Geometric Aspects of Twisted 3d Supersymmetric Gauge Theories

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To my grandparents

... und wir sind grundsätzlich geneigt zu behaupten, dass ... ohne eine beständige Fälschung der Welt durch die Zahl der Mensch nicht leben könnte. (F. Nietzsche)

“Ach”, sagte die Maus, “die Welt wird enger mit jedem Tag. Zuerst war sie so breit, daß ich Angst hatte, ich lief weiter und war glücklich, daß ich endlich rechts und links in der Ferne Mauern sah, aber diese langen Mauern eilen so schnell aufeinander zu, daß ich schon im letzten Zimmer bin, und dort im Winkel steht die Falle, in die ich laufe.” “Du mußt nur die Laufrichtung ändern”, sagte die Katze und fraß sie. (F. Kafka)

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Abstract

We study geometric aspects of 3d $\mathcal{N}=2$ and $\mathcal{N}=4$ supersymmetric gauge theories on the product of a line and a Riemann surface. Performing the topological twist on the Riemann surface, the theories preserve a supersymmetric quantum mechanics on the line. The quantum mechanics has an effective description where its target space is a moduli space of configurations that satisfy generalized vortex equations on the Riemann surface. We propose a construction of the space of supersymmetric ground states of selected $\mathcal{N}=2$ theories as a graded vector space in terms of a certain cohomology of the moduli spaces. This exhibits a rich dependence on deformation parameters compatible with the topological twist, including superpotentials, real mass parameters, and background vector bundles associated to flavour symmetries. By matching spaces of supersymmetric ground states, we perform new checks of 3d abelian mirror symmetry. We go on to the study of the twisted indices of a 3d $\mathcal{N}=4$ quiver gauge theories that have isolated vacua under generic mass and FI parameter deformations. These can be viewed as virtual Euler characteristics of the moduli spaces of generalized vortex equations, which in this case can be understood algebraically as quasi-maps to the Higgs branch. We demonstrate that this description agrees with the contour integral representation introduced in previous work. We then investigate 3d $\mathcal{N}=4$ mirror symmetry in this context, which implies an equality of enumerative invariants associated to mirror pairs of Higgs branches under the exchange of equivariant and degree counting parameters.

Introduction

In the last few decades, supersymmetry has been a powerful tool for the theoretical understanding of the behaviour of quantum field theories. While in non-supersymmetric quantum field theories phenomena in strongly-coupled regimes such as confinement can only be treated by means of heuristic arguments, supersymmetry can famously yield detailed and exact answers [1]. Often, the answers are based on remarkable mathematical theories or frameworks, which greatly benefit from the interaction with supersymmetric gauge theories. An example relevant to this thesis is the important results derived in the context of three- and four-manifolds invariants [2, 3, 4, 5, 6]. Other spectacular results close in spirit to this thesis have recently been obtained in the context of enumerative geometry [7, 8, 9], a field with a strong tradition of successes [10].

This thesis is devoted to the study of a specific class of supersymmetric gauge theories, namely 3d $\mathcal{N} = 2$ and $\mathcal{N} = 4$ theories topologically twisted on the product of a real line $\mathbb{R}$ and a Riemann surface Σ. In line with the general motivation for the study of supersymmetric quantum fields, our aim is to emphasize geometric aspects of the theories, and to show that a geometric point of view is beneficial for both physics and mathematics.

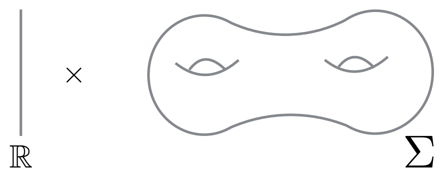

Figure 1: We consider 3d $\mathcal{N} = 2$ supersymmetric gauge theories on the product of a line and a Riemann surface Σ.

We achieve this by following a programme initiated in [11], which consists in interpreting the 3d supersymmetric gauge theories as an effective quantum mechanics on the line $\mathbb{R}$. The application of this point of view to our setup is new, and turns out to be particuarly fruitful. In the remainder of this introduction, we contextualise our research by highlighting important aspects of 3d supersymmetric gauge theories, and provide a summary of our contributions.

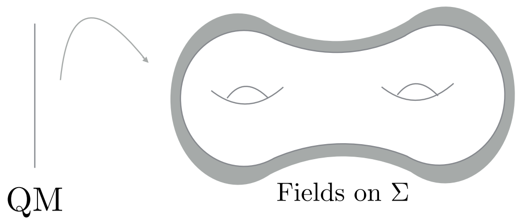

Figure 2: We interpret the system as a supersymmetric quantum mechanics on $\mathbb{R}$.

Background on (twisted) $\mathcal{N} = 2$ and $\mathcal{N} = 4$ 3d supersymmetric gauge theories

Although supersymmetric gauge theories in four dimensions may be thought to be more realistic than their three-dimensional counterparts, there are many interesting physical phenomena that are absent in the former but present in the latter. This is essentially because of the constraining power of holomorphy [12]. In fact, most of the stunning results of supersymmetric four-dimensional theories, such as the understanding of confinement, are based on the holomorphic dependence of the effective superpotential on complex parameters. Deformation parameters, in four dimensions, arise from expectation values of background chiral multiplets, and are therefore necessarily complex. This implies, for instance, that there cannot be any phase transition in these parameters. Some of the precluded phenomena can be accessed, without renouncing the power of supersymmetry, by studying theories in three dimensions.

In three-dimensional theories with at least $\mathcal{N} = 2$ supersymmetry there are natural real deformation parameters that correspond to expectation values of background vectormultiplets. In addition, unlike in four dimensions, three-dimensional theories allow for Chern-Simons terms. Notably, Chern-Simons terms were at the heart of the original physical construction of knot invariants [2], and have a long history of mathematical applications. They come together with discrete parameters, namely, the Chern-Simons levels, which are obviously also not subject to the constraints of holomorphy. The presence of all these parameters implies, in the first instance, the existence of remarkable phases in the moduli space of vacua of three-dimensional theories.

Moduli spaces of vacua In order to demonstrate this, and given the intricacy of the moduli spaces, it is best to consider an example. We follow [13]. Let us take a $\mathcal{N} = 2$ theory with a $U(1)$ gauge group and an unbroken $U(1)_R$ subgroup of the R-symmetry. We assume N matter fields ϕ_i of charge Q_i. To each $U(1)$ factor in the gauge group there is, in three dimension, a $U(1)$ topological symmetry, $U(1)_T$. This symmetry acts by rotating the dual photons γ, which are periodic scalars defined by $d\gamma = *dA$, where A is the gauge connection (notice that this is only possible in three dimensions). The unbroken maximal torus of the total symmetry group is $U(1)_R \times U(1)_T \times \prod_i^N U(1)_i/U(1)$, where the product over i is a flavour symmetry and we are dividing by gauge transformations. For each $U(1)$ factor in the flavour symmetry, we can turn on a real parameters m_i associated to background vectormultiplet for this factor. This is known as a 'real mass'. We also introduce a Fayet-Iliopoulos parameter ζ, which corresponds to a real mass for $U(1)_T$, and a Chern-Simons level k.

Notice that being bottom components of supermultiplets, the parameters ζ, m_i can only get one-loop renormalizations. The computation of the semi-classical potential shows that the effective real masses of the fields are

$$m_i(\sigma) = m_i + Q_i\sigma\,, \tag{1}$$

where σ is the real bottom component of the gauge multiplet. For $\sigma \neq 0$, ϕ_i therefore becomes massless at

$$\sigma_{Q_i} = -m_i/Q_i\,. \tag{2}$$

When integrating out matter fields, the parameters ζ and k get renormalisations

$$\begin{aligned} \zeta_{eff} &= \zeta + \frac{1}{2}\sum_{i=1}^{N} Q_i m_i \mathrm{sign}(m_i(\sigma)) \\ k_{eff} &= k + \frac{1}{2}\sum_{i=1}^{N} Q_i^2 \mathrm{sign}(m_i(\sigma))\,. \end{aligned} \tag{3}$$

The potential attains its minimum at

$$\sum_{i=1}^{N} 2\pi Q_i|\phi|^2 = \zeta_{eff} + k_{eff}\sigma\,,\ \ m_i(\sigma)Q_i = 0\,. \tag{4}$$

There are therefore three kinds of vacua:

- 'Higgs': the expectation value $\langle\phi_i\rangle$ of some ϕ_i is non-zero, $\sigma = \sigma_{Q_i}$ and so the gauge group is fully broken. There may be non-compact moduli spaces of Higgs vacua, but for generic mass parameters, they are isolated. They only exist if the RHS of (4) is positive;

- 'Coulomb': characterised by $\langle\phi_i\rangle = 0$ for all i, and $\zeta_{eff} = k_{eff} = 0$ so that there is a continuous moduli space;

- 'Topological': $\langle\phi_i\rangle = 0$ for all i, but $k_{eff} \neq 0$. They are isolated.

We will only consider non-abelian gauge theories in the presence of $\mathcal{N} = 4$ supersymmetry, so let us discuss this case briefly. When $\mathcal{N} = 4$, necessarily $k_{eff} = 0$ and there are only Higgs or Coulomb branch vacua. [14]. Typically, the moduli space of vacua is a union of 'branches' that can be products of the two types [15]. When only one type is present, say Higgs vacua, the branch is known as 'Higgs branch'. Analogously, there are also 'Coulomb' branches. Supersymmetry implies that both kinds of branches are hyperkähler manifolds. The Higgs branch has a neat geometric and exact description, which we are going to review at length in the bulk of this thesis. Briefly, it is a hyperkähler quotient which can be resolved by introducing FI parameters, and has isometries coming from the flavour symmetries that rotate hypermultiplets scalars. Real masses can be thought of as generators of these isometries. The Coulomb branch in flat space is classically spanned by monopole operators, which are operators constructed from the vectormultiplet scalars σ, and dual photons. Classically, it is

$$\mathcal{M}_C = (\mathbb{R}^3 \times S^1)^{\mathrm{rk}(G)}/W_G \tag{5}$$

where G is the gauge group and W_G is its associated Weyl group. However, the Coulomb branchs receives one-loop and nonperturbative quantum corrections, whose structure is in general unknown. This makes the rigorous definition of the Coulomb branch a substantial challenge. Notice that the classical expression manifests the isometries coming from rotating the dual photons. This means that the Coulomb branch has at least the topological symmetry group as an isometry group. In fact, the role of mass and FI parameters in interchanged in the Higgs and Coulomb branch.

Progress in understanding the structure of the Coulomb branch has been made for example in refs. [16, 15], and was part of the motivation for the program of [11] that we are following in this thesis. See also [17] for a proposed rigorous definition of the Coulomb branch in a large class of examples.

Mirror symmetry Another distinctive feature of three-dimensional $\mathcal{N} = 2$ is a duality known as 'mirror symmetry'. This was first discovered in ref. [18], but see also ref. [19] for the analogue in $\mathcal{N} = 2$ theories. The starting point is that three-dimensional theories have an intricate IR dynamics, which –unlike in the case of four dimensions– is present even in abelian cases [13]. Two theories can become identical in a non-trivial way at low energy scales. In this case, the theories need to share properties that are independent of the scale, and are therefore called 'mirror duals'.

Three-dimensional mirror symmetry can have very surprising consequences. For example, the moduli spaces of superymmetric vacua of mirror-dual theories need to be identical. It turns out that for $\mathcal{N} = 4$, the Coulomb and the Higgs branch of the moduli space of vacua are interchanged by the duality. Given their very different descriptions, this is very striking. It is the first and simplest instance of a physics-inspired mathematical duality known as 'symplectic duality', which can be enriched by studying other physical observables and deformations. It has profound mathematical implications, on which see for example [11, 20, 21].

The topological twist and the twisted index After this general introduction, we now turn to the setup which is the focus of the thesis, $\mathbb{R} \times \Sigma$. 3d $\mathcal{N} = 2$ supersymmetric gauge theories can be put on this geometry by means of a topological twist. We recall that the procedure of the twist involves redefining the Lorentz group $U(1)_L$ on the plane by mixing it with the $U(1)_R$ symmetry group. One then selects scalar nilpotent supercharges, so that the metric tensor on the plane is exact. Obervables which are closed under the action of the supercharge will be independent of the metric, and define a subsector of the theory that can be put on any Riemann surface. In the presence of $\mathcal{N} = 4$ supersymmetry there are two qualitatively distinct twists that utilise a $U(1)$ subgroup from each factor of the R-symmetry, which is a product of two groups $SU(2)_H \times SU(2)_C$ acting on hypermultiplet and vectormultiplet scalars respectively. These are commonly referred to simply as the 'H-twist' and 'C-twist'.

One of the basic observables of the twisted theory is the (graded) Witten index. We replace $\mathbb{R}$ by S^1 and compute

$$I = \mathrm{Tr}_{\mathcal{H}}(-1)^F e^{-2\pi\beta H} y^{J_f} \,, \tag{6}$$

where H is the Hamiltonian on the S^1, y represents 'fugacities' for the flavour symmetries of the theory (which for the moment we also take to include topological symmetries), and J_f are generators for the Cartan subgroup of the flavour symmetry group. Fugacities are, essentially, exponentiated real masses complexified by holonomies of background connections on S^1. The trace is over the Hilbert space $\mathcal{H}$ of states, and β is related to the radius on S^1. By standard arguments, in the case of $\mathcal{N} = 2$ the only states that contribute to the index are those satisfying

$$Q^2 = H - m_f \cdot J_f = 0, \tag{7}$$

where m_f is a real mass for flavour symmetry and Q is a supercharge. In the presence of $\mathcal{N} = 4$ supersymmetry, the constraints are stronger.

The twisted indices of three-dimensional supersymmetric gauge theories were first computed by Nekrasov and Shatashvili [22] in the context of the Bethe/Gauge correspondence. More recently, the twisted indices of 3d $\mathcal{N} = 2$ supersymmetric gauge theores have also been derived from UV localization on the classical Coulomb branch of the theory [23, 24, 25, 26]. The localization technique consist in writing some components of the action in terms of pieces that are exact with respect to the preserved supercharges. The bosonic parts of these actions are total squares. Because of Q-exactness, one can freely tune the parameters in front of the exact pieces, so that the path integral localizes to configurations that attain the minimum. We will not review the localization procedure in this thesis. Some foundational pieces of work can be found in refs. [27, 28]. For a more recent review, see e.g. [29].

The result of refs [25, 26] is a sum of contour integrals over the complexified maximal torus of the gauge group G. When G is a product of unitary groups, as is the case in this thesis, we have schematically

$$I(y, q) = \frac{1}{|W_G|} \sum_{\underline{\mathfrak{m}} \in \Lambda_G} (-q)^{\mathrm{Tr}(\underline{\mathfrak{m}})} \operatorname*{JK-Res}_{u=u*} du \; Z^{\text{class}} Z^{\text{1-loop}}(u, y) H^g(u, y) \,, \tag{8}$$

where the summation is over GNO quantized magnetic fluxes on Σ, or co-character lattice Λ_G. W_G is the Weyl group, q and y stand for the fugacities of Coulomb and Higgs branches respectively. The integrand is formed by classical and one-loop contributions from the matter content and the 'Hessian' H is related to the toplogy of the curve. The contribution from each flux sector is given by a Jeffrey-Kirwan residue that specifies the choice of contour. Notice that although the Jeffrey-Kirwan prescription finds its origin in the study of the cohomology of symplectic quotients [30], prima facie this interpretation is not manifest in supersymmetric localization computations. This was one of the motivation for our more geometric point of view.

Finally, we should mention a basic consequence of mirror symmetry –namely, the twisted indices of two mirror-dual theories need to be the same. In refs.[23, 25, 26] it was checked that this is indeed the case.

Vertex Operator Algebras and the Geoemtric Langlands Programme Many interesting aspects of three-dimensional theories, which we barely mention, also arise by viewing three-dimensional theories as boundary theories for four-dimensional theories. When $\mathcal{N} = 4$, the setup $\mathbb{R} \times \Sigma$ can be fitted in this way into the broader context of the Geometric Langlands Programme [31, 32, 33]. This has a wide range of mathematical applications, well beyond the scope of this thesis.

There is a single yet important point that we cannot avoid mentioning. This is a proposal put forward in refs. [34] and further studied in [35, 36]. Consider first the topologically twisted theory on $\mathbb{R}^+ \times \Sigma$, with a boundary at $\{0\} \times \Sigma$. In inserting local operators at the boundary, and imposing an asymptotic state at infinity, one gets a collection of correlation functions of local operators of a boundary Vertex Operator Algebra (VOA). This collection is consistent with the OPE, and corresponds by definition to the spaces of conformal blocks of the VOA. In this way, we get a map from the Hilbert spaces of a theory on $\mathbb{R} \times \Sigma$, (7), to the spaces of some conformal blocks of an associated VOA. This map is often an isomorphism. However, the explicit computation of the number of conformal blocks remains difficult.

Summary of our contributions

We now turn to our own contributions. Our starting point is that topologically twisted $\mathcal{N} = 2$ theories on $\mathbb{R} \times \Sigma$ preserve at least the algebra of a $\mathcal{N} = (0, 2)$ supersymmetric quantum mechanics on $\mathbb{R}$. In general, we would like to study observables of the three-dimensional theories in terms of an effective supersymmetric quantum mechanics, where neat geometric tools become available. As a first step, in this thesis we focus on the twisted indices (6) and the spaces of supersymmetric ground states (7).

A different localization locus and the generalized vortex equations Our first step towards a geometric description of the quantum mechanics amounts to introducing an additional Q-exact action, so that the path integral is dominated at given scales by configurations on Σ satisfying 'generalised vortex equations'. These take the schematic form

$$\begin{aligned} *F_A + e^2\left(\mu(\phi) - \tau - k_{eff}\sigma\right) = 0 \\ \bar{\partial}_A \phi = 0\,, \end{aligned} \tag{9}$$

where ϕ represents the chiral multiplet fields transforming in a given representation of the gauge group. μ is a moment map for the action of the gauge group on this representation. In this thesis, we always study theories for which $k_{eff} = 0$. The solutions to these equations form a moduli space $\mathfrak{M}$, which is a disjoint union of topologically distinct sectors labelled by the degree of the gauge bundle

$$\mathfrak{M} = \bigcup_{\mathfrak{m}\in\Lambda_T} \mathfrak{M}_{\mathfrak{m}}\ , \tag{10}$$

where Λ_T is the character lattice of the topological symmetry. The description of the moduli space depends on the combination $s = \tau e^2 \mathrm{Vol}(\Sigma)$ of a parameter τ valued in the Lie algebra of the topological symmetry, e^2, and the volume of the curve. τ plays the role of the FI parameter, and it can be seen as a FI parameter in the effective theory. Although this parameter appears in an exact deformation of the action, there can be an intricate wall-crossing behaviour in the parameter space. In this thesis, we formally take the limit $s \to \infty$ in a prescribed direction and avoid any wall-crossing behaviour. This limit, we argue, is related to the IR limit and is therefore relevant for mirror symmetry. The wall-crossing phenomenon and its relation with existing mathematical literature is being investigated, and will soon appear [37].

Twisted Hilbert spaces of $\mathcal{N} = 2$ theories We then move to the study of twisted 3d $\mathcal{N} = 2$ supersymmetric gauge theories. First of all, we describe in details how the system can be viewed as an effective quantum mechanics. We then study the Hilbert spaces of supersymmetric ground states $\mathcal{H}$ of a selected class of theories.

We already remarked that the ground states are charged under fermion number and flavour symmetries. As a consequence, the twisted Hilbert space transforms as a virtual representation of the flavour symmetry. Furthermore, it has a rich dependence on real and complex supersymmetric deformation parameters obtained by coupling to background vectormultiplets for flavour symmetries, in a way compatible with the topological twist on Σ. In particular, for a flavour symmetry G_f acting on chiral multiplets, we may turn on

1. A real mass parameters m_f valued in the Cartan subalgebra of G_F,

2. A background G_f-vector bundle E_f on Σ.

Analogous expression exist for topological symmetries.

We first focus on abelian $\mathcal{N} = 2$ supersymmetric gauge theories with $G = U(1)$. We view the theory as an effective supersymmetric quantum mechanics for each degree $\mathfrak{m}$, whose target space is a moduli space $\mathfrak{M}_{\mathfrak{m}}$ of solutions to the generalized vortex equations on Σ in the presence of the background vector bundle E_f. The equations are obtained as in (9), with the appropriate matter content and with the deformations induced by the background bundles. We tune the deformation parameters so that the theory presents only Higgs vacua. The generalized vortex moduli spaces $\mathfrak{M}_{\mathfrak{m}}$ we consider are Kähler becaue of $\mathcal{N} = 2$ supersymmetry, and may or may not be compact, depending on the matter content and flux $\mathfrak{m}$. Provided the moduli space is smooth, we propose that the supersymmetric ground states can be understood in terms of an L^2-cohomology

$$\mathcal{H}_{\mathfrak{m}} = H^{0,\bullet}_{\bar{\partial}_{m_f}+\delta}\left(\mathfrak{M}_{\mathfrak{m}}, \mathcal{F}_{\mathfrak{m}}\right), \tag{11}$$

where $\mathcal{F}_{\mathfrak{m}}$ denotes a $\mathbb{Z}_2$-graded vector bundle that receives contributions from supersymmetric Chern-Simons terms, a background line bundle for the topological flavour symmetry, and the quantization of fermion zero modes. The differential is a sum of a conjugated Dolbeault operator,

$$\bar{\partial}_{m_f} = e^{-h_f} \cdot \bar{\partial} \cdot e^{h_f}, \tag{12}$$

where the real superpotential $h_f = m_f \cdot \mu_f$ is constructed from the moment map μ_f for the action of the flavour symmetry G_f on $\mathcal{M}_{\mathfrak{m}}$, and an extra contribution δ from a 3d superpotential. This L^2-cohomology can depend intricately on the choice of background vector bundle E_f and real mass parameters m_f.

An important consistency check for our proposal is to reproduce the supersymmetric twisted index on $S^1 \times \Sigma$. We emphasize, however, that as expected from index computations the twisted Hilbert space exhibits information and a structure that go far beyond the supersymmetric twisted index:

- There can be dramatic cancellations in computing the supersymmetric index via (8), particularly on Riemann surfaces of genus $g > 0$.
- The supersymmetric twisted Hilbert space is sensitive to 3d superpotential deformations via the differential δ, which removes pairs of supersymmetric ground states whose contribution to the supersymmetric twisted index cancel out.
- The supersymmetric twisted Hilbert space may jump across hyperplanes in the space of real mass parameters m_f where there are non-compact massless degrees of freedom. On the other hand, the supersymmetric index is a meromorphic function with poles on these hyperplanes. The same remark applies to real FI parameters for topological flavour symmetries.

- The supersymmetric twisted Hilbert space depends on a choice of holomorphic vector bundle E_f on Σ for the flavour symmetry G_f, while the supersymmetric twisted index depends only on its Chern class. The same remark applies to background line bundles associated to topological flavour symmetries.

The supersymmetric twisted Hilbert space therefore has the potential to provide a more refined check of supersymmetric dualities such as 3d mirror symmetry.

In order to illustrate some of these points, we provide a brief appetizer. Let us consider a supersymmetric $U(1)$ Chern-Simons theory at level $+\frac{1}{2}$ with a chiral multiplet of charge $+1$. We show that the supersymmetric twisted Hilbert space in the $s \to +\infty$ limit is given by

$$\mathcal{H} = \bigoplus_{\mathfrak{m}=1-g}^{\infty} q^{\mathfrak{m}} \bigoplus_{j=0}^{\mathfrak{m}+g-1} \wedge^j(\mathbb{C}^g)\,, \tag{13}$$

where $g > 0$ is the genus of Σ and the parameter q keeps track of the grading by the topological flavour symmetry. Notice that there are non-vanishing contributions from an infinite number of fluxes, $\mathfrak{m} \geq 1-g$. On the other hand, the supersymmetric twisted index is a finite Laurent polynomial

$$I = q^{1-g}(1-q)^{g-1}\,, \tag{14}$$

with the contributions from fluxes $\mathfrak{m} \geq 0$ cancelling out in the trace. Nevertheless, we demonstrate that equation (13) agrees with the supersymmetric twisted Hilbert space of a single chiral multiplet with a positive real mass parameter. Furthermore, we extend this agreement to include a holomorphic line bundle for the flavour symmetry. This constitutes a new check of the simplest 3d mirror symmetry.

Twisted indices and Hilbert spaces of $\mathcal{N} = 4$ theories. As the methods we used for the Hilbert spaces of $\mathcal{N} = 2$ theories can fail in more complicated examples, we turn to the study of twisted indices of 3d $\mathcal{N} = 4$ supersymmetric gauge theories. The theories are more rigid and are open to neater mathematical interpretations.

The twisted indices can be regarded as the flavoured Witten index of the effective supersymmetric quantum mechanics on $\mathbb{R}$. As already mentioned in this introduction, $\mathcal{N} = 4$ theories possess two qualitatively different topological twists –the H-twist and C-twist. The topological twist results in an additional flavour symmetry $U(1)_t$. We can therefore upgrade (6) to

$$I^{H,C}(y,q,t) = \mathrm{Tr}_{\mathcal{H}_{H,C}}(-1)^F y^{J_H} q^{J_C} t^{J_t}\,, \tag{15}$$

where $\mathcal{H}_{H,C}$ is the Hilbert space of supersymmetric ground states on $S^1 \times \Sigma$ and

- J_H is the generator of the Cartan subalgebra of the Higgs branch flavour symmetry G_H, which in the $\mathcal{N} = 4$ context acts on the hypermultiplet scalars.

- J_C is the generator of the Cartan subalgebra of the Coulomb branch flavour symmetry G_C, which is realized as a topological symmetry in the UV.
- J_t is the generator of the combination $U(1)_t = U(1)_H - U(1)_C$ of R-symmetries, which commutes with the two supercharges preserved in both the H-twist and the C-twist.

We recall that in the context of superymmetric quantum mechanics, the Witten index [38]

$$I = \mathrm{Tr}_{\mathcal{H}}(-1)^F \ , \tag{16}$$

comes with a strong geometric interpretation. For example, in a 1d $\mathcal{N} = (0,2)$ sigma model to a compact target M endowed with a holomorphic vector bundle E, the Witten index can be identified with the holomorphic Euler characteristic

$$\chi\left(\mathcal{M}, K_M^{1/2} \otimes E\right) = \int_M \hat{A}(TM)\ \mathrm{ch}(E)\,. \tag{17}$$

In the presence of flavour symmetry, this can be promoted to a flavoured Witten index that computes the equivariant holomorphic Euler characteristic. We can therefore try to provide a geometric interpretation of the contour integral in equation (17) in terms of holomorphic Euler characteristics.

We focus on 3d $\mathcal{N} = 4$ superconformal quiver theories that have isolated massive vacua in the presence of generic mass and FI parameters. With our alternative localizing action, the path integral localizes on the $\mathcal{N} = 4$ version of (9). This takes the schematic form

$$\begin{gathered} *F_A + e^2\left(\mu_{\mathbb{R}} - 2[\varphi^\dagger, \varphi] - \tau\right) = 0 \\ \bar{\partial}_A X = 0 \quad \bar{\partial}_A Y = 0 \quad \bar{\partial}_A \varphi = 0 \\ \varphi \cdot X = 0 \quad \varphi \cdot Y = 0 \quad X \cdot Y = 0\ , \end{gathered} \tag{18}$$

where (X, Y) are the hypermultiplet scalar fields transforming in a quaternionic representation of G and φ is the vector multiplet complex scalar field in the adjoint representation. The solutions to these equations form a moduli space $\mathfrak{M}$, which is again a disjoint union of topologically distinct sectors labelled by the degree of the gauge bundle

$$\mathfrak{M} = \bigcup_{\mathfrak{m} \in \Lambda_C^\vee} \mathfrak{M}_{\mathfrak{m}}\ , \tag{19}$$

where $\Lambda_C^\vee$ is the character lattice of the Coulomb branch flavour symmetry G_C. In the $s \to \infty$ limit, we show that $\mathfrak{M}_{\mathfrak{m}}$ has an algebraic description as the moduli space of quasi-maps to the Higgs branch, $\Sigma \to \mathcal{M}_H$, of degree $\mathfrak{m}$ [39]. [1] More precisely, in the H-twist we recover the twisted quasi-maps to holomorphic symplectic quotients introduced in [45], while in the C-twist we find a generalization to arbitrary genus of a construction of [46].

[1] The moduli space of quasi-maps and their enumerative geometry have been discussed in various contexts, e.g., [40, 41, 8, 42, 43, 44].

In order to provide a concrete interpretation of the contour integral representation of the twisted index (8) in terms of the enumerative geometry of the moduli space $\mathfrak{M}$, we carefully study the massless fluctuations of the bosonic and fermionic fields around a point $p \in \mathfrak{M}$. From a mathematical viewpoint, these massless fluctuations can be identified with the virtual tangent bundle to the moduli space $\mathfrak{M}$ and give rise to perfect obstruction theories, which coincides with those considered in [45, 46]. We take the opportunity to remark that related constructions have also been extensively studied in [47, 46] in the context of the K-theoretic Donaldson-Thomas invariants of Calabi-Yau three-folds.

From this discussion, we argue that the localized path integral for the twisted index reproduces a generating function of virtual Euler characteristics of $\mathfrak{M}_{\mathfrak{m}}$ defined by

$$I^{H,C} = \sum_{\mathfrak{m}\in\Lambda_C^\vee} (-q)^{\mathfrak{m}} \int_{\mathfrak{M}_{\mathfrak{m}}} \hat{A}(T^{\mathrm{vir}}) \ . \tag{20}$$

In general, the moduli spaces $\mathfrak{M}_{\mathfrak{m}}$ are non-compact and these integrals are not well-defined. However, by turning on a real mass parameter with associated fugacity t, we can localize further to the compact fixed locus of the $U(1)_t$ symmetry. This fixed locus $\mathfrak{L} \subset \mathfrak{M}$ coincides with the moduli space of quasi-maps to a holomorphic Lagrangian $\mathcal{L}_H \subset \mathcal{M}_H$ known as the compact core. The virtual tangent bundle then decomposes on the fixed locus as

$$T^{\mathrm{vir}}|_{\mathfrak{L}_{\mathfrak{m}}} = T\mathfrak{L}_{\mathfrak{m}} + N_{\mathfrak{m}} \ , \tag{21}$$

where $T\mathfrak{L}_{\mathfrak{m}}$ is the virtual tangent bundle to the fixed locus and N is the virtual normal bundle. The path integral then reproduces the virtual Euler characteristic defined by localization with respect to the $U(1)_t$ action,

$$I^{H,C} = \sum_{\mathfrak{m}\in\Lambda_C^\vee} (-q)^{\mathfrak{m}} \int_{\mathfrak{L}_{\mathfrak{m}}} \frac{\hat{A}\left(T\mathfrak{L}_{\mathfrak{m}}\right)}{\mathrm{ch}\left(\widehat{\wedge}^{\bullet} N_{\mathfrak{m}}^\vee\right)} \ . \tag{22}$$

where the notation $\widehat{\wedge}^{\bullet}$ indicates the exterior algebra normalized by the square root of the determinant bundle. This gives a concrete geometric interpretation to the twisted index.

In order to perform explicit calculations, we can localize further to the fixed locus of the flavour symmetry G_H by turning on mass parameters with associated fugacity a, which play the role of equivariant parameters. Under our assumptions, we show that the fixed locus is a disjoint union of smooth compact spaces $\mathfrak{M}_{\underline{\mathfrak{m}},I}$, where I labels the fixed points on $\mathcal{M}_H$ and $\underline{\mathfrak{m}} \in \Lambda_G$ is a GNO quantized flux with $\mathrm{tr}(\underline{\mathfrak{m}}) = \mathfrak{m}$. Each component is given by a product of symmetric products of the curve Σ,

$$\mathfrak{M}_{\underline{\mathfrak{m}},I} = \prod_{a=1}^{\mathrm{rk}(G)} \mathrm{Sym}^{\mathfrak{n}_{I_a}} \Sigma \ , \tag{23}$$

where $\mathfrak{n}_{I_a}$'s are non-negative integers which depend on the twist and a component of the magnetic flux $\underline{\mathfrak{m}}$. On the fixed locus, the virtual tangent space decomposes

$$T^{\mathrm{vir}}\big|_{\mathcal{M}_{\underline{\mathfrak{m}},I}} = T\mathfrak{M}_{\underline{\mathfrak{m}},I} + N_{\underline{\mathfrak{m}},I} \ , \tag{24}$$

where $N_{\mathfrak{m},I}$ are the virtual normal bundles and non-zero weights under the $U(1)_t \times G_H$ action. The path integral then reproduces the equivariant virtual Euler characteristic via virtual localization,

$$I^{H,C} = \sum_{\underline{\mathfrak{m}} \in \Lambda_G} (-q)^{\mathfrak{m}} \sum_I \int_{\mathfrak{M}_{\underline{\mathfrak{m}},I}} \frac{\hat{A}(T\mathfrak{M}_{\underline{\mathfrak{m}},I})}{\mathrm{ch}(\widehat{\wedge}^{\bullet} N^{\vee}_{\underline{\mathfrak{m}},I})} \ . \tag{25}$$

The intersection theory on the symmetric product of a curve is well-known [48, 49, 50] allowing us to convert the expression (25) into a sum of the residue integrals. We show explicitly that this reproduces the contour integral representation of the twisted index (8). In particular the fixed loci of $U(1)_t \times G_H$ are in one-to-one correspondence with the poles selected by the Jeffrey-Kirwan residue integral. [2]

Sending $t \to 1$, the twisted index preserves four supercharges that generate a 1d $\mathcal{N} = (2,2)$ and $\mathcal{N} = (0,4)$ supersymmetric quantum mechanics in the H-twist and C-twist respectively. We can therefore add further exact terms to the localising action to further constrain the moduli space. In particular, the C-twisted index can be localised to the space of constant maps to the Higgs branch $\mathcal{M}_H$. In this limit, the virtual Euler characteristic is independent of q and reduces to the equivariant Rozansky-Witten invariants [4] of $\mathcal{M}_H$, associated with the three-manifold $S^1 \times \Sigma$,

$$I^C\big|_{t\to 1} = \int_{\mathcal{M}_H} \hat{A}(T\mathcal{M}_H)\ \mathrm{ch}\left(\widehat{\wedge}^{\bullet} T^*\mathcal{M}_H\right)^g \ . \tag{26}$$

On the other hand, the H-twisted index reduces to a generating function of the Euler classes of the G_H-fixed loci,

$$I^H\big|_{t\to 1} = \sum_{\underline{\mathfrak{m}} \in \Lambda_G} (-q)^{\mathfrak{m}} \sum_I (-1)^{\dim_{\mathbb{C}}(\mathfrak{M}_{\underline{\mathfrak{m}},I})} \int_{\mathfrak{M}_{\underline{\mathfrak{m}},I}} e(\mathfrak{M}_{\underline{\mathfrak{m}},I}) \ , \tag{27}$$

which is independent of the fugacity a.

As mentioned, an important feature of the class of 3d $\mathcal{N} = 4$ supersymmetric gauge theories we consider is the existence of mirror symmetry, which exchanges the H-twist and the C-twist of a dual pair of theories $\mathcal{T}$ and $\mathcal{T}^{\vee}$. This implies the following relation between the twisted indices of these theories,

$$I_H[\mathcal{T}](q,y,t) = I_C[\mathcal{T}^{\vee}](y,q,t^{-1}) \ , \tag{28}$$

This provides extremely non-trivial relationship between enumerative invariants of quasi-maps to pairs of Higgs branches $\mathcal{M}_H$ and $\mathcal{M}_H^{\vee}$ under the exchange of the degree counting parameters. It is a remarkable example of symplectic duality for quasi-map spaces.

Outline and Statement of Originality

The thesis is organized as follows. In chapter 1, we review and study supersymmetric quantum mechanical systems, emphasizing how the space of supersymmetric ground states depends on various

[2] The geometric interpretation of the twisted index for an $\mathcal{N} = 2$ supersymmetric Chern-Simons theory with an adjoint chiral multiplet has been studied in the references [51, 52].

types of deformation parameter. In chapter 2, we review and modify supersymmetric localization results. In chapter 3, we study the twisted Hilbert spaces of $\mathcal{N}=2$ supersymmetric gauge theories. In chapter 4, we turn to the investigation of the twisted indices of $\mathcal{N}=4$ supersymmetric gauge theories.

Chapters 1 and 3 are based on materials from the co-authored paper [53]. Chapters 2 and 4 are based on the co-autored pre-print [54]. The graduate work of the candidate also includes the unrelated paper [55]. Other more related papers are also due to appear soon [37, 56].

Contents

Chapter 1

Supersymmetric Quantum Mechanics

In this chapter, we treat the material about supersymmetric quantum mechanical systems that will be needed when studying $3d$ supersymmetric gauge theories on $\mathbb{R} \times \Sigma$. In section 1.1, we study $\mathcal{N} = (0, 2)$ supersymmetric quantum mechanics. These will arise in the presence of $\mathcal{N} = 2$ supersymmetry in 3d. In section 1.3, we briefly summarise aspects of $\mathcal{N} = 4$ quantum mechanics, which will arise in the presence of $\mathcal{N} = 4$ supersymmetry.

1.1 $\mathcal{N} = (0, 2)$ Supersymmetric Quantum Mechanics

We review and study supersymmetric quantum mechanics with supermultiplets that arise from the dimensional reduction of $\mathcal{N} = (0, 2)$ supermultiplets in two dimensions, emphasizing those aspects that will be important in applications to 3d $\mathcal{N} = 2$ theories on a $\mathbb{R} \times \Sigma$. For further background and examples of this class of supersymmetric quantum mechanics we refer the reader to [57, 58, 59]

1.1.1 Setup

An $\mathcal{N} = (0, 2)$ supersymmetric quantum mechanics has odd generators Q and $Q^\dagger$ that are adjoint with respect to a Hermitian inner product on the Hilbert space. We suppose the supersymmetric quantum mechanics has flavour symmetry G_f with conserved charges $J_f \in \mathfrak{t}_f^*$ and introduce an associated real mass parameter $m_f \in \mathfrak{t}_f$. Here $\mathfrak{t}_f$ is the Cartan subalgebra of G_f. The supersymmetry algebra is then

$$\begin{aligned} \{Q, Q\} &= 0 \\ \{Q, Q^\dagger\} &= H - m_f \cdot J_f \\ \{Q^\dagger, Q^\dagger\} &= 0\,, \end{aligned} \tag{1.1}$$

where H is the Hamiltonian.

We define supersymmetric ground states as those annihilated by $H - m_f \cdot J_f$. A standard argument shows that the spectrum of $H - m_f \cdot J_f$ is non-negative and that supersymmetric ground states are equivalently annihilated by both of the supercharges. We assume that the spectrum is gapped, in which case the supersymmetric ground states have another equivalent description as the cohomology of either supercharge Q or $Q^\dagger$.

The requirement that the spectrum is gapped may place constraints on the mass parameters m_f. We will denote the subspace of mass parameters where the spectrum is gapped by $\mathfrak{c}_f \subset \mathfrak{t}_f$. In all of the examples we are going to consider, $\mathfrak{c}_f$ will consist of a union of chambers $\mathfrak{c}_f = \bigcup_\alpha \mathfrak{c}_\alpha \subset \mathfrak{t}_f$ cut out by hyperplanes where there are non-compact massless degrees of freedom.

In this thesis, we will compute the space of supersymmetric ground states $\mathcal{H}$ as the cohomology of the supercharge $Q^\dagger$. Since $(-1)^F$ and J_f commute with the supercharges, the space of supersymmetric ground states is graded by Fermion number and flavour symmetry. Alternatively, we can say that it is a $\mathbb{Z}_2$-graded or virtual representation of the flavour symmetry G_f.

It will be important to understand how the supersymmetric ground states change as the real mass parameters m_f are varied. In our examples, the supercharges obey

$$\begin{aligned} \partial_{m_f} Q &= +[\mu_f, Q] \\ \partial_{m_f} Q^\dagger &= -[\mu_f, Q^\dagger]\,, \end{aligned} \tag{1.2}$$

where $\mu_f \in \mathfrak{t}_f^*$ is a Hermitian operator. This is an A-type deformation in the notation of [33]. In particular, the operator $\partial_{m_f} + \mu_f$ commutes with $Q^\dagger$ and descends to a complex flat Berry connection on the sheaf of supersymmetric ground states over $\mathfrak{c}_f \subset \mathfrak{t}_f$. Put simply, while the wavefunctions of the supersymmetric ground states will depend explicitly on the real mass parameters, $\mathcal{H}$ remains constant as a graded vector space, provided the spectrum remains gapped. Therefore, we associate a space of supersymmetric vacua $\mathcal{H}_\alpha$ to each chamber $\mathfrak{c}_\alpha$.

We will also encounter examples of B-type deformations of the supersymmetric quantum mechanics [33], where the supercharges depend holomorphically or anti-holomorphically on a set of complex parameters u,

$$\begin{aligned} \partial_u Q &= 0 \\ \partial_{\bar{u}} Q^\dagger &= 0\,. \end{aligned} \tag{1.3}$$

In particular, the derivative $\partial_{\bar{u}}$ commutes with $Q^\dagger$ and, provided the system is gapped, descends to a holomorphic Berry connection on the sheaf of supersymmetric ground states over the complex space parametrized by u.

The flavoured supersymmetric index is defined as a graded trace over the full Hilbert space of the supersymmetric quantum mechanics,

$$I = \mathrm{Tr}(-1)^F e^{-2\pi\beta H} e^{-2\pi i \beta a_f \cdot J_f}\,. \tag{1.4}$$

In the Euclidean path integral construction of the supersymmetric index, β is the radius of the circle and $e^{-2\pi i\beta a_f \cdot J_f}$ is a background Wilson line for the flavour symmetry. A standard argument shows that only supersymmetric ground states contribute to the supersymmetric index, and therefore, in each chamber $\mathfrak{c}_\alpha$ we obtain an expression

$$\begin{aligned} I_\alpha &= \mathrm{Tr}_{\mathcal{H}_\alpha}(-1)^F e^{-2\pi\beta(m_f+ia_f)\cdot J_f} \\ &= \mathrm{Tr}_{\mathcal{H}_\alpha}(-1)^F x^{J_f} \,, \end{aligned} \tag{1.5}$$

where

$$x = e^{-2\pi\beta(m_f+ia_f)} \tag{1.6}$$

is valued in the complexified maximal torus of the flavour symmetry G_f. The supersymmetric index can therefore be expressed as a graded trace over $\mathcal{H}_\alpha$.

The supersymmetric index I_α computed in equation (1.5) will in general yield a different Laurent polynomial in x in each chamber $\mathfrak{c}_\alpha$. However, they correspond to Laurent expansions of the same meromorphic function $I(x)$ in the different chambers $\mathfrak{c}_\alpha$ under the identification (1.6). This meromorphic function then has poles on the hyperplanes separating these chambers. In the case $\mathfrak{c}_f = \mathfrak{t}_f$, the supersymmetric index is a finite Laurent polynomial in x.

Finally, the supersymmetric index is insensitive to B-type deformations.

1.1.2 Geometric Model

We now consider a general class of supersymmetric quantum mechanics of the above type that arise from supersymmetric sigma models. The construction of these supersymmetric quantum mechanics has much in common with the construction of 2d $\mathcal{N} = (0, 2)$ supersymmetric sigma models [60].

We consider a supersymmetric sigma model specified by the following data:

- A complex manifold M with Hermitian metric.
- A $\mathbb{Z}$-graded Hermitian vector bundle F.
- A holomorphic differential $\delta : F \to F$ of degree $+1$ obeying $\delta^2 = 0$.

The full Hilbert space of the supersymmetric quantum mechanics consists of smooth square-integrable sections of

$$\Omega^{0,\bullet}(M) \otimes F \tag{1.7}$$

with respect to the Hermitian inner product

$$\langle \alpha, \beta \rangle = \int_M \bar{\alpha} \wedge *\beta \,. \tag{1.8}$$

Here, $*$ denotes the Hodge star operator on M and contraction along fiber directions using the Hermitian metric on F is understood.

The supersymmetric quantum mechanics has an R-symmetry transforming the supercharges Q, $Q^\dagger$ with charge -1, $+1$ respectively. Referring to the above geometric data, this R-symmetry can be identified with the sum of the form degree on the target space M and the $\mathbb{Z}$-grading on the Hermitian vector bundle F, modulo an additive constant.

In this thesis, we will only keep track of the Fermion number $(-1)^F$. In particular, we regard F as a $\mathbb{Z}_2$ graded vector bundle with decomposition $F = F_e \oplus F_o$ into even and odd components. The Fermion number $(-1)^F$ in the supersymmetric quantum mechanics is then given the sum of the form degree and the $\mathbb{Z}_2$-grading on F:

- $(-1)^q$ for a section of $\Omega^{0,q}(M) \otimes F_e$ and
- $(-1)^{q+1}$ for a section of $\Omega^{0,q}(M) \otimes F_o$.

Let G_f denote the group of isometries of M that lift to an equivariant action on F preserving its Hermitian metric and commuting with the holomorphic differential δ. This is the flavour symmetry of the supersymmetric quantum mechanics. At this point, we assume that M is Kähler and there exists a corresponding real moment map $\mu_f \in \mathfrak{t}_f^*$. We may then introduce an A-type deformation of the supersymmetric quantum mechanics by real mass parameters $m_f \in \mathfrak{t}_f$, which can be understood as a real superpotential

$$h_f = m_f \cdot \mu_f \,. \tag{1.9}$$

This superpotential is the moment map for the $U(1)_{m_f} \subset G_f$ isometry generated by the mass parameters m_f [61].

Let $\mathfrak{c}_f \subset \mathfrak{t}_f$ denote the mass parameters where the fixed locus of the $U(1)_{m_f}$ isometry of M is compact and the spectrum of the supersymmetric quantum mechanics is gapped. If M is non-compact, this is a disjoint union of chambers $\mathfrak{c}_f = \bigcup_\alpha \mathfrak{c}_\alpha$ cut out by hyperplanes. If M is compact, $\mathfrak{c}_f = \mathfrak{t}_f$.

The supercharges are identified with

$$\begin{aligned} Q &= \bar{\partial}^\dagger_{m_f} + \delta^\dagger \\ Q^\dagger &= \bar{\partial}_{m_f} + \delta \,. \end{aligned} \tag{1.10}$$

where

$$\begin{aligned} \bar{\partial}^\dagger_{m_f} &:= e^{h_f} \bar{\partial}^\dagger e^{-h_f} \\ \bar{\partial}_{m_f} &:= e^{-h_f} \bar{\partial} e^{h_f} \end{aligned} \tag{1.11}$$

and $\bar{\partial}, \bar{\partial}^\dagger$ denote respectively the twisted Dolbeault operator acting on sections of (1.7) and its adjoint with respect to the Hermitian inner product (1.8). Finally, $\delta^\dagger$ is the adjoint of holomorphic differential δ with respect to the Hermitian metric on F. Note that the supercharge $Q^\dagger$ depends holomorphically on deformations of the holomorphic vector bundle F and the differential δ : they are B-type deformations of the supersymmetric quantum mechanics.

These supercharges obey the supersymmetry algebra (1.1) with H schematically given by

$$H = \frac{1}{2}\Delta + (\partial h_f)(\bar{\partial} h_f) + m_f \cdot \Psi + \{\delta^\dagger, \delta\} \,, \tag{1.12}$$

where Δ is the Laplace operator, and $m_f \cdot \Psi$ is a term linear in m_f containing fermions and no derivatives. $m_f \cdot J_f$ can be seen as the generator of the $U(1)_{m_f}$ flavour symmetry.

Provided the mass parameters lie in $\mathfrak{c}_f \subset \mathfrak{t}_f$ and the spectrum of the supersymmetric quantum mechanics is gapped, the space of supersymmetric ground states can be identified with L^2-cohomology of the supercharge $Q^\dagger$, which we write schematically as

$$H^{0,\bullet}_{\bar{\partial}_{m_f}+\delta}(M, F) \,. \tag{1.13}$$

Due to the exponential dependence of the supercharge on the mass parameter m_f and the condition of square-normalizability, the computation of this cohomology for a non-compact target space M will generally yield a different space of supersymmetric ground states $\mathcal{H}_\alpha$ in each chamber $\mathfrak{c}_\alpha$. In other words, the space of supersymmetric ground states may jump across hyperplanes in $\mathfrak{t}_f$ where there are massless non-compact degrees of freedom[1] .

However, if M is compact, the spectrum of the supersymmetric quantum mechanics is gapped for any m_f and the space of supersymmetric vacua is constant on $\mathfrak{t}_f$. In this case, we can set $m_f = 0$ and identify the space of supersymmetric ground states with the regular hypercohomology,

$$\mathcal{H} = H^{0,\bullet}_{\bar{\partial}+\delta}(M, F) \,. \tag{1.14}$$

Let us finally consider the supersymmetric index in this class of supersymmetric quantum mechanics. The supersymmetric index I_α in each chamber computes the equivariant character of $\mathcal{H}_\alpha$ as a virtual representation of the flavour symmetry G_f. This index is independent of the differential and can be identified with an equivariant Euler character for L^2-cohomology classes of the conjugated Dolbeault operator $\bar{\partial}_{m_f}$ in equation (1.11). If M is compact, the supersymmetric index I coincides with the regular equivariant Euler character $\chi(M, F)$.

1.1.3 Examples

Chiral Multiplets

Our first example is a single chiral multiplet (ϕ, ψ) with a real mass parameter m_f for the $U(1)_f$ flavour symmetry. This model is a supersymmetric complex harmonic oscillator. In canonical quantization, the complex Fermion obeys $\{\psi, \bar{\psi}\} = 1$ and the supercharges take the form

$$\begin{aligned} Q &= \psi \left(-\frac{\partial}{\partial \phi} + m_f \bar{\phi} \right) \\ Q^\dagger &= \bar{\psi} \left(+\frac{\partial}{\partial \bar{\phi}} + m_f \phi \right) \,. \end{aligned} \tag{1.15}$$

[1]The cohomology groups 1.14 may in general be hard to compute directly. However, sometimes computations can be approached with the help of holomorphic instanton techniques. The starting observation is that in a given chamber the spaces of supersymmetric ground states, viewed as vector spaces, do not depend on the value of m_f. We can therefore carefully take a limit $m_f \to \infty$. In this limit, states are localised around fixed points, and they can be approximately described as if they were defined on flat space. This perturbative description needs to be supplemented by instanton corrections. These techniques were developed in [59]. We will go back to these in the forthcoming [56].

The supercharges obey (1.1) with

$$
\begin{aligned}
H &= -\frac{\partial^2}{\partial\phi\partial\bar{\phi}} + m_f^2|\phi|^2 - \frac{1}{2}m_f[\psi,\bar{\psi}] + m_f\kappa \\
J_f &= \phi\frac{\partial}{\partial\phi} - \bar{\phi}\frac{\partial}{\partial\bar{\phi}} + \frac{1}{2}[\psi,\bar{\psi}] + \kappa\,.
\end{aligned}
\tag{1.16}
$$

Note that while the supercharges and the combination $H - m_f J_f$ are unambiguous, H and J_f individually depend on a normal ordering constant κ, which can be understood as a supersymmetric Chern-Simons term for the $U(1)_f$ flavour symmetry [62, 63].

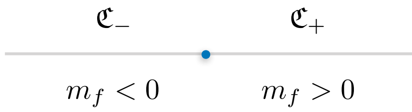

Figure 1.1: Chambers $\mathfrak{c}_+ = \{m_f > 0\}$ and $\mathfrak{c}_- = \{m_f < 0\}$ for a single chiral multiplet.

The supersymmetric ground states wavefunctions are annihilated by both supercharges. The gapped region of parameter space consists of two chambers $\mathfrak{c}_\pm \subset \mathfrak{t}_f = \mathbb{R}$ corresponding to $m_f > 0$ and $m_f < 0$ respectively - see figure 1.1. Choosing the Fock vacuum annihilated by ψ, the normalizable ground state wavefunctions are

$$
\begin{aligned}
\mathfrak{c}_+ &\quad : \quad e^{-h_f}\phi^n \\
\mathfrak{c}_- &\quad : \quad e^{h_f}\bar{\phi}^n\bar{\psi}\,,
\end{aligned}
\tag{1.17}
$$

where $n \geq 0$. Here we have defined a superpotential

$$
h_f = m_f|\phi|^2\,. \tag{1.18}
$$

The supersymmetric ground state wavefunctions can also be viewed as harmonic representatives of L^2-cohomology classes for the supercharge $Q^\dagger$. We denote the associated cohomology classes by $[\,\phi^n\,]$ in the chamber $\mathfrak{c}_+$ and $[\,\bar{\phi}^n\,]$ in the chamber $\mathfrak{c}_-$. In the second chamber, it is important to remember the presence of the Fermion $\bar{\psi}$, which is suppressed in our notation. Since the operator ϕ commutes with the supercharge $Q^\dagger$, it has a well-defined action on these cohomology classes,

$$
\begin{aligned}
\mathfrak{c}_+ &\quad : \quad \phi\cdot[\,\phi^n\,] = [\,\phi^{n+1}\,] \\
\mathfrak{c}_- &\quad : \quad \phi\cdot[\,\bar{\phi}^n\,] = [\,\bar{\phi}^{n-1}\,]\,,
\end{aligned}
\tag{1.19}
$$

which is compatible with the $U(1)_f$ flavour symmetry.

Although the supersymmetric ground state wavefunctions depend on m_f, as a vector space graded by $(-1)^F$ and $U(1)_f$, the space of supersymmetric ground states is constant in each chamber,

$$
\begin{aligned}
\mathcal{H}_+ &= x^{\kappa+\frac{1}{2}}\bigoplus_{j=0}^{\infty} x^j\mathbb{C} \\
\mathcal{H}_- &= -x^{\kappa-\frac{1}{2}}\bigoplus_{j=0}^{\infty} x^{-j}\mathbb{C}\,.
\end{aligned}
\tag{1.20}
$$

Here we have introduced a formal parameter $x \in \mathbb{C}^*$ to keep track of the $U(1)_f$ charge measured by J_f. Note that there is a choice of Fermion number for the Fock vacuum, and we have assigned Fermion number zero to the Fock vacuum annihilated by ψ.

The supersymmetric index computed in each chamber is

$$\begin{aligned} I_+ &= x^{\kappa+\frac{1}{2}} \sum_{j=0}^{\infty} x^j \\ I_- &= -x^{\kappa-\frac{1}{2}} \sum_{j=0}^{\infty} x^{-j} \,. \end{aligned} \tag{1.21}$$

Recalling that in computing the supersymmetric index we identify $x = e^{-2\pi\beta(m_f+iA_f)}$, this corresponds to the expansion of the same meromorphic function

$$I(x) = \frac{x^{\kappa+\frac{1}{2}}}{1-x} \tag{1.22}$$

in the appropriate regime, namely $|x| < 1$ in $\mathfrak{c}_+$ and $|x| > 1$ in $\mathfrak{c}_-$. This expression coincides with the 1-loop determinant for a chiral multiplet on a circle with a background supersymmetric Chern-Simons term for $U(1)_f$ at level κ.

This example can be extended to N chiral multiplets (ϕ_j, ψ_j) with flavour symmetry $G_f = U(N)$ and mass parameters $m_f = (m_1, \ldots, m_N) \in \mathfrak{t}_f$. There are now massless degrees of freedom on all coordinate hyperplanes in $\mathfrak{t}_f = \mathbb{R}^N$. Having removed these hyperplanes, the gapped region $\mathfrak{c}_f = \bigcup_\alpha \mathfrak{c}_\alpha$ consists of 2^N disjoint chambers

$$\mathfrak{c}_\alpha = \begin{cases} m_j > 0 & \alpha_j = + \\ m_j < 0 & \alpha_j = - \end{cases} , \tag{1.23}$$

labelled by a sign vector $\alpha = (\alpha_1, \ldots, \alpha_N)$.

The space of supersymmetric grounds states in each chamber is

$$\mathcal{H}_\alpha = \bigotimes_{j=1}^{N} \mathcal{H}_{\alpha_j} \tag{1.24}$$

where

$$\mathcal{H}_{\alpha_j} = \begin{cases} x_j^{\kappa+\frac{1}{2}} \bigoplus_{n=0}^{\infty} x_j^n \mathbb{C} & \alpha_j = + \\ -x_j^{\kappa-\frac{1}{2}} \bigoplus_{n=0}^{\infty} x_j^{-n} \mathbb{C} & \alpha_j = - \,. \end{cases} \tag{1.25}$$

We have chosen the same normal ordering constant κ for each chiral multiplet to preserve the underlying $G_f = U(N)$ flavour symmetry. As expected, the result reproduces the expansion of the supersymmetric index

$$\prod_{j=1}^{N} \frac{x_j^{\kappa+\frac{1}{2}}}{1-x_j} \tag{1.26}$$

in the appropriate regime $|x_j|^{\alpha_j} < 1$.

This model can be understood as a supersymmetric sigma model to $M = \mathbb{C}^N$ with the standard flat Kähler metric, supplemented by a Hermitian line bundle $F = K_{\mathbb{C}^N}^{\kappa+1/2}$. The flavour symmetry $G_f = U(N)$ corresponds to the isometries of $\mathbb{C}^N$. Introducing real mass parameters corresponds to a superpotential

$$h_f = \sum_j m_j |\phi_j|^2 \,, \tag{1.27}$$

which is the moment map for the $U(1)_{m_f}$ isometry generated by $m_f \in \mathfrak{t}_f$. The chambers $\mathfrak{c}_\alpha$ correspond to values of the mass parameters where the fixed locus of $U(1)_{m_f}$ is compact, namely the origin of $\mathbb{C}^N$.

Fermi Multiplets

Let us now consider a single Fermi multiplet (η, F) with a real mass parameter m_f for the $U(1)_f$ flavour symmetry. In canonical quantization, the complex Fermion obeys $\{\eta, \bar{\eta}\} = 1$ and

$$\begin{aligned} H &= \frac{m_f}{2}[\eta, \bar{\eta}] + m_f \kappa \\ J_f &= \frac{1}{2}[\eta, \bar{\eta}] + \kappa \,. \end{aligned} \tag{1.28}$$

The combination $H - m_f J_f = 0$ is again unambiguous, whereas H and J_f individually depend on a normal ordering constant κ.

In canonical quantization, we can choose a Fock vacuum or reference state $|0\rangle$ annihilated by the Fermion η and assign it Fermion number 0. The supersymmetric ground states are then $|0\rangle$ and $\bar{\eta}|0\rangle$ with flavour charge $\kappa + \frac{1}{2}$ and $\kappa - \frac{1}{2}$ respectively, as measured by J_f. We therefore find

$$\mathcal{H} = x^{\kappa+1/2}\mathbb{C} - x^{\kappa-1/2}\mathbb{C} \,, \tag{1.29}$$

in agreement with the supersymmetric index

$$I = x^{\kappa+1/2} - x^{\kappa-1/2} \,. \tag{1.30}$$

In the quantization of Fermi multiplets, there is a notational freedom to choose the Fock vacuum or reference state $|0\rangle$ to be annihilated by η or $\bar{\eta}$. In more complicated examples below, we will use this freedom to choose the representation that is most convenient for enumerating the supersymmetric ground states. The Fermion number assigned to this Fock vacuum is, however, meaningful and sets the Fermion number grading of supersymmetric ground states. This corresponds to an overall sign in the supersymmetric index. The reader is forewarned that we will typically omit the reference state $|0\rangle$ in our notation.

Superpotentials

We now present a number of examples that couple chiral and Fermi multiplets with holomorphic superpotentials and will reappear in computations relevant to 3d $\mathcal{N}=2$ theories in section 4.4.

Let us first consider a chiral multiplet (ϕ,ψ) coupled to a Fermi multiplet (η,F) with the following J-term superpotential

$$J(\phi)=u\phi\,, \tag{1.31}$$

where the complex mass parameter u can be regarded as a vacuum expectation value for a background chiral multiplet. The model preserves a $G_f=U(1)$ flavour symmetry under which (ϕ,ψ) have charge $+1$ and (η,F) have charge -1. We introduce a corresponding real mass parameter $m_f\neq 0$.

In canonical quantization, the supercharge

$$Q^\dagger=\left(Q^\dagger\right)^{(0)}+\left(Q^\dagger\right)^{(1)} \tag{1.32}$$

is a sum of two contributions

$$\left(Q^\dagger\right)^{(0)}=\left(\frac{\partial}{\partial\bar\phi}+m_f\phi\right)\bar\psi \qquad \left(Q^\dagger\right)^{(1)}=J(\phi)\eta\,, \tag{1.33}$$

where $\left(Q^\dagger\right)^{(0)}$ is the contribution from the chiral multiplet and $\left(Q^\dagger\right)^{(1)}$ is the additional contribution from the Fermi multiplet with J-term superpotential. Note that in this model there is both a real A-type parameter m_f and a complex B-type parameter u.

