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Cohomology in Topological String [1]

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ABSTRACT

We discuss some aspects to the BRST cohomology groups in topological string theory. We discuss the refined structure of the space of the topological Landau-Ginzburg models and find that the space is locally flat with some local singularities. The fusion rules in E_6 cohomology rings are explicitly expressed. After the models couple to topological gravity, a differential equation to the partition function, which is equivalent to the recursion relations associated with the Virasoro constraints, is set up. Making use of the differential equation, the relations of the correlation functions for the $q = 1$ models and $q > 1$ models for (p, q) minimal models are easily provided.

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Low dimensional string theory can be explained in three different approaches in the past few years. The concepts of continuous two dimensional induced gravity was elaborated by Polyakov[1] and has been developed in refs.[2]. The matrix models[3] may present some nonperturbative approaches to the discrete gravity[4], *e.g.* the nonperturbative partition function of two dimensional gravity is then given by the squire of the τ-function , satisfying the string equation, of the KdV hierarchy [5,6]. The third approach is based on the cohomological field theory[7]. In two dimension, the topological matter coupled to topological gravity is taken into account[8,9,10,11]. It has been shown that the topological theories are equivalent to the critical matrix models in the sense that both of them obey the string equation and the KdV hierarchy. For $K = 1$ one matrix models, the corresponding topological theory is pure topological gravity and the equivalence of them has explicitly verified [8,9]. It is also provided some evidences of the equivalence between multi-matrix models and topological matter coupled to topological gravity [9,11],*e.g.* the ADE series of topological minimal models or topological sigma models coupled to topological gravity.

There is a problem remaining to be solved in the above equivalence. As pointed out by the authors of ref.[5], in the matrix models, the string equation may be translated into the version of a loop equation which has a very clear geometrical interpretation[12]. However, the macroscopic loops [13] in the matrix models have not a well-defined correspondent in topological string, though the microscopic loops correspond to the physical operators in topological string. According to the formal definition of the macroscopic loops[11], the loops should belong to the BRST cohomology groups restricted to the boundary of the relevant Riemann surface. The groups relates to the observables in topological gravity as same as the Floer groups to the Donaldson's invariants[14]. In this proposal, we will search for the properties of the the BRST cohomology groups in topological string.

Before topological matter couples to gravity, it is useful to understand them in the absence of gravity. We focus on the topological Landau-Ginzburg(L-G) models here[11,15], which may be obtained by twisting the relevant $N = 2$ superconformal field theory[16]. The chiral rings in the $N = 2$ superconformal field theory[17] is identical to the BRST cohomology rings. For a given L-D potential, the Witten's index may be calculated and is equal to the 'Casson invariants' on the boundary, which is just the dimensions of the cohomology rings.

A two dimensional field theory may be regarded as a perturbation of the corresponding conformal field theory. Similarly, a two dimensional topological field theory may arise from a topological perturbation of a topological conformal field theory. All perturbed coupling constants form a phase space of topological field theories. As shown in ref.[11], the space of topological field theories are locally flat. By an analysis of the catastrophe theory, we will see that there are local singularities in the space of topological L-G models. Furthermore, there is a perturbed cohomology ring which has the same dimension as that in the unperturbed model and is overdetermined by unperturbed ring[11] . In this paper, we will present the structure of the perturbed cohomology ring in E_6 model explicitly.

After topological matter couples to the topological gravity, the structure of the BRST cohomology groups is vastly more complicated than the models in the absence of gravity. The factorization laws of the correlation functions without gravity are replaced by the recursion relations when the models coupled to gravity. The recursion relations associated with the Virasoro constraints in topological string theory have been deduced in refs.[8,9,10]. In the pure gravity case, the relations agree with those in the $K = 1$ critical one matrix models. And when the gravity is put into the theory, the relations agree with those in the $(p, 1)$ minimal models described by multi-matrix models. In order to obtain the results corresponding to the Kth multicritical one matrix models and to the (p, q) minimal models in multi-matrix models, the topological string theory should be perturbed by topological sources. By setting up a differential equation with respect to the partition function in perturbed theory, which is equivalent to the recursion relations in unperturbed models, it can be shown that the correlation functions in the Kth multicritical point for pure gravity and linking with the (p, q) minimal models may be represented by those in the $K = 1$ models and with the $(p, 1)$ models, respectively. The recursion relations of the correlation functions in topological theory coincide with those in matrix models.

