Fermilab

ENHANCEMENT OF DIFFUSION BY A NON-LINEAR FORCE

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Several observers[1,2] have recently speculated that the simultaneous presence of diffusion processes and the beam-beam interaction may lead to enhanced diffusion or beam loss greater than that present with either diffusion or the beam-beam interaction alone. To test these ideas we have written a computer code to simulate the effects of random diffusion and the beam-beam interaction. In this paper some first results of these simulations are presented. It is found that when the strength parameter of the beam-beam force $\Delta\nu$ includes a resonance within its tune width (see below) enhanced diffusion occurs.

In section 1 we outline the simulation of the beam-beam interaction and diffusion. In section 2 we describe some first simulation results, obtained with a 1-dimensional "weak-strong" non-linear(beam-beam) force. In section 3 we discuss features of these results and plans for future simulations.

1. Simulation procedure

In all of the simulations reported in this note, particle transport is calculated in three steps: a linear transport, a non-linear beam-beam kick, and a random diffusion kick. Particle motion through these steps is calculated for thousands of cycles to simulate beam storage for finite times.

Particle motion through the machine from interaction region to interaction region is simulated by a linear matrix calculation:

$$\begin{pmatrix} x \\ x' \\ y \\ y' \end{pmatrix} = \begin{pmatrix} M_x & 0 \\ 0 & M_y \end{pmatrix} \begin{pmatrix} x \\ x' \\ y \\ y' \end{pmatrix} \quad (1)$$

with the submatrices M_x and M_y given by:

$$M_x = \begin{pmatrix} \cos\mu_x + \alpha_x \sin\mu_x & \beta_x \sin\mu_x \\ \frac{-(1+\alpha_x^2)}{\beta_x} \sin\mu_x & \cos\mu_x - \alpha_x \sin\mu_x \end{pmatrix} \quad (2)$$

where α_x, β_x and μ_x are the usual Courant-Snyder functions. In this transport matrix x and y motion are completely decoupled, and the effects of any nonlinearities or dispersion in the lattice are not included.

The beam-beam interaction is simulated by adding a non-linear kick to the velocity:

$$\begin{pmatrix} x \\ x' \\ y \\ y' \end{pmatrix}_2 = \begin{matrix} x_1 \\ x_1' + F_x(x_1, y_1) \\ y_1 \\ y_1' + F_y(x_1, y_1) \end{matrix} \quad . \quad (3)$$

For the case of a beam-beam interaction caused by a cylindrically symmetric Gaussian beam we use

$$F_x(x,y) = -\frac{4\pi\ \Delta\nu}{\beta_o} \frac{\left(1 - e^{\frac{-(x^2+y^2)}{2\sigma^2}}\right)}{\left(\frac{x^2+y^2}{2\sigma^2}\right)} \ x \quad (4)$$

$$F_y(x,y) = -\frac{4\pi\ \Delta\nu}{\beta_o} \frac{\left(1 - e^{-\frac{(x^2+y^2)}{2\sigma^2}}\right)}{\left(\frac{x^2+y^2}{2\sigma^2}\right)} y \tag{4}$$

where $\Delta\nu$ is the "linear tune shift." Other forms for F_x, F_y, can be chosen to simulate other geometries. In the form of the beam-beam interaction chosen so far we have assumed that the longitudinal length of the interaction is zero, and that the collisions of the bunches are centered and "head-on". We have also chosen the "weak-strong" approximation, that is, the beam-beam force function is the same on every turn, and is not affected by changes in the calculated beam.

Diffusion is simulated by adding a random kick to the velocities on each turn:

$$\begin{aligned} x' &\rightarrow x' + \theta_x \cdot R_x \\ y' &\rightarrow y' + \theta_y \cdot R_y \end{aligned} \tag{5}$$

where θ_x, θ_y are maximum kick amplitudes and R_x, R_y are independent random numbers between -1 and +1, which are changed at each crossing.

The simulation procedure outlined above will be changed in the future in order to obtain more realistic simulations as results and discussions develop.