We assign Fermion number zero to the Fock vacuum annihilated by ψ and η. First, the cohomology of $\left(Q^\dagger\right)^{(0)}$ consists of the supersymmetric ground states of the chiral multiplet tensored with those of the Fermi multiplet,

$$\begin{aligned}\mathfrak{c}_+ \quad &: \quad [\,\phi^n\,]\,,\,[\,\phi^n\,]\,\bar\eta \qquad n\geq 0\\ \mathfrak{c}_- \quad &: \quad [\,\bar\phi^n\,]\,,\,[\,\bar\phi^n\,]\,\bar\eta \qquad n\geq 0\,.\end{aligned} \tag{1.34}$$

If $u=0$, the computation ends here and there is an infinite number of supersymmetric ground states. Assuming $u\neq 0$, a short spectral sequence argument shows that the cohomology of the total supercharge $Q^\dagger$ is equivalent to the cohomology of $\left(Q^\dagger\right)^{(1)}$ acting on the states (1.34). This is computed as follows:

- $\mathfrak{c}_+$: $\left(Q^\dagger\right)^{(1)}$ removes pairs $[\,\phi^{n+1}\,]$ and $[\,\phi^n\,]\,\bar\eta$ with $n\geq 0$ leaving only $[\,1\,]$.
- $\mathfrak{c}_-$: $\left(Q^\dagger\right)^{(1)}$ removes pairs $[\,\bar\phi^n\,]$ and $[\,\bar\phi^{n+1}\,]\,\bar\eta$ with $n\geq 0$ leaving only $[\,1\,]\,\bar\eta$.

We therefore find that for $u\neq 0$ there is a unique supersymmetric ground state and, setting normal ordering constants $\kappa=0$, $\mathcal{H}_\alpha=\mathbb{C}$ in both chambers $\alpha=\pm$.

Let us compare this result with the supersymmetric index. This is computed by multiplying the contributions from a chiral multiplet of charge $+1$ and a Fermi multiplet of charge -1, with the result $I = 1$ for both $u = 0$ and $u \neq 0$. In summary, the space of supersymmetric vacua is sensitive to the J-term superpotential whereas the supersymmetric index is not.

Let us now consider a second example with a pair of chiral multiplets ϕ_1, ϕ_2 coupled to a Fermi multiplet η with superpotential $J(\phi) = \phi_1\phi_2$[2]. This preserves a $U(1)_1 \times U(1)_2$ flavour symmetry whose charges can be chosen as follows,

	$U(1)_1$	$U(1)_2$
ϕ_1	1	0
ϕ_2	0	1
η	-1	-1

.

Introducing real mass parameters m_1, m_2 for the flavour symmetry, there are four chambers $\mathfrak{c}_\alpha \subset \mathfrak{t}_f = \mathbb{R}^2$ labelled by a sign vector $\alpha = (\alpha_1, \alpha_2)$ - see figure 1.2. We again choose the Fock vacuum annihilated by ψ_1, ψ_2 and η.

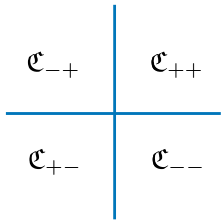

Figure 1.2: Chambers in the space of real mass parameters $(m_1, m_2) \in \mathfrak{t}_f = \mathbb{R}^2$ for two chiral multiplets ϕ_1, ϕ_2 and Fermi multiplet with superpotential $J = \phi_1\phi_2$.

As in our previous example, the supercharge is a sum of contributions from the chiral multiplets and the superpotential. The cohomology of $\left(Q^\dagger\right)^{(0)}$ is again the tensor product of supersymmetric ground states for the chiral and Fermi multiplets. Let us first consider the chamber $\mathfrak{c}_{++}$, in which we compute the cohomology of $\left(Q^\dagger\right)^{(1)} = \phi_1\phi_2\eta$ acting on

$$[\,\phi_1^{n_1}\phi_2^{n_2}\,]\,,\;[\,\phi_1^{n_1}\phi_2^{n_2}\,]\bar{\eta} \tag{1.35}$$

for $n_1, n_2 \geq 0$. The differential annihilates any $[\,\phi_1^{n_1}\phi_2^{n_2}\,]$ and sends the state $[\,\phi_1^{n_1}\phi_2^{n_2}\,]\bar{\eta}$ to $[\,\phi_1^{n_1+1}\phi_2^{n_2+1}\,]$. The remaining states in cohomology are

$$\begin{aligned} &[\,\phi_1^{n_1}\,] \qquad n_1 \geq 0 \\ &[\,\phi_2^{n_2}\,] \qquad n_2 > 0\,, \end{aligned} \tag{1.36}$$

and therefore

$$\mathcal{H}_{++} = \bigoplus_{n_1 \geq 0} x_1^{n_1}\mathbb{C} \oplus \bigoplus_{n_2 > 0} x_2^{n_2}\mathbb{C}\,, \tag{1.37}$$

where again we set $\kappa = 0$.

[2] As in our previous example, we could introduce a dimensionless complex parameter u in the superpotential. Since the Hilbert space of supersymmetric vacua will not depend on this parameter provided $u \neq 0$, we set $u = 1$ for convenience.

In the chamber $\mathfrak{c}_{+-}$, we compute the cohomology of $\left(Q^\dagger\right)^{(1)} = \phi_1\phi_2\eta$ acting on

$$[\,\phi_1^{n_1}\bar{\phi}_2^{n_2}\,]\,,\;[\,\phi_1^{n_1}\bar{\phi}_2^{n_2}\,]\bar{\eta} \tag{1.38}$$

with $n_1, n_2 \geq 0$. The differential annihilates any $[\,\phi_1^{n_1}\bar{\phi}_2^{n_2}\,]$ and sends the state $[\,\phi_1^{n_1}\bar{\phi}_2^{n_2}\,]\bar{\eta}$ to $[\,\phi_1^{n_1+1}\bar{\phi}_2^{n_2-1}\,]$ if it exists. The remaining states in cohomology are

$$\begin{aligned} &[\,\phi_1^{n_1}\,]\,\bar{\eta} \qquad &: \quad & n_1 \geq 0 \\ &[\,\bar{\phi}_2^{n_2}\,] \qquad &: \quad & n_2 \geq 0 \end{aligned} \tag{1.39}$$

and therefore

$$\mathcal{H}_{+-} = \bigoplus_{n_1 \geq 0} x_1^{n_1+1}\mathbb{C} \oplus \bigoplus_{n_2 \geq 0} (-x_2^{-n_2-1})\mathbb{C}\,. \tag{1.40}$$

There are similar results on regions $\mathfrak{c}_{-+}$ and $\mathfrak{c}_{--}$.

This is consistent with expanding the supersymmetric index

$$I = \frac{1 - x_1 x_2}{(1-x_1)(1-x_2)} \tag{1.41}$$

in the appropriate chambers. The supersymmetric index does not detect the presence of the J-term superpotential, except through the determination of the flavour symmetry and associated mass parameters. In particular, the contributions from states removed in pairs by $\left(Q^\dagger\right)^{(1)}$ cancel out in the supersymmetric index.

We consider one final example that will reappear in a three-dimensional problem in section 3.2.3. We introduce three chiral multiplets ϕ_1, ϕ_2, ϕ_3 coupled to three Fermi multiplets η_1, η_2, η_3 with J-term superpotentials

$$J_1 = \phi_2\phi_3 \qquad J_2 = \phi_3\phi_1 \qquad J_3 = \phi_1\phi_2\,. \tag{1.42}$$

This model in fact arises from a supersymmetric quantum mechanics with four supercharges with chiral multiplets Φ_1, Φ_2, Φ_3 and cubic superpotential $W = \Phi_1\Phi_2\Phi_3$. Here, we regard it as an $\mathcal{N} = (0,2)$ supersymmetric quantum mechanics with flavour symmetry

	$U(1)_T$	$U(1)_A$
ϕ_1	1	-1
ϕ_2	-1	-1
ϕ_3	0	2

.

Our notation and choice of charges is made with future applications in mind. Checking that the supersymmetric index is 1 is straighforward.

Introducing real mass parameters m_T and m_A, there are six chambers $\mathfrak{c}_\alpha \subset \mathfrak{t}_f = \mathbb{R}^2$ labelled by sign vectors $\alpha = (\alpha_1, \alpha_2, \alpha_3)$. The sign vectors $(+++)$ and $(---)$ are not allowed as the mass parameters of all three chirals must sum to zero. Let us concentrate here on the chamber $(+-+)$ corresponding to mass parameters $0 < m_A < m_T$. Following previous examples, we compute the cohomology of

$$\left(Q^\dagger\right)^{(1)} = \phi_1\phi_2\eta_3 + \phi_2\phi_3\eta_1 + \phi_3\phi_1\eta_2 \tag{1.43}$$

on states

$$[\,\phi_1^{n_1}\,\bar{\phi}_2^{n_2}\,\phi_3^{n_3}\,]\bar{\eta}_1^{s_1}\bar{\eta}_2^{s_2}\bar{\eta}_3^{s_3}\,. \tag{1.44}$$

where $n_1, n_2, n_3 \geq 0$ and $s_1, s_2, s_3 = 0, 1$. We have chosen the Fock vacuum annihilated by ψ_1, ψ_2, ψ_3 and η_1, η_2, η_3 and assigned it Fermion number one.

This example is simple enough to compute representatives of cohomology classes directly. A more systematic method is to split the supercharge into three terms and apply the method spectral sequences to compute the cohomology of the total complex in steps. We summarize representatives of the remaining cohomology classes and their contribution to the supersymmetric index below,

$\mathcal{H}_{+-+}$	I_{+-+}
$[\,\bar{\phi}_2^n\,]$	$+(qy)^{n+1}$
$[\,\phi_1\,\bar{\phi}_2^n\,]\,\bar{\eta}_1$	$-(qy)^{n+1}$
$[\,\phi_3^n\,]\,\bar{\eta}_1$	$-(y^2)^{n+1}$
$[\,\phi_3^{n+1}\,]\,\bar{\eta}_1\bar{\eta}_3$	$+(y^2)^{n+1}$
$[\,\phi_1^n\,]\,\bar{\eta}_3$	$-(qy^{-1})^{n+1}$
$[\,\phi_1^{n+1}\,]\,\bar{\eta}_1\bar{\eta}_3$	$+(qy^{-1})^{n+1}$
$[\,1\,]\,\bar{\eta}_1\bar{\eta}_3$	$+1$

,

where $n \geq 0$. Notice that we have chosen the normalization κ such that the vacuum has charge (qy). All of the contributions to the supersymmetric index cancel in pairs except for the final line, reproducing the expected result $I = 1$. A similar analysis can be performed in the remaining chambers.

A general model consists of N chiral multiplets (ϕ_j, ψ_j) and k Fermi multiplets (η^a, F^a) coupled by holomorphic superpotentials $J_a(\phi)$ and $E^a(\phi)$. The Hilbert space of supersymmetric ground states is the cohomology of

$$\left(Q^\dagger\right)^{(1)} = \eta^a J_a(\phi) + \bar{\eta}_a E^a(\phi) \tag{1.45}$$

acting on the tensor product of supersymmetric ground states for the individual chiral and Fermi multiplets. This can be understood as a supersymmetric sigma model to $M = \mathbb{C}^N$ together with the $\mathbb{Z}_2$-graded Hermitian vector bundle

$$F = K_{\mathbb{C}^N}^{1/2} \otimes \frac{\wedge^\bullet f^*}{\sqrt{\det f^*}}\,, \tag{1.46}$$

where f denotes the odd rank-k Hermitian vector bundle on $\mathbb{C}^N$ with fibers spanned by the complex fermions η^a. The holomorphic differential $\delta = \left(Q^\dagger\right)^{(1)}$ is given by the sum of contraction with the holomorphic section $\eta^a J_a(\phi)$ of f and the wedge product with the holomorphic section $\bar{\eta}_a E^a(\phi)$ of f^*.

1.2 Gauge Theory

A vectormultiplet in $\mathcal{N} = (0, 2)$ supersymmetric quantum mechanics contains a gauge field A_τ, a real scalar σ, and a real auxiliary field D, in addition to the complex fermions λ, $\tilde{\lambda}$. The real mass parameters introduced above can be regarded as coupling to a background vectormultiplet for the flavour symmetry G_f and turning on a vacuum expectation value $m_f = \langle \sigma_f \rangle$ for the scalar component.

We now consider dynamical vectormultiplets for a gauge symmetry G. We focus on $G = U(1)$ and introduce N chiral multiplets (ϕ_j, ψ_j) transforming with charge Q_j. We also introduce a real FI parameter $\zeta > 0$ and a supersymmetric Wilson line of charge q. These contribute ζD and $q(\sigma + iA_\tau)$ respectively to the lagrangian. Global anomaly cancellation requires

$$q - \frac{1}{2}\sum_{j=1}^{N} Q_j \in \mathbb{Z}\,. \tag{1.47}$$

This model will arise in computations of the supersymmetric twisted Hilbert space of three-dimensional gauge theories with $G = U(1)$ on $\Sigma = \mathbb{CP}^1$ in sections 3.2.4-3.2.5.

To compute the supersymmetric ground states, we introduce a supersymmetric sigma model onto configurations minimizing the Euclidean action in the 'geometric regime'. We first note that the auxiliary field can be eliminated by its equation of motion to give $D = e^2(\mu - \zeta)$, where $\mu = \sum_j Q_j |\phi_j|^2$ is the moment map for the $U(1)$ action on $\mathbb{C}^N$. The classical potential is then

$$U(\sigma, \phi) = \sigma^2 \sum_{j=1}^{N} |Q_j \phi_j|^2 + \frac{e^2}{2}(\mu - \zeta)^2 \tag{1.48}$$

Assuming $\zeta > 0$, the classical potential is minimized by configurations

$$\mu - \zeta = 0 \qquad \sigma = 0 \tag{1.49}$$

modulo constant $U(1)$ gauge transformations. At frequencies much smaller than $e^2\sqrt{\zeta}$, the system can be described by a supersymmetric sigma model to the Kähler quotient

$$M = \mu^{-1}(\zeta)/U(1)\,. \tag{1.50}$$

Since the dependence on e^2 and ζ is exact, provided the quotient M is smooth we expect the supersymmetric sigma model exactly to capture the space of supersymmetric ground states in the supersymmetric gauge theory. Note that for the quotient M to be smooth, we require

$$Q_j = \begin{cases} +1 & \text{for} \quad j = 1, \ldots, k \\ -1 & \text{for} \quad j = k+1, \ldots, N\,, \end{cases} \tag{1.51}$$

such that M is the total space of the line bundle $\mathcal{O}(-1)^{N-k} \to \mathbb{CP}^{k-1}$.

Let us specialize here to the compact case with N chiral multiplets of charge $Q_j = +1$. The flavour symmetry is $G_f = PSU(N)$, and the chiral multiplets transform in the projective representation of G_f obtained from the fundamental representation of $SU(N)$. In this case, we find a supersymmetric sigma model to $M = \mathbb{CP}^{N-1}$. In addition, there is is a Hermitian line bundle F with the following contributions:

- A contribution $K^{1/2}_{\mathbb{CP}^{N-1}} = \mathcal{O}(-\frac{N}{2})$ from quantizing chiral multiplet fermions.
- A contribution $\mathcal{O}(q)$ from the supersymmetric Wilson line.

The anomaly cancellation condition (1.47) ensures that the combination $F = \mathcal{O}(q - \frac{N}{2})$ is well-defined.

We can further introduce real mass parameters for the chiral multiplets $m_f = (m_1, \ldots, m_N)$ with $\sum_j m_j = 0$, which parametrize $\mathfrak{t}_f = \mathbb{R}^{N-1}$. In the supersymmetric sigma model, the real mass parameters introduce a real superpotential $h_f : \mathbb{CP}^{N-1} \to \mathbb{R}$ given by the moment map for the $U(1)_{m_f}$ isometry of $\mathbb{CP}^{N-1}$ generated by m_f.

Since the target of the supersymmetric sigma model is compact, the spectrum is always gapped and the space of supersymmetric ground states is constant over the whole parameter space $\mathfrak{t}_f = \mathbb{R}^{N-1}$. In particular, at $m_f = 0$ the space of supersymmetric ground states can be identified with the Dolbeault cohomology

$$\mathcal{H} = H^{0,\bullet}_{\bar\partial}(M, F)\,. \tag{1.52}$$

We introducing complex parameters $(x_1, \ldots, x_N)$ with $\prod_j x_j = 1$ parametrizing the complexified maximal torus of $G_f = PSU(N)$ to keep track of the grading by flavour symmetry. Then

$$\mathcal{H} = \begin{cases} S^{q-\frac{N}{2}}\left(\bigoplus_{j=1}^N x_j^{-1}\mathbb{C}\right) & q \geq \frac{N}{2} \\ \emptyset & -\frac{N}{2} < q < \frac{N}{2} \\ S^{-q-\frac{N}{2}}\left(\bigoplus_{j=1}^N x_j\mathbb{C}\right) & q \leq -\frac{N}{2}\,, \end{cases} \tag{1.53}$$

corresponding to symmetric powers of the fundamental and anti-fundamental representations of $G_f = PSU(N)$.

The supersymmetric index may be computed independently by localization in the supersymmetric gauge theory [57]. This computation reproduces the characters of the above representations of $G_f = PSU(N)$,

$$\begin{aligned} I &= \oint_\Gamma \frac{dz}{z} z^q \prod_{j=1}^N \frac{(zx_j)^{1/2}}{1 - zx_j} \\ &= \begin{cases} \chi_{S^{q-\frac{N}{2}}(\mathbb{C}^N)^*}(x_1, \ldots, x_N) & q \geq \frac{N}{2} \\ 0 & -\frac{N}{2} < q < \frac{N}{2} \\ \chi_{S^{-q-\frac{N}{2}}\mathbb{C}^N}(x_1, \ldots, x_N) & q \leq -\frac{N}{2}\,. \end{cases} \end{aligned} \tag{1.54}$$

The contour Γ surrounds the poles at $z = x_j^{-1}$ for all $j = 1, \ldots, N$. The contour integral expression for the supersymmetric index coincides with the computation of the holomorphic Euler character $\chi(M, F)$ using the Hirzebruch-Riemann-Roch theorem.

1.3 $\mathcal{N} = 4$ Supersymmetric Quantum Mechanics

We now briefly review the class of $\mathcal{N} = 4$ supersymmetric quantum mechanics that arise from topological twists of 3d $\mathcal{N} = 4$ theories on a Riemann surface.

1.3.1 Supersymmetry Algebra

We consider a supersymmetric quantum mechanics with $\mathcal{N} = 4$ supersymmetry and R-symmetry $SU(2)_R \oplus U(1)_r$. Below, it will be convenient to denote the corresponding Lie algebra by $\mathfrak{su}(2)_R \oplus \mathfrak{u}(1)_r$.

The supercharges Q^A, $\widetilde{Q}^A$ transform in the fundamental representation of $SU(2)_R$ and with weight $+\frac{1}{2}$, $-\frac{1}{2}$ under $U(1)_r$. They act on the Hilbert space such that $(Q^1)^\dagger = \widetilde{Q}^2$ and $(Q^2)^\dagger = -\widetilde{Q}^1$ with respect to the Hermitian inner product. The supercharges generate a supersymmetry algebra of the form

$$\begin{aligned}
&\{Q^A, Q^B\} = 0 \\
&\{Q^A, \widetilde{Q}^B\} = \epsilon^{AB} H + Z^{AB} \\
&\{\widetilde{Q}^A, \widetilde{Q}^B\} = 0
\end{aligned} \tag{1.55}$$

where H is the Hamiltonian operator and Z^{AB} are central charges transforming in the vector representation of $SU(2)_R$. A non-vanishing central charge will break $\mathfrak{su}(2)_R$ to a Cartan subalgbera. In what follows, we assume only $Z := Z^{12}$ is non-zero and write the corresponding maximal torus by $\mathfrak{u}(1)_R$.

1.3.2 Supersymmetric Ground States

We are primarily interested in the Hilbert space of supersymmetric ground states $\mathcal{H}$ annihilated by all of the supercharges Q^A, $\widetilde{Q}^A$. Such states are necessarily annihilated by both the Hamiltonian H and central charge Z^{12}. $\mathcal{H}$ therefore transforms as a representation of the full R-symmetry $\mathrm{SU}(2)_R \oplus U(1)_r$.

To understand the structure of supersymmetric ground states, it is useful to introduce a number of $\mathcal{N} = 2$ subalgebras. The supercharges Q^1 and Q^2 generate commuting subalgebras

$$\begin{aligned}
&\{Q^1, (Q^1)^\dagger\} = H + Z \\
&\{Q^2, (Q^2)^\dagger\} = H - Z \,,
\end{aligned} \tag{1.56}$$

while their sum $\mathcal{Q} := Q^1 + \widetilde{Q}^1$ generates a subalgebra $\{\mathcal{Q}, \mathcal{Q}^\dagger\} = 2H$ in which the central charge does not appear. We therefore obtain the unitarity bound $\langle H \rangle \geq |\langle Z \rangle|$.

We may define two spaces that saturate the bound

$$\begin{aligned}
&\mathcal{H}^+ = \{(H+Z)|\psi\rangle = 0\} \\
&\mathcal{H}^- = \{(H-Z)|\psi\rangle = 0\} \,.
\end{aligned} \tag{1.57}$$

Equivalently, $\mathcal{H}^+$ is the subspace annihilated by Q^1 and its adjoint while $\mathcal{H}^-$ is the subspace annihilated by Q^2 and its adjoint. They are graded by $\mathfrak{u}(1)_R \oplus \mathfrak{u}(1)_r$. The space supersymmetric ground states is the intersection $\mathcal{H} = \mathcal{H}^+ \cap \mathcal{H}^-$.

At the cost of manifest unitarity, it is often convenient to represent the spaces of states as the cohomology of a supercharge. Provided the spectrum of $H + Z$, $H - Z$ is gapped, $\mathcal{H}^\pm$ coincide with the cohomology of Q^1, Q^2 respectively. The space of supersymmetric ground states $\mathcal{H}$ is the cohomology of $\mathcal{Q}$. From this perspective the $\mathfrak{u}(1)_r$ grading is not manifest. However, introducing combinations $p = R + r$ and $q = R - r$, there is a spectral sequence converging to $\mathcal{H}$ such that $E_1^{p,q} = \mathcal{H}_+^{p,q}$ with differential $d_1 = \widetilde{Q}^1$.

If $Z = 0$ identically then the unitarity bound becomes $\langle H \rangle \geq 0$ and the supersymmetric ground states saturate this bound. If the spectrum of H is gapped, standard arguments show that the space of supersymmetric ground states $\mathcal{H}$ is the cohomology of any one of supercharges Q^1, Q^2 or $\mathcal{Q}$. In this case, the spectral sequence must collapse at the first step.

1.3.3 Kähler Sigma Models

Compact Kähler Target

First consider a supersymmetric sigma model defined by a smooth compact Kähler manifold M of complex dimension m. The Hilbert space consists of complex differential forms $\Omega^{p,q}(M, \mathbb{C})$ with Hermitian inner product

$$\langle \alpha, \beta \rangle = \int_M \bar{\alpha} \wedge *\beta \,. \tag{1.58}$$

The supercharges are

$$Q^1 = \bar{\partial} \qquad Q^2 = -\partial^\dagger \qquad \widetilde{Q}^1 = \partial \qquad \widetilde{Q}^2 = \bar{\partial}^\dagger \tag{1.59}$$

with Hamiltonian operator $H = \frac{1}{2}\Delta$ proportional to the Laplacian operator on M and vanishing central charge $Z^{12} = 0$.

The $\mathcal{N} = 4$ supersymmetry algebra neatly encodes the combination of Kähler relations and $\mathfrak{sl}(2)$ Lefschetz action on M. In particular, the Cartan generators of the R-symmetry $\mathfrak{su}(2)_R \oplus \mathfrak{u}(1)_r$ act by

$$R = \frac{1}{2}(p + q - m) \qquad r = \frac{1}{2}(p - q) \tag{1.60}$$

on (p, q)-forms.

Since $Z^{12} = 0$, the space of supersymmetric ground states $\mathcal{H} = \mathcal{H}^+$ can be understood as the cohomology of the supercharge $Q^1 = \bar{\partial}$, leading to an identification with Dolbeault cohomology $H^{p,q}_{\bar{\partial}}(M)$, or algebraically with the sheaf cohomology

$$\mathcal{H} = \bigoplus_{p,q=1}^{m} H^q(M, \wedge^p \Omega_M) \,. \tag{1.61}$$

This transforms as a representation of the R-symmetry with $\mathfrak{su}(2)_R$ acting by Lefschetz and $\mathfrak{u}(1)_r$ by $\frac{1}{2}(p-q)$, which is proportional to the Weil operator. For example, for a supersymmetric quantum mechanics with target $M = \mathbb{CP}^n$, we have $\mathcal{H} = \mathrm{Sym}^n V$ where V is the fundamental representation of $\mathfrak{su}(2)_R$ and weight $r = 0$.

Using instead the supercharge $\mathcal{Q} = Q^1 + \widetilde{Q}^1 = d$, we are lead to a description of the space of supersymmetric ground states $\mathcal{H}$ as the de Rham cohomology of M with complex coefficients.

The equivalence of these descriptions is guaranteed by the degeneration of the Hodge-to-de Rham spectral sequence for a compact Kähler manifold.

The Witten index of the supersymmetric quantum mechanics coincides with the Euler characteristic of M,

$$\mathrm{Tr}_{\mathcal{H}}(-1)^F = \sum_{p,q=1}^{m} (-1)^{p+q} h^{p,q} = \int_M e(T_M) \tag{1.62}$$

where the fermion number is $F = p + q = 2R + m$.

Mass Deformation

Now suppose that M has a Hamiltonian isometry group G_f with moment map $\mu : M \to \mathfrak{t}^*$. Then G_f is a flavour symmetry of the supersymmetric quantum mechanics. As in the $\mathcal{N} = (0, 2)$ case, turning on mass parameters $m_f \in \mathfrak{t}$ for the flavour symmetry amounts to introducing a real superpotential $h_f = m_f \cdot \mu_f$, which corresponds to the moment map for the infinitesimal $\mathfrak{u}(1)$ action generated by m_f. The critical locus of the moment map h_f coincides with the fixed locus of $\mathfrak{u}(1)$. It is a well-known fact that h_f is a perfect Morse-Bott function. In fact, the critical loci are necessarily Kähler and have a well-defined Morse index, which is always even[3].

In the presence of the mass parameter m_f, the supercharges of the supersymmetric quantum mechanics are deformed to

$$Q^1 = e^{-h_f}\,\bar{\partial}\,e^{h_f} \qquad Q^2 = -e^{h_f}\,\partial^\dagger\,e^{-h_f} \qquad \widetilde{Q}^1 = e^{-h_f}\partial\,e^{h_f} \qquad \widetilde{Q}^2 = e^{h_f}\bar{\partial}^\dagger\,e^{-h_f} \tag{1.63}$$

with the deformed Laplacian as an Hamiltonian and central charge $Z^{12} = m_f \cdot J_f$ where $J_f \in \mathfrak{t}_f$ are the conserved charges associated to the flavour symmetry.

As the central charge is now non-vanishing it is important to compute the space of supersymmetric ground states as the cohomology of $\mathcal{Q} = e^{-h_f}\,d\,e^{h_f}$. This generates a standard complex for the Morse-Bott function h_f. However, since h_f is perfect there are no differentials. The space of supersymmetric ground states can therefore be identified with the de Rham cohomology of the fixed locus.

Suppose the critical locus of h_f has components M_α. They are necessarily Kähler submanifolds with even Morse index $\nu(M_\alpha)$. Then the space of supersymmetric ground states is

$$\mathcal{H} = \sum_\alpha \mathcal{H}_\alpha \tag{1.64}$$

[3]Around an isolated fixed point, the action can be modelled as a torus action $T^{\mathrm{rk}(G_f)}$ on $\mathbb{C}^m$. Let the complex coordinates be $z_j = x_j + ix_j$. Assuming ρ_i to be the weights of this action, we have, around a fixed point,

$$m_f \cdot \mu(z_1, \dots, z_n) = \frac{1}{2}\sum_i |z_i|^2 \rho_i(m_f) = \frac{1}{2}\sum_i (x_i^2 + y_i^2)\rho_i(m_f)\,,$$

Clearly, the Hessian of this function has an even number of negative eigenvalues, and so the Morse index is even. This discussion can be generalised to non-isolated fixed points. In this case, one can show that the fixed loci are Kähler submanifolds, and that on a given submanifolds the Morse index is constant. Similar arguments as in the case of isolated fixed points go through. See for example ref. [64], section 10.5.6.

where

$$\mathcal{H}_\alpha = \oplus_{p,q} H^q(M_\alpha, \wedge^p T^*_{M_\alpha})\,. \tag{1.65}$$

The R-charges of the supersymmetric ground states in $\mathcal{H}_\alpha$ are given by

$$R = \frac{1}{2}(p+q+\nu(M_\alpha)-m) \qquad r = \frac{1}{2}(p-q) \tag{1.66}$$

Let us again consider the supersymmetric quantum mechanics with $M = \mathbb{CP}^{N-1}$. This has flavour symmetry $G = PSU(N)$ and therefore we can turn on real mass parameters $(m_1, \ldots, m_N) \in \mathfrak{t} = \mathbb{R}^{N-1}$ with $\sum_j m_j = 0$. For generic mass parameters, the critical locus of h_m consists of N isolated fixed points $\{p_j\}$ corresponding to coordinate lines in $\mathbb{CP}^{N-1}$. Supposing that the Kähler parameter of $\mathbb{CP}^{N-1}$ is $\zeta > 0$ then $h_m(p_j) = \zeta m_j$. If we order the mass parameters such that $m_1 < \cdots < m_N$ then

$$h_f(p_1) < h_f(p_2) < \cdots < h_f(p_N) \tag{1.67}$$

and $\nu(p_j) = 2j-2$. There is a single perturbative ground state $\Psi(p_j)$ associated to each fixed point with R-charges $R(p_j) = \frac{1}{2}(\nu(p_j) - (N-1)) = j - \frac{N+1}{2}$ and $r = 0$. We therefore find

$$\mathcal{H} = \mathbb{C}_{-\frac{N-1}{2},0} \oplus \cdots \oplus \mathbb{C}_{\frac{N-1}{2},0} \tag{1.68}$$

in agreement with our previous result.

1.3.4 Hyperkähler Sigma Models

We finally provide a minimal discussion about hyperkähler sigma models. These are labelled by a hyperkähler manifold endowed with complex structures (I, J, K) that satisfy the quaternion relations, together with a hyperholomorphic bundle. Obvious examples are the tangent or cotangent bundle of the manifolds. The four supercharges in this case are

$$Q^1 = \bar{\partial}_I \qquad Q^2 = \bar{\partial}_{-I} \qquad \widetilde{Q}^1 = -\bar{\partial}^\dagger_{-I} \qquad \widetilde{Q}^2 = \bar{\partial}^\dagger_I\,. \tag{1.69}$$

Here, $-I$ is the complex structure opposite to I. Their deformations induced by moment maps for an isometry group G_f read

$$Q^1 = e^{-h_f}\bar{\partial}_I e^{h_f} \qquad Q^2 = e^{h_f}\bar{\partial}_{-I} e^{-h_f} \qquad \widetilde{Q}^1 = -e^{-h_f}\bar{\partial}^\dagger_{-I} e^{h_f} \qquad \widetilde{Q}^2 = e^{h_f}\bar{\partial}^\dagger_I e^{-h_f}\,. \tag{1.70}$$

The supercharges satisfy the algebra (1.55) with

$$H = \frac{1}{2}\Delta_{m_f} - (m_f \cdot J_f)\ , \tag{1.71}$$

and $Z^{12} = 0$. The $\mathfrak{su}(2)_R$ symmetry has a neat interpretation. It simply rotates the complex structures of the hyperkähler manifold. This action commutes with the Laplacian, and turns into a $\mathfrak{su}(2)_R$ action on the Hilbert space of supersymmetric ground states[4].

[4]When M is compact, the action has been studied extensively in ref. [65]. Moreover, it is worth noticing that on the cohomology of hyperkähler manifolds there also is an extended $\mathfrak{so}(1,4)$ action [66], which is a composition of the Leftschetz actions in the three complex structures. In more physical terms, this can be understood from the point of view of a six dimensional supersymmetric sigma model reduced to a 1d quantum mechanics. In fact, the $\mathfrak{so}(1,4)$ action is nothing else than the residual 'internal' Lorentz symmetry in the other five dimensions [67].

Finally, in the presence of a mass gap, since there is no central charge in this model the Hilbert space of supersymmetric ground states can be identified (as a vector space) with the cohomology of the deformed Dolbeault operator in any complex structure.

1.3.5 Two Examples

One Free Chiral Multiplet

As a first example of a $\mathcal{N} = 4$ supersymmetric quantum mechanics, we start with a free chiral multiplet. With future applications in mind, we denote its fields by (X, ψ_X, η_Y, F_Y), where F_Y is an auxiliary field. The chiral multiplet has a $U(1)_f$ flavour symmetry, and we turn on a corresponding real mass m_f. The supercharges read, on shell

$$\begin{aligned} Q^1 &= \bar{\psi}_X \left(\frac{\partial}{\partial \bar{X}} + m_f X \right) \\ Q^2 &= \eta_Y \left(\frac{\partial}{\partial \bar{X}} + m_f X \right) \\ \widetilde{Q}^1 &= \bar{\eta}_Y \left(\frac{\partial}{\partial X} + m_f \bar{X} \right) \\ \widetilde{Q}^2 &= \psi_X \left(-\frac{\partial}{\partial X} + m_f \bar{X} \right) . \end{aligned} \tag{1.72}$$

They satisfy the same algebra of Kähler sigma models. In fact, this system can be thought of geometrically as an $N = 4$ quantum mechanics with target space $\mathbb{C}$, and the supercharges admit the interpretation (1.63).

The supercharges Q^1 and $\widetilde{Q}^2$ can be identified with the supercharges of a free $\mathcal{N} = (0, 2)$ chiral multiplet $Q^\dagger$ and Q (1.15). From the point of view of $\mathcal{N} = (0, 2)$ supersymmetry, the free chiral multiplet decomposes into a $(0, 2)$ chiral multiplet (X, ψ_X) and a Fermi multiplet (η_Y, F_Y).

Let us consider the space of supersymmetric ground states that are preserved by the four supercharges. It is easy to see that all the states annihilated by both Q and $Q^\dagger$ are not annihilated by the two remaining supercharges but one. Geometrically, this is the single representative of the deformed de Rahm cohomology of $\mathbb{C}$.

One Free Hypermultiplet

Our second and last example is a free hypermultiplet. We denote the fileds by (X, ψ_X, ψ_Y, Y). There is a $U(1)_f$ flavour symmetry that acts on X and Y with charges $+1$ and -1. Turning on a real mass for this symmetry, the supercharges read

$$\begin{aligned}
Q^1 &= \bar{\psi}_X \left(\frac{\partial}{\partial \bar{X}} + m_f X \right) + \bar{\psi}_Y \left(\frac{\partial}{\partial \bar{Y}} - m_f Y \right) \\
Q^2 &= -\bar{\psi}_Y \left(\frac{\partial}{\partial X} - m_f \bar{X} \right) + \bar{\psi}_X \left(\frac{\partial}{\partial Y} + m_f \bar{Y} \right) \\
\widetilde{Q}^1 &= -\psi_Y \left(\frac{\partial}{\partial \bar{X}} + m_f X \right) + \psi_X \left(\frac{\partial}{\partial \bar{Y}} - m_f Y \right) \\
\widetilde{Q}^2 &= -\psi_X \left(\frac{\partial}{\partial X} - m_f \bar{X} \right) - \psi_Y \left(\frac{\partial}{\partial Y} + m_f \bar{Y} \right) .
\end{aligned} \tag{1.73}$$

We could write covariantly

$$\begin{aligned}
Q^A &= -\bar{\psi}_X \left(\epsilon^{AB} \frac{\partial}{\partial \widetilde{X}^B} - m_f X^A \right) + \bar{\psi}_Y \left(\epsilon^{AB} \frac{\partial}{\partial X^B} - m_f \widetilde{X}^A \right) \\
\widetilde{Q}^A &= \psi_X \left(\epsilon^{AB} \frac{\partial}{\partial X^B} - m_f \widetilde{X}^A \right) + \psi_Y \left(\epsilon^{AB} \frac{\partial}{\partial \widetilde{X}^B} - m_f X^A \right) ,
\end{aligned} \tag{1.74}$$

where $X^A = (X, \bar{Y})$, $\widetilde{X}^A = (Y, -\bar{X})$. In this case, the supercharges satisfy the algebra of hyperkähler sigma models. We can in fact view a free hypermultiplet as an $\mathcal{N} = 4$ supersymmetric quantum mechanics with target space $T^*\mathbb{C}$.

Once again, Q^1 and $\widetilde{Q}^2$ obey the algebra of an $\mathcal{N} = (0, 2)$ quantum mechanics. From the perspective of $\mathcal{N} = (0, 2)$ supersymmetry, the free hypermultiplet decomposes into two chiral multiplets (X, ψ_X) and (Y, ψ_Y). As already mentioned, the space of supersymmetric ground states annihilated by the four supercharges can be identified with the cohomology of any of them.

There are two chambers $\mathfrak{c}_\pm \subset \mathfrak{t}_f = \mathbb{R}$ corresponding to $m_f > 0$ and $m_f < 0$. The same reasoning we used in 1.1.3 shows that the Hilbert space of four-supercharge preserving ground state is spanned by

$$\begin{aligned}
\mathfrak{c}_+ \quad &: \quad e^{-h_f} X^{n_1} \bar{Y}^{n_2} \bar{\psi}_Y \\
\mathfrak{c}_- \quad &: \quad e^{h_f} \bar{X}^{n_1} Y^{n_2} \bar{\psi}_X ,
\end{aligned} \tag{1.75}$$

where $n_1, n_2 \geq 0$. The real superpotential coming from the moment map is

$$h_f = m_f \left(|X|^2 - |Y|^2 \right) . \tag{1.76}$$

These states are representative for the deformed Dolbeault cohomology of $T^*\mathbb{C}$.

Chapter 2

Supersymmetric Localization Results

We now consider 3d $\mathcal{N} = 2$ theories with a topological twist on the product of a real line $\mathbb{R}$ and a compact connected Riemann surface Σ of genus g. This setup falls into the general class of supersymmetric backgrounds introduced in [68].

In this short chapter, we summarize and modify slightly some localisation results, in particular the computation of the twisted indices of the theories on $S^1 \times \Sigma$. The twisted indices of $\mathcal{N} = 2$ gauge theories have been studied extensively in [22, 23, 25, 26] from various perspectives, but we will mainly focus on the results obtained from UV Coulomb branch localisation [23, 25, 26]. Crucially, we introduce an additional exact action that leads to a different, and for us more appropriate, BPS locus.

2.1 Topological twist on $\mathbb{R} \times \Sigma$

We start with some elementary algebraic preliminaries. We use Euclidean $SU(2)$ spinor indices α, and assume that the theory has an unbroken $U(1)_R$ R-symmetry. In the absence of central charges, three-dimensional $\mathcal{N} = 2$ supersymmetric gauge theories enjoy the following supersymmetry algebra

$$\{Q_\alpha, \tilde{Q}_\beta\} = P_{\alpha\beta}\,, \tag{2.1}$$

where Q_α, $\tilde{Q}_\beta$ are the four supercharges of $U(1)_R$ charge $+1$ and -1 respectively. $P_{\alpha\beta}$ are the momentum generators. The topological twist we use is equivalent to the topological A-twist on Σ, which redefines the Lorentz group on the plane by mixing it with a fixed unbroken $U(1)_R \subset SU(2)_R$. Given a field of charge L under the U_{12} rotations on the plane, the twist assigns a new spin to the fields

$$L' = L + \frac{1}{2}R\,. \tag{2.2}$$

The twist preserves the supercharges that commute with the new Lorentz group. The resulting algebra is

$$\{Q, Q^\dagger\} = P_0\,, \tag{2.3}$$

where P_0 is the generator of translations along $\mathbb{R}$. These supersymmetry algebras can be deformed by central charges, something we will do soon[1].

2.2 3d $\mathcal{N} = 2$ Supermultiplets, Actions and BPS Loci

Consider now some 3d Chern-Simons-matter theory with a gauge group G and a chiral multiplet of R-charge r in a representation R of the gauge group. Following refs. [23, 69, 26, 25], after performing the topological twist we use a compact 'twisted field' notation for all the supermultiplets. We denote the fields in the vectormultiplet by

$$\mathcal{V} = (\sigma, A_\mu, \lambda, \bar{\lambda}_1, \Lambda_{\bar{1}}, \bar{\Lambda}, D)\,, \tag{2.4}$$

where σ is a real scalar, A_μ is the gauge connection, D is the auxiliary field, whereas $\lambda, \bar{\lambda}, \Lambda, \bar{\Lambda}$ are gauginos. We relegate the supersymmetry transformations of this and any other multiplet to A.1. In frame-index notation[2] , the standard Yang-Mills lagrangian is

$$\begin{aligned}\mathcal{L}_{\text{YM}} = \text{Tr}\Big[&\frac{1}{2}F_{01}F_{0\bar{1}} + \frac{1}{2}(-2iF_{1\bar{1}})^2 + \frac{1}{2}D^2 + \frac{1}{2}|D_\mu\sigma|^2 - i\bar{\lambda}D_0\lambda - i\bar{\Lambda}_{\bar{1}}D_0\Lambda_1 \\ &+2i\bar{\Lambda}_{\bar{1}}D_1\lambda - 2i\Lambda_1 D_{\bar{1}}\bar{\lambda} - i\bar{\Lambda}_{\bar{1}}[\sigma,\Lambda_1] + i\bar{\lambda}[\sigma,\lambda]\Big]\ .\end{aligned} \tag{2.5}$$

This action is exact with respect to the two supercharges Q, $Q^\dagger$ preserved by the topological twist on Σ. Using the fields of the vectormultiplet, we can also consider Chern-Simons actions

$$\mathcal{L}_{CS} = \frac{k}{4\pi}\text{Tr}\left(i\epsilon^{\mu\nu\rho}\left(A_\mu\partial_\mu A_\rho - \frac{2i}{3}A_\mu A_\nu A_\rho\right) - 2D\sigma + 2i\bar{\lambda}\lambda + 2i\bar{\Lambda}_{\bar{1}}\Lambda_1\right)\,, \tag{2.6}$$

which obviously are not exact. For each semi-simple factor, we will denote $k^{ab} := kh^{ab}$, where h^{ab} is the Killing form.

Besides the dynamical vectormultiplet, a crucial role will be played by background vectormultiplets for a maximal torus of any global symmetry group $T_f \subset G_f$, which we will turn on at appropriate times during our discussion. A background vectormultiplet $\mathcal{V}_f$ will in particular include

- Real scalar components m_f valued in the Cartan subalgebra $\mathfrak{t}_f$ of the flavour symmetry group G_f known as real masses.

[1] Depending on the choice of gauge, we might actually be forced to introduce some central charges, but for the sake of presentation we gloss over details in this summary. See [26].

[2] We have

$$e^0 = dx_3\ ,\quad e^1 = \sqrt{2g_{z\bar{z}}}dz\ ,\quad e^{\bar{1}} = \sqrt{2g_{z\bar{z}}}d\bar{z}\ ,$$

so that the metric on the Riemann surface is $ds^2 = e^1e^{\bar{1}} = 2g_{z\bar{z}}dzd\bar{z}$. We also define $F_{\mu\nu} = \partial_\mu A_\nu - \partial_\nu A_\mu - i[A_\mu, A_\nu]$, where $*F = -2iF_{1\bar{1}}$ is Hermitian. The holomorphic derivatives and the gauginos are $(\bar{\partial}_A, \bar{\Lambda}) = (D_{\bar{1}}e^{\bar{1}}, \Lambda_{\bar{1}}e^{\bar{1}})$ and $(\partial_A, \Lambda) = (D_1e^1, \Lambda_1e^1)$.

- A background connection A_f .

For instance, when the gauge group has an overall $U(1)$ factor, the theory has a $U(1)_T$ topological symmetry, and we can turn on a real mass of a background vectormultiplet. This mass is a real Fayet-Iliopoulos (FI) parameter. We can then add a contribution to the action which for $U(k)$ gauge groups is of the form

$$\mathcal{L}_{\rm FI}[V] = -\frac{i\zeta}{2\pi}\ \mathrm{Tr}(D)\ . \tag{2.7}$$

Next, let us introduce matter contributions. We denote the fields of a $\mathcal{N}=2$ twisted chiral multiplet by

$$\Phi = (\phi, \psi, \eta, F)\,. \tag{2.8}$$

We use the following lagrangian for the chiral multiplet,

$$\begin{aligned}\mathcal{L}_\Phi = {} & \phi^\dagger(-D_0^2 - 4D_1D_{\bar 1} + \sigma^2 + iD - 2iF_{1\bar 1})\phi - F^\dagger F \\ & - \frac{i}{2}\bar\psi(D_0+\sigma)\psi - 2i\bar\eta(D_0-\sigma)\eta + 2i\bar\psi D_1\eta - 2i\bar\eta D_{\bar 1}\psi \\ & - i\bar\psi\bar\lambda\phi + i\phi^\dagger\lambda\psi - 2i\phi^\dagger\Lambda_1\eta + 2i\bar\eta\bar\Lambda_{\bar 1}\phi\ ,\end{aligned} \tag{2.9}$$

which is also $Q, Q^\dagger$-exact. From chiral multiplets we can construct superpotential terms

$$\mathcal{L}_W[\Phi] + \mathcal{L}_{\bar W}[\Phi^\dagger] = \int d^2\theta\ W(\Phi) + \text{h.c.} \tag{2.10}$$

where W is a holomorphic functional of chiral multiplets with total $U(1)_R$-charge 2. Being super-descendants, superpotential terms are exact.

Summing the above lagrangian contributions gives

$$\mathcal{L} = \frac{1}{e^2}\mathcal{L}_{\rm YM} + \frac{1}{g^2}\mathcal{L}_\Phi + \frac{1}{g_W^2}\mathcal{L}_W + \mathcal{L}_{\rm FI} + \mathcal{L}_{\rm CS} \tag{2.11}$$

where we have inserted parameters e^2, g^2 and g_W^2 in front of the actions that are exact.

2.3 Twisted Indices and Contour Integral Formulae

Supersymmetric localization can be used to compute the twisted index of the theory on $S^1\times\Sigma$, which is defined as

$$I = \mathrm{Tr}_{\mathcal{H}}\ (-1)^F e^{2\pi i\zeta} e^{2\pi i m_f\cdot J_f}\ , \tag{2.12}$$

where $\mathcal{H}$ is the Hilbert space of states on Σ. By taking limits of (2.11) as the parameters e^2, g^2, g_W^2 tend to zero, we can localize the path integral to the critical loci of the combinations $\mathcal{L}_{\rm YM}$ and $\mathcal{L}_W$. By imposing a suitable reality condition for all the fields except for the auxiliary field, the path integral localizes to solutions of the following equations

$$*F_A = -iD\ ,\quad \bar\partial_A\phi = 0\ ,\quad d_A\sigma = 0\ ,\quad F_{01} = F_{0\bar 1} = 0\ ,\quad \sigma\cdot\phi = 0 \tag{2.13}$$

as found in [26, 25, 23]. Sometimes, we may also send g_W^2 to zero and impose the further constraint

$$\frac{\delta W}{\delta \Phi} = 0\,. \tag{2.14}$$

The solutions to these equations are shown in [23, 25, 26] to parametrize the complexified classical Coulomb branch $\mathfrak{M}_{C,\mathbb{C}}$. On the Coulomb branch, the gauge group is broken to a maximal torus. $\mathfrak{M}_{C,\mathbb{C}}$ is parametrized locally by the constant expectation values of the real scalar fields, σ_a, $a \in \{1,\dots,\mathrm{rk}(G)\}$, combined with flat connections for the gauge group on S^1 into the variables

$$u_a = i\beta(\sigma_a + i a_{0,a})\,, \tag{2.15}$$

where $a_{0,a}$ is the holonomy of a the gauge connection $A_{0,a}$ around S^1[3]. These parameters however may be identified by the action of the Weyl group, and so

$$\mathfrak{M}_{C,\mathbb{C}} \cong (\mathbb{C}^*)^{\mathrm{rk}(G)}/W_G\,. \tag{2.16}$$

After carefully integrating out the fermionic zero modes, the localized path integral can be written as a rk(G)-dimensional residue integral over the cover of $\mathfrak{M}_{C,\mathbb{C}}$ parametrized by the variables u_a. For a theory with chiral multiplets Φ^i in representations R^i of the gauge group, the result is

$$\begin{aligned} I = \frac{(2\pi i)^{\mathrm{rk}(G)}}{|W_G|} \sum_{\underline{\mathfrak{m}}\in\Lambda_G} (-q)^{\mathfrak{m}} \sum_{u_*=\{u_i\}} \Bigg(& \underset{u=u_*}{\text{JK-Res}}(Q_{u_*}(u),\eta) \\ & Z_{\text{classical}} Z^{\text{vector}}_{\text{1-loop}} \left(\prod_i Z^{\Phi_i}_{\text{1-loop}}\right) H^g \; d^{\mathrm{rk}(G)}u \Bigg)\,. \end{aligned} \tag{2.17}$$

Here, the first summation is over the GNO quantized flux $\underline{\mathfrak{m}}$ valued in the co-character lattice of the gauge group Λ_G. We also have written

$$q = e^{2\pi i\zeta}\,, \tag{2.18}$$

where it is understood that ζ, similarly as for the variables u, is complexified by background connections along S^1. We also set for simplicity $\beta = 1$ henceforth. The exponent $\mathfrak{m}$ is an element of the character lattice of the topological symmetry and can therefore be contracted with ζ. For $G = U(k)$, our main case of interest, $\mathfrak{m} = \mathrm{Tr}(\underline{\mathfrak{m}}) \in \mathbb{Z}$[4].

The integrand is written in terms of exponentiated variables, or fugacities, which are invariant under large gauge transformations. We set

$$x = e^{2\pi i u},\; y_i = e^{2\pi i m_{f,i}} \tag{2.19}$$

where $m_{f,i}$ are the real masses associated to $U(1)_i$ gauge groups, which are also complexified by holonomies around S^1. Incidentally, these complexifications –which arise from the compactification

[3] Notice that the parametrization is slightly different than in flat space.

[4] More generally, $\mathfrak{m}$ can be obtained from $\underline{\mathfrak{m}}$ via the identification of $\pi_1(G)$ with the quotient of Λ_G by the co-root lattice of G.

on S^1– are responsible for washing out the dependence of the indices on real parameters, which will still be detected by Hilbert spaces. We also denote by $\mathfrak{n}_i$ the background GNO flux induced by a background connection $A_{f,i}$.

The one-loop determinants evaluated at the BPS locus are

$$Z_{\text{class}} = \prod_{a,b} (x_a)^{k^{ab}\mathfrak{m}_b} \,, \tag{2.20}$$

$$Z^{\Phi_i}_{\text{1-loop}} = \prod_{\rho_i \in R^i} \left(\frac{x^{1/2} y_i^{1/2}}{1 - x^{\rho_i} y_i} \right)^{\rho_i(\mathfrak{m}) + \mathfrak{n}_i + (g-1)(r_i - 1)} \,, \tag{2.21}$$

and

$$Z^{\text{vector}}_{\text{1-loop}} = (-1)^{\sum_{\alpha>0} \alpha(\mathfrak{m})} \prod_{\alpha \in \Delta} (1 - x^\alpha)^{1-g} \,, \tag{2.22}$$

where Δ is the set of all roots of $\mathfrak{g}$, ρ are the weights of the representation R^i of G in which Φ^i transforms, whereas r_i is the R-charge of Φ^i.

The last term in (2.17) can be obtained from integrating out the gaugino zero modes $\Lambda_1, \Lambda_{\bar{1}}$:

$$H = \det_{ab} \left[k^{ab} + \sum_i \sum_{\rho_i} \rho^a \rho^b \frac{1}{2} \left(\frac{1 + x^{\rho^i} y_i}{1 - x^{\rho_i} y_i} \right) \right] \,. \tag{2.23}$$

The integrand of (2.17) has four types of singular hyperplanes in the domain of the u-integral, where each of the hyperplane H_Q is assigned a charge vector $Q \in \mathfrak{t}^*$:

- There exist potential singularities where a chiral multiplet becomes massless:

  $$H_{\rho_i} = \left\{ u \;\middle|\; \rho_i(u) \; + \nu_i \in \mathbb{Z} \right\} \tag{2.24}$$

  The order of the pole is $\rho(\underline{\mathfrak{m}}) + \mathfrak{n}_i + (g-1)(r_i - 1) + g$.

- For each $\alpha \in \Delta$, there exist potential singularities at

  $$H_\alpha = \left\{ u \;\middle|\; \alpha(u) \in \mathbb{Z} \right\} \,, \tag{2.25}$$

  where the W-boson becomes massless. This singularity corresponds to the boundary of the Weyl chamber, where the gauge symmetry enhances to a non-Abelian subgroup.

- Finally, the integrand can have a potential singularity at

  $$H_{Q_{a,\pm}} = \left\{ u \;\middle|\; u_a \to \pm i\infty \right\} \,. \tag{2.26}$$

  The behaviour of the integrand at infinity is governed by the charge of the monopole operators $T^\pm$ under the gauge and global symmetries in the theory [23, 25, 26], whose explicit form is not needed for this thesis.

The integral is given by a sum of the rk(G)-dimensional residue integral over the poles defined by the intersections of rk(G) singular hyperplanes. The JK-integral [70, 71, 72] is defined by the property

$$\begin{aligned}
&\underset{u=u_*}{\text{JK-Res}}(Q_{u_*}(u),\eta)\left[\frac{d^{\text{rk}(G)}u}{Q_1(u)\cdots Q_{\text{rk}(G)}(u)}\right] \\
&\qquad = \left[\begin{array}{cc} \dfrac{1}{|\det(Q_1,\cdots,Q_{\text{rk}(G)})|} & \text{if } \eta\in\text{Cone}(Q_1,\cdots,Q_{\text{rk}(G)}) \\ 0 & \text{else} \end{array}\right.
\end{aligned} \tag{2.27}$$

where Cone$(Q_1,\cdots,Q_r)$ is the positive cone spanned by the charge vectors $(Q_1,\cdots,Q_r)$, and $\eta\in\mathfrak{t}^*$ is an auxiliary parameter. This definition includes the charge vector $Q_\pm$ from the hyperplanes at asymptotic infinities (2.26). The charges $Q_{a,\pm}$ of these hyperplanes can be defined by examining the integral of the auxiliary field $\hat{D}$ in the large u region, and we will briefly comment on this in the appendix (A.2).

Note that the Jeffrey-Kirwan residue integral operation in (2.27) is ill-defined for the poles which intersect with the W-boson singularities (2.25). These singularities need to be properly resolved, and following [25, 26], we will exclude the residues from these poles in the final formula.

For $G=U(1)$ theory, one can show that the residue integral does not depend on the choice of η, due to the residue theorem. For non-Abelian gauge group, this point is more subtle, due to the singularities of the W-boson. We will briefly return on this point when necessary.

2.4 An Alternative BPS locus

In this thesis, we consider an alternative localising action akin to the one introduced in section 9 of reference [69] in the context of the twisted partition function of 2d $\mathcal{N}=(2,2)$ theories on S^2. In particular, we add a $(Q+Q^\dagger)$-exact term,

$$\mathcal{L}_{\text{H}}=\frac{1}{2i}\left(Q+Q^\dagger\right)\left[\left(\lambda+\bar{\lambda}\right)\left(\mu(\phi)-\tau\right)\right]\,, \tag{2.28}$$

whose bosonic part is

$$\mathcal{L}_{\text{H}}^{\text{bos}}=i\text{Tr}\left[\left(D-2F_{1\bar{1}}\right)\left(\mu(\phi)-\tau\right)\right]\,. \tag{2.29}$$

Here $\mu(\phi)$ is a moment map for the gauge action on the representation R. We emphasize that the parameter $\tau\in\mathfrak{t}^*$ is distinct from the physical 3d FI parameter ζ introduced in equation (2.7). In fact, as we will explain later, the combination it couples to can be identified with an auxiliary field in the 1d quantum mechanics on $\mathbb{R}$. Therefore, τ can be viewed as a 1d FI parameter in the quantum mechanics. We then replace the vectormultiplet action by

$$\frac{1}{e^2}\mathcal{L}_{\text{YM}}\ \rightarrow\ \frac{1}{t^2}\left(\frac{1}{e^2}\mathcal{L}_{\text{YM}}+\mathcal{L}_{\text{H}}\right) \tag{2.30}$$

and consider the limit as $t, g \to 0$ such that $t/g \to 0$, while keeping e finite. After integrating out the auxiliary field, the path integral localizes to configurations solving the following set of 'generalized vortex equations' on Σ,

$$\begin{gathered} *F_A + e^2\left(\mu_{\mathbb{R}}(\phi) - \tau_{eff} + \sigma k_{eff}\right) = 0\ , \quad \bar{\partial}_A \phi = 0 \\ d_A \sigma = 0\ , \quad \sigma \cdot \phi = 0\ , \quad \frac{\delta W}{\delta \Phi} = 0 \end{gathered} \tag{2.31}$$

where it is understood that σ and $\bar{\partial}_A$ act in the appropriate representation. Notice that when integrating out massive matter, the 1d FI parameter τ and the Chern-Simons level k may be affected by one-loop renormalization, as in (3), and we have therefore introduced a subscript 'eff'. In this thesis, however, we will be interpreted in cases where this renormalization is immaterial, and k_{eff} is zero[5]. This is always the case for $\mathcal{N} = 4$ theories.

Most of our efforts will be centred on finding an effective description of the 3d theories around the BPS locus (2.31). The result of the supersymmetric localization computation of the index is mildly affected by the exact term (2.30). In fact, it modifies the charges of the hyperplanes in (2.26). For $G = U(1)$, we show in the appendix A.2 that the new charges are given by

$$Q_{\pm} = \frac{2\pi\mathfrak{m}}{e^2} - \text{vol}(\Sigma)\tau\ , \tag{2.32}$$

for each GNO flux sector $\mathfrak{m} \in \mathbb{Z}$, where τ is the parameter we introduced in (2.28). There is therefore a natural choice η_0 for the parameter η in (2.27)

$$\eta = -\frac{2\pi\mathfrak{m}}{e^2} + \text{vol}(\Sigma)\tau := \eta_0\ , \tag{2.33}$$

so that the residue integral (2.17) does not select poles involving the hyperplane at asymptotic boundaries.

[5]In principle, also the coupling e is renormalized, but with for us less important consequences. To lighten the notation, we will not express this renormalization.

Chapter 3

Twisted Hilbert Spaces of 3d $\mathcal{N} = 2$ Supersymmetric Gauge Theories

The setup $\mathbb{R} \times \Sigma$ preserves the same supersymmetry as an $\mathcal{N} = (0, 2)$ supersymmetric quantum mechanics on $\mathbb{R}$ of the type considered in chapter 1. As anticipated in the introduction, the aim of this thesis is to embrace the quantum-mechanical perspective, and to provide a geometric interpretation of physical observables such as twisted indices and twisted Hilbert spaces of supersymmetric ground states.

In this chapter, we begin our study by considering the Hilbert spaces of supersymmetric ground states of some selected $\mathcal{N} = 2$ theories. We emphasize once again that the twisted Hilbert spaces contain more information than the twisted indices, which correspond to graded traces thereof. Besides cancellations due to the grading, the Hilbert spaces are subject to an intricate dependence on supersymmetric deformation parameters such as real masses and background connections. In particular, checks of mirror symmetry can be performed over the full moduli spaces of deformation parameters.

The chapter is organized as follows. We start by explaining how the three-dimensional supermultiplets decompose into those of the supersymmetric quantum mechanics, and introduce an effective description that captures the twisted Hilbert spaces of supersymmetric ground states (section 3.1). Supersymmetric twisted Hilbert spaces are then computed explicitly in a number of examples (section 3.2). Finally, as some of these examples are related to each other via three-dimensional mirror symmetry, we check that the duality holds for the twisted Hilbert spaces (section 3.3).

3.1 Effective Quantum-mechanical Description

We noted in passing in the previous chapter that the supersymmetry algebra preserved by 3d $\mathcal{N} = 2$ Chern-Simons-matter theories on $\mathbb{R} \times \Sigma$ is the same as the algebra of a $\mathcal{N} = (0, 2)$ supersymmetric quantum mechanics (1.1)

$$\{Q, Q^\dagger\} = P_0 - m_f \cdot J_f \,. \tag{3.1}$$

In this pivotal section, we exploit this fact and introduce a description for the 3d theory in terms of an effective 1d quantum mechanics valued in field configurations fluctuating around the BPS locus (2.31).

3.1.1 Decomposing Supermultiplets

Chiral Multiplets

Consider first a 3d chiral multiplet Φ of R-charge r transforming in a unitary representation R_f of a flavour symmetry G_f. Recall that we can introduce two standard deformation parameters associated to the flavour symmetry that are compatible with the topological twist on Σ: a real mass parameters $m_f \in \mathfrak{t}_f$ and a background gauge connection A_f. A background connection induces a holomorphic bundle E_f on Σ, determined by its $(0,1)$-component.