Let Σ be a compact Riemann surface without boundary and S_0 denote the action of a topological string theory,*i.e.* topological matter coupled to topological gravity. This is a cohomological field theory and then , by construction, there is a local BRST symmetry with associated charge Q . Although the local BRST cohomology is trivial, this trivality needs not to be worry since there is an obstruction to make a left-right decomposition of the fields globally on a general Riemann surface with genus g such that it is not considered in common. Instead of the BRST cohomology, the non-trivial equivariant cohomology may be taken into account[18]. The BRST operator may be divided into two parts in general: the first part denotes Q_s which generates a global supersymmetry in both of the matter and the gravity sectors and the second part generates the ordinary Faddeev-Popov ghost. The nilpotent operator Q_s is the coboundary operator of the equivariant cohomology. Let $\sigma_{a,i}$ be the operators in this topological quantum field theory, then a perturbed topological string theory may be formulated by the following action

$$S = S_0 - \sum_{a,i} t_{a,i} \int_\Sigma \sigma_{a,i} \tag{1}$$

where the $t_{a,i}$ are known as coupling constants.The partition function, thus, in genus g, reads

$$F_g(t) = \int \mathcal{D}X exp\{-S\} \tag{2}$$

where X denotes all fields in S_0. The total partition function is given by the sum of F_g over topology of the Riemann surfaces

$$F(t) = \sum_{g=0}^{\infty} F_g(t) \tag{3}$$

The correlation functions in the perturbed model are defined as

$$<< \sigma_{d_1,i_1} \ldots \sigma_{d_n,i_n} >>= \frac{\partial}{\partial t_{d_1,i_1}} \ldots \frac{\partial}{\partial t_{d_n,i_n}} F(t) \tag{4}$$

The unperturbed correlation functions are naturally

$$< \sigma_{d_1,i_1} \ldots \sigma_{d_n,i_n} >=<< \sigma_{d_1,i_1} \ldots \sigma_{d_n,i_n} >> |_{\{t_{a,i}=0\}} \tag{5}$$

Now, if the Riemann surface has a non-empty boundary B which is homeomorphic to $S^1 \cup \ldots \cup S^1$, then the Hamiltonian formalism may be studied near the boundary. The neighborehood around the boundary B looks like a product manifold $B \times R^1$. Hence, as done by Witten on four manifolds[8], we can find that another operator $\overline{Q}_s$ linking with charge Q_s and the Hamiltonian H by

$$\{Q_s, \overline{Q}_s\} = 2H \tag{6}$$

The stress-tensor $T_{\alpha\beta}$ of the topological string may be obtained by the functional derivative of the action S_0 with respect to the metric of the Riemann surface and the stress-tensor has just the version of Q_s coboundary

$$T_{\alpha\beta} = \{Q_s, G_{\alpha\beta}\} \tag{7}$$

where G is a certain functional to the fields in the theory and with the ghost number -1. Near the boundary, the Hamiltonian is defined as

$$H = \int_B dx T_{00} \tag{8}$$

Thus, we can construct an operator $\overline{Q}_s$ obeying (6) simply by

$$\overline{Q}_s = 2 \int_B dx G_{00} \tag{9}$$

By the construction of the action S_0, there exists a time reversal symmetry T in the theory. And, under T,

$$\begin{aligned} \overline{Q}_s &\to Q_s \\ Q_s &\to -\overline{Q}_s \end{aligned} \tag{10}$$

Therefore, the fact $Q_s^2 = 0$ implies $\overline{Q}_s^2 = 0$ also. Thus, the cohomology groups have a description of an analogue of the Hodge theory. The elements in the cohomology groups are defined by the relation

$$Q_s|\psi >= 0, \quad |\psi >\equiv |\psi > +Q_s|\lambda > \tag{11}$$

and the 'Hodge conditions'[3]

$$\overline{Q}_s|\psi >= H|\psi >= 0 \tag{12}$$

It is well known that the cohomology groups describe the ground states of the Hilbert space $\mathcal{H}$ in topological string theory. For any state $|\Phi >$ in $\mathcal{H}$, it can be decomposed, in terms of Hodge decomposition theorem, as

$$|\Phi >= |\psi_0 > +Q_s|\psi_1 > +\overline{Q}_s|\psi_2 > \tag{13}$$

where $|\psi_0 >$ is the 'harmonic form' corresponding to H. If $|\psi >$ is Q_s closed, then

$$|\psi >= |\psi_0 > +Q_s|\psi_1 > \tag{14}$$

The connection of the BRST cohomology groups and the observables in topological string theory has the following interpretation. Let $\mathcal{O}$ be a product of local operators $\sigma_{a,i}$, $|\Psi(X_B) >$ be a state in Hilbert space $\mathcal{H}$ on the boundary B, where X_B represents the restriction of the whole collection of fields in topological string to B. The path integral with boundary condition $|\Psi(X_B) >$ is just

$$Z(\mathcal{O}, \Psi) = \int (\mathcal{D}X)\ exp(-S_0) \cdot \mathcal{O} \cdot \Psi(X_B) \tag{15}$$

