aligned} \tag{6.4}$$

Note that we do not consider bosonic transformations yet. We will add them only to the final result.

Let us recall the salient points of the limiting procedure that we introduced in chapter 3. To take the limit we use a limiting parameter ω which we will send to infinity. The rescaling of the symmetry parameters follows from the Inönü–Wigner contraction of the related symmetry generators. This means for example that we will require

$$\epsilon_\pm = \frac{\omega^{\mp 1/2}}{\sqrt{2}}\,(\eta_1 \pm \gamma_0\eta_2)\,. \tag{6.5}$$

It remains to find the scalings of all other fields which are not determined a priori by the procedure, like for example the scalings of the auxiliary fields. Those scalings are relatively easy to find and it turns out that we need to use

$$\chi_\pm = \frac{\omega^{-1\pm 1/2}}{\sqrt{2}}\,(\chi_1 \pm \gamma_0\chi_2)\,, \tag{6.6}$$

[2] This (rigid) limit coincides with the non-relativistic limit performed in [65].

for the two spinors, while we need to take

$$\tilde{\varphi}_i = \frac{1}{\omega}\,\varphi_i\,, \qquad \tilde{F}_i = -\frac{1}{\omega}\,F_i\,, \tag{6.7}$$

for the scaling of the bosons. Note that the scaling of the spinor variables (6.6) differs from the rescaling of the gravitini (which of course scale in the same way as the symmetry parameters $\epsilon_\pm$ (6.5)). The effect is that they transform oppositely under boost transformations, i.e. the role of plus and minus spinors is exchanged, see (6.9).

Using (6.5)–(6.7) in (6.4) we can derive the transformation rules for finite ω. After calculating the transformation rules in the limit $\omega \to \infty$ we drop the tildes and find

$$\begin{aligned}
\delta\varphi_1 &= \bar{\epsilon}_+\chi_+ + \bar{\epsilon}_-\chi_-\,,\\
\delta\varphi_2 &= \bar{\epsilon}_+\gamma^0\chi_+ - \bar{\epsilon}_-\gamma^0\chi_-\,,\\
\delta\chi_+ &= \frac{1}{4}\,\gamma^0\epsilon_+\,\partial_t\varphi_1 + \frac{1}{4}\,\epsilon_+\,\partial_t\varphi_2 + \frac{1}{4}\,\gamma^i\epsilon_-\,\partial_i\varphi_1 + \frac{1}{4}\,\gamma^{i0}\epsilon_-\,\partial_i\varphi_2\\
&\quad + \frac{1}{4}\,\epsilon_-\,F_1 + \frac{1}{4}\,\gamma_0\epsilon_-\,F_2\,,\\
\delta\chi_- &= \frac{1}{4}\,\gamma^i\epsilon_+\,\partial_i\varphi_1 - \frac{1}{4}\,\gamma^{i0}\epsilon_+\,\partial_i\varphi_2 + \frac{1}{4}\,\epsilon_+\,F_1 - \frac{1}{4}\,\gamma_0\epsilon_+\,F_2\,,\\
\delta F_1 &= \bar{\epsilon}_+\gamma^i\partial_i\chi_+ + \bar{\epsilon}_+\gamma^0\partial_t\chi_- + \bar{\epsilon}_-\gamma^i\partial_i\chi_-\,,\\
\delta F_2 &= \bar{\epsilon}_+\gamma^{i0}\partial_i\chi_+ + \bar{\epsilon}_+\partial_t\chi_- - \bar{\epsilon}_-\gamma^{i0}\partial_i\chi_-\,.
\end{aligned} \tag{6.8}$$

Together with the bosonic transformation rules, which we refrain from giving here but which can be obtained easily by similar techniques, the transformation rules (6.8) realize the rigid Bargmann superalgebra. Next, we promote this multiplet to a representation of the rigid Schrödinger superalgebra by assigning transformations under the Schrödinger transformations that are not contained in the Bargmann superalgebra. After that, we couple the multiplet to the fields of Schrödinger supergravity. Following standard techniques of coupling matter to supergravity we find for the bosonic transformations

$$
\begin{aligned}
\delta\varphi_1 &= w\,\Lambda_D\varphi_1 + \frac{2w}{3}\,\rho\,\varphi_2\,,\\
\delta\varphi_2 &= w\,\Lambda_D\varphi_2 - \frac{2w}{3}\,\rho\,\varphi_1\,,\\
\delta\chi_+ &= \frac{1}{4}\,\lambda^{ab}\gamma_{ab}\chi_+ - \frac{1}{2}\,\lambda^a\gamma_{a0}\chi_- + (w-1)\,\Lambda_D\chi_+ - \big(\frac{2w}{3}+1\big)\,\gamma_0\chi_+\,\rho\,,\\
\delta\chi_- &= \frac{1}{4}\,\lambda^{ab}\gamma_{ab}\chi_- + w\,\Lambda_D\chi_- + \big(\frac{2w}{3}+1\big)\,\gamma_0\chi_-\,\rho\,,\\
\delta F_1 &= (w-1)\,\Lambda_D F_1 + 2\,\big(\frac{w}{3}+1\big)\,\rho\,F_2\,,\\
\delta F_2 &= (w-1)\,\Lambda_D F_2 - 2\,\big(\frac{w}{3}+1\big)\,\rho\,F_1\,,
\end{aligned}
\tag{6.9}
$$

while the fermionic transformation rules are given by

$$
\begin{aligned}
\delta\varphi_1 &= \bar\epsilon_+\chi_+ + \bar\epsilon_-\chi_-\,,\\
\delta\varphi_2 &= \bar\epsilon_+\gamma^0\chi_+ - \bar\epsilon_-\gamma^0\chi_-\,,\\
\delta\chi_+ &= \frac{1}{4}\,\gamma^0\epsilon_+\,\tau^\mu\hat D_\mu\varphi_1 + \frac{1}{4}\,\epsilon_+\,\tau^\mu\hat D_\mu\varphi_2 + \frac{1}{4}\,\gamma^a\epsilon_-\,e^\mu{}_a\hat D_\mu\varphi_1\\
&\quad + \frac{1}{4}\,\gamma^{a0}\epsilon_-\,e^\mu{}_a\hat D_\mu\varphi_2 + \frac{1}{4}\,\epsilon_-\,F_1 + \frac{1}{4}\,\gamma_0\epsilon_-\,F_2 - \frac{w}{4}\,\gamma^0\eta\,\varphi_1 - \frac{w}{4}\,\eta\,\varphi_2\,,\\
\delta\chi_- &= \frac{1}{4}\,\gamma^a\epsilon_+\,e^\mu{}_a\hat D_\mu\varphi_1 - \frac{1}{4}\,\gamma^{a0}\epsilon_+\,e^\mu{}_a\hat D_\mu\varphi_2 + \frac{1}{4}\,\epsilon_+\,F_1 - \frac{1}{4}\,\gamma_0\epsilon_+\,F_2\,,\\
\delta F_1 &= \bar\epsilon_+\gamma^a e^\mu{}_a\hat D_\mu\chi_+ + \bar\epsilon_+\gamma^0\tau^\mu\hat D_\mu\chi_- + \bar\epsilon_-\gamma^a e^\mu{}_a\hat D_\mu\chi_- - (w+1)\,\bar\eta\,\gamma^0\chi_-\,,\\
\delta F_2 &= \bar\epsilon_+\gamma^{a0}e^\mu{}_a\hat D_\mu\chi_+ + \bar\epsilon_+\tau^\mu\hat D_\mu\chi_- - \bar\epsilon_-\gamma^{a0}e^\mu{}_a\hat D_\mu\chi_- - (w+1)\,\bar\eta\,\chi_-\,.
\end{aligned}
\tag{6.10}
$$

The covariant derivatives that appear in (6.10) can be deduced from the transformation rules (6.9) and (6.10). For the bosonic fields they are given by

$$
\begin{aligned}
\hat D_\mu\varphi_1 &= \partial_\mu\varphi_1 - w\,b_\mu\,\varphi_1 - \frac{2w}{3}\,r_\mu\,\varphi_2 - \bar\psi_{\mu+}\chi_+ - \bar\psi_{\mu-}\chi_-\,,\\
\hat D_\mu\varphi_2 &= \partial_\mu\varphi_2 - w\,b_\mu\,\varphi_2 + \frac{2w}{3}\,r_\mu\,\varphi_1 - \bar\psi_{\mu+}\gamma^0\chi_+ + \bar\psi_{\mu-}\gamma^0\chi_-\,,\\
\hat D_\mu F_1 &= \partial_\mu F_1 - (w-1)\,b_\mu\,F_1 - 2\,\big(\frac{w}{3}+1\big)\,r_\mu\,F_2 - \bar\psi_{\mu+}\gamma^a e^\rho{}_a\hat D_\rho\chi_+\\
&\quad - \bar\psi_{\mu+}\gamma^0\tau^\rho\hat D_\rho\chi_- - \bar\psi_{\mu-}\gamma^a e^\rho{}_a\hat D_\rho\chi_- + (w+1)\,\bar\phi_\mu\gamma^0\chi_-\,,\\
\hat D_\mu F_2 &= \partial_\mu F_2 - (w-1)\,b_\mu\,F_2 + 2\,\big(\frac{w}{3}+1\big)\,r_\mu\,F_1 - \bar\psi_{\mu+}\gamma^{a0}e^\rho{}_a\hat D_\rho\chi_+\\
&\quad - \bar\psi_{\mu+}\tau^\rho\hat D_\rho\chi_- + \bar\psi_{\mu-}\gamma^{a0}e^\rho{}_a\hat D_\rho\chi_- + (w+1)\,\bar\phi_\mu\chi_-\,,
\end{aligned}
\tag{6.11}
$$

while for the covariant derivatives of the fermions we find

$$\begin{aligned}
\hat{D}_\mu\chi_+ &= D_\mu\chi_+ + \frac{1}{2}\,\omega_\mu{}^a\gamma_{a0}\chi_- - (w-1)\,b_\mu\chi_+ + \big(\frac{2w}{3}+1\big)\,r_\mu\,\gamma_0\chi_+ \\
&\quad - \frac{1}{4}\,\gamma^0\psi_{\mu+}\,\tau^\rho\hat{D}_\rho\varphi_1 - \frac{1}{4}\,\psi_{\mu+}\,\tau^\rho\hat{D}_\rho\varphi_2 - \frac{1}{4}\,\gamma^a\psi_{\mu-}\,e^\rho{}_a\hat{D}_\rho\varphi_1 \\
&\quad - \frac{1}{4}\,\gamma^{a0}\psi_{\mu-}\,e^\rho{}_a\hat{D}_\rho\varphi_2 - \frac{1}{4}\,\psi_{\mu-}\,F_1 - \frac{1}{4}\,\gamma_0\psi_{\mu-}\,F_2 \\
&\quad + \frac{w}{4}\,\gamma^0\phi_\mu\,\varphi_1 + \frac{w}{4}\,\phi_\mu\,\varphi_2\,, \\
\hat{D}_\mu\chi_- &= D_\mu\chi_- - w\,b_\mu\,\chi_- - \big(\frac{2w}{3}+1\big)\,r_\mu\,\gamma_0\chi_- - \frac{1}{4}\,\gamma^a\psi_{\mu+}\,e^\rho{}_a\hat{D}_\rho\varphi_1 \\
&\quad + \frac{1}{4}\,\gamma^{a0}\psi_{\mu+}\,e^\rho{}_a\hat{D}_\rho\varphi_2 - \frac{1}{4}\,\psi_{\mu+}\,F_1 + \frac{1}{4}\,\gamma_0\psi_{\mu+}\,F_2\,.
\end{aligned} \tag{6.12}$$

This completes our derivation of the first non-relativistic matter multiplet. In section 6.2 we will use this scalar multiplet to derive an new off-shell formulation of Newton–Cartan supergravity.

6.1.2 The vector multiplet

The $\mathcal{N}=2$ vector multiplet in three dimensions contains a vector, a physical scalar, two spinors and an auxiliary scalar (C_μ,ρ,λ_i,D). Using the three-dimensional epsilon symbol we can define a new "dual" vector $V_\mu = \varepsilon_\mu{}^{\nu\rho}\,\partial_\nu C_\rho$ which obeys

$$\partial^\mu V_\mu = 0\,, \tag{6.13}$$

and has the dimension of an auxiliary field. In terms of (ρ,λ_i,V_μ,D) we have the following supersymmetry transformation rules

$$\begin{aligned}
\delta\rho &= \varepsilon^{ij}\,\bar{\eta}_i\lambda_j\,, \\
\delta\lambda_i &= -\frac{1}{2}\,\gamma^\mu\eta_i\,V_\mu - \frac{1}{2}\,\varepsilon^{ij}\eta_j\,D - \frac{1}{4}\,\gamma^\mu\varepsilon^{ij}\eta_j\,\partial_\mu\rho\,, \\
\delta D &= \frac{1}{2}\,\varepsilon^{ij}\,\bar{\eta}_i\,\gamma^\mu\partial_\mu\lambda_j\,, \\
\delta V_\mu &= \frac{1}{2}\,\delta^{ij}\,\bar{\eta}_i\,\gamma_\mu{}^\nu\partial_\nu\lambda_j\,.
\end{aligned} \tag{6.14}$$

Note that the constraint (6.13) is needed to close the algebra on the relativistic multiplet.

Next, we perform the non-relativistic limiting procedure. We have to find the scalings of the fields starting again with (6.5). We define new spinors

$$\lambda_\pm = \frac{\omega^{-1\pm1/2}}{\sqrt{2}}\,(\lambda_1 \pm \gamma_0\lambda_2)\,, \tag{6.15}$$

and the bosonic field

$$\phi = \frac{\rho}{\omega}\,. \tag{6.16}$$

Furthermore, we find it useful to introduce the new fields

$$\begin{aligned} S &= -\frac{1}{\omega}\,V_0 - D\,, \qquad\qquad F = \frac{1}{\omega^3}\,V_0 - \frac{1}{\omega^2}\,D\,, \\ C_i &= \frac{1}{\omega}\,\big(V_i + \frac{1}{2}\,\varepsilon^{ij}\,\partial_j\rho\big)\,. \end{aligned} \tag{6.17}$$

In the limit $\omega \to \infty$ this leads to the following supersymmetry transformations:

$$\begin{aligned} \delta\phi &= \bar{\epsilon}_+\gamma^0\lambda_+ - \bar{\epsilon}_-\gamma^0\lambda_-\,, \\ \delta\lambda_+ &= \frac{1}{4}\,\epsilon_+\,\partial_t\phi - \frac{1}{2}\,\gamma_0\epsilon_+\,S + \frac{1}{2}\,\gamma^{i0}\epsilon_-\,\partial_i\phi - \frac{1}{2}\,\gamma^i\epsilon_-\,C_i\,, \\ \delta S &= \frac{1}{2}\,\bar{\epsilon}_+\,\partial_t\lambda_+ - \bar{\epsilon}_-\gamma^{i0}\partial_i\lambda_+ - \frac{1}{2}\,\bar{\epsilon}_-\,\partial_t\lambda_-\,, \\ \delta C_i &= \bar{\epsilon}_-\gamma^{ij}\partial_j\lambda_- + \frac{1}{2}\,\bar{\epsilon}_+\gamma^{i0}\partial_t\lambda_-\,, \\ \delta\lambda_- &= -\frac{1}{2}\,\gamma^i\epsilon_+\,C_i + \frac{1}{2}\,\gamma_0\epsilon_-\,F\,, \\ \delta F &= \bar{\epsilon}_+\gamma^{i0}\partial_i\lambda_-\,. \end{aligned} \tag{6.18}$$

To prove closure one has to use the constraint

$$\partial^i C_i = \frac{1}{2}\,\partial_t F\,, \tag{6.19}$$

which follows from inserting the definitions (6.17) in the relativistic constraint (6.13) and sending $\omega \to \infty$.

