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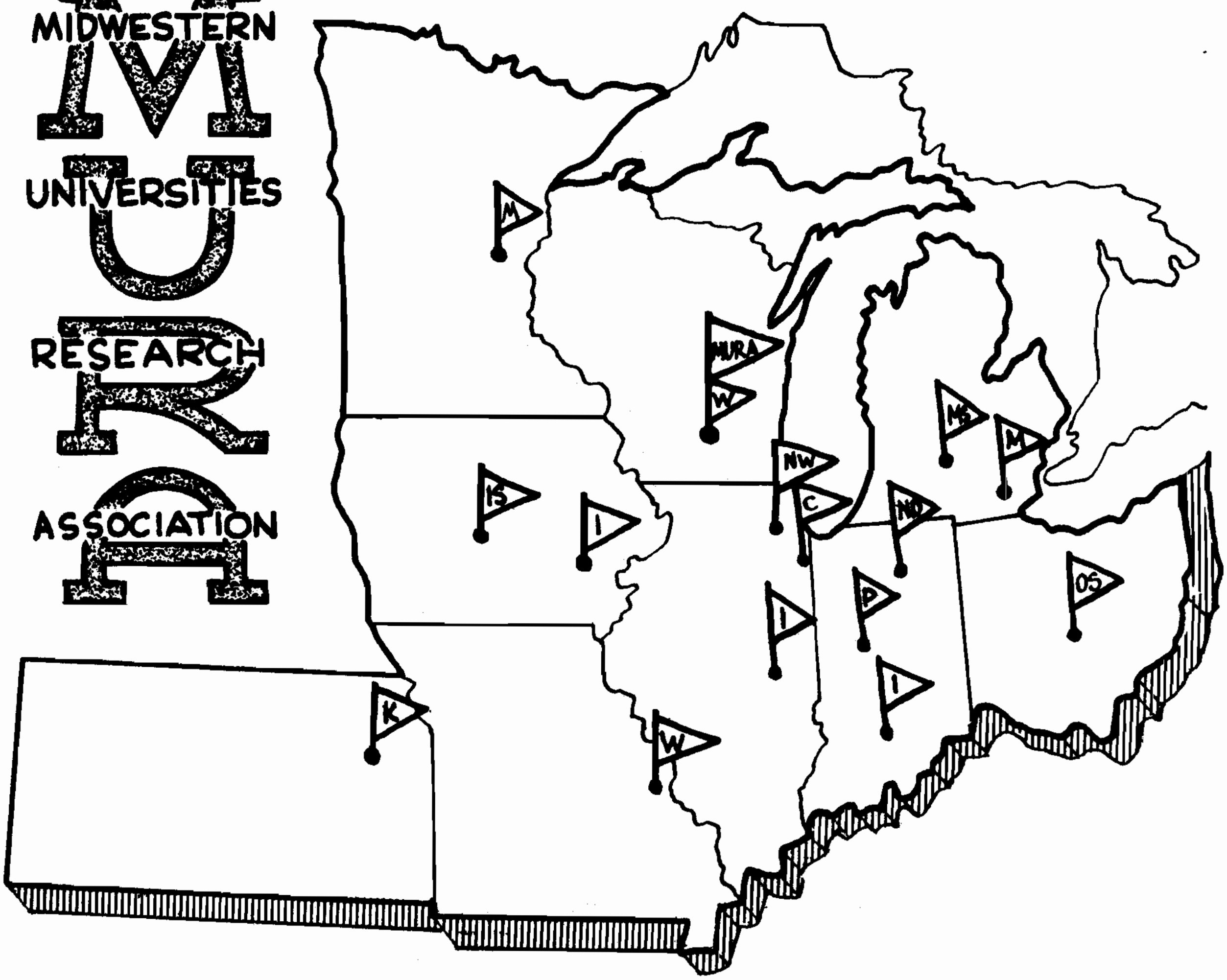

REMARKS ON THE DESIGN OF FFAG ACCELERATORS

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ABSTRACT

A review is given of the factors involved in the design of multi-Bev proton FFAG accelerators, including magnet weights, equilibrium orbits, linear and non-linear betatron oscillations, misalignments and rf. The general features of radial and spiral sector accelerators which emerge from consideration of these factors are discussed.