The condition that (15) is topologically invariant implies that $|\Psi >$ represents a Q_s cohomology class. Thus, in (15) we obtain an observable with value in the Q_s cohomology groups of B.

The equation (15) may be rewritten as a more interesting version which perhaps relates closely to the loop equations. Let B consists of incoming string B_1 and outgoing string B_2. Giving a state $|\Psi(X_{B_1}) >$ of the boundary values of the fields on the B_1. Then a state $|\Psi'(X') >$ of the values of the fields on the B_2 may be determined as

$$|\Psi'(X') >= \int_{(X_{B_2}=X')} (\mathcal{D}X)\ exp(-S_0) \cdot \mathcal{O} \cdot |\Psi(X_{B_1}) > \tag{16}$$

[3]We have assume the positivity of the scalar product in the cohomology group.

It is obvious that $|\Psi >\rightarrow |\Psi' >$ a morphism of the tensor product of the Q_s cohomology groups of the B_1 to that of the B_2. This morphism tells us the following two facts. First, because (16) describes a topological change of the string, it should link closely with the loop equations. For example, for pure gravity, let $H^*(B_i), i = 1,2$, be the BRST cohomology groups of $B_i \sim S^1$ computed with the particular metrices, the length of B_i, l_i, the loops $|w(l_i)>= \sum_{m=0} \frac{l_i^{m+1/2}}{\Gamma(m+1/2)}|\sigma_m >$ belong to $H^*(B_i, l_i) = \bigoplus_{m=0} H^m(B_i, l_i)$. In principle, the loop equations with respect to $w(l)$ may arise from (16). Unfortunately, we are not able to obtain an exact connection between (16) and the loop equation in the present proposal. Second, it may be shown that the morphisms from $H^*(B, g^{(1)})$ to $H^*(B, g^{(2)})$computed with the metrices $g^{(i)}$, are isomorphism. This implies that the BRST groups of B are topologically invariant up to the isomorphisms.

In the above discussions, we know that there exists the Hodge analogue in the theory only near the boundary of the Riemann surface or the surface with a version of product manifold $B \times R$. But, if we consider a topological conformal field theory on a Riemann surface, the Hodge analogue of the BRST cohomology groups always exists. The stress tensor $T_{\alpha\beta}$ in the topological conformal field theory is traceless even the fields in the theory take arbitrary values.

In a topological conformal field theory, the BRST operator Q_s may be divided into left- and right-moving operators

$$Q_s = Q_L + Q_R \tag{17}$$

where

$$Q_L = \oint Q(z) \tag{18}$$

with a holomorphic current $Q(z)$ and Q_R is defined by an analogue of (18) with $z \rightarrow \bar{z}$. Here, $z = t + ix$ and $\bar{z} = t - ix$. Thus, the properties of the BRST cohomology groups discussed above hold on the whole Riemann surface. Moreover, $T_{\alpha\beta}, G_{\alpha\beta}, Q$ and $\overline{Q}$ may be written as the sum of left- and right-moving operators and then the BRST cohomology are identified with the Dolbeault cohomology in a suitable complex manifold.

In the absence of gravity, paying the attention to left-mover, $T(z)$,$G(z)$, $Q(z)$ and an addition bosonic U(1) current $J(z)$ (ghost number current) form a twisted $N = 2$ superconformal algebra[16,17,11]. There is a unique consistent commutator algebra in the Laurent expansion modes of those fields. So, besides the analogue properties mentioned from (11) to (14), the Hilbert space $\mathcal{H}$ is graded by the U(1) charge q

$$J_0|\Phi >= q|\Phi > \tag{19}$$

The commutators of the U(1) current modes show the presence of the background charge d. Thus, the conditions (11) and (12) defining the cohomology groups imply that the cohomology class may be represented by the 'harmonic forms' corresponding to $L_0 = \{Q_0, G_0\} = H$. Those harmonic forms are one-to-one corresponding to the so-called 'chiral primary fields' in the $N = 2$ superconformal field theory. For those fields, their U(1) charges q have rational values in the interval

$$0 \le q \le d \tag{20}$$

The set of chiral primary fields forms a ring called chiral ring. There are certain well known properties of the chiral rings.

