# No-dipole-hair theorem for higher-dimensional static black holes

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## Abstract

We prove that static black holes in $n$-dimensional asymptotically flat spacetime cannot support nontrivial electric $p$-form field strengths when $(n+1)/2 \leq p \leq n-1$. This implies, in particular, that static black holes cannot possess dipole hair under these fields.

## 1 Introduction

Motivated by superstring theory, higher dimensional black holes has been attracted much attention [1]. In higher-dimensional static (electro)vacuum spacetime, the uniqueness thoerem holds. But, if one puts matter fields, it's highly nontrivial whether uniqueness theorem holds or not.

In this article we prove the impossibility of dipole hair for static black holes. The proof follows the one employed in the uniqueness theorem of higher-dimensional static black holes [2–5].

Together with the gauge dipole, we will also consider the inclusion of scalar fields and scalar hair. Bekenstein proved that a static black hole can not have scalar hair in four dimensions [6]. This no-hair theorem is easily extended to higher dimensions since the dimensionality does not enter into the proof. However, this type of proof cannot be applied to systems where the scalar field couples to higher form fields.

## 2 No-dipole-hair theorem

We consider $n$-dimensional asymptotically flat solutions of theories described by the class of Lagrangians

$$\mathcal{L} = R - \frac{1}{2}(\partial\phi)^2 - \frac{1}{p!}e^{-\alpha\phi}H_{(p)}^2, \tag{1}$$

where $R$ is the $n$-dimensional Ricci scalar, $\phi$ is a dilaton with coupling $\alpha$, and $H_{(p)}$ is the field strength of a $(p-1)$-form field potential $B_{(p-1)}$,

$$H_{(p)} = dB_{(p-1)}. \tag{2}$$

Since we are interested in asymptotically flat spacetimes, we take $p \leq n-1$. A form field with $p = n$ does not have any dynamical degree of freedom and behaves like a cosmological constant, which would prevent asymptotic flatness.

We only consider electric fields of $H_{(p)}$. Note that via electric-magnetic duality we can always trade a magnetic charge or dipole under $H_{(p)}$ for an electric one under $H_{(n-p)}$ . However, we do not consider the possibility of simultaneous presence of dipoles and monopole charges of electric and magnetic type, *e.g.*, in $n = p+2$ one can have solutions with both magnetic monopole charge and electric dipole of $H_{(p)}$.

The metric of a static spacetime can be written as

$$ds^2 = g_{MN}dx^M dx^N = -V^2(x^i)dt^2 + g_{ij}(x^k)dx^i dx^j, \tag{3}$$

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where $x^i$ are spatial coordinates on $x^0 = t =$const. surfaces $\Sigma$. In these coordinates, the event horizon is located at $V = 0$, *i.e.*, the Killing horizon. The static ansatz for the $(p-1)$-form potential is of the form

$$B_{(p-1)} = \varphi_{i_1 \cdots i_{p-2}}(x^k) dt \wedge dx^{i_1} \wedge \cdots \wedge dx^{i_{p-2}}. \tag{4}$$

Then the only nontrivial component of the field strength is $H_{0i_1 \cdots 1_{p-1}}$. The metric components and the potential do not depend on $t$.

We shall prove the following theorem:

*No-dipole-hair theorem: The only static, asymptotically flat black hole solution for the theories (1) with electric p-form field strength, with $(n+1)/2 \le p \le n-1$, is the Schwarzschild-Tangherlini solution.*

From the Einstein equation we have

$${}^{(n-1)}R = \frac{e^{-\alpha\phi}}{(p-1)! V^2} H_0{}^{i_1 \cdots i_{p-1}} H_{0i_1 \cdots i_{p-1}} + \frac{1}{2}(D\phi)^2 \tag{5}$$

and

$$D^2 V = \frac{n-p-1}{(n-2)(p-1)!} \frac{e^{-\alpha\phi}}{V} H_0{}^{i_1 \cdots i_{p-1}} H_{0i_1 \cdots i_{p-1}}. \tag{6}$$

where $D_i$ is the covariant derivative with respect to $g_{ij}$ and ${}^{(n-1)}R$ is Ricci scalar on $\Sigma$.