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**On leave from the State University of Iowa.

I. INTRODUCTION

Very little has appeared in MURA reports concerning the application of theory to the actual design of FFAG accelerators. In some ways, such material might be classified as "lore", making it very difficult to commit to paper. There are no unique answers to most design problems, but rather a thicket of competing factors from which the designer must choose some optimal compromise. The following is an attempt to exhibit these competing factors, not only to show what is known and to give some feeling for the compromises involved in a practical accelerator, but also to show the directions in which further work is needed. In the following sections the pertinent factors will be discussed separately, after which we shall try to indicate their combined effect on the design of an accelerator.

We shall restrict our attention to large proton accelerators with frequency-modulated accelerating voltage (sometimes called FFAG synchrotrons, though they are much closer in principle to synchrocyclotrons). The designer of an electron accelerator or a fixed frequency cyclotron faces quite different problems from those encountered in a large proton machine. In the electron case, the damping effects of the synchrotron radiation become of great importance above about 100 Mev. In the cyclotron case, the number of waves per revolution of the betatron oscillations changes during acceleration, which raises many problems not germane to a multi-Bev proton accelerator.

II. MAGNETIC FIELD PATTERNS

Let us limit our discussion to the case where the magnetic field is symmetric about the median plane, except for misalignment and field error effects. The more general case where such symmetry does not exist has been explored only in a preliminary way;[1] there are certainly interesting particle orbits, but the magnetic field patterns appear to be very complicated.

When median plane symmetry exists, it is customary to write the median plane field in cylindrical coordinates as

$$\begin{cases} B_r = B_\theta = 0 \\ B_z = -B_o\left(\frac{r}{r_o}\right)^k \sum_{n=0}^{\infty} \left(g_n \cos n\Phi + f_n \sin n\Phi\right) \\ \Phi = K \ln\left(\frac{r}{r_o}\right) - N\theta \end{cases} \qquad (2.1)$$

The field off the median plane can be found from (2.1). B_z can be expressed as an even power series in z/r, while B_r and B_θ can be expressed as odd power series in z/r.[2] The minus sign is written explicitly so that B_o will be positive to guide a particle of positive charge around the accelerator in the increasing θ (counterclockwise) direction.

The quantity k, the relative radial gradient of the average field, is in many ways similar to the quantity n of alternating gradient theory. The differences come because in an FFAG accelerator there can be large differences between the accelerator radius r and the magnetic radius of curvature ρ.

The quantity K is related to the angle ζ at which field spirals cut radial lines by

$$\frac{K}{N} = \tan \zeta \tag{2.2}$$

K is called $1/w$ in many MURA reports; $2\pi w$ is essentially the radial wavelength of the spirals.

In (2.1), the scale of the Fourier coefficients g_n and f_n can be chosen at will by choice of the constant B_o, so that only ratios of g_n and f_n are physically meaningful. Many of the focusing properties of FFAG orbits are determined to good approximation by the quantities

$$\left\{ \begin{aligned} F^2 &= \sum_{n=1}^{\infty} (g_n^2 + f_n^2) \\ G^2 &= \frac{1}{2} \sum_{n=1}^{\infty} \frac{g_n^2 + f_n^2}{n^2 N^2} \end{aligned} \right. , \tag{2.3}$$

where F is called the "flutter." $g_o F$ is then a measure of the size of the spatially-varying field relative to the constant field. For spiral sector accelerators $g_o F \sim 1$, while for radial sector accelerators, where $K = 0$, $g_o F \gtrsim 6$. The Ohkawa two-way accelerator[3] is just the limiting case of a radial sector accelerator* with $g_o = 0$ and some azimuth θ about which the magnetic field is odd in θ.