a) As we have mentioned, each primary chiral field ϕ_i is one-to-one corresponding to a harmonic form $|\phi_i>$. Especially, a unique vacuum state $|0>$ with $q = 0$ corresponds to the identity operator. All harmonic states form the physical Hilbert space $\mathcal{H}_{phys}$ on which the metric η^{ij} is defined by the inverse of the matrix of two-point functions on the Riemann sphere

$$<\phi_i\phi_j>_0 = \eta_{ij} \tag{21}$$

where the states $|\phi_j>$ and $<\phi_i|$ are defined by

$$\begin{aligned} |\phi_j> &= \phi(0)|0> \\ <\phi_i| &= <\infty|\phi_i(\infty) \end{aligned} \tag{22}$$

and the state $<\infty|$ carrying U(1) charge $q = d$ is dual to $|0>$.

b) The correlation functions

$$<\phi_{i_1}\ldots\phi_{i_s}>_\Sigma \tag{23}$$

may be factorized by insertion of the sum

$$\mathbf{1}_{phys} = \sum_{i,j} |\phi_i> \eta^{ij} <\phi_j| \tag{24}$$

c) The set of the primary chiral fields ϕ_i forms a ring implies that

$$\phi_i \times \phi_j = \sum_k c_{ij}^k \phi_k \tag{25}$$

where $c_{ij}^k = \eta^{kl} c_{ijl} = \eta^{kl} <\phi_i\phi_j\phi_l>_0$. The metric $\eta_{ij} = c_{0ij}$ is special coefficient.

d) Moveover, the four point function may be factorized as

$$<\phi_i\phi_j\phi_k\phi_l>_0 = \sum_m c_{ij}^m c_{mlk} \tag{26}$$

and the commutative and associative laws in the ring are equivalent to the condition

$$\sum_m c_{ij}^m c_{mkl} = \sum_m c_{ik}^m c_{mjl} \tag{27}$$

Now, let us consider the perturbation of the topological conformal field theory. Similar to (1), the perturbed topological field theory has the action

$$S(t) = S_0 - \sum_n t_n \int_\Sigma \phi_n \tag{28}$$

where S_0 is the action of a topological conformal field theory and the t_n are coupling constants. The perturbed correlation functions relate to the unperturbed functions by

$$<< \phi_{i_1}...\phi_{i_s} >>_\Sigma = < \phi_{i_1}...\phi_{i_s} \cdot exp(\sum_n t_n \int_\Sigma \phi_n) >_\Sigma \tag{29}$$

It is easy to see that the perturbed correlation functions still obey all properties of that of unperturbed theory. Therefore, the perturbed three point functions in genus zero

$$c_{ijk}(t) = << \phi_i \phi_j \phi_k >>_0 \tag{30}$$

can be used to define a commutative and associative ring. The fusion coefficients (30) have the following properties:[11]

(i) The two point functions are independent of the coupling constants

$$c_{0ij} = \eta_{ij} \tag{31}$$

(ii) The coefficients satisfy

$$\partial_m c_{ijk}(t) = \partial_k c_{ijm}(t) \tag{32}$$

In ref.[11], the authors have pointed out that, for $d < 1$ topological minimal models, the perturbed ring may be overdetermined by (31) and (32) as well as

(iii) The $c_{ijk}(t)$ is a polynomial of finite order p in the t_n's, where p equals to the minimum of i, j and k.

(iv) at $t_n = 0$ the ring respects the U(1) charge conservation of the twisted $N = 2$ models.

By use of the information about correlation functions, the topological ADE minimal models have been studied in ref.[11]. Here, we present some comments on the results in ref.[11]. The data of the topological ADE models was translated into that in the topological L-G models. We know right away the Witten's index in topological L-G models associates with the index of the

BRST operator Q_s. When the Riemann surface has a non-empty boundary the index defines a 'Casson invariant' on the boundary. The indices of ADE series are just equal to the dimensions of the rings, *i.e.* for A_{k+1}, D_n, E_6, E_7 and E_8, $Ind\ Q_S$ are $k+1, n, 6, 7$ and 8, respectively.