2. First Results of the Beam-Beam Simulations

In the cases studied in this note the particle motions are reduced to one-dimensional (1-D) motions by setting $x \equiv x' \equiv 0$ in equations 1-5. The interactions are studied by generating a set of particles in an initial gaussian distribution and following their motion through a large number of turns. For most cases discussed in this note the number of particles N_p is 100 and the number of turns N_T is 200,000. We calculate the emittance as a function of time, where the emittance is defined as

$$\varepsilon = 6\sqrt{<(y-y_o)^2> \cdot <(y'-y_o')^2>} \quad , \tag{6}$$

and the averages are over the total number of particles N_p. Phase space plots are also generated, and the distributions of the particles can be studied.

We present the results in 4 categories, depending on whether or not the beam-beam kick and/or the diffusion kick is non-zero.

(A) No beam-beam and no diffusion ("linear" and "quiet"): In this case the beam behavior is trivial with particle positions exactly repeating themselves whenever $n_T\mu_i = p\, 2\pi$ where n_T is the number of turns and p is any integer. It has been checked that the program does produce this result. There is no change in emittance and no change in the particle distribution.

(B) Beam-beam kick and no diffusion ("non-linear" and "quiet"): Particle motion is affected by the beam-beam interaction, but there is no significant increase in beam size. The major change in particle motion is in the phase of motion as the tune is shifted from μ to $\mu + 2\pi\ \Delta\nu \equiv 2\pi\ \nu_o$ as $y \to 0$ where the beam-beam tune shift is largest ($\mu \to \mu$ as $y \to \infty$). In figure 1 we show beam emittance as a function of time (number of turns) for $\nu_o = 0.4$ and $\Delta\nu = 0$, 0.005, 0.02, 0.5 and 0.10. From these cases and others we find that there is no increase in time ("diffusion") for any values of ν_o and $\Delta\nu$.

The phase space distributions are distorted by the nonlinear force and this provides the increase in the scatter of measured emittance with large tune shifts shown in figure 1.

(C) No beam-beam and a "diffusion" kick:

In this case the measured emittance increases linearly with time. The increase can be calculated, obtaining

$$\begin{aligned}\varepsilon(n_T) &= \varepsilon_o + 6n_T\beta \frac{\langle(\Delta y')^2\rangle}{2} \\ &= \varepsilon_o + n_T\beta\,\theta_y^2\end{aligned} \qquad (7)$$

where n_T is the number of turns and ε_o the initial emittance and where ε is calculated using equation 6. Typical cases of such emittance increase are shown in figures 2 and 3 in the cases with $\Delta\nu = 0$. The increase in this and similar cases agrees with equation 7.

We can define a diffusion coefficient through equation 7

$$D_o = \frac{\partial\varepsilon}{\partial t} = \frac{\beta}{T}(\theta_y)^2 \qquad (8)$$

where T is the time associated with one turn.

(D) Beam-beam interaction and diffusion kick ("non-linear" and "noisy"):

Our first simulations have showed some interesting effects connecting diffusion and the non-linear interaction. Figure 3 shows a typical set of results in which the tune at vanishing amplitude ν_o is kept constant at a value of 0.2 and the beam-beam parameter $\Delta\nu$ is varied. For $\Delta\nu$ = 0, 0.005, 0.02, 0.03 and 0.033 the diffusion is constant and agrees with equation (8) within expected statistical accuracy. For $\Delta\nu$ = 0.04, 0.06 and 0.10

the diffusion has doubled, approximately.

This increase is shown in figures 2 and 3 and also in Table 1A, where we have tabulated the diffusion coefficient D and the enhancement factor x_E ($D = x_E D_0$) as a function of $\Delta\nu$ and D_o. D is calculated using a least squares fit, solving the equation:

$$\varepsilon(n) = \varepsilon_o + D\frac{n}{100{,}000}$$

where we have set 100,000 turns equal to one unit of time. Table 1A includes cases with $\Delta\nu$ ranging from 0.0 to 0.10, for two different values of D_o, all with $\nu_o = 0.2$.

We note the following features of the simulations:

1. There is no measurable diffusion enhancement ($x_E \cong 1$) for $\Delta\nu \lesssim .033$. For $\Delta\nu \gtrsim .04$ the diffusion is roughly doubled ($x_E \cong 2$). The diffusion enhancement is roughly constant for all $\Delta\nu \gtrsim 0.4$. This seems to imply that measurable diffusion enhancement ($x_E \gtrsim 1.2$) occurs when the resonant tune 1/6 = .1666 is within the beam-beam tune spread, and implies that the enhancement does not change greatly with tune spread providing only that the major resonance is within the tune spread.