An effect of taking the non-relativistic limit is that there exists a consistent truncation of this multiplet. As this simplifies the calculation that we will perform in the next section considerably, we impose

$$C_i = 0\,, \qquad\qquad F = 0\,, \qquad\qquad \lambda_- = 0\,. \tag{6.20}$$

This results in the following representation of the rigid Bargmann superalgebra:

$$\begin{aligned} \delta\phi &= \bar{\epsilon}_+\gamma^0\lambda_+\,, \\ \delta\lambda_+ &= \frac{1}{4}\,\epsilon_+\,\partial_t\phi - \frac{1}{2}\,\gamma_0\epsilon_+\,S + \frac{1}{2}\,\gamma^{i0}\epsilon_-\,\partial_i\phi\,, \\ \delta S &= \frac{1}{2}\,\bar{\epsilon}_+\,\partial_t\lambda_+ - \bar{\epsilon}_-\gamma^{i0}\partial_i\lambda_+\,. \end{aligned} \tag{6.21}$$

While this multiplet looks like a scalar multiplet and appears to be simpler than (6.8), its relation to the relativistic vector multiplet manifest itself in the following way. Due to the redefinition (6.17) the auxiliary field S is related to the zero component of the vector field. As a consequence of this the auxiliary fields transforms non-trivially under Galilean boost. This can already be seen in the rigid transformations but we will only give the bosonic transformations when we couple (6.21) to the Schrödinger supergravity. After coupling to supergravity the bosonic transformations read

$$\begin{aligned}
\delta\phi &= w\,\Lambda_D\phi\,,\\
\delta\lambda &= \frac{1}{4}\,\lambda^{ab}\gamma_{ab}\lambda + (w-1)\,\Lambda_D\,\lambda - \rho\,\gamma_0\lambda\,,\\
\delta S &= (w-2)\,\Lambda_D\,S - \frac{1}{2}\,\varepsilon^{ab}\lambda^a\,e^\mu{}_b\hat{D}_\mu\phi\,,
\end{aligned} \tag{6.22}$$

while the fermionic ones take the form

$$\begin{aligned}
\delta\phi &= \bar{\epsilon}_+\gamma^0\lambda\,,\\
\delta\lambda &= \frac{1}{4}\,\epsilon_+\,\tau^\mu\hat{D}_\mu\phi + \frac{1}{2}\,\gamma^{a0}\epsilon_-\,e^\mu{}_a\hat{D}_\mu\phi - \frac{1}{2}\,\gamma_0\epsilon_+\,S - \frac{w}{4}\,\eta\,\phi\,,\\
\delta S &= \frac{1}{2}\,\bar{\epsilon}_+\tau^\mu\hat{D}_\mu\lambda - \bar{\epsilon}_-\gamma^{a0}e^\mu{}_a\hat{D}_\mu\lambda - \frac{w-1}{2}\,\bar{\eta}\,\lambda\,.
\end{aligned} \tag{6.23}$$

Note the odd transformation of S under local Lorentz boosts, see eq. (6.22). This makes clear the vector multiplet origin of (6.22) and (6.23). In the formulas above we use the covariant derivatives

$$\begin{aligned}
\hat{D}_\mu\phi &= \partial_\mu\phi - \bar{\psi}_{\mu+}\gamma^0\lambda - w\,b_\mu\,\phi\,,\\
\hat{D}_\mu\lambda &= \partial_\mu\lambda - \frac{1}{4}\,\omega_\mu{}^{ab}\gamma_{ab}\lambda - (w-1)\,b_\mu\lambda + r_\mu\,\gamma_0\lambda + \frac{w}{4}\,\phi_\mu\,\phi\\
&\quad - \frac{1}{4}\,\psi_{\mu+}\,\tau^\nu\hat{D}_\nu\phi - \frac{1}{2}\,\gamma^{a0}\psi_{\mu-}\,e^\nu{}_a\hat{D}_\nu\phi + \frac{1}{2}\,\gamma_0\psi_{\mu+}\,S\,,\\
\hat{D}_\mu S &= \partial_\mu S + 2\,b_\mu\,S - \frac{1}{2}\,\bar{\psi}_{\mu+}\tau^\rho\hat{D}_\rho\lambda + \bar{\psi}_{\mu-}\gamma^{a0}e^\rho{}_a\hat{D}_\rho\lambda\\
&\quad + \frac{1}{2}\,\varepsilon^{ab}\,\omega_\mu{}^a\,e^\rho{}_b\hat{D}_\rho\phi + \frac{w-1}{2}\,\bar{\lambda}\,\phi_\mu\,.
\end{aligned} \tag{6.24}$$

This finishes our derivation of a second non-relativistic matter multiplet. In the following section we will use those two matter multiplets to derive two inequivalent formulations of Newton–Cartan supergravity with torsion. Before doing so we will give an overview of the multiplets, the matter and the supergravity ones, that we have discussed this far and which provide for us the basis of a non-relativistic superconformal tensor calculus, see table 6.1.

Note that if we were to add a third column to this table for Z-weight we would

Overview of non-relativistic multiplets

multiplet	field	type	D-weight	R-weight
Schrödinger	τ_μ	time-like vielbein	2	0
	$e_\mu{}^a$	spatial vielbein	1	0
	m_μ	Z gauge-field	0	0
	r_μ	R gauge-field	0	0
	b	"D gauge-field"	-2	0
	$\psi_{\mu+}$	Q_+ gravitino	1	-1
	$\psi_{\mu-}$	Q_- gravitino	0	1
Scalar	φ_1	physical scalar	w	$\frac{2w}{3}$
	φ_2	physical scalar	w	$-\frac{2w}{3}$
	χ_+	spinor	$w-1$	$-\frac{2w}{3}-1$
	χ_-	spinor	w	$\frac{2w}{3}+1$
	F_1	auxiliary scalar	$w-1$	$\frac{2w}{3}+2$
	F_2	auxiliary scalar	$w-1$	$-\frac{2w}{3}-2$
Vector	ϕ	physical scalar	w	0
	λ	spinor	$w-1$	-1
	S	auxiliary	$w-2$	0

Table 6.1: *Properties of three-dimensional non-relativistic multiplets.*

find only zeros. The fact that we do not know how to realize the Schrödinger algebra on a field that is charged under Z is one of the obstacles that prevents us from discussing an on-shell supergravity theory with torsion. We will come back to this issue in the conclusion section and in the last chapter.

6.2 Newton–Cartan supergravity with torsion

At this point we have at our disposal a "conformal" supergravity theory and matter multiplets which we can use to fix some of the gauge-symmetries. This enables us to use superconformal techniques to derive non-relativistic supergravity multiplets. The Schrödinger tensor calculus naturally leads to a Newton–Cartan supergravity theory with torsion, i.e. the curl of the gauge-field of local time-translations τ_μ is non-zero. The origin of the torsion is the spatial part b_a of the dilatation gauge-field. Unlike in the relativistic case, this spatial part cannot be shifted away by a special conformal transformation. Instead, it is a dependent gauge-field whose presence leads to torsion.

In this section we show how the extra symmetries of the Schrödinger superalgebra that are not contained in the Bargmann superalgebra, i.e. dilatations D, special conformal transformations K, S-supersymmetry and possibly R-symmetry, can be eliminated by using a compensating matter multiplet. First, we eliminate the special conformal transformations by setting

$$b = \tau^\mu b_\mu = 0\,. \tag{6.25}$$

The induced compensating transformation is (5.29). This step is independent of which compensating multiplet we use. In the following we shall use both, the scalar and the vector multiplet from the previous section. In analogy to the relativistic case, we refer to the resulting off-shell formulations as the "old minimal" one when we use a compensating scalar multiplet and the "new minimal" formulation when the compensator multiplet is the vector multiplet.

6.2.1 The "old minimal" formulation

In this subsection we choose the scalar multiplet, whose transformation rules can be found in (6.9) and (6.10), as the compensator multiplet. Like in the relativistic case we eliminate both physical scalars thus gauge-fixing dilatations and also the local $U(1)$ R-symmetry. One of the fermions is used to get rid of the conformal supersymmetry:

$$\left.\begin{aligned} \varphi_1 &= 1 : \\ \varphi_2 &= 0 : \end{aligned}\right\} \qquad \text{fixes dilatations and } R\text{-symmetry}\,, \tag{6.26}$$

$$\chi_+ = 0 : \qquad \text{fixes conformal } S\text{-supersymmetry}\,. \tag{6.27}$$

The compensating transformations are given by

$$\Lambda_D = -\frac{1}{w}\,\bar\epsilon_-\chi_-\,, \qquad \rho = -\frac{3}{2w}\,\bar\epsilon_-\gamma^0\chi_-\,, \tag{6.28}$$

and

$$\begin{aligned} \eta = &-\frac{1}{w}\,\epsilon_+\,\tau^\mu\bar\psi_{\mu-}\chi_- + \gamma_0\epsilon_+\,\tau^\mu\Big(\frac{2}{3}\,r_\mu + \frac{1}{w}\,\bar\psi_{\mu-}\gamma^0\chi_-\Big) \\ &-\gamma^{a0}\epsilon_-\Big(b_a + \frac{1}{w}\,e^\mu{}_a\,\bar\psi_{\mu-}\chi_-\Big) - \gamma^a\epsilon_-\,e^\mu{}_a\Big(\frac{2}{3}\,r_\mu + \frac{1}{w}\,\bar\psi_{\mu-}\gamma^0\chi_-\Big) \\ &+\frac{1}{w}\,\gamma_0\epsilon_-\,F_1 - \frac{1}{w}\,\epsilon_-\,F_2 - \frac{2}{w}\,\lambda^a\gamma_a\chi_-\,. \end{aligned} \tag{6.29}$$

Hence, we end up with the field content given in eq. (6.2). The transformation rules of all fields are given by those of the Schrödinger background fields, see chapter 5, and eqs. (6.9) and (6.10) with the compensating transformations (5.29), (6.28) and (6.29) all taken into account. Note that because of the lengthy nature of the compensating transformations (5.29) and especially (6.29) the transformation rules of fields that transform non-trivially under S-supersymmetry and special conformal transformations proliferate quite quickly. We find the following bosonic transformations for the independent fields:

$$\begin{aligned}
\delta\tau_\mu &= 0\,, \\
\delta e_\mu{}^a &= \lambda^a{}_b\, e_\mu{}^b + \tau_\mu\, \lambda^a\,, \\
\delta m_\mu &= \partial_\mu\sigma + \lambda^a e_\mu{}^a\,, \\
\delta r_\mu &= -\frac{3}{4w}\,\lambda^a\,\bar{\psi}_{\mu+}\gamma^a\chi_-\,, \\
\delta F_1 &= 0\,, \\
\delta F_2 &= 0\,,
\end{aligned} \tag{6.30}$$

and

$$\begin{aligned}
\delta\psi_{\mu+} &= \frac{1}{4}\,\lambda^{ab}\gamma_{ab}\psi_{\mu+} + \frac{2}{w}\,\tau_\mu\,\lambda^a\gamma_a\chi_-\,, \\
\delta\psi_{\mu-} &= \frac{1}{4}\,\lambda^{ab}\gamma_{ab}\psi_{\mu-} - \frac{1}{2}\,\lambda^a\gamma_{a0}\psi_{\mu+} + \frac{1}{w}\,e_\mu{}^a\,\lambda^b\,\gamma_a\gamma_{b0}\chi_-\,, \\
\delta\chi_- &= \frac{1}{4}\,\lambda^{ab}\gamma_{ab}\chi_-\,.
\end{aligned} \tag{6.31}$$

Note the non-trivial boost transformation of the gauge-field r_μ in (6.30). Due to complicated compensating transformation (6.29) the supersymmetry transformations are much longer. They are given by

$$\begin{aligned}
\delta\tau_\mu &= \frac{1}{2}\,\bar{\epsilon}_+\gamma^0\psi_{\mu+}\,, \\
\delta e_\mu{}^a &= \frac{1}{2}\,\bar{\epsilon}_+\gamma^a\psi_{\mu-} + \frac{1}{2}\,\bar{\epsilon}_-\gamma^a\psi_{\mu+}\,, \\
\delta m_\mu &= \bar{\epsilon}_-\gamma^0\psi_{\mu-}\,, \\
\delta r_\mu &= -\frac{3}{8}\,\bar{\epsilon}_+\phi_\mu + \frac{1}{4}\,\bar{\epsilon}_+\gamma^0\psi_{\mu+}\,\tau^\rho r_\rho + \frac{1}{4}\,\bar{\epsilon}_-\gamma^{a0}\psi_{\mu+}\,e^\rho{}_a r_\rho + \frac{3}{8}\,\bar{\epsilon}_-\gamma^{a0}\psi_{\mu+}\,b_a \\
&\quad + \frac{3}{8w}\,\bar{\epsilon}_-\gamma^0\psi_{\mu+}\,F_1 - \frac{3}{8w}\,\bar{\epsilon}_-\psi_{\mu+}\,F_2 - \frac{3}{8w}\,\bar{\epsilon}_+\gamma^a\psi_{\rho-}\,\tau^\rho\,\bar{\psi}_{\mu+}\gamma^a\chi_- \\
&\quad + \frac{3}{8w}\,\bar{\epsilon}_-\psi_{\rho-}\,e^\rho{}_a\,\bar{\psi}_{\mu+}\gamma^{a0}\chi_- - \frac{3}{8w}\,\bar{\epsilon}_-\gamma^0\psi_{\rho-}\,e^\rho{}_a\,\bar{\psi}_{\mu+}\gamma^a\chi_-\,,
\end{aligned} \tag{6.32}$$

for the "physical" bosonic fields. The auxiliary fields F_i and all fermionic fields transformed under S-supersymmetry, hence we get more elaborate expressions.

For the fermions we find

$$
\begin{aligned}
\delta\psi_{\mu+} &= D_\mu\epsilon_+ - e_\mu{}^a\, b_a\, \epsilon_+ + \big(r_\mu - \frac{2}{3}\,\tau_\mu\tau^\rho r_\rho\big)\,\gamma_0\epsilon_+ + \frac{2}{3}\,\gamma^a\epsilon_-\,\tau_\mu\, e^\rho{}_a\, r_\rho \\
&\quad + \gamma^{a0}\epsilon_-\,\tau_\mu\, b_a - \frac{1}{w}\,\gamma_0\epsilon_-\,\tau_\mu\, F_1 + \frac{1}{w}\,\epsilon_-\,\tau_\mu\, F_2 - \frac{1}{w}\,\psi_{\mu+}\,\bar\epsilon_-\chi_- \\
&\quad + \frac{3}{2w}\,\gamma_0\psi_{\mu+}\,\bar\epsilon_-\gamma^0\chi_- + \frac{1}{w}\,\tau_\mu\,\gamma^a\chi_-\,\bar\epsilon_+\gamma^a\psi_{\rho-}\,\tau^\rho \\
&\quad - \frac{1}{w}\,\tau_\mu\,\gamma^{a0}\chi_-\,\bar\epsilon_-\psi_{\rho-}\, e^\rho{}_a + \frac{1}{w}\,\tau_\mu\,\gamma^a\chi_-\,\bar\epsilon_-\gamma^0\psi_{\rho-}\, e^\rho{}_a\,, \\
\delta\psi_{\mu-} &= D_\mu\epsilon_- - r_\mu\,\gamma_0\epsilon_- + \frac{1}{2}\,\omega_\mu{}^a\gamma_{a0}\epsilon_+ - \frac{1}{3}\,\gamma^a\epsilon_+\, e_\mu{}^a\tau^\rho\, r_\rho \\
&\quad - \frac{1}{3}\,\gamma^a\gamma^{b0}\epsilon_+\, e_\mu{}^a e^\rho{}_b\, r_\rho + \frac{1}{2}\,\gamma^a\gamma^b\epsilon_-\, e_\mu{}^a\, b_b - \frac{1}{2w}\,\gamma^a\epsilon_-\, e_\mu{}^a\, F_1 \\
&\quad - \frac{1}{2w}\,\gamma_{a0}\epsilon_-\, e_\mu{}^a\, F_2 - \frac{3}{2w}\,\gamma_0\psi_{\mu-}\,\bar\epsilon_-\gamma^0\chi_- \\
&\quad - \frac{1}{2w}\,\gamma^a\gamma^{b0}\chi_-\,\bar\epsilon_+\gamma^b\psi_{\rho-}\, e_\mu{}^a\,\tau^\rho - \frac{1}{2w}\,\gamma^a\gamma^b\chi_-\,\bar\epsilon_-\psi_{\rho-}\, e_\mu{}^a e^\rho{}_b \\
&\quad - \frac{1}{2w}\,\gamma^a\gamma^{b0}\chi_-\,\bar\epsilon_-\gamma^0\psi_{\rho-}\, e_\mu{}^a e^\rho{}_b\,, \\
\delta\chi_- &= -\frac{w}{6}\,\gamma^{a0}\epsilon_+\, e^\mu{}_a\, r_\mu - \frac{w}{4}\,\gamma^a\epsilon_+\, b_a - \frac{1}{3w}\,\epsilon_-\,\bar\chi_-\chi_- + \frac{1}{4}\,\epsilon_+\, F_1 \\
&\quad - \frac{1}{4}\,\gamma_0\epsilon_+\, F_2 - \frac{1}{4}\,\gamma^a\gamma^b\chi_-\,\bar\epsilon_+\gamma^b\psi_{\mu-}\, e^\mu{}_a\,.
\end{aligned}
\tag{6.33}
$$