The information of the perturbed chiral ring of the topological ADE models is also translated into the perturbed version of the topological L-D models. For example, the perturbed ring of A_{k+1} models may be derived from the perturbed potential

$$W(x,t) = \frac{x^{k+2}}{k+2} - \sum_{i=0}^{k} g_i(t)x^i \tag{33}$$

The functions $g_i(t)$ are given functions of the coupling constants t_j. For instance, in A_3 model, $g_0(t) = t_0 - \frac{1}{2}t_2^2$, $g_1(t) = t_1$ and $g_2(t) = t_2$, and in A_4 model, $g_0(t) = t_0 - t_3 t_2$, $g_1(t) = t_1 - t_3^2$, $g_2(t) = t_2$ and $g_3(t) = t_3$. The property (i) implies that $\frac{\partial}{\partial t_l}\eta_{ij} = 0$, *i.e.* the phase space of the topological field theories is locally flat. We can also see that there are some local singularities in the space. Taking the A_{k+1} models as examples, the catastrophe manifold M_k of the A_{k+1} models, which includes in$R \times R^k$, is defined by

$$W'(x,t) = x^{k+1} - \sum_{i=1}^{k} i g_i(t) x^{i-1} = 0 \tag{34}$$

The catastrophe set is the subset C_k of M_k

$$C_k = \{(g_i(t), x) \in R \times R^k | W' = W'' = \dots = W^{(k)} = 0\} \tag{35}$$

The phase graph of the theory is described by the bifurcation set B_k, the projection of the C_k onto R^k. In bifurcation B^k, there are some local singularities,*e.g.* for A_3, M_2,C_2 and B_2 are exhibited in figure 1. The point (0,0) in B_3 is the second multicritical point and the set $B_2 - \{(0,0)\}$ gives the set of the first critical points. For A_4, the bifurcation B_3 set in figure 2 describes the phase graph in the model. The third,second and first multicritical points are expressed as the sharp points, curves and planes, respectively, in B_3.

By use of the properties $(i) - (iv)$ and the commutative and associative laws, we have explicitly calculated the fusion rules in the ring of the E_6 model, which list in the tables 1-6 (with $c_{ijk} = c_{ij}^{10-k}$). Those fusion rules are consistent with the L-G superpotential given in ref.[11]. For E_7 and E_8 models, the corresponding fusion coefficients may be derived in a similar way.

Now, turn our attention to topological string theory. We discuss the topological conformal matter coupled to topological gravity. Because this type of topological string theories has conformal symmetry, the BRST charge Q_s may be divided into the sum of left- and right-moving charges. Therefore, we have still a Hodge analogue in the topological string theory. Although the structure of the BRST cohomology groups become complicated since the factorization of the correlation functions is no longer valid, we can obtain some data about the cohomology, *e.g.* the index of the charge Q_s and the recursion relations of the correlation function.

In order to obtain the index of Q_s, we look first at pure gravity. Topological gravity in two dimension can start from the moduli space $\mathcal{M}$ of flat gauge fields A on a compact Riemann surface Σ. In terms of the topology, *i.e.* the genus, of the surface, the gauge group choices $SO(3)$, $ISO(2)$ or $SL(2,R)$ with respect to $g = 0, 1$ or $g > 1$. Pure gravity action is conformally invariant and Q_s invariant. As known for us, the Hilbert space of the theory may be decomposed into $\mathcal{H} = \mathcal{H}^+ \oplus \mathcal{H}^-$ according to the eigenvalues of $(-1)^F$ where F is the fermion number operator. The physical subspace $\mathcal{H}_{phys}$ corresponds to the Q_s cohomology groups. Assume now the Riemann surface has a boundary B. The moduli space of the flat connections restricted to B may parametrized by

$$\mathcal{M}|_B = \{\mathcal{P}Tr\ exp\{i \oint_B A\} | \forall\ flat\ A\} / \mathcal{G} \tag{36}$$

Thereby, the dimension of the moduli space $\mathcal{M}|_B$ is zero. Hence, the Q_s cohomology group restricted to B is one dimension. This implies that the flat connection is isolated on B and there being no bosonic and fermionic zero modes. The quantization will give rise to a unique quantum ground state for each isolated flat connection. The value of U(1) charge for this state must be determined by computing the fermion normal ordering constant. The index of Q_s is expressed as

$$Ind\ Q_s = \sum_i (-1)^{q_i} \tag{37}$$

with q_i are the U(1) charges of the ground states.