2. We have calculated diffusion enhancement for two very different values of D_o (.008 and .032). The diffusion enhancement factor x_E appears to be independent of D_o.

3. In the cases tested to date the change in emittance seems to remain linearly increasing with time whether or not the diffusion is "resonance-enhanced". These cases have so far been limited to a few hundred thousand turns and to an increase in emittance by a factor of ~4.

4. Although diffusion enhancement is relatively constant for $\Delta\nu \gtrsim .04$, the particle phase-space distributions change significantly. For $\Delta\nu = .04$, much of the enhancement is due to a few particles kicked to very large amplitudes, whereas for $\Delta\nu = 0.10$ the enhancement seems to be distributed throughout the particle distribution.

In Figure 4 we show the variation of x_E with D_o and $\Delta\nu$.

In table IB we show results of other simulations for various values of D_o, ν_o, $\Delta\nu$. Diffusion enhancement occurs when the major resonances 1/4 (.25) and 1/8 (.125) are within the tune spreads. Enhancement by the $\nu = 0.25$ resonance is much larger ($x_E \cong 6$), and enhancement by the $\nu = .125$ resonance is somewhat less ($x_E < 1.5$) In the cases considered to date only ~100 particle trajectories have been followed; statistical inaccuracy makes it difficult to notice enhancement with $x_E \lesssim 1.2$. We have not yet identified enhancement due to resonances of order higher than eight. We have not yet determined whether diffusion enhancement caused by a particular resonance is strongly dependent on ν_o. The cases with $\nu_o = 0.175$ imply some dependence.

3. Discussion and Summary

In these first simulations we have limited ourselves to a one-dimensional, "weak-strong" simulation with only 100 particles tracked for a few hundred thousand turns. In two (or three) dimensions the situation becomes much more complex, and the simple identification of diffusion enhancement with resonant tunes in 1-D may be more difficult in higher dimensions. We plan to explore 2-D effects soon.

We do not yet completely understand the nature of the diffusion enhancement in one dimension. We have not fully explored or explained the dependence of diffusion enhancement on ν_o, $\Delta\nu$, D_o, the beam-beam force shape, and time. Future numerical and analytic studies will explore the details of this effect and may provide analytic methods of calculating the enhancement.

In the work to date, we have begun exploration of the relationship between a non-linear periodic force ("beam-beam") and the increase in the mean-square emittance with time ("diffusion") due to the beam-beam force and/or random processes. We have found that when the beam-beam tune shift includes a major low-order resonance significant enhancement of diffusion due to random processes can occur.

References

1. F. Mills, private communication
2. A. Ruggiero, "The Theory of a Non-Linear System in the Presence of Noise", February 1980

Table 1A

Diffusion as a function of tune shift with $\nu_o = 0.20$. In each simulation we have chosen $\varepsilon_o = 0.02$, a total number of particles of 100, and 200,000 turns of calculation.

D_o (calculated diffusion)	$\Delta\nu$ (tune shift)	D (measured diffusion)	x_E (enhancement factor)
.008	0.	.00765	0.96
.008	0.005	.00886	1.11
.008	0.02	.00902	1.1
.008	0.03	.00675	0.85
.008	0.033	.0111	1.39
.008	0.04	.0174	2.18
.008	0.05	.0199	2.49
.008	0.06	.0141	1.76
.008	0.10	.0178	2.23
.032	0.0	.0302	0.94
.032	0.03	.0383	1.20
.032	0.04	.0820	2.56
.032	0.05	.0612	1.91
.032	0.06	.0654	2.04
.032	0.10	.0770	2.41

Table 1B

Other cases of diffusion simulation with various values of D_o, ν_o, $\Delta\nu$. A typical case has $\varepsilon_o = 0.02$, $\beta_o = 2$, $N_p = 100$ and is tracked for 200,000 turns.

ν_o	$\Delta\nu$	D_o	D	x_E
0.30	0.0	.008	.00691	0.86
0.30	0.04	.008	.0101	1.26
0.30	0.06	.008	.0595	7.44
0.30	0.08	.008	.0322	4.03
0.30	0.10	.008	.0254	3.18
0.15	0.02	.032	.0297	0.93
0.15	0.04	.032	.0394	1.23
0.175	0.005	.032	.0296	0.93
0.175	0.02	.032	.0394	1.23
0.175	0.04	.032	.0346	1.08
0.175	0.06	.032	.0670	2.09

Figure 1: Beam emittance E as a function of time for beam-beam interaction strengths $\Delta\nu$ = 0, 0.02, 0.05, 0.10. There is no diffusion in these cases. In all these cases we have used ν_0 = 0.2, β_0 = 2. m.