For the F_i we obtain

$$
\begin{aligned}
\delta F_1 &= \bar\epsilon_+\gamma^0\tau^\mu\hat D_\mu\chi_- + \bar\epsilon_-\gamma^a e^\mu{}_a\hat D_\mu\chi_- + \frac{2}{w}\,\bar\epsilon_-\chi_-\, F_1 - \frac{2}{w}\,\bar\epsilon_-\gamma^0\chi_-\, F_2 \\
&\quad - \frac{1}{4}\,\bar\epsilon_+\gamma^a\psi_{\mu-}\, e^\mu{}_a\, F_1 + \frac{1}{4}\,\bar\epsilon_+\gamma^{a0}\psi_{\mu-}\, e^\mu{}_a\, F_2 + \frac{w}{4}\,\bar\epsilon_+\gamma^{a0}\phi_\mu\, e^\mu{}_a \\
&\quad + \frac{1}{2}\,\bar\epsilon_+\gamma^a\gamma_{b0}\chi_-\, e^\mu{}_a\,\omega_\mu{}^b - \frac{w}{6}\,\bar\epsilon_+\gamma^a\psi_{\mu+}\, e^\mu{}_a\tau^\rho\, r_\rho \\
&\quad - \frac{w}{6}\,\bar\epsilon_+\gamma^a\gamma^{b0}\psi_{\mu-}\, e^\mu{}_a e^\rho{}_b\, r_\rho + \frac{2(w+1)}{3}\,\bar\epsilon_+\chi_-\,\tau^\mu\, r_\mu \\
&\quad - \frac{2(w+1)}{3}\,\bar\epsilon_-\gamma^{a0}\chi_-\, e^\mu{}_a\, r_\mu + \frac{w}{4}\,\bar\epsilon_+\gamma^a\gamma^b\psi_{\mu-}\, e^\mu{}_a\, b_b \\
&\quad + (w+1)\,\bar\epsilon_-\gamma^a\chi_-\, b_a + \frac{1}{4}\,\bar\epsilon_+\gamma^a\gamma^{b0}\chi_-\,\bar\psi_{\rho-}\gamma^b\psi_{\mu+}\, e^\mu{}_a\tau^\rho \\
&\quad - \frac{1}{4}\,\bar\epsilon_+\chi_-\,\bar\psi_{\rho-}\psi_{\mu-}\, e^\mu{}_a e^\rho{}_a + \frac{1}{4}\,\bar\epsilon_+\gamma^{ab0}\chi_-\,\bar\psi_{\rho-}\gamma^0\psi_{\mu-}\, e^\mu{}_a e^\rho{}_b\,,
\end{aligned}
\tag{6.34}
$$

and

$$\begin{aligned}\delta F_2 = {} & \bar\epsilon_+ \gamma^0 \tau^\mu \hat D_\mu \chi_- - \bar\epsilon_- \gamma^{a0} e^\mu{}_a \hat D_\mu \chi_- + \frac{2}{w}\,\bar\epsilon_- \chi_-\, F_2 + \frac{2}{w}\,\bar\epsilon_- \gamma^0 \chi_-\, F_1 \\ & - \frac{1}{4}\,\bar\epsilon_+ \gamma^{a0} \psi_{\mu -}\, e^\mu{}_a\, F_1 - \frac{1}{4}\,\bar\epsilon_+ \gamma^a \psi_{\mu -}\, e^\mu{}_a\, F_2 - \frac{w}{4}\,\bar\epsilon_+ \gamma^0 \phi_\mu\, e^\mu{}_a \\ & - \frac{1}{2}\,\bar\epsilon_+ \gamma^a \gamma^b \chi_-\, e^\mu{}_a \omega_\mu{}^b - \frac{w}{6}\,\bar\epsilon_+ \gamma^{a0} \psi_{\mu +}\, e^\mu{}_a \tau^\rho\, r_\rho \\ & - \frac{w}{6}\,\bar\epsilon_+ \gamma^a \gamma^b \psi_{\mu +}\, e^\mu{}_a e^\rho{}_b\, r_\rho - \frac{2(w+1)}{3}\,\bar\epsilon_+ \gamma^0 \chi_-\, \tau^\mu\, r_\mu \\ & - \frac{2(w+1)}{3}\,\bar\epsilon_- \gamma^a \chi_-\, e^\mu{}_a\, r_\mu - \frac{w}{4}\,\bar\epsilon_+ \gamma^a \gamma^{b0} \psi_{\mu -}\, e^\mu{}_a\, b_b \\ & - (w+1)\,\bar\epsilon_- \gamma^{a0} \chi_-\, b_a + \frac{1}{4}\,\bar\epsilon_+ \gamma^a \gamma^b \chi_-\, \bar\psi_{\rho -} \gamma^b \psi_{\mu +}\, e^\mu{}_a \tau^\rho \\ & + \frac{1}{4}\,\bar\epsilon_+ \gamma^0 \chi_-\, \bar\psi_{\rho -} \psi_{\mu -}\, e^\mu{}_a e^\rho{}_a + \frac{1}{4}\,\bar\epsilon_+ \gamma^{ab} \chi_-\, \bar\psi_{\rho -} \gamma^0 \psi_{\mu -}\, e^\mu{}_a e^\rho{}_b\,. \end{aligned} \tag{6.35}$$

These are only the transformations of the independent fields. Those of the dependent fields $\omega_\mu{}^{ab}$, $\omega_\mu{}^a$, f_μ, b_a and ϕ_μ would be even longer, which is why we refrain from denoting them at all. They can be derived easily from (5.6), (5.8), (5.36) and (5.37). Note that in the transformations of $\omega_\mu{}^a$ and ϕ_μ one should also take into account the new expressions for curvatures of the gravitini $\psi_{\mu -}$ and of r_μ, see also the next section were we do work out those transformations for the dependent fields.

6.2.2 The "new minimal" formulation

In this subsection we choose the vector multiplet, see eqs. (6.22) and (6.23), as the compensator multiplet. The gauge-fixing of dilatations and the special conformal S-supersymmetry is done by imposing

$$\begin{aligned} \phi = 1 \ : & \qquad \text{fixes dilatations}\,, \\ \lambda = 0 \ : & \qquad \text{fixes } S\text{-supersymmetry}\,, \end{aligned} \tag{6.36}$$

and the resulting compensating gauge transformations are

$$\Lambda_D = 0\,, \qquad\qquad \eta = -\frac{2}{w}\,\gamma_0 \epsilon_+\, S - 2\,\gamma^{a0} \epsilon_-\, b_a\,. \tag{6.37}$$

At this point we are left with the symmetries of the Bargmann superalgebra, eqs. (5.1) and (3.50), plus an extra $U(1)$ R-symmetry. These symmetries are realized on the set of independent fields of the "new minimal" Newton–Cartan supergravity theory given in eq. (6.3). This theory is the non-relativistic version of the three-dimensional $\mathcal{N} = (2,0)$ new minimal Poincaré supergravity theory, with torsion. The truncation to zero torsion is discussed in the following section, where we will relate this theory to the one we have derived in chapter 3.

The bosonic symmetry transformations are

$$\begin{aligned}
\delta\tau_\mu &= 0\,,\\
\delta e_\mu{}^a &= \lambda^a{}_b\, e_\mu{}^b + \tau_\mu\, \lambda^a\,,\\
\delta m_\mu &= \partial_\mu\sigma + \lambda^a e_\mu{}^a\,,\\
\delta r_\mu &= \partial_\mu\rho\,,\\
\delta S &= -\frac{1}{2}\,\varepsilon^{ab}\lambda^a b_b\,,
\end{aligned} \tag{6.38}$$

and

$$\begin{aligned}
\delta\psi_{\mu+} &= \frac{1}{4}\,\lambda^{ab}\gamma_{ab}\psi_{\mu+} - \gamma_0\psi_{\mu+}\,\rho\,,\\
\delta\psi_{\mu-} &= \frac{1}{4}\,\lambda^{ab}\gamma_{ab}\psi_{\mu-} - \frac{1}{2}\,\lambda^a\gamma_{a0}\psi_{\mu+} + \gamma_0\psi_{\mu+}\,\rho\,.
\end{aligned} \tag{6.39}$$

Note that S transforms non-trivially under Galilean boosts. This transformation is proportional to b_a, i.e. to torsion. The supersymmetry transformations including the compensating terms that follow (6.37) are given by

$$\begin{aligned}
\delta\tau_\mu &= \frac{1}{2}\,\bar\epsilon_+\gamma^0\psi_{\mu+}\,,\\
\delta e_\mu{}^a &= \frac{1}{2}\,\bar\epsilon_+\gamma^a\psi_{\mu-} + \frac{1}{2}\,\bar\epsilon_-\gamma^a\psi_{\mu+}\,,\\
\delta m_\mu &= \bar\epsilon_-\gamma^0\psi_{\mu-}\,,\\
\delta r_\mu &= \frac{3}{4}\,\bar\epsilon_-\gamma^{a0}\psi_{\mu+}\, b_a - \frac{3}{8}\,\bar\epsilon_+\phi_\mu - \frac{3}{4w}\,\bar\epsilon_+\gamma^0\psi_{\mu+}\, S\,,\\
\delta S &= \frac{w}{8}\,\bar\epsilon_+\phi_\mu\,\tau^\mu + \frac{w}{4}\,\bar\epsilon_+\gamma^{a0}\psi_{\mu-}\,\tau^\mu\, b_a - \frac{1}{4}\,\bar\epsilon_+\gamma^0\psi_{\mu+}\, S\\
&\quad - \frac{w}{4}\,\bar\epsilon_-\gamma^{a0}\phi_\mu\, e^\mu{}_a - \frac{w}{2}\,\bar\epsilon_-\gamma^a\gamma^b\psi_{\mu-}\, e^\mu{}_a\, b_b - \frac{1}{2}\,\bar\epsilon_-\gamma^a\psi_{\mu+}\, e^\mu{}_a\, S\,,
\end{aligned} \tag{6.40}$$

and

$$\begin{aligned}
\delta\psi_{\mu+} &= D_\mu\epsilon_+ - \epsilon_+\, e_\mu{}^a\, b_a + \gamma_0\epsilon_+\, r_\mu + \frac{2}{w}\,\gamma_0\epsilon_+\,\tau_\mu\, S + 2\,\gamma^{a0}\epsilon_-\,\tau_\mu\, b_a\,,\\
\delta\psi_{\mu-} &= D_\mu\epsilon_- - \gamma_0\epsilon_-\, r_\mu + \gamma^a\gamma^b\epsilon_-\, e_\mu{}^a\, b_b + \frac{1}{2}\,\gamma_{a0}\epsilon_+\,\omega_\mu{}^a + \frac{1}{w}\gamma_a\epsilon_+\, e_\mu{}^a\, S\,.
\end{aligned} \tag{6.41}$$

In the new minimal formulation the transformations of the dependent fields $\omega_\mu{}^{ab}$, $\omega_\mu{}^a$, b_a and ϕ_μ are not too difficult. Therefore, and in order to proceed with the truncation in the next section, we also denote them here. The bosonic transformations are

$$\begin{aligned}
\delta\omega_\mu{}^{ab} &= \partial_\mu\lambda^{ab}\,,\\
\delta\omega_\mu{}^{a} &= \partial_\mu\lambda^{a} - \omega_\mu{}^{ab}\lambda^b + \lambda^a\, e_\mu{}^b\, b_b + e_\mu{}^a\lambda^b\, b_b + \lambda^a{}_b\,\omega_\mu{}^b\,,\\
\delta b_a &= \lambda^a{}_b\, b_b\,,\\
\delta\phi_\mu &= \frac{1}{4}\,\lambda^{ab}\gamma_{ab}\phi_\mu - \gamma_0\phi_\mu\,\rho - \psi_{\mu+}\,\lambda^a\, b_a\,.
\end{aligned} \tag{6.42}$$

and

$$\begin{aligned}
\delta\omega_\mu{}^{ab} &= -\frac{1}{4}\,\bar\epsilon_+\gamma^{ab0}\phi_\mu + \frac{1}{2w}\,\bar\epsilon_+\gamma^{ab}\psi_{\mu+}\,S + \bar\epsilon_-\gamma^{[a}\psi_{\mu+}\,b^{b]}\,,\\
\delta\omega_\mu{}^{a} &= \bar\epsilon_-\gamma^0\hat\psi_\mu{}^a{}_- + \frac{1}{4}\,e_\mu{}^b\,\bar\epsilon_+\gamma^b\hat\psi^a{}_{0-} + \frac{1}{4}\,\bar\epsilon_+\gamma^a\hat\psi_{\mu0-} - \frac{1}{w}\,\bar\epsilon_+\gamma^{a0}\psi_{\mu-}\,S\\
&\quad - \frac{1}{w}\,\bar\epsilon_-\gamma^{a0}\psi_{\mu+}\,S - 2\,\varepsilon^{ab}\,\bar\epsilon_-\psi_{\mu-}\,b_b + e_\mu{}^b e^\rho{}_a\,\bar\epsilon_-\gamma^0\gamma^b\gamma^c\psi_{\rho-}\,b_c\\
&\quad - \frac{1}{2}\,e_\mu{}^b e^\rho{}_a\,\bar\epsilon_-\gamma^b\big(\phi_\rho + \frac{2}{w}\,\gamma_0\psi_{\rho+}\,S\big) + \frac{1}{2}\,e_\mu{}^a\,\tau^\rho\,\bar\epsilon_+\gamma^b\psi_{\rho+}\,b_b\,,\\
\delta b_a &= -\frac{1}{2}\,\bar\epsilon_+\gamma^b\psi_{\mu-}\,e^\mu{}_b\,b_a - \frac{1}{2}\,\bar\epsilon_+\gamma^0\psi_{\mu+}\,\tau^\mu\,b_a - \frac{1}{4}\,\bar\epsilon_+\gamma^0\phi_\mu\,e^\mu{}_a\\
&\quad - \frac{1}{2w}\,\bar\epsilon_+\psi_{\mu+}\,e^\mu{}_a\,S\,,\\
\delta\phi_\mu &= \epsilon_+\,f_\mu - \frac{2}{3}\,\gamma^0\epsilon_+\,\big[\hat R_{\mu0}(R) + \frac{3}{2}\,\tau^\nu\,\bar\psi_{[\mu-}\gamma^{a0}\psi_{\nu]+}\,b_a - \frac{3}{4w}\,\tau^\nu\,\bar\psi_{[\mu+}\gamma^0\psi_{\nu]+}\,S\big]\\
&\quad + \frac{4}{3}\,\gamma^a\epsilon_-\,\big[\hat R_{\mu a}(R) + \frac{3}{2}\,e^\nu{}_a\,\bar\psi_{[\mu-}\gamma^{b0}\psi_{\nu]+}\,b_b - \frac{3}{4w}\,e^\nu{}_a\,\bar\psi_{[\mu+}\gamma^0\psi_{\nu]+}\,S\big]\\
&\quad - \big(D_\mu + e_\mu{}^a\,b_a + r_\mu\,\gamma_0\big)\big(\frac{2}{w}\,\gamma_0\epsilon_+\,S + 2\,\gamma^{b0}\epsilon_-\,b_b\big)\,.
\end{aligned} \tag{6.43}$$

Here, we used the covariant curvatures of the independent gauge-fields $\psi_{\mu-}$ and r_μ, which are

$$\begin{aligned}
\hat\psi_{\mu\nu-} &= 2\,\partial_{[\mu}\psi_{\nu]-} - \frac{1}{2}\,\omega_{[\mu}{}^{ab}\gamma_{ab}\psi_{\nu]-} - 2\,r_{[\mu}\,\gamma_0\psi_{\nu]-} + \omega_{[\mu}{}^a\,\gamma_{a0}\psi_{\nu]+}\\
&\quad + 2\,\gamma^a\gamma^b\psi_{[\nu-}\,e_{\mu]}{}^a b_b + \frac{2}{w}\,\gamma_a\psi_{[\nu+}\,e_{\mu]}{}^a\,S\,,\\
\hat R_{\mu\nu}(R) &= 2\,\partial_{[\mu}r_{\nu]} + \frac{3}{4}\,\bar\psi_{[\mu+}\phi_{\nu]} - \frac{3}{2}\,\bar\psi_{[\mu-}\gamma^{a0}\psi_{\nu]+}\,b_a + \frac{3}{4w}\,\bar\psi_{[\mu+}\gamma^0\psi_{\nu]+}\,S\,.
\end{aligned} \tag{6.44}$$

The expression for the special conformal gauge-field f_μ can be found in (5.35). We did not derive its transformation rule because no independent field transforms to f_μ and therefore its variation is not needed for any checks on e.g. closure of the commutator algebra.