The asymptotic behavior of $V, g_{ij}$, and $H_{(p)}$ is

$$V = 1 - \frac{m}{r^{n-3}} + O(1/r^{n-2}) \tag{7}$$

$$g_{ij} = \delta_{ij}\Big(1 + \frac{2}{n-3}\frac{m}{r^{n-3}}\Big) + O(1/r^{n-2}) \tag{8}$$

$$H_{0i_1 \cdots i_{p-1}} = O(1/r^{n-p+1}). \tag{9}$$

Observe that the falloff of $H_{(p)}$ is the appropriate one for a dipole field, or higher multipole components. In our proof this decay rate could be relaxed to that of a monopole field, $O(1/r^{n-p})$. However, as we explained in the introduction, when $p > 2$ electric monopole charges are incompatible with asymptotic flatness.

We also assume regularity on the event horizon. To this effect, we compute the curvature invariant

$$\begin{aligned} R_{MNKL}R^{MNKL} &= {}^{(n-1)}R_{ijkl}{}^{(n-1)}R^{ijkl} \\ &\quad + \frac{4(n-2)}{(n-3)V^2\rho^2}[k_{ab}k^{ab} + k^2 + \mathcal{D}_a\rho\mathcal{D}^a\rho]. \end{aligned} \tag{10}$$

Here we have used the fact that the spatial metric can be written as

$$g_{ij}dx^i dx^j = \rho^2 dV^2 + h_{ab}dx^a dx^b, \tag{11}$$

where $x^a$ is the coordinate on the level surfaces of $V$. $\mathcal{D}_a$ is the covariant derivative with respect to $h_{ab}$. $k_{ab}$ is the extrinsic curvature of $V =$const. surface and $\rho := |D^i V D_i V|^{-1/2}$. Then, from Eq. (10), one can easily see that

$$k_{ab}|_{V=0} = \mathcal{D}_a\rho|_{V=0} = 0 \tag{12}$$

hold on the event horizon. From the Einstein equation, we can also easily see that regularity implies $H_{0i_1 \cdots i_{p-1}} = 0$ on the event horizon; see Eq. (5).

Let us consider the conformal transformation defined by

$$\tilde{g}_{ij} = \Omega_\pm^2 g_{ij} \tag{13}$$

where

$$\Omega_{\pm} = \left(\frac{1 \pm V}{2}\right)^{\frac{2}{n-3}} =: \omega_{\pm}^{\frac{2}{n-3}}. \tag{14}$$

This conformal transformation is the same as the one employed in the proof for the vacuum case [2, 4]. Now we have two manifolds, $(\tilde{\Sigma}^{+}, \tilde{g}^{+})$ and $(\tilde{\Sigma}^{-}, \tilde{g}^{-})$. The Ricci scalar of $\tilde{\Sigma}^{\pm}$ is

$$\Omega_{\pm}^{2}\,{}^{(n-1)}\tilde{R}_{\pm} = \frac{1}{(p-1)!}\frac{e^{-\alpha\phi}}{V^2}\frac{\lambda_{\pm}}{\omega_{\pm}}H_0{}^{i_1\cdots i_{p-1}}H_{0i_1\cdots i_{p-1}} + \frac{1}{2}(D\phi)^2, \tag{15}$$

where

$$\lambda_{\pm} := \frac{1 \mp \frac{3n-4p-1}{n-3}V}{2}. \tag{16}$$

Since $0 \le V \le 1$, the $\lambda_{\pm}$ are positive-definite if

$$\frac{n+1}{2} \le p \le n-1. \tag{17}$$

Under this condition the positivity of ${}^{(n-1)}\tilde{R}_{\pm}$ follows. We will use this result later.

On $\tilde{\Sigma}^{+}$ the asymptotic behavior of the metric becomes

$$\tilde{g}^{+}_{ij} = \Big(1 + O(1/r^{n-2})\Big)\delta_{ij} \tag{18}$$

and therefore the ADM mass vanishes there. On $\tilde{\Sigma}^{-}$, the metric behaves like

$$\begin{aligned} \tilde{g}^{-}_{ij}dx^i dx^j &= \frac{(m/2)^{4/(n-3)}}{r^4}\delta_{ij}dx^i dx^j + O(1/r^5) \\ &= (m/2)^{4/(n-3)}(d\rho^2 + \rho^2 d\Omega^2_{n-2}) + O(\rho^5), \end{aligned} \tag{19}$$

where we set $\rho := 1/r$. From this, we see that infinity on $\Sigma$ corresponds to a point, which we denote as $q$.