The higher harmonics g_n and f_n usually decrease at least as rapidly as $1/n$ in FFAG accelerators. Their effect on the motion is not large, except that they can excite non-linear resonances. The most notable exeption to the rapid decrease of f_n and g_n with n occurs when long straight sections are introduced in an accelerator. Then the period N of the magnet structure

*Spiraling can be added to the two-way accelerator to improve vertical focusing.

is reduced to a smaller value, N', called the "superperiod." The N'^{th} harmonic of the quide field (now Fourier analyzed for a complete revolution of the accelerator) is in general much smaller than the N^{th}. FFAG structures with superperiods have not been completely analyzed at this time and we will exclude them from our discussion.

The three MURA accelerators built to date are scaling accelerators. In a scaling accelerator, k, K, and the coefficients g_n and f_n are all constants, independent of radius. A scaling accelerator is the simplest means of insuring that ν_x and ν_y, the number of radial and vertical betatron oscillation waves per revolution (usually called "frequencies") are independent of energy. The orbits of particles of different momenta are geometrically similar to each other. If r_o and r_1 are the radii of corresponding points on orbits of total momentum p_o and p_1 respectively, then

$$\frac{p_1}{p_o} = \left(\frac{r_1}{r_o}\right)^{k+1} \qquad (2.4)$$

III. MAGNET WEIGHTS

In the magnets of the first two FFAG accelerators the radial gradient k was achieved by backwinding the energizing currents of the magnet across the polefaces to terminate magnetomotive force surfaces appropriately. In order that the flutter be independent of radius, i.e., that the field scale, the vertical gap was constructed to be proportional to the radius. The gap is larger at high field than at low, since $k > 0$, which increases the backwound current necessary to achieve a given k.

In higher field accelerators, this scaling pole is highly wasteful of both gap and current. If the radial field variation is made instead by shaping iron, note that the shaping must be three dimensional to keep the flutter independent of radius. This three-dimensional machining is more critical in a large accelerator than in a small one because the vertical gap is smaller relative to the magnet length, so that higher field harmonics are not as small. Achieving the radial gradient by pure iron shaping is again wasteful of gap, because the gap at low energies is now very large and will hold particles which will not undergo adiabatic damping quickly enough as their radii increase, so that they will strike the upper and lower walls during acceleration.

At this time it appears that some compromise between iron shaping and backwound current is the optimum way to build FFAG magnets. Iron shaping can be used to best advantage in saving windings (and therefore power) at the high field end. For example, in the MURA 40 Mev two-way electron accelerator, the non-scaling pole which gives the field shape at high field extends over only about 15% of the radial aperture, but saves a factor of over 2 in backwound current.

In AG synchrotrons the large gradient limits the guide field on the central orbit because it gives rise to much larger fields at the chamber edges. There is an analogous effect in FFAG accelerators. The poles must extend radially about one vertical gap length beyond the maximum energy orbit in order to keep the field good (scaling) on that orbit. Since the field increases as r^k, there are in this region considerably larger fields than those the particles see. This difficulty is particularly pronounced in spiral sector accelerators.

According to present ideas, radial and spiral sector magnets are not too different in their external appearance. The much larger flutter of radial sector accelerators is achieved by having half the magnets negative field magnets. The spiral field is achieved by iron spirals on the polefaces which are forward and back wound to increase the flutter. Flux travels radially in the poles in spiral sector accelerators, not along the spirals. It is believed that short (of order one foot) radial straight sections can be introduced in spiral sector accelerators for rf cavities, backwinding return currents, etc., if care is taken.