Performing the topological twist on Σ, the three-dimensional chiral multiplet decomposes into the following supermultiplets in the supersymmetric quantum mechanics:

- A 1d $\mathcal{N} = (0,2)$ chiral multiplet (ϕ, ψ) valued in smooth sections of E_Φ,
- A 1d $\mathcal{N} = (0,2)$ Fermi multiplet (η, F) valued in smooth sections of $\Omega^{0,1}_\Sigma \otimes E_\Phi$.

Here

$$E_\Phi := K_\Sigma^{r/2} \otimes E_f \tag{3.2}$$

and K_Σ is the canonical bundle on Σ. Notice that this may require a choice of a spin structure on Σ, and different choices are related by tensoring E_f with a flat line bundle on Σ. The holomorphic bundle E_Φ inherits a Hermitian metric from that on the canonical bundle on Σ and the Hermitian metric on the vector space of the unitary representation R_f.

The action for the chiral multiplet was presented in (2.9). In addition to the standard kinetic term contributions in the supersymmetric quantum mechanics from these supermultiplets, this action contains terms that come from E-type superpotentials for the Fermi multiplet,

$$E = \bar{\partial}_{A_f} \phi \,, \tag{3.3}$$

where $\bar{\partial}_{A_f}$ denotes the holomorphic structure on E_Φ. Note that the E-term superpotential transforms in the same way as the Fermi multiplet (η, F), as required for supersymmetry. These terms are

$$\begin{aligned} ||E||^2 &= \int_\Sigma \partial_{A_f} \bar{\phi} \wedge * \bar{\partial}_{A_f} \phi \\ \mathrm{Re}\, \bar{\eta} \frac{\partial E}{\partial \phi} \psi &= \mathrm{Re} \int_\Sigma \bar{\eta} \wedge \bar{\partial}_{A_f} \psi \,, \end{aligned} \tag{3.4}$$

where again contraction using the Hermitian metric on E_Φ is understood. The E-type superpotential provides kinetic terms with derivatives along Σ, which correspond to the equation

$$\bar{\partial}_{A_f} \phi = 0 \tag{3.5}$$

in (2.31) and its analogue for the fermionic partners. From the point of view of the supercharge, the E-term contribution to $Q^\dagger$ is proportional to

$$\int_\Sigma \bar{\eta} \wedge E = \int_\Sigma \bar{\eta} \wedge \bar{\partial}_{A_f} \phi \,, \tag{3.6}$$

where contraction using the Hermitian metric on E_Φ is understood. The choice of holomorphic structure $\bar{\partial}_{A_f}$ on E_Φ is therefore a B-type parameter in the supersymmetric quantum mechanics.

Finally, real mass parameters generate the real superpotential

$$h_f = \int_\Sigma *(m_f \cdot \mu_f) \,, \tag{3.7}$$

where $\mu_f \in \mathfrak{t}_f^*$ is the moment map for the action of G_f on the unitary representation R_f. In the presence of real mass parameters, the supercharges of the supersymmetric quantum mechanics are formally conjugated by the exponential factor e^{h_f}, as in section 1.1.2. This is an A-type deformation of the supersymmetric quantum mechanics.

3d Superpotentials

The above model can be deformed by a 3d superpotential $W(\Phi)$ preserving the R-symmetry used to perform the topological twist on Σ. A superpotential will place restrictions on the flavour symmetry G_f and therefore on the allowed background vector bundle E_f and real mass parameters m_f.

The superpotential must have R-charge $+2$ and will therefore transform as a section of the canonical bundle on the curve Σ in the twisted theory. From the point of view of the quantum mechanics, it is equivalent to the introduction of a J-term superpotential

$$\mathcal{J}_W = \frac{\delta W}{\delta \Phi} \,, \tag{3.8}$$

which transforms as a smooth section of $\Omega^{1,0}_\Sigma \otimes \bar{E}_\Phi$. This generates an additional contribution to the supercharge $Q^\dagger$ proportional to

$$\int_\Sigma \mathcal{J}_W \wedge \eta = \int_\Sigma \frac{\delta W}{\delta \Phi} \wedge \eta \,, \tag{3.9}$$

where contraction using the Hermitian metric on E_Φ is understood. The superpotential has the effect of enforcing

$$\frac{\delta W}{\delta \Phi} = 0 \,, \tag{3.10}$$

with the higher fermionic corrections. The complex parameters in the 3d superpotential therefore become B-type deformation parameters in the supersymmetric quantum mechanics.

Gauge Theory

Now consider a 3d $\mathcal{N}=2$ vectormultiplet for a compact group G. Performing the topological twist on Σ, the 3d $\mathcal{N}=2$ vectormultiplet decomposes into the following multiplets in the supersymmetric quantum mechanics:

- A 1d $\mathcal{N}=(0,2)$ vectormultiplet consisting of A_0, σ, and auxiliary field $D^{1\mathrm{d}} := D + *F$ where $*$ is the Hodge operator on Σ.
- A 1d $\mathcal{N}=(0,2)$ chiral multiplet valued with complex scalar component given by the $(0,1)$-form component of the gauge connection on Σ.

In the supersymmetric quantum mechanics, the vectormultiplet is associated to the infinite-dimensional group $\mathcal{G}$ of gauge transformations $g : \Sigma \to G$.

Let us consider a dynamical vectormultiplet with gauge symmetry G together with chiral multiplets Φ of R-charge r transforming in a unitary representation R of G. We suppose the chiral multiplets transform in a unitary representation R_f of a residual flavour symmetry G_f. We can again introduce deformation parameters m_f and E_f associated to the flavour symmetry G_f. In the supersymmetric quantum mechanics, the chiral multiplets decompose as above with

$$E_\Phi = K_\Sigma^{r/2} \otimes E \otimes E_f \tag{3.11}$$

where E, E_f are the holomorphic vector bundles associated to the representations R, R_f. This follows from the fact that the parameters m_f, E_f can be understood as vacuum expectation values for a background vectormultiplet for the flavour symmetry G_f. Preserving both supercharges of the supersymmetric quantum mechanics requires $D_f^{1\mathrm{d}} = 0$ and therefore we should also turn on a compensating auxiliary field $iD_f = - * F_f$ given by the curvature of E_f.

If G contains Abelian factors, we can turn on a 3d real FI parameter ζ as well as a 1d FI parameter τ. As previously noted, the former can be understood as a background expectation value $m_T = \zeta$ for the scalar component of a topological flavour symmetry $U(1)_T$. We also introduce a background holomorphic line bundle L_T.

Finally, for each Abelian or simple factor in G, we can introduce a supersymmetric Chern-Simons term. For example, the contribution to the supersymmetric quantum mechanics from a $G = U(1)$ Chern-Simons term at level k is

$$\frac{k}{2\pi}\int_\Sigma (\sigma + iA_0)F - \frac{k}{2\pi}\int_\Sigma *\sigma D^{1d} \,. \tag{3.12}$$

In general, we can introduce various mixed Chern-Simons contributions between gauge, flavour and R-symmetries.

3.1.2 Effective Quantum Mechanics

In the previous section, we rephrased the topological twist of a 3d $\mathcal{N}=2$ theory on $\mathbb{R}\times\Sigma$ as an infinite dimensional supersymmetric quantum mechanics on $\mathbb{R}$. We now introduce an effective supersymmetric quantum mechanics that captures the space of supersymmetric ground states. We focus exclusively on regimes where there are only 'Higgs branch' vacua in the sense of [13]. In this case, the effective supersymmetric quantum mechanics is a sigma model onto the moduli space of vortex-like configurations (2.31) that minimize the effective Euclidean action, in line with the philosophy of [73, 31, 74].

Chiral Multiplets

Let us return to the model with chiral multiplets transforming a unitary representation R_f of a flavour symmetry G_f from section 3.1.1. The Euclidean action is minimized by time-independent configurations that minimize the potential of the supersymmetric quantum mechanics

$$U=\int_\Sigma \|\bar\partial_{A_f}\phi\|^2 := \int_\Sigma \partial_{A_f}\bar\phi\wedge *\bar\partial_{A_f}\phi\,, \tag{3.13}$$

which is induced by the E-type superpotential term in equation (3.4). Such configurations therefore satisfy $\bar A_f\phi=0$, as in (2.31). In addition, time-independent solutions of the Fermi multiplet equations of motion obey $\bar\partial_{A_f}\bar\eta=0$.

We therefore consider an effective supersymmetric quantum mechanics consisting of a finite number of fluctuations:

- Chiral multiplets (ϕ,ψ) valued in $H^0(E_\Phi)$
- Fermi multiplets (η,F) valued in $H^1(E_\Phi)$.

Let us define the number of chiral and Fermi multiplet fluctuations by

$$n_C := h^0(E_\Phi) \qquad n_F := h^1(E_\Phi)\,. \tag{3.14}$$

The difference is determined by the Riemann-Roch theorem,

$$n_C - n_F = c_1(E_\Phi) - \mathrm{rk}(E_\Phi)(1-g)\,, \tag{3.15}$$

and depends on the background holomorphic vector bundle only through $c_1(E_\Phi)$ and $\mathrm{rk}(E_\Phi)$. In contrast, the individual numbers of fluctuations may depend on the particular choice of holomorphic vector bundle with these parameters.

Introducing mass parameters $m_f\in\mathfrak{t}_f$, we can quantize the chiral and Fermi multiplet fluctuations as in section 1.1.3. We obtain a supersymmetric twisted Hilbert space $\mathcal{H}_\alpha$ in each chamber $\mathfrak{c}_\alpha\subset\mathfrak{t}_f$ where all of the fluctuations are massive. Furthermore, each $\mathcal{H}_\alpha$ will jump as the holomorphic vector bundle E_f is varied whenever the numbers n_C and n_F change.

A supersymmetric Chern-Simons term for the flavour symmetry descends to a Chern-Simons term in the supersymmetric quantum mechanics and shifts the flavour grading of supersymmetric ground states. For example, for $G_f = U(1)$ the contribution (3.12) to the Lagrangian of the supersymmetric quantum mechanics is

$$(m_f + iA_{f,0})k\mathfrak{m}_f \tag{3.16}$$

where $\mathfrak{m}_f = c_1(E_f)$ is the flavour flux on Σ. This shifts the flavour conserved charge by $J_f \to J_f + k\mathfrak{m}_f$. This can be further supplemented by a superpotential $W(\Phi)$. In this case, we will assume that we can first quantize the fluctuations as above and then implement the J-term superpotential (3.8) arising from $W(\Phi)$ in the supersymmetric quantum mechanics of these fluctuations.

We explicitly construct the supersymmetric twisted Hilbert spaces $\mathcal{H}_\alpha$ for a single chiral multiplet with a supersymmetric Chern-Simons term for the $G_f = U(1)$ flavour symmetry in section 3.2.1. We then proceed to examples involving superpotentials in sections 3.2.2 and 3.2.3.

Abelian Gauge Theories

We now return to supersymmetric gauge theory. At this point, we specialize to $G = U(1)$ with a supersymmetric Chern-Simons term at level k. We will introduce parameters ζ, L_T associated to the $U(1)_T$ topological symmetry and parameters m_f, E_f associated to the flavour symmetry G_f acting on chiral multiplets. We also introduce a 1d FI parameter τ.

We consider configurations minimizing the action. We first set $m_f = 0$ and then later turn the mass parameters back on in the effective supersymmetric sigma model description. Configurations minimizing the Euclidean effective action are solutions to

$$\begin{gathered} \frac{1}{e^2} * F_A + \mu = \xi_{eff}(\sigma) \\ d_A\sigma = 0 \qquad \bar{\partial}_A\phi = 0 \qquad \sigma \cdot \phi = 0 \end{gathered} \tag{3.17}$$

modulo gauge transformations $g : \Sigma \to G$. Here

$$\xi_{eff} := \tau + \sigma k_{eff} \tag{3.18}$$

is a combination of τ and the 1-loop quantum corrected parameter k_{eff}, which is piecewise constant functions of σ. Finally, μ is the moment map for the action of $G = U(1)$ on the unitary representation R. In particular, the vectormultiplet scalar σ is real and therefore must be a constant on Σ.

The equations (3.17) admit a variety of different solutions depending on the supersymmetric Chern-Simons level k, the matter content and the value of τ, similarly to the vacua of the theory. In this thesis, we focus exclusively on regimes with 'Higgs branch' solutions, characterized by $\sigma = 0$, a non-vanishing expectation value for ϕ and the gauge symmetry completely broken. We can then focus our attention on solutions of the generalized vortex equations

$$\frac{1}{e^2} * F + \mu = \tau \qquad \bar{\partial}_A\phi = 0\,, \tag{3.19}$$

modulo gauge transformations. We note that these equations play an important role in the A-twist of 2d $\mathcal{N} = (2,2)$ gauge theories [60, 75].

The moduli space of solutions to the generalized vortex equations (3.19) is a disjoint union of components

$$\mathfrak{M} = \bigcup_{\mathfrak{m}} \mathfrak{M}_{\mathfrak{m}} \tag{3.20}$$

labelled by the flux

$$\mathfrak{m} = \frac{1}{2\pi} \int_{\Sigma} F \in \mathbb{Z} \tag{3.21}$$

and each component $\mathfrak{M}_{\mathfrak{m}}$ is finite-dimensional. Since the dependence on the gauge coupling is exact, we expect the spectrum of supersymmetric ground states of $U(1)_T$ topological charge $\mathfrak{m}$ to be captured by an effective supersymmetric quantum mechanics that is a sigma model with target space $\mathfrak{M}_{\mathfrak{m}}$.

The twisted Hilbert space of supersymmetric ground states has the form

$$\mathcal{H} = \bigoplus_{\mathfrak{m} \in \mathbb{Z}} q^{\mathfrak{m}} \mathcal{H}_{\mathfrak{m}} \,, \tag{3.22}$$

where we have introduced a parameter $q \in \mathbb{C}^*$ to measure the charge under the topological symmetry $U(1)_T$. In going over to the twisted index on $S^1 \times \Sigma$ with a circle of radius β, this parameter corresponds to the exponential of ζ as in (2.18), with ζ taken in the same chamber as τ.

The flavour symmetry G_f descends to an isometry of $\mathfrak{M}_{\mathfrak{m}}$ and becomes a flavour symmetry in the effective supersymmetric quantum mechanics and each $\mathcal{H}_{\mathfrak{m}}$ transforms as a virtual representation of G_f. Introducing real mass parameters generates a superpotential in the effective supersymmetric sigma model equal to the moment map h_f for the $U(1)_{m_f} \subset G_f$ isometry generated by m_f. This has the effect of formally conjugating the supercharges by the exponential of h_f.

From the general structure of supersymmetric quantum mechanics summarized in section 1.1.2, we propose that the supersymmetric ground states with topological charge $\mathfrak{m}$ are computed by the L^2-cohomology

$$\mathcal{H}_{\mathfrak{m}} = H^{0,\bullet}_{\bar{\partial}_{m_f}+\delta}(\mathfrak{M}_{\mathfrak{m}}, \mathcal{F}_{\mathfrak{m}}) \tag{3.23}$$

where $\mathcal{F}_{\mathfrak{m}}$ is an appropriate $\mathbb{Z}_2$ graded Hermitian vector bundle on $\mathfrak{M}_{\mathfrak{m}}$. The vector bundle $\mathcal{F}_{\mathfrak{m}}$ receives contributions from the following sources:

- A universal contribution of the square root of the canonical bundle $\sqrt{K_{\mathfrak{M}_{\mathfrak{m}}}}$ from quantizing the fermions in chiral multiplets parametrizing $\mathfrak{M}_{\mathfrak{m}}$.

- There are contributions from Fermi multiplet zero modes, which are solutions to $\bar{\partial}_A \eta = 0$ in the background of a solution to the vortex equations. They transform as sections of a holomorphic vector bundle $\mathfrak{f}$ over the moduli space $\mathfrak{M}_{\mathfrak{m}}$. There is then a contribution to $\mathcal{F}_{\mathfrak{m}}$ from quantizing these fluctuations,

$$\bigoplus_i (-1)^i \frac{\wedge^i \mathfrak{f}^*}{\sqrt{\mathrm{Det}(\mathfrak{f}^*)}} \,. \tag{3.24}$$

- A supersymmetric Chern-Simons term at level k for gauge symmetry $G = U(1)$ contributes an additional factor $K_{\mathfrak{M}_{\mathfrak{m}}}^{-k}$ [76].
- A background holomorphic line bundle L_T for the topological flavour symmetry on Σ induces a line bundle $\widetilde{L}_T$ on the moduli space $\mathfrak{M}_{\mathfrak{m}}$. This corresponds to an electric impurity in the language of [77]. Alternatively, the holomorphic line bundle $\widetilde{L}_T$ can be constructed mathematically from the universal line bundle on $\Sigma \times \mathfrak{M}_{\mathfrak{m}}$ and the Deligne pairing [1].

Finally, as in the case of chiral multiplets, a 3d superpotential $W(\Phi)$ generates an additional contribution δ to the differential.

Let us briefly comment on the dependence on the mass parameters m_f, mirroring the discussion in supersymmetric quantum mechanics of section 1.1.2. If the moduli space $\mathfrak{M}_{\mathfrak{m}}$ is non-compact, the cohomology (3.23) will yield a different result in each chamber $\mathfrak{c}_\alpha \subset \mathfrak{t}_f$ separated by walls where there are massless non-compact fluctuations. On the other hand, if $\mathfrak{M}_{\mathfrak{m}}$ is compact we may set $m_f = 0$ and identify the supersymmetric twisted Hilbert space with the hypercohomology

$$\mathcal{H}_{\mathfrak{m}} = H^{0,\bullet}_{\bar{\partial}+\delta}(\mathfrak{M}_{\mathfrak{m}}, \mathcal{F}_{\mathfrak{m}})\,. \tag{3.25}$$

Finally, the background holomorphic vector bundle E_f is a B-type deformation parameter in the effective supersymmetric quantum mechanics and therefore $\mathfrak{M}_{\mathfrak{m}}$, $\mathcal{F}_{\mathfrak{m}}$ and $\mathcal{H}_{\mathfrak{m}}$ may jump as this is deformed.

3.1.3 Vortex Moduli Spaces

The structure of the moduli spaces $\mathfrak{M}_{\mathfrak{m}}$ depends intricately on the gauge and R-charges of the chiral multiplets, the genus g of Σ, and the background holomorphic vector bundle E_f. In this section, we nevertheless attempt to make some general comments on their structure that will be used in examples in section 3.2. We also offer more systematic details in appendix A.3.

First, the vortex moduli space can be understood as an infinite-dimensional Kähler quotient. The group of gauge transformations $g : \Sigma \to G$ acts on the infinite-dimensional flat Kähler manifold $\mathfrak{P}_{\mathfrak{m}}$ parametrized by pairs $(\bar{\partial}_A, \phi)$ with moment map

$$\mu_{\mathcal{G}} := \frac{1}{e^2} * F + \mu \tag{3.26}$$

Imposing the complex equation $\bar{\partial}_A \phi = 0$ defines a Kähler submanifold $\mathfrak{V}_{\mathfrak{m}} \subset \mathfrak{P}_{\mathfrak{m}}$ with moment map given by the restriction of the above. The moduli space of solutions to the generalized vortex equations is then the symplectic quotient $\mathfrak{M}_{\mathfrak{m}} = \mu_{\mathcal{G}}^{-1}(\tau)/\mathcal{G}$.

[1] We provide some justification of these claims in the appendices. In appendix A.3, we summarize the construction of the holomorphic line bundle $\widetilde{L}_T$ using the universal line bundle on $\Sigma \times \mathfrak{M}_{\mathfrak{m}}$ and Deligne pairing. In appendix A.5, we show that the curvature of this line bundle agrees with the 'dirty connection' of the electric impurity introduced in [77].

It will be useful to introduce an algebraic description of the moduli spaces $\mathfrak{M}_{\mathfrak{m}}$. This is obtained by imposing the complex equation $\bar{\partial}_A\phi = 0$ as above but replacing the real moment map equation by a stability condition on the pair $(\bar{\partial}_A, \phi)$ and dividing by complexified gauge transformations $g : \Sigma \to G_{\mathbb{C}}$. A point in $\mathfrak{M}_{\mathfrak{m}}$ is then specified by the following data:

- A holomorphic line bundle E of degree $\mathfrak{m}$.
- A holomorphic section ϕ of E_Φ subject to a stability condition.

Some details concerning stability conditions can be found in appendix A.3. In this chapter, we will only specify them in examples in section 3.2.

Finally, the algebraic description makes it clear that there is a holomorphic map

$$j : \mathfrak{M}_{\mathfrak{m}} \longrightarrow J_\Sigma \tag{3.27}$$

to the Picard variety parametrizing the holomorphic line bundle E, whose fibers are toric varieties. This holomorphic map is not generally surjective. However, when $\mathfrak{m}$ is sufficiently large it become a holomorphic fibration whose structure can be useful for computing the cohomology groups (3.23). We will study an explicit example of this phenomenon in section 3.2.5. If $g = 0$ then J_Σ is a point and we recover the description of the vortex moduli space as a toric variety [75].

3.2 Examples

3.2.1 1 Chiral Multiplet

We first consider a single chiral multiplet Φ of integer R-charge r. We introduce a real mass parameter m_f and a background holomorphic line bundle L_f for the $U(1)_f$ flavour symmetry. For later convenience, we define

$$L_\Phi := K_\Sigma^{r/2} \otimes L_f\,, \tag{3.28}$$

where if necessary (as anticipated) we assume a choice of spin structure on Σ. We will return on this sublety shortly.

The effective supersymmetric quantum mechanics has the following supermultiplets of charge $+1$ under $U(1)_f$:

- Chiral multiplets (ϕ, ψ) valued in $H^0(L_\Phi)$.
- Fermi multiplets (η, F) valued in $H^1(L_\Phi)$.

Let us denote the number of chiral and Fermi multiplet fluctuations by $n_C = h^0(L_\Phi)$ and $n_F = h^1(L_\Phi)$ respectively. The difference is fixed by the Riemann-Roch theorem,

$$\begin{aligned} n_C - n_F &= \mathfrak{m}_\Phi - g + 1 \\ &= \mathfrak{m}_f + (r-1)(g-1). \end{aligned} \tag{3.29}$$

where $\mathfrak{m}_f$, $\mathfrak{m}_\Phi$ are the degrees of L_f, L_Φ respectively. For extreme values of the degree, the numbers of fluctuations are fixed

$$\begin{aligned} \mathfrak{m}_\Phi > 2g-2 \quad &\Rightarrow \quad \begin{cases} n_C = \mathfrak{m}_\Phi - g + 1 \\ n_F = 0 \end{cases} \\ \mathfrak{m}_\Phi < 0 \quad &\Rightarrow \quad \begin{cases} n_C = 0 \\ n_F = -\mathfrak{m}_\Phi + g - 1 \,. \end{cases} \end{aligned} \tag{3.30}$$

However, in the intermediate region $0 \leq \mathfrak{m}_\Phi \leq 2g-2$, the individual numbers of fluctuations depend on the choice of background line bundle L_Φ and will jump at loci in the Jacobian parametrizing L_Φ.

This dependence can be expressed in a precise manner. Let $\mathrm{Pic}^{\mathfrak{m}_f}(\Sigma)$ be the Picard variety parametrizing isomorphism classes of holomorphic line bundles of degree $\mathfrak{m}_f$. We can construct a Poincaré line bundle

$$\mathcal{P} \to \mathrm{Pic}^{\mathfrak{m}_f}(\Sigma) \times \Sigma \tag{3.31}$$

with the property that restriction to a point on $\mathrm{Pic}^{\mathfrak{m}_f}(\Sigma)$ is a holomorphic line bundle in the isomorphism class corresponding to that point. We can then define a perfect complex of sheaves on $\mathrm{Pic}^{\mathfrak{m}_f}(\Sigma)$ by means of the derived pushforward

$$\mathcal{V}^\bullet = R^\bullet \pi_*(\mathcal{P}) \,, \tag{3.32}$$

where $\pi : \mathrm{Pic}^{\mathfrak{m}}(\Sigma) \times \Sigma \to \mathrm{Pic}^{\mathfrak{m}}(\Sigma)$ is the projection. These sheaves have the property that their stalks are the complex vector spaces $\mathcal{V}^\bullet$. They are not generally locally free, corresponding to the fact that the dimension of these vector spaces may jump as the holomorphic line bundle is deformed. We can then construct the space of supersymmetric ground states as a complex of sheaves

$$\sqrt{\det \mathcal{V}^\bullet} \otimes \mathrm{Sym}(\mathcal{V}^\bullet) \tag{3.33}$$

on the Picard variety.

There are, however, subtle issues with this construction. The first issue is that the choice of Poincaré line bundle $\mathcal{P}$ is not unique due to $\mathbb{C}^*$ automorphisms of holomorphic line bundles on Σ. In particular, there is an ambiguity $\mathcal{P} \to \mathcal{P} \otimes \pi^*(\mathcal{R})$ where $\mathcal{R}$ is any holomorphic line bundle on $\mathrm{Pic}^{\mathfrak{m}_f}(\Sigma)$. This implies that the Poincaré line bundles can be seen as choices of trivializations of a gerbe on $\mathrm{Pic}^{\mathfrak{m}_f}(\Sigma)$, as explained for example in [32]. Since the supersymmetric ground states are charged under $U(1)_f$, and so transform under the $\mathbb{C}^*$ automorphisms of L_f, the spaces of supersymmetric ground states transform as complexes of sheaves over this gerbe.

The second issue is that, as already mentioned, we may be forced to make a choice of spin structure on Σ. This would correspond to the choice of a base point on the Picard gerbe. Since any two choices differ by a flat bundle, we may change our choice of base point by tensoring the Poincaré line bundle by the pull-back of a flat line bundle on Σ. This transformation is a simple example of a Fourier-Mukai transform.

The structure of the Hilbert space not only depends on the number of fluctuations, but also on the choice of vacuum and on the chambers for the real masses. We will choose the Fock vacuum of Fermion number zero to be annihilated by the fermions ψ and $\bar{\eta}$. This Fock vacuum then has charge $+\frac{1}{2}(n_C - n_F)$ under $U(1)_f$. The chambers to consider are two: $\mathfrak{c}_+ = \{m_f > 0\}$ and $\mathfrak{c}_- = \{m_f < 0\}$. Quantizing the fluctuations as in section 1.1.3, we find

$$\begin{aligned} \mathcal{H}_+ &= x^{\frac{n_C - n_F}{2}} \bigoplus_{j=0}^{\infty} x^j \left(\bigoplus_{p+q=j} S^p \mathbb{C}^{n_C} \otimes \wedge^q \mathbb{C}^{n_f} \right) \\ \mathcal{H}_- &= (-1)^{n_C - n_F} x^{-\frac{n_C - n_F}{2}} \bigoplus_{j=0}^{\infty} x^{-j} \left(\bigoplus_{p+q=j} S^p \mathbb{C}^{n_C} \otimes \wedge^q \mathbb{C}^{n_f} \right) . \end{aligned} \tag{3.34}$$

This reproduces the expansions of the supersymmetric twisted index

$$I = \left(\frac{x^{1/2}}{1-x} \right)^{n_C - n_F} \tag{3.35}$$

in the regions $|x| < 1$ and $|x| > 1$ respectively. Note that the supersymmetric twisted index depends only on the difference $n_C - n_F$ and therefore on the degree $\mathfrak{m}_\Phi$. On the other hand, $\mathcal{H}_\pm$ will jump as the background line bundle L_Φ is varied over the Picard gerbe.

Let us now consider the special case $\Sigma = \mathbb{CP}^1$. The background holomorphic line bundle is now fixed $L_\Phi = \mathcal{O}(\mathfrak{m}_\Phi)$ with $\mathfrak{m}_\Phi = \mathfrak{m}_f - r$ and there are either chiral or Fermi multiplet fluctuations,

$$\begin{aligned} \mathfrak{m}_\Phi \geq 0 \quad &\Rightarrow \quad \begin{cases} n_C = \mathfrak{m}_\Phi + 1 \\ n_F = 0 \end{cases} \\ \mathfrak{m}_\Phi < 0 \quad &\Rightarrow \quad \begin{cases} n_C = 0 \\ n_F = |\mathfrak{m}_\Phi| - 1 \, . \end{cases} \end{aligned} \tag{3.36}$$

In this case, the effective supersymmetric quantum mechanics has an additional flavour symmetry $U(1)_\epsilon$ transforming the homogeneous coordinates of $\mathbb{CP}^1$ by

$$(z, w) \to (\xi^{1/2} z, \xi^{-1/2} w) \, . \tag{3.37}$$

This induces an action on the chiral and Fermi multiplet fluctuations. For example, for $\mathfrak{m}_\Phi \geq 0$ the holomorphic sections $z^{\mathfrak{m}_\Phi - j} w^j$ transform with weight $\xi^{\frac{\mathfrak{m}_\Phi}{2} - j}$ for $j = 0, \ldots, \mathfrak{m}_\Phi$. The vector space of chiral multiplet fluctuations therefore decomposes in the following way as a representation of $U(1)_\epsilon$,

$$\xi^\rho \mathbb{C}^{\mathfrak{m}_\Phi + 1} = \xi^{\frac{\mathfrak{m}_\Phi}{2}} \mathbb{C} \oplus \xi^{\frac{\mathfrak{m}_\Phi}{2} - 1} \mathbb{C} \oplus \cdots \oplus \xi^{-\frac{\mathfrak{m}_\Phi}{2}} \mathbb{C} \, , \tag{3.38}$$

where

$$\rho = \left(\frac{\mathfrak{m}_\Phi}{2}, \frac{\mathfrak{m}_\Phi - 1}{2}, \ldots, -\frac{\mathfrak{m}_\Phi}{2} \right) \tag{3.39}$$

is the appropriate Weyl vector. Similar arguments apply for the Fermi multiplet fluctuations when $\mathfrak{m}_\Phi < 0$.

In the region $m_f > 0$, we find

$$\mathcal{H}_+ = \begin{cases} x^{\frac{\mathfrak{m}_\Phi+1}{2}} \bigoplus_{j=0}^{\infty} x^j \, S^j(\xi^\rho \mathbb{C}^{\mathfrak{m}_\Phi+1}) & \mathfrak{m}_\Phi \geq 0 \\ x^{\frac{\mathfrak{m}_\Phi+1}{2}} \bigoplus_{j=0}^{|\mathfrak{m}_\Phi|-1} x^j \wedge^j (\xi^\rho \mathbb{C}^{|\mathfrak{m}_\Phi|-1}) & \mathfrak{m}_\Phi < 0\,. \end{cases} \tag{3.40}$$

The supersymmetric index is consequently

$$I = \begin{cases} \prod_{j=0}^{\mathfrak{m}_\Phi} \frac{(\xi^{\frac{\mathfrak{m}_\Phi}{2}-j}x)^{1/2}}{1-\xi^{\frac{\mathfrak{m}_\Phi}{2}-j}x} & \mathfrak{m}_\Phi \geq 0 \\ \prod_{j=0}^{|\mathfrak{m}_\Phi|-2} \frac{1-\xi^{\frac{|\mathfrak{m}_\Phi|}{2}-1-j}x}{(\xi^{\frac{|\mathfrak{m}_\Phi|}{2}-1-j}x)^{1/2}} & \mathfrak{m}_\Phi < 0\,, \end{cases} \tag{3.41}$$

which can be combined into the uniform expression

$$\frac{x^{\frac{\mathfrak{m}_\Phi}{2}}}{(\xi^{-\frac{\mathfrak{m}_\Phi}{2}}x, \xi)_{\mathfrak{m}_\Phi+1}}\,, \tag{3.42}$$

where $(a,q)_n := \prod_{k=0}^{n-1}(1-aq^k)$ is the finite Q-Pochhammer symbol. This is in agreement with the 1-loop determinant from supersymmetric localization [23], which is a refinement of the one considered in chapter 2.

3.2.2 XY Model

We now consider a pair of chiral multiplets Φ_1, Φ_2 with a quadratic superpotential $W = \Phi_1\Phi_2$ and complementary R-charges $r_1 + r_2 = 2$. There is a $U(1)_f$ flavor symmetry under which Φ_1 and Φ_2 have charges $Q_1 = +1$ and $Q_2 = -1$ respectively and we introduce a corresponding real mass parameter by m_f. We will focus here on $\mathfrak{c}_+ = \{m_f > 0\}$ since the opposite chamber can be obtained by interchanging Φ_1 and Φ_2.

We fix a background line bundle L_f of degree $\mathfrak{m}_f$ for the $U(1)_f$ flavour symmetry and define $L_{\Phi_j} := K_\Sigma^{r_j/2} \otimes L_f^{Q_j}$. The chiral multiplet Φ_j contributes $n_{j,c} = h^0(L_{\Phi_j})$ chiral multiplets and $n_{j,f} = h^1(L_{\Phi_j})$ Fermi multiplets to the supersymmetric quantum mechanics. Combining Serre duality with $r_1 + r_2 = 2$, we have

$$n_1 := n_{1,c} = n_{2,f} \qquad n_2 := n_{2,c} = n_{1,f} \tag{3.43}$$

and furthermore from the Riemann-Roch theorem

$$\begin{aligned} n_1 - n_2 &= (r_1 - 1)(g-1) + \mathfrak{m}_f \\ &= -(r_2 - 1)(g-1) + \mathfrak{m}_f\,. \end{aligned} \tag{3.44}$$

Let us denote the chiral multiplet fluctuations by $\phi_{1,a}$, $\phi_{2,a'}$ and the Fermi multiplet fluctuations by $\eta_{1,a'}$, $\eta_{2,a}$ where $a = 1, \ldots, n_1$ and $a' = 1, \ldots, n_2$. The superpotential $W = \Phi_1\Phi_2$ induces the following J-type superpotentials

$$J_{\eta_{1,a'}} = \phi_{2,a'} \qquad J_{\eta_{2,a}} = \phi_{1,a}\,. \tag{3.45}$$

for the Fermi multiplets in the supersymmetric quantum mechanics.

In quantizing these fluctuations, we choose the Fock vacuum with Fermion number zero to be annihilated by the fermions $\psi_{1,a}$, $\psi_{2,a'}$ and $\eta_{1,a'}$, $\eta_{2,a}$. This Fock vacuum is uncharged under $U(1)_f$. The supersymmetric twisted Hilbert space is then the cohomology of the supercharge

$$\begin{aligned}(Q^\dagger)^{(1)} &= \sum_{a'=1}^{n_2} J_{\eta_{1,a'}}\eta_{1,a'} + \sum_{a=1}^{n_1} J_{\eta_{2,a}}\eta_{2,a} \\ &= \sum_{a'=1}^{n_2} \phi_{2,a'}\eta_{1,a'} + \sum_{a=1}^{n_1} \phi_{1,a}\eta_{2,a}\end{aligned} \tag{3.46}$$

acting on wavefunctions

$$\Big[\prod_{a=1}^{n_1} \phi_{1,a}^{k_a} \prod_{a'=1}^{n_2} \bar{\phi}_{2,a'}^{k_{a'}}\Big] \prod_{a'=1}^{n_2} \bar{\eta}_{1,a'}^{s_{a'}} \prod_{a=1}^{n_1} \bar{\eta}_{2,a}^{s_a}\,, \tag{3.47}$$

where $k_a, k_{a'} \in \mathbb{Z}_{\geq 0}$ and $s_1, s_2 \in \{0,1\}$ and we use the notation for representatives of supersymmetric ground states introduced in section 1.1.3. The result is essentially $n_1 + n_2$ copies of the supersymmetric quantum mechanics considered there, consisting of a chiral multiplet ϕ and a Fermi multiplet η, with superpotential $J = \phi$. Here, n_1 chiral multiplets have positive real mass and n_2 chiral multiplets have negative real mass.

There is a single supersymmetric ground state in the cohomology of $(Q^\dagger)^{(1)}$ with $k_a = k_{a'} = 0$, $s_a = 0$ and $s_{a'} = 1$ for all $a = 1, \ldots, n_1$ and $a' = 1, \ldots, n_2$. This corresponds to the fermions $\prod_{a'} \bar{\psi}_{2,a'} \prod_a \bar{\eta}_{1,a'}$ acting on our choice of Fock vacuum. The supersymmetric ground state has vanishing $U(1)_f$ charge and Fermion number zero, and so we have $\mathcal{H}_+ = \mathbb{C}$.

The supersymmetric index computed using localization is a product of contributions from the chiral multiplets Φ_1 and Φ_2,

$$I = \left(\frac{x^{1/2}}{1-x}\right)^{(r_1-1)(g-1)+\mathfrak{m}} \left(\frac{x^{-1/2}}{1-x^{-1}}\right)^{(r_2-1)(g-1)-\mathfrak{m}} = (-1)^{(r_2-1)(g-1)-\mathfrak{m}}\,, \tag{3.48}$$

which is independent of the three-dimensional superpotential $W = \Phi_1\Phi_2$. This agrees with the single supersymmetric ground state up to a sign related to our choice of Fermion number for the Fock vacuum.

3.2.3 XYZ Model

Now consider three chiral multiplets Φ_1,Φ_2,Φ_3 with cubic superpotential $W = \Phi_1\Phi_2\Phi_3$ and R-charges such that $\sum_{j=1}^3 r_j = 2$. The flavour symmetry is

$$G_f = (\prod_{j=1}^{3} U(1)_j)/U(1)_D \tag{3.49}$$

where $U(1)_D$ is the diagonal combination.

It is convenient to represent the charges of the chiral multiplets Φ_i under $U(1)_j$ by a flavour charge matrix Q^i_j satisfying the following constraints:

- $\sum_{i=1}^{3} Q_j^i = 0$ ensures that the superpotential has zero charge under $U(1)_j$;
- $\sum_{j=1}^{3} Q_j^i = 0$ ensures no fields are charged under the diagonal combination $U(1)_D$;
- $\mathrm{rk}(Q) = 2$ ensures the full flavour symmetry is manifest.

A simple choice of charge matrix is

$$Q = \begin{pmatrix} 1 & \text{-1} & 0 \\ 0 & 1 & \text{-1} \\ \text{-1} & 0 & 1 \end{pmatrix} . \tag{3.50}$$

We can turn on corresponding real mass parameters m_j such that $\sum_{j=1}^{3} m_j = 0$.

Furthermore, we can introduce background line bundles L_j for each $U(1)_j$ such that

$$L_1 \otimes L_2 \otimes L_3 = \mathcal{O}_\Sigma . \tag{3.51}$$

and define $L_{\Phi_i} = K_\Sigma^{r_j/2} \otimes \prod_j L_j^{Q_j^i}$. The chiral multiplet Φ_i then leads to $n_{i,c} = h^0(L_{\Phi_i})$ chiral multiplet fluctuations and $n_{i,f} = h^1(L_{\Phi_i})$ Fermi multiplet fluctuations. From the Riemann-Roch theorem, these numbers satisfy

$$n_{i,c} - n_{i,f} = (r_i - 1)(g-1) + \sum_j Q_j^i \mathfrak{m}_j \tag{3.52}$$

and therefore

$$\sum_i (n_{i,c} - n_{i,f}) = 1 - g , \tag{3.53}$$

where $\mathfrak{m}_j$ denotes the degree of L_j.

Combining Serre duality and equation (3.51) we find

$$H^0(\Sigma, L_{\Phi_1} \otimes L_{\Phi_2}) \cong H^1(\Sigma, L_{\Phi_3})^* . \tag{3.54}$$

In particular, we have $n_{1,c}+n_{2,c} = n_{3,f}$ together with cyclic permutations of this relation. The superpotential then induces J-type superpotentials in the effective supersymmetric quantum mechanics associated to the natural maps

$$H^0(\Sigma, L_{\Phi_1}) \otimes H^0(\Sigma, L_{\Phi_2}) \to H^0(\Sigma, L_{\Phi_1} \otimes L_{\Phi_2}) \cong H^1(\Sigma, L_{\Phi_3})^* \tag{3.55}$$

and their cyclic permutations, which are given by multiplication of holomorphic sections. We remark that these multiplication maps have a rich structure and play a pivotal rôle in the theory of algebraic curves [78].

The structure of the effective supersymmetric quantum mechanics intricately depends on the flux $\mathfrak{m}$ and background line bundles L_A and L_T. We will first consider the case $g > 0$ choosing for simplicity to turn off background line bundles for flavour symmetries, before returning to perform a more systematic analysis for $g = 0$ in the presence of non-vanishing background fluxes.

Example: $g \geq 1$

For concreteness, let us specialize to $r_3 = 2$ and $r_1 = r_2 = 0$ and rename $\Phi_1 = X$, $\Phi_2 = Y$ and $\Phi_3 = Z$. Furthermore, let us remove the redundant $U(1)$ flavour symmetry and write the flavour symmetry as $G_f = U(1)_T \times U(1)_A$ with charges

	$U(1)_T$	$U(1)_A$
X	1	-1
Y	-1	-1
Z	0	2

.

The assignment of charges has been selected for a later comparison with $U(1)$ supersymmetric QED.

Turning off background line bundles for flavour symmetries, we have $L_X = L_Y = \mathcal{O}_\Sigma$ and $L_Z = K_\Sigma$ and therefore we find a supersymmetric quantum mechanics with

$$\begin{aligned} n_{c,X} = n_{c,Y} = n_{f,Z} = 1 \\ n_{f,X} = n_{f,Y} = n_{c,Z} = g \end{aligned} \tag{3.56}$$

and superpotentials

$$J_{\eta_{x,\alpha}} = yz_\alpha \qquad J_{\eta_{y,\alpha}} = xz_\alpha \qquad J_{\eta_z} = xy \tag{3.57}$$

where $\alpha = 1, \ldots, g$. As an aside, we mention that non-trivial background line bundles would necessarily introduce J-terms that are sums of monomials, complicating the computation of the supersymmetric ground states. The special case $g = 1$ has already appeared as an example of supersymmetric quantum mechanics in section 1.1.3.

With applications to 3d mirror symmetry in mind, we will consider the chamber $\mathfrak{c}_{+-+} = \{0 < m_A < m_T\}$ inside the space of mass parameters $\mathfrak{t}_f = \mathbb{R}^2$. The Hilbert space of supersymmetric ground states in this chamber should reproduce the expansion of supersymmetric twisted index

$$I = y^{2g-2} \left[\frac{(1-qy^{-1})(1-q^{-1}y^{-1})}{(1-y^2)} \right]^{g-1} \tag{3.58}$$

for $|q| < |y| < 1$, where we have introduced fugacities q and y for the flavour symmetries $U(1)_T$ and $U(1)_A$ respectively.

In order to enumerate the supersymmetric ground states in this model, it is convenient to choose the Fock vacuum annihilated by the fermions ψ_x, ψ_y, $\psi_{z,\alpha}$ and $\eta_{x,\alpha}$, $\eta_{y,\alpha}$, η_z for all $\alpha = 1, \ldots, g$. This Fock vacuum is uncharged under the flavour symmetry and here we assign it Fermion number $(-1)^F = -1$ to match the supersymmetric twisted index. The supersymmetric ground states correspond to the cohomology of

$$(Q^\dagger)^{(1)} = \sum_{\alpha=1}^{g} yz_\alpha \eta_{x,\alpha} + \sum_{\alpha=1}^{g} xz_\alpha \eta_{y,\alpha} + xy\eta_z \tag{3.59}$$

acting on wavefunctions

$$[\, x^n \, \bar{y}^m \prod_{\alpha=1}^{g} z_\alpha^{l_\alpha}] \prod_{\alpha=1}^{g} \bar{\eta}_{x,\alpha}^{s_\alpha} \prod_{\alpha=1}^{g} \bar{\eta}_{y,\alpha}^{t_\alpha} \bar{\eta}_z^{r} \, , \tag{3.60}$$

where $n, m, l_\alpha \in \mathbb{Z}_{\geq 0}$ and $s_\alpha, t_\alpha, r \in \{0, 1\}$. Since the flavour symmetry generators commute with the supercharge $(Q^\dagger)^{(1)}$, we can enumerate the supersymmetric ground states with fixed flavour charge separately. We find that the computation of the cohomology of $(Q^\dagger)^{(1)}$ splits into three qualitatively different regions depending on the $U(1)_T$ charge q_T. This is illustrated in figure 3.1.

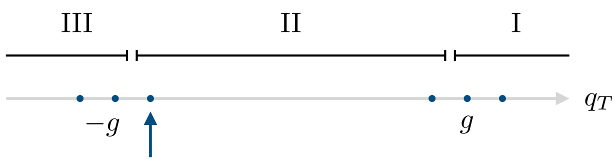

Figure 3.1: We separate our discussion of the supersymmetric ground states for the XYZ model into three regions depending on the $U(1)_T$ flavour symmetry charge q_T. An arrow marks the point in region II where we obtain a complete result.

First, there are no supersymmetric ground states of charge $q_T \leq -g$ and therefore the contributions to the supersymmetric twisted Hilbert space vanishes in region III,

$$\mathcal{H}_{(q_T)} = \emptyset \qquad q_T \leq -g\,. \tag{3.61}$$

Next, we consider $q_T \geq g$ corresponding to region I and find the following representatives of the supersymmetric ground states with weights

$$\begin{aligned} &\left[x^n\, \bar{y}^{q_T-1}\right] \bar{\eta}_{x,i_1} \cdots \bar{\eta}_{x,i_n} &&: \quad +(-1)^n q^{q_T} y^{q_T} \\ &\left[x^{n+q_T-1}\right] \bar{\eta}_z\, \bar{\eta}_{x,i_1}, \cdots, \bar{\eta}_{x,i_n} &&: \quad -(-1)^n q^{q_T} y^{-q_T} \end{aligned} \tag{3.62}$$

for $n = 0, \ldots, g$ and therefore

$$\mathcal{H}_{(q_T)} = q^{q_T}(y^{q_T} - y^{-q_T}) \otimes \wedge^\bullet(\mathbb{C}^g) \qquad q_T \geq g\,. \tag{3.63}$$

The twisted index should therefore vanish in region I due to the summation over contributions from an exterior algebra,

$$\wedge^\bullet(\mathbb{C}^g) \longrightarrow \sum_{j=0}^{g} (-1)^j \binom{g}{j} = 0 \tag{3.64}$$

for $g > 0$. These results are consistent with the supersymmetric twisted index (3.58), which is a finite Laurent polynomial starting at order $\mathcal{O}(q^{1-g})$ and ending at $\mathcal{O}(q^{g-1})$.

In region II, where the supersymmetric twisted index is non-vanishing, the computation of the supersymmetric vacua is more intricate. For $g = 1$, the computation is summarized as an example in section 1.1.3. The supersymmetric ground state representatives and their weights are

$$\begin{aligned} &\left[z^n\right] \bar{\eta}_x && -\, y^{2n+2} \\ &\left[z^n\right] \bar{\eta}_x \bar{\eta}_z && +\, y^{2n} \end{aligned} \tag{3.65}$$

for all $n \geq 0$ and thus

$$\mathcal{H}_{(0)} = (1 - y^2) \bigoplus_{n \geq 0} y^{2n} \mathbb{C} \qquad \text{if} \quad g = 1\,. \tag{3.66}$$

This reproduces the supersymmetric index, which is 1.

For $g>1$, we have not obtained a systematic closed form for representatives of the supersymmetric ground states in region II. However, the sector $q_T = 1-g$ is a generalization of the above computation at $g=1$ and we find

$$\begin{aligned}
&\left[z_1^{n_1}\cdots z_g^{n_g}\right]\bar\eta_{x_1}\cdots\bar\eta_{x_g} && +(-1)^g q^{1-g} y^{g+1+2\sum_\alpha n_\alpha} \\
&\left[z_1^{n_1}\cdots z_g^{n_g}\right]\bar\eta_{x_1}\cdots\bar\eta_{x_g}\bar\eta_z && -(-1)^g q^{1-g} y^{g-1+2\sum_\alpha n_\alpha}
\end{aligned} \tag{3.67}$$

for any $n_1,\ldots,n_g \geq 0$ and therefore

$$\mathcal{H}_{(1-g)} = (-1)^{g-1} q^{1-g} y^{g-1}(1-y^2)\, S^\bullet\left(y^2\mathbb{C}^g\right) . \tag{3.68}$$

This reproduces the correct contribution to the supersymmetric twisted index (3.58) with $q_T = 1-g$,

$$I_{(1-g)} = (-1)^{g-1} q^{1-g}\left(\frac{y}{1-y^2}\right)^{g-1} . \tag{3.69}$$

Example: $g=0$

We now consider $g=0$ and introduce non-trivial holomorphic line bundles $L_A = \mathcal{O}(\mathfrak{m}_A)$ and $L_T = \mathcal{O}(\mathfrak{m}_T)$ for the flavour symmetries such that

$$\begin{aligned}
L_{\Phi_1} &= \mathcal{O}(\mathfrak{m}_T - \mathfrak{m}_A) \\
L_{\Phi_2} &= \mathcal{O}(-\mathfrak{m}_T - \mathfrak{m}_A) \\
L_{\Phi_3} &= \mathcal{O}(2\mathfrak{m}_A - 2) .
\end{aligned} \tag{3.70}$$

The supersymmetric twisted index is

$$I = \left(\frac{y}{1-y^2}\right)^{2\mathfrak{m}_A-1}\left(\frac{q^{1/2}y^{-1/2}}{1-qy^{-1}}\right)^{\mathfrak{m}_T-\mathfrak{m}_A+1}\left(\frac{q^{-1/2}y^{-1/2}}{1-q^{-1}y^{-1}}\right)^{-\mathfrak{m}_T-\mathfrak{m}_A+1} . \tag{3.71}$$

In contrast to $g>0$, for any given $(\mathfrak{m}_A,\mathfrak{m}_T)$ each 3d chiral multiplet leads to either chiral or Fermi multiplet fluctuations in the supersymmetric quantum mechanics, but not both. Now, a J-term contribution to the supercharge $(Q^\dagger)^{(1)}$ can only exist when two 3d chiral multiplets have chiral multiplet fluctuations and one has Fermi multiplet fluctuations. This happens when the fluxes that obey either

1. $\mathfrak{m}_T \geq \mathfrak{m}_A \geq 1$, or
2. $\mathfrak{m}_A \leq 0$ and $|\mathfrak{m}_T| \leq -\mathfrak{m}_A$.

Outside of these regions, the supersymmetric ground states are obtained simply by quantizing the chiral and Fermi multiplet fluctuations. We now spell out the supersymmetric ground states in two such regions needed to perform checks of mirror symmetry in section 3.3. As above, we consider the chamber $\mathfrak{c}_{+-+} = \{0 < m_A < m_T\}$ in the space of real mass parameters.

First, we consider fluxes obeying the constraints $\mathfrak{m}_A \geq 1$, $|\mathfrak{m}_T| < \mathfrak{m}_A$. Here, L_{Φ_1}, L_{Φ_2} have negative degree and contribute only Fermi multiplet fluctuations, whereas L_{Φ_3} has non-negative degree and contributes only chiral multiplet fluctuations. There is no possibility of a J-term superpotential and therefore we find a tensor product of chiral and Fermi multiplet Fock spaces,

$$\mathcal{H} = q^{\mathfrak{m}_T} y^{3\mathfrak{m}_A-2} S^\bullet(y^2\mathbb{C}^{2\mathfrak{m}_A-1}) \otimes \wedge^\bullet(qy^{-1}\mathbb{C}^{-\mathfrak{m}_T+\mathfrak{m}_A-1}) \otimes \wedge^\bullet(q^{-1}y^{-1}\mathbb{C}^{\mathfrak{m}_T+\mathfrak{m}_A-1})\,, \tag{3.72}$$

reproducing the supersymmetric twisted index (3.71). For future comparison with mirror symmetry, it is useful to expand this result in powers of q,

$$\begin{aligned}\mathcal{H} =& y^{3\mathfrak{m}_A-2} S^\bullet(y^2\mathbb{C}^{2\mathfrak{m}_A-1}) \otimes \\ &\otimes \bigoplus_{\mathfrak{m}=-\mathfrak{m}_T-\mathfrak{m}_A+1}^{-\mathfrak{m}_T+\mathfrak{m}_A-1} q^{\mathfrak{m}_T+\mathfrak{m}} \left(\bigoplus_{i-k=\mathfrak{m}} \wedge^i(y^{-1}\mathbb{C}^{-\mathfrak{m}_T+\mathfrak{m}_A-1}) \otimes \wedge^k(y^{-1}\mathbb{C}^{\mathfrak{m}_T+\mathfrak{m}_A-1}) \right)\,.\end{aligned} \tag{3.73}$$

Second, we consider $\mathfrak{m}_A \leq 0$, $\mathfrak{m}_T > -\mathfrak{m}_A$. Here, L_{Φ_1} has non-negative degree and contributes only chiral multiplet fluctuations, whereas Φ_2 and Φ_3 have negative degree and contribute only Fermi multiplet fluctuations. There are again no possible J-term superpotentials. We therefore have

$$\mathcal{H} = q^{\mathfrak{m}_T} y^{3\mathfrak{m}_A-2} S^\bullet(qy^{-1}\mathbb{C}^{\mathfrak{m}_T-\mathfrak{m}_A+1}) \otimes \wedge^\bullet(q^{-1}y^{-1}\mathbb{C}^{\mathfrak{m}_T+\mathfrak{m}_A-1}) \otimes \wedge^\bullet(y^2\mathbb{C}^{-2\mathfrak{m}_A+1})\,, \tag{3.74}$$

reproducing the supersymmetric twisted index (3.71). Expanding again in q, this becomes

$$\begin{aligned}\mathcal{H} =& q^{\mathfrak{m}_T} y^{3\mathfrak{m}_A-2} \wedge^\bullet (y^2\mathbb{C}^{-2\mathfrak{m}_A+1}) \otimes \\ &\otimes \bigoplus_{\mathfrak{m}=-\mathfrak{m}_T-\mathfrak{m}_A+1}^{\infty} q^{\mathfrak{m}} \left(\bigoplus_{k=0}^{\mathfrak{m}_T+\mathfrak{m}_A-1} S^{\mathfrak{m}-k}(y^{-1}\mathbb{C}^{\mathfrak{m}_T-\mathfrak{m}_A+1}) \otimes \wedge^k(y^{-1}\mathbb{C}^{\mathfrak{m}_T+\mathfrak{m}_A-1}) \right)\,.\end{aligned} \tag{3.75}$$

3.2.4 $U(1)_{\frac{1}{2}} + 1$ Chiral Multiplet

We now consider supersymmetric Chern-Simons theory with $G = U(1)$ at level $k = +\frac{1}{2}$ together with a single chiral multiplet Φ of charge $+1$ and R-charge $+1$.

In the supersymmetric quantum mechanics, the chiral multiplet Φ decomposes into a chiral multiplet and Fermi multiplet valued in sections of the holomorphic line bundle L_Φ and $\Omega^{0,1}_\Sigma \otimes L_\Phi$ respectively, where

$$L_\Phi = K_\Sigma^{1/2} \otimes L\,. \tag{3.76}$$

The supersymmetric quantum mechanics localizes to solutions of the following system of equations on Σ,

$$* F + e^2\left(\bar{\phi}\cdot\phi - \xi_{eff}(\sigma)\right) = 0 \qquad \bar{\partial}_A\phi = 0 \qquad \sigma\phi = 0\,, \tag{3.77}$$

where σ is constant and

$$\xi_{eff}(\sigma) = \begin{cases} \tau + \sigma & \sigma > 0 \\ \tau & \sigma \leq 0 \end{cases} \tag{3.78}$$

is the combination of effective parameters introduced in equation (3.18).

By integrating (3.77) over Σ we find that there are 'Higgs branch' vortex solutions with $\sigma = 0$ and ϕ non-vanishing provided

$$\tau > \frac{2\pi\mathfrak{m}}{e^2\mathrm{Vol}(\Sigma)}\,, \tag{3.79}$$

where

$$\mathfrak{m} = c_1(L) = \frac{1}{2\pi}\int_\Sigma F \tag{3.80}$$

and $\mathrm{Vol}(\Sigma)$ is the volume of Σ. Notice that the right-hand-side takes the same form as (2.33). Here we take the limit $s \to +\infty$ for

$$s := \tau e^2 \mathrm{Vol}(\Sigma) \tag{3.81}$$

where this condition is satisfied for any flux $\mathfrak{m}$. an effective description of the supersymmetric quantum mechanics as a sigma model onto the moduli space $\mathfrak{M}_\mathfrak{m}$ of solutions to the vortex equations

As explained in section (3.1.3), $\mathfrak{M}_\mathfrak{m}$ has a complex algebraic description given in terms of pairs (L,ϕ) given by a holomorphc line bundle L and a holomorphic section ϕ. In appendix A.3.1, we explain that these pairs can be parametrised by the zeros of ϕ, which (counting multiplicities) are $\mathfrak{m}_\Phi$. In turns, the zeros are parametrised by the $\mathfrak{m}_\Phi$-symmetric product of Σ

$$\mathfrak{M}_\mathfrak{m} = \begin{cases} \mathrm{Sym}^{\mathfrak{m}_\Phi}\Sigma & \text{if} \quad \mathfrak{m}_\Phi \geq 0 \\ \emptyset & \text{if} \quad \mathfrak{m}_\Phi < 0\,. \end{cases} \tag{3.82}$$

Physically, this is the statement that vortices in this theory have no internal moduli and the symmetric product simply parametrizes the positions of the vortices on Σ. Notice that the preimage of a point under (3.27) is the projective space of holomorphic sections, $j^{-1}(L_\Phi) = \mathbb{P}H^0(L_\Phi)$. The structure of this map is in general intricate since the dimension $h^0(L_\Phi)$ may jump at loci in the Jacobian. However, if $\mathfrak{m}_\Phi > 2g-2$ the dimension $h^0(L_\Phi) = \mathfrak{m}_\Phi - g + 1$ is constant and this holomorphic map becomes a holomorphic fibration with fiber $\mathbb{CP}^{\mathfrak{m}_\Phi - g}$.

The contribution to the twisted Hilbert space of supersymmetric vacua from flux $\mathfrak{m}$ is captured by a supersymmetric sigma model to $\mathfrak{M}_\mathfrak{m}$ and therefore

$$\mathcal{H}_\mathfrak{m} = \begin{cases} H^{0,\bullet}_{\bar\partial}(\mathfrak{M}_\mathfrak{m}, \mathcal{F}_\mathfrak{m}) & \mathfrak{m} \geq 1-g \\ \emptyset & \mathfrak{m} < 1-g \end{cases} \tag{3.83}$$

where $\mathcal{F}_\mathfrak{m}$ is a Hermitian line bundle on $\mathfrak{M}_\mathfrak{m}$ [2]. The line bundle $\mathcal{F}_\mathfrak{m}$ receives contributions from the following sources:

- There is a universal contribution $K^{1/2}_{\mathfrak{M}_\mathfrak{m}}$ from quantizing fermions in chiral multiplets parametrizing $\mathfrak{M}_\mathfrak{m}$.
- The supersymmetric Chern-Simons term at level $+\frac{1}{2}$ contributes a factor $\mathcal{K}^{-1/2}_{\mathfrak{M}_\mathfrak{m}}$.

[2]An important consistency check is that the remaining massless fermions in chiral multiplets transform in the tangent bundle to $\mathfrak{M}_\mathfrak{m}$. In appendix A.4, we provide a short argument why this claim. This can be thought of as a simple example of the more general phenomenon that we are going to investigate in the next chapter.

- Introducing a background line bundle L_T for the topological symmetry induces a line bundle $\widetilde{L}_T$ on the moduli space $\mathfrak{M}_{\mathfrak{m}}$ in the supersymmetric quantum mechanics, as explained in section 3.1.2.

The universal contribution and that of the supersymmetric Chern-Simons term cancel out, leaving $\mathcal{F}_{\mathfrak{m}} = \widetilde{L}_T$.

For $\tau < 0$ we find 'topological' solutions to equations (3.77) with $\phi = 0$, $\sigma = -\tau$ and unbroken gauge symmetry as $\tau e^2 \to 0$. We expect that the supersymmetric ground states are captured by an effective supersymmetric Chern-Simons theory in this regime, and hope to return to this in future work.

Example: $g > 0$

The supersymmetric ground states in equation (3.83) can be computed by applying the Künneth formula for the $\mathfrak{m}_\Phi$-fold product of curves $\Sigma^{\mathfrak{m}_\Phi}$, and then imposing invariance under permutations to compute cohomology classes on the symmetric product regarded as a quotient $\mathrm{Sym}^{\mathfrak{m}_\Phi}\Sigma = \Sigma^{\mathfrak{m}_\Phi}/S_{\mathfrak{m}_\Phi}$.