Let us construct a new manifold $(\tilde{\Sigma}, \tilde{g}_{ij}) := (\tilde{\Sigma}^{+}, \tilde{g}^{+}_{ij}) \cup (\tilde{\Sigma}^{-}, \tilde{g}^{-}_{ij}) \cup \{q\}$ by gluing the two manifolds $(\tilde{\Sigma}^{+}, \tilde{g}^{+}_{ij})$ and $(\tilde{\Sigma}^{-}, \tilde{g}^{-}_{ij})$ along the surface $V = 0$ and adding the point $q$. The calculations above imply that $(\tilde{\Sigma}, \tilde{g}_{ij})$ has zero mass and non-negative Ricci scalar. Note also that near the point $q$ (which corresponds to $r \to \infty$) we have ${}^{(n-1)}\tilde{R}_{-} = O(r^{-(n-3)})$, so $\tilde{\Sigma}^{-}$ is regular at $q$. Thus $\tilde{\Sigma}$ is a Riemannian manifold with non-negative Ricci scalar and zero ADM mass. Then, by the positive energy theorem [8], $\tilde{\Sigma}$ is flat. So the metric $\tilde{g}_{ij}$ is flat and

$$H_{0i_1\cdots i_{p-1}} = 0 \quad \text{and} \quad \phi = \text{const} \tag{20}$$

hold. That is, asymptotically flat static black holes in $n$ dimensions cannot support an electric dipole $p$-form field strength with $p$ in the range (17), nor a nontrivial scalar field.

Once we have ruled out the possibility of nontrivial $p$-form and scalar fields, the problem is exactly the same as in vacuum and the results of [2] imply the uniqueness of the Schwarzschild-Tangherlini solution. For the sake of completeness, we briefly review this argument.

We have seen that $\tilde{\Sigma}^{+}$ must be flat space. In addition, we can check that the extrinsic curvature of the surface $V = 0$ on $\tilde{\Sigma}^{+}$ is proportional to its induced metric with a constant coefficient. According to Kobayashi and Nomizu [9], such a surface in flat space is spherically symmetric. Next, we define the function $v$ by

$$v = \frac{2}{1+V}. \tag{21}$$

It is easy to see that it is a harmonic function on flat space $\tilde{\Sigma}^{+}$, that is,

$$\partial^2 v = 0. \tag{22}$$

The boundary corresponding to the horizon is spherically symmetric. So the problem is reduced to the familiar one of an electrostatic potential with spherical boundary in flat space. We can easily see that the level surfaces of $v$ are spherically symmetric in the full region of $\tilde{\Sigma}^+$. So we have shown that $\Sigma$ is spherically symmetric and then the spacetime must be the Schwarzschild-Tangherlini spacetime. This completes our proof.

# 3 Outlook

We have proven a no-dipole-hair theorem for $p$-form fields with $p$ in the range (17). The proof can be straightforwardly extended to theories containing several electric form fields $H_{(p_i)}$ of different rank $p_i$, each with its own coupling $\alpha_i$ to the dilaton, as long as each of the $p_i$ satisfies (17).

As mentioned above, the upper bound on $p$ is a natural one given the requirement of asymptotic flatness. But the physical motivation for the lower bound, if any, is unclear. Could static black holes support dipoles when $p < (n+1)/2$? The answer when $p = 2$ is known: the uniqueness theorem of [3] affirms that a static black hole can have electric monopole charge, but not any higher multipole. However, here we are more interested in $p > 2$ where monopoles are not allowed. For instance, could there be static black holes in $n \geq 6$ with electric three-form, *i.e.*, string, dipole? The heuristic argument presented in the introduction would seem to run counter to this possibility, but maybe this argument misses a way to balance or cancel the tension of dipole sources that does not involve centrifugal forces. If this were the case it would be a striking new feature of static black holes afforded by higher dimensions. Alternatively, and more simply, maybe our no-dipole-hair theorem can be strengthened to rule out all $p$-form dipoles whenever $p \leq n - 1$. This issue seems worthy of further investigation.

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