6.3 Truncation to zero torsion

In the previous section we derived a Newton–Cartan supergravity multiplet with non-zero torsion. This needs to be contrasted with the Newton–Cartan supergravity theories of chapters 3 and 4 which have zero torsion. To see the difference it is instructive to compare the curvature of local time translations for the theories with and without torsion. Indicating the curvature of the torsionfull theory with $\mathcal{R}(H)$ and the one of the zero-torsion theory with $\hat{R}(H)$ we have

$$\begin{aligned} \mathcal{R}_{\mu\nu}(H) &= 2\,\partial_{[\mu}\tau_{\nu]} - 4\,b_{[\mu}\,\tau_{\nu]} - \frac{1}{2}\,\bar{\psi}_{[\mu+}\gamma^0\psi_{\nu]+}\,, \\ \hat{R}_{\mu\nu}(H) &= 2\,\partial_{[\mu}\tau_{\nu]} - \frac{1}{2}\,\bar{\psi}_{[\mu+}\gamma^0\psi_{\nu]+}\,. \end{aligned} \tag{6.45}$$

Note that the space-space components of both curvatures are the same. The difference is in the time-space component. In the torsionfull case, setting the time-space component to zero, is a conventional constraint that is used to solve for the spatial part b_a of the dilatation gauge-field, whereas in the torsionless case it represents an un-conventional constraint. Indeed, we have

$$b_a = \frac{1}{2}\,\hat{R}_{a0}(H)\,, \tag{6.46}$$

and therefore setting the torsion to zero, i.e.

$$b_a = 0\,, \tag{6.47}$$

leads to the un-conventional constraint $\hat{R}_{a0}(H) = 0$ in the torsionless theory.

This points us to an interesting observation: the existence of a non-trivial truncation of the old minimal and the new minimal Newton–Cartan supergravity multiplets constructed in the previous section 6.2. Indeed, we shall show in this section how we can reduce the new minimal torsionfull theory constructed in subsection 6.2.2 to the known new minimal torsionless Newton–Cartan supergravity theory of chapter 3.

We will now investigate the consequences imposing the constraint (6.47). It is convenient to use explicit expression for the gauge-field of S-supersymmetry ϕ_μ, which simplifies to

$$\phi_\mu = \gamma^{a0}\hat{\psi}_{a\mu-} - \frac{2}{w}\,\gamma_0\psi_{\mu+}\,S\,, \tag{6.48}$$

when we use the curvatures and constraints that we introduce below. The only dependent gauge-fields of the Newton–Cartan supergravity theory are the connection fields for spatial rotations and Galilean boosts. For the supersymmetry transformation rules of the independent fields we find

$$\begin{aligned} \delta\tau_\mu &= \frac{1}{2}\,\bar{\epsilon}_+\gamma^0\psi_{\mu+}\,, \\ \delta e_\mu{}^a &= \frac{1}{2}\,\bar{\epsilon}_+\gamma^a\psi_{\mu-} + \frac{1}{2}\,\bar{\epsilon}_-\gamma^a\psi_{\mu+}\,, \\ \delta m_\mu &= \bar{\epsilon}_-\gamma^0\psi_{\mu-}\,, \\ \delta r_\mu &= -\frac{3}{8}\,\bar{\epsilon}_+\gamma^{a0}\hat{\psi}_{a\mu-} - \frac{3}{2w}\,\bar{\epsilon}_+\gamma^0\psi_{\mu+}\,S\,, \\ \delta S &= \frac{w}{8}\,\bar{\epsilon}_+\gamma^{a0}\hat{\psi}_{a0-}\,, \end{aligned} \tag{6.49}$$

and

$$\begin{aligned} \delta\psi_{\mu+} &= D_\mu\epsilon_+ + \gamma_0\epsilon_+\,r_\mu + \frac{2}{w}\,\gamma_0\epsilon_+\,\tau_\mu\,S\,, \\ \delta\psi_{\mu-} &= D_\mu\epsilon_- - \gamma_0\epsilon_-\,r_\mu + \frac{1}{2}\,\gamma_{a0}\epsilon_+\,\omega_\mu{}^a + \frac{1}{w}\,\gamma_a\epsilon_+\,e_\mu{}^a\,S\,. \end{aligned} \tag{6.50}$$

The covariant curvatures and derivatives of the new minimal Newton–Cartan supergravity theory are now given by (6.45) and

$$\begin{aligned} \hat{R}_{\mu\nu}{}^a(P) &= 2\,\partial_{[\mu}e_{\nu]}{}^a - 2\,\omega_{[\mu}{}^{ab}e_{\nu]}{}^b - 2\,\omega_{[\mu}{}^a\tau_{\nu]} - \bar{\psi}_{[\mu+}\gamma^a\psi_{\nu]-}\,, \\ \hat{R}_{\mu\nu}(Z) &= 2\,\partial_{[\mu}m_{\nu]} - \bar{\psi}_{[\mu-}\gamma^0\psi_{\nu]-}\,, \\ \hat{R}_{\mu\nu}(R) &= 2\,\partial_{[\mu}r_{\nu]} + \frac{3}{2w}\,\bar{\psi}_{[\mu+}\gamma^0\psi_{\nu]+}\,S + \frac{3}{4}\,\bar{\psi}_{[\mu+}\gamma^{a0}\hat{\psi}_{a\nu]-}\,, \\ \hat{D}_\mu S &= \partial_\mu S - \frac{w}{8}\,\bar{\psi}_{\mu+}\gamma^{a0}\hat{\psi}_{a0-}\,, \\ \hat{\psi}_{\mu\nu+} &= 2\,\partial_{[\mu}\psi_{\nu]+} - \frac{1}{2}\,\omega_{[\mu}{}^{ab}\gamma_{ab}\psi_{\nu]+} - 2\,\gamma_0\psi_{[\mu+}\,r_{\nu]} - \frac{4}{w}\,\gamma_0\psi_{[\mu+}\,\tau_{\nu]}\,S\,, \\ \hat{\psi}_{\mu\nu-} &= 2\,\partial_{[\mu}\psi_{\nu]-} - \frac{1}{2}\,\omega_{[\mu}{}^{ab}\gamma_{ab}\psi_{\nu]-} + 2\,\gamma_0\psi_{[\mu-}\,r_{\nu]} + \omega_{[\mu}{}^a\gamma_{a0}\psi_{\nu]+} \\ &\quad - \frac{2}{w}\,\gamma_a\psi_{[\mu+}\,e_{\nu]}{}^a\,S\,. \end{aligned} \tag{6.51}$$

As we explained at the beginning of this section, the zero-torsion constraint (6.47) may convert a conventional constraint into an un-conventional one. If this happens we have to check if the supersymmetry variation of this un-conventional constraint leads to further constraints. To perform this check we need the transformation rules of the dependent connection gauge-fields which reduce to

$$\begin{aligned}\delta\omega_\mu{}^{ab} &= -\frac{1}{2}\,\bar\epsilon_+\gamma^{[a}\hat\psi^{b]}{}_{\mu-} + \frac{1}{w}\,\bar\epsilon_+\gamma^{ab}\psi_{\mu+}\,S\,,\\ \delta\omega_\mu{}^a &= \bar\epsilon_-\gamma^0\hat\psi_\mu{}^a{}_- + \frac{1}{4}\,e_\mu{}^b\,\bar\epsilon_+\gamma^b\hat\psi^a{}_{0-} + \frac{1}{4}\,\bar\epsilon_+\gamma^a\hat\psi_{\mu0-}\\ &\quad - \frac{1}{w}\,\bar\epsilon_+\gamma^{a0}\psi_{\mu-}\,S - \frac{1}{w}\,\bar\epsilon_-\gamma^{a0}\psi_{\mu+}\,S\,.\end{aligned} \tag{6.52}$$

The corresponding curvatures read

$$\begin{aligned}\hat R_{\mu\nu}{}^{ab}(J) &= 2\,\partial_{[\mu}\omega_{\nu]}{}^{ab} + \bar\psi_{[\mu+}\gamma^{[a}\hat\psi^{b]}{}_{\nu]-} - \frac{1}{w}\,\bar\psi_{[\mu+}\gamma^{ab}\psi_{\nu]+}\,S\,,\\ \hat R_{\mu\nu}{}^a(G) &= 2\,\partial_{[\mu}\omega_{\nu]}{}^a - 2\,\omega_{[\mu}{}^{ab}\omega_{\nu]}{}^b - 2\,\bar\psi_{[\mu-}\gamma^0\hat\psi_{\nu]}{}^a{}_- - \frac{1}{2}\,e_{[\nu}{}^b\bar\psi_{\mu]+}\gamma^b\hat\psi^a{}_{0-}\\ &\quad - \frac{1}{2}\,\bar\psi_{[\mu+}\gamma^a\hat\psi_{\nu]0-} + \frac{2}{w}\,\bar\psi_{[\mu+}\gamma^{a0}\psi_{\nu]-}\,S\,,\end{aligned} \tag{6.53}$$

in agreement with (3.89).

We are now ready to discuss the constraint structure of the truncated theory. Some of the curvatures did not change, hence we can immediately infer that

$$\hat R_{ab}(R) = 0\,, \qquad\qquad \frac{3}{4}\,\varepsilon^{ab}\,\hat R_{\mu\nu}{}^{ab}(J) = \hat R_{\mu\nu}(R)\,. \tag{6.54}$$

The constraints $\hat R_{\mu\nu}{}^a(P) = 0$ and $\hat R_{\mu\nu}(Z) = 0$ are identities when we insert the expressions for the connection gauge fields, i.e. they are conventional constraints. More importantly though, we find new constraints. This is due to the fact that we imposed $\hat R_{a0}(H) = 0$, which is an example of a conventional constraint (necessary to solve for the spatial part b_a of the dilatation gauge field) that gets converted into an un-conventional constraint. Together with the constraint $\hat R_{ab}(H) = 0$ which reads the same in the torsionfull as well as in the torsionless case, we find $\hat R_{\mu\nu}(H) = 0$. Supersymmetry variations of this constraint reveal the following additional constraints:

$$\xrightarrow{Q_-}\qquad \hat\psi_{ab-} = 0 \tag{6.55}$$

$$\hat R_{\mu\nu}(H) = 0 \quad \xrightarrow{Q_+} \quad \hat\psi_{\mu\nu+} = 0 \quad \xrightarrow{Q_+} \quad \hat R_{\mu\nu}{}^{ab}(J) = \frac{4}{w}\,\varepsilon^{ab}\,\tau_{[\mu}\hat D_{\nu]}\,S\,. \tag{6.56}$$

Further transformations only lead to Bianchi identities. By combing the constraints (6.56) with (6.54) we furthermore derive that

$$-\frac{6}{w}\,\hat D_{[\mu}\big(\tau_{\nu]}\,S\big) = 2\,\hat D_{[\mu}r_{\nu]}\,. \tag{6.57}$$

This constraint implies that up to an arbitrary constant the R-symmetry gauge-field

r_μ is determined by τ_μ and S. In fact, when we set

$$r_\mu = -\frac{3}{w}\,\tau_\mu\, S\,, \tag{6.58}$$

the truncated theory precisely leads to the off-shell Newton–Cartan multiplet that was presented in chapter 3. Furthermore, by making the redefinition

$$r_\mu = -V_\mu - \frac{1}{w}\,\tau_\mu\, S\,, \tag{6.59}$$

one obtains precisely the off-shell multiplet that was obtained when taking the limit of the new minimal Poincaré multiplet.

6.4 A brief summary

This chapter served various purposes. We introduced matter multiplets and thus showed how we can couple non-relativistic matter to non-relativistic supergravity backgrounds. Secondly, we derived another generalization of Newton–Cartan supergravity, namely the theory with torsion. Lastly, we showed that there is no obstacle in taking over the ideas of the superconformal tensor calculus used in the relativistic setting and applying them to our non-relativistic supergravity theories.

Note that we did not derive an on-shell Newton–Cartan supergravity theory with torsion. It would be interesting to see how one can go on-shell in the presence of torsion. Unlike the truncation to zero torsion that we performed in section 6.3 this is not a straightforward thing to do. The problem is that even in the bosonic case the equations of motion describing Newton–Cartan gravity with torsion have not been written down so far.[3] Even in the absence of torsion the equations of motion have only been written down under the assumption that the curvature of spatial rotations is zero [80]. It is not difficult to write down the equations of motion for the case that this curvature is nonzero but the price one has to pay is that one has to add extra terms to the equation of motion proposed in [80] that break the invariance under central charge transformations [29]. In the non-relativistic superconformal approach we are using in this work this requires the introduction of *two* compensating scalars: one for dilatations and one for the central charge transformations. Unlike the compensating scalar for dilatations, the compensating scalar for central charge transformations should transform non-trivially under central charge transformations and hence should be part of a different multiplet. The construction of such a multiplet is different from our investigations in section 6.1 and goes beyond the scope of this thesis.

[3] A systematic approach to construct such an equation of motion is part of the analysis in [102].

7

Conclusion

In this chapter we will recap the results that we have obtained and take another look at generalizations and extensions thereof. For a large part this will be but a repetition of the outlook given in each of the previous chapters.

7.1 Summary of the thesis

In this thesis we constructed non-relativistic theories of supergravity in three space-time dimensions. Two different methods were used for that purpose. Gauging techniques, which are known to provide one way towards supergravity in the relativistic case [95], and which were used in the derivation of the first Newton–Cartan supergravity too, featured in chapters 4 and 5. A limiting and contraction procedure was developed in chapter 3 and later used in chapter 4 to re-derive the result obtained previously by gauging techniques and in chapter 6 the limit was used to construct non-relativistic matter multiplets.

Before summarizing the result let us comment a bit more on the main characteristics and differences of those two methods. For comparison, we have highlighted some of those features in table 7.1. Let us focus first on the gauging techniques. It

Methods to obtain non-relativistic theories

	gauging	limit
results	NH supergravity, Schrödinger supergravity,	NC (super-)gravity, off-shell supergravity, NH supergravity,
drawbacks	not easy: close algebra on g.c.t. no off-shell formulation	starting point is crucial need correct constraints

Table 7.1: *In this table we enlist the main results achieved using both methods and their drawbacks. We use the abbreviations NC for Newton–Cartan and NH for Newton–Hooke, while g.c.t. stands for general coordinate transformations.*

is straightforward to realize an algebra on a set of gauge-fields. The challenge is to make those gauge-fields transform under general coordinate transformations in the correct way. To achieve this one usually has to imposes curvature constraints to identify local translations with diffeomorphisms. These curvature constraints are constraints on the theory and their consequences have to be investigated carefully. Certainly, one option is to set the curvatures of all gauge-fields to zero, but then we are left with a "trivial" theory only, i.e. all fields are pure gauge. The main question in this approach is thus if we can find a set of non-trivial curvature constraints that make the superalgebra close (with diffeomorphisms).[1] Ideally, those same constraints will also allow us to solve for all dependent gauge-fields in terms of the independent ones. The drawback of this approach is that it does not enable one to find, in a straightforward manner, multiplets that contain auxiliary fields.

[1] Also, if we use Newton–Cartan structures to describe e.g. phenomena in condensed matter theory, these constraints should comply with expectations from those condensed matter models, e.g. we might *not* want to set torsion to zero.

However, as we showed in chapter 6, Schrödinger techniques might be one option to circumvent this obstacle.

In conclusion, the starting point here is easy and the difficulty lies in converting this into a geometric theory. We should also point out that, while we sometimes made use of both techniques, ultimately all supergravity theories in this thesis were derived using gauging techniques.

In contrast, the difficulty of the limiting procedure lies in finding the correct starting point. From then on, one just needs to apply the procedure and see what it leads to. It might just be that one finds the limit does not work. For example, in order to avoid divergent terms when taking the limit, we might be forced to put too many constraints on the independent fields of the theory. Then, one could try to tweak the starting point, or one must deduce that the theory under consideration does not have this kind of non-relativistic limit.