This argument will rely on the following construction of the line bundle $\widetilde{L}_T$ associated to the $U(1)_T$ topological symmetry, which we expand upon in appendix A.3.5. The first step is to construct a line bundle on the direct product $\Sigma^{\mathfrak{m}_\Phi}$,

$$L_T^{\boxtimes\,\mathfrak{m}_\Phi} := \bigotimes_j \pi_j^* L_T \,, \tag{3.84}$$

where

$$\pi_j : \Sigma^{\mathfrak{m}_\Phi} \to \Sigma \tag{3.85}$$

is the projection onto the j-th factor. This is invariant under permutations and descends to a line bundle $\widetilde{L}_T$ on the symmetric product $\mathrm{Sym}^{\mathfrak{m}_\Phi}\Sigma$. In particular, this construction shows that $c_1(\widetilde{L}_T) = \mathfrak{m}_T\eta$ where $\mathfrak{m}_T$ is the degree of L_T and $\eta \in H^{1,1}(\mathfrak{M}_{\mathfrak{m}})$ is the class constructed from the Kähler form on Σ.

With the above construction in hand, we can proceed to compute the twisted Hilbert space following arguments in [79], but including higher degree cohomology. First, a short spectral sequence argument shows that the cohomology of the symmetric product in equation (3.83) can be identified with the $S_{\mathfrak{m}_\Phi}$-invariant part of the cohomology $H^{0,\bullet}_{\bar\partial}(\Sigma^{\mathfrak{m}_\Phi}, L_T^{\boxtimes \mathfrak{m}_\Phi})$. The latter can be computed using the Künneth decomposition, with the result

$$H^{0,j}_{\bar\partial}(\mathrm{Sym}^{\mathfrak{m}_\Phi}\Sigma, \widetilde{L}_T) = \mathrm{S}^{\mathfrak{m}_\Phi - j} H^0(L_T) \otimes \wedge^j H^1(L_T)\,. \tag{3.86}$$

We therefore find

$$\mathcal{H} = \bigoplus_{\mathfrak{m}=1-g}^{\infty} q^{\mathfrak{m}} \bigoplus_{i+j=\mathfrak{m}+g-1} S^i H^0(L_T) \otimes \wedge^j H^1(L_T)\,. \tag{3.87}$$

where the parameter q is introduced to keep track of charge under the topological flavour symmetry and we remind the reader that $\mathfrak{m}_\Phi = \mathfrak{m} + g - 1$.

Introducing the notation $n_C := h^0(L_T)$ and $n_F := h^1(L_T)$ with

$$n_C - n_F = \mathfrak{m}_T - g + 1\,, \tag{3.88}$$

the supersymmetric twisted index can be computed from the graded trace over the twisted Hilbert space (3.87) as follows,

$$\begin{aligned} I &= \sum_{\mathfrak{m}=1-g}^{\infty} q^{\mathfrak{m}} \binom{n_C - n_F + \mathfrak{m} - g}{\mathfrak{m} - g + 1} \\ &= q^{1-g} \left(\frac{1}{1-q} \right)^{\mathfrak{m}_T - g + 1} \end{aligned} \tag{3.89}$$

This agrees with the contour integral formula from localization in the supersymmetric gauge introduced in section 2.3, with the choice of auxiliary parameter (2.33). In fact, the JK contour integral formula reduces to

$$\sum_{\mathfrak{m}\in\mathbb{Z}} (-q)^{\mathfrak{m}} \int_{\mathrm{JK}} \frac{dx}{2\pi i x} \frac{x^{\mathfrak{m}+\mathfrak{m}_T}}{(1-x)^{\mathfrak{m}+g}}\,, \tag{3.90}$$

where the JK-contour surrounds in this case the pole at $z = 1$, modulo an overall sign $(-1)^g$. The twisted supersymmetric index can be identified with the generating function for equivariant Euler characters

$$I = \sum_{\mathfrak{m}=1-g}^{\infty} (-q)^{\mathfrak{m}} \chi(\mathfrak{M}_{\mathfrak{m}}, \mathcal{F}_{\mathfrak{m}})\,. \tag{3.91}$$

From this perspective, the contour integral (3.90) from supersymmetric localization reproduces the computation of the equivariant Euler characters using Hirzebruch-Riemann-Roch [80]. This is an elementary instance of, and the inspiration for, the index computations we will perform in the next chapter of this thesis.

A special case summarized in the introduction is $L_T = \mathcal{O}_\Sigma$, where $n_C = 1$ and $n_F = g$, and therefore

$$\mathcal{H} = \bigoplus_{\mathfrak{m}=1-g}^{\infty} q^{\mathfrak{m}} \bigoplus_{j=0}^{\mathfrak{m}+g-1} \wedge^j(\mathbb{C}^g) \tag{3.92}$$

with supersymmetric twisted index $I = q^{1-g}(1-q)^{g-1}$. Note that while the twisted Hilbert space contains an infinite number of supersymmetric ground states, the supersymmetric twisted index truncates to a finite number of terms for $g \geq 1$ due to the complete cancellation in the sum over the exterior algebra when $\mathfrak{m} > 0$.

Example: $g = 0$

Let us now analyze $g = 0$ in more detail. The vortex moduli space $\mathfrak{M}_{\mathfrak{m}} = \mathbb{CP}^{\mathfrak{m}-1}$ is now the complex projective space parametrizing non-vanishing holomorphic sections of $L_\Phi = \mathcal{O}(\mathfrak{m} - 1)$

modulo constant complex gauge transformations. This is consistent with the above construction, since

$$\mathrm{Sym}^{\mathfrak{m}-1}\mathbb{CP}^1 \cong \mathbb{CP}^{\mathfrak{m}-1}\,. \tag{3.93}$$

The supersymmetric quantum mechanics admits a finite-dimensional gauged linear sigma model description in terms of $\mathfrak{m}$ chiral multiplets of charge $+1$ under a $U(1)$ gauge symmetry.

In this case the contributions to $\mathcal{F}_{\mathfrak{m}}$ have a more straightforward interpretation:

- The contribution to the Lagrangian of the supersymmetric quantum mechanics from a supersymmetric Chern-Simons term at level k is

$$\frac{k}{2\pi}(\sigma + iA_0)\int_\Sigma F = (\sigma + iA_0)k\mathfrak{m}\,. \tag{3.94}$$

  This is a supersymmetric Wilson line of charge $k\mathfrak{m}$ in the finite-dimensional gauged quantum mechanics for $\mathbb{CP}^{\mathfrak{m}-1}$ and so contributes the line bundle $\mathcal{O}(k\mathfrak{m})$. For a supersymmetric Chern-Simons term at level $k = +\frac{1}{2}$ we find a contribution $\mathcal{O}(+\frac{\mathfrak{m}}{2})$.

- Turning on $L_T = \mathcal{O}(\mathfrak{m}_T)$ for the topological symmetry contributes

$$\frac{1}{2\pi}(\sigma + iA_0)\int_\Sigma F_T = (\sigma + iA_0)\mathfrak{m}_T\,, \tag{3.95}$$

  which is a supersymmetric Wilson line of charge $\mathfrak{m}_T$ in the finite-dimensional gauged supersymmetric quantum mechanics, and therefore the line bundle $\widetilde{L}_T = \mathcal{O}(\mathfrak{m}_T)$ on $\mathbb{CP}^{m-1}$.

Combining these contributions with the universal contribution $K^{1/2}_{\mathbb{CP}^{m-1}} = \mathcal{O}(-\frac{m}{2})$, we find $\mathcal{F} = \mathcal{O}(\mathfrak{m}_T)$ to be in agreement with our previous arguments.

Including the parameter ξ for the additional grading on $\Sigma = \mathbb{CP}^1$, the supersymmetric Hilbert space now has the form

$$\begin{aligned}\mathcal{H} &= \bigoplus_{\mathfrak{m}=1}^{\infty} q^{\mathfrak{m}} H^{0,\bullet}_{\bar\partial}(\mathbb{CP}^{\mathfrak{m}-1}, \mathcal{O}(\mathfrak{m}_T)) \\ &= \bigoplus_{\mathfrak{m}=1}^{\infty} q^{\mathfrak{m}} \begin{cases} S^{\mathfrak{m}_T}(\xi^\rho\mathbb{C}^{\mathfrak{m}}) & \text{if} \quad \mathfrak{m}_T \geq 0 \\ \emptyset & \text{if} \quad -\mathfrak{m} < \mathfrak{m}_T < 0 \\ S^{|\mathfrak{m}_T|-\mathfrak{m}}(\xi^\rho\mathbb{C}^{\mathfrak{m}}) & \text{if} \quad \mathfrak{m}_T \leq -\mathfrak{m} \end{cases}.\end{aligned} \tag{3.96}$$

3.2.5 $U(1)$ SQED

We now consider a $U(1)$ gauge theory with two chiral multiplets Φ, $\tilde\Phi$ of charge $+1$, -1 respectively. The theory has both a $U(1)_T$ topological and a $U(1)_A$ axial flavour symmetry, and we can introduce corresponding background line bundles L_T and L_A of degrees $\mathfrak{m}_T$ and $\mathfrak{m}_A$. The charge assignments are summarized below,

	$U(1)_G$	$U(1)_T$	$U(1)_A$	$U(1)_R$
Φ	1	0	1	1
$\tilde\Phi$	-1	0	1	1

.

The supersymmetric twisted index (2.17) in this case is given by

$$
\begin{aligned}
I &= \sum_{\mathfrak{m}\in\mathbb{Z}} (-q)^{\mathfrak{m}} \int_{\mathrm{JK}} \frac{dx}{2\pi i\, x} x^{\mathfrak{m}_T} \left(\frac{x^{\frac{1}{2}}y^{\frac{1}{2}}}{1-xy}\right)^{\mathfrak{m}+\mathfrak{m}_A} \left(\frac{x^{-\frac{1}{2}}y^{\frac{1}{2}}}{1-x^{-1}y}\right)^{-\mathfrak{m}+\mathfrak{m}_A} \left(\frac{1-y^2}{(1-xy)(1-y/x)}\right)^{g} \\
&= (-1)^{\mathfrak{m}_T} \left(\frac{y}{1-y^2}\right)^{2\mathfrak{m}_A+g-1} \left(\frac{q^{1/2}y^{-1/2}}{1-qy^{-1}}\right)^{\mathfrak{m}_T-\mathfrak{m}_A+1-g} \left(\frac{q^{-1/2}y^{-1/2}}{1-q^{-1}y^{-1}}\right)^{-\mathfrak{m}_T-\mathfrak{m}_A+1-g} ,
\end{aligned}
\tag{3.97}
$$

where the contour JK is specified by the Jeffrey-Kirwan prescription of (2.27).

In the supersymmetric quantum mechanics the chiral multiplets Φ, $\tilde{\Phi}$ decompose into chiral and Fermi multiplets, transforming as scalar and $(0,1)$-form sections of the holomorphic line bundles

$$
L_\Phi := K_\Sigma^{1/2} \otimes L \otimes L_A \,, \qquad L_{\tilde{\Phi}} := K_\Sigma^{1/2} \otimes L^{-1} \otimes L_A \,, \tag{3.98}
$$

of degrees

$$
\mathfrak{m}_\Phi = \mathfrak{m} + \mathfrak{m}_A + g - 1 \,, \qquad \mathfrak{m}_{\tilde{\Phi}} = -\mathfrak{m} + \mathfrak{m}_A + g - 1 \,. \tag{3.99}
$$

First setting the mass parameter for the axial flavour symmetry to zero, $m_A = 0$, the supersymmetric quantum mechanics localizes to solutions of the following system of equations on Σ,

$$
* F_A + e^2 \left(\bar{\phi} \cdot \phi - \bar{\tilde{\phi}} \cdot \tilde{\phi} - \tau_{eff} \right) = \qquad \bar{\partial}_A \phi = \bar{\partial}_A \phi = 0 \qquad \sigma\phi = \sigma\tilde{\phi} = 0 \,, \tag{3.100}
$$

where $\tau_{eff} = \tau$ when the axial mass parameter vanishes. In the limit $s = e^2\tau \mathrm{Vol}\,(\Sigma) \to +\infty$ there are 'Higgs branch' vortex solutions with $\sigma = 0$ for each $\mathfrak{m}$.

We therefore consider an effective supersymmetric sigma model for each flux $\mathfrak{m} \in \mathbb{Z}$ whose target is the moduli space $\mathfrak{M}_\mathfrak{m}$ of solutions to equation (3.100). This moduli space has an algebraic description in terms of the following holomorphic data:

- A holomorphic line bundle L of degree $\mathfrak{m}$.
- A pair of holomorphic sections $\phi \in H^0(\Sigma, L_\Phi)$ and $\tilde{\phi} \in H^0(\Sigma, L_{\tilde{\Phi}})$, where ϕ is required to be non-vanishing.

The structure of the moduli space and the effective supersymmetric quantum mechanics depends intricately on the flux $\mathfrak{m}$ and background line bundles L_A and L_T. We will first consider the case $g > 0$ choosing for simplicity to turn off background line bundles for flavour symmetries, before returning to perform a more systematic analysis for $g = 0$ in the presence of non-vanishing background fluxes.

Genus $g > 0$

Let us then consider $g > 0$ with trivial background line bundles $L_T = L_A = \mathcal{O}_\Sigma$. The structure of the supersymmetric quantum mechanics depends on the flux $\mathfrak{m}$. We separate our discussion into the three regions shown in figure 3.2.

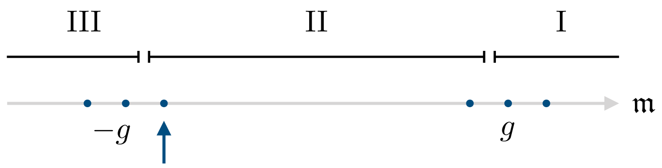

Figure 3.2: We separate our computation of the supersymmetric vacua for supersymmetric QED into three different regions depending on the flux $\mathfrak{m} \in \mathbb{Z}$. An arrow marks the point in region II discussed in the main text.

Region I corresponds to $\mathfrak{m} \geq g$. In this region,

$$\begin{aligned} h^0(L_\Phi) &= \mathfrak{m} \qquad & h^0(L_{\tilde{\Phi}}) &= 0 \\ h^1(L_\Phi) &= 0 \qquad & h^1(L_{\tilde{\Phi}}) &= \mathfrak{m}\,, \end{aligned} \tag{3.101}$$

independent of the line bundle L. There are therefore exactly $\mathfrak{m}$ chiral multiplet fluctuations from Φ and $\mathfrak{m}$ Fermi multiplet fluctuations from $\tilde{\Phi}$. The underlying moduli space of vortices is again a symmetric product

$$\mathfrak{M} = \mathrm{Sym}^{\mathfrak{m}_\Phi}\Sigma\,. \tag{3.102}$$

Since $\mathfrak{m}_\Phi > 2g-2$, the symmetric product is a holomorphic fibration $\mathfrak{M}_\mathfrak{m} \to J_\Sigma$ with fiber $\mathbb{CP}^{\mathfrak{m}-1}$. The Fermi multiplet fluctuations from $\tilde{\Phi}$ then transform as a holomorphic section of the vector bundle $\mathcal{F} = \mathcal{O}(1) \otimes \mathbb{C}^\mathfrak{m}$ over each fiber.

The existence of the holomorphic fibration implies that the space of supersymmetric ground states factorizes into contributions from the fibre and base,

$$\mathcal{H}_\mathfrak{m} = \mathcal{H}_\mathfrak{m}^{(b)} \otimes \mathcal{H}_\mathfrak{m}^{(f)}\,. \tag{3.103}$$

The contribution from the base is

$$\begin{aligned} \mathcal{H}_\mathfrak{m}^{(b)} &= \bigoplus_{q=0}^{g} H^{0,q}_{\bar\partial}(J_\Sigma) \\ &= \sum_{q=0}^{g} \wedge^q(\mathbb{C}^g)\,. \end{aligned} \tag{3.104}$$

The contribution from the fiber is

$$\begin{aligned} \mathcal{H}_\mathfrak{m}^{(f)} &= \bigoplus_{p=0}^{\mathfrak{m}-1}\bigoplus_{\alpha=1}^{\mathfrak{m}} H^{0,p}_{\bar\partial}\left(\mathbb{CP}^{\mathfrak{m}-1}, K^{1/2}_{\mathbb{CP}^{\mathfrak{m}-1}} \otimes \frac{\wedge^\alpha\mathcal{F}}{\sqrt{\det\mathcal{F}}}\right) \\ &= \bigoplus_{p=0}^{\mathfrak{m}-1}\bigoplus_{\alpha=1}^{\mathfrak{m}} H^{0,p}_{\bar\partial}\left(\mathbb{CP}^{\mathfrak{m}-1}, \mathcal{O}(\alpha-\mathfrak{m})\right) \otimes \wedge^\alpha(\mathbb{C}^\mathfrak{m}) \\ &= H^{0,0}_{\bar\partial}(\mathbb{CP}^{\mathfrak{m}-1}, \mathcal{O}) \oplus H^{0,\mathfrak{m}-1}_{\bar\partial}(\mathbb{CP}^{m-1}, \mathcal{O}(-\mathfrak{m})) \end{aligned} \tag{3.105}$$

Putting these contributions together and introducing parameters q and y to keep track of the $U(1)_T$ and $U(1)_A$ symmetries respectively, we find

$$\mathcal{H}_\mathfrak{m} = (-1)^\mathfrak{m} q^\mathfrak{m} (y^\mathfrak{m}\mathbb{C} - y^{-\mathfrak{m}}\mathbb{C}) \otimes \wedge^\bullet(\mathbb{C}^g)\,. \tag{3.106}$$

In section 3.3.2, we will demonstrate that this result is in complete agreement with 3d mirror symmetry.

An immediate consequence of the factorization into contributions from the fiber and base is that the supersymmetric twisted index should vanish in region I due to the complete cancellation

$$\wedge^\bullet(\mathbb{C}^g) \longrightarrow \sum_{j=0}^{g}(-1)^j \binom{g}{j} = 0 \tag{3.107}$$

for $g > 0$. This is consistent with the supersymmetric localization result (3.97), which for $\mathfrak{m}_T = \mathfrak{m}_A = 0$ is a finite Laurent polynomial in q with maximum power q^{g-1}.

Region II corresponds to $-g < \mathfrak{m} < g$. While

$$\begin{aligned} h^0(L_\Phi) - h^1(L_\Phi) &= \mathfrak{m} \\ h^0(L_{\tilde{\Phi}}) - h^1(L_{\tilde{\Phi}}) &= -\mathfrak{m}\,, \end{aligned} \tag{3.108}$$

the numbers of chiral and Fermi multiplet fluctuations may jump as the holomorphic line bundle L varies over the Jacobian J_Σ. The computation of the supersymmetric vacua is more difficult in the region and we do not present a general answer.

An exception is $\mathfrak{m} = 1-g$, which is marked with an arrow in figure 3.2. In this case, the holomorphic line bundle L_Φ has degree $\mathfrak{m}_\Phi = 0$. The stability condition requires the existence of a non-zero holomorphic section ϕ, so we conclude that

$$L_\Phi = \mathcal{O}_\Sigma \qquad L_{\tilde{\Phi}} = K_\Sigma \tag{3.109}$$

corresponding to $L = K_\Sigma^{-1/2}$. Recall that there is a holomorphic map $\mathfrak{M}_\mathfrak{m} \to J_\Sigma$ to the Jacobian parametrizing L. In this case, the map is particularly simple: there is one non-vanishing fiber over a single point in the Jacobian. This fiber is parametrized by chiral multiplet fluctuations of the meson $M = \Phi\tilde{\Phi}$ valued in $H^0(K_\Sigma)$, and the moduli space is therefore $\mathfrak{M}_\mathfrak{m} = \mathbb{C}^g$. In addition there is a Fermi multiplet fluctuation $\eta \in H^1(K_\Sigma)$.

Since the moduli space is non-compact, it is essential to turn on a real mass parameter m_A for the $U(1)_A$ axial flavour symmetry. We will choose $m_A > 0$. Noting that the meson fluctuations have charge $+2$ under the axial flavour symmetry, we conclude that the twisted Hilbert space for $\mathfrak{m} = 1-g$ is

$$\mathcal{H}_{(1-g)} = q^{1-g} y^{g-1}(1-y^2) \bigoplus_{n\geq 0} y^{2n} S^n(\mathbb{C}^g)\,. \tag{3.110}$$

The contribution to the supersymmetric twisted index is therefore

$$\begin{aligned} I_{1-g} &= q^{1-g} y^{g-1}(1-y^2) \sum_{n\geq 0} y^{2n} \binom{n+g-1}{g-1} \\ &= q^{1-g}\left(\frac{y}{1-y^2}\right)^{g-1}. \end{aligned} \tag{3.111}$$

This agrees with the result from supersymmetric localization (3.97) for $\mathfrak{m}_T = \mathfrak{m}_A = 0$.

Finally, region III corresponds to $\mathfrak{m} \leq -g$. In this region, $h^0(L_\Phi) = 0$. This is incompatible with the stability condition and therefore there are no supersymmetric ground states in this region. The result is consistent with the localization expression (3.97), which for $\mathfrak{m}_T = \mathfrak{m}_A = 0$ is a finite Laurent polynomial with minimum power q^{1-g}.

Genus $g = 0$

For genus zero, we perform a more systematic analysis with non-vanishing background fluxes $\mathfrak{m}_A$, $\mathfrak{m}_T$ for the flavour symmetries. The holomorphic line bundles on $\Sigma = \mathbb{CP}^1$ associated to the chiral multiplets are

$$L_\Phi := \mathcal{O}(\mathfrak{m} + \mathfrak{m}_A - 1) \qquad L_{\tilde{\Phi}} := \mathcal{O}(-\mathfrak{m} + \mathfrak{m}_A - 1)\,. \tag{3.112}$$

To guide the reader through our analysis, we summarize the structure of moduli spaces $\mathfrak{M}_\mathfrak{m}$ and corresponding contributions to the twisted Hilbert space in the $(\mathfrak{m}, \mathfrak{m}_A)$ plane in figure 3.3.

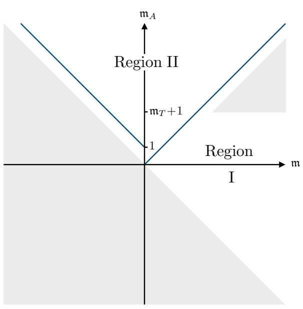

Figure 3.3: Summary of the structure of the moduli space and supersymmetric ground states in the $(\mathfrak{m}, \mathfrak{m}_A)$ plane. There are no supersymmetric ground states in the shaded regions.

First, we note that in order to satisfy the stability condition we require that $h^0(L_\Phi) > 0$. This immediately implies that the moduli space is empty and there are no supersymmetric ground states in the region $\mathfrak{m} + \mathfrak{m}_A \leq 0$. This corresponds to the large shaded region in figure 3.3.

Now assuming that $\mathfrak{m} + \mathfrak{m}_A > 0$, the structure of the moduli space $\mathfrak{M}_\mathfrak{m}$ critically depends on the combination $-\mathfrak{m} + \mathfrak{m}_A$, which determines whether $\tilde{\Phi}$ generates chiral or Fermi multiplet fluctuations in the supersymmetric quantum mechanics. We therefore consider two distinct regions separated by the line $\mathfrak{m} = \mathfrak{m}_A$, as shown in figure 3.3.

Region I

We first consider $-\mathfrak{m} + \mathfrak{m}_A \leq 0$. This region is characterized by chiral multiplet fluctuations from Φ and Fermi multiplet fluctuations from $\tilde{\Phi}$:

- $n_{C,\Phi} = \mathfrak{m} + \mathfrak{m}_A > 0$.
- $n_{F,\tilde{\Phi}} = \mathfrak{m} - \mathfrak{m}_A \geq 0$.

- $n_{F,\Phi} = n_{C,\tilde{\Phi}} = 0$.

The moduli space of vortices is therefore parametrized by the non-vanishing chiral multiplet fluctuations of Φ modulo complex rescaling,

$$\mathcal{M}_{\mathfrak{m}} = \mathbb{CP}^{\mathfrak{m}+\mathfrak{m}_A-1}. \tag{3.113}$$

The Fermi multiplet fluctuations $\bar{\eta}_{\tilde{\Phi}}$ have gauge charge $+1$ and therefore transform as sections of the holomorphic vector bundle $\mathcal{F} := \mathcal{O}(1) \otimes \mathbb{C}^{\mathfrak{m}-\mathfrak{m}_A}$ over $\mathfrak{M}_{\mathfrak{m}}$.

The contribution to the twisted Hilbert space is therefore given by[3]

$$\mathcal{H}_{\mathfrak{m}} = H^{0,\bullet}_{\bar{\partial}}\left(\mathfrak{M}_{\mathfrak{m}}, K^{1/2}_{\mathfrak{M}_{\mathfrak{m}}} \otimes \frac{\wedge^{\bullet}\mathcal{F}}{\sqrt{\det(\mathcal{F})}} \otimes \widetilde{L}_T\right) \tag{3.114}$$

where $\widetilde{L}_T$ is the holomorphic line bundle on the moduli space induced by the holomorphic line bundle $L_T = \mathcal{O}(\mathfrak{m}_T)$ on the curve associated to the topological symmetry. First, we notice that the combination

$$K^{1/2}_{\mathfrak{M}_{\mathfrak{m}}} \otimes \frac{1}{\sqrt{\det(\mathcal{F})}} = \mathcal{O}(-\mathfrak{m})\,. \tag{3.115}$$

Second, the same argument as section 3.2.4 shows that $\widetilde{L}_T = \mathcal{O}(\mathfrak{m}_T)$. Assembling the various pieces, we find

$$\begin{aligned}\mathcal{H}_{\mathfrak{m}} &= H^{\bullet}\left(\mathfrak{M}_{\mathfrak{m}}, \mathcal{O}(\mathfrak{m}_T - \mathfrak{m}) \otimes \wedge^{\bullet}\mathcal{F}\right) \\ &= \bigoplus_{i=0}^{\mathfrak{m}-\mathfrak{m}_A} H^{\bullet}\left(\mathbb{CP}^{\mathfrak{m}+\mathfrak{m}_A-1}, \mathcal{O}(\mathfrak{m}_T - \mathfrak{m} + i)\right) \otimes \wedge^{i}(\mathbb{C}^{\mathfrak{m}-\mathfrak{m}_A})\,.\end{aligned} \tag{3.116}$$

Including the flavour symmetry and Fermion number grading, we find

$$\begin{aligned}\mathcal{H}_{\mathfrak{m}} =& q^{\mathfrak{m}} y^{\mathfrak{m}} \bigoplus_{i=0}^{-\mathfrak{m}_A-\mathfrak{m}_T} S^{-\mathfrak{m}_T-\mathfrak{m}_A-i}(y\mathbb{C}^{\mathfrak{m}+\mathfrak{m}_A}) \otimes \wedge^{i}(-y^{-1}\mathbb{C}^{\mathfrak{m}-\mathfrak{m}_A}) \\ &\oplus (-1)^{\mathfrak{m}+\mathfrak{m}_A-1}\, q^{\mathfrak{m}} y^{-\mathfrak{m}_A} \bigoplus_{\max\{0,-\mathfrak{m}_T+\mathfrak{m}\}}^{\mathfrak{m}-\mathfrak{m}_A} S^{-\mathfrak{m}+\mathfrak{m}_T+i}(y^{-1}\mathbb{C}^{\mathfrak{m}+\mathfrak{m}_A}) \otimes \wedge^{i}(-y^{-1}\mathbb{C}^{\mathfrak{m}-\mathfrak{m}_A})\,,\end{aligned} \tag{3.117}$$

where the first line arises from the degree zero cohomology and the second from the maximum degree $\mathfrak{m}+\mathfrak{m}_A-1$ cohomology. It follows from the above formula that there are no supersymmetric ground states when both $\mathfrak{m} > \mathfrak{m}_A$ and $\mathfrak{m}_A > \mathfrak{m}_T$. This is the shaded part of region I on the right of Figure 3.3.

The contribution to the supersymmetric twisted index is therefore

$$\begin{aligned}I_{\mathfrak{m}} = q^{\mathfrak{m}} &\sum_{i=0}^{-\mathfrak{m}_T-\mathfrak{m}_A} y^{\mathfrak{m}-\mathfrak{m}_A-\mathfrak{m}_T-2i}(-1)^{i}\binom{-\mathfrak{m}_T+\mathfrak{m}-1-i}{-\mathfrak{m}_A-\mathfrak{m}_T-i}\binom{\mathfrak{m}-\mathfrak{m}_A}{i} \\ &+ q^{\mathfrak{m}} \sum_{\max\{0,-\mathfrak{m}_T+\mathfrak{m}\}}^{\mathfrak{m}-\mathfrak{m}_A} y^{\mathfrak{m}-\mathfrak{m}_A-\mathfrak{m}_T-2i}(-1)^{\mathfrak{m}+\mathfrak{m}_A-1+i}\binom{\mathfrak{m}_A+\mathfrak{m}_T-1+i}{\mathfrak{m}_T-\mathfrak{m}+i}\binom{\mathfrak{m}-\mathfrak{m}_A}{i}\,,\end{aligned} \tag{3.118}$$

which agrees with the coefficient of $q^{\mathfrak{m}}$ in the expansion of the supersymmetric twisted index (3.97) with $g = 0$, up to an overall sign $(-1)^{\mathfrak{m}_A}$.

[3] This computation can be encoded in the cohomology of superprojective space $\mathbb{CP}^{\mathfrak{m}+\mathfrak{m}_A-1|\mathfrak{m}-\mathfrak{m}_A-1}$.

Region II

We now consider the remaining region $\mathfrak{m}_A > |\mathfrak{m}|$. This is characterized by chiral multiplet fluctuations from both Φ and $\tilde{\Phi}$. The vortex moduli space is therefore a toric variety given by the total space of the vector bundle

$$\mathfrak{M}_{\mathfrak{m}} = \mathcal{O}(-1)^{\mathfrak{m}_A - \mathfrak{m}} \to \mathbb{CP}^{\mathfrak{m}+\mathfrak{m}_A-1} \, . \tag{3.119}$$

The base is parametrized by the $\mathfrak{m}+\mathfrak{m}_A$ chiral multiplets from Φ modulo holomorphic gauge transformations and the fiber by the $-\mathfrak{m}+\mathfrak{m}_A$ chiral multiplets from $\tilde{\Phi}$.

This moduli space is non-compact but the $U(1)_A$ flavour symmetry acts on the moduli space $\mathfrak{M}_{\mathfrak{m}}$ with fixed locus given by the compact zero section $\mathbb{CP}^{\mathfrak{m}+\mathfrak{m}_A-1}$. Therefore, turning on a real mass parameter m_A for the flavour symmetry $U(1)_A$, the effective supersymmetric quantum mechanics is gapped and the contribution to the twisted Hilbert space is

$$\mathcal{H}_{\mathfrak{m}} = H^{0,\bullet}_{\bar{\partial}_{m_A}} (\mathfrak{M}_{\mathfrak{m}}, \sqrt{K_{\mathfrak{M}_{\mathfrak{m}}}} \otimes \tilde{L}_T) \, . \tag{3.120}$$

where $\bar{\partial}_{m_A}$ is the Dolbeault operator deformed by the real mass parameter m_A and L^2-cohomology classes are understood.

We will not attempt a direct computation here, except in the special case $\mathfrak{m} = -\mathfrak{m}_A + 1$, corresponding to the blue line in figure 3.3. Here, the base of the fibration in equation (3.119) collapses to a point and the moduli space consists of the fiber $\mathfrak{M}_{\mathfrak{m}} = \mathbb{C}^{2\mathfrak{m}_A-1}$ parametrized by fluctuations of the meson $M = \Phi\tilde{\Phi}$. Quantizing these fluctuations, we find

$$\mathcal{H}_{\mathfrak{m}} = y^{2\mathfrak{m}_A-1-\mathfrak{m}_T} \bigoplus_{j=0}^{\infty} S^j \left(y^2 \mathbb{C}^{2\mathfrak{m}_A-1} \right) \, , \tag{3.121}$$

where y^2 is the weight of each coordinate on $\mathfrak{M}_{\mathfrak{m}} = \mathbb{C}^{2\mathfrak{m}_A-1}$ and the overall contribution $y^{2\mathfrak{m}_A-1-\mathfrak{m}_T}$ is the weight of the Fock vacuum induced by $K_{\mathfrak{M}_{\mathfrak{m}_A}} \otimes \widetilde{L}_T$. This agrees with the contribution to the supersymmetric twisted index

$$I_{\mathfrak{m}} = (-1)^{1+\mathfrak{m}_A} y^{-\mathfrak{m}_T} \left(\frac{y}{1-y^2} \right)^{2\mathfrak{m}_A-1} \, , \tag{3.122}$$

modulo a sign $(-1)^{\mathfrak{m}_A+1}$.

We finally provide two complete examples of fixed background fluxes $(\mathfrak{m}_A, \mathfrak{m}_T)$ by summing the contributions from the entire range of $\mathfrak{m}$.

- $\mathfrak{m}_A = 0$, $\mathfrak{m}_T = 0$

  There are no supersymmetric ground states with $\mathfrak{m} \leq 0$. Summing the contributions from $\mathfrak{m} > 0$, it follows from eq. (3.117) that

$$\mathcal{H} = \bigoplus_{\mathfrak{m}=1}^{\infty} q^{\mathfrak{m}} (y^{\mathfrak{m}}\mathbb{C} - y^{-\mathfrak{m}}\mathbb{C}) \tag{3.123}$$

with supersymmetric twisted index

$$I = \sum_{\mathfrak{m}=1}^{\infty} q^{\mathfrak{m}}(y^{\mathfrak{m}} - y^{-\mathfrak{m}}) = \frac{1-y^2}{y^2\left(1-\frac{1}{qy}\right)\left(1-\frac{q}{y}\right)} \,. \tag{3.124}$$

- $\mathfrak{m}_A = 1$, $\mathfrak{m}_T = 0$

  There are no supersymmetric ground states with $\mathfrak{m} < 0$ as the moduli space is empty. For $\mathfrak{m} > 0$, the moduli space is $\mathfrak{M}_{\mathfrak{m}} = \mathbb{CP}^{\mathfrak{m}}$ but nevertheless the contribution (3.118) to the twisted Hilbert space vanishes. Finally, the non-vanishing contribution from $\mathfrak{m} = 0$ gives

$$\mathcal{H} = y\, S^{\bullet}(y^2\mathbb{C}) \tag{3.125}$$

  with supersymmetric twisted index

$$I = \frac{y}{1-y^2} \,. \tag{3.126}$$

3.3 Mirror Symmetry

In this section, we will compare results obtained in section 3.2 to provide new checks of two basic instances of 3d mirror symmetry, which is an infrared duality. Naturally, we can probe the infrared regime of the theory by taking the limit of large volume of the curve $\mathrm{Vol}\,(\Sigma) \to +\infty$. The volume of the curve enters our equations in the dimensionless combination

$$s = \frac{\tau e^2 \mathrm{Vol}(\Sigma)}{2\pi} \tag{3.127}$$

introduced in (3.81). The deep infrared regime can therefore be reached by taking $s \to \infty$ in a chamber that is determined by τ.

3.3.1 Particle-Vortex Duality

We consider the following pair:

- I. Chiral multiplet with R-charge $r = 0$ and supersymmetric Chern-Simons terms $k_{ff} = k_{fR} = -\frac{1}{2}$ for the $U(1)_f$ flavour symmetry.
- II. $U(1)$ Chern-Simons theory at level $k = +\frac{1}{2}$ with one chiral multiplet of charge $+1$ and R-charge $r = 1$.

In theory II, the monopole operator of topological charge -1 is gauge neutral and can be identified with the chiral multiplet in theory I. The $U(1)_f$ flavour symmetry of theory I is identified with the $U(1)_T$ topological symmetry of theory II and therefore we make the identifications $m_f = \zeta$ and $x = q$.

We first compare the supersymmetric twisted Hilbert spaces for $g \geq 1$. Let us first recall the supersymmetric twisted Hilbert space of theory I with a background line bundle L_f for the $U(1)_f$ flavour symmetry in the chamber $\mathfrak{c}_+ = \{m_f > 0\}$. From equation (3.34) we find

$$\begin{aligned} \mathcal{H}_+^{(\mathrm{I})} &= x^{-\frac{\mathfrak{m}_f}{2}} x^{-\frac{g-1}{2}} \cdot x^{\frac{\mathfrak{m}_f - g + 1}{2}} \bigoplus_{j=0}^{\infty} x^j \left(\bigoplus_{p+q=j} S^p H^0(L_f) \otimes \wedge^q H^1(L_f) \right) \\ &= x^{1-g} \bigoplus_{j=0}^{\infty} x^j \left(\bigoplus_{p+q=j} S^p H^0(L_f) \otimes \wedge^q H^1(L_f) \right) , \end{aligned} \tag{3.128}$$

where $h^0(L_f) - h^1(L_f) = \mathfrak{m}_f - g + 1$. The additional factor $x^{-\frac{\mathfrak{m}_f}{2}} x^{-\frac{g-1}{2}}$ arises from the shift in the charge of the Fock vacuum induced by the mixed supersymmetric Chern-Simons terms $k_{ff} = k_{fR} = -\frac{1}{2}$.

Now recall the supersymmetric twisted Hilbert space of theory II in the chamber $\mathfrak{c}_+ = \{\tau > 0\}$. Introducing a background line bundle L_T for the topological symmetry, from equation (3.87) we find

$$\begin{aligned} \mathcal{H}_+^{(\mathrm{II})} &= \bigoplus_{\mathfrak{m} \geq 1-g} q^{\mathfrak{m}} \left(\bigoplus_{p+q=\mathfrak{m}+g-1} S^p H^0(L_T) \otimes \wedge^q H^1(L_T) \right) \\ &= q^{1-g} \bigoplus_{j=0}^{\infty} q^j \left(\bigoplus_{p+q=j} S^p H^0(L_T) \otimes \wedge^q H^1(L_T) \right) \end{aligned} \tag{3.129}$$

where $h^0(L_T) - h^1(L_T) = \mathfrak{m}_T - g + 1$. Under the identifications $m_f = \zeta$, $x = q$ and $L_f = L_T$, we find complete agreement between the supersymmetric twisted Hilbert spaces.

This constitutes a more refined check of mirror symmetry than the supersymmetric index. First, $\mathcal{H}$ contains an infinite number of supersymmetric ground states whereas the index I truncates to a finite Laurent polynomial. Second, we find agreement between the supersymmetric Hilbert space as we vary the background line bundle $L_f = L_T$ over the Jacobian J_Σ. The index cannot detect this phenomenon, as it depends only on the degree of $\mathfrak{m}_f = \mathfrak{m}_T$.

Let us briefly consider the genus zero case, including the parameter ξ for the additional flavour symmetry $U(1)_\epsilon$ rotating $\Sigma = \mathbb{CP}^1$. For simplicity, we assume that $\mathfrak{m} \geq 0$. The supersymmetric twisted Hilbert spaces are

$$\begin{aligned} \mathcal{H}_+^{(\mathrm{I})} &= \bigoplus_{j=1}^{\infty} x^j S^{j-1}(\xi^\rho \mathbb{C}^{\mathfrak{m}+1}) \\ \mathcal{H}_+^{(\mathrm{II})} &= \bigoplus_{j=1}^{\infty} x^j S^{\mathfrak{m}}(\xi^\rho \mathbb{C}^j) , \end{aligned} \tag{3.130}$$

which agree due to the following isomorphism of graded vector spaces,

$$S^{j-1}(\xi^\rho \mathbb{C}^{\mathfrak{m}+1}) \cong S^{\mathfrak{m}}(\xi^\rho \mathbb{C}^j) . \tag{3.131}$$

3.3.2 XYZ $\leftrightarrow$ SQED

We now turn to the investigation of the following pair:

- I. Three chiral multiplets with cubic superpotential $W = XYZ$ and charge matrix:

	$U(1)_T$	$U(1)_A$	$U(1)_R$
X	1	-1	0
Y	-1	-1	0
Z	0	2	2

.

- II. $U(1)$ supersymmetric gauge theory with chiral multiplets Φ, $\tilde{\Phi}$ with charge matrix:

	$U(1)$	$U(1)_T$	$U(1)_A$	$U(1)_R$
Φ	1	0	1	1
$\tilde{\Phi}$	-1	0	1	1

.

Here we identify from the outset the $U(1)_T \times U(1)_A$ flavour symmetry. Under 3d mirror symmetry, Z is mapped to the gauge invariant combination $\Phi\tilde{\Phi}$ whereas X,Y are mapped to monopole operators in the supersymmetric gauge theory of charge ± 1 under the topological symmetry.

Let us first consider $g > 0$. We will not attempt an exhaustive analysis, but specialize to $L_A = L_T = \mathcal{O}_\Sigma$ and real mass parameters in the chamber

$$\mathfrak{c}_{+-+} = \{0 < m_A < m_T\} \tag{3.132}$$

where we have obtained explicit results on both sides. This corresponds to an expansion of the supersymmetric twisted index in the region $|q| < |y| < 1$.

First, there are no supersymmetric ground states of topological charge $q_T \leq g$ on both sides. Second, we consider the supersymmetric ground states of charge $q_T \geq g$, whose contributions to the supersymmetric index vanish. Comparing equations (3.63) and (3.106) we find the contribution to $\mathcal{H}_{+-+}$ in both theories is

$$q^{q_T}(y^{q_T}\mathbb{C} - y^{-q_T}\mathbb{C}) \otimes \wedge^\bullet(\mathbb{C}^g) \qquad q_T \geq g \tag{3.133}$$

modulo an overall sign. Note that $q_T = \mathfrak{m}$. Finally, in the intermediate region $-g < q_T < g$ we have only computed explicitly the supersymmetric ground states with minumum charge $q_T = 1-g$. Comparing equations (3.68) and (3.110), we find that the contribution to $\mathcal{H}_{+-+}$ in both theories is

$$q^{1-g}y^{g-1}(\mathbb{C} - y^2\mathbb{C}) \otimes \bigoplus_{n=0}^{\infty} y^{2n} S^n(\mathbb{C}^g) \,. \tag{3.134}$$

In regions where we can independently compute the supersymmetric ground states, we therefore find agreement with mirror symmetry.

We now study the case $g = 0$ with background fluxes $\mathfrak{m}_T$, $\mathfrak{m}_A$ for the flavour symmetry. We separate the analysis into characteristic regions in the $(\mathfrak{m}_T, \mathfrak{m}_A)$ plane:

- In the region $\mathfrak{m}_A \leq 0$, $\mathfrak{m}_T \geq -\mathfrak{m}_A$ we have computed the supersymmetric twisted Hilbert space in both theory I and II. These computations agree, but there are no cancellations in passing to the supersymmetric twisted index in this case, so we cannot provide a stronger check of mirror symmetry.

- In the region $\mathfrak{m}_A \leq 0$, $0 \leq \mathfrak{m}_T < -\mathfrak{m}_A$, we have computed the supersymmetric twisted Hilbert space of theory II in equation (3.117). In this case, there are cancellations between bosons and fermions in computing the supersymmetric twisted index. However, we have not performed an independent computation for theory I. Nevertheless, it is possible to check that the supersymmetric ground states in (3.117) form a subset of those for the three chiral multiplets X, Y, Z in the absence of a superpotential. This is consistent with the presence of a non-vanishing J-term differential in supersymmetric quantum mechanics for theory I in this region and equation (3.117) can be regarded as a prediction for its cohomology.

- In the other direction, we can use our result for supersymmetric ground states of theory I to make a prediction for the supersymmetric ground states in the unknown region $\mathfrak{m}_A \geq 1$, $|\mathfrak{m}_T| < \mathfrak{m}_A$ of theory II. From equation (3.73),

$$\begin{aligned}\mathcal{H}^{(I)}_{+-+} =& q^{\mathfrak{m}_T} y^{3\mathfrak{m}_A-2} S^\bullet(y^2\mathbb{C}^{2\mathfrak{m}_A-1})\otimes \\ &\otimes \bigoplus_{\mathfrak{m}=-\mathfrak{m}_T-\mathfrak{m}_A+1}^{-\mathfrak{m}_T+\mathfrak{m}_A-1} q^{\mathfrak{m}} \left(\bigoplus_{i-k=\mathfrak{m}} \wedge^i(y^{-1}\mathbb{C}^{-\mathfrak{m}_T+\mathfrak{m}_A-1}) \otimes \wedge^k(y^{-1}\mathbb{C}^{\mathfrak{m}_T+\mathfrak{m}_A-1}) \right) .\end{aligned} \tag{3.135}$$

  This should reproduce the cohomology expressed in equation (3.120),

$$\mathcal{H}^{(II)}_{+-+} = H^{0,\bullet}_{\bar{\partial}_{m_A}}(\mathfrak{M}_{\mathfrak{m}}, \sqrt{K_{\mathfrak{M}_{\mathfrak{m}}}} \otimes \widetilde{L}_T)\,, \tag{3.136}$$

  where

$$\mathfrak{M}_{\mathfrak{m}} = \mathcal{O}(-1)^{\mathfrak{m}_A-\mathfrak{m}} \to \mathbb{CP}^{\mathfrak{m}+\mathfrak{m}_A-1}\,. \tag{3.137}$$

  In the degenerate case $\mathfrak{m} = -\mathfrak{m}_A + 1$ where the base collapses to a point, equation (3.135) correctly reproduces the cohomology of the unique fibre $\mathfrak{M}_{\mathfrak{m}} = \mathbb{C}^{2\mathfrak{m}_A-1}$. It would be interesting to understand how to compute this cohomology in the general case.

Finally, demonstrating agreement between regions where there are no supersymmetric ground states is straightforward. Taking this into consideration, we are able to chart the supersymmetric ground states on almost the whole range of parameters $(\mathfrak{m}_T, \mathfrak{m}_A, \mathfrak{m})$, aside from a small region that depends on $\mathfrak{m}_T$.

Chapter 4

Twisted Indices of $\mathcal{N} = 4$ Gauge Theories and Enumerative Geometry of Quasi-maps

In the previous chapter, we introduced a description of 3d $\mathcal{N} = 2$ Chern-Simons-matter theories in terms of a 1d effective quantum mechanics. We focussed on the spaces of supersymmetric ground states of some selected theories, unveiling rich structures and dependences on background parameters that were hidden by index computations. The target spaces of the quantum mechanics, however, are in general too complicated and loosely defined to be amenable to similar techniques.

In this chapter, we therefore impose two restrictions. First, we consider a specific class of $\mathcal{N} = 4$ quiver gauge theories. Second, we prioritize the computation of the twisted indices. $\mathcal{N} = 4$ supersymmetry is restrictive, but it is known to have deep connections with cutting-edge mathematics in the context of symplectic duality, see for example [81, 82, 83]. Our aim is to give the indices a mathematical interpretation in terms of the enumerative geometry of spaces of quasi-maps. In this way, we will be able both to compute some of the indices more easily, and to make predictions about the enumerative invariants in the context of symplectic duality.

The chapter is organised as follows. In section 4.1, we define the class of 3d $\mathcal{N} = 4$ theories we are going to study and we provide some background. In particular, we summarize the construction of one of the moduli spaces of vacua, the Higgs branch $\mathcal{M}_H$, in algebraic terms. In section 4.2, we review the procedure of topological reduction of the 3d $\mathcal{N} = 4$ theories on $\mathbb{R} \times \Sigma$, and study the moduli space of solutions to the BPS equations $\mathfrak{M}$ in terms of quasi-maps to $\mathcal{M}_H$. More specifically, we study the massless fluctuations of the bosonic and fermionic fields at a point on the moduli space and reconstruct the virtual tangent bundle T^{vir} over $\mathfrak{M}$. From this discussion, we provide a geometric interpretation of the contour integral formula as the virtual Euler characteristics constructed from T^{vir}. In section 4.3, we study the reduced moduli space that preserves four supercharges and discuss the relation to the twisted indices evaluated in the limit $t \to 1$. In section 4.4, we explore the geometric interpretations of the twisted indices in concrete examples. Finally, in section 4.5, we study the implications of mirror symmetry in this context, and explicitly check the proposed dualities for some theories in the limit $t \to 1$. This implies a notion of symplectic duality for quasi-maps invariants.

4.1 Background and Notation

4.1.1 Quiver Gauge Theories

A renormalizable 3d $\mathcal{N} = 4$ supersymmetric gauge theory is specified by a compact group G and a linear quaternionic representation Q –we refer the reader to [15, 83] for a summary and further background. In this chapter, we will focus on unitary quiver gauge theories. Introducing an index $I = 1, \ldots, L$ labelling the nodes of the quiver, this corresponds to the choice

$$G = \prod_I U(V_I) \qquad Q = T^*M \tag{4.1}$$

where

$$M = \bigoplus_I \mathrm{Hom}(W_I, V_I) \oplus \bigoplus_{I \leq J} \mathrm{Hom}(V_I, V_J) \otimes Q_{IJ} \,. \tag{4.2}$$

is a unitary representation of G. Here V_I, W_I denote complex vector spaces while Q_{IJ} are multiplicities. In physical parlance, there is a dynamical vectormultiplet for the gauge group G and

- Q_{II} hypermultiplets in the adjoint representation of $U(V_I)$,
- Q_{IJ} hypermultiplets in the bifundamental representation of $U(V_I) \times U(V_J)$ for $I < J$,
- and $\dim_{\mathbb{C}} W_I$ hypermultiplets in the fundamental representation of $U(V_J)$.

An example is the single node quiver with $V = \mathbb{C}^{N_c}$, $W = \mathbb{C}^{N_f}$ and unitary representation $M = \mathrm{Hom}(W, V)$. This is supersymmetric QCD with $G = U(N_c)$ and N_f fundamental hypermultiplets, as illustrated in figure 4.1. In the following sections 4.2 and 4.3, we will formulate our constructions for a general unitary quiver (subject to an assumption explained in section 4.1.2) but our explicit examples in section 4.4 will be almost exclusively supersymmetric QCD.

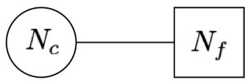

Figure 4.1: Quiver for $U(N_c)$ supersymmetric QCD with N_f fundamental hypermultiplets

In what follows, we use Euclidean $SU(2)$ spinor indices α in addition to spinor indices A, $\dot{A}$ for the $SU(2)_H \times SU(2)_C$ R-symmetry, with uniform conventions summarized in Appendix A.1.

With this notation, the vectormultiplet includes a gauge connection $A_{\alpha\beta}$, scalar fields $\phi^{\dot{A}\dot{B}}$, and gauginos $\lambda_\alpha^{A\dot{A}}$ transforming in the adjoint representation of G. The hypermultiplets contains complex scalars X_A and fermionic spinors $\Psi_\alpha^{\dot{A}}$ transforming in the unitary representation M.

It will be convenient to decompose the supermultiplets under a fixed maximal torus $U(1)_H \times U(1)_C$ of the R-symmetry. The vectormultiplet scalars decompose into real and complex components σ, φ, $\varphi^\dagger$ transforming with $U(1)_C$ charge $0, +1, -1$ respectively, while the hypermultiplet scalars decompose into a pair of complex scalars X, Y transforming with $U(1)_H$ charge $+\frac{1}{2}$. The charges of these fields are shown in table 4.1.

	G	$U(1)_H$	$U(1)_C$	$U(1)_t$
σ	Adj	0	0	0
φ	Adj	0	+1	−1
X	M	$+\frac{1}{2}$	0	$+\frac{1}{2}$
Y	M^*	$+\frac{1}{2}$	0	$+\frac{1}{2}$

Table 4.1: Summary of gauge and R-symmetry representations.

The flavour symmetry is a product $G_H \times G_C$ where:

- G_H acts on the hypermultiplets and coincides with the unitary transformations of M that act independently of G, forming an exact sequence
$$G \hookrightarrow U(M) \to G_H \to 0\,. \tag{4.3}$$
- G_C contains topological symmetry $U(1)^L$ under which monopole operators are charged. This may be enhanced in the IR to a non-Abelian group with maximal torus $U(1)^L$.

We turn on associated real mass deformations valued in the Cartan subalgebras $\mathfrak{t}_H$, $\mathfrak{t}_C$ of the flavour symmetry factors:

- Real mass parameters $m \in \mathfrak{t}_H$ are vacuum expectation values for the real scalar in a background vectormultiplet for G_H.
- Real FI parameters $\zeta \in \mathfrak{t}_C$ are vacuum expectation values for the real scalar in a background twisted vectormultiplet for G_C.

In principle, we could also turn on complex FI parameters, but we do not consider them in this thesis.

In supersymmetric QCD, $G_H = PSU(N_f)$ and $G_C = U(1)$, enhanced to $G_C = SU(2)$ when $N_f = 2N_c$. Correspondingly, we introduce real mass parameters $m = (m_1, \ldots, m_{N_f}) \in \mathbb{R}^{N_f-1}$ satisfying $\sum_j m_j = 0$ and a single FI parameter $\zeta \in \mathbb{R}$.

It will also be important to introduce a real mass parameter that breaks $\mathcal{N} = 4$ to $\mathcal{N} = 2$ supersymmetry. Given the maximal torus $U(1)_H \times U(1)_C$ with generators T_H, T_C, we may decompose the supermultiplets under the $\mathcal{N} = 2$ supersymmetry commuting with the $U(1)_t$ generated by
$$T_t = T_H - T_C\,. \tag{4.4}$$

From this perspective, $U(1)_t$ is a distinguished flavour symmetry. We can then choose an integer R-symmetry for the $\mathcal{N}=2$ supersymmetry algebra generated by $R_H = 2T_H$ or $R_C = 2T_C$. This choice is important when performing a topological twist on $S^1 \times \Sigma$.

From the perspective of $\mathcal{N}=2$ supersymmetry σ transforms in a vectormultiplet, while φ, X, Y transform in chiral multiplets whose charges are summarized in table 4.1. There are also superpotentials

$$W_I = \mathrm{Tr}_{V_I}(\varphi XY) \tag{4.5}$$

at each node whose R-charges are always $+2$. The real mass parameters m are now obtained by coupling to a background $\mathcal{N}=2$ vectormultiplet for the flavour symmetry G_H while ζ is an FI parameter for the dynamical $\mathcal{N}=2$ vectormultiplet.

We can now explicitly break to $\mathcal{N}=2$ supersymmetry by introducing a real mass parameter m_t for the distinguished $U(1)_t$ flavour symmetry. This is the mass deformation mentioned in the introduction to this thesis and, as anticipated there, it will play an important role in this chapter as a localization parameter.

4.1.2 Moduli Spaces of Vacua

The moduli space of vacua of 3d $\mathcal{N}=4$ supersymmetric gauge theory includes a Higgs branch and a Coulomb branch, denoted by $\mathcal{M}_H$ and $\mathcal{M}_C$ respectively. They are both hyper-Kähler, such that the R-symmetries $SU(2)_H$, $SU(2)_C$ rotate the complex structure on $\mathcal{M}_H$, $\mathcal{M}_C$. Furthermore, the flavour symmetries G_H, G_C act by tri-Hamiltonian isometries of $\mathcal{M}_H$, $\mathcal{M}_C$.

The choice of maximal torus $U(1)_H \times U(1)_C$ selects a complex structure on $\mathcal{M}_H$ and $\mathcal{M}_C$. From this point of view, they are Kähler manifolds equipped with holomorphic symplectic forms of weight $+1$ under Kähler isometries $U(1)_H$, $U(1)_C$. The flavour symmetries G_H, G_C act by Hamiltonian isometries of $\mathcal{M}_H$, $\mathcal{M}_C$ that leave invariant the holomorphic symplectic form.

In this chapter, we make a crucial assumption that the supersymmetric quiver gauge theory flows to a superconformal fixed point and has isolated massive vacua when generic real mass and FI parameters are turned on. This translates into the assumption that $\mathcal{M}_H$, $\mathcal{M}_C$ are conical symplectic resolutions with isolated fixed points under infinitesimal T_H, T_C transformations. Furthermore, $\mathfrak{t}_H$, $\mathfrak{t}_C$ describe Kähler resolution parameters for $\mathcal{M}_C$, $\mathcal{M}_H$ under the identifications

$$\mathfrak{t}_H = H^2(\mathcal{M}_C, \mathbb{R}) , \qquad \mathfrak{t}_C = H^2(\mathcal{M}_H, \mathbb{R}) . \tag{4.6}$$

In more physical terms:

- The mass parameters $m \in \mathfrak{t}_H$ are resolution parameters for $\mathcal{M}_C$ and generate an infinitesimal Hamiltonian isometry of $\mathcal{M}_H$,

- The FI parameters $\zeta \in \mathfrak{t}_C$ are resolution parameters for $\mathcal{M}_H$ and generate an infinitesimal Hamiltonian isometry of $\mathcal{M}_C$.

This assumption will permeate our considerations on $S^1 \times \Sigma$, allowing explicit computations to be performed while encompassing an infinite and rich class of examples. A further motivation is that such theories transform straightforwardly under 3d mirror symmetry and play an important role in connection with symplectic duality [81, 82]. For further motivation and background we refer the reader to reference [83]. We will return to this connection in section 4.5.

4.1.3 Higgs Branch Geometry

The Higgs branch is particularly important for consideration of the twisted index on $S^1 \times \Sigma$. We therefore explain its construction in more detail now. We first set the mass parameters $m = 0$. The classical vacuum configurations are solutions to

$$\begin{aligned} \mu_{\mathbb{R}} - \zeta = 0 \qquad & \mu_{\mathbb{C}} = 0 \\ \sigma \cdot X = 0 \qquad & \varphi \cdot X = 0 \qquad \varphi^\dagger \cdot X = 0 \\ \sigma \cdot Y = 0 \qquad & \varphi \cdot Y = 0 \qquad \varphi^\dagger \cdot Y = 0 \\ [\sigma, \varphi] = 0 \qquad & [\varphi, \varphi^\dagger] = 0\,, \end{aligned} \tag{4.7}$$

modulo gauge transformations. Here it is understood that vectormultiplet scalars act on (X, Y) in the representation T^*M. Finally,

$$\mu_{\mathbb{R}} = X \cdot X^\dagger - Y^\dagger \cdot Y \qquad \mu_{\mathbb{C}} = X \cdot Y \tag{4.8}$$

are the real and complex moment maps for the G action on T^*M.

Equations (4.7) may be decomposed into contributions from each node labelled by an index $I = 1, \ldots, L$. Here we are employing shorthand notation such as $\zeta = \{\zeta_1, \ldots, \zeta_L\}$ and $\mu_{\mathbb{R}} = \{\mu_{\mathbb{R},1}, \ldots, \mu_{\mathbb{R},L}\}$ to express the contributions from all of the nodes simultaneously.

For future applications, it is useful to reconsider the vacuum equations in the language of $\mathcal{N} = 2$ supersymmetry. From this perspective, the vacuum equations are

$$\begin{aligned} & \mu_{\mathbb{R}} - 2[\varphi^\dagger, \varphi] - \zeta = 0 \\ & \varphi \cdot X = 0 \qquad \varphi \cdot Y = 0 \qquad \mu_{\mathbb{C}} = 0 \\ & \sigma \cdot X = 0 \qquad \sigma \cdot Y = 0 \qquad [\sigma, \varphi] = 0\,, \end{aligned} \tag{4.9}$$

where the first line contains the D-term equations and the second line the F-term equations associated to the superpotential $W = \mathrm{Tr}_V(\varphi X Y)$. Note that the D-term equation involves an additional commutator compared to (4.7). However, by squaring the D-term equation and imposing the F-term equations,

$$\|\mu_{\mathbb{R}} - 2[\varphi^\dagger, \varphi] - \zeta\|^2 = \|\mu_{\mathbb{R}} - \zeta\|^2 + 4\|[\varphi^\dagger, \varphi]\|^2 + 2\|\varphi \cdot X^\dagger\|^2 + 2\|\varphi \cdot Y^\dagger\|^2\,, \tag{4.10}$$

which requires $[\varphi^\dagger, \varphi] = 0$ separately and recovers the remaining equations in (4.7).

Hyper-Kähler Quotient

Under our assumptions, the FI parameter ζ can be chosen such that G acts freely on solutions to the vacuum equations (4.7). This typically requires that the FI parameter lie in the complement of hyperplanes

$$\zeta \in \mathbb{R}^L \backslash \cup_\alpha H_\alpha \,, \tag{4.11}$$

which split the parameter space $\mathfrak{t}_C = \mathbb{R}^L$ into chambers. In supersymmetric QCD, this means we assume that $N_f \geq N_c$ and $\zeta > 0$ or $\zeta < 0$.

The implies $\sigma = \varphi = 0$ on solutions of the vacuum equations, which would otherwise generate unbroken gauge transformations. The remaining equations then describe the Higgs branch as a smooth hyper-Kähler quotient

$$\mathcal{M}_{\zeta,H} := T^*M /\!/\!/_\zeta \, G \,, \tag{4.12}$$

which is a Nakajima quiver variety [84, 85]. We note that the holomorphic symplectic form on the Higgs branch is independent of ζ within each chamber, while the real symplectic form or Kähler structure depends explicitly on ζ.