The generalization of (37) to the string theory with topological matter is straightforward

$$Ind\ Q_s = dim\mathcal{R} \cdot \sum_i (-1)^{q_i} \tag{38}$$

where $dim\mathcal{R}$ is the dimension of the corresponding chiral ring $\mathcal{R}$.

The recursion relations, associated with the Virasoro constraints, of the correlation functions in ADE series of topological strings have been derived in ref.[10], based on the way to solved topological gravity in ref.[9]. For pure gravity, the recursion relations read[9]

$$
\begin{aligned}
&< \sigma_{n+1} \prod_{a=1}^{s} \sigma_{d_a} >_g \\
&= \sum_{a=1}^{s} (2d_a + 1) < \sigma_{d_a+n} \prod_{b\neq a}^{s} \sigma_{d_b} >_g \\
&+ \sum_{k=1}^{n} \{ < \sigma_{k-1}\sigma_{n-k} \prod_{a=1}^{s} \sigma_{d_a} >_{g-1} \\
&+ \frac{1}{2} \sum_{S=X\cup Y, g=g_1+g_2} < \sigma_{k-1} \prod_{a\in X} \sigma_{d_a} >_{g_1} < \sigma_{n-k} \prod_{b\in Y} \sigma_{d_b} >_{g_2} \}
\end{aligned} \tag{39}
$$

which cioncide with those in the $K = 1$ critical one matrix models[5]. As the puncture equation may be translated into the string equation[8], the recursion relations (39) are equivalent to the following differential equation for the perturbed partition function

$$
\begin{aligned}
\frac{\partial F(t)}{\partial t_{n+1}} &= \sum_{a=0} (2a+1) t_a \frac{\partial F(t)}{\partial t_{a+n}} \\
&+ \sum_{k=1} \{ \frac{\partial^2 F(t)}{\partial t_{k-1} \partial t_{n-k}} \\
&+ \frac{1}{2} \frac{\partial F(t)}{\partial t_{k-1}} \frac{\partial F(t)}{\partial t_{n-k}} \}
\end{aligned} \tag{40}
$$

It becomes easy to relate the $K = 1$ critical models to the $K > 1$ multicritical models. To show this point, we carry the $t_{d_1}, ..., t_{d_s}$ derivatives on the both hand sides of (40) and take $t_1 = \frac{1}{3}, t_K = -\frac{1}{2K+1}$ and other t_a vanish, then we have

$$
\begin{aligned}
&< \sigma_{n+K} \prod_{a=1}^{s} \sigma_{d_a} >_g^K = \sum_{a=1}^{s} (2d_a + 1) < \sigma_{d_a+n} \prod_{b\neq a}^{s} \sigma_{d_b} >_g^K \\
&+ \sum_{k=1}^{n} \{ < \sigma_{k-1}\sigma_{n-k} \prod_{a=1}^{s} \sigma_{d_a} >_{g-1}^K \\
&+ \frac{1}{2} \sum_{S=X\cup Y, g=g_1+g_2} < \sigma_{k-1} \prod_{a\in X} \sigma_{d_a} >_{g_1}^K < \sigma_{n-k} \prod_{b\in Y} \sigma_{d_b} >_{g_2}^K \}
\end{aligned} \tag{41}
$$

which are just the recursion relations in the Kth multicritical matrix models . Therefore, we know that

$$
< \sigma_{d_1} ... \sigma_{d_s} >_g^K = < \sigma_{d_1} ... \sigma_{d_s} \, exp\{ \frac{1}{3} \int_{\Sigma} \sigma_1 - \frac{1}{2K+1} \int_{\Sigma} \sigma_K \} >_g \tag{42}
$$

This relation has been obtained by Distler for $g = 0$ and Dijkgraaf and Witten for $g = 1$[8].