Let us now turn to the results that we have obtained. To provide an overview over the different non-relativistic supergravity theories discussed in this thesis we summarize them in table 7.2. The first line, on-shell Newton–Cartan supergravity,

Overview of supergravities

	field content	limit of	algebra gauged
NC	τ_μ, $e_\mu{}^a$, m_μ, $\psi_{\mu\pm}$	on-shell Poincaré	$\mathcal{N}=2$ Bargmann
off-shell NC	*NC* plus S	new minimal multiplet	($\mathcal{N}=2$ *Bargmann*)
NH	*NC*	AdS supergravity	$\mathcal{N}=(2,0)$ NH
Schrödinger	*NC* plus r_μ	no limit	Schrödinger
TNC “old”	*NC* plus r_μ, F_1, F_2, χ	no limit (known)	($\mathcal{N}=2$ *Bargmann*)
TNC “new”	*NC* plus r_μ, S	no limit (known)	($\mathcal{N}=2$ *Bargmann*)

Table 7.2: *In this table we review the non-relativistic supergravities of this thesis and some of their features. The abbreviations NC, NH and AdS stand for Newton–Cartan, Newton–Hooke and anti-de Sitter, respectively. TNC “old”/“new” refer to the old/new minimal formulation of Newton–Cartan supergravity with torsion. The algebras in brackets indicate that the theory is not obtained by gauging techniques, but nevertheless realizes this algebra.*

is the first example of a theory of non-relativistic supergravity. This thesis is very much built on that result and aimed to generalize it in various directions, which we have depicted in figure 1.2 in the introduction. Our generalizations given in the first column of table 7.2, comprise the following extentions. The cosmological extension that we called Newton–Hooke supergravity featured in chapter 4. A “conformal” extension that we dubbed Schrödinger supergravity was presented in chapter 5. Finally, an extension with non-vanishing torsion and two novel off-shell formulations of the original Newton–Cartan supergravity theory of [77] were the subject of chapter 6 (and also chapter 3).

The field content of the various theories, given in the second column of table 7.2, always consists of

$$\left(\tau_\mu, e_\mu{}^a, m_\mu, \psi_{\mu\pm}\right) + \text{more}\,, \tag{7.1}$$

i.e. at least the fields of on-shell Newton–Cartan supergravity. Several extensions then include more fields which are either auxiliary, or in the case of Schrödinger supergravity where there are more symmetries, extra gauge-fields. Recall that we argued in chapter 6 that one can eliminate the gauge-field r_μ (we also used the notation V_μ) of R-symmetry when considering the truncation to zero torsion of the new minimal formulation. Thus, the difference in field content of the two new minimal formulations in line two and six of table 7.2.

In those cases where we also discussed the connection to a relativistic theory, via the limiting procedure, we indicated this "parent" theory in the third column of table 7.2. As we mentioned in chapter 5 there is no way to apply the limiting procedure in the case for Schrödinger supergravity. In contrast, we do not claim that no such limit exists for the torsional Newton–Cartan supergravity theories. However, taking the limit as discussed in chapter 3, especially for the new minimal theory, imposes that the theory be torsionless. Therefore we did not indicate related relativistic theories here.

The purpose of the last column is to depict the symmetry algebra that the multiplet realizes. In most cases that means that this theory is obtained by gauging said algebra, as shown in many examples in our work. Only for the off-shell formulations this is not possible in a straightforward way.

In hindsight we realize that not all extensions are on the same footing, as suggested by figure 1.2 in the introduction. Having gone through the gauge-fixings and other truncations (on-shell/off-shell and torsion/no torsion) in detail in chapter 6, we understand the relation between all those extensions much better now. It turns out that the figure should look more like figure 7.1. The form of this diagram is also suggestive of extensions of this work. Can we first go on-shell and then truncate to zero torsion? What is off-shell formulation of Newton–Hooke supergravity? We come back to these points in the following section, when we provide an outlook on future work.

The construction of non-relativistic (conformal) supergravity enabled us to construct non-relativistic matter multiplets too. This was part of our analysis in chapter 6, where we also made use of our limiting procedure. Let us repeat part of the table where the field content and weights of the fields of the matter multiplets were summarized, see table 7.3. These are the only matter multiplets that we constructed so far but there are certainly others. Indeed, the construction of one particular matter multiplet with a (scalar) field that is charged under central charge symmetry is part

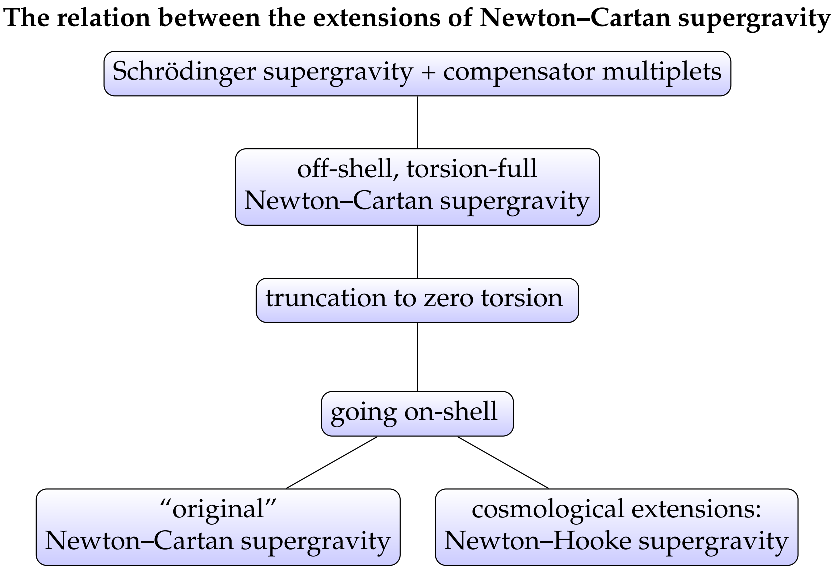

Figure 7.1: *The relation of the different supergravity theories discussed in this thesis.*

of one very interesting extension of this work. We come back to this issue in the next section, when discussing in more detail the possible extensions of our work.

Finally, in the last chapter we also showed that the techniques from the superconformal tensor calculus can be applied in the non-relativistic setting in a straightforward way. This enabled us to re-derive and extend all of the previous results on torsionless supergravities. It also provided a very efficient way to construct torsionfull theories.

This finishes our summary of the results that we have obtained in this thesis. In the following, we will expand on possible extensions of this work.

7.2 Outlook

To start, let us come back to some issues which we have just found in the previous section. For example, looking at the table 7.2 we realize that we do not have an on-shell formulation of torsional Newton–Cartan supergravity. The problem here is that we do not have a compensating matter multiplet that contains a field which transforms under the central charge symmetry. Remember, if $\hat{R}_{\mu\nu}{}^{ab}(J) \neq 0$ the equation of motion is not invariant under boosts. Say, we add extra terms that solve this problem, then we obtain an equation of motion that is not invariant under

Overview of non-relativistic matter multiplets

multiplet	field	type	D-weight	R-weight
Scalar	φ_1	physical scalar	w	$\frac{2w}{3}$
	φ_2	physical scalar	w	$-\frac{2w}{3}$
	χ_+	spinor	$w-1$	$-\frac{2w}{3}-1$
	χ_-	spinor	w	$\frac{2w}{3}+1$
	F_1	auxiliary scalar	$w-1$	$\frac{2w}{3}+2$
	F_2	auxiliary scalar	$w-1$	$-\frac{2w}{3}-2$
Vector	ϕ	physical scalar	w	0
	λ	spinor	$w-1$	-1
	S	auxiliary	$w-2$	0

Table 7.3: *Properties of three-dimensional non-relativistic matter multiplets.*

central charge transformations. In the bosonic case, it suffices to use a Stückelberg field to overcome this problem, see e.g. [29]. In the supersymmetric case, this Stückelberg field must be part of a multiplet. Hence, a matter multiplet with a field that transforms under central charge symmetry is definitely worth looking for.

This is were this possible extension connects with the exploration of non-relativistic superspace. Indeed, finding more realizations of the Bargmann or Schrödinger superalgebra would be part of an extension of this work. It would be of interest to investigate more in non-relativistic superspace. A very rewarding, and hugely simplifying, project would be to describe non-relativistic supersymmetric physics in an equally elegant superspace formulation as it is possible in the relativistic case, see e.g. [149]. Of course there is no need to refrain from the component formulation.

In this work we considered only pure, simple Newton–Cartan supergravity. Some extensions comprise for example considering more extended supersymmetry algebras, or more general non-relativistic matter-coupled supergravity. Matter-coupled supergravities would lead towards the non-relativistic analogue of the geometries that one encounters in the relativistic matter-coupled supergravity theories. It would be interesting to find out what the analogue of a Kähler target space is.

In this work we consider non-relativistic superalgebras with two supercharges to be minimal. How do we extend this with more supercharges? Perhaps it makes sense to consider any number bigger than two, but maybe we should only take $\mathcal{N}$ to be a multiple of two if we want to retain similar structures in the algebra.

In the last chapter we have shown that techniques that are used in the relativistic case can be put to use in the non-relativistic case too. Apparently this works for superconformal techniques, so why not for other ones. In particular, we might ask if there is a non-relativistic version of localization techniques. In fact, the construction of off-shell multiplets throughout this work was an effort in this direction. Off-shell formulations are needed to put (rigid) supergravity on a curved background, see e.g. [76]. So obviously, now that we have those off-shell formulations one extension can consist of answering the question on what backgrounds we can put a non-relativistic theory of supergravity. In a next step, one can ask if we can apply localization techniques to calculate partition functions for non-relativistic supersymmetric theories on curved backgrounds.

An equally obvious but probably more "physical" extension would be to go to higher dimensions. Using techniques presented in this thesis we might aim to construct Newton–Cartan supergravity in four space-time dimensions, a goal which has remained elusive this far. The limiting procedure put forward in chapter 3 might be useful for that purpose.

In fact, one can find several applications for the non-relativistic limiting procedure that we developed here. For example, one could set out to derive the non-relativistic string [101] in a way similar to the point-particle limit that we took in chapter 3. To do so, one would first have to find a suitable extension of the Poincaré algebra whose Inönü–Wigner contraction leads to the extended Galilei algebra of [64].

The procedure need not be used to take non-relativistic limits only. The basic idea of the limiting procedure goes beyond that. It should be clear from the analysis of chapter 3 that, naively, all we need to perform the limit is the contraction of one algebra to another one. There possibly exist similar limits that descend from the ultra-relativistic contraction of the Poincaré algebra, which leads to the Carroll algebra. Ideally, such a contraction procedure would lead to an ultra-relativistic version of gravity, such as the theory of Carroll gravity introduced by Hartong [61]. Perhaps, such a limit could also be used to derive the Carroll (super-)particle in a curved background, see [59, 60].

It is not clear whether every algebra contraction can be translated into a contraction at the level of the field theory representing that algebra. Moreover, certain non-relativistic symmetry algebras cannot be viewed as contractions of relativistic ones. An example of such an algebra is given by the Schrödinger algebra. However, the Bargmann and the Schrödinger algebra can be obtained as light-like reductions of relativistic algebras [150, 151]. Perhaps one can define a different sort of contraction or limiting procedure related to such kind of reductions, which would give rise to (torsional) Newton–Cartan structures as presented in [29, 80]. In view of the recent applications of torsional Newton–Cartan geometry in non-relativistic holography [15, 25–28], it would be interesting to investigate this case in more detail.

This brings us back to the putative applications of our results in condensed matter theory. Recall that our search for off-shell formulations was motivated by the long-term goal to investigate localization techniques in the non-relativistic context. If such techniques can indeed be applied, they might prove very useful in yielding exact results (partition functions, which lead to response functions, etc. . .) for non-relativistic field theories.

For example, one piece of the action of Son's effective model for the quantum

Hall effect [30] remains undetermined in that work. As he points out, symmetry arguments certainly restrict the possible terms that can appear as extra contributions to the action. Perhaps requiring supersymmetry can put enough restrictions to determine the possible terms even more.

List of publications

[I] D. Grumiller, N. Johansson, and T. Zojer,
"Short-cut to new anomalies in gravity duals to logarithmic conformal field theories," *JHEP* **1101** (2011) 090, `arXiv:1010.4449 [hep-th]`.

[II] M. Bertin, D. Grumiller, D. Vassilevich, and T. Zojer,
"Generalised massive gravity one-loop partition function and AdS/(L)CFT," *JHEP* **1106** (2011) 111, `arXiv:1103.5468 [hep-th]`.

[III] N. Johansson, A. Naseh, and T. Zojer,
"Holographic two-point functions for 4d log-gravity," *JHEP***1209** (2012) 114, `arXiv:1205.5804 [hep-th]`.

[IV] E. A. Bergshoeff, S. de Haan, W. Merbis, J. Rosseel, and T. Zojer,
"Three-dimensional tricritical gravity," *Phys.Rev.***D86** (2012) 064037, `arXiv:1206.3089 [hep-th]`.

[V] T. Zojer,
"On one-loop partition functions of three-dimensional critical gravities," *Class.Quant.Grav.***30** (2013) 075005, `arXiv:1210.6887 [hep-th]`.

[VI] D. Grumiller, W. Riedler, J. Rosseel, and T. Zojer,
"Holographic applications of logarithmic conformal field theories," *J.Phys.***A46** (2013) 494002, `arXiv:1302.0280 [hep-th]`.

[VII] E. A. Bergshoeff, M. Kovačević, L. Parra, J. Rosseel, Y. Yin, and T. Zojer, "New massive supergravity and auxiliary fields," *Class.Quant.Grav.* **30** (2013) 195004, arXiv:1304.5445 [hep-th].

[VIII] E. Bergshoeff, M. Kovačević, L. Parra, and T. Zojer, "A new road to massive gravity?," **PoS** *Corfu2012* (2013) 053.

[IX] E. Bergshoeff, J. Gomis, M. Kovačević, L. Parra, J. Rosseel, and T. Zojer, "Non-relativistic superparticle in a curved background," *Phys.Rev.***D90** (2014) 065006, arXiv:1406.7286 [hep-th].

[X] E. Bergshoeff, J. Rosseel, and T. Zojer, "Newton–Cartan (super)gravity as a non-relativistic limit," *Class.Quant.Grav.* **32** (2015) 205003, arXiv:1505.02095 [hep-th].

[XI] E. Bergshoeff, J. Rosseel, and T. Zojer, "Newton–Cartan supergravity with torsion and Schrödinger supergravity," *JHEP* **1511** (2015) 180, arXiv:1509.04527 [hep-th].

References

[1] A. Einstein, “Die Grundlage der allgemeinen Relativitätstheorie,” *Annalen der Physik* **49** (1916) 769–822. (in German).

[2] E. Cartan, “Sur les variétés à connexion affine et la théorie de la relativité généralisée (première partie),” *Ann. École Norm. Sup.* **40** (1923) 325–412. link to full text (in French).

[3] E. Cartan, “Sur les variétés à connexion affine et la théorie de la relativité généralisée (première partie) (Suite),” *Ann. École Norm. Sup.* **41** (1924) 1–25. link to full text (in French).

[4] C. Duval, “On Galileian isometries,” *Class.Quant.Grav.* **10** (1993) 2217–2222, arXiv:0903.1641 [math-ph].

[5] D. Son and M. Wingate, “General coordinate invariance and conformal invariance in nonrelativistic physics: Unitary Fermi gas,” *Annals Phys.* **321** (2006) 197–224, arXiv:cond-mat/0509786 [cond-mat].

[6] Y. Nishida and D. T. Son, “Nonrelativistic conformal field theories,” *Phys.Rev.* **D76** (2007) 086004, arXiv:0706.3746 [hep-th].

[7] D. Son, “Toward an AdS/cold atoms correspondence: A Geometric realization of the Schrodinger symmetry,” *Phys.Rev.* **D78** (2008) 046003, arXiv:0804.3972 [hep-th].

[8] M. Geracie, D. T. Son, C. Wu, and S.-F. Wu, “Spacetime Symmetries of the Quantum Hall Effect,” *Phys.Rev.* **D91** (2015) 045030, arXiv:1407.1252 [cond-mat.mes-hall].

[9] B. Bradlyn and N. Read, “Low-energy effective theory in the bulk for transport in a topological phase,” *Phys. Rev.* **B91** (2015) 125303, arXiv:1407.2911 [cond-mat.mes-hall].

[10] T. Brauner, S. Endlich, A. Monin, and R. Penco, “General coordinate invariance in quantum many-body systems,” *Phys. Rev.* **D90** (2014) 105016, arXiv:1407.7730 [hep-th].

[11] K. Jensen, "On the coupling of Galilean-invariant field theories to curved spacetime," arXiv:1408.6855 [hep-th].

[12] K. Jensen, "Aspects of hot Galilean field theory," *JHEP* **1504** (2015) 123, arXiv:1411.7024 [hep-th].