The assumption of section 4.1.2 requires that

$$\nu : \mathcal{M}_{H,\zeta} \to \mathcal{M}_{H,0} \tag{4.13}$$

is a conical symplectic resolution. The inverse image

$$\mathcal{L}_{H,\zeta} := \nu^{-1}(0) \tag{4.14}$$

is then a compact holomorphic Lagrangian known as the 'compact core'. This has a convenient Kähler quotient description, which reads as follows. The choice of chamber selects a holomorphic Lagrangian splitting $T^*M = L \oplus L^*$, corresponding to a decomposition of the hypermultiplet fields (X_L, Y_L) where $Y_L = 0$ on the compact core. We then have

$$\mathcal{L}_{H,\zeta} = L /\!/_\zeta \, G = \{\mu_{\mathbb{R}}|_L = 0\}/G \,. \tag{4.15}$$

We frequently fix a chamber and omit the dependence on ζ, writing $\mathcal{M}_H$ and $\mathcal{L}_H$ respectively for the Higgs branch and its compact core.

In supersymmetric QCD, this assumption requires that $N_f \geq 2N_c$. In this case, the Higgs branch is a cotangent bundle to the grassmannian of N_c-planes in N_f complex dimensions, $\mathcal{M}_H = T^*G(N_c, N_f)$. The map (4.13) is the Springer resolution of the nilpotent cone closure $\bar{\mathcal{N}}_\rho \subset \mathfrak{sl}(N_f, \mathbb{C})$ labelled by $\rho^T = (N_c, N_f - N_c)$. The compact core $\mathcal{L}_H = G(N_c, N_f)$ is the grassmannian base, where:

- In the chamber $\zeta > 0$, $\mathcal{L}_H$ is characterized by the decomposition $(X_L, Y_L) = (X, Y)$ and corresponds to configurations with $Y = 0$.
- In the chamber $\zeta < 0$, $\mathcal{L}_H$ is characterized by the decomposition $(X_L, Y_L) = (Y, -X)$ and corresponds to configurations with $X = 0$.

Algebraic Description

The Higgs branch has an algebraic description as a holomorphic symplectic quotient, obtained by omitting the D-term equation in favour of an appropriate stability condition, and by dividing by complex gauge transformations.

Starting from $(X,Y) \in T^*M$, solutions of the F-term equation cut out the subspace $\mu_{\mathbb{C}}^{-1}(0) \subset T^*M$. We then impose a stability condition depending on the chamber of $\zeta \in \mathbb{R}^L \setminus \cup_\alpha H_\alpha$ and quotient by complex gauge transformations $G_{\mathbb{C}}$. Under our assumptions, stability coincides with semi-stability and we obtain a smooth quotient,

$$\mathcal{M}_H = \mu_{\mathbb{C}}^{-1}(0)^s / G_{\mathbb{C}} \,. \tag{4.16}$$

We do not describe the stability condition for a general quiver, and instead focus later on the example of supersymmetric QCD [1].

This provides an algebraic description of the tangent bundle to $\mathcal{M}_H$, which will reappear in section 4.2.4. Considering small fluctuations of the hypermultiplets $(\delta X, \delta Y)$ compatible with the F-term equation, modulo infinitesimal complex gauge transformations, generates the following complex

$$0 \longrightarrow \mathfrak{g}_{\mathbb{C}} \xrightarrow{\alpha} T^*M \xrightarrow{\beta} \mathfrak{g}_{\mathbb{C}}^* \longrightarrow 0 \tag{4.17}$$

of trivial $G_{\mathbb{C}}$-equivariant vector bundles on T^*M. The maps

$$\alpha : \delta g \mapsto (\delta g \cdot X, \delta g \cdot Y) \qquad \beta : (\delta X, \delta Y) \mapsto X \cdot \delta Y + \delta X \cdot Y \tag{4.18}$$

at a point $(X,Y) \in T^*M$ correspond to infinitesimal complex gauge transformations and the differential of the complex moment map respectively. On restriction to the stable locus $\mu_{\mathbb{C}}^{-1}(0)^s$, α is injective and β surjective, and equation (4.17) descends to a complex of vector bundles on $\mathcal{M}_H$ whose cohomology is the tangent bundle,

$$T\mathcal{M}_H = \mathrm{Ker}(\beta)/\mathrm{Im}(\alpha) \,. \tag{4.19}$$

In supersymmetric QCD in the chamber $\zeta > 0$, the stable locus consists of solutions where X has maximal rank and defines a complex N_c-plane in $W = \mathbb{C}^{N_f}$. The holomorphic symplectic quotient

$$\{\, X, Y \,|\, X \cdot Y = 0, \, \mathrm{rk}(X) = N_c \,\} / GL(N_c, \mathbb{C}) \tag{4.20}$$

provides an algebraic description of $\mathcal{M}_H = T^*G(N_c, N_f)$. The tangent bundle is the cohomology of the complex

$$0 \longrightarrow \mathrm{Hom}(\mathcal{V},\mathcal{V}) \xrightarrow{\alpha} T^*\mathrm{Hom}(\mathcal{W},\mathcal{V}) \xrightarrow{\beta} \mathrm{Hom}(\mathcal{V},\mathcal{V}) \longrightarrow 0 \,, \tag{4.21}$$

where $\mathcal{V}$ is the tautological complex vector bundle with fiber $V = \mathbb{C}^{N_c}$ and $\mathcal{W}$ is the trivial complex vector bundle with fiber W. The maps are the infinitesimal complex gauge transformation $\alpha : \delta g \mapsto (\delta g X, -Y \delta g)$ and the differential of the complex moment map $\beta : (\delta X, \delta Y) \mapsto \delta X Y + X \delta Y$.

[1] An account of the appropriate stability condition for a general quiver that is close to the perspective taken here can be found in [86].

4.1.4 Mass Parameters and Fixed Loci

We now consider the fate of the Higgs branch vacua in the presence of real mass parameters m_t and m associated to flavour symmetries $U(1)_t$ and G_H respectively.

$U(1)_t$ Mass Parameter

The mass parameter m_t is a vacuum expectation value for a background $\mathcal{N}=2$ vectormultiplet for the flavour symmetry $U(1)_t$. Accordingly, the supersymmetric vacuum equations (4.7) are modified by replacing $\sigma \to \sigma + m_t$ (acting in the appropriate representation). More precisely,

$$\sigma \cdot X + \frac{m_t}{2} X = 0 \qquad \sigma \cdot Y + \frac{m_t}{2} Y = 0 \qquad [\sigma, \varphi] - m_t \varphi = 0 \tag{4.22}$$

in view of the charges presented in table 4.1. The remaining supersymmetric vacua correspond to configurations (X, Y, φ) solving the modified vacuum equations, for which there exists a σ such that the combined infinitesimal gauge and $U(1)_t$ transformation generated by σ and m_t leaves the configuration invariant.

Such configurations are found by setting $Y_L = 0$ where $T^*M = L \oplus L^*$ is the Lagrangian splitting introduced above. It is useful to note that under the combined gauge and $U(1)_t$ transformation that leaves this configuration invariant, the hypermultiplet fields (X_L, Y_L) transform with weight $(0,1)$. This property could be used to characterize the holomorphic Lagrangian splitting.

Geometrically, the remaining supersymmetric vacua correspond to the fixed locus of the $U(1)_t$ Kähler isometry of $\mathcal{M}_H$ generated by the mass parameter m_t. From the discussion above, this coincides with the compact core,

$$\mathcal{M}_H^{U(1)_t} = \mathcal{L}_H \, . \tag{4.23}$$

In the algebraic description, the $U(1)_t$ isometry becomes a $\mathbb{C}^*$ action that transforms the holomorphic symplectic form with weight $+1$. This will play an important role in the definition of the enumerative invariants to be considered in section 4.2.

For example, in supersymmetric QCD with $N_f \geq 2N_c$ in the chamber $\zeta > 0$, the mass deformation requires $\sigma = -\frac{m_t}{2} 1_{N_c}$ and $Y = 0$. Indeed, $U(1)_t$ acts on the fibres of $\mathcal{M}_H = T^*G(N_c, N_f)$ with weight $+1$ such that the remaining supersymmetric vacua coincide with the compact core, $\mathcal{M}_H^{U(1)_t} = G(N_c, N_f)$.

G_H Mass Parameters

Let us now add real mass parameters $m \in \mathfrak{t}_H$ by turning on a vacuum expectation value for a background $\mathcal{N}=2$ vectormultiplet for the G_H flavour symmetry. The vacuum equations (4.7) are modified by

$$\sigma \to \sigma + m + m_t \, , \tag{4.24}$$

where again it is understood that the mass parameters act in the appropriate representation of $U(1)_t \times G_H$. The remaining vacua now correspond to configurations (X, Y, φ) solving the modified vacuum equations, for which there exists a σ such that the combined infinitesimal gauge and $G_H \times U(1)_t$ transformation generated by σ and $m + m_t$ leaves the configuration invariant.

Geometrically, the remaining vacua correspond to the fixed locus of the $T_H \times U(1)_t$ isometry of $\mathcal{M}_H$ generated $m + m_t$. The assumption of section 4.1.2 requires that for generic mass parameters m, the fixed locus be a set of isolated points

$$\mathcal{M}_H^{T_H \times U(1)_t} = \{v_I\} \,. \tag{4.25}$$

The fixed points necessarily lie in the compact core. Each massive vacuum corresponds to a configuration of $\mathrm{rk}(G)$ non-vanishing hypermultiplet fields chosen from X_L, which we denote collectively by $\{Z_a\}$. We note that in the algebraic description, T_H is promoted to a $(\mathbb{C}^*)^{\mathrm{rk}(G)}$ action leaving the holomorphic symplectic form invariant.

In supersymmetric QCD the flavour symmetry $G_H = PSU(N_f)$ acts by Kähler isometries on $\mathcal{M}_H = T^*G(N_c, N_f)$. Turning on generic mass parameters $m = \{m_1, \ldots, m_{N_f}\}$ obeying $\sum_{i=1}^{N_f} m_i = 0$, there are $\binom{N_f}{N_c}$ massive supersymmetric vacua labelled by distinct subsets $I = \{i_1, \ldots, i_{N_c}\} \subset \{1, \ldots, N_f\}$ where

$$v_I \quad : \quad \sigma_a = m_{i_a} \qquad \varphi_a = 0 \qquad Z_a = X^a{}_{i_a} \,. \tag{4.26}$$

They are the fixed points of a generic $T_H \times U(1)_t$ isometry of $\mathcal{M}_H$ and coincide with the coordinate hyperplanes in the grassmannian base $\mathcal{L}_H = G(N_c, N_f)$.

4.2 $\mathcal{N} = 4$ Twisted Theories on $\mathbb{R} \times \Sigma$

Having defined the class of theories of interest to us, we turn to the compactification on $\mathbb{R} \times \Sigma$. $\mathcal{N} = 4$ theories admit two topological twists, the Rozansky-Witten twist and its mirror. On a general three-manifold, these twist can be performed by using the $SU(2)_C$ and $SU(2)_H$ R-symmetries respectively; in our configuration, it is sufficient to use their $U(1)_C$ and $U(1)_H$ subalgebras[2]. The resulting topologically twisted theories are special cases of the $\mathcal{N} = 2$ supersymmetric gauge theories on $\mathbb{R} \times \Sigma$ considered in chapter 2.

In this section, we summarize the important aspects of the possible topological twists and of the localization results. We then take the effective quantum-mechanical perspective, and study the fluctuations around a point on the moduli space of solutions to the BPS equation (2.31). From this we will provide a general relation between the twisted indices and enumerative invariants of the moduli space. An important motivation for the work in this chapter was to understand the geometric origin of the Jeffrey-Kirwan contour prescription (2.17), as in the original mathematical constructions [70], and we provide some explanations in this direction.

[2]For more details on the mirror of the Rozansky-Witten twist, see ref. [87].

4.2.1 Algebraic Preliminaries

With the real mass and FI parameter deformations we introduced above, the flat space supersymmetry algebra is

$$\{Q_\alpha^{A\dot{A}}, Q_\beta^{B\dot{B}}\} = \epsilon^{AB}\epsilon^{\dot{A}\dot{B}}P_{\alpha\beta} - \epsilon_{\alpha\beta}\epsilon^{AB}\delta^{(\dot{A}\dot{1}}\delta^{\dot{B})\dot{2}}(m\cdot J_H) - \epsilon_{\alpha\beta}\epsilon^{\dot{A}\dot{B}}\delta^{(A1}\delta^{B)2}(\zeta\cdot J_C) \tag{4.27}$$

where $Q_\alpha^{A\dot{A}}$ denotes the supercharges, $P_{\alpha\beta}$ is the momentum generator, and J_H, J_C are the conserved charges associated to the flavour symmetry. Notice that we have to symmetrize over the indices A and B in the Kronecker deltas, and the real masses break the R-symmetry to a maximal torus $U(1)_H \times U(1)_C$[3]. Under our conventions, the supercharges $Q_\alpha^{1\dot{A}}$, $Q_\alpha^{2\dot{A}}$ have $U(1)_H$ charge $+\frac{1}{2}$, $-\frac{1}{2}$, while the supercharges $Q_\alpha^{A\dot{1}}$, $Q_\alpha^{A\dot{2}}$ have $U(1)_C$ charge $+\frac{1}{2}$, $-\frac{1}{2}$.

We now perform the topological twist on $\mathbb{R}\times\Sigma$ using $U(1)_H$ or $U(1)_C$, and regard the system as a supersymmetric quantum mechanics on $\mathbb{R}$.

H-Twist

In the H-twist, we restrict to supercharges commuting with the diagonal combination of $U(1)_H \subset SU(2)_H$ and the $U(1)_{12}$ rotations in the $x^{1,2}$-plane. The invariant supercharges are

$$Q^{\dot{A}} := Q_1^{1\dot{A}} \qquad \widetilde{Q}^{\dot{A}} := Q_2^{2\dot{A}} \tag{4.28}$$

and generate the $\mathcal{N}=(2,2)$ supersymmetric quantum mechanics

$$\begin{aligned}
&\{Q^{\dot{A}}, Q^{\dot{B}}\} = 0\\
&\{Q^{\dot{A}}, \widetilde{Q}^{\dot{B}}\} = \epsilon^{\dot{A}\dot{B}}(P_3 - \zeta\cdot J_C) - \delta^{(\dot{A}\dot{1}}\delta^{\dot{B})\dot{2}} m\cdot J_H\\
&\{\widetilde{Q}^{\dot{A}}, \widetilde{Q}^{\dot{B}}\} = 0
\end{aligned} \tag{4.29}$$

with Hamiltonian $H = P_3 - \zeta\cdot J_C$ and central charge $Z = -m\cdot J_H$. In particular, we identify $U(1)_R = U(1)_C$ and $U(1)_r = U(1)_H$. Notice that this fits into the $\mathcal{N}=4$ quantum mechanics of section 1.3, and especially into the algebra of the Kähler model.

C-Twist

In the C-twist, we restrict ourselves to supercharges commuting with the diagonal combination of $U(1)_C$ and the $U(1)_{12}$ rotations in the $x^{1,2}$-plane. The invariant supercharges are

$$Q^{A} := Q_1^{A\dot{1}} \qquad \widetilde{Q}^{A} := Q_2^{A\dot{2}} \tag{4.30}$$

and generate the $\mathcal{N}=(0,4)$ supersymmetric quantum mechanics

$$\begin{aligned}
&\{Q^{A}, Q^{B}\} = 0\\
&\{Q^{A}, \widetilde{Q}^{B}\} = \epsilon^{AB}(P_3 - m\cdot J_H) - \delta^{(A1}\delta^{B)2}\zeta\cdot J_C\\
&\{\widetilde{Q}^{A}, \widetilde{Q}^{B}\} = 0
\end{aligned} \tag{4.31}$$

[3] The fully covariant expression involves complex masses, which we do not consider in this thesis.

with Hamiltonian $H = P_3 - m \cdot J_H$ and central charge $Z = -\zeta \cdot J_C$. In particular, we identify $U(1)_R = U(1)_C$ and $U(1)_r = U(1)_H$. This algebra is akin to the hyperkähler model introduced in section 1.3.

Breaking to $\mathcal{N} = (0, 2)$

Turning on the real mass parameter m_t, both twists preserve a common 1d $\mathcal{N} = (0, 2)$ subalgebra that commutes with the $U(1)_t$ symmetry. The preserved supercharges are

$$Q := Q_1^{1\dot{1}} \qquad \widetilde{Q} := Q_2^{2\dot{2}} \,, \tag{4.32}$$

and they satisfy the $\mathcal{N} = (0, 2)$ algebra

$$\begin{aligned} \{Q, Q\} &= 0 \\ \{Q, Q^\dagger\} &= P_3 - m \cdot J_H - \zeta \cdot J_C - m_t \cdot J_t \\ \{Q^\dagger, Q^\dagger\} &= 0 \,. \end{aligned} \tag{4.33}$$

From the perspective of 3d $\mathcal{N} = 2$ supersymmetry, we are performing topological twists on Σ using the integer valued R-symmetries generated by R_H and R_C.

4.2.2 $\mathcal{N} = 4$ Localising Actions

The localization procedure of $\mathcal{N} = 4$ theories can be considered a special case of $\mathcal{N} = 2$ theories. Thus, to study the BPS loci we simply have to state how $\mathcal{N} = 4$ multiplets decompose into $\mathcal{N} = 2$ multiplets. We have

- The $\mathcal{N} = 4$ vectormultiplet decomposes into an $\mathcal{N} = 2$ vectormultiplet V and an $\mathcal{N} = 2$ chiral multiplet $\Phi_\varphi = (\varphi, \psi_\varphi, \eta_\varphi, F_\varphi)$ in the adjoint representation .
- The $\mathcal{N} = 4$ hypermultiplet decomposes into a pair of $\mathcal{N} = 2$ chiral multiplets denoted by $\Phi_X = (X, \psi_X, \eta_X, F_X)$ and $\Phi_Y = (Y, \psi_Y, \eta_Y, F_Y)$ transforming in the unitary representations M and M^* respectively .

The details of these decompositions can be found in appendix A.1. The theory is endowed with a $\mathcal{N} = 2$ superpotential $W = \langle Y, \Phi \cdot X\rangle$ of R-charge $+2$.

It is important to note that on the curve Σ, accounting for the R-charges summarized in table 4.1, the chiral multiplets mentioned above transform as sections of the associated bundles

$$\begin{aligned} P_\varphi &:= (P \times_G \mathfrak{g}) \otimes K_\Sigma^{1-r} \\ P_X &:= (P \times_G M) \otimes K_\Sigma^{r/2} \\ P_Y &:= (P \times_G M^*) \otimes K_\Sigma^{r/2} \,, \end{aligned} \tag{4.34}$$

where

$$r := \begin{cases} 1 & \text{H-twist} \\ 0 & \text{C-twist} \end{cases} \tag{4.35}$$

and P is the gauge bundle (which is a principal G-bundle). The $\mathcal{N}=4$ version of the generalized vortex equations (2.31) reads

$$\begin{gathered} *F_A + e^2\left(\mu_{\mathbb{R}} - 2[\varphi^\dagger,\varphi] - \tau\right) = 0 \quad d_A\sigma = 0 \\ \bar\partial_A X = 0 \quad \bar\partial_A Y = 0 \quad \bar\partial_A \varphi = 0 \\ \varphi\cdot X = 0 \quad \varphi\cdot Y = 0 \quad X\cdot Y = 0 \\ \sigma\cdot\varphi = 0 \quad \sigma\cdot X = 0 \quad \sigma\cdot Y = 0 \end{gathered} \tag{4.36}$$

where it is understood that σ, φ and $\bar\partial_A$ act in the appropriate representation. Notice that $\mu_{\mathbb{R}}$ is the real moment map associated to the representation T^*M. The extra term arises from the $\mathcal{N}=2$ chiral adjoint multiplet present in the $\mathcal{N}=4$ vectormultiplet, by analogy with the Higgs branch description (4.10). In the following section, we explain that the algebraic description of the solutions to these equations coincide with that of 'quasi-maps'.

4.2.3 The Vortex Moduli Space

We now consider the moduli space of solutions to the generalized vortex equations (4.36) for the class of supersymmetric theories introduced in section 4.1. Recall that we consider quiver gauge theories with $G=\prod_{I=1}^L U(N_I)$.

First, solutions of the generalized vortex equations form topologically distinct sectors labelled by the flux

$$\mathfrak{m}_I := \frac{1}{2\pi}\int_\Sigma \mathrm{Tr}(F_I)\,. \tag{4.37}$$

We can equivalently write $\mathfrak{m}_I = c_1(\mathcal{V}_I)$ where $\mathcal{V}_I$ denotes the vector bundle on Σ in the fundamental representation of $U(N_I)$. We use shorthand notation $\mathfrak{m} := \{\mathfrak{m}_I\}\in\mathbb{Z}^L$.

The allowed fluxes $\mathfrak{m}\in\mathbb{Z}^L$ generate a lattice in the Lie algebra of the Abelian part of G. The latter can be identified with the dual of the Cartan subalgebra the Coulomb branch flavour symmetry, $\mathfrak{t}_C^\vee \cong \mathbb{R}^L$. The flux lattice is then naturally identified with the character lattice

$$\Lambda_C^\vee := \mathrm{Hom}(T_C, U(1))\,. \tag{4.38}$$

The homomorphism $\zeta \mapsto e^{2\pi i\langle\zeta,\mathfrak{m}\rangle}$ arises in the contribution to the path integral from the FI parameter. Through the identification (4.6) the flux lattice is equivalently

$$\Lambda_C^\vee \simeq H_2(\mathcal{M}_H,\mathbb{Z})\,, \tag{4.39}$$

which suggests that solutions of the generalized vortex equations are related to holomorphic maps $\Sigma\to\mathcal{M}_H$ of degree $\mathfrak{m}$. We will explain below in what sense this is realized.

Second, the parameter $\tau\in\mathbb{R}^L$ appearing in the generalized vortex equations (4.36) arises from an exact contribution to the Lagrangian. In what follows, we always choose this parameter to lie in the same connected component or chamber of the parameter space $\mathbb{R}^L \backslash \cup_\alpha H_\alpha$ as the physical FI parameter ζ.

In general, we expect an intricate dependence of the moduli space of solutions on τ, e^2 and $\mathrm{Vol}(\Sigma)$. As the combination $s = \tau e^2 \mathrm{Vol}(\Sigma)$ (already introduced in (3.81)) is varied, the number of supported vortices may jump, and we expect to observe wall-crossing pheonomena[4]. In order to obtain a uniform description of the moduli space of solutions to (4.36) for all fluxes $\mathfrak{m} \in \mathbb{Z}^L$, we will send the parameter $s \to \infty$ within the appropriate chamber of $\mathbb{R}^L \backslash \cup_\alpha H_\alpha$ defined by τ. In this limit, vortices are point-like and an arbitrary number of them can be supported on the curve.

When $s \to \infty$, the magnetic flux is therefore concentrated at a finite set of points P on Σ. Provided we restrict to $\Sigma - P$, the magnetic flux may be neglected in the first line of equation (4.36) and therefore

$$\mu_{\mathbb{R}} - 2[\varphi^\dagger, \varphi] = \tau \, . \tag{4.40}$$

This is identical to the D-term equation in the $\mathcal{N} = 2$ supersymmetry description of the Higgs branch described in equation (4.9). Under the assumptions of section 4.1.2, solutions of the generalized vortex equations therefore have the property that, for each point in $\Sigma - P$, $\sigma = \varphi = 0$ and they determine a point on $\mathcal{M}_{H,\tau}$. Together with the remaining equations in (4.36) this is sufficient to determine that $\sigma = \varphi = 0$ everywhere.

In the $s \to \infty$ limit, it is therefore sufficient to restrict our attention to the following system of equations

$$\begin{gathered} *F_A + e^2 \left(\mu_{\mathbb{R}} - \tau\right) = 0 \\ \bar{\partial}_A X = 0 \qquad \bar{\partial}_A Y = 0 \qquad X \cdot Y = 0 \end{gathered} \tag{4.41}$$

whose solutions with a fixed degree $\mathfrak{m} \in \mathbb{Z}^L$ describe holomorphic maps $\Sigma \to \mathcal{M}_H$ away from a finite set of points on Σ. Let us then denote the moduli space of solutions to the generalized vortex equations (4.41) modulo gauge transformations by $\mathfrak{M}$. As explained above, this is a disjoint union of topologically distinct components,

$$\mathfrak{M} = \bigcup_{\mathfrak{m} \in \Lambda_C^\vee} \mathfrak{M}_{\mathfrak{m}} \, . \tag{4.42}$$

We emphasize that the moduli space encompasses both boson and fermion zero modes. More precisely, the moduli space is parametrized by the vacuum expectation values of both 1d $\mathcal{N} = (0,2)$ chiral multiplets and 1d $\mathcal{N} = (0,2)$ Fermi multiplets. In the following section, we explain that the algebraic description of the solutions to these equations coincide with that of 'quasi-maps'.

[4] We will discuss the moduli space of gauge theories at finite τ and the wall-crossing phenomena in upcoming work [37].

4.2.4 Algebraic Description

To understand the vortex moduli spaces $\mathfrak{M}_{\mathfrak{m}}$ and the mathematical interpretation of the twisted index, we consider an algebraic description of the moduli space of generalized vortex equations (4.41) in the limit $s \to \infty$. We show that this description coincides with two variations of moduli spaces of stable quasi-maps $\Sigma \to \mathcal{M}_H$ in the H-twist and C-twist respectively.

As for the Higgs branch, the algebraic description of the moduli space $\mathfrak{M}_{\mathfrak{m}}$ is found schematically by removing the D-term vortex equation from (4.41) in favour of a stability condition and dividing by complex gauge transformations (under which the equation $X \cdot Y$ is invariant). A solution is then represented by the following holomorphic data:

- A holomorphic $G_{\mathbb{C}}$-bundle E on Σ;
- Holomorphic sections X, Y of the associated holomorphic vector bundles E_X, E_Y subject to the complex moment map constraint $\mu_{\mathbb{C}} = X \cdot Y = 0$;
- Subject to a stability condition;

and modulo complex gauge transformations. We refer to a collection of such algebraic data as (E, X, Y). This associates to each point on Σ a point in $\mu_{\mathbb{C}}^{-1}(0) \subset T^*M$. We can therefore regard this algebraic data as a twisted holomorphic map $\Sigma \to \mu_{\mathbb{C}}^{-1}(0)$ of degree $\mathfrak{m}$.

Let us now consider the stability condition arising from the vortex equation,

$$* F_A + e^2 (\mu_{\mathbb{R}} - \tau) = 0 \,. \tag{4.43}$$

The determination of the relevant stability condition depends intricately on the choice of parameter τ and has been studied extensively in particular examples [88, 89, 90].

The $s \to \infty$ limit leads to a simplification in the stability condition: away from a finite set of points on Σ the curvature term in equation (4.43) can be ignored and the image of the map $\Sigma \to \mu_{\mathbb{C}}^{-1}(0)$ determined by the algebraic data must lie in the stable locus $\mu_{\mathbb{C}}^{-1}(0)^s$. This is precisely the stability condition introduced in [39, 46, 45] to define quasi-maps $\Sigma \to \mathcal{M}_H$. Accounting for the R-charges as in (4.34), in the C-twist we therefore have an algebraic description of $\mathfrak{M}_{\mathfrak{m}}$ as the moduli space of quasi-maps $\Sigma \to \mathcal{M}_H$ of degree $\mathfrak{m} \in \mathbb{Z}^L$ as considered in [46] for the special case $\Sigma = \mathbb{CP}^1$. In the H-twist, we have a similar algebraic description as twisted quasi-maps as described in [45].

4.2.5 Virtual Tangent Bundle

We can further study this identification by computing the massless fluctuations around solutions of the generalized vortex equations (4.36). By supersymmetry preserved on $S^1 \times \Sigma$, these fluctuations must organize into supermultiplets of 1d $\mathcal{N} = (0, 2)$ supersymmetry. We will demonstrate that the massless fluctuations reproduce the structure of virtual tangent bundles or perfect obstruction theories for $\mathfrak{M}_{\mathfrak{m}}$ considered in [39, 46, 45].

Let us fix a point on the moduli space represented by the algebraic data (E, X, Y). Then each of the three-dimensional chiral multiplets $\phi = X, Y, \varphi$ generates a pair of 1d $\mathcal{N} = (0,2)$ supermultiplet fluctuations at this point:

- Chiral multiplets: $(\delta\phi, \psi_\phi) \in H^0(E_\phi)$.
- Fermi multiplets: $(\eta_\phi) \in H^1(E_\phi)$.

In addition, the three-dimensional vectormultiplet contributes a chiral multiplet fluctuation $(\delta\bar{A}, \bar{\Lambda}) \in H^1(E_V)$, where E_V is the holomorphic vector bundle associated with the adjoint representation, corresponding to deformations of the holomorphic vector bundle E via the derivative operator $\bar{\partial}_A$, and a Fermi multiplet $\lambda \in H^0(E_V)$ corresponding to infinitesimal holomorphic gauge transformations.

Not all of these fluctuations remain massless. First, let us fix the holomorphic vector bundle E and consider fluctuations of the hypermultiplets (X, Y). For the scalar fluctuations $(\delta X, \delta Y)$, linearisation of the complex moment map equation $X \cdot Y = 0$ generates the complex of vector spaces

$$H^0(E_V) \xrightarrow{\alpha^0} H^0(E_X \oplus E_Y) \xrightarrow{\beta^0} H^1(E_\varphi)^* \ , \tag{4.44}$$

where the map $\alpha^0 : \delta g \mapsto (\delta g \cdot X, \delta g \cdot Y)$ is an infinitesimal complex gauge transformation and $\beta^0 : (\delta X, \delta Y) \to X \cdot \delta Y + \delta X \cdot Y$ is the differential of the complex moment map. The massless fluctuations of the complex scalars lie in $\mathrm{Ker}(\beta^0)/\mathrm{Im}(\alpha^0)$. We note that under our assumptions α^0 is injective.

The same result must hold for the fermion components (ψ_X, ψ_Y) of the chiral multiplets by 1d $\mathcal{N} = (0,2)$ supersymmetry but it is illuminating to check this explicitly. This can be understood from the Yukawa couplings with the Fermi multiplet fluctuations $\lambda \in H^0(E_V)$ and $\eta_\varphi \in H^1(E_\varphi)$. First, there is

$$\int_\Sigma * \langle \lambda, \psi_X \cdot X^\dagger \rangle + \int_\Sigma * \langle \lambda, Y^\dagger \cdot \psi_Y \rangle \, . \tag{4.45}$$

Here we exceptionally denote by $\langle \cdot, \cdot \rangle$ the pairing between the lie algebra $\mathfrak{g}$ and its dual $\mathfrak{g}^*$. Other contractions are implicit. Let us suppose that the fermion fluctuations take the form $(\psi_X, \psi_Y) = (\epsilon \cdot X, \epsilon \cdot Y)$ for some fermion $\epsilon \in H^0(E_V)$, meaning they lie in the image of α^0. Then the above contributions are proportional to

$$\int_\Sigma * \langle \mu_{\mathbb{R}} (X, Y) \, , \lambda \epsilon \rangle. \tag{4.46}$$

By the stability condition, the real moment map cannot vanish identically on Σ and therefore this coupling generates a mass for the fermions ϵ and λ. We conclude that the fermion fluctuations (ψ_X, ψ_Y) in the image of α^0 become massive. Second, the superpotential (4.5) generates the Yukawa couplings

$$\int_\Sigma \langle \, \eta_\varphi \, , \, X \cdot \psi_Y + \psi_X \cdot Y \, \rangle \ . \tag{4.47}$$

It is clear that if the fermion fluctuations satisfy $X \cdot \psi_Y + \psi_X \cdot Y = 0$, then the sum of these couplings vanishes and these fluctuations are massless. Otherwise they pair up with η_φ to become massive. We therefore conclude that the remaining massless fluctuations (ψ_X, ψ_Y) lie in $\mathrm{Ker}(\beta^0)/\mathrm{Im}(\alpha^0)$. In addition there are Fermi multiplet fluctuations $\bar{\eta}_\varphi$ in the cokernel of β^0. In sum, there are massless 1d $\mathcal{N} = (0,2)$ fluctuations given by the cohomology of the complex (4.44).

Let us now return to considering fluctuations of the holomorphic bundle E via the derivative operator $\bar{\partial}_A$. Deformations of the holomorphic vector bundle E correspond to elements in $H^1(E)$. However, these deformations must be such that (X, Y) remain holomorphic sections, meaning they lie in the kernel of the map

$$\alpha^1 : H^1(E_V) \longrightarrow H^1(E_X \oplus E_Y) \ , \tag{4.48}$$

where $\alpha^1 : \delta\bar{A} \to (\delta\bar{A} \cdot X, \delta\bar{A} \cdot Y)$. The same condition must hold for the fermion component of the chiral multiplet $(\delta\bar{A}, \bar{\Lambda})$, but it is again illuminating to show this directly. This follows by noting that the Yukawa couplings

$$\int_\Sigma \langle \bar{\Lambda}, X\bar{\eta}_X \rangle + \int_\Sigma \langle \bar{\Lambda}, Y\bar{\eta}_Y \rangle \tag{4.49}$$

vanish when the fermion $\bar{\Lambda}$ lies in the kernel of α^1.

Finally, let us consider the chiral multiplet fluctuations $(\delta\varphi, \psi_\varphi) \in H^0(E_\varphi)$. The complex scalar fluctuations must obey $\delta\varphi \cdot X = 0$ and $\delta\varphi \cdot Y = 0$, which means that they lie in the co-kernel of the map

$$\beta^1 : H^1(E_X \oplus E_Y) \to H^0(E_\varphi)^* \, , \tag{4.50}$$

where $\beta^1 : (A, B) \to X \cdot B + A \cdot Y$. Under our assumptions, φ vanishes identically on solutions to the generalized vortex equations and therefore β^1 is surjective. Once again, the same condition must hold for the fermion components of the supermultiplet. This time we consider the remaining Yukawa couplings

$$\int_\Sigma \langle \, \psi_\varphi \, , \, X \cdot \eta_Y + \eta_X \cdot Y \, \rangle \, , \tag{4.51}$$

which shows that the combination of fermion fluctuations $X \cdot \eta_Y + \eta_X \cdot Y$ that are not kernel of β^1 pair up with the fluctuations ψ_φ and become massive.

To sum up, the massless fluctuations around a point on the moduli space $\mathfrak{M}$ of quasi-maps $\Sigma \to \mathcal{M}_H$ represented by algebraic data (E, X, Y) are encoded in the cohomology of the following pair of complexes

$$\begin{aligned} H^0(E) \xrightarrow{\alpha^0} H^0(E_X \oplus E_Y) \xrightarrow{\beta^0} H^1(E_\varphi)^* \\ H^1(E) \xrightarrow{\alpha^1} H^1(E_X \oplus E_Y) \xrightarrow{\beta^1} H^0(E_\varphi)^* \ . \end{aligned} \tag{4.52}$$

This can be promoted to a complex of $G_H \times U(1)_t$ equivariant sheaves on the moduli space $\mathfrak{M}$ using the universal construction on $\mathfrak{M} \times \Sigma$. The starting point is the universal G-bundle $\mathcal{P} \to \mathfrak{M} \times \Sigma$. We then have

$$R\pi^\bullet(\mathcal{P}) \xrightarrow{\alpha} R\pi^\bullet(\mathcal{P}_X \oplus \mathcal{P}_Y) \xrightarrow{\beta} R\pi^\bullet(\mathcal{P}_\varphi)^* \ , \tag{4.53}$$

where $\pi : \mathfrak{M} \times \Sigma \to \mathfrak{M}$ is the other projection and the associated vector bundles $\mathcal{P}_X, \mathcal{P}_Y, \mathcal{P}_\varphi$ are defined as before using the pullback $\mathcal{K} = f^* K_\Sigma$ where $f : \mathfrak{M} \times \Sigma \to \Sigma$ is the projection. Note that this mirrors the structure of the complex whose cohomology computes the tangent bundle to $\mathcal{M}_H$ outlined in section 4.1.3. In the remainder of the thesis, we will mainly refer to T^{vir} as the equivariant K-theory class of the complex (4.53).

This construction coincides with the perfect obstruction theory constructed in [46] for $\Sigma = \mathbb{CP}^1$ in the C-twist and [45] in H-twist on a general curve Σ of genus g. The two obstruction theories have remarkably different features. The obstruction theories for the H-twist is symmetric, meaning that there is an isomorphism between the complex in degree 0 in (4.53) and the dual of the complex in degree 1. This implies that the virtual dimension of the moduli space is zero. In the C-twist the obstruction fails to be symmetric unless the curve is elliptic, so that the canonical bundle is trivial. A Hirzebruch-Riemann-Roch computation shows that

$$\dim_{\text{vir}}(\mathfrak{M}_{\mathfrak{m}}) = \begin{cases} 0 & \text{H-twist} \\ \dim(\mathcal{M}_H)(1-g) & \text{C-twist}\,. \end{cases} \tag{4.54}$$

The difference between the two twists will be particularly manifest when we attempt to give an interpretation of the twisted indices.

4.2.6 Mass Parameters and Fixed Loci

The moduli spaces $\mathfrak{M}_{\mathfrak{m}}$ introduced above are in general expected to be non-compact. The presence of massless non-compact fluctuations would render the computation of the twisted index on $S^1 \times \Sigma$ ill-defined. To remedy this, we introduce real mass parameters for flavour symmetries that, as for the Higgs branch in section (4.1.3), will cut down the moduli space to the fixed locus of this flavour symmetry.

The mass parameter for the $U(1)_t$ symmetry associated to the breaking to $\mathcal{N} = 2$ supersymmetry is enough to ensure the twisted index on $S^1 \times \Sigma$ is well-defined and identify its mathematical interpretation. Further introduction of mass parameters for G_H will make the twisted index explicitly computable in our localization scheme.

$U(1)_t$ Mass Parameters

Let us introduce the mass parameter m_t for $U(1)_t$. The effect of this deformation is to replace $\sigma \to \sigma + m_t$ in the generalized vortex equations (4.36), where m_t acts with the appropriate weight according to table 4.1. The remaining moduli space of solutions is the fixed locus of the $U(1)_t$ action on $\mathfrak{M}_{\mathfrak{m}}$.

First recall from section 4.1.4 that turning on the mass parameter m_t restricts the Higgs branch to a compact holomorphic Lagrangian known as the compact core $\mathcal{M}_H^{U(1)_t} = \mathcal{L}_H$. This is characterized by a holomorphic Lagrangian splitting $T^*M = L \oplus L^*$ such that the hypermultiplet fields in L^* vanish on the fixed locus and $\mathcal{L}_H = L//_\zeta G$.

Similarly, solutions of the generalized vortex equations invariant under $U(1)_t$ correspond to configurations where the hypermultiplet fields in L^* vanish and correspond algebraically to twisted quasi-maps $\Sigma \to \mathcal{L}_H$ to the compact core. We denote the fixed locus of the moduli space by $\mathfrak{M}_{\mathfrak{m}}^{U(1)_t} = \mathfrak{L}_{\mathfrak{m}}$. Upon restriction to the fixed locus, the virtual tangent bundle splits into two pieces

$$H^\bullet(E) \xrightarrow{\alpha} H^\bullet(E_L) \qquad H^\bullet(E_{L^*}) \xrightarrow{\beta} H^{1-\bullet}(E_\varphi)^* \ . \tag{4.55}$$

transforming with weight 0 and +1 respectively under $U(1)_t$. They can be identified with the virtual tangent bundle to $\mathfrak{L}_{\mathfrak{m}}$ and the virtual normal bundle respectively. At the level of K-theory classes we have

$$T^{\text{vir}}\big|_{\mathfrak{L}} = T^{\text{vir}}_{\mathfrak{L}} + tN \ , \tag{4.56}$$

where $t = e^{2\pi i m_t}$ is the equivariant parameter for the $U(1)_t$ symmetry.

In the H-twist, the tangent and normal fluctuations at the fixed locus are related by Serre duality

$$H^\bullet(E_L) = H^{1-\bullet}(E_{L^*})^* \ , \quad H^\bullet(E) = H^{1-\bullet}(E_\varphi)^* \tag{4.57}$$

and $N^{\text{vir}} = -\left(T^{\text{vir}}_{\mathfrak{L}}\right)^\vee$ as K-theory classes. When the Higgs branch is a cotangent bundle we expect that the extended moduli space including fermionic fluctuations is actually a shifted cotangent bundle $T^*[-1]\mathfrak{L}$. In the C-twist, the virtual normal bundle N^{vir} can be identified with the class of the complex

$$H^\bullet(E \otimes K_\Sigma)^* \longrightarrow H^\bullet(E_L \otimes K_\Sigma)^* \tag{4.58}$$

by an application of Serre duality.

G_H Mass Parameters

Let us now introduce real mass parameters $m \in \mathfrak{t}_H$ and consider localization with respect to $T_H \subset G_H$. Under our assumption that fixed points $\{v_I\}$ of $\mathcal{M}_H$ are isolated, the fixed locus in $\mathfrak{M}$ corresponds to a union of $\mathfrak{M}_I$, where the gauge group G is broken to its maximal torus

$$G \to U(1)^{\text{rk}(G)} \ . \tag{4.59}$$

Then the associated degree $\mathfrak{m}$ vector bundle E decomposes into the sum of line bundles

$$E = L_1 \oplus \cdots \oplus L_{\text{rk}(G)} \ , \tag{4.60}$$

where $\deg(L_i) = \mathfrak{m}_i$. The r-vector $\underline{\mathfrak{m}} = (\mathfrak{m}_1, \cdots, \mathfrak{m}_{\text{rk}(G)})$ is valued in the co-character lattice Λ_G of the gauge group G, and satisfies the relation $\text{Tr}(\underline{\mathfrak{m}}) = \mathfrak{m}$. This implies that each fixed locus $\mathfrak{M}_I$ can be further decomposed into

$$\mathfrak{M}_I = \bigcup_{\underline{\mathfrak{m}} \in \Lambda_G} \mathfrak{M}_{\underline{\mathfrak{m}},I} \ . \tag{4.61}$$

Furthermore, at each fixed locus labeled by I, there are exactly $\mathrm{rk}(G)$ non-vanishing chiral multiplet fields Z_a, $a = 1, \ldots, \mathrm{rk}(G)$, which corresponds to the isolated vacua v_I in (4.26). Then each component of the fixed locus $\mathfrak{M}_{\underline{\mathfrak{m}},I}$ parametrizes the holomorphic line bundle L_a together with non-vanishing holomorphic section Z_a, which can be identified with the $\mathrm{rk}(G)$-fold product of symmetric products of a curve Σ

$$\mathfrak{M}_{\underline{\mathfrak{m}},I} = \prod_{a=1}^{\mathrm{rk}(G)} \mathrm{Sym}^{\mathfrak{m}_a + r(g-1)} \Sigma \ , \tag{4.62}$$

where r is the R-charge. This is a compact smooth Kähler manifold of complex dimension $\mathfrak{m} + \mathrm{rk}(G) r(g-1)$.

Now the massless fluctuations transform in the tangent bundle to the fixed loci $T\mathfrak{M}_{\underline{\mathfrak{m}},I}$ and the remaining fluctuations are massive. This corresponds to a decomposition of the virtual tangent bundle

$$T^{\mathrm{vir}}|_{\mathfrak{M}_{\underline{\mathfrak{m}},I}} = T\mathfrak{M}_{\underline{\mathfrak{m}},I} + N_{\underline{\mathfrak{m}},I} \ , \tag{4.63}$$

where the virtual normal bundle $N_{\underline{\mathfrak{m}},I}$ encodes the fluctuations that have become massive upon turning on the mass parameter. These two contributions are known as the 'fixed' and 'moving' parts and are characterized as those transforming with trivial weight and non-trivial weight under the $T_H \times U(1)_t$ transformation generated by the mass parameters m_H, m_t.

4.2.7 Evaluating the Partition Function

The path integral of the twisted index computes the generating function of the equivariant virtual Euler characteristic of the moduli spaces $\mathfrak{M}_{\mathfrak{m}}$. This is defined by the following integral

$$\chi(\mathfrak{M}, \hat{\mathcal{O}}_{\mathrm{vir}}) = \sum_{\mathfrak{m} \in \Lambda_C^\vee} (-q)^{\mathfrak{m}} \int_{\mathfrak{M}_{\mathfrak{m}}} \hat{A}(T_{\mathrm{vir}}) \ , \tag{4.64}$$

where $\hat{A}(T_{\mathrm{vir}})$ is the A-roof genus of the virtual tangent bundle. This quantity has been extensively studied in [47, 46] in the context of the enumerative geometry of curves in Calabi-Yau five-folds. The analogous construction for the four-dimensional Vafa-Witten invariants has been recently studied in [91].

Due to the non-compactness of $\mathfrak{M}_{\mathfrak{m}}$, this formula should be evaluated with a proper virtual localization theorem. Let us first consider the localization with respect to the $U(1)_t$ action, which leads to the expression

$$\begin{aligned} \chi(\mathfrak{M}, \hat{\mathcal{O}}_{\mathrm{vir}}) &= \sum_{\mathfrak{m} \in \Lambda_C^\vee} (-q)^{\mathfrak{m}} \int_{[\mathfrak{L}_{\mathfrak{m}}]} \frac{\hat{A}\left(T^{\mathrm{vir}}_{\mathfrak{L}_{\mathfrak{m}}}\right)}{\mathrm{ch}\left(\widehat{\wedge}^\bullet N^\vee_{\mathfrak{m}}\right)} \\ &= \sum_{\mathfrak{m} \in \Lambda_C^\vee} (-q)^{\mathfrak{m}} \int_{[\mathfrak{L}_{\mathfrak{m}}]} \hat{A}\left(T^{\mathrm{vir}}_{\mathfrak{L}_{\mathfrak{m}}}\right) \mathrm{ch}\left(\widehat{S}^\bullet N^\vee_{\mathfrak{m}}\right) \ . \end{aligned} \tag{4.65}$$

Here we introduced the symmetrized exterior and symmetric algebras

$$\widehat{S}^\bullet V := (\det V)^{1/2} \otimes S^\bullet V \ , \quad \widehat{\wedge}^\bullet V := (\det V)^{-1/2} \otimes \wedge^\bullet V \ , \tag{4.66}$$

where

$$S^\bullet V = \bigoplus_{i \geq 0} S^i V \ , \quad \wedge^\bullet V = \bigoplus_{i \geq 0} (-1)^i \wedge^i V \tag{4.67}$$

are the symmetric and exterior algebra of V. In the H-twist, the identification of the virtual normal bundle with $N = -(T^{\text{vir}}_{\mathfrak{L}})^\vee$ means we can also interpret the twisted index as a symmetrized virtual χ_y-genus with $y = -t$.

These integrals can be explicitly evaluated by a further localization with respect to $T_H \subset G_H$. In turning on the real mass parameters m, we have seen that the solutions of the BPS equations are restricted to the fixed locus $\mathfrak{M}^T$, which is a disjoint union of the smooth compact fixed loci. Let us denote the inclusion by $\sigma_{\underline{\mathfrak{m}},I} : \mathfrak{M}_{\underline{\mathfrak{m}},I} \hookrightarrow \mathfrak{M}$. Then the integral decomposes as a sum of contributions from the distinct components of the fixed locus

$$\begin{aligned} \chi(\mathfrak{M}, \hat{\mathcal{O}}_{\text{vir}}) &= \sum_{\mathfrak{m} \in \Lambda_C^\vee} (-q)^{\mathfrak{m}} \sum_I \int_{\mathfrak{M}_{\underline{\mathfrak{m}},I}} \frac{\hat{A}\left(T\mathfrak{M}_{\underline{\mathfrak{m}},I}\right)}{\text{ch}\left(\widehat{\wedge}^\bullet N^\vee_{\underline{\mathfrak{m}},I}\right)} \\ &= \sum_{\underline{\mathfrak{m}} \in \Lambda_G} (-q)^{\mathfrak{m}} \sum_I \int_{\mathfrak{M}_{\underline{\mathfrak{m}},I}} \hat{A}\left(T\mathfrak{M}_{\underline{\mathfrak{m}},I}\right) \ \text{ch}(\widehat{S}^\bullet N^\vee_{\underline{\mathfrak{m}},I}) \ . \end{aligned} \tag{4.68}$$

We note the individual contributions from the components of the fixed locus may be interpreted as the index of the Dirac operator on the smooth space $\mathfrak{M}_{\underline{\mathfrak{m}},I}$ twisted by a complex of holomorphic vector bundles represented by $\widehat{S}^\bullet N^\vee_{\underline{\mathfrak{m}},I}$. This is expected form of the partition function of a finite-dimensional $\mathcal{N} = (0,2)$ supersymmetric quantum mechanics with target space $\mathfrak{M}_{\underline{\mathfrak{m}},I}$.

As discussed in the previous section, under our assumptions, the fixed loci are smooth products of symmetric products and these integrals can be evaluated explicitly. We will explore an extensive set of examples in section 4.4.

4.2.8 Relation to Contour Integral Formulae

The main focus of this chapter is to provide a concrete geometric interpretation of the twisted indices of 3d $\mathcal{N} = 4$ theories on $S^1 \times \Sigma$. For the class of $\mathcal{N} = 4$ theories we consider in this chapter, the twisted index (2.12) becomes

$$I = \text{Tr}_{\mathcal{H}} \ (-1)^F e^{2\pi i \zeta \cdot J_C} e^{2\pi i m \cdot J_H} t^{J_t} = \sum_{\mathfrak{m} \in \Lambda_C^\vee} \text{Tr}_{\mathcal{H}_\mathfrak{m}} \ (-1)^F (-q)^{\mathfrak{m}} e^{2\pi i m \cdot J_H} t^{J_t} \ , \tag{4.69}$$

where $\mathcal{H}$ is the Hilbert space of states on Σ. This can be decomposed into topological sectors labelled by $\mathfrak{m} \in \Lambda_C^\vee$, which is a selection parameter in the quantum mechanics. We defined $q = e^{2\pi i \zeta}$ and multiplied by $(-1)^{\mathfrak{m}}$ for each topological sector for future convenience. m indicates real masses for the Higgs branch symmetry G_H.

In the $\mathcal{N}=4$ case, we can write the one-loop determinants presented in section 2.3 as follows

$$Z^{\text{vector}}_{\text{1-loop}}(u) = (2i\sin\pi m_t)^{(g-1)(2r-1)\mathrm{rk}(G)} \prod_{\alpha\in\Delta} \frac{(2i\sin\pi\alpha(u))^{\alpha(\underline{\mathfrak{m}})+(g-1)(2r-1)}}{(2i\sin\pi(\alpha(u)-m_t))^{\alpha(\underline{\mathfrak{m}})-(g-1)(2r-1)}} \tag{4.70}$$

and

$$Z^{\text{hyper}}_{\text{1-loop}}(u,m) = \prod_i \prod_{\rho\in M} \frac{\left(2i\sin\pi\left(-\rho(u)-m_i+\frac{m_t}{2}\right)\right)^{\rho(\underline{\mathfrak{m}})-(g-1)(r-1)}}{\left(2i\sin\pi\left(\rho(u)+m_i+\frac{m_t}{2}\right)\right)^{\rho(\underline{\mathfrak{m}})+(g-1)(r-1)}} , \tag{4.71}$$

where Δ is the set of all roots of $\mathfrak{g}$ and ρ is the weights in a complex representation M of G. The Hessian reads

$$H(u,m) = \det_{ab}\left[H^{\text{vector}}_{ab}(u) + H^{\text{hyper}}_{ab}(u,m)\right] , \tag{4.72}$$

where

$$H^{\text{vector}}_{ab}(u) = \sum_{\alpha\in\Delta} \alpha^a\alpha^b \frac{\cos\pi(\alpha(u)-m_t)}{2i\sin\pi(\alpha(u)-m_t)} \tag{4.73}$$

and

$$H^{\text{hyper}}_{ab}(u,m) = \sum_i \sum_{\rho\in M} \rho^a\rho^b \left(\frac{\cos\pi(\rho(u)+m_i+m_t/2)}{2i\sin\pi(\rho(u)+m_i+m_t/2)} + \frac{\cos\pi(-\rho(u)-m_i+m_t/2)}{2i\sin\pi(-\rho(u)-m_i+m_t/2)}\right) . \tag{4.74}$$

The integration contour is given by the Jeffrey-Kirwan prescription (2.27), which depends on the auxiliary parameter $\eta \in \mathfrak{t}^*$. We identified a natural choice in (2.33)

$$\eta = -\frac{2\pi\mathfrak{m}}{e^2} + \mathrm{Vol}(\Sigma)\tau := \eta_0 . \tag{4.75}$$

It follows that the residue integral (2.17) does not include the poles involving the hyperplane at asymptotic boundaries.

Notice that by integrating the D-term equation (4.41) over Σ, we obtain

$$\int_\Sigma * \mu_{\mathbb{R}} = \eta_0 . \tag{4.76}$$

From this relation we can check that the poles that pass the JK condition with the choice (4.75) are in one-to-one correspondence with the fixed loci of the moduli space described in section 4.2.6[5]. Furthermore, the poles that involve the hyperplanes coming from adjoint chiral multiplets do not contribute to the integral as the residues of such poles always vanish due to the order of zeros in the numerator. Therefore, for the class of theories we consider, the non-trivial contributions are from the residue integrals which consist of type of hyperplanes coming from hypermultiplets only. They correspond to the fixed loci

$$\sum_I \prod_{a=1}^{\mathrm{rk}(G)} \mathrm{Sym}^{\mathfrak{m}_{I_a}+r(g-1)}\Sigma , \tag{4.77}$$

[5] As already anticipated in section 2.3, we exclude the poles coming from the W-bosons. Once we exclude these poles, the final result may of course depend on the choice of η. For the theories we consider in this chapter, however, we will show in section 4.4.4 that the uniform choice $\eta = \eta_0 > 0$ with the residues from the W-boson singularities excluded reproduces the correct integral representation of the Euler characteristics of the moduli spaces in the $\tau \to \infty$ chamber.

parameterized by the sum of the line bundles (4.60) and non-vanishing sections thereof. This discussion gives a geometric interpretation of the contour expressions, which we extensively study with various examples in section 4.4. In particular, using the intersection theory of the symmetric product of a curve Σ studied in [48, 49], the contour integral expressions of the twisted indices can be converted to the equivariant integrals computing the virtual Euler characteristics discussed in section 4.2.7. This provides a powerful way to compute enumerative invariants of moduli spaces of quasi-maps.

4.3 The Limit $t \to 1$

As discussed in section 4.2.1, compactifying 3d $\mathcal{N} = 4$ theories on a Riemann surface preserves 1d $\mathcal{N} = (2,2)$ or $\mathcal{N} = (0,4)$ supersymmetry in the H- and C-twist respectively. So far, we have considered a localization scheme which preserves a $\mathcal{N} = (0,2)$ subalgebra only. Once we turn off the $U(1)_t$ mass parameter, we can add various exact terms to the localising action with respect to the supercharges that do not commute with the $U(1)_t$ symmetry. This further constrains the BPS moduli space and the twisted indices in the limit $t \to 1$ are expected to provide a geometric invariant for a reduced moduli space.

As we will see, the localization scheme which preserves four supercharges turns out to be most powerful in the C-twist, where we can reduce the bosonic BPS moduli space to the Higgs branch itself, and the twisted indices can be interpreted as the Rozansky-Witten invariants [4] of the Higgs branch $\mathcal{M}_H$. From the 3d mirror symmetry that exchanges the C- and H-twist, these considerations imply remarkable statements relating invariants of very differently-looking spaces, which we elaborate on in section 4.5.

The notation for the fields and the supersymmetry algebra used in this section are summarized in appendix A.1.

4.3.1 C-twist

Let us start from the C-twist. In addition to the localizing action (2.11) with the term (2.30), we can write down additional Q-exact terms using the four supercharges in the $\mathcal{N} = (0,4)$ algebra:

$$\frac{1}{t_C^2}\mathcal{L}_{C,\text{vector}} = \widetilde{Q}^2\left(\widetilde{\lambda}_1 V^\dagger\right) + Q^2\left(\lambda_1 V^\dagger\right) + \widetilde{Q}^1\left(\widetilde{\lambda}_2 V^\dagger\right) + Q^1\left(\lambda_2 V^\dagger\right) \ , \tag{4.78}$$

where

$$V = \frac{1}{4t_C^2}\left(\widetilde{Q}^2\widetilde{\lambda}_1 - Q^2\lambda_1\right) - \frac{1}{4t_C^2}\left(\widetilde{Q}^1\widetilde{\lambda}_2 - Q^1\lambda_2\right) \ . \tag{4.79}$$

The bosonic part of this action is a total square

$$\frac{1}{t_C^2}\| * F_A - 2e^2[\varphi^\dagger, \varphi]\|^2 \ . \tag{4.80}$$

If we take the limit $t_C \to 0$, the field configuration of the vector multiplet localizes to the intersection of (4.36) and

$$* F_A - 2e^2[\varphi^\dagger, \varphi] = 0 \ . \tag{4.81}$$

For the hypermultiplet, we can add

$$\frac{1}{s_C^2}\mathcal{L}_{C,\text{hyper}}[V,X] = \frac{1}{s_C^2}\left(-\widetilde{Q}^B(\psi_z Q_B\widetilde{\psi}_{\bar{z}} + \psi_{\bar{z}} Q_B\widetilde{\psi}_z) - Q^B(\psi_z \widetilde{Q}_B\widetilde{\psi}_z + \psi_{\bar{z}}\widetilde{Q}_B\widetilde{\psi}_{\bar{z}})\right) \ . \tag{4.82}$$

The bosonic part of this action is

$$\frac{1}{s_C^2}\mathcal{L}^{\text{bosonic}}_{C,\text{hyper}} = 4D_1 X_B D_{\bar{1}}\widetilde{X}^B + 4D_{\bar{1}}X_B D_1\widetilde{X}^B + \varphi X_B\widetilde{X}^B\varphi^\dagger + \varphi^\dagger X_B\widetilde{X}^B\varphi \ . \tag{4.83}$$

Taking $s_C \to 0$, the path integral localizes to the equations

$$D_1 X_A = D_{\bar{1}}X_A = \varphi\cdot X_B = \varphi^\dagger\cdot X_B = 0 \ , \tag{4.84}$$

which in particular implies that X^A's are covariantly constant on Σ.

Combining these results, we can define the bosonic C-twisted $\mathcal{N}=4$ moduli space $\mathcal{M}_{\mathcal{N}=4}$ to be the space of field configurations (A, φ, X_B) satisfying the following set of equations:

$$\begin{aligned}
& * F_A - 2[\varphi^\dagger, \varphi] = 0 \ , \\
& \bar{\partial}_A\varphi = 0 \ , \\
& d_A X^B = 0 \ , \\
& \varphi\cdot X_B = \varphi^\dagger\cdot X_B = 0 \ , \\
& \mu_{\mathbb{R}} - \tau = 0 \ , \ \mu_{\mathbb{C}} = 0 \ .
\end{aligned} \tag{4.85}$$

Note that the equations for the vector multiplet fields (A, φ) alone define the *Hitchin moduli space* [92] associated with the gauge group G. For the class of theories that we are interested in, the BPS equations (4.85) imply $\varphi = 0$. Furthermore, the real moment map condition, together with the condition that the sections X_A are covariantly constant implies that the vector bundle E must be trivial. Therefore the bosonic moduli space reduces to the Higgs branch $\mathcal{M}_H$ itself. Let us now look at the various contribution in the virtual tangent bundle. The first complex (the deformation space) in (4.52) reduces to

$$\mathfrak{g}_{\mathbb{C}} \xrightarrow{\alpha} M\oplus M^* \xrightarrow{\beta} \mathfrak{g}^*_{\mathbb{C}} \ , \tag{4.86}$$

which defines the tangent space of $\mathcal{M}_H$. Similarly, the second complex becomes

$$\mathbb{C}^g\otimes\left[\mathfrak{g}_{\mathbb{C}} \xrightarrow{\alpha} M\oplus M^* \xrightarrow{\beta} \mathfrak{g}^*_{\mathbb{C}}\right] \ . \tag{4.87}$$

This can be identified with the g copies of the tangent bundle $T\mathcal{M}_H$. In the limit $t\to 1$, the virtual Euler characteristic gets contributions from the zero-flux sector only, and is therefore independent of q. In particular, we recover the holomorphic Euler characteristic of $\mathcal{M}_H$ valued in $\left(\widehat{\wedge}^\bullet T^*\mathcal{M}_H\right)^g$,

$$\chi(\mathcal{M}, \hat{\mathcal{O}}_{\text{vir}})\big|_{t\to 1} = \chi\left(\mathcal{M}_H, \left(\widehat{\wedge}^\bullet T^*\mathcal{M}_H\right)^g\right) \ , \tag{4.88}$$

which is the Rozansky-Witten invaraint on $\Sigma \times S^1$ associated with the Higgs branch $\mathcal{M}_H$. Notice that in this limit the virtual dimension (4.54) is manifest. This relation between the twisted indices and the Rozansky-Witten invariants has also been studied in [93].