The results (41) and (42) for pure gravity may be generalized to topological string with minimal matter. In the topological string theory, the recursion relations associated with the Virasoro constraints read

$$
\begin{aligned}
&< \sigma_{n+1,1} \prod_{(a,i)} \sigma_{a,i} >_g = \sum_{(a,i)} (ap+i) < \sigma_{a+n,i} \prod_{(b,j)\neq(a,i)} \sigma_{b,j} >_g \\
&+ \sum_{s=1}^{n} \sum_{j,k} \eta^{jk} \{ < \sigma_{s-1,j}\sigma_{n-s,k} \prod_{(a,i)} \sigma_{a,i} >_{g-1} \\
&+ \frac{1}{2} \sum_{S=X\cup Y, g=g_1+g_2} < \sigma_{s-1,j} \prod_{(a,i)\in X} \sigma_{a,i} >_{g_1} < \sigma_{n-s,k} \prod_{(a,i)\in Y} \sigma_{a,i} >_{g_2} \}
\end{aligned} \tag{43}
$$

The operators$\sigma_{a,i} = \sigma_a(\phi_i)$ are the product of the basic operators σ_a in pure gravity and the chiral primary fields ϕ_i in topological minimal models. Those relations are same as that in multi-matrix models associated with the (p,q) minimal models at the $q = 1$ critical point. Translate (43) into a differential equation for the perturbed partition function , the equation reads

$$
\begin{aligned}
\frac{\partial F(t)}{\partial t_{n+1,1}} &= \textstyle\sum_{a,i}(ap+i)t_{a,i}\frac{\partial F(t)}{\partial t_{a+n,i}} \\
&+ \sum_{s=1}^{n} \eta^{jk} \{ \frac{\partial^2 F(t)}{\partial t_{s-1,j}\partial t_{n-s,k}} \\
&+ \frac{1}{2}\frac{\partial F(t)}{\partial t_{s-1,j}}\frac{\partial F(t)}{\partial t_{n-s,k}} \}
\end{aligned} \tag{44}
$$

For a set of (a,i), carrying the $t_{a,i}$ derivatives on the both hand sides of (44) and taking $t_{1,1} = \frac{1}{p+1}, t_{1,q} = -\frac{1}{p+q}$ and other $t_{a,i}$ vanish, we obtain the recursion relations at general q critical

$$
\begin{aligned}
&< \sigma_{n+1,q} \prod_{(a,i)} \sigma_{a,i} >_g^q = \sum_{(a,i)} (ap+i) < \sigma_{a+n,i} \prod_{(b,j)\neq(a,i)} \sigma_{b,j} >_g^q \\
&+ \sum_{s=1}^{n} \sum_{j,k} \eta^{jk} \{ < \sigma_{s-1,j}\sigma_{n-s,k} \prod_{(a,i)} \sigma_{a,i} >_{g-1}^q \\
&+ \frac{1}{2} \sum_{S=X\cup Y, g=g_1+g_2} < \sigma_{s-1,j} \prod_{(a,i)\in X} \sigma_{a,i} >_{g_1}^q < \sigma_{n-s,k} \prod_{(a,i)\in Y} \sigma_{a,i} >_{g_2}^q \}
\end{aligned} \tag{45}
$$

Of course, the equation (42) generalizes to

$$
< \sigma_{d_1,i_1} \ldots \sigma_{d_s,i_s} >_g^q = < \sigma_{d_1,i_1} \ldots \sigma_{d_s,i_s} \; exp\{ \frac{1}{p+1} \int_\Sigma \sigma_{1,1} - \frac{1}{p+q} \int_\Sigma \sigma_{1,q} \} >_g \tag{46}
$$

In conclusion, we have investigated the BRST cohomology in topological string theory. In the absence of gravity, the phase structures of topological L-G models corresponding to the ADE series have been depicted. We also portray the fusion rules in E_6 cohomology ring. For the models with gravity, we exhibit the connections of the recursion relations of correlation function among the different critical points. We are not involved in other L-G models besides the

ADE series. For example, we may be interested in the models corresponding to P_8, X_9 and J_{10}, which have the dimension $d = 1$ of the target manifold. Once gravity coupled to matter, the W constraints should be considered. Here, we do not touch upon that. We will discuss those issues in the next stage.

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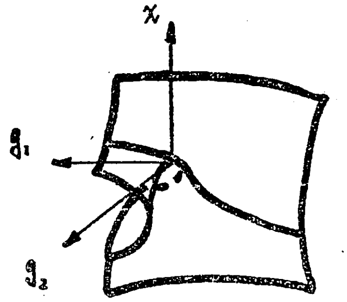
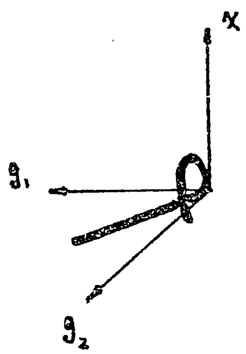
FIGURE CAPTIONS

Figure 1:(a) the catastrophe manifold M_2; (b) the catastrophe set C_2; (c) the bifurcation B_2.