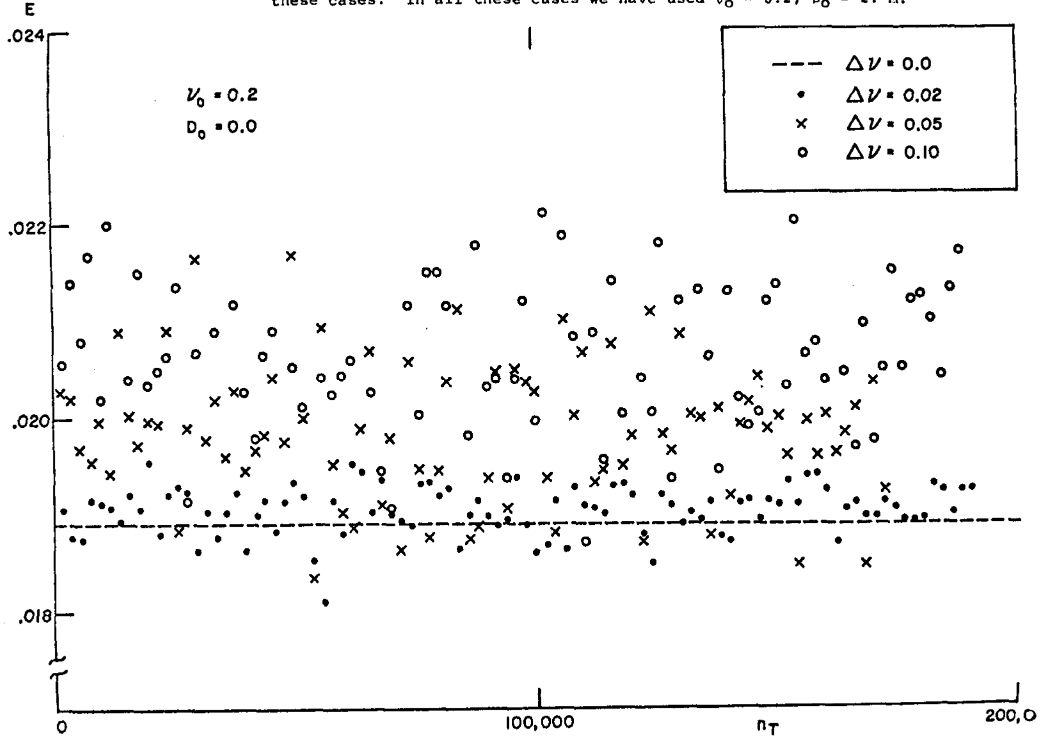

Figure 2: Beam emittance E as a function of time n_T with diffusion. We have set ν_0 = 0.2 and included cases with $\Delta\nu$ = 0., 0.02, 0.05, 0.10.