4.3.2 H-twist

Similarly, for the H-twist, we can write down additional Q-exact terms using the four supercharges in the $\mathcal{N} = (2,2)$ algebra. We choose

$$\frac{1}{t_H^2}\mathcal{L}_{H,\text{vector},1} = Q^1\left(\lambda_{\dot{2}}V^\dagger\right) + Q^{\dot{2}}\left(\widetilde{\lambda}_1 V^\dagger\right) \ , \tag{4.89}$$

where

$$V = \frac{1}{4t_H^2}\left(Q^{\dot{1}}\lambda_2 - Q^{1}\widetilde{\lambda}_1\right) . \tag{4.90}$$

As in the C-twist, the bosonic part is a sum of squares, but now takes the form

$$\| * F_A + iD\|^2, \tag{4.91}$$

where D is the auxiliary field for the $\mathcal{N} = 2$ vector multiplet. Solving the equation of motion for the D-term, and taking the limit $t_H \to 0$ gives rise to the condition

$$* F_A + e^2(\mu_{\mathbb{R}} - \tau) = 0 . \tag{4.92}$$

Therefore the H-twisted moduli space for the vector multiplet on Σ can be viewed as the intersection of solutions to (4.36) and (4.92), which can be written as[6]

$$\begin{aligned}
&* F_A + e^2(\mu_{\mathbb{R}} - \tau) = 0 \ , \\
&\bar{\partial}_A X_B = 0 \ , \\
&\bar{\partial}_A \varphi = [\varphi^\dagger, \varphi] = 0 \ , \\
&\varphi \cdot X_B = 0 \ , \\
&\mu_{\mathbb{C}} = 0 \ .
\end{aligned} \tag{4.96}$$

Here, $\bar{\partial}_A$ is the Dolbeault operator induced by the gauge connection A, whereas B is a $SU(2)$ index. Since under our assumptions φ vanishes on the moduli space, the bosonic moduli space remains the same as in the $\mathcal{N} = 2$ case. However, since φ decouples from the D-term equation, the derivation of the stability condition simplifies.

[6]Notice that there are interesting additional terms that could be added to the action. For example,

$$\frac{1}{t_H^2}\mathcal{L}_{H,\text{vector},2} = \frac{1}{4t_H^2}Q^{\dot{2}}\left(\widetilde{\Lambda}_{\bar{1},\dot{2}}V^\dagger\right) \tag{4.93}$$

where

$$V = Q^{\dot{2}}\widetilde{\Lambda}_{\bar{1},\dot{2}} \, , \tag{4.94}$$

whose bosonic part is

$$\|D_{\bar{1}}\varphi\|^2 . \tag{4.95}$$

This forces φ and $\varphi^\dagger$ to be covariantly constant on Σ, not just covariantly holomorphic. We could also add terms coming from the hypermultiplet, giving $\varphi^\dagger \cdot X_A = \varphi \cdot X_A^\dagger = 0$.

As mentioned, the additional supercharges do not commute with $U(1)_t$ and therefore we consider the limit $t \to 1$ of the twisted index. In this limit, the virtual χ_t-genus greatly simplifies to the generating function of the integral of the Euler class of the fixed loci $\mathfrak{L}_\mathfrak{m}$ of the $U(1)_t$ action

$$\chi(\mathfrak{M}, \hat{\mathcal{O}}_{\text{vir}})\big|_{t\to 1} = \sum_{\mathfrak{m}\in\Lambda_C^\vee} (-q)^{\mathfrak{m}} (-1)^{\dim_{\text{vir}}(\mathfrak{L}_\mathfrak{m})} \int_{[\mathfrak{L}_\mathfrak{m}]} e\left(T^{\text{vir}}_{\mathfrak{L}_\mathfrak{m}}\right) \ . \tag{4.97}$$

For the class of the theories we consider, the localization with respect to the Higgs branch flavour symmetry G_H provides an alternative expression for the index in the $t \to 1$ limit. Since the fixed loci $\mathfrak{M}_{\underline{\mathfrak{m}},I}$ with respect to $T_H \subset G_H$ are smooth and compact, the expression (4.97) can be explicitly evaluated by a computation of the sum of the Euler characteristic of the fixed loci:

$$\chi(\mathfrak{M}, \hat{\mathcal{O}}_{\text{vir}})\big|_{t\to 1} = \sum_{\underline{\mathfrak{m}}\in\Lambda_G} (-q)^{\mathfrak{m}} \sum_I (-1)^{\dim_{\mathbb{C}}(\mathfrak{M}_{\underline{\mathfrak{m}},I})} \int_{\mathfrak{M}_{\underline{\mathfrak{m}},I}} e(\mathfrak{M}_{\underline{\mathfrak{m}},I}) \ , \tag{4.98}$$

As discussed in the paper [57], the supersymmetric ground states in the effective quantum mechanics that preserve $\mathcal{N} = (2,2)$ supersymmetries are singlet under the flavour symmetry G_H. This agrees with the result (4.98), which is independent of the equivariant parameters m.

4.4 Examples

In this section, we apply the strategy outlined above to some concrete examples. We explicitly prove that the virtual Euler characteristics of the appropriate moduli spaces of quasi-maps, computed via equivariant localization (4.68), reproduces the contour integral formulae of the twisted indices derived in [26, 23] and summarized in 4.2.8. For each example, we also discuss and verify interpretations that become available in the $t \to 1$ limit, where $\mathcal{N} = 4$ supersymmetry is restored, as anticipated in the previous section.

4.4.1 Free Hypermultiplets

We start the study of our examples by briefly collecting some facts about the free hypermultiplet, since they are going to be useful in view of mirror symmetry. In $\mathcal{N} = 2$ language, the hypermultiplet corresponds to two chiral multiplets Φ_X and Φ_Y, which have a $U(1)_H$ flavour symmetry, and which are charged as follows:

	$U(1)_H$	$U(1)_t$
X	$+1$	$\frac{1}{2}$
Y	-1	$\frac{1}{2}$

For an arbitrary R-charge r, the index reads

$$I = \left(\frac{\left(t^{1/2}x\right)^{1/2}}{1 - t^{1/2}x}\right)^{\mathfrak{m}_H + \mathfrak{m}_t + (r-1)(g-1)} \left(\frac{\left(t^{1/2}/x\right)^{1/2}}{1 - t^{1/2}/x}\right)^{-\mathfrak{m}_H + \mathfrak{m}_t + (r-1)(g-1)} , \tag{4.99}$$

where $\mathfrak{m}_H$ and $\mathfrak{m}_t$ are the degrees of the line bundles L_H and L_t, and x and t are the fugacities for $U(1)_H$ and $U(1)_t$ respectively. When $t \neq 1$, the two factors in (4.99) correspond to the indices of Φ_X and Φ_Y. The contribution of each $\mathcal{N}=2$ chiral multiplet can be understood from the point of view of the 1d $\mathcal{N}=(0,2)$ quantum mechanics precisely as discussed in section 1.1.3. In particular, the first factor is the index of a 1d quantum mechanics with $h^0(\Sigma, K_\Sigma^{r/2}\otimes L_H\otimes L_t)$ chirals and $h^1(\Sigma, K_\Sigma^{r/2}\otimes L_H\otimes L_t)$ fermi multiplets, whose difference is controlled by the Riemann-Roch theorem. The same (but with the appropriate charges) holds for the second factor. In the $t\to 1$ limit, and turning off background line bundles for flavour symmetries in the C-twist, or L_t only in the H-twist, the $\mathcal{N}=(0,2)$ multiplets recombine into free 1d $\mathcal{N}=4$ chiral multiplets (H-twist) or 1d $N=4$ hypermultiplets (C-twist). These were studied in sections 1.3.5 and 1.3.5 respectively.

4.4.2 SQED[1]

Let us consider a $U(1)$ gauge theory with a hypermultiplet which has the following charges. This is the $\mathcal{N}=4$ version of $\mathcal{N}=2$ SQED considered in section 3.2.5, and has, in addition, the adjoint chiral multiplet included in the $\mathcal{N}=4$ vectormultiplet[7]

$$\begin{array}{c|cc|cc} & U(1)_G & U(1)_t & U(1)_H & U(1)_C \\ \hline X & 1 & \frac{1}{2} & \frac{1}{2} & 0 \\ Y & -1 & \frac{1}{2} & \frac{1}{2} & 0 \\ \varphi & 0 & -1 & 0 & 1 \end{array} , \tag{4.100}$$

The $\mathcal{N}=2$ BPS equations become

$$\begin{aligned} &* F_A + e^2(XX^\dagger - Y^\dagger Y - \tau) = 0 \ , \\ &\bar\partial_A X = \bar\partial_A Y = 0 \ , \ X\cdot Y = 0 \ , \\ &\bar\partial_A \varphi = 0 \ , \ \varphi\cdot X = \varphi \cdot Y = 0 \ . \end{aligned} \tag{4.101}$$

The moduli space of solutions to the above equations is a disjoint union of topological components

$$\mathfrak{M} = \bigcup_{\mathfrak{m}\in\mathbb{Z}} \mathfrak{M}_\mathfrak{m} \ , \tag{4.102}$$

indexed by the degree of the holomorphic line bundles L associated to the connection A. X and Y are holomorphic sections of $L\otimes K^{r/2}$ and $L^{-1}\otimes K^{r/2}$ respectively. Integrating the D-term equation over Σ, we can check that X is non-vanishing, provided

$$\tau > \frac{2\pi\mathfrak{m}}{e^2\mathrm{Vol}(\Sigma)} \ . \tag{4.103}$$

Note that this condition is equivalent to the choice $\eta_0>0$ in the twisted index computation (4.75). Since X is a holomorphic section of a line bundle, the number of zeros of X on Σ is finite and equal to the degree of $L\otimes K^{r/2}$. The remaining BPS equations imply that $Y=\varphi=0$. Therefore the moduli space in this chamber is

$$\mathfrak{M}_\mathfrak{m}^+ = \left\{(A,X) \mid * F + e^2(XX^\dagger - \tau) = 0 \ , \ \bar\partial_A X = 0\right\} / U(1)_G \ . \tag{4.104}$$

[7]Strictly speaking this theory falls short of the class we have previously defined in section 4.1.2. However, the resolved Higgs branch is well-defined (it is a point) and the computations are still possible. This example contains the basic building blocks needed for more elaborate examples.

This space defines the moduli space of Abelian vortices, which we have already encountered in section 3.2.4. Introducing the notation

$$\Sigma_n := \mathrm{Sym}^n \Sigma \,, \tag{4.105}$$

we have

$$\mathfrak{M}^+_{\mathfrak{m}} = \Sigma_{n_+} \ , \quad n_+ = \mathfrak{m} + r(g-1) \ . \tag{4.106}$$

The chamber $\tau < \frac{2\pi\mathfrak{m}}{e^2 \mathrm{Vol}(\Sigma)}$ can be treated in a similar way. The bosonic moduli space is constructed from non-vanishing sections Y (when they exist) and their corresponding line bundles L, whereas X is set to zero. In this chamber the bosonic moduli space becomes

$$\mathfrak{M}^-_{\mathfrak{m}} = \Sigma_{n_-} \ , \quad n_- = -\mathfrak{m} + r(g-1) \ . \tag{4.107}$$

For concreteness, we will work in the chamber (4.103) for all the flux sectors $\mathfrak{m} \in \mathbb{Z}$ by formally sending $\tau \to \infty$, and omit the superscript $+$ from $\mathfrak{M}^+_{\mathfrak{m}}$.

H-twist In order to compute the index using virtual localization, we need to study the virtual tangent space to $\mathfrak{M}_{\mathfrak{m}}$. In the H-twist, the physical fluctuations around the bosonic moduli space are given by

$$\begin{aligned}
(\delta X, \psi_X) \in H^0(L \otimes K_\Sigma^{1/2}) \ , \quad (\eta_X, F_X) \in H^1(L \otimes K_\Sigma^{1/2}) \ , \\
(\delta Y, \psi_Y) \in H^0(L^{-1} \otimes K_\Sigma^{1/2}) \ , \quad (\eta_Y, F_Y) \in H^1(L^{-1} \otimes K_\Sigma^{1/2}) \ , \\
(\varphi, \psi_\varphi) \in H^0(\mathcal{O}) \ , \quad (\eta_\varphi, F_\varphi) \in H^1(\mathcal{O})
\end{aligned} \tag{4.108}$$

The virtual tangent space restricted to a point D of the moduli space (4.52) therefore corresponds to the cohomology of the following two complexes:

$$\begin{aligned}
H^0(\mathcal{O}) \xrightarrow{\alpha^0} H^0((\mathcal{O}(D) \oplus \mathcal{O}(D)^{-1}) \otimes K_\Sigma^{1/2}) \xrightarrow{\beta^0} H^1(\mathcal{O})^* \ , \\
H^1(\mathcal{O}) \xrightarrow{\alpha^1} H^1((\mathcal{O}(D) \oplus \mathcal{O}(D)^{-1}) \otimes K_\Sigma^{1/2}) \xrightarrow{\beta^1} H^0(\mathcal{O})^* \ ,
\end{aligned} \tag{4.109}$$

where the map α is defined as multiplication by $(X, -Y)$, while β is defined by taking an inner product with (Y, X). Since Y vanishes identically on the moduli space $\mathfrak{M}_{\mathfrak{m}}$, these complexes split into two pieces each:

$$\begin{aligned}
H^0(\mathcal{O}) \xrightarrow{\alpha^0} H^0(\mathcal{O}(D) \otimes K_\Sigma^{1/2}) \ , \quad H^0(\mathcal{O}(D)^{-1} \otimes K_\Sigma^{1/2}) \xrightarrow{\beta^0} H^1(\mathcal{O})^* \ , \\
H^1(\mathcal{O}) \xrightarrow{\alpha^1} H^1(\mathcal{O}(D) \otimes K_\Sigma^{1/2}) \ , \quad H^1(\mathcal{O}(D)^{-1} \otimes K_\Sigma^{1/2}) \xrightarrow{\beta^1} H^0(\mathcal{O})^* \ .
\end{aligned} \tag{4.110}$$

Let us first consider the cohomology of the two complexes on the left hand side. The maps α^0 and α^1 are injective and surjective respectively and therefore the cohomology can be written as

$$T_D \mathfrak{M}_{\mathfrak{m}} = \ker(\alpha^1) \oplus H^0(\mathcal{O}(D) \otimes K_\Sigma^{1/2}) / \mathrm{im}(\alpha^0) \ , \tag{4.111}$$

which corresponds to the tangent space of the symmetric product at point D. It follows that some of the massless fermionic fluctuations at point D encoded by the complexes span the tangent space to the bosonic moduli space, as explained in appendix A.3.4. By Serre duality, it is then easy to see that the combination of the two complexes on the right of (4.110) define the cotangent space $T^*\mathfrak{M}_\mathfrak{m}$ over the moduli space. Thus the virtual tangent space restricted on $\mathfrak{M}_\mathfrak{m}$ is given by

$$T_{\text{vir}}\big|_{\mathfrak{M}_\mathfrak{m}} = T\mathfrak{M}_\mathfrak{m} - T^*\mathfrak{M}_\mathfrak{m} \ , \tag{4.112}$$

where the second factor has weight t under the $U(1)_t$ action. Hence, we can identify the virtual Euler characteristic with the holomorphic Euler characteristic valued in the exterior powers of the tangent bundle, which can be identified with the χ_t-genus of the moduli space $\mathfrak{M}_\mathfrak{m}$. This can be computed from the ordinary index theorem:

$$\chi(\mathfrak{M}_\mathfrak{m}, \hat{\mathcal{O}}_{\text{vir}}) = \chi_t(\mathfrak{M}_\mathfrak{m}) = \int_{\mathfrak{M}_\mathfrak{m}} \hat{A}(T\mathfrak{M}_\mathfrak{m}) \ \text{ch}(\widehat{\wedge}^{\bullet} t \ T\mathfrak{M}_\mathfrak{m}) \ , \tag{4.113}$$

where $\widehat{S}^{\bullet}$ and $\widehat{\wedge}^{\bullet}$ are the normalized symmetric and exterior product defined in (4.66). [8]

In order to relate this expression to the twisted index computation, we have to introduce classes over symmetric products as well as some useful identities. First of all, we introduce standard generators of the cohomology ring of the symmetric product Σ_n following [48], and as summarized in appendixA.3.3:

$$\xi_i, \xi_i' \in H^1(\Sigma_n, \mathbb{Z}) \ , \quad \eta \in H^2(\Sigma_n, \mathbb{Z}) \ . \tag{4.114}$$

We also define the combination

$$\sigma_i = \xi_i \xi_i' \ , \quad i = 1, \cdots, g \quad \text{and} \quad \sum_{i=1}^{g} \sigma_i = \sigma \ . \tag{4.115}$$

The generators ξ_i and ξ_i' anticommute with each other and commute with η. The Chern class of the tangent bundle $T\Sigma_n$ is computed in [48]:

$$c(T\Sigma_n) = (1+\eta)^{n-2g+1} \prod_{i=1}^{g} (1 + \eta - \sigma_i) \ , \tag{4.116}$$

from which we obtain the Todd class:

$$\text{td}(T\Sigma_n) = \left(\frac{\eta}{1-e^{-\eta}}\right)^{n-2g+1} \prod_{i=1}^{g} \frac{\eta - \sigma_i}{1 - e^{-\eta+\sigma_i}} \ . \tag{4.117}$$

This formula can be simplifed by means of the following useful identity due to Don Zagier [49]. For any power series $h(\eta)$ on Σ_n, we have the identity

$$\begin{aligned} h(\eta)^{n-2g+1} \prod_{i=1}^{g} h(\eta - \sigma_i) &= h(\eta)^{n-g+1} \prod_{i=1}^{g} \left(1 - \sigma_i \frac{h'(\eta)}{h(\eta)}\right) \\ &= h(\eta)^{n-g+1} \exp\left(-\sigma \frac{h'(\eta)}{h(\eta)}\right) \ , \end{aligned} \tag{4.118}$$

[8] In standard notation for the Hirzebruch χ_y-genus, this is $t^{-\dim(\mathfrak{M}_\mathfrak{m})/2}\chi_{-t}(\mathfrak{M}_\mathfrak{m})$.

which follow from $\sigma_i^2 = 0$. If we choose $h(\eta) = \frac{\eta}{1-e^{-\eta}}$, we get

$$\mathrm{td}(T\Sigma_n) = \left(\frac{\eta}{1-e^{-\eta}}\right)^{n-g+1} \exp\left(\frac{\sigma}{e^{\eta}-1} - \frac{\sigma}{\eta}\right) . \tag{4.119}$$

The $\hat{A}$ genus of the tangent bundle can be obtained from the Todd class (4.119):

$$\begin{aligned}\hat{A}(T\mathfrak{M}_{\mathfrak{m}}) &= e^{-c_1(T\mathfrak{M}_{\mathfrak{m}})/2}\, \mathrm{td}(T\mathfrak{M}_{\mathfrak{m}}) \\ &= \left(\frac{\eta e^{-\eta/2}}{1-e^{-\eta}}\right)^{n-g+1} \exp\left(\frac{\sigma(e^{\eta}+1)}{2(e^{\eta}-1)} - \frac{\sigma}{\eta}\right) ,\end{aligned} \tag{4.120}$$

with $n = n_+ = \mathfrak{m} + g - 1$. Finally, the Chern character of the exterior powers of the tangent bundle can be obtained from (4.116). We find

$$\begin{aligned}\mathrm{ch}(\widehat{\wedge}^{\bullet} t\, T\mathfrak{M}_{\mathfrak{m}}) = &\left(e^{\pi i m_t} - e^{-\pi i m_t}\right)^{g-1} \left(e^{-\eta/2+\pi i m_t} - e^{\eta/2-\pi i m_t}\right)^{n-2g+1} \\ &\prod_{i=1}^{g} \left(e^{\pi i m_t - (\eta-\sigma_i)/2} - e^{-\pi i m_t + (\eta-\sigma_i)/2}\right) ,\end{aligned} \tag{4.121}$$

where $t = e^{2\pi i m_t}$. Again using the identity (4.118), we can simplify the expression to

$$\begin{aligned}\mathrm{ch}(\widehat{\wedge}^{\bullet} t\, T\mathfrak{M}_{\mathfrak{m}}) = &\left(e^{\pi i m_t} - e^{-\pi i m_t}\right)^{g-1} \left(e^{-\eta/2+\pi i m_t} - e^{\eta/2-\pi i m_t}\right)^{n-g+1} \\ &\exp\left(-\frac{\sigma(1+e^{-\eta+2\pi i m_t})}{2(1-e^{-\eta+2\pi i m_t})}\right) .\end{aligned} \tag{4.122}$$

Combining all these expressions, we now have

$$\begin{aligned}\chi(\mathfrak{M}_{\mathfrak{m}}, \hat{\mathcal{O}}_{\mathrm{vir}}) = \left(e^{\pi i m_t} - e^{-\pi i m_t}\right)^{g-1} &\int_{\Sigma_n} \left(\frac{\eta\left(e^{-(\eta/2-\pi i m_t)} - e^{(\eta/2-\pi i m_t)}\right)}{e^{\eta/2} - e^{-\eta/2}}\right)^{n-g+1} \\ &\exp\left(\frac{\sigma(e^{\eta}+1)}{2(e^{\eta}-1)} - \frac{\sigma}{\eta} - \frac{\sigma(1+e^{-\eta+2\pi i m_t})}{2(1-e^{-\eta+2\pi i m_t})}\right) .\end{aligned} \tag{4.123}$$

The integral can be converted into the residue integral using the following identity, also due to Don Zagier [49]. For any power series $A(\eta)$ and $B(\eta)$, one can show that

$$\int_{\Sigma_n} A(\eta) e^{\sigma B(\eta)} = \operatorname*{res}_{u=0} du\, \frac{A(u)(1+uB(u))^g}{u^{n+1}} . \tag{4.124}$$

Note that this formula holds also for $n = 0$ where $\Sigma_n = \mathrm{pt}$. Using this identity, we find

$$\begin{aligned}\chi_t(\mathfrak{M}_{\mathfrak{m}}) = 2\pi i \left(e^{\pi i m_t} - e^{-\pi i m_t}\right)^{g-1} &\operatorname*{res}_{u=0} \left(\frac{e^{-\pi i(u-m_t)} - e^{\pi i(u-m_t)}}{e^{\pi i u} - e^{-\pi i u}}\right)^{n-g+1} \\ \cdot &\left(\frac{e^{2\pi i u}+1}{2(e^{2\pi i u}-1)} - \frac{1+e^{2\pi i(-u+m_t)}}{2(1-e^{2\pi i(-u+m_t)})}\right)^{g} .\end{aligned} \tag{4.125}$$

This exactly reproduces the integral formula of the twisted index in the chamber $\tau > \frac{2\pi\mathfrak{m}}{e^2 \mathrm{Vol}(\Sigma)}$. One can check that the residue is non-zero in the region

$$-g+1 \leq \mathfrak{m} \leq g-1 . \tag{4.126}$$

This is consistent with the geometric observation that Σ_n becomes a holomorphic fibration over the Jacobian with fiber $\mathbb{CP}^{\mathfrak{m}-1}$ when $\mathfrak{m} > g-1$, see appendix A.3.3. In fact, the cohomology of Σ_n

factorizes into $H^\bullet(\Sigma_n) = H^\bullet(\mathbb{CP}^{\mathfrak{m}-1}) \otimes H^\bullet(\mathrm{Jac}[g])$ and therefore the index vanishes in this region since $\chi_t(\mathrm{Jac}[g]) = 0$. Multiplying by the weight $(-q)^{\mathfrak{m}}$ for each flux sector and summing over $\mathfrak{m}$, we have

$$\chi_t(\mathfrak{M}) = \sum_{\mathfrak{m}\in\mathbb{Z}} (-q)^{\mathfrak{m}} \chi_t(\mathfrak{M}_{\mathfrak{m}}) = (-q)^{-g+1} \left[(1-qt^{-1/2})(1-qt^{1/2})\right]^{g-1} \ , \tag{4.127}$$

which agrees with the generating function of the χ_t genus of the symmetric product of Σ computed in [48], up to an overall sign. Notice that as dictated by mirror symmetry, this also agrees with the index of the C-twist of the free hypermultiplet in the absence of background fluxes, see (4.99).

In the limit $t \to 1$, because of the the relation $\hat{A}(TM)\mathrm{ch}(\widehat{\wedge}^\bullet TM) = (-1)^{\dim_{\mathbb{C}} M} e(M)$ the virtual Euler characteristic becomes

$$\chi_t(\mathfrak{M})\big|_{t\to 1} = (-1)^{g-1} \sum_{\mathfrak{m}\in\mathbb{Z}} q^{\mathfrak{m}} \int_{\mathfrak{M}_{\mathfrak{m}}} e(\mathfrak{M}_{\mathfrak{m}}) = (-1)^{g-1} q^{-g+1} (1-q)^{2(g-1)} \ . \tag{4.128}$$

This reproduces the generating function of the Euler characteristic of the symmetric product of Σ.

C-twist In the case of the C-twist, the underlying moduli space is $\mathfrak{M}_{\mathfrak{m}} = \Sigma_{\mathfrak{m}}$. The fluctuations of the various fields on $\mathfrak{M}_{\mathfrak{m}}$ can be written as follows:

$$\begin{aligned} (\delta X, \psi_X) \in H^0(L) \ , \ \ (\eta_X, F_X) \in H^1(L) \ , \\ (\delta Y, \psi_Y) \in H^0(L^{-1}) \ , \ \ (\eta_Y, F_Y) \in H^1(L^{-1}) \ , \\ (\varphi, \psi_\varphi) \in H^0(K_\Sigma) \ , \ \ (\eta_\varphi, F_\varphi) \in H^1(K_\Sigma) \end{aligned} \tag{4.129}$$

where $\deg(L) = \mathfrak{m}$. In this case, the virtual tangent bundle at a point (4.52) coincides with[9]

$$\begin{aligned} H^0(\mathcal{O}) \longrightarrow H^0((L \oplus L^{-1})) \longrightarrow H^1(K_\Sigma)^* \\ H^1(\mathcal{O}) \longrightarrow H^1((L \oplus L^{-1})) \longrightarrow H^0(K_\Sigma)^* \ . \end{aligned} \tag{4.130}$$

Y vanishes identically in the chamber (4.106) and the complex split in various pieces. Note furthermore that $H^0(L^{-1})$ is empty when $\mathfrak{m} > 0$. Let us first assume $\mathfrak{m} > 0$. Then the virtual tangent bundle restricted to the bosonic moduli space can be written as

$$T_{\mathrm{vir}}\big|_{\mathfrak{M}_{\mathfrak{m}}} = T\mathfrak{M}_{\mathfrak{m}} + N_{\mathfrak{m}} \ , \tag{4.131}$$

where $T\mathfrak{M}_{\mathfrak{m}}$ is again the tangent space of the underlying moduli space defined by the complexes

$$H^0(\mathcal{O}) \longrightarrow H^0(L) \ , \quad H^1(\mathcal{O}) \longrightarrow H^1(L) \ . \tag{4.132}$$

The second component $N_{\mathfrak{m}}$ is the contribution from the normal bundle, which can be obtained from the cohomology of the remaining complex

$$H^0(L^{-1}) \to H^1(K_\Sigma)^* \ , \quad H^1(L^{-1}) \to H^0(K_\Sigma)^* \ , \tag{4.133}$$

[9] We omit the details about the maps, which we have already spelled out for the H-twist.

which defines a smooth vector bundle whose class is

$$[N_{\mathfrak{m}}] = -[H^\bullet(L \otimes K_\Sigma)^*] + [H^\bullet(K_\Sigma)^*] \ . \tag{4.134}$$

Therefore, for the C-twist, the virtual Euler characteristic computes the holomorphic Euler characteristic valued in $\widehat{S}^\bullet N_{\mathfrak{m}}^\vee$:

$$\chi(\mathfrak{M}_{\mathfrak{m}}, \hat{\mathcal{O}}_{\text{vir}}) = \int_{\mathfrak{M}_{\mathfrak{m}}} \hat{A}(T\mathfrak{M}_{\mathfrak{m}}) \wedge \text{ch}(\widehat{S}^\bullet N_{\mathfrak{m}}^\vee) \ . \tag{4.135}$$

Note that this can be extended to $\mathfrak{m} = 0$, where the moduli space is a point and the virtual tangent space is trivial.

The characteristic classes of the normal bundle $N_{\mathfrak{m}}$ are be most easily computed by means of a universal construction. This is summarized in appendix A.3.5. We introduce a universal divisor

$$\Delta \subset \Sigma \times \text{Sym}^{\mathfrak{m}}\Sigma \tag{4.136}$$

of degree $\mathfrak{m}$. This is defined by the property that if we restrict to an effective divisor D on $\Sigma \simeq \Sigma \times \{D\}$, we have

$$\Delta|_{\Sigma\times\{D\}} = D \times \{D\} \ , \tag{4.137}$$

which implies

$$\mathcal{O}(\Delta)|_{\Sigma\times\{D\}} = \mathcal{O}(D) \ . \tag{4.138}$$

Let us denote by π and f the projection onto each of the factors:

$$\begin{tikzcd} & \Sigma \times \text{Sym}^{\mathfrak{m}}\Sigma \arrow[dl, "\pi"'] \arrow[dr, "f"] & \\ \text{Sym}^{\mathfrak{m}}\Sigma & & \Sigma \end{tikzcd} \tag{4.139}$$

Then

$$\begin{aligned} R^0\pi_* \left(\mathcal{O}(\Delta) \otimes f^*M\right)|_D &= H^0(\Sigma, \mathcal{O}(D) \otimes M) \,, \\ R^1\pi_* \left(\mathcal{O}(\Delta) \otimes f^*M\right)|_D &= H^1(\Sigma, \mathcal{O}(D) \otimes M) \,, \end{aligned} \tag{4.140}$$

for any line bundle M on Σ, where $R^\bullet\pi_*$ is the derived pushforward. For the sake of simplicity, we will denote it by π_*. In particular, we can write the class of the vector bundle $N_{\mathfrak{m}}$ in (4.134) as

$$[N_{\mathfrak{m}}] = -[\pi_* \left(\mathcal{O}(\Delta) \otimes f^*K_\Sigma\right)^*] + [H^\bullet(K_\Sigma)^*] \ . \tag{4.141}$$

The Grothendieck-Riemann-Roch formula tells us

$$\text{td}(T\mathfrak{M}_{\mathfrak{m}}) \ \text{ch}\left(\pi_* \left(\mathcal{O}(\Delta) \otimes f^*K_\Sigma\right)\right) = \pi_* \left[\text{td}(\Sigma \times \Sigma_{\mathfrak{m}}) \ \text{ch}\left(\mathcal{O}(\Delta) \otimes f^*K_\Sigma\right)\right] \ . \tag{4.142}$$

Using $\pi_*\text{td}(\Sigma \times \Sigma_{\mathfrak{m}}) = \text{td}(\Sigma_{\mathfrak{m}}) \wedge \pi_*\text{td}(\Sigma)$, we find

$$\text{ch}\left(\pi_* \left(\mathcal{O}(\Delta) \otimes f^*K_\Sigma\right)\right) = \pi_* \left[\text{td}(\Sigma) \wedge \text{ch}\left(\mathcal{O}(\Delta) \otimes f^*K_\Sigma\right)\right] \ . \tag{4.143}$$

The cohomology class of Δ on the product $\Sigma \times \Sigma_{\mathfrak{m}}$ is computed in [50] using the Künneth decomposition. We can write a class $\delta \in H^2(\Sigma \times \Sigma_{\mathfrak{m}}, \mathbb{Z})$ as

$$\delta = \delta^{2,0} + \delta^{1,1} + \delta^{0,2} \ , \tag{4.144}$$

where $\delta^{i,j}$ is an element of $H^i(\Sigma) \otimes H^j(\Sigma_{\mathfrak{m}})$. The result is

$$\delta = \mathfrak{m}\eta_\Sigma + \gamma + \eta \ , \tag{4.145}$$

where η_Σ is the Kähler class on Σ, and γ is an element of $H^1(\Sigma) \otimes H^1(\mathrm{Sym}^{\mathfrak{m}}\Sigma)$. One can check that they satisfy $\eta_\Sigma^2 = \eta_\Sigma\gamma = \gamma^3 = 0$ and $\gamma^2 = -2\eta_\Sigma\sigma$. Using these identities, we find

$$\mathrm{ch}(\mathcal{O}(\Delta)) = e^\eta + \mathfrak{m}\eta_\Sigma e^\eta - \eta_\Sigma \sigma e^\eta + \gamma e^\eta \ . \tag{4.146}$$

The remaining factors in (4.143) can be easily computed:

$$\mathrm{td}(\Sigma) = 1 + (1-g)\eta_\Sigma \ , \quad \mathrm{ch}(f^* K_\Sigma) = 1 + 2(g-1)\eta_\Sigma \ . \tag{4.147}$$

Combining all these expressions, we find

$$\mathrm{ch}\left(\pi_* \left(\mathcal{O}(\Delta) \otimes f^* K_\Sigma\right)\right) = (\mathfrak{m} - \sigma + g - 1)t^{-1}e^\eta \ , \tag{4.148}$$

where $t = e^{2\pi i m_t}$. From this expression, we obtain the Chern class of this bundle. Using $\sigma_i^2 = 0$, we can rewrite (4.148) as

$$\mathrm{ch}\left(\pi_* \left(\mathcal{O}(\Delta) \otimes f^* K_\Sigma\right)\right) = \left[(\mathfrak{m}-1)t^{-1}e^\eta + \sum_{i=1}^{g} t^{-1} e^{\eta - \sigma_i}\right] , \tag{4.149}$$

which implies

$$c\left(\pi_* \left(\mathcal{O}(\Delta) \otimes f^* K_\Sigma\right)\right) = (1 + \eta - 2\pi i m_t)^{\mathfrak{m}-1} \prod_{i=1}^{g} (1 + \eta - 2\pi i m_t - \sigma_i). \tag{4.150}$$

Applying the identity (4.118), we arrive at the expression

$$\mathrm{ch}(\widehat{S}^\bullet N_{\mathfrak{m}}^\vee) = (e^{\pi i m_t} - e^{-\pi i m_t})^{1-g} (e^{-(\eta/2 - \pi i m_t)} - e^{(\eta/2 - \pi i m_t)/2})^{\mathfrak{m}+(g-1)} \exp\left[-\frac{\sigma(e^{\eta - 2\pi i m_t} - 1)}{2(e^{\eta - 2\pi i m_t} - 1)}\right] \ . \tag{4.151}$$

Multiplying all the contributions, the holomorphic Euler characteristic can now be written as

$$\begin{aligned}\chi(\mathfrak{M}_{\mathfrak{m}}, \hat{\mathcal{O}}_{\mathrm{vir}}) = (e^{\pi i m_t} - e^{-\pi i m_t})^{1-g} \int_{\mathcal{M}_{\mathfrak{m}}} \left(\frac{\eta e^{-\eta/2}}{1 - e^{-\eta}}\right)^{\mathfrak{m}-g+1} (e^{-(\eta/2 - \pi i m_t)} - e^{(\eta/2 - \pi i m_t)})^{\mathfrak{m}+(g-1)} \\ \wedge \exp\left[-\frac{\sigma}{\eta} + \frac{\sigma(e^\eta + 1)}{2(e^\eta - 1)} - \frac{\sigma(e^{\eta - 2\pi i m_t} + 1)}{2(e^{\eta - 2\pi i m_t} - 1)}\right] \ .\end{aligned} \tag{4.152}$$

Using the identity (4.124), we can convert this formula into the residue integration

$$\begin{aligned}\chi(\mathfrak{M}_{\mathfrak{m}}, \hat{\mathcal{O}}_{\mathrm{vir}}) = 2\pi i (e^{\pi i m_t} - e^{-\pi i m_t})^{1-g} \operatorname*{res}_{u=0} \frac{\left(e^{\pi i(-u+m_t)} - e^{\pi i(u - m_t)}\right)^{\mathfrak{m}+g-1}}{\left(e^{\pi i u} - e^{-\pi i u}\right)^{\mathfrak{m}-g+1}} \\ \cdot \left(\frac{e^{2\pi i u} + 1}{2(e^{2\pi i u} - 1)} - \frac{e^{2\pi i(u - m_t)} + 1}{2(e^{2\pi i(u - m_t)} - 1)}\right)^g \ ,\end{aligned} \tag{4.153}$$

which exactly reproduces the twisted index computation. One can check that

$$\chi(\mathfrak{M}_{\mathfrak{m}}, \hat{\mathcal{O}}_{\text{vir}}) = \begin{cases} 1 , & \mathfrak{m} = 0 \\ 0 , & \mathfrak{m} \neq 0 \end{cases} \tag{4.154}$$

Notice that this result is also compatible with the $t \to 1$ limit as described below (4.88), as well as with the result of the H-twist of the free hypermultiplet in the absence of background fluxes (4.99), in accordance with mirror symmetry.

4.4.3 SQED$[N]$

Let us now generalize the previous discussion to a $U(1)$ gauge theory with N fundamental hypermultiplets. These theories have non-trivial Higgs-branch flavour symmetry, and they satisfy the conditions spelled out in 4.1.2 provided $N \geq 2$. We assume the following charge assignment:

$$\begin{array}{c|ccc|cc} & U(1)_G & U(1)_t & SU(N)_H & U(1)_H & U(1)_C \\ \hline X & 1 & \frac{1}{2} & \overline{\square} & \frac{1}{2} & 0 \\ Y & -1 & \frac{1}{2} & \square & \frac{1}{2} & 0 \\ \varphi & 0 & -1 & 0 & 0 & 1 \end{array} . \tag{4.155}$$

$\mathcal{N} = 2$ moduli space

The BPS moduli space $\mathfrak{M}$ that preserves $\mathcal{N} = 2$ supersymmetry is given by triples (A, X, Y) which satisfy following equations:

$$* F_A + e^2 \left(XX^\dagger - Y^\dagger Y - \tau \right) = 0 \ , \quad \bar{\partial}_A X_i = \bar{\partial}_A Y_i = 0 \ , \quad \sum_{i=1}^{N} X_i Y_i = 0 \ , \tag{4.156}$$

modulo $U(1)$ gauge transformations. As in SQED[1], the moduli space of solutions decomposes into topological sectors

$$\mathfrak{M} = \bigcup_{\mathfrak{m} \in \mathbb{Z}} \mathfrak{M}_{\mathfrak{m}} \ , \tag{4.157}$$

where $\mathfrak{m}$ is the degree of the gauge bundle. We will work in the limit $s \to +\infty$, where we recall $s = e^2 \text{Vol}(\Sigma)$ so that the moduli space is uniformly described with non-vanishing X. As explained in section 4.2.4, the algebraic description of the moduli space coincides with the space of stable quasi-maps into the Higgs branch $T^*\mathbb{CP}^{N-1}$ (C-twist) or twisted stable quasi-maps (H-twist).

In order to compute the index, we consider the action of the Higgs branch flavour symmetry $G_H = SU(N)$ and apply the localization principle to the diagonal subgroup

$$\text{diag}\,(y_1 \ , \cdots , y_N) \in T_H \ , \quad \prod_{i=1}^{N} y_i = 1 \ . \tag{4.158}$$

The variables a_i's are chosen to be completely generic, so that we have $a_i \neq a_j$ for any pair $i, j = 1, \cdots, N$. The subgroup acts on the moduli space as

$$t_H : (d_A, \{X_i, Y_i\}) \to \left(d_A, \{a_i X_i, a_i^{-1} Y_i\} \right) \ . \tag{4.159}$$

In addition to the T_H action, we can also consider the action of $U(1)_t$ which acts on X and Y as multiplication by $t^{1/2} = e^{\pi i m_t}$. The fixed loci of (4.158) is determined up to the action of the gauge symmetry. In our limit, the fixed locus is a disjoint union of N components

$$\mathfrak{M}_{\mathfrak{m}}^{\text{fixed}} = \bigcup_{i=1}^{N} \mathfrak{M}_{\mathfrak{m}}^{(i)} \ , \tag{4.160}$$

which are defined by setting all the bosonic fields to zero except for one of the X_i's:

$$\mathfrak{M}_{\mathfrak{m}}^{(i)} = \left\{(A, X_i) | * F_A + e^2(X_i X_i^\dagger - \tau) = 0 \ , \ \bar{\partial}_A X_i = 0\right\} / \ U(1)_G \ . \tag{4.161}$$

Note that $\mathfrak{M}_{\mathfrak{m}}^{(i)}$ can be again identified with a symmetric power of the curve Σ

$$\mathfrak{M}_{\mathfrak{m}}^{(i)} = \Sigma_n \ , \quad n = \mathfrak{m} + r(g-1) \ , \tag{4.162}$$

and that for each fixed locus there exists an inclusion

$$\sigma_i : \mathfrak{M}_{\mathfrak{m}}^{(i)} \to \mathfrak{M}_{\mathfrak{m}} \ . \tag{4.163}$$

From now on, we understand the moduli space algebraically and work with its virtual tangent space. The virtual tangent space at a generic point on $\mathfrak{M}_{\mathfrak{m}}$ is given by the cohomology of the following complexes:

$$\begin{aligned} &H^0(\mathcal{O}) \xrightarrow{\alpha^0} H^0(M_X \oplus M_Y) \xrightarrow{\beta^0} H^1(K_\Sigma^{1-r})^* \ , \\ &H^1(\mathcal{O}) \xrightarrow{\alpha^1} H^1(M_X \oplus M_Y) \xrightarrow{\beta^1} H^0(K_\Sigma^{1-r})^* \ . \end{aligned} \tag{4.164}$$

Here we defined

$$M_X = \bigoplus_{i=1}^{N} L \otimes K_\Sigma^{r/2} \quad \text{and} \quad M_Y = \bigoplus_{i=1}^{N} L^{-1} \otimes K_\Sigma^{r/2} \ , \tag{4.165}$$

where each summand has weight $y_i t^{1/2}$ and $y_i^{-1} t^{1/2}$ respectively under the action of $T_H \times U(1)_t$. We recall that the maps are defined by

$$\begin{aligned} &\alpha : \epsilon \mapsto (\epsilon X_1, \cdots, \epsilon X_N, -\epsilon Y_1, \cdots, -\epsilon Y_N) \\ &\beta : (A_1, \cdots, A_N, B_1, \cdots, B_N) \mapsto \sum_{i=1}^{N} A_i Y_i + B_i X_i \ . \end{aligned} \tag{4.166}$$

We notice that if we restrict to points in a component of the fixed locus $\mathfrak{M}_{\mathfrak{m}}^{(i)}$, the complexes split into various pieces. From the first line of (4.164), we have

$$\begin{aligned} H^0(\mathcal{O}) \longrightarrow H^0(L \otimes K_\Sigma^{r/2}) \ , \quad H^0(L^{-1} \otimes K_\Sigma^{r/2}) \longrightarrow H^1(K^{1-r})^* \ , \\ H^0\left(\bigoplus_{j\neq i}^{N} (L \oplus L^{-1}) \otimes K_\Sigma^{r/2}\right) \longrightarrow 0 \ . \end{aligned} \tag{4.167}$$

From the second line, we obtain similar complexes with the degree shifted by one:

$$\begin{aligned} H^1(\mathcal{O}) \longrightarrow H^1(L \otimes K_\Sigma^{r/2}) \ , \quad H^1(L^{-1} \otimes K_\Sigma^{r/2}) \longrightarrow H^0(K_\Sigma^{1-r})^* \longrightarrow 0 \ , \\ H^1\left(\bigoplus_{j\neq i}^{N} (L \oplus L^{-1}) \otimes K_\Sigma^{r/2}\right) \longrightarrow 0 \ . \end{aligned} \tag{4.168}$$

As explained around (4.111), we can identify the first complex of (4.167) and (4.168) as the tangent space of the fixed locus $\mathfrak{M}_{\mathfrak{m}}^{(i)}$. Therefore the virtual tangent space restricted to the fixed locus can be written as

$$T_{\mathrm{vir}}\big|_{\mathfrak{M}_{\mathfrak{m}}^{(i)}} = T\mathfrak{M}_{\mathfrak{m}}^{(i)} + N_{\mathfrak{m}}^{(i)} \ . \tag{4.169}$$

The second piece corresponds to the contributions of the virtual normal bundle $N_{\mathfrak{m}}^{(i)}$, which have a non-zero weight under the action of $T_H \times U(1)_t$. The class of the virtual normal bundle is

$$[N_{\mathfrak{m}}^{(i)}] = [H^\bullet(L^{-1} \otimes K_\Sigma^{r/2})] - [H^\bullet(K_\Sigma^{1-r})^*] + \left[H^\bullet\left(\bigoplus_{j\neq i}^{N}(L \oplus L^{-1}) \otimes K_\Sigma^{r/2}\right)\right] \ . \tag{4.170}$$

The first two terms

$$[H^\bullet(L^{-1} \otimes K_\Sigma^{r/2})] - [H^\bullet(K_\Sigma^{1-r})^*] := \widetilde{N}_{\mathfrak{m}}^{(i)} \ , \tag{4.171}$$

are the contributions of the multiplet Y^i and the vector multiplet, whose Chern classes were computed in the last section. The last summand of (4.170) contains contributions from the hypermultiplets (X_j, Y_j) with $j \neq i$, which have non-zero weights $(y_{ji}, y_{ji}^{-1}t)$ under the action $t_H \times U(1)_t$. We will denote the contributions from these fields as $N_{\mathfrak{m}}^{(i),X_j} := [H^\bullet(L \otimes K_\Sigma^{r/2})]$ and $N_{\mathfrak{m}}^{(i),Y_j} := [H^\bullet(L^{-1} \otimes K_\Sigma^{r/2})]$ for $j \neq i$. Now the equivariant virtual Euler characteristic can be written as

$$\begin{aligned} \chi(\mathfrak{M}_{\mathfrak{m}}, \hat{\mathcal{O}}_{\mathrm{vir}}) &= \sum_{i=1}^{N} \int_{\mathfrak{M}_{\mathfrak{m}}^{(i)}} \hat{A}(T\mathfrak{M}_{\mathfrak{m}}^{(i)}) \ \mathrm{ch}(\widehat{S}^\bullet N_{\mathfrak{m}}^{(i)\vee}) \\ &= \sum_{i=1}^{N} \int_{\mathfrak{M}_{\mathfrak{m}}^{(i)}} \hat{A}(T\mathfrak{M}_{\mathfrak{m}}^{(i)}) \ \mathrm{ch}(\widehat{S}^\bullet \widetilde{N}_{\mathfrak{m}}^{(i)\vee}) \ \prod_{j\neq i}^{N} \left[\mathrm{ch}(\widehat{S}^\bullet N_{\mathfrak{m}}^{(i),X_j\vee}) \ \mathrm{ch}(\widehat{S}^\bullet N_{\mathfrak{m}}^{(i),Y_j\vee})\right] \ . \end{aligned} \tag{4.172}$$

The Chern class of $[N_{\mathfrak{m}}^{(i),X_j}]$ and $[N_{\mathfrak{m}}^{(i),Y_j}]$ for $j \neq i$ can also be computed from the universal construction discussed in the last section. We can derive

$$c\left(N_{\mathfrak{m}}^{(i),X_j}, m_{ji}\right) = (1 + \eta + 2\pi i m_{ji})^{n-2g+1} \prod_{a=1}^{g} (1 + \eta + 2\pi i m_{ji} - \sigma_a) \ . \tag{4.173}$$

where we defined $a_i = e^{2\pi i m_i}$ and $m_{ij} := m_i - m_j$. Using the identity (4.118), we can write the Chern characteristics of the symmetric powers as

$$\mathrm{ch}\left(\widehat{S}^\bullet N_{\mathfrak{m}}^{(i),X_j\vee}\right) = \left(\frac{e^{(-\eta/2+\pi i m_{ji})}}{1 - e^{-(\eta - 2\pi i m_{ji})}}\right)^{\mathfrak{m}+(r-1)(g-1)} \exp\left[\frac{\sigma(e^{\eta-2\pi i m_{ji}} + 1)}{2(e^{\eta-2\pi i m_{ji}} - 1)}\right] \ . \tag{4.174}$$

From the multiplet Y_j $(j \neq i)$,

$$\begin{aligned} \mathrm{ch}\left(\widehat{S}^\bullet N_{\mathfrak{m}}^{(i),Y_j\vee}\right) &= \left(e^{(-\eta/2+\pi i m_{ji}+\pi i m_t)} - e^{\eta/2-\pi i m_{ji}-\pi i m_t}\right)^{\mathfrak{m}-(r-1)(g-1)} \\ &\quad \cdot \exp\left[\frac{\sigma(e^{-\eta+2\pi i(m_{ji}+m_t)} + 1)}{2(e^{-\eta+2\pi i(m_{ji}+m_t)} - 1)}\right] \ . \end{aligned} \tag{4.175}$$

The contribution from $\mathrm{ch}\left(\widehat{S}^\bullet \widetilde{N}_{\mathfrak{m}}^{(i)}\right)$ is the same as the normal bundle contribution studied in the last example. We have

$$\begin{aligned} \mathrm{ch}\left(\widehat{S}^\bullet \widetilde{N}_{\mathfrak{m}}^{(i)\vee}\right) &= \left(e^{\pi i m_t} - e^{-\pi i m_t}\right)^{(2r-1)(g-1)} \left(e^{\pi i m_t - \eta/2} - e^{-\pi i m_t + \eta/2}\right)^{\mathfrak{m}-(r-1)(g-1)} \\ &\quad \cdot \exp\left[\frac{\sigma(e^{-\eta+2\pi i m_t} + 1)}{2(e^{-\eta+2\pi i m_t} - 1)}\right] \ . \end{aligned} \tag{4.176}$$

In converting this expression into the residue integral using (4.124), we find that the equivariant virtual Euler characteristic can be written as

$$\begin{aligned}\chi(\mathfrak{M}_{\mathfrak{m}},\hat{\mathcal{O}}_{\mathrm{vir}}) = 2\pi i &\left(t^{1/2}-t^{-1/2}\right)^{(2r-1)(g-1)} \\ &\cdot \sum_{i=1}^{N} \operatorname*{res}_{u=m_i} du \prod_{j=1}^{N} \frac{\left(e^{\pi i(-u+m_j+m_t)}-e^{\pi i(u-m_j-m_t)}\right)^{\mathfrak{m}-(g-1)(r-1)}}{\left(e^{\pi i(u-m_j)}-e^{\pi i(-u+m_j)}\right)^{\mathfrak{m}+(g-1)(r-1)}} \\ &\cdot \left[\sum_{j=1}^{N}\left(\frac{1+e^{2\pi i(-u+m_j)}}{2(1-e^{2\pi i(-u+m_j)})}+\frac{1+e^{2\pi i(u-m_j-m_t)}}{2(1-e^{2\pi i(u-m_j-m_t)})}\right)\right]^{g} .\end{aligned} \tag{4.177}$$

This again reproduces the integral representation of the twisted index computation.

The $t\to 1$ limit

H-twist For the H-twist, the expression (4.177) (with $r=1$) can be understood as the virtual χ_t genus of the $U(1)_t$-fixed locus

$$\mathfrak{L} = \bigoplus_{\mathfrak{m}\in\mathbb{Z}} \mathfrak{L}_{\mathfrak{m}} \ , \tag{4.178}$$

where $\mathfrak{L}_{\mathfrak{m}}$ can be identified as a space of degree $\mathfrak{m}$ twisted quasi-maps to the compact core $\mathbb{CP}^{N-1}$, the base of Higgs branch $\mathcal{M}_H$. This space is parametrized by the solution (A, X_i) to the equations

$$* F_A + e^2\left(XX^\dagger-\tau\right)=0 \ , \ \bar\partial_A X = 0 \ , \tag{4.179}$$

modulo $U(1)$ gauge tranformations.

The H-twisted $\mathcal{N}=4$ moduli space for this theory is identical to that of the $\mathcal{N}=2$ moduli space defined in (4.157). In the limit $t\to 1$, we recover the expression for the integral of the virtual Euler class of the fixed locus $\mathfrak{L}$ inside the moduli space. This quantity can be directly computed using the alternative localization scheme with respect to the $T_H\subset G_H$ action. Then the index can be written as a sum of the Euler characteristics of the smooth compact fixed loci $\mathfrak{M}^{(i)}_{\mathfrak{m}}$ defined in (4.161). We have

$$\begin{aligned}\sum_{\mathfrak{m}\in\mathbb{Z}}(-q)^{\mathfrak{m}}\chi(\mathfrak{M}_{\mathfrak{m}},\hat{\mathcal{O}}_{\mathrm{vir}})\big|_{t\to 1} &= (-1)^{g-1}\sum_{\mathfrak{m}\in\mathbb{Z}} q^{\mathfrak{m}} \sum_{i=1}^{N}\int_{\mathfrak{M}^{(i)}_{\mathfrak{m}}} e(\mathfrak{M}^{(i)}_{\mathfrak{m}}) \\ &= (-1)^{g-1}N(q^{1/2}-q^{-1/2})^{2(g-1)} \ ,\end{aligned} \tag{4.180}$$

which correctly reproduces the generating function for the Euler characteristic of the N copies of the $\mathrm{Sym}^{\mathfrak{m}}\Sigma$. Note that the residue integration at each i is independent of the equivariant parameters $\{a_i\}$, which agrees with the fact that the Hilbert space of the effective quantum mechanics is the de Rham cohomology [57].

C-twist For the C-twist, imposing $\mathcal{N}=4$ BPS equations trivializes the line bundle L, and the moduli space parametrizes the the solutions $(\{X_i,Y_i\})$ to the equation

$$XX^\dagger - Y^\dagger Y = \tau \ , \ \ \sum_{i=1}^{N} X_iY_i = 0 \ , \tag{4.181}$$

for constant X_i and Y_i, modulo $U(1)$ gauge transformation. This is the resolution of the Higgs branch $\mathcal{M}_H = T^*(\mathbb{CP}^{N-1})$ inside the $\mathcal{N}=2$ moduli space $\mathfrak{M}$.

The $t \to 1$ limit with $r = 0$ of the result (4.177) can be understood as the Rozansky-Witten invariant computing the holomorphic Euler characteristic of $\mathcal{M}_H$ valued in the vector bundle $\left(\widehat{\wedge}^{\bullet} T^*\mathcal{M}_H\right)^g$:

$$\chi(\mathfrak{M}, \hat{\mathcal{O}}_{\mathrm{vir}})\big|_{t\to 1} = \int_{\mathcal{M}_H} \hat{A}\left(T\mathcal{M}_H\right) \wedge \mathrm{ch}\left[\left(\widehat{\wedge}^{\bullet} T^*\mathcal{M}_H\right)^{\otimes g}\right] . \tag{4.182}$$

$\mathcal{M}_H$ is non-compact and this expression can be evaluated from the equivariant localization with respect to the $T_H \subset G_H$ action. Let us consider the action of g_H defined in (4.158). When $\tau > 0$, the fixed loci are N isolated points, where i-th fixed point is defined by $X_i \neq 0$ and all the other bosonic fields are identically zero. From the fixed point formula we arrive at the expression

$$\begin{aligned} \chi(\mathfrak{M}, \hat{\mathcal{O}}_{\mathrm{vir}})(y \to 1) &= \sum_{i=1}^{N} \prod_{j\neq i} \left(\frac{e^{-\pi i m_{ij}}}{1-e^{-2\pi i m_{ij}}}\right)^2 \prod_{j\neq i} \left(e^{\pi i m_{ij}} - e^{-\pi i m_{ij}}\right)^{2g} \\ &= \sum_{i=1}^{N} \prod_{j\neq i} \left(e^{-\pi i m_{ij}} - e^{\pi i m_{ij}}\right)^{2(g-1)} . \end{aligned} \tag{4.183}$$

4.4.4 SQCD$[N_c, N_f]$

We can generalize our previous analysis to non-Abelian gauge groups, provided that the fixed loci of the moduli space are products of symmetric products of the curve Σ. In this section we present the simplest example, which is SQCD$[N_c, N_f]$ where $N_f \geq 2N_c$, as discussed in section 4.1.2. The fields of the theory are charged as follows:

	$U(N_c)_G$	$U(1)_t$	$SU(N)_H$	$U(1)_H$	$U(1)_C$
X	$\square$	$\frac{1}{2}$	$\overline{\square}$	$\frac{1}{2}$	0
Y	$\overline{\square}$	$\frac{1}{2}$	$\square$	$\frac{1}{2}$	0
φ	adj	-1	$\mathbf{1}$	0	1

(4.184)

$\mathcal{N}=2$ moduli space

The $\mathcal{N}=2$ BPS equations for SQCD$[N_c, N_f]$ read

$$\begin{aligned} &* F_A + e^2\left(XX^\dagger - Y^\dagger Y - 2[\varphi^\dagger, \varphi] - \tau\right) = 0 \\ &\bar{\partial}_A X = \bar{\partial}_A Y = D_{\bar{z}}\varphi = 0 \\ &\varphi \cdot X = \varphi \cdot Y = X \cdot Y = 0 . \end{aligned} \tag{4.185}$$

The moduli space of solutions to BPS equations modulo gauge transformations can be decomposed into topological sectors labelled by the degree of the holomorphic bundle in the fundamental representation associated to the gauge bundle P:

$$\mathfrak{M} = \bigcup_{\mathfrak{m}\in\Lambda_C^\vee} \mathfrak{M}_{\mathfrak{m}} . \tag{4.186}$$

We again consider the limit $s \to +\infty$, where $s = e^2 \mathrm{Vol}(\Sigma)$. It follows from the discussion in section 4.2.3 that by using a Hitchin-Kobayashi correspondence, the moduli space has the follwing algebraic description for every $\mathfrak{m}$. A point of the moduli space in the component $\mathfrak{M}_{\mathfrak{m}}$ is given by

- A holomorphic $GL(N_c, \mathbb{C})$-bundle E of degree $\mathfrak{m}$;
- Holomorphic sections (X, Y) of associated bundles E_X and E_Y, corresponding to N_f copies of the fundamental and anti-fundamental representation respectively;
- Subject to the complex moment map condition $Y \cdot X = 0$;
- Subject to the stability condition that X has generically maximal rank on Σ;
- Modulo complexified gauge transformations.

This can be thought of as the space of stable quasi-maps into the Higgs branch $\mathcal{M}_H = T^*G(N_c, N_f)$ (C-twist) or twisted stable quasi maps (H-twist).

Let us consider the fixed points of a maximal torus T_H of the flavour symmetry, which locally acts as

$$X \mapsto X t_H, \ , \ Y \mapsto t_H Y \ , \ \ \bar{\partial}_{A_{\mathbb{C}}} \mapsto \bar{\partial}_{A_{\mathbb{C}}} \ , \tag{4.187}$$

for t_H represented as a diagonal $N_f \times N_f$ matrix, and with $A_{\mathbb{C}}$ the connection on the holomorphic bundle $E_{\mathbb{C}}$. The fixed points are solutions to the equations

$$g_{\mathbb{C}} X = X t_H \ , \ Y g_{\mathbb{C}} = t_H Y \ , \ g_{\mathbb{C}}^{-1} d_{A_{\mathbb{C}}} g_{\mathbb{C}} = d_{A_{\mathbb{C}}} \ , \tag{4.188}$$

for an element of the complex gauge transformation $g_{\mathbb{C}} \in G_{\mathbb{C}}$. Given the stability condition on X, $g_{\mathbb{C}}$ must act non-trivially. From the last equation of (4.188), $E_{\mathbb{C}}$ decomposes at fixed points as a direct sum of line bundles

$$E_{\mathbb{C}} = L_1 \oplus \cdots \oplus L_{N_c} \, . \tag{4.189}$$

Let us denote $\mathfrak{m}_a = \deg(L_a)$, which satisfy

$$\mathfrak{m} = \sum_a \mathfrak{m}_a \in H_2(\mathcal{M}_H, \mathbb{Z}) \cong \mathbb{Z} \ . \tag{4.190}$$

The associated bundles E_X and E_Y decompose accordingly

$$\begin{aligned} E_X &\cong (L_1 \oplus \cdots \oplus L_{N_c})^{\oplus N_f} \otimes K^{r/2} \\ E_Y &\cong \left(L_1^{-1} \oplus \cdots \oplus L_{N_c}^{-1}\right)^{\oplus N_f} \otimes K^{r/2} \, . \end{aligned} \tag{4.191}$$

For later convenience, we also note that on any fixed locus E_V and E_Φ decompose as [10]

$$E_V \cong \left(\mathcal{O}^{N_c} \oplus \bigoplus_{a \neq b} L_a \otimes L_b^{-1} \right) \quad \text{and} \quad E_\Phi \cong \left(\mathcal{O}^{N_c} \oplus \bigoplus_{a \neq b} L_a \otimes L_b^{-1} \right) \otimes K_\Sigma^{1-r} \, . \tag{4.193}$$

[10]In fact, the complexified Lie algebra decomposes under the adjoint action as

$$\mathfrak{g}_{\mathbb{C}} = \mathfrak{t}_{\mathbb{C}} \oplus \bigoplus_{\alpha \in \mathfrak{g}_{\mathbb{C}}} \mathfrak{g}_{\mathbb{C}\alpha} \ , \tag{4.192}$$

where the summands can be identified with diagonal matrices (over which the adjoint action of $\mathfrak{t}_{\mathbb{C}} \cong (a_1, \cdots, a_N)$ is trivial) and matrices with one single off-diagonal entry e_{ij} (the action corresponding to $x_{ij} \mapsto a_i a_j^{-1} x_{ij}$).