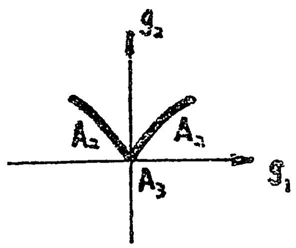
Figure 2: the bifurcation B_3.



(a)



(b)



(c)

Figure 1

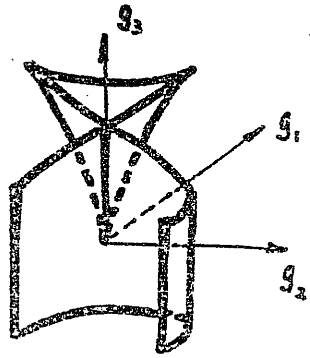


Figure 2

i\j	0	3	4	6	7	10
0	0	0	0	0	0	1
3		0	0	0	1	0
4			0	1	0	0
6				0	0	0
7					0	0
10						0

table 1. The cofficients c_{0ij}.
Here the symmetry of i, j, k
in c_{ijk} has been considered.

i\j	3	4	6	7	10
3	0	1	t_{10}	0	$\frac{1}{3}t_{10}^3 + t_6$
4		0	0	$\frac{1}{2}t_{10}^2$	$t_{10}t_7$
6			t_7	$\frac{1}{6}t_{10}^3 + t_6$	$t_3 + \frac{1}{2}t_{10}^2 t_7$
7				$t_{10}t_7$	$\frac{1}{2}t_{10}^2 t_6 + t_{10}t_4 + \frac{1}{2}t_7^2$
10					$t_{10}t_3 + t_7 t_4 + t_{10}t_7 t_6$

table 2. The cofficients c_{3ij}.

i\j	4	6	7	10
4	t_{10}	0	t_7	$\frac{1}{2}t_{10}^4 + t_4$
6		$\frac{1}{3}t_{10}^3$	$t_{10}t_7$	$t_{10}^2 t_6 + \frac{1}{2}t_7^2$
7			$\frac{1}{12}t_{10}^4 + t_{10}t_6$ $+t_4$	$\frac{1}{3}t_{10}^3 t_7 + t_{10}t_3$ $+t_7 t_6$
10				$\frac{2}{3}t_{10}^3 t_4 + \frac{1}{2}t_{10}^2 t_7^2$ $+t_7 t_3 + t_6^2 t_{10}$

table 3. The cofficients c_{4ij}.

i\j	6	7	10
6	$2t_{10}t_6$	$t_{10}^2t_7+t_3$	$\frac{1}{18}t_{10}^6+t_{10}^2t_4+t_{10}t_7^2+t_6$
7		$\frac{1}{12}t_{10}^5+t_{10}^2t_6$ $+t_{10}t_4+t_7^2$	$\frac{5}{12}t_{10}^4t_7+\frac{1}{2}t_{10}^2t_3+2t_{10}t_7t_3$ $+t_7t_4$
10			$\frac{1}{3}t_{10}^5t_6+\frac{5}{6}t_{10}^3t_7^2+t_{10}t_7t_3$ $+2t_{10}t_6t_4+t_7^2t_6$

table 4. The cofficients c_{6ij}.

i\j	7	10
7	$t_{10}^3t_7+t_{10}t_3+2t_7t_6$	$\frac{1}{36}t_{10}^7+\frac{5}{12}t_{10}^4t_6+\frac{1}{3}t_{10}^3t_4+\frac{3}{2}t_{10}^2t_7^2$ $+t_{10}t_6^2+t_7t_3+t_6t_4$
10		$\frac{7}{36}t_{10}^6t_7+\frac{5}{3}t_{10}^3t_7t_6+t_{10}^2t_7t_4+t_{10}t_6t_3$ $+t_{10}t_7^3+t_7t_6^2+t_4t_3$

$$c_{101010}=\frac{7}{12}t_{10}^5t_7^2+\frac{5}{6}t_{10}^2t_6^2+t_{10}^2t_4^2+\frac{5}{2}t_{10}^2t_7^2t_6+t_{10}t_7^2t_4+t_{10}t_3^2+\frac{1}{4}t_7^4+t_7t_6t_3+t_6^2t_4$$

table 5. The cofficients c_{7ij} and c_{101010}.