[13] K. Jensen and A. Karch, "Revisiting non-relativistic limits," *JHEP* **1504** (2015) 155, arXiv:1412.2738 [hep-th].

[14] K. Jensen, "Anomalies for Galilean fields," arXiv:1412.7750 [hep-th].

[15] J. Hartong, E. Kiritsis, and N. A. Obers, "Field Theory on Newton-Cartan Backgrounds and Symmetries of the Lifshitz Vacuum," *JHEP* **08** (2015) 006, arXiv:1502.00228 [hep-th].

[16] M. Geracie, K. Prabhu, and M. M. Roberts, "Curved non-relativistic spacetimes, Newtonian gravitation and massive matter," *J. Math. Phys.* **56** (2015) 103505, arXiv:1503.02682 [hep-th].

[17] M. Geracie, K. Prabhu, and M. M. Roberts, "Fields and fluids on curved non-relativistic spacetimes," *JHEP* **08** (2015) 042, arXiv:1503.02680 [hep-th].

[18] O. Andreev, M. Haack, and S. Hofmann, "On Nonrelativistic Diffeomorphism Invariance," *Phys.Rev.* **D89** no. 6, (2014) 064012, arXiv:1309.7231 [hep-th].

[19] R. Banerjee, A. Mitra, and P. Mukherjee, "A new formulation of non-relativistic diffeomorphism invariance," *Phys.Lett.* **B737** (2014) 369–373, arXiv:1404.4491 [gr-qc].

[20] R. Banerjee, A. Mitra, and P. Mukherjee, "Localization of the Galilean symmetry and dynamical realization of Newton-Cartan geometry," *Class.Quant.Grav.* **32** (2015) 045010, arXiv:1407.3617 [hep-th].

[21] O. Andreev, "More On Nonrelativistic Diffeomorphism Invariance," *Phys.Rev.* **D91** no. 2, (2015) 024035, arXiv:1408.7031 [hep-th].

[22] R. Banerjee, A. Mitra, and P. Mukherjee, "General algorithm for nonrelativistic diffeomorphism invariance," *Phys.Rev.* **D91** no. 8, (2015) 084021, arXiv:1501.05468 [gr-qc].

[23] A. Mitra, "Weyl rescaled Newton-Cartan geometry from the localization of Galilean and scale symmetries," arXiv:1508.03207 [hep-th].

[24] R. Banerjee and P. Mukherjee, "New approach to nonrelativistic diffeomorphism invariance and its applications," arXiv:1509.05622 [gr-qc].

[25] M. H. Christensen, J. Hartong, N. A. Obers, and B. Rollier, "Torsional Newton-Cartan Geometry and Lifshitz Holography," *Phys.Rev.* **D89** (2014) 061901, arXiv:1311.4794 [hep-th].

[26] M. H. Christensen, J. Hartong, N. A. Obers, and B. Rollier, "Boundary Stress-Energy Tensor and Newton-Cartan Geometry in Lifshitz Holography," *JHEP* **1401** (2014) 057, arXiv:1311.6471 [hep-th].

[27] J. Hartong, E. Kiritsis, and N. A. Obers, "Lifshitz spacetimes for Schrödinger holography," *Phys.Lett.* **B746** (2015) 318–324, arXiv:1409.1519 [hep-th].

[28] J. Hartong, E. Kiritsis, and N. A. Obers, "Schrödinger Invariance from Lifshitz Isometries in Holography and Field Theory," *Phys. Rev.* **D92** (2015) 066003, arXiv:1409.1522 [hep-th].

[29] E. A. Bergshoeff, J. Hartong, and J. Rosseel, "Torsional Newton-Cartan Geometry and the Schrödinger Algebra," *Class.Quant.Grav.* **32** (2015) 135017, arXiv:1409.5555 [hep-th].

[30] D. T. Son, "Newton-Cartan Geometry and the Quantum Hall Effect," arXiv:1306.0638 [cond-mat.mes-hall].

[31] C. Hoyos and D. T. Son, "Hall Viscosity and Electromagnetic Response," *Phys.Rev.Lett.* **108** (2012) 066805, arXiv:1109.2651 [cond-mat.mes-hall].

[32] A. G. Abanov and A. Gromov, "Electromagnetic and gravitational responses of two-dimensional noninteracting electrons in a background magnetic field," *Phys. Rev.* **B90** (2014) 014435, arXiv:1401.3703 [cond-mat.str-el].

[33] A. Gromov and A. G. Abanov, "Thermal Hall Effect and Geometry with Torsion," *Phys.Rev.Lett.* **114** (2015) 016802, arXiv:1407.2908 [cond-mat.str-el].

[34] X. Wen and A. Zee, "Shift and spin vector: New topological quantum numbers for the Hall fluids," *Phys.Rev.Lett.* **69** (1992) 953–956.

[35] C. Hoyos, S. Moroz, and D. T. Son, "Effective theory of chiral two-dimensional superfluids," *Phys. Rev.* **B89** (2014) 174507, arXiv:1305.3925 [cond-mat.quant-gas].

[36] S. Moroz and C. Hoyos, "Effective theory of two-dimensional chiral superfluids: gauge duality and Newton-Cartan formulation," *Phys.Rev.* **B91** no. 6, (2015) 064508, arXiv:1408.5911 [cond-mat.quant-gas].

[37] B. Carter and I. Khalatnikov, "Canonically covariant formulation of Landau's Newtonian superfluid dynamics," *Rev.Math.Phys.* **6** (1994) 277–304.

[38] B. Carter and N. Chamel, "Covariant analysis of newtonian multi-fluid models for neutron stars: I milne-cartan structure and variational formulation," *Int.J.Mod.Phys.* **D13** (2004) 291–326, arXiv:astro-ph/0305186 [astro-ph].

[39] B. Carter and N. Chamel, "Covariant analysis of Newtonian multi-fluid models for neutron stars: 2 Stress - energy tensors and virial theorems," *Int. J. Mod. Phys.* **D14** (2005) 717–748, arXiv:astro-ph/0312414 [astro-ph].

[40] B. Carter and N. Chamel, "Covariant analysis of Newtonian multi-fluid models for neutron stars. 3. Transvective, viscous, and superfluid drag dissipation," *Int. J. Mod. Phys.* **D14** (2005) 749–774, arXiv:astro-ph/0410660 [astro-ph].

[41] N. Banerjee, S. Dutta, and A. Jain, "Null Fluids - A New Viewpoint of Galilean Fluids," arXiv:1509.04718 [hep-th].

[42] A. Jain, "Galilean Anomalies and Their Effect on Hydrodynamics," arXiv:1509.05777 [hep-th].

[43] M. Taylor, "Non-relativistic holography," arXiv:0812.0530 [hep-th].

[44] K. Balasubramanian and J. McGreevy, "Gravity duals for non-relativistic CFTs," *Phys.Rev.Lett.* **101** (2008) 061601, arXiv:0804.4053 [hep-th].

[45] S. A. Hartnoll, "Lectures on holographic methods for condensed matter physics," *Class.Quant.Grav.* **26** (2009) 224002, arXiv:0903.3246 [hep-th].

[46] A. Bagchi and R. Gopakumar, "Galilean Conformal Algebras and AdS/CFT," *JHEP* **0907** (2009) 037, arXiv:0902.1385 [hep-th].

[47] G. 't Hooft, "Dimensional reduction in quantum gravity," arXiv:gr-qc/9310026 [gr-qc].

[48] L. Susskind, "The World as a hologram," *J.Math.Phys.* **36** (1995) 6377–6396, arXiv:hep-th/9409089 [hep-th].

[49] A. Castro, M. R. Gaberdiel, T. Hartman, A. Maloney, and R. Volpato, "The Gravity Dual of the Ising Model," *Phys.Rev.* **D85** (2012) 024032, arXiv:1111.1987 [hep-th].

[50] M. R. Gaberdiel and R. Gopakumar, "Minimal Model Holography," *J.Phys.* **A46** (2013) 214002, arXiv:1207.6697 [hep-th].

[51] D. Grumiller, W. Riedler, J. Rosseel, and T. Zojer, "Holographic applications of logarithmic conformal field theories," *J.Phys.* **A46** (2013) 494002, arXiv:1302.0280 [hep-th].

[52] J. D. Brown and M. Henneaux, "Central Charges in the Canonical Realization of Asymptotic Symmetries: An Example from Three-Dimensional Gravity," *Commun.Math.Phys.* **104** (1986) 207–226.

[53] S. Detournay, T. Hartman, and D. M. Hofman, "Warped Conformal Field Theory," *Phys. Rev.* **D86** (2012) 124018, arXiv:1210.0539 [hep-th].

[54] D. M. Hofman and B. Rollier, "Warped Conformal Field Theory as Lower Spin Gravity," *Nucl. Phys.* **B897** (2015) 1–38, arXiv:1411.0672 [hep-th].

[55] C. Duval, G. W. Gibbons, and P. A. Horvathy, "Conformal Carroll groups," *J. Phys.* **A47** (2014) 335204, arXiv:1403.4213 [hep-th].

[56] A. Bagchi, S. Chakrabortty, and P. Parekh, "Tensionless Strings from Worldsheet Symmetries," arXiv:1507.04361 [hep-th].

[57] C. Duval, G. W. Gibbons, P. A. Horvathy, and P. M. Zhang, "Carroll versus Newton and Galilei: two dual non-Einsteinian concepts of time," *Class. Quant. Grav.* **31** (2014) 085016, arXiv:1402.0657 [gr-qc].

[58] C. Duval, G. W. Gibbons, and P. A. Horvathy, "Conformal Carroll groups and BMS symmetry," *Class. Quant. Grav.* **31** (2014) 092001, arXiv:1402.5894 [gr-qc].

[59] E. Bergshoeff, J. Gomis, and G. Longhi, "Dynamics of Carroll Particles," *Class.Quant.Grav.* **31** (2014) 205009, arXiv:1405.2264 [hep-th].

[60] E. Bergshoeff, J. Gomis, and L. Parra, "The Symmetries of the Carroll Superparticle," arXiv:1503.06083 [hep-th].

[61] J. Hartong, "Gauging the Carroll Algebra and Ultra-Relativistic Gravity," *JHEP* **08** (2015) 069, arXiv:1505.05011 [hep-th].

[62] J. Gomis and H. Ooguri, "Nonrelativistic closed string theory," *J.Math.Phys.* **42** (2001) 3127–3151, arXiv:hep-th/0009181 [hep-th].

[63] U. H. Danielsson, A. Guijosa, and M. Kruczenski, "IIA/B, wound and wrapped," *JHEP* **0010** (2000) 020, arXiv:hep-th/0009182 [hep-th].

[64] J. Brugues, T. Curtright, J. Gomis, and L. Mezincescu, "Non-relativistic strings and branes as non-linear realizations of Galilei groups," *Phys.Lett.* **B594** (2004) 227–233, arXiv:hep-th/0404175 [hep-th].

[65] J. Gomis, K. Kamimura, and P. K. Townsend, "Non-relativistic superbranes," *JHEP* **0411** (2004) 051, arXiv:hep-th/0409219 [hep-th].

[66] J. Gomis and F. Passerini, "Rotating solutions of non-relativistic string theory," *Phys. Lett.* **B617** (2005) 182–192, arXiv:hep-th/0411195 [hep-th].

[67] J. Gomis, J. Gomis, and K. Kamimura, "Non-relativistic superstrings: A New soluble sector of AdS(5) x S**5," *JHEP* **0512** (2005) 024, arXiv:hep-th/0507036 [hep-th].

[68] J. Gomis, F. Passerini, T. Ramirez, and A. Van Proeyen, "Non relativistic Dp branes," *JHEP* **10** (2005) 007, arXiv:hep-th/0507135 [hep-th].

[69] B. S. Kim, "World Sheet Commuting beta gamma CFT and Non-Relativistic String Theories," *Phys. Rev.* **D76** (2007) 106007, arXiv:0708.4261 [hep-th].

[70] B. S. Kim, "Non-relativistic superstring theories," *Phys. Rev.* **D76** (2007) 126013, arXiv:0710.3203 [hep-th].

[71] X. Bekaert and K. Morand, "Connections and dynamical trajectories in generalised Newton-Cartan gravity I. An intrinsic view," arXiv:1412.8212 [hep-th].

[72] X. Bekaert and K. Morand, "Connections and dynamical trajectories in generalised Newton-Cartan gravity II. An ambient perspective," arXiv:1505.03739 [hep-th].

[73] D. Tong and C. Turner, "The Quantum Hall Effect in Supersymmetric Chern-Simons Theories," arXiv:1508.00580 [hep-th].

[74] V. Pestun, "Localization of gauge theory on a four-sphere and supersymmetric Wilson loops," *Commun.Math.Phys.* **313** (2012) 71–129, arXiv:0712.2824 [hep-th].

[75] M. Marino, "Lectures on localization and matrix models in supersymmetric Chern-Simons-matter theories," *J.Phys.* **A44** (2011) 463001, arXiv:1104.0783 [hep-th].

[76] G. Festuccia and N. Seiberg, "Rigid Supersymmetric Theories in Curved Superspace," *JHEP* **1106** (2011) 114, arXiv:1105.0689 [hep-th].

[77] R. Andringa, E. A. Bergshoeff, J. Rosseel, and E. Sezgin, "3D Newton-Cartan Supergravity," *Class.Quant.Grav.* **30** (2013) 205005, arXiv:1305.6737 [hep-th].

[78] S. R. Coleman and J. Mandula, "All Possible Symmetries of the S Matrix," *Phys.Rev.* **159** (1967) 1251–1256.

[79] R. Haag, J. T. Lopuszanski, and M. Sohnius, "All Possible Generators of Supersymmetries of the s Matrix," *Nucl.Phys.* **B88** (1975) 257.

[80] R. Andringa, E. Bergshoeff, S. Panda, and M. de Roo, "Newtonian Gravity and the Bargmann Algebra," *Class.Quant.Grav.* **28** (2011) 105011, arXiv:1011.1145 [hep-th].

[81] A. Bagchi, R. Gopakumar, I. Mandal, and A. Miwa, "GCA in 2d," *JHEP* **1008** (2010) 004, arXiv:0912.1090 [hep-th].

[82] C. Hagen, "Scale and conformal transformations in galilean-covariant field theory," *Phys.Rev.* **D5** (1972) 377–388.

[83] U. Niederer, "The maximal kinematical invariance group of the free Schrödinger equation.," *Helv.Phys.Acta* **45** (1972) 802–810.

[84] R. Puzalowski, "Galilean supersymmetry," *Acta Phys.Austriaca* **50** (1978) 45.

[85] G. De Franceschi and F. Palumbo, "Spontaneous supersymmetry breaking and superconductivity in a nonrelativistic model," *Nucl.Phys.* **B162** (1980) 478.

[86] T. Clark and S. Love, "Nonrelativistic supersymmetry," *Nucl.Phys.* **B231** (1984) 91.

[87] J. Beckers and V. Hussin, "Dynamical Supersymmetries of the Harmonic Oscillator," *Phys.Lett.* **A118** (1986) 319–321.

[88] J. Beckers, D. Dehin, and V. Hussin, "Symmetries and Supersymmetries of the Quantum Harmonic Oscillator," *J.Phys.* **A20** (1987) 1137–1154.

[89] J. P. Gauntlett, J. Gomis, and P. Townsend, "Supersymmetry and the physical phase space formulation of spinning particles," *Phys.Lett.* **B248** (1990) 288–294.

[90] J. de Azcarraga and D. Ginestar, "Nonrelativistic limit of supersymmetric theories," *J.Math.Phys.* **32** (1991) 3500–3508.

[91] M. Leblanc, G. Lozano, and H. Min, "Extended superconformal Galilean symmetry in Chern-Simons matter systems," *Annals Phys.* **219** (1992) 328–348, `arXiv:hep-th/9206039 [hep-th]`.

[92] C. Duval and P. Horvathy, "On Schrödinger superalgebras," *J.Math.Phys.* **35** (1994) 2516–2538, `arXiv:hep-th/0508079 [hep-th]`.

[93] O. Bergman and C. B. Thorn, "SuperGalilei invariant field theories in (2+1)-dimensions," *Phys.Rev.* **D52** (1995) 5997–6007, `arXiv:hep-th/9507007 [hep-th]`.

[94] O. Bergman, "Galilean supersymmetry in (2+1)-dimensions," *Int.J.Mod.Phys.* **A12** (1997) 1173–1182, `arXiv:hep-th/9609098 [hep-th]`.

[95] A. H. Chamseddine and P. C. West, "Supergravity as a Gauge Theory of Supersymmetry," *Nucl.Phys.* **B129** (1977) 39.