Fixed points are labelled by N_c-subsets $I = \{i_1, \cdots, i_{N_c}\} \subset \{1, \cdots, N_f\}$ so that the only non-vanishing sections are

$$X^a_{i_a} \neq 0\,, \tag{4.194}$$

and fixed loci reduce to disjoint unions of N_c copies of symmetric products

$$\mathfrak{M}^T_{\mathfrak{m}} = \bigsqcup_{\substack{(\mathfrak{m}_1,\cdots,\mathfrak{m}_{N_c}) \\ \sum_a \mathfrak{m}_a = \mathfrak{m}}} \left(\prod_{a=1}^{N_c} \mathrm{Sym}^{\mathfrak{m}_a + r(g-1)} \Sigma \right), \tag{4.195}$$

where as usual $r = 1, 0$ for H- and C-twist respectively.

The breaking of the gauge bundle into a sum of Abelian contributions makes the generalisation from SQED$[N]$ to SQCD$[N_c, N_f]$ rather straightforward, and we will therefore be brief, mainly working at the level of K-theory classes. We will work on the component $\underline{\mathfrak{m}} = (\mathfrak{m}_1, \cdots, \mathfrak{m}_{N_c})$ of the fixed locus I, which we denote $\mathfrak{M}_{I,\underline{\mathfrak{m}}}$.

Over $\mathfrak{M}_{I,\underline{\mathfrak{m}}}$, the virtual tangent bundle decomposes into the following contributions

$$\begin{aligned} \left[T_{\mathrm{vir}}|_{\mathfrak{M}_{I,\underline{\mathfrak{m}}}}\right] = &\sum_{a=1}^{N_c} \left(-[H^\bullet(\mathcal{O})] + \left[H^\bullet \left(L_a^{i_a} \otimes K_\Sigma^{r/2} \right) \right] \right) \\ &+ \sum_{a=1}^{N_c} \sum_{\substack{j=1 \\ j\neq i_a}}^{N_f} \Bigg[\underbrace{H^\bullet \left(L_a^j \otimes K_\Sigma^{r/2} \right)}_{:=NX^a_j} \Bigg] + \sum_{a=1}^{N_c} \sum_{j=1}^{N_f} \Bigg[\underbrace{H^\bullet \left(\left(L_a^j\right)^{-1} \otimes K_\Sigma^{r/2} \right)}_{:=NY^a_j} \Bigg] \\ &- \sum_{a=1}^{N_c} \Bigg[\underbrace{H^\bullet\left(K_\Sigma^r\right)}_{:=N\Phi_{aa}} \Bigg] - \sum_{a\neq b} \Bigg[\underbrace{H^\bullet \left(L_a^{-1} \otimes L_b \otimes K_\Sigma^r \right)}_{:=N\Phi_{ab}} \Bigg] - \sum_{a\neq b} \Bigg[\underbrace{H^\bullet (L_a \otimes L_b^{-1})}_{:=NV_{ab}} \Bigg]\,. \end{aligned} \tag{4.196}$$

The first line includes all contributions tangent to the fixed locus (fixed part), whereas all other contributions are normal (moving part). In order to express these contributions in terms of characteristic classes over the fixed locus $\prod_{a=1}^{N_c} \Sigma_{n_a}$, let us first define the generators of cohomology class as follows:

$$\eta_a \in H^2(\Sigma_{n_a}, \mathbb{Z}) \;\text{ and }\; \sigma^{ab} = \sum_{i=1}^{g} \xi^a_i \xi'^b_i\;,\quad \xi^a_i, \xi'^a_i \in H^1(\Sigma_{n_a}, \mathbb{Z})\;. \tag{4.197}$$

Then from the fixed part, we obtain the tangent bundle over the fixed locus which contributes

$$\hat{A}\left(T\mathfrak{M}_{I,\underline{\mathfrak{m}}}\right) = \prod_{a=1}^{N_c} \left(\frac{\eta_a e^{-\eta_a/2}}{1 - e^{-\eta_a}} \right)^{\mathfrak{m}_a + (r-1)(g-1)} \exp\left(\frac{\sigma^{aa}(e^{\eta_a}+1)}{2(e^{\eta_a}-1)} - \frac{\sigma^{aa}}{\eta_a} \right)\,. \tag{4.198}$$

The contributions from the moving part $N_{I,\underline{\mathfrak{m}}}$ can be summarized as

$$\prod_{a=1}^{N_c} \left[\prod_{\substack{j=1 \\ j\neq i_a}}^{N_f} \mathrm{ch}(\widehat{S}^\bullet NX^{a\vee}_j) \prod_{j=1}^{N_f} \mathrm{ch}(\widehat{S}^\bullet NY^{a\vee}_j) \prod_{b=1}^{N_c} \mathrm{ch}(\widehat{\wedge}^\bullet N\Phi^\vee_{ab}) \right] \prod_{a\neq b} \mathrm{ch}(\widehat{\wedge}^\bullet NV^\vee_{ab})\,. \tag{4.199}$$

The arguments (4.174)-(4.176) can be straightforwardly generalized to obtain the contribution from the hypermultiplets:

$$\mathrm{ch}(\widehat{S}^\bullet NX^{a\vee}_j) = \left(\frac{e^{-\eta_a/2 + \pi i m_{ji_a}}}{1 - e^{-\eta_a + 2\pi i m_{ji_a}}} \right)^{\mathfrak{m}_a + (r-1)(g-1)} \exp\left[\frac{\sigma^{aa}(e^{\eta_a - 2\pi i m_{ji_a}} + 1)}{2(e^{\eta_a - 2\pi i m_{ji_a}} - 1)} \right] \tag{4.200}$$

and

$$\begin{aligned}\mathrm{ch}(\widehat{S}^\bullet NY_j^{a\vee}) = &\left(e^{-\eta_a/2+\pi i(m_{ji_a}+m_t)} - e^{\eta_a/2-\pi i(m_{ji_a}+m_t)}\right)^{\mathfrak{m}_a-(r-1)(g-1)} \\ &\cdot \exp\left[\frac{\sigma^{aa}(e^{-\eta_a+2\pi i(m_{ji_a}+m_t)}+1)}{2(e^{-\eta_a+2\pi i(m_{ji_a}+m_t)}-1)}\right].\end{aligned} \tag{4.201}$$

The contribution from the multiplet in the adjoint representation, NV_{ab} and $N\Phi_{ab}$ can be written as classes on $\mathrm{Sym}^{\mathfrak{m}_a+r(g-1)}\Sigma \times \mathrm{Sym}^{\mathfrak{m}_b+r(g-1)}\Sigma$. We computed the characteristic classes of these contributions in appendix A.6. To summarize, the vector multiplet contribution is

$$\begin{aligned}\prod_{a\neq b}\mathrm{ch}(\widehat{\wedge}^\bullet NV_{ab}^\vee) = &\prod_{a\neq b}\left(e^{(-\eta_a+\eta_b)/2-\pi i(m_{i_a}-m_{i_b})} - e^{(\eta_a-\eta_b)/2+\pi i(m_{i_a}-m_{i_b})}\right)^{-\mathfrak{m}_a+\mathfrak{m}_b+1-g} \\ &\exp\left[(\sigma^{aa}+\sigma^{bb}-\sigma^{ab}-\sigma^{ba})\frac{(e^{-\eta_a+\eta_b-2\pi i(m_{i_a}-m_{i_b})}+1)}{2(e^{-\eta_a+\eta_b-2\pi i(m_{i_a}-m_{i_b})}-1)}\right].\end{aligned} \tag{4.202}$$

Note that the exponential terms in (4.202) with positive and negative root α cancel each other out, and we are left with a simple expression

$$\prod_{a\neq b}\mathrm{ch}(\widehat{\wedge}^\bullet NV_{ab}^\vee) = (-1)^{\sum_{\alpha>0}\alpha(\underline{\mathfrak{m}})}\prod_{a\neq b}\left(e^{(-\eta_a+\eta_b)/2-\pi i(m_{i_a}-m_{i_b})} - e^{(\eta_a-\eta_b)+\pi i(m_{i_a}-m_{i_b})}\right)^{1-g}. \tag{4.203}$$

The contribution from the adjoint chiral $N\Phi_{ab}$ can be similarly written as

$$\begin{aligned}\prod_{a,b=1}^{N_c}\mathrm{ch}(\widehat{\wedge}^\bullet N\Phi_{ab}^\vee) = &\prod_{a,b=1}^{N_c}\left(e^{(\eta_a-\eta_b)+\pi i(m_{i_a}-m_{i_b})+\pi i m_t} - e^{(-\eta_a+\eta_b)/2-\pi i(m_{i_a}-m_{i_b})-\pi i m_t}\right)^{\mathfrak{m}_a-\mathfrak{m}_b-(1-2r)(g-1)} \\ &\exp\left[(\sigma^{aa}+\sigma^{bb}-\sigma^{ab}-\sigma^{ba})\frac{e^{\eta_a+\eta_b+2\pi i(m_{i_a}-m_{i_b})+2\pi i m_t}-1}{2(e^{\eta_a-\eta_b+2\pi i(m_{i_a}-m_{i_b})+2\pi i m_t}-1)}\right]\end{aligned} \tag{4.204}$$

We can now compute the equivariant virtual Euler characteristic, which by (4.68) can be written as

$$\begin{aligned}&\chi(\mathfrak{M},\hat{\mathcal{O}}_{\mathrm{vir}}) \\ &=\sum_{\mathfrak{m}\in\mathbb{Z}}(-q)^{\mathfrak{m}}\sum_{\substack{(\mathfrak{m}_1,\cdots,\mathfrak{m}_{N_c})\\ \sum_a\mathfrak{m}_a=\mathfrak{m}}}\sum_{I\subset\{1,\cdots,N_f\}}\int_{\mathfrak{M}_{I,\underline{\mathfrak{m}}}}\hat{A}(T\mathfrak{M}_{I,\underline{\mathfrak{m}}})\mathrm{ch}(\widehat{S}^\bullet N_{I,\underline{\mathfrak{m}}}^\vee).\end{aligned} \tag{4.205}$$

where by (4.199), the integral can be expanded as

$$\int_{\mathfrak{M}_{I,\underline{\mathfrak{m}}}}\hat{A}(T\mathfrak{M}_{I,\underline{\mathfrak{m}}})\prod_{a\in I}\left[\prod_{\substack{j=1\\ j\neq i_a}}^{N_f}\mathrm{ch}(\widehat{S}^\bullet NX_j^{a\vee})\prod_{j=1}^{N_f}\mathrm{ch}(\widehat{S}^\bullet NY_j^{a\vee})\prod_{b=1}^{N_c}\mathrm{ch}(\widehat{\wedge}^\bullet N\Phi_{ab}^\vee)\right]\prod_{a\neq b}\mathrm{ch}(\widehat{\wedge}^\bullet NV_{ab}^\vee). \tag{4.206}$$

Combining the result from (4.198)-(4.204), these contributions are equal to

$$\int_{\mathfrak{M}_{I,\underline{\mathfrak{m}}}}\left(\prod_{a\in I}\eta_a^{\mathfrak{m}_a+(r-1)(g-1)}\right)A_I(\eta_1,\cdots,\eta_{N_c})\exp\left[\sum_{a,b}^{N_c}\sigma^{ab}B_{I,ab}(\eta_1,\cdots,\eta_{N_c})\right]. \tag{4.207}$$

where

$$
\begin{aligned}
A_I(\eta_1,\cdots,\eta_{N_c}) = &\prod_{a\in I}\prod_{j=1}^{N_f}\left[\left(\frac{e^{-\eta_a/2+\pi i m_{j i_a}}}{1-e^{-\eta_a+2\pi i m_{j i_a}}}\right)^{\mathfrak{m}_a+(r-1)(g-1)}\right.\\
&\left.\cdot\left(e^{-\eta_a/2+\pi i(m_{j i_a}+m_t)}-e^{(\eta_a/2-\pi i(m_{j i_a}+m_t)}\right)^{\mathfrak{m}_a-(r-1)(g-1)}\right]\\
&\prod_{\substack{a,b\in I\\ a\neq b}}\left(e^{(-\eta_a+\eta_b)/2-\pi i(m_{i_a}-m_{i_b})}-e^{(\eta_a-\eta_b)/2+\pi i(m_{i_a}-m_{i_b})}\right)^{-\mathfrak{m}_a+\mathfrak{m}_b+1-g}\\
&\prod_{a,b\in I}\left(e^{(\eta_a-\eta_b)/2+\pi i(m_{i_a}-m_{i_b})+\pi i m_t}-e^{(-\eta_a+\eta_b)/2-\pi i(m_{i_a}-m_{i_b})-\pi i m_t}\right)^{\mathfrak{m}_a-\mathfrak{m}_b-(1-2r)(g-1)}
\end{aligned}
\tag{4.208}
$$

and

$$
B_{I,ab}(\eta_1,\cdots,\eta_{N_c}) = H_{I,ab}(\eta_1,\cdots,\eta_{N_c}) - \delta_{ab}\eta_a^{-1}\ , \tag{4.209}
$$

where H_{ab} is given by the expression

$$
\begin{aligned}
H_{I,ab} = \delta_{ab}&\left[\sum_{j=1}^{N_f}\frac{1+e^{-\eta_a+2\pi i m_{j i_a}}}{2(1-e^{-\eta_a+2\pi i m_{j i_a}})}+\sum_{c\neq a}^{N_c}\frac{1+e^{\eta_a-\eta_c+2\pi i(m_{i_a}-m_{i_c})+2\pi i m_t}}{2(1-e^{(\eta_a-\eta_c+2\pi i(m_{i_a}-m_{i_c})+2\pi i m_t)})}\right.\\
&\left.+\sum_{j=1}^{N_f}\frac{1+e^{\eta_a-2\pi i(m_{j i_a}+m_t)}}{2(1-e^{\eta_a-2\pi i(m_{j i_a}+m_t)})}+\sum_{c\neq a}^{N_c}\frac{1+e^{\eta_c-\eta_a+2\pi i(m_{i_c}-m_{i_a})+2\pi i m_t}}{2(1-e^{\eta_c-\eta_a+2\pi i(m_{i_c}-m_{i_a})+2\pi i m_t})}\right]\\
+(1-\delta_{ab})&\left[\frac{1+e^{\eta_a-\eta_b+2\pi i(m_{i_a}-m_{i_b})+2\pi i m_t}}{2(1-e^{\eta_a-\eta_b+2\pi i(m_{i_a}-m_{i_b})+2\pi i m_t})}+\frac{1+e^{\eta_b-\eta_a-2\pi i(m_{i_a}-m_{i_b})+2\pi i m_t}}{2(1-e^{\eta_b-\eta_a-2\pi i(m_{i_a}-m_{i_b})+2\pi i m_t})}\right]\ .
\end{aligned}
\tag{4.210}
$$

The last expression in (4.205) can be converted to a product of residue integrals as in the Abelian examples. We show in appendix A.6 that the identity (4.124) over Σ_n can be generalized to integrals over $\prod_{i=1}^{N}\Sigma_n$: For any power series $A(\eta_1,\cdots,\eta_{N_c})$ and $B_{ab}(\eta_1,\cdots,\eta_{N_c})$ on $\prod_{i=1}^{N_c}\Sigma_{n_i}$, we have

$$
\begin{aligned}
&\int_{\prod_{i=1}^{N_c}\Sigma_{n_i}} A(\eta_1,\cdots,\eta_{N_c})\exp\left[\sum_{a,b=1}^{N_c}\sigma^{ab}B_{ab}(\eta_1,\cdots,\eta_{N_c})\right]\\
&=\operatorname*{res}_{u_1=0}\cdots\operatorname*{res}_{u_{N_c}=0}\frac{A(u_1,\cdots u_{N_c})}{u_1^{n_1+1}\cdots u_{N_c}^{n_{N_c}+1}}\left[\det_{ab}\left(\delta_{ab}+u_aB_{ab}(u_1,\cdots,u_{N_c})\right)\right]^g\ .
\end{aligned}
\tag{4.211}
$$

Then the integral (4.205) becomes

$$
\begin{aligned}
&\chi(\mathfrak{M},\hat{\mathcal{O}}_{\mathrm{vir}})\\
&=\sum_{\underline{\mathfrak{m}}\in\mathbb{Z}^{N_c}}(-q)^{\sum_{a=1}^{N_c}\mathfrak{m}_a}\sum_{I\subset\{1,\cdots,N_f\}}\operatorname*{res}_{u_1=0}\cdots\operatorname*{res}_{u_{N_c}=0}A_I(u_1,\cdots,u_{N_c})\left[\det_{ab}H_{I,ab}(u_1,\cdots,u_{N_c})\right]^g\ .
\end{aligned}
\tag{4.212}
$$

By a redefinition of the integration variables $u_a\to u_a-m_{i_a}+m_t/2$ for each summand labelled by

I, the integral can be rewritten as

$$
\begin{aligned}
&\chi(\mathfrak{M}, \hat{\mathcal{O}}_{\text{vir}}) \\
&= (2\pi i)^{N_c} \sum_{\underline{\mathfrak{m}} \in \mathbb{Z}^{N_c}} (-q)^{\sum_{a=1}^{N_c} \mathfrak{m}_a} \sum_{I \subset \{1, \cdots, N_f\}} \left(\prod_{a=1}^{N_c} \underset{u_a = m_{i_a} - m_t/2}{\text{res}} \right) \mathbf{A}(u_1, \cdots, u_{N_c}) \left[\det_{ab} \mathbf{H}_{ab}(u_1, \cdots, u_{N_c}) \right]^g .
\end{aligned}
\tag{4.213}
$$

where $\mathbf{A}$ and $\mathbf{H}$ is

$$
\begin{aligned}
\mathbf{A}(u_1, \cdots, u_{N_c}) = &\prod_{a=1}^{N_c} \prod_{j=1}^{N_f} \frac{(e^{\pi i(-u_a + m_j + m_t/2)} - e^{\pi i(u_a - m_j - m_t/2)})^{\mathfrak{m}_a - (r-1)(g-1)}}{\left(e^{\pi i(u_a - m_j + m_t/2)} - e^{\pi i(-u + m_j - m_t/2)}\right)^{\mathfrak{m}_a + (r-1)(g-1)}} \\
&\prod_{\substack{a,b=1 \\ a \neq b}}^{N_c} \left(e^{\pi i(-u_a + u_b)} - e^{\pi i(u_a - u_b)} \right)^{-\mathfrak{m}_a + \mathfrak{m}_b + 1 - g} \\
&\prod_{a,b=1}^{N_c} \left(e^{\pi i(u_a - u_b + m_t)} - e^{\pi i(-u_a + u_b - m_t)} \right)^{\mathfrak{m}_a - \mathfrak{m}_b - (1-2r)(g-1)}
\end{aligned}
\tag{4.214}
$$

and

$$
\begin{aligned}
\mathbf{H}_{ab}(u_1, \cdots, u_{N_c}) = \delta_{ab} &\left[\sum_{j=1}^{N_f} \frac{1 + e^{2\pi i(-u_a + m_j + m_t/2)}}{2(1 - e^{2\pi i(-u_a + m_j + m_t/2)})} + \sum_{c \neq a}^{N_c} \frac{1 + e^{2\pi i(u_a - u_c + m_t)}}{2(1 - e^{2\pi i(u_a - u_c + m_t)})} \right. \\
&\left. + \sum_{j=1}^{N_f} \frac{1 + e^{2\pi i(u_a - m_j + m_t/2)}}{2(1 - e^{2\pi i(u_a - m_j + m_t/2)})} + \sum_{c \neq a}^{N_c} \frac{1 + e^{2\pi i(u_c - u_a + m_t)}}{2(1 - e^{2\pi i(u_c - u_a + m_t)})} \right] \\
+ (1 - \delta_{ab}) &\left[\frac{1 + e^{2\pi i(u_a - u_b + m_t)}}{2(1 - e^{2\pi i(u_a - u_b + m_t)})} + \frac{1 + e^{2\pi i(u_b - u_a + m_t)}}{2(1 - e^{2\pi i(u_b - u_a + m_t)})} \right] .
\end{aligned}
\tag{4.215}
$$

Finally, it is straightforward to show that the residue integral together with the choice of fixed point is the equivalent to the Jeffrey-Kirwan residue integral of the integrand with the choice $\eta > 0$:

$$
\sum_{I \subset \{1, \cdots, N_f\}} \left(\prod_{a=1}^{N_c} \underset{u_a = m_{i_a} - m_t/2}{\text{res}} \right) = \frac{1}{N!} \sum_{u_* = \{u_i\}} \underset{u = u_*}{\text{JK-Res}}(Q_{u_*}(u), \eta > 0) . \tag{4.216}
$$

Therefore we again proved that the equivariant virtual Euler characteristic of the moduli space reproduces the twisted indices computation. This procedure can be generalized to the class of the theories defined in section 4.1.2.

The limit $t \to 1$

H-twist The $\mathcal{N} = 4$ BPS equation for the H-twist is given by

$$
\begin{aligned}
&* F_A + e^2 \left(X X^\dagger - Y^\dagger Y - \tau \right) = 0 , \\
&\bar{\partial}_A X = \bar{\partial}_A Y = \bar{\partial}_A \varphi = [\varphi^\dagger, \varphi] = 0 , \\
&X \cdot \varphi = \varphi \cdot Y = X \cdot Y = 0 ,
\end{aligned}
\tag{4.217}
$$

modulo $U(N_c)$ gauge transformation. If we consider localization with respect to the $U(1)_t$ action, the fixed locus $\mathfrak{M}^{U(1)_t}$ is parametrized by the solutions (A, X) to the equations

$$
* F_A + e^2 \left(X X^\dagger - \tau \right) = 0 , \quad \bar{\partial}_A X = 0 , \tag{4.218}
$$

modulo $U(N_c)$ gauge transformations, which can be identified as the space of twisted quasi-maps to the compact core inside the Higgs branch $\mathcal{M}_H = T^*G(N_c, N_f)$, which we denote by $\mathfrak{L}$. Similarly to the SQED example, the index (4.213) with $r = 1$ can also be thought of as the virtual χ_t genus

$$\chi(\mathfrak{M}, \hat{\mathcal{O}}_{\text{vir}})|_{r=1} = \chi_t^{\text{vir}}(\mathfrak{L}) \ . \tag{4.219}$$

In the limit $t \to 1$, the index reduces to the generating function of the integral of the virtual Euler class. This can be evaluated using the virtual localization with respect to the T_H action (4.187):

$$\chi(\mathfrak{M}, \hat{\mathcal{O}}_{\text{vir}})\big|_{\substack{r=1 \\ t\to 1}} = (-1)^{N_c(g-1)} \sum_{\underline{\mathfrak{m}}\in\mathbb{Z}^{N_c}} q^{\mathfrak{m}} \sum_{\substack{(\mathfrak{m}_1,\cdots,\mathfrak{m}_{N_c}) \\ \sum_a \mathfrak{m}_a = \mathfrak{m}}} \sum_I \int_{\mathfrak{M}_{\underline{\mathfrak{m}},I}} e(T\mathfrak{M}_{\underline{\mathfrak{m}},I}) \ . \tag{4.220}$$

Summing over $\mathfrak{m}$, we have

$$\chi(\mathfrak{M}, \hat{\mathcal{O}}_{\text{vir}})\big|_{\substack{r=1 \\ t\to 1}} = (-1)^{N_c(g-1)} \binom{N_f}{N_c} \left(q^{1/2} - q^{-1/2}\right)^{2N_c(g-1)} \ , \tag{4.221}$$

using the generating function of the Euler numbers for a symmetric product (4.128).

C-twist As we studied in section 4.3, imposing the $\mathcal{N} = 4$ BPS equations trivializes the vector bundle E, and the moduli space reduces to the resolution of the Higgs branch $\mathcal{M}_H = T^*G(N_c, N_f)$. The index in the $t \to 1$ limit then computes the equivariant Rozansky-Witten invariants of the target $\mathcal{M}_H$. This can be directly computed from the $\mathfrak{m} = 0$ sector of the expression (4.205), taking the $t \to 1$ limit and extracting the constant term in the power series expansion of the characteristic classes. This procedure gives

$$\chi(\mathfrak{M}, \hat{\mathcal{O}}_{\text{vir}})\big|_{\substack{r=0 \\ t\to 1}} = \sum_I \prod_{\substack{i\in I \\ j\in I^\vee}} \left(e^{-\pi i m_{ij}} - e^{\pi i m_{ij}}\right)^{2(g-1)} \ , \tag{4.222}$$

where $I^\vee$ is the complement of the index set I in $\{1, \cdots, N_f\}$.

4.5 Mirror Symmetry

4.5.1 Symplectic Duality for Twisted Stable Quasi-maps

A distinctive feature of the class of the theories we consider in this chapter is mirror symmetry. We can find a pair of UV theories $\mathcal{T}$ and $\mathcal{T}^\vee$, which are dual under exchange of the following pairs of objects and parameters:

$$\begin{aligned} \text{H-twist} &\leftrightarrow \text{C-twist} \\ \mathcal{M}_H &\leftrightarrow \mathcal{M}_C \\ G_H &\leftrightarrow G_C \\ \{m_i\} &\leftrightarrow \{\zeta_i\} \\ t &\leftrightarrow t^{-1} \ . \end{aligned} \tag{4.223}$$

Recall that the duality holds when the theory flows to the deep infrared. As discuss in section (3.3), this limit can be viewed as the $s \to \infty$ limit, taken in a chamber determined by τ. It follows that the twisted indices computed in this chamber are expected to exhibit the duality exchanging the parameter as in (4.223).

Given the interpretation of the twisted indices we have offered in this chapter, mirror symmetry implies an extremely non-trivial relation between two generating functions of enumerative invariants of twisted stable quasi-maps into a conical symplectic resolution of the Higgs branch $\mathcal{M}_H$. In fact, mirror symmetry implies two relations

$$I_H(\zeta, m, t)[\mathcal{T}] = I_C(m, \zeta, t^{-1})[\mathcal{T}^\vee] \ , \tag{4.224}$$

and

$$I_C(\zeta, m, t)[\mathcal{T}] = I_H(m, \zeta, t^{-1})[\mathcal{T}^\vee] \ . \tag{4.225}$$

In particular, this exchanges the equivariant parameters m and the degree counting parameters ζ of two generating functions. We may call this *symplectic duality for stable quasi-maps.*

The simplest example is the theory SQED[2], which is a self-dual theory $\mathcal{T} = \mathcal{T}^\vee$. The generating functions for the first few genera are explicitly computed in [26]. For example, the generating function for the H-twist with $g = 2$ is

$$I_H(q, y, t)\big|_{g=2} = -\frac{(1+t)\left[t(y+y^{-1}-2)(q+q^{-1}-2)+4(1-t)^2\right]}{t^{1/2}(t-y)(t-y^{-1})} \ , \tag{4.226}$$

where $y = e^{2\pi i(m_1-m_2)}$ and $q = e^{2\pi i \zeta}$. This can now be interpreted as the generating function for the equivariant virtual χ_t genus of $\mathfrak{L}$, where $\mathfrak{L}$ can be identified with the space of twisted stable quasi-maps into $\mathbb{P}^1$. On the other hand, for the C-twist, we have

$$I_C(q, y, t)\big|_{g=2} = -\frac{(1+t)[t(y+y^{-1}-2)(q+q^{-1}-2)+4(1-t)^2]}{t^{1/2}(t-q)(t-q^{-1})} \ . \tag{4.227}$$

This corresponds to the generating function of the virtual Euler characteristic for stable quasi-maps. It agrees with (4.226) by exchanging $q \leftrightarrow y$ and $t \leftrightarrow t^{-1}$ as expected.

4.5.2 Mirror Symmetry for the $\mathcal{N} = 4$ Index

As studied in section 4.3, the twisted indices drastically simplify in the limit $t \to 1$. The H-twisted indices in this limit can be identified with the sum over the integrals of the Euler class of the fixed loci

$$I_H(q) := I_H(q, a, t)|_{t\to 1} = \sum_{\underline{\mathrm{m}} \in \Lambda_G^\vee} (-q)^{\mathrm{m}} \sum_I (-1)^{\dim_{\mathbb{C}} \mathfrak{M}_{\underline{\mathrm{m}}, I}} \int_{M_{\underline{\mathrm{m}}, I}} e\left(\mathfrak{M}_{\underline{\mathrm{m}}, I}\right) \ , \tag{4.228}$$

which is independent of the equivariant parameters a. On the other hand, C-twisted indices receive contributions from the degree zero sector only and are therefore independent of q in the limit $t \to 1$. The index in this limit computes the Rozansky-Witten invariants

$$I_C(a) := I_C(q, a, t)|_{t\to 1} = \int_{\mathcal{M}_H} \hat{A}\left(T\mathcal{M}_H\right) \ \mathrm{ch}\left[\left(\widehat{\wedge}^\bullet T^* \mathcal{M}_H\right)^{\otimes g}\right] \ , \tag{4.229}$$

where $\mathcal{M}_H$ is the resolved Higgs branch.

Mirror symmetry, or symplectic duality, takes a simple form in this limit

$$I_H(q)[\mathcal{T}] = I_C(y)[\mathcal{T}^\vee] \ , \tag{4.230}$$

with the identification $q = y$. Below, we show explicitly that this holds for Abelian theories and for $T[SU(N)]$.

4.5.3 Abelian Gauge Theories

Abelian 3d $\mathcal{N} = 4$ gauge theories are specified by a rank-r gauge group $\prod_{a=1}^{r} U(1)_a$ and N hypermultiplets (X^i, Y^i) with charges $(Q_i^a, -Q_i^a)$ under $U(1)_a$. We will denote maximal tori of the symmetry groups G_H and G_C by $T_H = \prod_{b=1}^{N-r} U(1)_b$ and $T_C = \prod_{c=1}^{r} U(1)^c$ respectively. The charges of the hypermultiplets will be similarly denoted by $(q_i^b, -q_i^b)$.

The vectors $\vec{q}^b$ are defined only modulo the vectors $\vec{Q}^a$; together, they form a basis for $\mathbb{R}^N$. It is customary to assign to an abelian theory the $N \times N$ matrix

$$\mathbf{Q} = \begin{pmatrix} Q \\ q \end{pmatrix} \ . \tag{4.231}$$

The condition $\det(\mathbf{Q}) = 1$ ensures that the basis is minimal. We turn on real masses $\{m_1, \ldots, m_{N-r}\} \in \mathfrak{t}_H \cong \mathbb{R}^{N-r}$, and generic FI parameter $\vec{\zeta} \in \mathfrak{t}_C \cong \mathbb{R}^r$, and assume that the fixed points of the resolved Higgs branch under the actions generated by the real masses are isolated.

We claim that isolated fixed points of the Higgs branch are given by a selection of

- r hypermultiplets whose charge matrix is non-degenerate, represented by a multi-index $I = \{i_1 \ldots, i_r\}$,
- a sign vector $\{\alpha_{i_1}, \ldots, \alpha_{i_r}\}$, $\alpha_{i_j} \in \{+, -\}$.

This can be seen as follows. The fixed-point equations are

$$\begin{gathered} \left(\sum_{a=1}^{r} Q_a^i \sigma_a + \sum_{b=1}^{N-r} q_b^i m_b \right) X^i = 0 \\ \left(\sum_{a=1}^{r} Q_a^i \sigma_a + \sum_{b=1}^{N-r} q_b^i m_b \right) Y^i = 0 \, , \end{gathered} \tag{4.232}$$

for generic mass parameters $\{m_1, \ldots, m_{N-r}\}$ and for all $i \in \{1, \ldots, N\}$. We can solve for unique σ_a's only provided r hypermultiplets are non-vanishing, and the associated matrix

$$Q_I := \left(Q_a^{\alpha_{i_j}} \right)_a^j \tag{4.233}$$

is non-degenerate. In addition, we have the complex and real moment-map conditions

$$\begin{gathered} \sum_{j=1}^{r} Q_a^{i_j} X^{i_j} Y^{i_j} = 0 \\ \sum_{j=1}^{r} \left(Q_a^{i_j} |X^{i_j}|^2 - Q_a^{i_j} |Y^{i_j}|^2 \right) = \zeta_a \, , \quad a \in \{1, \ldots, r\} \, . \end{gathered} \tag{4.234}$$

The first equation tells us that for each i_j either X^{i_j} or Y^{i_j} needs to vanish at a fixed point. The second equation can always be satisfied by choosing X^{i_j} or Y^{i_j} appropriately. In fact, Q_I being non-degenerate, the vectors $\vec{Q}^{i_j}$, $j \in \{1, \ldots, r\}$ form a basis of $\mathbb{R}^r$. We can therefore write ζ uniquely as a linear combination of these vectors, with either positive or negative coefficients. If the coefficient of $\vec{Q}^{i_j}$ is positive, we set $|X^{i_j}|^2$ to the value of the coefficient and Y^{i_j} to zero, otherwise $-|Y^{i_j}|^2$ to the value of the coefficient, and X^{i_j} to zero. Solutions are therefore indexed by a sign vector which encodes the choice at each fixed point, as claimed.

H-twist

To each fixed point on the Higgs branch there corresponds a fixed locus in the space of quasi-maps of degree $\mathfrak{m}$ to the Higgs branch, $\mathfrak{M}_{\mathfrak{m},I}$. Under our conventions, it is easy to see that the fixed locus reads

$$\mathfrak{M}_{\mathfrak{m},I} = \sum_{\substack{(\mathfrak{m}_1,\ldots,\mathfrak{m}_r) \\ \in \mathbb{Z}^r}} \prod_{a=1}^{r} q_a^{\mathfrak{m}_a} \prod_{j=1}^{r} \mathrm{Sym}^{\sum_a \alpha_{i_j} Q_a^{i_j} \mathfrak{m}_a + (g-1)} (\Sigma) \;, \; \sum_{a=1}^{r} \mathfrak{m}_a = \mathfrak{m} \,. \tag{4.235}$$

In order to compute the index, we can first redefine the labels $\mathfrak{m}_a \mapsto \widetilde{\mathfrak{m}}_j$ as follows

$$\sum_a \alpha_{i_j} Q_a^{i_j} \mathfrak{m}_a + (g-1) := \widetilde{\mathfrak{m}}_j \,, \tag{4.236}$$

and then use the generating functions of Euler classes of symmetric products as we did in previous examples. Notice that the change of variables is possible because each matrix Q_I must be non-singular. Then letting

$$q^{Q_I^{-1}{}_j} := \prod_a q_a{}^{Q_I{}^a_i} \,, \tag{4.237}$$

we find, up to an overall sign

$$I_{H,I} = \prod_{i \in I} \left(\frac{q^{Q_I^{-1}{}_j/2}}{1 - q^{Q_I^{-1}{}_j}} \right)^{2(1-g)} \,. \tag{4.238}$$

C-twist

In the C-twist, in order to apply a fixed-point formula, we simply need to find the weights of the normal fluctuations at fixed points. Given a fixed point I, these correspond to the weights of the hypermultiplets indexed by the complement of I in $\{1, \ldots, N\}$, which we denote $I^\vee$, at the fixed point I. In order to obtain them, we first need to solve for the σ_a's in (4.232)

$$\vec{\sigma}_I = -\left(Q_I^T\right)^{-1} q_I^T \vec{m} \tag{4.239}$$

From this, we get that the weights of the hypermultiplets of charge $(\vec{Q}^i, \vec{q}^i)$, $i \in I^\vee$ at the fixed point I are given by

$$\left(\mathbf{q}_{I^\vee}\right)_i \vec{m} \,, \tag{4.240}$$

where

$$\mathbf{q}_{I^\vee} := q^T_{I^\vee} - Q^T_{I^\vee}\left(Q^T_I\right)^{-1} q^T_I\,. \tag{4.241}$$

Consequently, the index at the fixed point I reads

$$\mathcal{I}_I = \prod_{i\in I^\vee}\left(\frac{y^{(\mathbf{q}_{I^\vee})_i/2}}{1-y^{(\mathbf{q}_{I^\vee})_i}}\right)^{2(g-1)} \tag{4.242}$$

whith $y^{(\mathbf{q}_{I^\vee})_i} = \prod_b y_b^{(\mathbf{q}_{I^\vee})^b_i}$.

Abelian Mirror Symmetry

We are now ready to discuss abelian mirror symmetry. At the level of charge matrices, it takes a simple form [94, 15]. Using tildes for mirror variables as above, we have

$$\widetilde{\mathbf{Q}} := \begin{pmatrix}\widetilde{q}\\ \widetilde{Q}\end{pmatrix} = \begin{pmatrix}Q\\ q\end{pmatrix}^{-1,T}\,,\quad \widetilde{\zeta}_a = m_a \quad \widetilde{m}_a = \zeta_a\,. \tag{4.243}$$

We would like to verify that the H-twisted index of a theory, provided the above identifications hold, coincides with the C-twisted index of the mirror theory. Select a fixed point I, and assume without loss of generality that $I = \{1,\dots r\}$. Notice that

$$\mathbf{Q}\widetilde{\mathbf{Q}}^T = \mathbf{1}_{N,N} \tag{4.244}$$

implies

$$\left(\begin{array}{c|c} Q_I & Q_{I^\vee}\\ \hline q_I & q_{I^\vee}\end{array}\right)\left(\begin{array}{c|c} \widetilde{q}^T_I & \widetilde{Q}^T_I\\ \hline \widetilde{q}^T_{I^\vee} & \widetilde{Q}^T_{I^\vee}\end{array}\right) = \mathbf{1}_{N,N} \tag{4.245}$$

where we have also decomposed $\widetilde{\mathbf{Q}}$ in blocks of the same size of those in $\mathbf{Q}$. From this equation, we can infer that

$$\begin{aligned} q_I\widetilde{Q}^T_I + q_{I^\vee}\widetilde{Q}^T_{I^\vee} &= 0\\ Q_I\widetilde{Q}^T_I + Q_{I^\vee}\widetilde{Q}^T_{I^\vee} &= 0\,. \end{aligned} \tag{4.246}$$

It follows from (4.246) that

$$\left(q_{I^\vee} - q_I Q_I^{-1} Q_{I^\vee}\right)\widetilde{Q}^T_{I^\vee} = \mathbf{1}_{N-r,N-r}\,, \tag{4.247}$$

or

$$\widetilde{Q}_{I^\vee}\mathbf{q}_{I^\vee} = \mathbf{1}_{N-r,N-r}\,. \tag{4.248}$$

Chasing through the definitions and the results of the twisted indices, we see that from the contribution of a fixed-point of the C-twisted index labelled by I we get a contribution of a fixed point of the mirror H-twisted index labelled by $I^\vee$, and vice-versa. Therefore, mirror symmetry in this context holds at the level of the index fixed-point by fixed-point.

4.5.4 $T[SU(N)]$

We now perform a similar analysis with $T[SU(N)]$ theories. These theories can be represented as the quiver in Figure 4.2. They play an important role in the S-duality of the half-BPS boundary conditions in $\mathcal{N}=4$ Yang-Mills theory [95].

The twisted indices of $T[SU(N)]$ quiver depends on the FI-parameters $\zeta_{i=1,\cdots,N-1}$ for each factor of the gauge group $U(1)\times\cdots\times U(N-1)$ and mass parameters $m_{i=1,\cdots,N}$ for the $PSU(N)$ flavour symmetry, which satisfies $\sum_{i=1}^{N} m_i = 0$. The theories in this class are known to be self dual under exchanging $\epsilon_i \leftrightarrow m_i$ for all i, where $\zeta_i = \epsilon_i - \epsilon_{i+1}$.

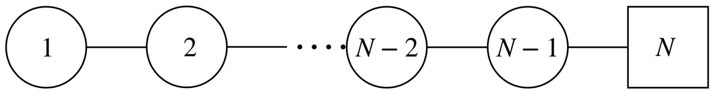

Figure 4.2: $T[SU(N)]$ theory

H-twist The moduli space of $T[SU(N)]$ theory can be decomposed into the topological sectors weighted by the FI-parameters $q_i = e^{2\pi i \zeta_i}$:

$$\mathcal{M} = \bigcup_{\substack{(\mathfrak{m}_1,\cdots,\mathfrak{m}_{N-1})\\ \in\mathbb{Z}^{N-1}}} q_1^{\mathfrak{m}_1} q_2^{\mathfrak{m}_2}\cdots q_{N-1}^{\mathfrak{m}_{N-1}}\ \mathcal{M}_{\mathfrak{m}_1,\cdots,\mathfrak{m}_{N-1}}\ . \tag{4.249}$$

For each factor of the gauge group labelled by $a=1,\cdots,N-1$, we denote by

$$\mathfrak{m}_a = \frac{1}{2\pi}\int_\Sigma \mathrm{Tr}\,(F_a) \in \mathbb{Z}\ , \tag{4.250}$$

the degree of the vector bundle E_a of rank a, associated with the $U(a)$ gauge bundle P_a. Let us denote X_a^{a+1}, Y_{a+1}^a by the bi-fundamental fields between a-th and $a+1$-th nodes, where X_a^{a+1} can be regarded as a $a\times(a+1)$ matrix whose components are $X_{a(k_a)}^{a+1(k_{a+1})}$, and similarly for the Y. Then the $\mathcal{N}=4$ moduli space for the H-twist is given by the space of solutions $(A_1,\cdots A_{N-1},X,Y)$ to the equations

$$\begin{aligned}&*F_a + e^2\left(X_a^{a+1}X_a^{a+1\dagger} - Y_{a+1}^{a}{}^\dagger Y_{a+1}^a - X_{a-1}^{a}{}^\dagger X_{a-1}^a + Y_a^{a-1}Y_a^{a-1\dagger} - \tau_a\right) = 0\ ,\\ &\bar\partial_A X_a^{a+1} = \bar\partial_A Y_{a+1}^a = 0\ ,\quad X_a^{a+1}Y_{a+1}^a = 0\ ,\quad \text{for } a=1,\ldots,N-1\ ,\end{aligned} \tag{4.251}$$

modulo $U(1)\times\cdots\times U(N-1)$ gauge transformation.

Let us consider the chamber where all τ_a's are sufficiently large. As in the previous examples, we perform the equivariant localization with respect to the action of the flavour symmetry $g_H \in PSU(N_f)$, by turning on the mass parameters m_i. If we keep these parameters generic, on the fixed locus, each factor of the gauge group is abelianized

$$U(a) \to U(1)^a\ ,\quad \text{for } a=1,\cdots,N-1\ , \tag{4.252}$$

and accordingly the vector bundle E_a is decomposed into the sum of the line bundle

$$E_a = L_{a(1)} \oplus \cdots \oplus L_{a(a)} \ , \tag{4.253}$$

on the fixed loci, where $\deg(L_{a(k)}) = \mathfrak{m}_{a(k)}$ with $\mathfrak{m}_a = \sum_{k=1}^{a} \mathfrak{m}_{a(k)}$. Then the moduli space reduces to a disjoint union of $N!$ fixed loci labelled by a set of the index sets

$$\{I_1, \cdots, I_{N-1}\} \ , \quad \text{where } I_a \subset \{1, \cdots, a+1\} \ , \quad |I_a| = a \ . \tag{4.254}$$

If we denote the element of the index set I_a by $i_{a(k_a)}$ with $k_a = 1, \cdots, a$, then on the fixed locus $\{I_1, \cdots, I_{N-1}\}$, the only non-vanishing bosonic fields in the chiral multiplets are

$$X_{a(k_a)}^{a+1(i_{a(k_a)})} \neq 0 \ , \quad \text{for} \ \ a = 1, \cdots, N-1 \ . \tag{4.255}$$

To simplify the notation, let us make a choice of fixed locus defined by $I_a = \{1, \cdots, a\}$ for all a, where the only non-vanishing bosonic fields in the chiral multiplets are $X_{a(k_a)}^{a+1(k_a)}$ for all $k_a = 1, \cdots, a$. All other fixed points can be obtained by an action of the $PSU(N)$ flavour symmetry. The fixed locus is described by the equations

$$\begin{aligned} &* F_{a(i_a)} + e^2 \left(|X_{a(i_a)}^{a+1(i_a)}|^2 - |X_{a-1(i_a)}^{a(i_a)}|^2 - \tau_a \right) = 0 \ , \\ &\bar{\partial}_A X_{a(i_a)}^{a+1(i_a)} = 0 \ , \quad \text{for} \ \ i_a = 1, \ldots a. \end{aligned} \tag{4.256}$$

modulo $\prod_{a=1}^{N-1} U(1)^a$ gauge symmetries, where we defined $X_{a-1(a)}^{a(i_a)} = 0$. Here $X_{a(i_a)}^{a+1(i_a)}$ is a holomorphic section of the line bundle $L_{a(i_a)} \otimes L_{a+1(i_a)}^{-1} \otimes K^{1/2}$. Therefore the fixed locus can be described as the $1 + \cdots + N - 1$ copies of the symmetric product:

$$\mathfrak{M}_{I_a} = \prod_{a=1}^{N-1} \prod_{i_a=1}^{a} \Sigma_{\mathfrak{m}_{a(i_a)} - \mathfrak{m}_{a+1(i_a)} + g - 1} \ . \tag{4.257}$$

Then the contribution of this fixed locus to the virtual Euler characteristic in the limit $t \to 1$ (4.228) becomes

$$\sum_{\substack{\mathfrak{m}_{a(k_a)} \in \mathbb{Z} \\ \text{for } k_a = 1, \cdots, a-1 \\ a = 1, \cdots, N-1}} q_1^{\mathfrak{m}_{1(1)}} q_2^{\mathfrak{m}_{2(1)} + \mathfrak{m}_{2(2)}} \cdots q_{N-1}^{\sum_{i=1}^{N-1} \mathfrak{m}_{N-1(i)}} \int_{\mathfrak{M}_{I_a}} \prod_{a=1}^{N-1} \prod_{i_a=1}^{a} e\left(\Sigma_{\mathfrak{m}_{a(i_a)} - \mathfrak{m}_{a+1(i_a)} + g - 1} \right) \ . \tag{4.258}$$

It is convenient to change the summation variable as

$$\mathfrak{m}_{a(k_a)} \to \mathfrak{m}_{a(k_a)} + \sum_{r=a+1}^{N-1} \mathfrak{m}_{r(k_a)} \ , \quad \text{for all} \ \ a, k_a \tag{4.259}$$

the expression (4.258) then becomes

$$\sum_{\substack{\mathfrak{m}_{a(k_a)} \in \mathbb{Z} \\ \text{for } k_a = 1, \cdots, a-1 \\ a = 1, \cdots, N-1}} \prod_{a=1}^{N-1} (q_1 \cdots q_a)^{\mathfrak{m}_{a(1)}} (q_2 \cdots q_a)^{\mathfrak{m}_{a(2)}} \cdots (q_a)^{\mathfrak{m}_{a(a)}} \int_{\mathfrak{M}_{I_a}} \prod_{a=1}^{N-1} \prod_{i_a=1}^{a} e\left(\Sigma_{\mathfrak{m}_{a(i_a)} + g - 1} \right) \ . \tag{4.260}$$

Let us redefine

$$q_i = e^{2\pi i(\epsilon_i - \epsilon_{i+1})} \ , \quad \text{with} \ \sum_{i=1}^{N} \epsilon_i = 0 \ , \tag{4.261}$$

and sum over $1 + \cdots + N - 1$ copies of integers $\mathfrak{m}_{a(i_a)}$. Using the relation (4.128), we find a simple expression

$$\prod_{i \neq j}^{N} \left(e^{\pi i(\epsilon_i - \epsilon_j)} - e^{\pi i(-\epsilon_i + \epsilon_j)} \right)^{(g-1)} \tag{4.262}$$

for the fixed locus $\mathfrak{M}_{I_a}$. Since the result does not depend on the equivariant parameters $\{m_i\}$, the contributions from $N!$ fixed locus are the same. Therefore we conclude

$$I_H(\{\epsilon_i\}) = (-1)^{N(N-1)(g-1)/2} N! \prod_{i<j}^{N} \left(e^{\pi i(\epsilon_i - \epsilon_j)} - e^{\pi i(-\epsilon_i + \epsilon_j)} \right)^{2(g-1)} \ . \tag{4.263}$$

Note that the result shows the structure of the full $SU(N)$ Coulomb branch symmetry G_C enhanced from the UV topological symmetry $U(1)^{N-1}$.

C-twist Once we impose the $\mathcal{N} = 4$ BPS equation for the C-twist, the associated vector bundle $E_1 \oplus \cdots \oplus E_{N-1}$ trivialises. In the large s limit, the moduli space reduces to the resolved Higgs branch $\mathcal{M}_{H,\tau}$, which can be identified as a cotangent space of a flag variety.

Similarly to the H-twist, the fixed loci of the g_H action are given by the choice of the index set $\{I_1, \cdots, I_{N-1}\}$, where $I_a = \{i_{a(1)}, \cdots, i_{a(N)}\} \subset \{1, \cdots, a+1\}$, for all a. Each fixed locus is an isolated point, characterised by the non-vanishing bi-fundamental chiral fields

$$X_{a(k_a)}^{a+1(i_{a(k_a)})} \neq 0 \ . \tag{4.264}$$

The C-twisted index in the limit $t \to 1$ gets contribution from the $\mathfrak{m} = 0$ sector only. It is straightforward to compute (4.229) equivariantly at each fixed points, which gives the expression

$$\begin{aligned} I_C(\{m_i\}) = \sum_{\{(I_1, \cdots, I_{N-1})\}} & \prod_{\substack{i \in I_{N-1} \\ j \in I_{N-1}^\vee}} \left(e^{-\pi i m_{ij}/2} - e^{\pi i m_{ij}/2} \right)^{2(g-1)} \cdot \\ & \cdot \prod_{\substack{i \in I_{N-2} \\ j \in I_{N-2}^\vee}} \left(e^{-\pi i m_{ij}/2} - e^{\pi i m_{ij}/2} \right)^{2(g-1)} \\ & \cdots \prod_{\substack{i \in I_1 \\ j \in I_1^\vee}} \left(e^{-\pi i m_{ij}/2} - e^{\pi i m_{ij}/2} \right)^{2(g-1)} \ , \end{aligned} \tag{4.265}$$

where the summation is over $N!$ choices of the fixed locus. $I_{a-1}^\vee$ is defined as the complement of I_{a-1} inside the index set $\{1, \cdots, a+1\}$. Note that each term in the summation is invariant under the Weyl group W_{G_H} of the flavour symmetry and therefore the contributions from all the fixed loci are identical. The expression simplifies to

$$I_C(\{m_i\}) = N! \prod_{i<j} \left(e^{-\pi i m_{ij}/2} - e^{\pi i m_{ij}/2} \right)^{2(g-1)} \ . \tag{4.266}$$

Comparing two expressions (4.263) and (4.266), we find an agreement

$$I_C(\{\epsilon_i\})[T[SU(N)]] = I_H(\{m_i\})[T[SU(N)]] \tag{4.267}$$

up to an overall sign, under the identification of the parameters $\epsilon_i = m_i$, $\forall i$. This agrees with the self-dual property of $T[SU(N)]$ theories.

Appendix

A.1 Supersymmetry Algebra

In this section, we summarize the supersymmetry algebra of twisted $\mathcal{N}=4$ and $\mathcal{N}=2$ theories. We start by fixing our notation for the $\mathcal{N}=4$ vector multiplet and the $\mathcal{N}=4$ chiral multiplet, and the respective decomposition into $\mathcal{N}=2$ multiplets. We then address the H- and C- twist algebras in turns. Our convention is that

$$\epsilon^{12} = -\epsilon_{12} = \epsilon^{\dot{1}\dot{2}} = -\epsilon_{\dot{1}\dot{2}} = 1 \tag{A.1}$$

and for contractions between vector and spinor indices we use standard Pauli matrices $(\sigma^\mu)_\alpha{}^\beta$.

The $\mathcal{N}=4$ vector multiplet consists of the fields

$$V_{\mathcal{N}=4} = \left(A_\mu, \lambda_\alpha^{A\dot{B}}, \varphi^{\dot{A}\dot{B}}, D^{AB}\right) \ . \tag{A.2}$$

The fields are subject to the following reality condition:

$$D_{AB} = \left(D^{AB}\right)^\dagger \ , \ \varphi_{\dot{A}\dot{B}} = -\left(\varphi^{\dot{A}\dot{B}}\right)^\dagger \ . \tag{A.3}$$

The multiplet decomposes into an $\mathcal{N}=2$ vector multiplet $V = (A_\mu, \sigma, \lambda, \bar{\lambda}, \Lambda_1, \bar{\Lambda}_{\bar{1}}, D)$ and an $\mathcal{N}=2$ chiral multiplet $\Phi_\varphi = (\varphi, \psi_\varphi, \eta_\varphi, F_\varphi)$ in the adjoint representation with the identification

$$\begin{aligned}
&\lambda = \frac{1}{2}\lambda_2^{\dot{2}\dot{2}} \ , \bar{\lambda} = \ \frac{1}{2}\lambda_1^{\dot{1}\dot{1}} \ , \ \Lambda_1 = \lambda_1^{\dot{2}\dot{2}} \ , \bar{\Lambda}_{\bar{1}} = \lambda_2^{\dot{1}\dot{1}} \ , \\
&\sigma = \varphi_{\dot{1}\dot{2}} \ , \ D = D_{12} \ , \\
&\varphi^\dagger = -\frac{1}{2}\varphi_{\dot{2}\dot{2}} \ , \ \varphi = \frac{1}{2}\varphi_{\dot{1}\dot{1}} \ , \bar{\psi}_\varphi = \lambda_1^{\dot{2}\dot{1}}, \ \psi_\varphi = \lambda_1^{\dot{1}\dot{2}} \ , \\
&\bar{\eta}_\varphi = \lambda_2^{\dot{1}\dot{2}} \ , \eta_\varphi = \lambda_2^{\dot{2}\dot{1}} \ , \ F_\varphi^\dagger = -D_{22}, \ F_\varphi = D_{11} \ .
\end{aligned} \tag{A.4}$$

The $\mathcal{N}=4$ hypermultiplet consists of fields

$$H_{\mathcal{N}=4} = \left(X_A, X^{A\dagger}, \psi_{\alpha\dot{A}}, \bar{\psi}_{\alpha\dot{A}}\right) \tag{A.5}$$

In terms of $\mathcal{N}=2$ fields, we identify

$$(X, Y^\dagger) = (X_1, X_2), \ (Y, X^\dagger) = (X^{2\dagger}, X^{1\dagger}) \ , \tag{A.6}$$

with similar identifications for superpartners.