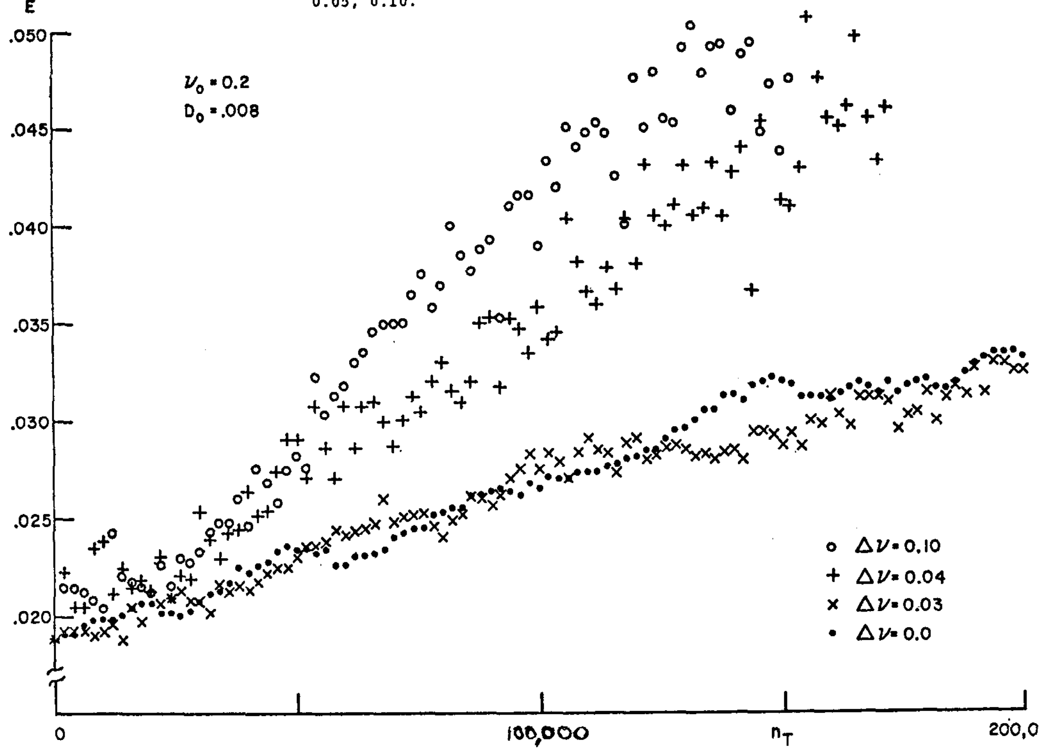

Figure 3: Beam emittance as a function of time n_T with the diffusion kick twice as large as in figure 2, which means an initial diffusion four times as large.

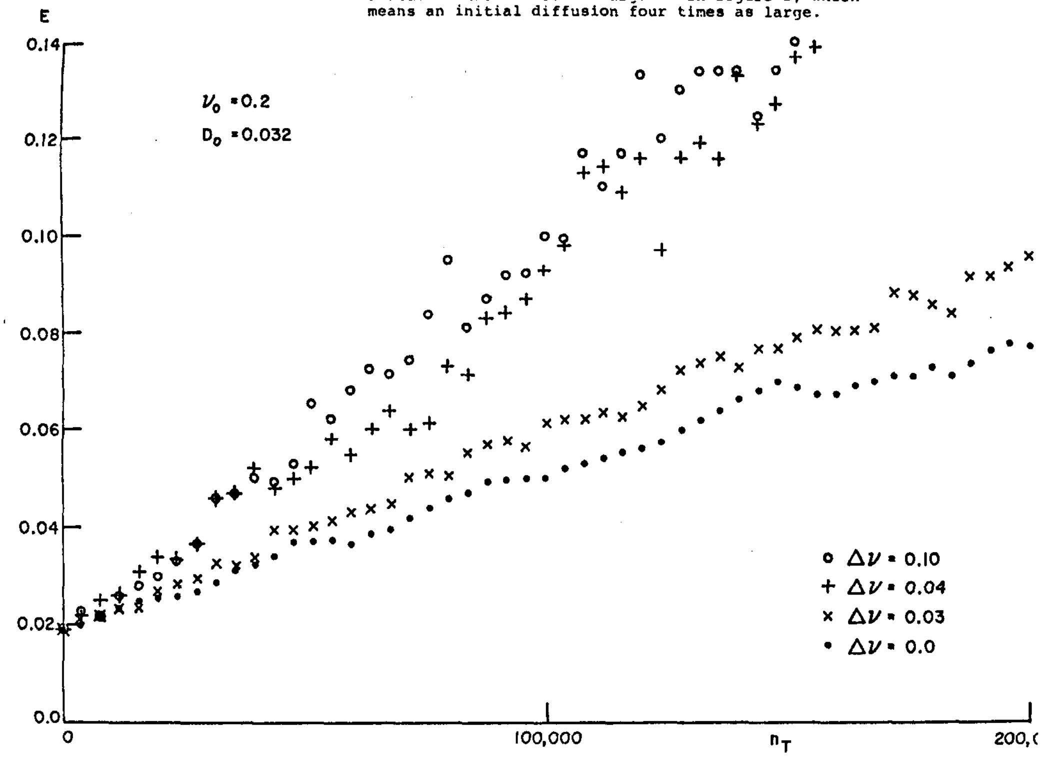

Figure 4: Dit. on enhancement x_E as a function of beam-beam kick strength $\Delta\nu$, for diffusion constant D_o = .008 and .032.