[96] E. Inonu and E. P. Wigner, "On the Contraction of groups and their represenations," *Proc.Nat.Acad.Sci.* **39** (1953) 510–524.

[97] D. Z. Freedman and A. Van Proeyen, "Supergravity," *Cambridge University Press* (2012) .

[98] V. Bargmann, "On Unitary ray representations of continuous groups," *Annals Math.* **59** (1954) 1–46.

[99] J.-M. Lévy-Leblond, "Galilei group and Galilean invariance," *Group Theory and Applications (Loebl Ed.), Acad. Press, New York* **II** (1972) 222.

[100] A. Trautmann, "Foundations and Current Problems of General Relativity," *Brandeis Summer Institute, Lectures on General Relativity, Prentice-Hall* (1965) .

[101] R. Andringa, E. Bergshoeff, J. Gomis, and M. de Roo, "'Stringy' Newton-Cartan Gravity," *Class.Quant.Grav.* **29** (2012) 235020, `arXiv:1206.5176 [hep-th]`.

[102] H. Afshar, E. A. Bergshoeff, A. Mehra, P. Parekh, and B. Rollier, "Hořava-Lifshitz gravity and Schrödinger scalar field theories,". Work in progress, UG-15-55.

[103] J. Hartong and N. A. Obers, "Hořava-Lifshitz gravity from dynamical Newton-Cartan geometry," *JHEP* **07** (2015) 155, `arXiv:1504.07461 [hep-th]`.

[104] P. Horava, "Membranes at Quantum Criticality," *JHEP* **03** (2009) 020, `arXiv:0812.4287 [hep-th]`.

[105] P. Horava, "Quantum Gravity at a Lifshitz Point," *Phys. Rev.* **D79** (2009) 084008, `arXiv:0901.3775 [hep-th]`.

[106] M. Spradlin, A. Strominger, and A. Volovich, "Les Houches lectures on de Sitter space," in *Unity from duality: Gravity, gauge theory and strings. Proceedings, NATO Advanced Study Institute, Euro Summer School, 76th session, Les Houches, France, July 30-August 31, 2001*, pp. 423–453. 2001. `arXiv:hep-th/0110007 [hep-th]`. `http://alice.cern.ch/format/showfull?sysnb=2277605`.

[107] S. Deser, R. Jackiw, and G. 't Hooft, "Three-Dimensional Einstein Gravity: Dynamics of Flat Space," *Annals Phys.* **152** (1984) 220.

[108] S. Deser and R. Jackiw, "Three-Dimensional Cosmological Gravity: Dynamics of Constant Curvature," *Annals Phys.* **153** (1984) 405–416.

[109] S. Giddings, J. Abbott, and K. Kuchar, "Einstein's theory in a three-dimensional space-time," *Gen. Rel. Grav.* **16** (1984) 751–775.

[110] E. Bergshoeff, J. Rosseel, and T. Zojer, "Newton–Cartan (super)gravity as a non-relativistic limit," *Class. Quant. Grav.* **32** (2015) 205003, `arXiv:1505.02095 [hep-th]`.

[111] E. Bergshoeff, J. Gomis, M. Kovacevic, L. Parra, J. Rosseel, and T. Zojer, "The Non-Relativistic Superparticle in a Curved Background," *Phys.Rev.* **D90** (2014) 065006, `arXiv:1406.7286 [hep-th]`.

[112] A. Achucarro and P. Townsend, "A Chern-Simons Action for Three-Dimensional anti-De Sitter Supergravity Theories," *Phys.Lett.* **B180** (1986) 89.

[113] A. Achucarro and P. Townsend, "Extended Supergravities in $d = (2{+}1)$ as Chern-Simons Theories," *Phys.Lett.* **B229** (1989) 383.

[114] P. S. Howe, J. Izquierdo, G. Papadopoulos, and P. Townsend, "New supergravities with central charges and Killing spinors in (2+1)-dimensions," *Nucl.Phys.* **B467** (1996) 183–214, `arXiv:hep-th/9505032 [hep-th]`.

[115] R. Casalbuoni, "The Classical Mechanics for Bose-Fermi Systems," *Nuovo Cim.* **A33** (1976) 389.

[116] L. Brink and J. Schwarz, "Quantum Superspace," *Phys.Lett.* **B100** (1981) 310–312.

[117] W. Siegel, "Hidden Local Supersymmetry in the Supersymmetric Particle Action," *Phys.Lett.* **B128** (1983) 397.

[118] K. Kuchar, "Gravitation, geometry, and nonrelativistic quantum theory," *Phys.Rev.* **D22** (1980) 1285–1299.

[119] R. De Pietri, L. Lusanna, and M. Pauri, "Standard and generalized Newtonian gravities as 'gauge' theories of the extended Galilei group. I. The standard theory," *Class.Quant.Grav.* **12** (1995) 219–254, `arXiv:gr-qc/9405046 [gr-qc]`.

[120] E. Witten, "Twistor - Like Transform in Ten-Dimensions," *Nucl.Phys.* **B266** (1986) 245.

[121] G. W. Moore and P. C. Nelson, "Anomalies in Nonlinear σ Models," *Phys.Rev.Lett.* **53** (1984) 1519.

[122] C. Hull and A. Van Proeyen, "Pseudoduality," *Phys.Lett.* **B351** (1995) 188–193, `arXiv:hep-th/9503022 [hep-th]`.

[123] G. Dautcourt, "Die Newtonsche Gravitationstheorie als strenger Grenzfall der Allgemeinen Relativitätstheorie," *Acta.Phys.Polon.* **25** (1964) 637.

[124] H. Kuenzle, "Covariant Newtonian limit of Lorentz spacetimes," *Gen.Rel.Grav.* **7** (1976) 445.

[125] M. Le Ballac and J. Levy-Léblond, "Galilean electromagnetism," *Nuovo Cimento* **B14** (1973) 217.

[126] H. Bacry and J. Levy-Leblond, "Possible kinematics," *J.Math.Phys.* **9** (1968) 1605–1614.

[127] G. Gibbons and C. Patricot, "Newton-Hooke space-times, Hpp waves and the cosmological constant," *Class.Quant.Grav.* **20** (2003) 5225, arXiv:hep-th/0308200 [hep-th].

[128] J. Brugues, J. Gomis, and K. Kamimura, "Newton-Hooke algebras, non-relativistic branes and generalized pp-wave metrics," *Phys.Rev.* **D73** (2006) 085011, arXiv:hep-th/0603023 [hep-th].

[129] J. Lukierski, P. Stichel, and W. Zakrzewski, "Acceleration-enlarged symmetries in nonrelativistic space-time with a cosmological constant," *Eur.Phys.J.* **C55** (2008) 119–124, arXiv:0710.3093 [hep-th].

[130] J. Gomis and J. Lukierski, "Enlarged NH symmetries: Particle Dynamics and Gauge Symmetries," *Phys.Lett.* **B664** (2008) 107–111, arXiv:0803.3965 [hep-th].

[131] C. Duval and P. Horvathy, "Conformal Galilei groups, Veronese curves, and Newton-Hooke spacetimes," *J.Phys.* **A44** (2011) 335203, arXiv:1104.1502 [hep-th].

[132] A. Galajinsky and I. Masterov, "Remarks on l-conformal extension of the Newton-Hooke algebra," *Phys.Lett.* **B702** (2011) 265–267, arXiv:1104.5115 [hep-th].

[133] M. Sakaguchi and K. Yoshida, "Non-relativistic AdS branes and Newton-Hooke superalgebra," *JHEP* **0610** (2006) 078, arXiv:hep-th/0605124 [hep-th].

[134] A. Galajinsky, "N=2 superconformal Newton-Hooke algebra and many-body mechanics," *Phys.Lett.* **B680** (2009) 510–515, arXiv:0906.5509 [hep-th].

[135] A. Galajinsky, "Conformal mechanics in Newton-Hooke spacetime," *Nucl.Phys.* **B832** (2010) 586–604, arXiv:1002.2290 [hep-th].

[136] Y.-h. Gao, "Symmetries, matrices, and de Sitter gravity," *Advanced Lectures in Mathematics* **I** (2008) 271–310, arXiv:hep-th/0107067 [hep-th].

[137] E. Bergshoeff, J. Rosseel, and T. Zojer, "Newton–Cartan supergravity with torsion and Schrödinger supergravity," *JHEP* **11** (2015) 180, arXiv:1509.04527 [hep-th].

[138] J. A. de Azcarraga and J. Lukierski, "Galilean Superconformal Symmetries," *Phys. Lett.* **B678** (2009) 411–415, arXiv:0905.0141 [math-ph].

[139] M. Sakaguchi, "Super Galilean conformal algebra in AdS/CFT," *J. Math. Phys.* **51** (2010) 042301, arXiv:0905.0188 [hep-th].

[140] A. Bagchi and I. Mandal, "Supersymmetric Extension of Galilean Conformal Algebras," *Phys.Rev.* **D80** (2009) 086011, arXiv:0905.0580 [hep-th].

[141] M. Sakaguchi and K. Yoshida, "More super Schrödinger algebras from $psu(2,2|4)$," *JHEP* **0808** (2008) 049, arXiv:0806.3612 [hep-th].

[142] K. S. Stelle and P. C. West, "Minimal auxiliary fields for supergravity," *Phys. Lett.* **B74** (1978) 330.

[143] S. Ferrara and P. van Nieuwenhuizen, "The auxiliary fields of supergravity," *Phys. Lett.* **B74** (1978) 333.

[144] M. Kaku and P. K. Townsend, "Poincaré supergravity as broken superconformal gravity," *Phys. Lett.* **B76** (1978) 54.

[145] A. Das, M. Kaku, and P. K. Townsend, "A unified approach to matter coupling in Weyl and Einstein supergravity," *Phys. Rev. Lett.* **40** (1978) 1215.

[146] M. F. Sohnius and P. C. West, "An alternative minimal off-Shell version of N=1 supergravity," *Phys. Lett.* **B105** (1981) 353.

[147] B. de Wit and M. Rocek, "Improved tensor multiplets," *Phys.Lett.* **B109** (1982) 439.

[148] S. Ferrara, L. Girardello, T. Kugo, and A. Van Proeyen, "Relation Between Different Auxiliary Field Formulations of $N = 1$ Supergravity Coupled to Matter," *Nucl.Phys.* **B223** (1983) 191.

[149] S. J. Gates, M. T. Grisaru, M. Rocek, and W. Siegel, "Superspace Or One Thousand and One Lessons in Supersymmetry," *Front. Phys.* **58** (1983) 1–548, `arXiv:hep-th/0108200 [hep-th]`.

[150] C. Duval, G. Burdet, H. P. Kunzle, and M. Perrin, "Bargmann Structures and Newton–Cartan Theory," *Phys. Rev.* **D31** (1985) 1841–1853.

[151] C. Duval, G. W. Gibbons, and P. Horvathy, "Celestial mechanics, conformal structures and gravitational waves," *Phys. Rev.* **D43** (1991) 3907–3922, `arXiv:hep-th/0512188 [hep-th]`.

Samenvatting

De zwaartekracht is een ongetwijfeld alomtegenwoordige, fundamentele kracht in de natuur. Ze is zo gewoon voor ons dat we ons nauwelijks een wereld zonder haar kunnen voorstellen. Uiteraard werd de zwaartekracht ook als eerste onderworpen aan onderzoek door natuurkundigen. Newton en Kepler waren de eersten die de beweging van deeltjes en planeten, die gravitationele interactie ondergaan, succesvol beschreven. Echter geloven we tegenwoordig dat de Newtoniaanse theorie maar een effectieve beschrijving van een andere, meer fundamentele theorie is. Deze algemene relativiteitstheorie werd formeel voor het eerst beschreven door Einstein precies honderd jaar geleden.

De theorie van Einstein berust slechts op enkele pijlers, uitspraken die beperkingen vastleggen, bepaalde wetten die de theorie moet gehoorzamen. Een daarvan is dat de theorie onafhankelijk van het coördinatensysteem, dat gebruikt wordt om de theorie te beschrijven, moet zijn. In de hedendaagse (theoretische, maar niet uitsluitend) natuurkunde is dit een voor de hand liggend ding om te doen. Dat wil zeggen dat men onveranderlijkheid van de theorie en haar voorspellingen onder algemene coördinatentransformaties vereist. Echter, achteraf gezien vragen we ons af waarom niemand deze vraag aan de theorie van Newton gericht had. Zelfs vóór de ontdekking van de algemene relativiteitstheorie had men het concept van diffeomorfismeinvariantie op een theorie, die tot nu toe alleen op een zodanige wijze geformuleerd was dat ze invariant onder Galileïsche coördinatentransformaties is, kunnen richten.

Op dat moment kan dit voldoende zijn geweest. Fysische modellen werden meestal beschouwd door zogenaamde vrije-val, of inertiaalstelsel. Dit zijn coördinatensystemen waarin een waarnemer geen enkele zwaartekracht voelt. Verschillende niet-relativistische inertiaalstelsels worden vervolgens gerelateerd door Galileïsche coördinatentransformaties. (In de relativistische context zijn dit de Lorentz transformaties.) Galileïsche transformaties verbinden twee verschillende coördinatensystemen via *constante* rotaties, boosts en verschuivingen. Daarentegen zijn willekeurige algemene coördinatentransformaties niet onderworpen aan enige beperking, zoals bestendigheid van de symmetrie parameters. Zoals eerder vermeld, beschouwt men in de context van niet-relativistische theorieën meestal alleen Galileïsche transformaties. Echter, in principe moet er een beschrijving van elke niet-relativistische theorie, en dus ook van de Newtoniaanse zwaartekracht, die invariant onder willekeurige transformaties van de coördinaten is, bestaan.

Dus, wat is de coördinatenonafhankelijke beschrijving van de Newtoniaanse zwaartekracht? Slechts een paar jaar na de ontdekking van Einsteins algemene relativiteitstheorie werd deze vraag uiteindelijk beantwoord. Het werd ontdekt door de Franse wiskundige Cartan en de theorie is nu bekend onder de naam Newton–Cartan zwaartekracht. Het is deze coördinatenonafhankelijke beschrijving van de Newtoniaanse zwaartekracht die we voornamelijk behandelen in dit proefschrift.

Merk op dat terwijl de Newton–Cartan theorie wel één van de pilaren waarop de Einsteinse theorie gebaseerd is gehoorzaamt, wijkt ze van de algemene relativiteitstheorie in een ander cruciaal aspect af. Namelijk, in de Newton–Cartan theorie bestaat geen limiterende snelheid. We bespreken dit onderwerp ook in dit proefschrift en we onderzoeken hoe de theorie van Einstein wordt vervormd wanneer we het snelheidslimiet, de lichtsnelheid, naar oneindig laten gaan. Zoals we laten zien, biedt dit een manier om de Newton–Cartan theorie van de algemene relativiteitstheorie af te leiden.

Recente belangstelling voor het beschrijven van theorieën die invariant onder niet-relativistische algemene coördinatentransformaties zijn, is aangewakkerd door ontwikkelingen in de theorie van de gecondenseerde materie. Dit leidde tot meer systematische studies, met name hoe niet-relativistische veldentheorieën aan Newton–Cartan achtergronden koppelen. Er is opgemerkt dat naast de "gewone" ijkvelden, de tijd-achtige en ruimtelijke vielbein, een extra vectorveld moet worden toegevoegd om niet-relativistische veldentheorieën consistent aan willekeurige gekromde achtergronden te koppelen. In het algemeen laten deze onderzoeken ook een ander belangrijk aspect van de achtergrond geometrie zien: dat men de torsie van de niet-relativistische zwaartekracht theorie niet mag beperken.

De noodzaak om een extra vectorijkveld aan de Newton–Cartan velden toe te voegen was bijvoorbeeld gerealiseerd door Son. Het gebruik van Newton–Cartan structuren is verplicht in zijn effectieve actie voor het Quantum Hall effect. Dit model is vanzelf voorzien van een zogenaamde Wen–Zee term welke de koppeling tussen het extra ijkveld en de kromming beschrijft. Dit codeert de Hall viscositeit en beschrijft dus meer universele kenmerken van het Quantum Hall effect dan alleen de gekwantiseerd Hall geleidbaarheid. Door middel van een soortgelijke koppeling met een $U(1)$ ijkveld en een Wen–Zee term verkrijgt men een ook een effectieve actie voor chirale superfluïde vloeistof. Het werd gerealiseerd dat Newton–Cartan structuren een bijzonder nette manier bieden om de actie in een covariante vorm te schrijven, en dus bieden ze een handige kader om, bijvoorbeeld, de energiestroom van de superfluïde vloeistof te berekenen. Newton–Cartan structuren zijn ook essentieel in het beschrijven van Newtoniaanse vloeistoffen. In het bijzonder werden dergelijke modellen gebruikt om de effecten van superfluïditeit in Neutronensterren te beschrijven. Een studie van de niet-relativistische vloeistoffen, of "Galileïsche vloeistoffen", dat wil zeggen hydrodynamica op Galileïsche achtergronden, maakt ook gebruik van Newton–Cartan structuren.