A.1.1 H-twist

In both twists, we use the following convenient notation for supersymmetry transformations. We introduce a spinor $\zeta_\alpha^{A\dot{A}}$, and the supersymmetry transformation of a field $\mathcal{F}$ is under the $\mathcal{N}=4$ supercharge $Q_\alpha^{A\dot{A}}$ is the coefficent of $\zeta_\alpha^{A\dot{A}}$ in $\delta\mathcal{F}$. The four supercharges preserved under the H-twist can be represented by

$$\zeta_1^{1\dot{A}} := \zeta_H^{\dot{A}} \ , \quad \zeta_2^{2\dot{A}} := \widetilde{\zeta}_H^{\dot{A}} \ . \tag{A.7}$$

Note that the $\mathcal{N}=2$ subalgebra is generated by $\zeta_H^{\dot{1}}$ and $\widetilde{\zeta}_H^{\dot{2}}$. For convenience, we redefine the vector multiplet fermions as

$$\lambda_1^{1\dot{A}} = \lambda^{\dot{A}} \ , \ \lambda_2^{2\dot{A}} = \bar{\lambda}^{\dot{A}} \ , \ \lambda_{1,1\dot{A}} = \bar{\Lambda}_{\bar{1},\dot{A}} \ , \ \lambda_{2,2\dot{A}} = \Lambda_{1,\dot{A}} \ . \tag{A.8}$$

The supersymmetry transformation of the vector multiplet is given by

$$\begin{aligned}
\delta A_0 &= \frac{i}{2}\widetilde{\zeta}_H^{\dot{A}}\lambda_{\dot{A}} - \frac{i}{2}\zeta_H^{\dot{A}}\bar{\lambda}_{\dot{A}} \ , \\
\delta A_1 &= i\widetilde{\zeta}_H^{\dot{A}}\Lambda_{\dot{A},1} \ , \\
\delta A_{\bar{1}} &= -i\zeta_H^{\dot{A}}\bar{\Lambda}_{\dot{A},\bar{1}} \ , \\
\delta\varphi_{\dot{A}\dot{B}} &= \frac{1}{2}\left(\zeta_{H,\dot{A}}\bar{\lambda}_{\dot{B}} + \widetilde{\zeta}_{H,\dot{A}}\lambda_{\dot{B}} + \zeta_{H,\dot{B}}\bar{\lambda}_{\dot{A}} + \widetilde{\zeta}_{H,\dot{B}}\lambda_{\dot{A}}\right) \ , \\
\delta D_{11} &= -i\widetilde{\zeta}_H^{\dot{A}}\left(2D_1\bar{\lambda}_{\dot{A}} - D_0\bar{\Lambda}_{\bar{1},\dot{A}}\right) - i\widetilde{\zeta}_H^{\dot{B}}\left[\bar{\Lambda}_{\bar{1}}^{\dot{C}}, \varphi_{\dot{B}\dot{C}}\right] \ , \\
\delta D_{22} &= i\zeta_H^{\dot{A}}\left(2D_{\bar{1}}\lambda_{\dot{A}} - D_0\Lambda_{1,\dot{A}}\right) - i\zeta_H^{\dot{B}}\left[\Lambda_1^{\dot{C}}, \varphi_{\dot{B}\dot{C}}\right] \ , \\
\delta D &= \frac{i}{2}\zeta_H^{\dot{B}}\left(2D_{\bar{1}}\bar{\Lambda}_{\bar{1},\dot{B}} + D_0\bar{\lambda}_{\dot{B}} + \left[\bar{\lambda}^{\dot{C}}, \varphi_{\dot{B}\dot{C}}\right]\right) + \frac{i}{2}\widetilde{\zeta}_H^{\dot{B}}\left(2D_1\Lambda_{1,\dot{B}} + D_0\lambda_{\dot{B}} - \left[\lambda^{\dot{C}}, \varphi_{\dot{B}\dot{C}}\right]\right) \ , \\
\delta\lambda_{\dot{A}} &= -\left(2F_{1\bar{1}} - D\right)\zeta_{H,\dot{A}} - i\zeta_H^{\dot{B}}D_0\varphi_{\dot{B}\dot{A}} + \frac{i}{2}\zeta_{H,\dot{D}}\left[\varphi_{\dot{A}}^{\ \dot{C}}, \varphi_{\dot{C}}^{\ \dot{D}}\right] \ , \\
\delta\bar{\lambda}_{\dot{A}} &= \left(2F_{1\bar{1}} - D\right)\widetilde{\zeta}_{H,\dot{A}} + i\widetilde{\zeta}_H^{\dot{B}}D_0\varphi_{\dot{B}\dot{A}} + \frac{i}{2}\widetilde{\zeta}_{H,\dot{D}}\left[\varphi_{\dot{A}}^{\ \dot{C}}, \varphi_{\dot{C}}^{\ \dot{D}}\right] \ , \\
\delta\Lambda_{1,\dot{A}} &= 2F_{0\bar{1}}\zeta_{\dot{A}} - 2i\zeta^{\dot{B}}D_{\bar{1}}\varphi_{\dot{B}\dot{A}} + D_{22}\widetilde{\zeta}_{\dot{A}} \ , \\
\delta\bar{\Lambda}_{\bar{1},\dot{A}} &= 2F_{01}\widetilde{\zeta}_{\dot{A}} + 2i\widetilde{\zeta}^{\dot{B}}D_1\varphi_{\dot{B}\dot{A}} + D_{11}\zeta_{\dot{A}} \ .
\end{aligned} \tag{A.9}$$

The supersymmetry transformations of the hypermultiplet can be written as

$$\begin{aligned}
\begin{pmatrix} \delta X_1 \\ \delta X_2 \end{pmatrix} &= \begin{pmatrix} -\widetilde{\zeta}_H^{\dot{B}}\psi_{1,\dot{B}} \\ -\zeta_H^{\dot{B}}\psi_{2,\dot{B}} \end{pmatrix} \ , \\
\begin{pmatrix} \delta\widetilde{X}^1 \\ \delta\widetilde{X}^2 \end{pmatrix} &= \begin{pmatrix} -\zeta_H^{\dot{B}}\bar{\psi}_{2,\dot{B}} \\ \widetilde{\zeta}_H^{\dot{B}}\bar{\psi}_{1,\dot{B}} \end{pmatrix} \ , \\
\begin{pmatrix} \delta\psi_{1,\dot{A}} \\ \delta\psi_{2,\dot{A}} \end{pmatrix} &= \begin{pmatrix} -i\zeta_{H,\dot{A}}D_0X_1 - 2i\widetilde{\zeta}_{H,\dot{A}}D_1X_2 - i\zeta_H^{\dot{B}}X_1\varphi_{\dot{B}\dot{A}} \\ i\widetilde{\zeta}_{H,\dot{A}}D_0X_2 - 2i\zeta_{H,\dot{A}}D_{\bar{1}}X_1 - i\widetilde{\zeta}_H^{\dot{B}}X_2\varphi_{\dot{B}\dot{A}} \end{pmatrix} \ , \\
\begin{pmatrix} \delta\bar{\psi}_{1,\dot{A}} \\ \delta\bar{\psi}_{2,\dot{A}} \end{pmatrix} &= \begin{pmatrix} i\zeta_{H,\dot{A}}D_0X^{2\dagger} - 2i\widetilde{\zeta}_{H,\dot{A}}D_1X^{1\dagger} - i\zeta_H^{\dot{B}}X^{2\dagger}\varphi_{\dot{B},\dot{A}} \\ i\widetilde{\zeta}_{H,\dot{A}}D_0X^{1\dagger} + 2i\zeta_{H,\dot{A}}D_{\bar{1}}X^{2\dagger} + i\widetilde{\zeta}_H^{\dot{B}}X^{1\dagger}\varphi_{\dot{B}\dot{A}} \end{pmatrix} \ .
\end{aligned} \tag{A.10}$$

A.1.2 C-twist

The four supercharges preserved under the C-twist can be written as

$$\zeta_1^{A,\dot{1}} = \zeta_C^A \ , \ \zeta_2^{A,\dot{2}} = \widetilde{\zeta}_C^A \ . \tag{A.11}$$

The $\mathcal{N}=2$ subalgebra is generated by ζ_C^1 and $\widetilde{\zeta}_C^2$. For the vector multiplet, we define

$$\begin{aligned}
&\varphi = \frac{1}{2}\varphi_{1\dot{1}} \ , \ \varphi^\dagger = -\frac{1}{2}\varphi_{2\dot{2}} \ , \\
&\lambda_1^{A\dot{1}} = \lambda^A \ , \ \lambda_2^{A\dot{2}} = \bar{\lambda}^A \ , \ \lambda_{1,A\dot{1}} = \bar{\Lambda}_{\bar{1},A} \ , \ \lambda_{2,A\dot{2}} = \Lambda_{1,A} \ .
\end{aligned} \tag{A.12}$$

The supersymmetry transformation of the vector multiplet is given by

$$
\begin{aligned}
\delta A_0 &= \frac{i}{2}\widetilde{\zeta}_C^A\lambda_A - \frac{i}{2}\zeta_C^A\bar{\lambda}_A \ , \\
\delta A_1 &= i\widetilde{\zeta}_C^A\Lambda_{A,1} \ , \\
\delta A_{\bar{1}} &= -i\zeta_C^A\bar{\Lambda}_{A,\bar{1}} \\
\delta\sigma &= \frac{1}{2}\zeta_C^A\bar{\lambda}_A - \frac{1}{2}\widetilde{\zeta}_C^A\lambda_A \ , \\
\delta\varphi &= -\frac{1}{2}\widetilde{\zeta}_C^A\bar{\Lambda}_{\bar{1},A} \ , \\
\delta\varphi^\dagger &= \frac{1}{2}\zeta_C^A\Lambda_{1,A} \ , \\
\delta D^{AB} &= i\zeta_C^A\left(D_{\bar{1}}\bar{\Lambda}_{\bar{1}}^B + \frac{1}{2}D_0\bar{\lambda}^B - \frac{1}{2}\left[\bar{\lambda}^B,\sigma\right] - \left[\Lambda_1^B,\varphi\right]\right) + (A\leftrightarrow B) \ , \\
&\quad - i\widetilde{\zeta}_C^A\left(D_1\Lambda_1^B + \frac{1}{2}D_0\lambda^B - \frac{1}{2}\left[\lambda^B,\sigma\right] - \left[\bar{\Lambda}_{\bar{1}}^B,\varphi^\dagger\right]\right) + (A\leftrightarrow B) \ . \\
\delta\lambda_A &= \left(-2F_{1\bar{1}} - iD_0\sigma - 2i[\varphi,\varphi^\dagger]\right)\zeta_{C,A} + 4i\widetilde{\zeta}_{C,A}D_1\varphi_1 - D_A{}^B\zeta_{C,B} \ , \\
\delta\bar{\lambda}_A &= \left(2F_{1\bar{1}} - iD_0\sigma + 2i[\varphi,\varphi^\dagger]\right)\widetilde{\zeta}_{C,A} + 4i\zeta_{C,A}D_{\bar{1}}\varphi_{\bar{1}} - D_A{}^B\widetilde{\zeta}_{C,B} \ , \\
\delta\bar{\Lambda}_{\bar{1},A} &= \left(2F_{10} + 2iD_1\sigma\right)\widetilde{\zeta}_{C,A} - 2iD_0\varphi\widetilde{\zeta}_{C,A} - 2i\zeta_{C,A}\left[\sigma,\varphi\right] \ , \\
\delta\Lambda_{1,A} &= \left(2F_{\bar{1}0} - 2iD_{\bar{1}}\sigma\right)\zeta_{C,A} - 2iD_0\varphi^\dagger\zeta_{C,A} - 2i\widetilde{\zeta}_{C,A}[\sigma,\varphi^\dagger] \ .
\end{aligned}
\tag{A.13}
$$

For the hypermultiplet, we define

$$
\begin{aligned}
&\psi_1^{\dot{1}} = \chi \ , \ \psi_2^{\dot{2}} = \eta \ , \ \psi_1^{\dot{2}} = \psi_1 \ , \ \psi_2^{\dot{1}} = -\psi_{\bar{1}} \ , \\
&\bar{\psi}_1^{\dot{1}} = \bar{\chi} \ , \ \bar{\psi}_2^{\dot{2}} = \bar{\eta} \ , \ \bar{\psi}_1^{\dot{2}} = \bar{\psi}_1 \ , \ \bar{\psi}_2^{\dot{1}} = -\bar{\psi}_{\bar{1}} \ .
\end{aligned}
\tag{A.14}
$$

We have

$$
\begin{aligned}
\delta X_A &= \widetilde{\zeta}_{C,A}\chi + \zeta_{C,A}\eta \ , \\
\delta X^{A\dagger} &= \widetilde{\zeta}_C^A\bar{\chi} + \zeta_C^A\bar{\eta} \ , \\
\begin{pmatrix} \delta\chi \\ \delta\psi_{\bar{1}} \end{pmatrix} &= \begin{pmatrix} -i\zeta_C^B(D_0+\sigma)X_B \\ 2i\zeta_C^B D_{\bar{1}}X_B - 2i\widetilde{\zeta}_C^B X_B\varphi^\dagger \end{pmatrix} \ , \\
\begin{pmatrix} \delta\psi_1 \\ \delta\eta \end{pmatrix} &= \begin{pmatrix} -2i\widetilde{\zeta}_C^B D_1X_B + 2i\zeta_C^B X_B\varphi \\ i\widetilde{\zeta}_C^B(D_0+\sigma)X_B \end{pmatrix} \ , \\
\begin{pmatrix} \delta\bar{\chi} \\ \delta\bar{\psi}_{\bar{1}} \end{pmatrix} &= \begin{pmatrix} i\zeta_{C,B}(D_0-\sigma)X^{B\dagger} \\ -2i\zeta_{C,B}D_{\bar{1}}X^{B\dagger} - 2i\widetilde{\zeta}_{C,B}X^{B\dagger}\varphi^\dagger \end{pmatrix} \ , \\
\begin{pmatrix} \delta\bar{\psi}_1 \\ \delta\bar{\eta} \end{pmatrix} &= \begin{pmatrix} 2i\widetilde{\zeta}_{C,B}D_1X^{B\dagger} + 2i\zeta_{C,B}X^{B\dagger}\varphi \\ -i\widetilde{\zeta}_{C,B}(D_0-\sigma)X^{B\dagger} \end{pmatrix} \ .
\end{aligned}
\tag{A.15}
$$

A.2 Localization in the Large $|u|$ Regions

In this section, we show that the residue integrals involving the hyperplanes of type (2.26) do not contribute to the integral with the choice (2.33). Let us consider the localizing action we take for the vector multiplet

$$
\frac{1}{t^2}\left[\frac{1}{e^2}\mathcal{L}_{\rm YM} + \mathcal{L}_{\rm H}\right] \ , \tag{A.16}
$$

which is modified from the localizing action used in [23, 25, 26] by an additional term $\mathcal{L}_{\mathrm{H}}$. We take the limit $t \to 0$ with e finite so that the localisation locus for the vector multiplet is given by

$$* F + iD = 0 \ , \quad D = -i\left(\mu(\phi) - \tau\right) \ , \quad \sigma = \text{constant} \ , \tag{A.17}$$

and therefore the path integral localizes to the finite dimensional integral of the Cartan zero modes $u = i\beta(\sigma + a_0)$. As discussed in [23, 25, 26], it is convenient to allow a constant non-BPS mode $\hat{D}$ such that the auxiliary field localizes to the field configuration which satisfies

$$* F + iD = i\hat{D} \ , \quad \text{where } \hat{D} \in \mathbb{R}^r \ . \tag{A.18}$$

Then the contour integral expression can be derived from the algebra of the zero mode multiplets $\mathcal{V} = (u, \bar{u}, \lambda_0, \bar{\lambda}_0, \hat{D})$.

The modified Q-exact action (A.16) affects the $\hat{D}$ integrals in the large $|u|$ region. The boundary integral for a given η in the neighbourhood of the hyperplane (2.26) is governed by the expression

$$\begin{aligned} I_{\text{asymp}}(\eta) = \sum_{\mathfrak{m} \in \mathbb{Z}} q^{\mathfrak{m}} \lim_{t \to 0} \oint_{u \to \pm i\infty} du \int_{\mathbb{R} + i\delta} \frac{d\hat{D}}{\hat{D}} \\ g_{\mathfrak{m}}(u, m, \hat{D}) \ \exp\left[\frac{\beta \mathrm{Vol}(\Sigma)}{2t^2 e^2}\hat{D}^2 - \frac{i\beta}{t^2}\left(-\frac{2\pi\mathfrak{m}}{e^2} + \mathrm{Vol}(\Sigma)\tau\right)\hat{D}\right] \ , \end{aligned} \tag{A.19}$$

where $g(u, m, \hat{D})$ is the one-loop contribution with the non-zero $\hat{D}$ background, which reduces to the integrand of the expression (2.17) at $\hat{D} = 0$. Here $\delta \in \mathfrak{t}$ is introduced as a regulator of the $\hat{D}$ integral, which is chosen in such a way that it satisfies $\eta(\delta) < 0$ for a choice of $\eta \in \mathfrak{t}^*$ in the definition of the JK-residue integral. [57, 23] The integral can be performed by rescaling $\hat{D} \to t^2\hat{D}$ and taking the limit $t \to 0$. We find

$$\begin{aligned} I_{\text{asymp}}(\eta) &= \sum_{\mathfrak{m} \in \mathbb{Z}} q^{\mathfrak{m}} \oint_{u \to \pm i\infty} du \int_{\mathbb{R} + i\delta} \frac{d\hat{D}}{\hat{D}} \ g_{\mathfrak{m}}(u, m, 0) \ e^{-i\beta\left(-\frac{2\pi\mathfrak{m}}{e^2} + \mathrm{Vol}(\Sigma)\tau\right)\hat{D}} \\ &= -2\pi i \ \mathrm{sgn}(\eta) \sum_{\mathfrak{m} \in \mathbb{Z}} q^{\mathfrak{m}} \ \Theta\left[\eta\left(\frac{2\pi\mathfrak{m}}{e^2} - \mathrm{Vol}(\Sigma)\tau\right)\right] \oint_{u \to \pm i\infty} du \ g_{\mathfrak{m}}(u, m, 0) \ . \end{aligned} \tag{A.20}$$

If we assign the charges to the pole at infinity at each flux sector $\mathfrak{m}$ as

$$Q_{\pm\infty} = \frac{2\pi\mathfrak{m}}{e^2} - \mathrm{Vol}(\Sigma)\tau \ , \tag{A.21}$$

we can write

$$I_{\text{asymp}}(\eta) = -2\pi i \ \mathrm{sgn}(\eta) \sum_{\mathfrak{m} \in \mathbb{Z}} q^{\mathfrak{m}} \ \underset{u \to \pm\infty}{\text{JK-Res}}(Q_{\pm\infty}(u), \eta) \ g_{\mathfrak{m}}(u, m, 0) \ du \tag{A.22}$$

A.3 Some Mathematical Background on Vortices and Symmetric Products

Generalized vortex equations on a Riemann surface have been extensively studied, and their moduli spaces of solutions have been given an algebraic description by means of an extension of the classical

Hitchin-Kobayashi correspondence [96, 97, 89, 98]. This correspondence relates holomorphic vector bundles that satisfy a stability condition to Einstein-Hermitian vector bundles. We recall that the latter are complex vector bundles endowed with a Hermitian metric, whose curvature (seen as an endomorphism of the tangent bundle) is a constant times the identity operator. Similarly, the generalized vortex equations can be formulated as equations for the existence of a specific hermitian metric on a complex vector bundle, and Einstein-Hermitian metrics can be interpreted as a special case of these.

The aim of this appendix is to summarize and develop the main notions concerning generalized vortex equations needed in the bulk of the thesis.

A.3.1 Abelian vortex equations

Let us start with the simplest example. Consider an hermitean line bundle L endowed with a smooth unitary connection A and a smooth section ϕ of L. Let $\mathfrak{N}_d$ denote the space of pairs (A, ϕ) that are solutions to the vortex equations on Σ,

$$\begin{aligned} \frac{1}{e^2} * F_A + |\phi|^2 &= \tau \\ \bar{\partial}_A \phi &= 0\,, \end{aligned} \tag{A.23}$$

where F_A is the curvature of the connection A and $\bar{\partial}_A$ is the holomorphic structure on L inherited from d_A and the complex structure on Σ. Furthermore, let $\mathcal{G}$ be the group of gauge transformations, $\mathcal{G} = \mathrm{Hom}(\Sigma, U(1))$. The moduli space of vortices is defined by the quotient

$$\mathfrak{M}_d := \mathfrak{N}_d \,/\, \mathcal{G}\,. \tag{A.24}$$

This can be understood as an infinite-dimensional Kähler quotient. First, the space of pairs (A, ϕ) is an infinite-dimensional Kähler manifold with flat metric

$$g = \frac{1}{4\pi} \int_\Sigma \left(\frac{1}{e^2} \delta A \wedge * \delta A + * |\delta\phi|^2 \right) d\Sigma \tag{A.25}$$

inherited from the metric on Σ and the Hermitian metric on the line bundle L. The second vortex equation $\bar{\partial}_A \phi = 0$ defines a Kähler submanifold $\mathfrak{V}_d$ of this space, on which gauge transformations act with moment map

$$\frac{1}{e^2} * F_A + \mu(\phi)\,, \tag{A.26}$$

where $\mu(\phi) = |\phi|^2$. We can therefore express the vortex moduli space as an infinite-dimensional Kähler quotient $\mathfrak{M}_d = \mathfrak{V}_d \,//\, \mathcal{G}$.

In our computations, we will make use of a Hitchin-Kobayashi correspondence and express the moduli space of solutions algebraically. First, we notice that by integrating the first vortex equation in (A.23), a necessary condition for the existence of solutions is

$$\tau \geq \frac{2\pi d}{e^2 \mathrm{Vol}(\Sigma)}\,. \tag{A.27}$$

We assume the strict version of this inequality in what follows. It is then clear that the section ϕ cannot vanish everywhere on Σ, which is is the simplest instance of a stability condition.

The general strategy of the Hitchin-Kobayashi correspondence for vortices on Σ is to replace (A.23) with its respective stability condition, and then to take the quotient of the solution to the remaining one by complex gauge transformations $\mathcal{G}_{\mathbb{C}} = \mathrm{Hom}(\Sigma, \mathbb{C}^*)$. The precise statement in this case [99] is that given a pair (A, ϕ) such that ϕ is a non-vanishing holomorphic section of L, in each complexified gauge orbit there exists one pair satisfying (A.23), which is unique up to $U(1)$ gauge transformations $\mathcal{G}$. Furthemore, provided the strict version of (A.27) holds, any solution can be written in this way.

The relation to the classical Hitchin-Kobayashi correspondence comes from the fact that (A.23) can be viewed as an equation for a hermitian metric h, intead of a connection A. This is because given a complex structure $\bar{\partial}_A$ and a hermitian metric L there is a unique connection A, the Chern connection, compatible with both structures. The proof relies on this point of view, and can be applied also to the case of more general gauge groups. Finally, we remark that this construction can be viewed as an infinite-dimensional analogue of the Kempff-Ness theorem, applied to the Kähler quotient $\mathfrak{M}_d = \mathfrak{V}_d \,//\, \mathcal{G}$.

The Hitchin-Kobayashi correspondece implies that the moduli space of solutions to the vortex equations can be parametrized by pairs (L, ϕ), where L is a holomorphic line bundle of degree d and ϕ is a non-vanishing holomorphic section of L. There is a map from this space to the symmetric product of the curve $\mathrm{Sym}^d\Sigma$. In fact, this parametrizes degree d divisors on Σ, and the map is given by taking the divisor of zeros of ϕ

$$D = p_1 + \ldots + p_d \,. \tag{A.28}$$

From a physical perspective, the points $p_1, \ldots, p_d$ correspond to the positions of the vortex centres. It turns out that the hermitian line bundle can be recovered by means of the map

$$\begin{aligned} j : \mathrm{Sym}^d\Sigma &\to \mathrm{Pic}^d(\Sigma) \cong J_\Sigma \\ : \{D\} &\mapsto \mathcal{O}_\Sigma(D) \ . \end{aligned} \tag{A.29}$$

The connection A is then defined uniquely. Thus, we have an isomorphism

$$\mathfrak{M}_d \cong \mathrm{Sym}^d\Sigma \,. \tag{A.30}$$

We notice that the map j has remarkable properties. Provided $d \geq 2g - 1$, it is a holomorphic fibration with fiber $\mathbb{CP}^{d-g}$ given by the projective space of global sections $\mathbb{P}H^0(\Sigma, L)$, or equivalently the complete linear system associated to $D = p_1 + \cdots + p_d$. For $d < 2g - 1$, the structure of this map is studied in the context of Brill-Noether theory.

A.3.2 $U(N)$ Vortices with Fundamental Matter

We now extend our discussion about the Hitchin-Kobayashi correspondnece to $U(N)$ vortices with fundamental matter. Let E be a holomorphic vector bundle with structure group $U(N)$, endowed with a d-bar operator $\bar{\partial}_E$. Let $\phi \in H^0(\Sigma, E)$, that is

$$\bar{\partial}_E \phi = 0\,. \tag{A.31}$$

As explained in the section about Abelian vortices, it is convenient to view the vortex equation as an equation for the metric h. For fundamental matter, we have the $\mathfrak{u}(2)^*$-valued equation

$$*_h F - e^2 \left(\phi \cdot \phi^\dagger\right)^h + e^2 \tau = 0\,, \tag{A.32}$$

where we emphasize the metric dependence. In analogy to the $U(1)$ case, we would like to derive a stability condition from this equation. Our presentation is based on [100]. We again integrate over Σ and we get

$$\mu(E) \leq \frac{e^2 \tau \mathrm{Vol}\,(\Sigma)}{2\pi}\,, \tag{A.33}$$

where any bundle E $\mu(E) := \frac{\deg(E)}{\mathrm{rank}(E)}$ is the *slope*. This is a first necessary condition, which we are now going to refine. Suppose there is a given holomorhic subbundle $E' \subset E$. As smooth complex vector bundles, we have

$$E =_{\text{smoothly}} E' \oplus (E/E')\,, \tag{A.34}$$

but this might not be true holomorphically. In fact, let $(\vec{e}_1, \vec{e}_2)$ be a holomorphic unitary frame so that $\vec{e}_1$ is a basis for E'. Let $D_{\bar{\partial}_E, h}$ be the metric connection (the connection induced by the metric and the Dolbeault operator $\bar{\partial}_E$). We have

$$D_{\bar{\partial}_E, h} e_a = A_{ab} e_b \tag{A.35}$$

where

$$A = \begin{pmatrix} A' & B \\ -B^\dagger & A^\perp \end{pmatrix}\,. \tag{A.36}$$

Here A' is the metric connection that arises from the restriction of h and $\bar{\partial}_E$ to E' and $A^\perp$ gives a connection on the complement of E'. B is a $(1,0)$-form which is interepreted as the second fundamental form of the embedding $E' \hookrightarrow E$ (that is, it computes the extrinsic curvature of E' in E) and $A^\dagger$ is its conjugate transpose[1]. In obvious notation, we can compute

$$F_h = dA - A \wedge A = \begin{pmatrix} F' + B \wedge B^\dagger & * \\ * & F^\perp + B^\dagger \wedge B \end{pmatrix}\,. \tag{A.37}$$

Importantly, a quick computation in local coordinates shows that

$$\begin{aligned} &\int_\Sigma \mathrm{Tr}\left(*B \wedge B^\dagger\right) d\Sigma \geq 0\,, \\ &\int_\Sigma \mathrm{Tr}\left(*B^\dagger \wedge B\right) d\Sigma = -\int_\Sigma \mathrm{Tr}\left(*B \wedge B^\dagger\right) d\Sigma \leq 0\,, \end{aligned} \tag{A.38}$$

[1] $B = (1-\pi) D_{\bar{\partial}_E, h} \pi$ where π is the projection onto L.

where the only important thing to keep in mind is that B is of type $(1,0)$. Now, we can also write (A.32) in local coordinates as

$$\begin{pmatrix} *F^{'} + *B \wedge B^{\dagger} & * \\ * & F^{\perp} + *B^{\dagger} \wedge B \end{pmatrix} + e^2 \begin{pmatrix} \phi^{'} \otimes \phi^{'*} - \tau & * \\ * & \phi^{\perp} \otimes \phi^{\perp *} - \tau \end{pmatrix} = 0\,. \tag{A.39}$$

Taking the trace of the upper left component and integrating over the curve, we get that

$$\mu(E') \leq \frac{e^2 \tau \mathrm{Vol}(\Sigma)}{2\pi}\,, \tag{A.40}$$

with equality if and only if

$$\int_{\Sigma} \mathrm{Tr}\left(*B \wedge B^{\dagger}\right) d\Sigma = 0\,. \tag{A.41}$$

By definition, if the above equation holds, then

$$E \cong_{\mathrm{hol}} E' \oplus (E/E') \tag{A.42}$$

holomorphically. Now suppose that $\phi \in H^0(\Sigma, E')$. Then, taking the trace of the lower-right component of (A.39), we similarly get

$$\mu\left(E/E'\right) \geq \frac{e^2 \tau \mathrm{Vol}(\Sigma)}{2\pi} \tag{A.43}$$

with equality if and only if (A.41) holds, E/L is holomorphic and (A.42) holds holomorphically.

We can now summarize the above findings as follows. Let

$$\begin{aligned} \mu_M &:= \sup\{\mu\left(E\right),\ \mu\left(L\right)\ \mid L \text{ holomorhic subbundle of } E\}\,, \\ \mu_m &:= \inf\{(E/L)\ \mid L \text{ holomorhic subbundle of } E, \phi \in H^0(\Sigma, L)\}\,. \end{aligned} \tag{A.44}$$

Further, define the following notion of stability for pairs (E, ϕ)

Definition 1 *A pair (E,ϕ) is* stable *if and only if*

$$\mu_M < \frac{e^2 \tau \mathrm{Vol}(\Sigma)}{2\pi} < \mu_m\,.$$

Then we have the following

Lemma 1 *If there is a metric h satisfying the equations* (A.32)*, then either the pair (E,ϕ) is* stable *or $E =_{hol} E' \oplus (E/E')$ with $\phi \in H^0\left(\Sigma, E'\right)$. In the latter case, the pair (E',ϕ) satisfies the inequality*

$$\mu(E') < \frac{e^2 \tau \mathrm{Vol}(\Sigma)}{2\pi}$$

and the holomorphic bundle E/E' satisfies

$$\mu\left(E/E'\right) = \frac{e^2 \tau \mathrm{Vol}(\Sigma)}{2\pi}\,.$$

In [100] the converse is also proven.

Finally, we consider the $s \to \infty$ limit, where $s = e^2\tau\mathrm{Vol}(\Sigma)$ In this limit, the stability condition simplifies drastically. First of all, notice that in this limit the lower bound is obviously satisfied. As for the upper bound, it is easy to see that it immediately implies that ϕ cannot be contained in any subbundle of E. But this means that generically ϕ has maximal rank. This discussion can be generalized to matter fields in both the fundamental and anti-fundamental representation at no cost. The result in the large s limit remains the same. For other representations, more sophisticated techniques are needed [88, 89].

A.3.3 Cohomology of Symmetric Products

The symmetric product of a curve $\mathrm{Sym}^d\Sigma$, and especially its cohomology, plays a central role in this thesis. The simplest way to understand the latter is via the isomorphism

$$H^\bullet(\mathrm{Sym}^d\Sigma, \mathbb{K}) \cong H^\bullet(\Sigma^d, \mathbb{K})^{S_d}, \tag{A.45}$$

where $\mathbb{K}$ is any field. The right-hand side consists of permutation-invariant elements in the cohomology of the d-fold product of Σ.

Let us introduce standard generators for the cohomology ring of Σ,

$$\gamma_i \in H^{1,0}(\Sigma, \mathbb{K}), \quad \tilde{\gamma}_i \in H^{0,1}(\Sigma, \mathbb{K}), \quad \beta \in H^2(\Sigma, \mathbb{K}), \tag{A.46}$$

where $i = 1, \ldots, g$.

They induce cohomology classes in the d-fold product of Σ,

$$\begin{aligned} \gamma_{i,j} &= 1 \otimes \cdots \otimes 1 \otimes \gamma_i \otimes 1 \otimes \cdots \otimes 1 \in H^{1,0}(\Sigma^d, \mathbb{K}) \\ \tilde{\gamma}_{i,j} &= 1 \otimes \cdots \otimes 1 \otimes \tilde{\gamma}_i \otimes 1 \otimes \cdots \otimes 1 \in H^{0,1}(\Sigma^d, \mathbb{K}) \\ \beta_j &= 1 \otimes \cdots \otimes 1 \otimes \beta_i \otimes 1 \otimes \cdots \otimes 1 \in H^{1,1}(\Sigma^d, \mathbb{K}), \end{aligned} \tag{A.47}$$

where the generator appears in the j-th factor. The classes

$$\xi_i = \sum_{j=1}^d \gamma_{i,j}, \quad \xi'_i = \sum_{j=1}^d \gamma_{i,j}, \quad \eta = \sum_{j=1}^d \beta_j \tag{A.48}$$

then descend to $H^\bullet(\Sigma^d, \mathbb{K})^{S_d}$, and in fact generate it. Clearly, the generators ξ_i and ξ'_i anticommute with each other and commute with η. There is also one last ring relation:

$$\xi_{i_1}\xi_{i_2}\ldots\xi_{i_a}\xi'_{j_1}\xi'_{j_2}\ldots\xi'_{j_b}\left(\xi_{k_1}\xi'_{k_1} - \eta\right)\ldots\left(\xi_{k_c}\xi'_{k_c} - \eta\right)\eta^q = 0 \tag{A.49}$$

provided

$$a + b + 2c + q = n + 1, \tag{A.50}$$

For more details, see the standard reference [80].

From this discussion, it follows that as a graded vector space, we have

$$\mu^p \nu^q H^{p,q}\left(\mathrm{Sym}^d(\Sigma)\right) \cong \bigoplus_{i=0}^{\min(p,q)} S^i(\mu\nu\mathbb{C}) \otimes \wedge^{p-i}(\mu\mathbb{C}^g) \wedge \wedge^{q-i}(\nu\mathbb{C}^g) \tag{A.51}$$

where μ, ν are grading parameters. It follows that

$$\sum_{d\in\mathbb{N}} x^d \left(\mu\nu \sum_{p,q} H^{p,q}\left(\mathrm{Sym}^d(\Sigma)\right)\right) \cong S^\bullet(x\mathbb{C} \oplus \mu\nu x\mathbb{C}) \otimes \wedge^\bullet(\mu x\mathbb{C}^g) \wedge \wedge^\bullet(\nu x\mathbb{C}^g) \tag{A.52}$$

for another grading parameter x. By taking a graded trace, we get that $h^{p,q}\left(\mathrm{Sym}^d(\Sigma)\right)$ is the coefficient of $x^d\mu^p\nu^q$ in the series expansion of

$$\frac{(1+\mu x)^g(1+\nu x)^g}{(1-x)(1-\mu\nu x)} \tag{A.53}$$

around $x=0$. Restricting to the grading by the fermion number, which amounts to setting $\mu=\nu=-1$, we can derive a generating function for the Euler-Poincaré characteristic

$$\sum_{d\in\mathbb{N}} x^d \left(\sum_{k=0}^{d}(-1)^k H^k_{dR}\left(\mathrm{Sym}^d(\Sigma)\right)\right) = (1-x)^{2(g-1)}\,. \tag{A.54}$$

It is also useful to understand how the cohomology of a symmetric product is induced from the fibration structure when $d \geq 2g-1$. The cohomology of a fibration with compact fibres can be computed using the Serre spectral sequence. In this particular example, the Serre spectral sequence collapses immediately and

$$H^\bullet(\mathrm{Sym}^d(\Sigma), \mathbb{K}) \cong H^\bullet(\mathbb{CP}^{d-g}, \mathbb{K}) \otimes H^\bullet(J_\Sigma, \mathbb{K}). \tag{A.55}$$

Since the Jacobian is isomorphic to a $2g$-dimensional torus, its cohomology is an exterior algebra, whose generators are the classes ξ_i, ξ_i' introduced above. The cohomology of the fiber $\mathbb{CP}^{d-g}$ is generated by the Chern class of the dual of the tautological bundle, which is identified with the class η.

A.3.4 Tangent Spaces of the Symmetric Producs

We recall the construction of the tangent space of $\mathrm{Sym}^d(\Sigma)$. We take the perspective that this is the space parameterizing divisors on Σ. The tangent space at a divisor D is given by

$$T_D\mathfrak{M}_\mathfrak{m} = H^0(\Sigma, \mathcal{O}(D)/\mathcal{O})\,. \tag{A.56}$$

This has a simple explanation when the divisor $D = p_1 + \cdots + p_d$ consists of separated points and therefore

$$T_D\mathfrak{M}_\mathfrak{m} = \times_{i=1}^{\mathfrak{m}_\Phi} T_{p_i}\Sigma\,. \tag{A.57}$$

Each cotangent space $T^*_p\Sigma$ can be identified with $\mathfrak{m}_p/(\mathfrak{m}_p)^2$, where $\mathfrak{m}_p$ is the ideal of holomorphic functions vanishing at p. From this point of view, the dual of $\mathfrak{m}_p/(\mathfrak{m}_p)^2$ is then the space of residues of meromorphic functions with a simple pole at p, since there is a pairing given by multiplication and evaluation. These residues are exactly parametrized by (A.56).

Let us now write L for the holomorphic line bundle induced by $O(D)$. From the short exact sequence

$$0 \longrightarrow \mathcal{O} \longrightarrow L \longrightarrow L_\Phi/\mathcal{O} \longrightarrow 0\,, \tag{A.58}$$

we see that the tangent space (A.56) fits into a long exact sequence

$$0 \longrightarrow H^0(\mathcal{O}) \xrightarrow{\alpha} H^0(L) \xrightarrow{\beta} H^0(L/\mathcal{O}) \xrightarrow{\gamma} H^1(\mathcal{O}) \xrightarrow{\delta} H^1(L) \longrightarrow 0\,. \tag{A.59}$$

Notice that both maps α and δ are inherited from multiplication by the holomorphic section $\phi(D)$ whose zeros are parametrized by the divisor D. For fixed D, equation (A.59) is a long exact sequence of vector spaces, which splits. In particular,

$$H^0(L_\Phi) = \mathrm{im}(\beta) \oplus \mathrm{coker}(\beta)\,, \tag{A.60}$$

and so we can reconstruct the tangent space $T_D\mathfrak{M}_\mathfrak{m}$ from $\mathrm{im}(\beta)$ and $\mathrm{coker}(\beta)$ separately. This means that we can consider the following two short exact sequences

$$\begin{aligned} &0 \longrightarrow H^0(\mathcal{O}) \xrightarrow{\alpha} H^0(L) \xrightarrow{\beta} \mathrm{im}(\beta) \longrightarrow 0 \\ &0 \longrightarrow \mathrm{coker}(\beta) \xrightarrow{\gamma} H^1(\mathcal{O}) \xrightarrow{\delta} H^1(L) \longrightarrow 0\,. \end{aligned} \tag{A.61}$$

The first line is the Euler sequence for

$$j^{-1}(L_\Phi) = \mathbb{P}H^0(L)\,, \tag{A.62}$$

where $j : \mathrm{Sym}^d(\Sigma) \to J_\Sigma$ is the holomorphic map to the Jacobian parametrizing L. Furthermore, in the case $d > 2g-2$, $H^1(L_\Phi) = 0$ and so $\mathrm{coker}(\beta) = H^1(\mathcal{O})$. The map γ becomes surjective and corresponds to the derivative of the projection j.

A.3.5 Universal Constructions

A natural way to obtain geometric objects on $\mathrm{Sym}^d\Sigma$ is to use universal constructions on the product $\mathrm{Sym}^d\Sigma \times \Sigma$. We first explain this in terms of divisors. The subset

$$\Delta = \{(D,p) \in \mathrm{Sym}^d\Sigma \times \Sigma \mid p \in D\} \tag{A.63}$$

is called the universal effective divisor on $\mathrm{Sym}^d\Sigma \times \Sigma$. An immediate consequence of the definition is that Δ cuts out on $\{D\} \times \Sigma$ precisely the divisor D on Σ. Conversely, given a point $p \in \Sigma$ the intersection of the universal divisor with $\mathrm{Sym}^d\Sigma \times \{p\}$ is $\widetilde{D}_p \times \{p\}$, where the divisor $\widetilde{D}_p$ is the image of the inclusion

$$\begin{aligned} i_p &: \mathrm{Sym}^{d-1}(\Sigma) \hookrightarrow \mathrm{Sym}^d(\Sigma) \\ i_p &: D \mapsto D + p\,. \end{aligned} \tag{A.64}$$

The divisor $\widetilde{D}_p$ defines a class in $H^2(\mathrm{Sym}^d(\Sigma), \mathbb{Z})$ which coincides with the cohomology class η defined in equation (A.48).

This universal construction can be lifted to a universal construction for line bundles. In fact, the divisor Δ defines a universal line bundle $\mathcal{L} = \mathcal{O}(\Delta)$ on $\mathrm{Sym}^d\Sigma \times \Sigma$. In the context of the $U(1)$ vortex equations, this can be described in terms of the space of pairs (A, ϕ), as follows. Let $\pi_{\mathfrak{N}_d} : \mathfrak{N}_d \times \Sigma \to \Sigma$ denote the projection on the second factor. Then the universal line bundle is

$$\mathcal{L} \cong (\pi^*_{\mathfrak{N}_d} L)/\mathcal{G}, \tag{A.65}$$

where $L = \mathcal{O}(D)$ and the group of gauge transformations $\mathcal{G}$ acts on both the space of solutions to the vortex solutions $\mathfrak{N}_d$ and the line bundle L [101].

We now summarize some important properties of the universal line bundle. First, for each point $[A, \phi] \in \mathfrak{M}_d$ there is an isomorphism of $U(1)$-bundles

$$\mathcal{L}|_{[A,\phi]} \cong L\,. \tag{A.66}$$

Second, $\mathcal{L}$ has a natural connection called the universal connection, which we will denote $\mathcal{A}$. It is induced by the connection on

$$\pi^*_{\mathfrak{N}_d} L \to \mathfrak{N}_d \times \Sigma, \tag{A.67}$$

which is trivial in the $\mathfrak{N}_d$ directions and tautological in the Σ directions, in that on $(A, \phi) \times \Sigma$ it acts exactly like A. In order to pass to the quotient in equation (A.65) and explain further properties of the universal connection $\mathcal{A}$, we first need to make use of standard constructions in gauge theory.

We first note that the space of solutions (A, ϕ) to the vortex equations is naturally a principal $\mathcal{G}$-bundle over the moduli space, $\mathfrak{N}_d \to \mathfrak{N}_d/\mathcal{G} = \mathfrak{M}_d$. The tangent space can therefore be decomposed as a direct sum

$$T\mathfrak{N}_d = T\mathfrak{N}_{d,\mathrm{vert}} \oplus T\mathfrak{N}_{d,\mathrm{hor}}\,, \tag{A.68}$$

where the vertical subspace $T\mathfrak{N}_{d,\mathrm{vert}}$ is canonically defined as the subspace tangent to the fibres of $\mathfrak{N}_d \to \mathfrak{M}_d$: they correspond to infinitesimal gauge transformations. However, without additional structure the horizontal subspace $T\mathfrak{N}_{d,\mathrm{hor}}$ is not canonical, and such a decomposition is equivalent to a choice of connection one-form $\theta \in \Omega^1(\mathfrak{N}_d, \mathfrak{g})$ where $\mathfrak{g}$ is the Lie agebra of $\mathcal{G}$. Given such a connection one-form, the horizontal subspace is defined by

$$T_{(A,\phi)}\mathfrak{N}_{\mathrm{hor}} = \{(\dot{A}, \dot{\phi}) \in T_{(A,\phi)}\mathfrak{N} \mid \theta_{(A,\phi)}(\dot{A}, \dot{\phi}) = 0\ \}, \tag{A.69}$$

where we follow the standard convention of denoting tangent vectors by $(\dot{A}, \dot{\phi})$. Conversely, any choice of horizontal subspace gives rise to a connection one-form

$$\alpha(\dot{A}, \dot{\phi}) = (d_A\alpha, i\alpha \cdot \phi), \tag{A.70}$$

is the action of an infinitesimal gauge transformation α.

We can now define the universal connection $\mathcal{A}$. The connection is uniquely determined by the action of its curvature $F_{\mathcal{A}}$ on tangent vectors $(\dot{A}, \dot{\phi}, u) \in T_{(A,\phi,p)}(\mathfrak{M}_d \times \Sigma)$. This is given by

$$\begin{aligned}
&F_{\mathcal{A}}((0,0,u_1),(0,0,u_2))|_{(A,\phi,p)} = F_A(u_1,u_2)(p) \\
&F_{\mathcal{A}}((\dot{A},\dot{\phi},0),(0,0,u))|_{(A,\phi,p)} = \dot{A}(u)|_p \\
&F_{\mathcal{A}}(((\dot{A}_1,\dot{\phi}_1),0),((\dot{A}_2,\dot{\phi}_2),0))|_{(A,\phi,p)} = F_\theta((\dot{A}_1,\dot{\phi}_1),(\dot{A}_2,\dot{\phi}_2))|_p \,.
\end{aligned} \tag{A.71}$$

Here we are abusing notation somewhat by conflating tangent vectors to $\mathfrak{M}_d$ and $\mathfrak{N}_d$. The universal connection is therefore tautological in the Σ directions and implements Gauss' law constraint in the $\mathfrak{M}_d$ directions. At the level of cohomology classes,

$$\begin{aligned}
-\frac{1}{2\pi}[F_{\mathcal{A}}]|_{[A,\phi]\times\Sigma} &= d \in H^2(\Sigma,\mathbb{Z}) = \mathbb{Z}\,; \\
-\frac{1}{2\pi}[F_{\mathcal{A}}]|_{\mathfrak{M}_d\times\{p\}} &= \eta \in H^2(\mathrm{Sym}^d(\Sigma),\mathbb{Z})\,,
\end{aligned} \tag{A.72}$$

which follows from the fact that the connection is tautological along Σ and the restriction of $\mathcal{L}$ to $\mathfrak{M}_d$ is the holomorphic line bundle induced by the divisor $\tilde{D}_p$ with class η.

A.3.6 Line Bundles and Deligne Pairing

The universal construction provides a natural way of obtaining a line bundle $\widetilde{L}_T$ on the moduli space $\mathfrak{M}_d = \mathrm{Sym}^d(\Sigma)$ starting from a line bundle L_T on Σ.[2] We consider the following diagramme

$$\begin{tikzcd}
 & \mathrm{Sym}^d(\Sigma)\times\Sigma \arrow[dl, "\pi"'] \arrow[dr, "p"] & \\
\mathrm{Sym}^d(\Sigma) & & \Sigma\,,
\end{tikzcd}$$

and a line bundle L_T on Σ. Then we have two obvious holomorphic line bundles on $\mathrm{Sym}^d(\Sigma)\times\Sigma$: the pull-back p^*L_T and the universal line bundle $\mathcal{L}_{\mathcal{T}}$. As explained in reference [79], we can then produce a holomorphic line bundle $\widetilde{L}_T$ on $\mathrm{Sym}^d(\Sigma)$ known as the *Deligne pairing*,

$$\widetilde{L}_T = \langle \mathcal{L}, p^* L_T \rangle\,. \tag{A.73}$$

We will not provide the details of this construction here. However, it is important to point out that given connections on L_T and $\mathcal{L}$ the following equality between the curvatures holds,

$$F_{\widetilde{A}} = \frac{1}{2\pi}\int_\Sigma F_{\mathcal{A}} \wedge F_A\,. \tag{A.74}$$

Reference [79] provides an alternative description of the Deligne pairing in this particular instance. The first step is to construct a line bundle on the direct product Σ^d,

$$L_T^{\boxtimes\,\mathfrak{m}_\Phi} := \bigotimes_j \pi_j^* L_T\,, \tag{A.75}$$

[2] We write L_T for the line bundle because in all our uses of this fact, L_T is a background bundle for the topological symmetry.

where

$$\pi_j : \Sigma^d \to \Sigma \tag{A.76}$$

is the projection onto the j-th factor. This is invariant under permutations and descends to a line bundle $\widetilde{L}_T$ on the symmetric product $\mathrm{Sym}^d\Sigma$. In particular, this construction shows that $c_1(\widetilde{L}_T) = d\eta$ where d_T is the degree of L_T and $\eta \in H^{1,1}(\mathfrak{M}_d)$ is the class constructed from the Kähler form on Σ.

A.4 $U(1)_{1/2}$ with One Chiral: Tangent Bundle from Massless Fermions

In section 3.2.4, we stated that the contribution to the space of supersymmetric vacua with flux $\mathfrak{m}$ is captured by a supersymmetric sigma model to the moduli space $\mathfrak{M}_{\mathfrak{m}} = \mathrm{Sym}^{\mathfrak{m}_\Phi}(\Sigma)$, where $\mathfrak{m}_\Phi = \mathfrak{m} + g - 1 > 0$. An important consistency check is that the massless Fermion fluctuations transform in the tangent space of $\mathfrak{M}_{\mathfrak{m}}$.

In this appendix, we show that $\ker(\beta)$ and $\mathrm{coker}(\beta)$ in (A.61) arise from Yukawa couplings in the original 3d supersymmetric gauge theory, and that the massless fermionic fluctuations are therefore valued in the tangent space to $\mathfrak{M}_{\mathfrak{m}}$.

In order to recover $\ker(\beta)$, we first note that terms in the first short exact sequence in (A.61) correspond to the fermions:

- $H^0(L_\Phi)$: fermions ψ in the $\mathcal{N} = (0,2)$ chiral multiplet obtained from decomposition of the 3d chiral multiplet Φ.
- $H^0(\mathcal{O})$: gauginos λ in the $\mathcal{N} = (0,2)$ vectormultiplet.

In the supersymmetric quantum mechanics, there is a Yukawa coupling proportional to

$$\int_\Sigma \bar{\phi}\Lambda_0 \wedge *\psi\,. \tag{A.77}$$

We now note that since the map $\alpha : H^0(\mathcal{O}) \to H^0(L_\Phi)$ corresponds to multiplication by ϕ, a Fermion element in the image of α has the form $\psi = \phi\psi'$ where $\psi' \in H^0(\mathcal{O})$. On the image of α, the Yukawa coupling then becomes

$$\psi'\Lambda_0 \int_\Sigma *|\phi|^2, \tag{A.78}$$

which generates a mass term for ψ'. As a consequence, we find that

$$\mathrm{im}(\beta) = H^0(L_\Phi)/H^0(\mathcal{O}) \tag{A.79}$$

parametrizes the remaining massless fluctuations.

The second short exact sequence in (A.61) arizes from the remaining fermions:

- $H^1(L_\Phi)$: fermions η in the $\mathcal{N} = (0,2)$ Fermi multiplet obtained from decomposition of the 3d chiral multiplet Φ.

- $H^1(\mathcal{O})$: gauginos $\bar{\Lambda}$ in the $\mathcal{N} = (0,2)$ chiral multiplet containing the covariant derivative $\bar{\partial}_A$.

In the supersymmetric quantum mechanics, there is a Yukawa coupling proportional to

$$\int_\Sigma \bar{\Lambda} \wedge *\langle \phi, \bar{\eta} \rangle \,. \tag{A.80}$$

The second short exact sequence (A.61) implies that the image of coker(β) in $H^1(\mathcal{O})$ is ker(δ). Since the map $\delta : H^1(\mathcal{O}) \to H^1(L_\Phi)$ corresponds to multiplication by the holomorphic section ϕ, a Fermion fluctuation $\bar{\Lambda}$ is in its kernel if and only if the product $\bar{\Lambda}\phi$ vanishes in the cohomology $H^1(L_\Phi)$. Equivalently, $\bar{\Lambda}\phi = \bar{\partial}\lambda$ for some $\lambda \in H^0(L_\Phi)$. This is the case if and only if the Yukawa coupling (A.80) vanishes for each $\bar{\eta}$ and the Fermion remains massless.

A.5 Background Line Bundles and Electric Impurities

In this appendix, we explain how a background line bundle L_T for a topological flavour symmetry can be understood as an electric impurity in the gauge theory. In particular, we show that the curvature of the 'dirty connection' introduced in [77] coincides with the curvature of the Deligne pairing (A.74).

We introduce local coordinates $\{X^a\}$ on the moduli space $\mathfrak{M}_d$ and let $A(x,X)$, $\phi(x,X)$ denote a solution of the vortex equations corresponding to the point in the moduli space with local coordinates X^a. We also introduce a local coordinate x on Σ. In the notation of Appendix A.3, this corresponds to a smooth section $s : \mathfrak{M}_d \to \mathfrak{N}_d$ of the principal bundle $\mathfrak{N}_d$. We therefore use a shorthand notation $s(X) = (A(x,X), \phi(x,X))$.

Let us now consider a tangent vector

$$\frac{\partial}{\partial X^a} \in T_X \mathcal{M}_d \,. \tag{A.81}$$

The push-forward of this tangent vector to $\mathfrak{V}_d$ is given by

$$s_* \left(\frac{\partial}{\partial X^a} \right) = \left(\frac{\partial A}{\partial X^a}\bigg|_X , \frac{\partial \phi}{\partial X^a}\bigg|_X \right) \in T_{s(X)} \mathfrak{N}_d \,. \tag{A.82}$$

This tangent vector need not be horizontal with respect to the connection one-form $\theta \in \Omega^1(\mathcal{V}_d, \mathfrak{G})$ and therefore we obtain $\mathfrak{G}$-valued functions on $\mathcal{M}_d \times \Sigma$,

$$\theta_{s(X)} \left(s_* \frac{\partial}{\partial X^a}\bigg|_X \right) = \alpha_a(x,X) \,. \tag{A.83}$$

It follows that $s^*\theta = \alpha_a(x,X) \mathrm{d}X^a$ is a connection on $\mathfrak{M}_d$ with covariant derivative

$$\begin{aligned} \delta_a A &= \left(\frac{\partial}{\partial X^a} + d\alpha_a \right) A \\ \delta_a \phi &= \left(\frac{\partial}{\partial X^a} + i\alpha_a \right) \phi \,. \end{aligned} \tag{A.84}$$

The universal connection on $\mathfrak{M}_d \times \Sigma$ can then be expressed

$$\mathcal{A}(X,x) = \alpha_a(x,X)\mathrm{d}X^a + A(X,x)\,. \tag{A.85}$$

In reference [77] the bosonic part of the action for the effective supersymmetric quantum mechanics of the collective coordinates $\{X^a\}$ is given by

$$S = \int dx_0 \; g_{ab}(X)\frac{dX^a}{dx_0}\frac{dX^b}{dx_0}\,. \tag{A.86}$$

Here x_3 is the euclidean time coordinate on $\mathbb{R}$ and the metric on $\mathfrak{M}_d$ is the one inherited from (A.25)

An electric impurity amounts to adding a term $(\sigma+iA_0)f$ to the original gauge theory lagrangian, where f is an arbitrary function on Σ and σ is the scalar component of the vectormultiplet. It is shown in [77] that this results in a 'dirty connection' $\tilde{A}_a(X)$ in the effective sigma-model action,

$$S = \int dx_0 \; g_{ab}(X)\frac{dX^a}{dx_0}\frac{dX^b}{dx_0} + \tilde{A}_a(X)\frac{dX^a}{dx_0}\,. \tag{A.87}$$

given by

$$\tilde{A}_a(X) = \int_\Sigma \alpha_a(X) \wedge *f\,. \tag{A.88}$$

On the other hand, introducing a background line bundle L_T for a topological flavour symmetry on Σ amounts to adding the following contribution to the Lagrangian of the supersymmetric gauge theory,

$$\frac{1}{2\pi}(\sigma + iA_0) * F_T\,, \tag{A.89}$$

where F_T is the curvature of the connection on the holomorphic line bundle L_T. This is equivalent to an electric impurity with $f = \frac{1}{2\pi} * F_T$. The corresponding dirty connection is

$$\widetilde{A}_{T,a}(X) = \frac{1}{2\pi}\int_\Sigma F_T \wedge \alpha_a(X)\,, \tag{A.90}$$

which can be written more invariantly using the universal connection (A.85) as follows

$$\widetilde{A}_T(X) = \frac{1}{2\pi}\int_\Sigma F_T \wedge \mathcal{A}(X)\,. \tag{A.91}$$

The curvature of the dirty connection is

$$F_{\widetilde{A}_T}(X) = \frac{1}{2\pi}\int_\Sigma F_T \wedge F_{\mathcal{A}}(X)\,. \tag{A.92}$$

We therefore see that the curvature of the dirty connection agrees with the curvature of the holomorphic line bundle $\widetilde{L}_T$ constructed using Deligne pairing in equation (A.74).

A.6 Characteristic Classes on a Fixed Locus and their Integration

In this Appendix, we derive expressions for various characteristic classes of bundles on

$$\mathfrak{M}^T_{(\mathfrak{m}_1,\cdots,\mathfrak{m}_k)} = \prod_{i=1}^{k} \mathrm{Sym}^{\mathfrak{m}_i}\Sigma \ . \tag{A.93}$$

and their integration, as needed in section 4.4. In particular, we would like to compute (4.202) and prove the integration formula (4.211).

We start by considering the universal divisor

$$\Delta \subset \Sigma \times \mathrm{Sym}^{\mathfrak{m}}\Sigma \tag{A.94}$$

of degree $\mathfrak{m}$, which was introduced in (4.136). We denote by f and π the projections onto Σ and $\mathrm{Sym}^{\mathfrak{m}}\Sigma$ respectively. Following [48], we denote classes on Σ by

$$e_1, \cdots, e_g, e'_1, \cdots, e'_g \in H^1(\Sigma, \mathbb{Z}) \ , \quad \eta_\Sigma \in H^2(\Sigma, \mathbb{Z}) \tag{A.95}$$

and as explained in the main body, standard classes on $\mathrm{Sym}^{\mathfrak{m}}\Sigma := \Sigma_{\mathfrak{m}}$ by

$$\xi_i, \xi'_i \in H^1(\Sigma_{\mathfrak{m}}, \mathbb{Z}), \quad \eta \in H^2(\Sigma_{\mathfrak{m}}, \mathbb{Z}) \ . \tag{A.96}$$

By the Künneth decomposition, the class of the universal divisor can be written as

$$[\Delta] = \mathfrak{m}\eta_\Sigma + \gamma + \eta \tag{A.97}$$

where $\gamma = \sum_{i=1}^{g} \xi'_i e_i - \xi_i e'_i$. By the ring relations of the cohomology of the symmetric product (A.49) and the standard relations on the curve, we have $\gamma^2 = -2\sigma\eta_\Sigma$. The Grothendieck-Riemann-Roch theorem then impies that for any $q \in \mathbb{Z}$

$$\mathrm{td}(\Sigma_{\mathfrak{m}})\mathrm{ch}[\pi_*\mathcal{O}(q\Delta)] = \pi_*[\mathrm{td}(\Sigma \times \Sigma_{\mathfrak{m}})\mathrm{ch}(\mathcal{O}(q\Delta))] \ . \tag{A.98}$$

From this we obtain

$$\begin{aligned} \mathrm{ch}[\pi_*\mathcal{O}(q\Delta)] &= \pi_*[\mathrm{td}(\Sigma)\mathrm{ch}(q\Delta)] \\ &= \pi_*(1 + (1-g)\eta_\Sigma)\exp(q\mathfrak{m}\eta_\Sigma + q\gamma + q\eta) \\ &= \pi_*(1 + (1-g)\eta_\Sigma)(1 + q\mathfrak{m}\eta_\Sigma)(1 + q\gamma - q^2\sigma\eta_\Sigma)e^{q\eta} \\ &= (q\mathfrak{m} + (1-g) - q^2\sigma)e^{q\eta} \ . \end{aligned} \tag{A.99}$$

In order to compute the expression (4.202), we first compute the Chern class of the line bundle $\pi_*(\mathcal{O}(-\Delta_a) \otimes \mathcal{O}(\Delta_b))$ on $\mathrm{Sym}^{\mathfrak{m}_a}\Sigma \times \mathrm{Sym}^{\mathfrak{m}_b}\Sigma$, where for the sake of notational simplicity we omit the pullbacks by the projections on $\mathrm{Sym}^{\mathfrak{m}_a}\Sigma$, $\mathrm{Sym}^{\mathfrak{m}_b}\Sigma$. We can compute

$$\begin{aligned} \mathrm{ch}[\pi_*\mathcal{O}(-\Delta_a) \otimes \mathcal{O}(\Delta_b)] &= \pi_*[\mathrm{td}(\Sigma)\mathrm{ch}(\mathcal{O}(-\Delta_a) \otimes \mathcal{O}(\Delta_b))] \\ &= \pi_*(1 + (1-g)\eta_\Sigma)\exp[-\mathfrak{m}_a\eta_\Sigma - \gamma_a - \eta_a + \mathfrak{m}_b\eta_\Sigma + \gamma_b + \eta_b] \\ &= [-\mathfrak{m}_a + \mathfrak{m}_b + (1-g) - (\sigma^{aa} + \sigma^{bb} - \sigma^{ab} - \sigma^{ba})]e^{-\eta_a + \eta_b} \ , \end{aligned} \tag{A.100}$$

where

$$\sigma_i^{ab} = \xi_i^a \xi_i'^b \ , \quad \sigma^{ab} = \sum_{i=1}^{g} \sigma_i^{ab} \ , \tag{A.101}$$

which satisfy the relation $\gamma_a\gamma_b = -2\sigma^{ab}\eta_\Sigma$. From this we obtain

$$c[\pi_*(\mathcal{O}(-\Delta_a)\otimes\mathcal{O}(\Delta_b))] = (1-\eta_a+\eta_b)^{-\mathfrak{m}_a+\mathfrak{m}_b-(g-1)} \exp\left[-\frac{\sigma^{aa}+\sigma^{bb}-\sigma^{ab}-\sigma^{ba}}{1-\eta_a+\eta_b}\right] \ . \tag{A.102}$$

Let us define a function

$$h(\eta_a-\eta_b) = e^{(-\eta_a+\eta_b+t)/2} - e^{(\eta_a-\eta_b-t)/2} \ . \tag{A.103}$$

Then using the relation $(\sigma_i^{ab})^2 = 0$, we can show that the contribution from the class $[N\mathcal{M}^{\mathcal{V}_{ba}}] = [H^\bullet(L_a^{-1}\otimes L_b)]$ can be written as[3]

$$\mathrm{ch}(\hat{\wedge}^\bullet N\mathcal{M}^{\mathcal{V}_{ba}}) = h(\eta_a-\eta_b)^{-m_a+\mathfrak{m}_b-(g-1)} \exp\left[(\sigma^{aa}+\sigma^{bb}-\sigma^{ab}-\sigma^{ba})\frac{h'(\eta_a-\eta_b)}{h(\eta_a-\eta_b)}\right] \ . \tag{A.104}$$

We would now like to prove (4.211), which is a generalization of the formula by Don Zagier (4.124) to integrals over $\mathfrak{M}^T = \prod_{a=1}^k \mathrm{Sym}^{n_a}\Sigma$. We want to show that for any function $A(\eta_1,\cdots,\eta_k)$ and $B(\eta_1,\cdots,\eta_k)$, we have

$$\begin{aligned} &\int_{\mathfrak{M}^T} A(\eta_1,\cdots,\eta_k)\exp\left[\sum_{a,b=1}^k \sigma^{ab}B_{ab}(\eta_1,\cdots,\eta_k)\right] \\ &= \operatorname*{res}_{u_1=0}\cdots\operatorname*{res}_{u_k=0} \frac{A(u_1,\cdots u_k)}{u_1^{n_1+1}\cdots u_k^{n_k+1}} \left[\det_{ab}\left(\delta_{ab}+u_aB_{ab}(u_1,\cdots,u_k)\right)\right]^g \ . \end{aligned} \tag{A.105}$$

This can be demonstrated as follows. First we notice that

$$\begin{aligned} \exp\left[\sum_{a=1,b=1}^k \sigma^{ab}B_{ab}\right] &= \prod_{i=1}^g\prod_{a=1}^k\prod_{b=1}^k \exp\left(\sigma_i^{ab}B_{ab}\right) \\ &= \prod_{i=1}^g\prod_{a=1}^k\prod_{b=1}^k \left(1+\sigma_i^{ab}B_{ab}\right) \\ &= \prod_{i=1}^g\left(\sum_{p=0}^k \sum_{\substack{\text{all }\{a_1,\cdots,a_p\}\\ \subset\{1,\cdots,k\}}} \sum_{\substack{\text{all }\{b_1,\cdots,b_p\}\\ \subset\{1,\cdots,k\}}} \prod_{l=1}^p\prod_{m=1}^p \sigma_i^{a_lb_m}B_{a_lb_m}\right) \ , \end{aligned} \tag{A.106}$$

where we used $\left(\sigma_i^{ab}\right)^2 = 0$ as well as the fact that σ's with different indices commute. We can then make use of the identity [49]

$$1 = \int_{\Sigma_{n_a}} \eta_a^{n_a}\left(\prod_{i\in I}\eta_a^{-1}\sigma_i^{aa}\right) \tag{A.107}$$

and its straightforward generalizations to products of symmetric products. They imply that the only monomials contained in (A.106) surviving integration are the ones for which the subsets $\{b_1,\cdots,b_p\}$

[3]In the Appendix, we omit the weights under the action of the flavour symmetry T_H for the sake of simplicity, which can be reintroduced easily.

are permutations of the $\{a_1, \cdots, a_p\}$. Let us denote by S_p the permutation group of p elements and suppose there is an $s \in S_p$ so that $s(a_i) = b_i$. Then

$$\prod_{l=1}^{k} \sigma_i^{a_l s(a_l)} = \mathrm{sgn}(s) \prod_{l=1}^{k} \sigma_i^{a_l a_l} \,. \tag{A.108}$$

Therefore,

$$\begin{aligned}
&\int_{\mathfrak{M}^T} A(\eta_1, \cdots, \eta_k) \prod_{i=1}^{g} \left[\sum_{p=0}^{k} \sum_{\substack{\text{all } \{a_1,\cdots,a_p\} \\ \subset\{1,\cdots,k\}}} \sum_{\substack{\text{all } \{b_1,\cdots,b_p\} \\ \subset\{1,\cdots,k\}}} \prod_{l=1}^{p} \prod_{m=1}^{p} \left(\sigma_i^{a_l b_m} \eta_{a_l}^{-1} \right) \eta_{a_l} B_{a_l b_m} \right] \\
&= \operatorname*{res}_{u_1=0} \cdots \operatorname*{res}_{u_k=0} \frac{A(u_1, \cdots u_k)}{u_1^{n_1+1} \cdots u_k^{n_k+1}} \prod_{i=1}^{g} \left[\sum_{p=0}^{k} \sum_{\substack{\text{all } \{a_1,\cdots,a_p\} \\ \subset\{1,\cdots,k\}}} \sum_{s\in S_p} \mathrm{sgn}(s) \prod_{l=1}^{p} u_{a_l} B_{a_l s(a_l)} \right] \qquad \text{(A.109)} \\
&= \operatorname*{res}_{u_1=0} \cdots \operatorname*{res}_{u_k=0} \frac{A(u_1, \cdots u_k)}{u_1^{n_1+1} \cdots u_k^{n_k+1}} \left[\sum_{p=0}^{k} \sum_{\substack{\text{all } \{a_1,\cdots,a_p\} \\ \subset\{1,\cdots,k\}}} \sum_{s\in S_p} \mathrm{sgn}(s) \prod_{l=1}^{p} u_{a_l} B_{a_l s(a_l)} \right]^g ,
\end{aligned}$$

where in the first line we formally divided and multiplied by η_{a_l} with respect to (A.106). By means of the Leibniz expansion of the determinant, this coincides with (A.105), as required.

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