Sommige onderzoeken in de theorie van de gecondenseerde materie worden gemotiveerd door de opkomst van nieuwe "holografische" technieken. Het holografische principe, dat een overeenstemming van zwaartekracht theorieën in anti-de Sitter ruimte-tijd en conforme veldentheorieën voorspeld, de AdS/CFT correspondentie, ligt aan de basis van deze technieken. Enkele eenvoudige, maar inzichtelijke, manifestaties van deze dualiteit werden gevonden in modellen in de drie-dimensionale ruimte-tijd, voortbouwend op een werk van Brown en Henneaux, welke inderdaad een van de voorlopers van deze dualiteit is. Terwijl, zoals we eerder hebben gezegd, dergelijke technieken al worden toegepast, moet nog veel worden onderzocht wat de algemeenheid van deze correspondentie betreft. Het onderzoek naar het vinden van niet-relativistische realisaties van deze dualiteit is een poging in deze richting.

Newton–Cartan-achtige structuren, in feite duale structuren die meer gerelateerd zijn aan de ultra-relativistische natuurkunde dan aan niet-relativistische natuurkunde, komen ook in kromgetrokken conforme veldentheorieën te voorschijn. Vermoedelijk spelen ze ook een rol bij de spanningsloze limiet van de snaartheorie. De spanningsloze limiet betekent het nemen van een ultra-relativistische limiet, dat tot "Carrollische" fysica en ruimte-tijd leidt. Deze zijn in vele opzichten duaal aan Newton–Cartan structuren. Andere, in feite tegenovergestelde, niet-relativistische limieten van de snaartheorie zijn ook overwogen. Niet-relativistische snaartheorie werd bestudeerd als een mogelijk oplosbare sector in snaartheorie of M-theorie. Deze onderzoeken zijn ook belangrijk voor de toepassingen van het holografische principe in het kader van de theorie van gecondenseerde materie die we eerder hebben vermeld. Bovendien, als we ons werk willen verbinden met snaartheorie of M-theorie lijkt het alleen maar logisch dat we supersymmetrische uitbreidingen van Newton–Cartan structuren zouden bestuderen. Daarom zijn we in dit proefschrift vooral genteresseerd in *supersymmetrische uitbreidingen van Newton–Cartan structuren*.

Terwijl wij (niet-relativistische) supersymmetrie verderop gaan bespreken, laat ons eerst onze motivaties voor onze interesse in deze extensie bespreken. Een van de onderzoeken over het Quantum Hall effect maakt bijvoorbeeld gebruik van een (niet-relativistisch) supersymmetrisch model. Verdere motivatie, en een van de belangrijkste redenen voor deze stelling in het algemeen, is gerelateerd aan de zeer succesvolle toepassing van lokalisatietechnieken in de relativistische context. De toepassingen van lokalisatietechnieken zijn nuttig om exacte resultaten voor relativistische supersymmetrische veldentheorieën te krijgen. Misschien kunnen we op dezelfde manier ook succesvolle resultaten in de niet-relativistische context verwachten. De werkwijze berust op de koppeling van niet-relativistische veldentheorieën aan willekeurige supersymmetrische achtergronden. Om dit te doen moet men natuurlijk niet-relativistische supergravitatietheorieën bestuderen om dergelijke achtergronden te kunnen beschrijven. Echter, de constructie die we in gedachten hebben maakt gebruik van de zogenaamde off-shell formuleringen van supergravitatie, een kenmerk dat de enige tot nu toe bekende theorie van niet-relativistische supergravitatie niet bezit. Geschikte uitbreidingen van Newton–Cartan supergravitatie zouden moeten worden nagegaan om te zien of deze technieken ook tot niet-relativistische theorieën uitgebreid kunnen worden.

We bespraken de belangrijkste drijfveren voor onze belangstelling voor niet-relativistische fysica en verder dat Newton–Cartan structuren voor een handige manier van aanduiding van de niet-relativistische achtergrond zorgen. Bovendien gaven we een motivatie voor het overwegen van supersymmetrische theorieën. Laten we nu terugkeren naar het punt waar we begonnen: symmetrieën.

Te eisen dat een theorie, Newtoniaanse zwaartekracht in dit geval, invariant is onder algemene coördinatetransformaties betekent dat ze invariant hoort te zijn onder een speciaal soort symmetrietransformatie. Symmetrieën spelen een belangrijke rol in de hedendaagse theoretische natuurkunde. Zij leggen sterke beperkingen op aan een theorie, bijvoorbeeld omdat ze de aard van de interactietermen beperken. Dit is van belang aangezien bepaalde symmetrieën van een theorie vaak te vinden zijn in experimenten en daardoor de onderliggende "fundamentele" theorie veel beter kan worden beschreven. Alle (kwantum) theorieën die worden gebruikt om het standaardmodel van deeltjesfysica te beschrijven zijn ijktheorieën. Dat betekent dat ze invariant zijn onder een zekere symmetrie. De symmetrie groepen zijn $U(1)$ voor de elektrodynamica, $SU(2)$ voor de zwakke interacties en $SU(3)$ voor de sterke wisselwerking. Op een bepaalde manier kan de algemene relativiteitstheorie ook worden opgevat als een ijktheorie. Het is de ijktheorie van diffeomorfismen. Deze functie, het ijktheoretische aspect van de algemene relativiteitstheorie, is van bijzonder belang voor ons.

Met het oog op een dergelijke veelheid aan toepassingen is het een interessante vraag wat de meest algemene symmetrie (groep) is die we kunnen toestaan. Dit is uitgebreid bestudeerd en één van de meest gevierde stellingen door Coleman en Mandula stelt dat een fysieke theorie ten hoogste invariant kan zijn onder de conforme groep. Alle andere symmetrieën moeten "interne" symmetrieën zijn, dat betekent dat de fysieke waarneembaarheid altijd een scalaire representatie van die symmetrieën moet zijn. Deze stelling geldt niet voor één uitzondering: ze geldt niet voor fermionische symmetrieën. Zoals later bleek uit onderzoek van Haag, Lopuszanski en Sohnius is de toevoeging van supersymmetrie, of superconforme symmetrieën, echt de meest algemene mogelijkheid. Dit wijst op een fundamentele aard van supersymmetrie, en dient als een extra motivatie voor ons bestuderen van supersymmetrische theorieën in het verband met Newton–Cartan structuren in dit proefschrift.

Laten we nu onze focus meer op niet-relativistische symmetrieën zetten. Om te begrijpen hoe een theorie invariant onder niet-relativistische diffeomorfismen kan worden moet men natuurlijk weten wat niet-relativistische diffeomorfismen zijn. Dit is ook onderdeel van het huidige onderzoek. Een benadering bestaat uit het nabootsen van enkele ideeën die ook op de algemene relativiteitstheorie toegepast kunnen worden. De algemene relativiteitstheorie kan als een ijktheorie van de Poincaré algebra worden gezien. Dus kan men proberen om een niet-relativistische versie ervan te ontlenen door het ijken van een niet-relativistische symmetrie algebra in plaats van een relativistische symmetrie algebra. Deze aanpak werd succesvol voortgezet en het werd ook aangetoond hoe de ijktheorie van de Bargmann algebra naar Newtoniaanse zwaartekracht kan worden verminderd door het vastleggen van de ijking. De Newton–Cartan theorie was dus gekoppeld aan de Newtoniaanse zwaartekracht en het werd duidelijk aangetoond dat het verschil tussen deze theorieën juist "de hoeveelheid van symmetrie" die we toestaan is. We gebruiken dezelfde ideeën, dat wil zeggen ijktechnieken, om een aantal nieuwe (superzwaartekracht) theorieën af te leiden in dit proefschrift.

In wat voor soort niet-relativistische symmetrieën zijn we geïnteresseerd? We hebben al Galileïsche symmetrieën en ook de Bargmann algebra besproken. Net als in de relativistische geval kunnen wij deze met nog meer symmetrie generatoren uitbreiden. In het relativistische geval leidt dit tot conforme symmetrie. In het niet-relativistische geval hebben we echter de keuze tussen twee verschillende "conforme" uitbreidingen van de Galilei algebra. De Galileïsche conforme algebra is het niet-relativistische analoog van de conforme algebra en de andere mogelijkheid is de Schrödinger algebra. We zullen kiezen om met de laatstgenoemde te werken, want deze is de enige mogelijke uitbreiding van de Bargmann algebra. Dit betekent vooral dat we voor een eindige massa kunnen zorgen. De aanvullende symmetrie van de Bargmann algebra (ten opzichte van de Galilei algebra) wordt vaak geïnterpreteerd in verband met het behoud van massa (en in de non-relativistische context betekent dit ook het behoud van het aantal deeltjes). De Schrödinger algebra omvat de symmetrie generatoren die tot de symmetrie van de Schrödinger vergelijking leiden (vandaar ook de naam). De Schrödinger transformaties laten ook de eenvoudige actie van de niet-relativistische punt-deeltje invariant.

Dus, in de geest van Coleman–Mandula voegen we "conforme" extensies toe. Maar in dit proefschrift in het bijzonder zullen we ook in een andere toepassing van de niet-relativistische symmetrieën geïnteresseerd zijn, namelijk de toevoeging van supersymmetrie. Boven gaven we een aantal "fysische" redenen om Newton–Cartan theorieën te bestuderen en ook voor het bestuderen van supersymmetrie in deze context. Een andere motivatie komt uit de analyse van de relativistische context. In het relativistische geval zijn supersymmetrie of superconforme symmetrieën de meest algemene symmetrieën die we kunnen toelaten voor een theorie. In het niet-relativistische geval bestaan er echter geen argumenten vergelijkbaar met die van Coleman–Mandula of Haag–Lopuszanski–Sohnius, maar het is nog steeds interessant om te zien of we zelfs niet-relativistische theorieën kunnen vinden die supersymmetrisch zijn.

Niet-relativistische supersymmetrie algebra's zijn niet per se moeilijk om te bouwen en enkele eenvoudige veldentheorieën die deze symmetrieën realiseren zijn ook gevonden. De eerste *lokale* verwezenlijking van niet-relativistische supersymmetrie stamt slechts uit het jaar 2013. Er zijn geen (bekende) obstakels die het vinden van dergelijke theorieën in principe zouden voorkomen. Echter, het blijkt dat dit geen gemakkelijke taak is en men is geconfronteerd met heel wat moeilijkheden. Dit gaf aanleiding om onszelf eerst bekend te maken met de bouw van voorbeelden in een eenvoudiger setting. Dergelijke eenvoud kan bijvoorbeeld komen door theorieën in lagere dimensies te overwegen. Enkele eenvoudige realisaties van het holografische principe worden ook bestudeerd in drie ruimte-tijd dimensies en een supersymmetrisch theorie die wordt gebruikt als een effectieve beschrijving van de Quantum Hall effect berust ook op een driedimensionale model. Daarom zijn de niet-relativistische supergravitatietheorieën die we in dit proefschrift zullen bouwen theorieën in drie ruimte-tijd dimensies. We hopen dat dit ook tot een beter begrip en een verklaring van de mechanismen in het geval van vierdimensionale theorieën leidt.

Kortom, dit proefschrift gaat over de niet-relativistische supergravitatietheorieën in drie ruimte-tijd dimensies. De belangstelling voor dergelijke theorieën volgt uit effectieve modellen in de theorie van gecondenseerde materie en mogelijke toepassingen van de lokalisatie technieken. Het feit dat we in drie dimensies werken volgt uit een eenvoudiger motto van ons: eenvoud eerst.

In eerste instantie richten we ons op de vraag hoe we deze niet-relativistische supergravitatietheorien bouwen. Hier gebruiken we twee verschillende technieken voor. De eerste is afgeleid van de relativistische superzwaartekracht en het feit dat deze door ijken van de Poincaré superalgebra kan worden verkregen. Met toepassing van soortgelijke technieken kunnen wij niet-relativistische supergravitatietheorieën vinden door het ijken van niet-relativistische superalgebra's. De andere methode om niet-relativistische supergravitaties af te leiden is een soort van niet-relativistisch limiet dat pas is ontwikkeld in dit proefschrift.

Er zijn bijzondere redenen waarom we twee in plaats van slechts één methode gebruiken om niet-relativistische supergravitatietheorieën te verkrijgen. IJktechnieken, in het bijzonder in drie dimensies, leiden tot voorstellingen die alleen ijkvelden bevatten. Dit betekent dat ijktechnieken niet tot multipletten met extra (hulp) velden, die typisch voor de zogenaamde off-shell formuleringen voor supergravitatie zijn, leiden. Al gaat het uitvoeren van de toepassing van de lokalisatietechnieken buiten het bestek van dit proefschrift, streven we wel naar de constructie van off-shell formuleringen voor hun vermeende gebruik. In een eenvoudige manier om supergravitatie aan gekromde achtergronden te koppelen zijn we namelijk genoodzaakt om bepaalde (niet nul!) waarden voor de hulpvelden te kiezen. Vandaar de behoefte aan formuleringen die dergelijke velden bevatten.

In dit proefschrift introduceren we ook bepaalde technieken om off-shell formuleringen te verkrijgen via ijktechnieken. Echter, deze berekening zal afhangen van het bestaan van materie multipletten en we gebruiken onze limiet procedure om deze te verkrijgen. We maken hierbij gebruik van een niet-relativistische versie van de superconforme tensorrekening om off-shell formuleringen van driedimensionale niet-relativistische supergravitatie te verkrijgen. Het blijkt dat deze structuren vergelijkbaar zijn met de relativistische zaak.

Off-shell formuleringen van niet-relativistische superzwaartekracht zijn niet de enige extensies die we overwegen. Een andere generalisatie bestaat bijvoorbeeld uit "kosmologische" extensies. We vinden deze door het ijken van een niet-relativistische symmetrie algebra met een "kosmologische constante". We kijken ook naar theorieën die een "maximale" aantal symmetrieën bezitten. Daartoe construeren we een supersymmetrische Schrödinger uitbreiding van de drie-dimensionale niet-relativistische supergravitatie.

Een belangrijk aspect bij het koppelen van niet-relativistische veldentheorieën aan willekeurige achtergronden is dat de torsie van de achtergrond theorie niet beperkt mag zijn, dat wil zeggen dat de kromming van het ijkveld van tijds-verschuivingen onbeperkt moet zijn. Wij zullen dan ook zijn vooral geïnteresseerd zijn in extenties met torsie.

Tot slot breiden we de theorie die werd gepresenteerd in 2013 uit in de richtingen weergegeven in figuur 2. In feite legt dit proefschrift ook uit dat niet alle uit-

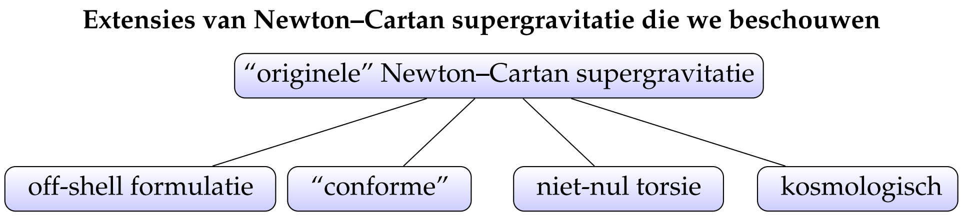

Figure 2: *Extensies van de "originele" drie-dimensionale, $\mathcal{N} = 2$, nul-torsie, on-shell, Newton–Cartan supergravitietheorie die we in dit proefschrift beschouwen.*

breidingen op dezelfde voet staan en we leggen in detail uit hoe ze middels ijkbevestigingen en andere afknottingen (on-shell / off-shell en torsie / geen torsie) aan elkaar gerelateerd zijn. Ook tonen we aan dat de technieken van de superconforme tensorrekening op de niet-relativistische theorie op een eenvoudige manier kan worden toegepast. Dit stelt ons in staat om alle voorgaande resultaten op torsievrije supergravitaties uit te breiden. Het levert ook een zeer efficiënte manier op om theorieën met torsie te construeren.

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Thomas Zojer
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