We can give a rough estimate of the amount of iron in the magnets. The flux carried per unit azimuthal length from the back leg to a radius r is

$$\Phi = \int_0^r B_z \, r dr = \frac{r_o B_o}{k + 2} \left(\frac{r}{r_o}\right)^{k + 2} , \tag{3.1}$$

where B_o is the field at radius r_o. The iron thickness necessary to carry this flux with a maximum flux density $B_{max}^{(1)}$ is

$$t_p = \frac{B_o}{B_{max}^{(1)}} \quad \frac{r_o}{k + 2} \left(\frac{r}{r_o}\right)^{k + 2} \tag{3.2}$$

and the volume of iron in the poles is

$$V_p \cong 4\pi \int_0^r r \, t_p \, dr$$

$$\cong \frac{4\pi B_o r_o^3}{B_{max}^{(1)} (k + 2)(k + 4)} \left(\frac{r_{max}}{r_o}\right)^{k + 4} \tag{3.3}$$

where r_{max} is the maximum good field radius. We have neglected the iron in the spirals.

If the back leg is to have a flux density $B_{max}^{(2)}$, its thickness is

$$t_B = \frac{B_o}{B_{max}^{(2)}} \quad \frac{r_o}{k + 2} \quad \left(\frac{r_{max}}{r_o}\right)^{k+2} , \tag{3.4}$$

The total volume of iron in the back leg is

$$V_B \cong 2 \pi r_{max} \cdot 2 t_B^2$$

$$\cong \frac{4 \pi r_o^3}{(k + 2)^2} \left(\frac{B_o}{B_{max}^{(2)}}\right)^2 \left(\frac{r_{max}}{r_o}\right)^{2(k+2)} \tag{3.5}$$

Note that both V_p and V_B are inversely proportional to k^2, for $k \gg 1$. The total volume of iron is approximately

$$V \cong \frac{4 \pi r_{max}^3}{k^2} \quad \frac{B_{max}}{B_{max}^{(1)}} \left[1 + \frac{B_{max} B_{max}^{(1)}}{B_{max}^{(2)}} \right] , \tag{3.6}$$

by taking $r = r_{max}$.

To this volume must be added extra iron for spirals (if any), for mechanical strength at the low field end and for extra back leg to go around the forward windings. The gap size affects the weight only in this last term. It has, of course, a large effect on the power required.

The copper and power costs of FFAG accelerators cannot be neglected in this discussion. For example, if it desired to stack beams, it is probably necessary to provide clearing electrodes to sweep out the negative charges which accumulate in the stacked positive beam and lower the space charge limit. The problems of holding the necessary high electric fields in a strong magnetic field mean that the magnet gap at the high field end must be enlarged to accommodate the electrodes. These problems are particularly acute in the two-way accelerator because of the large radius and low ν_y. In the 15 Bev two-way accelerator proposed by MURA in 1958, only 6 cm. of gap was available for particle motion out of a total of 15 cm.

IV. EQUILIBRIUM ORBIT

Because of the variation of the guide field with θ, the curvature of a particle of given total momentum varies with θ. Therefore the equilibrium orbit, which is that orbit on which a particle will remain as long as its total momentum does not change, is not a circle, but has a sinusoidal (or "scalloped") motion about some average circle. It is customary to discuss the motion in terms of the relative deviations x and y from a circle of radius r_o defined by

$$\begin{cases} r = r_o\,(1 + x) \\ z = r_o\, y \end{cases} \tag{4.1}$$

The equations of motion are

$$\begin{cases} (x'\,Z)' = (1 + x)\,Z - \alpha \left[(1 + x)\dfrac{B_z}{-B_o} - y'\,\dfrac{B_\theta}{-B_o}\right] \\[2ex] (y'\,Z)' = \alpha \left[x'\,\dfrac{B_\theta}{-B_o} - (1 + x)\,\dfrac{B_r}{-B_o}\right] \end{cases} \tag{4.2}$$

where $Z = \left[(1 + x)^2 + x'^2 + y'^2\right]^{-1/2}$, primes denote derivatives with respect to θ and

$$\alpha = \frac{e\, r_o\, B_o}{c\, p} \tag{4.3}$$

is in a scaling accelerator a dimensionless constant. Since the y equation is homogeneous, the equilibrium orbit lies in the median plane. The equilibrium orbit is periodic and can be expanded in a Fourier series

$$x_e = \sum_{n=0}^{\infty} (a_n \cos n N \theta + b_n \sin n N \theta) \cdot \tag{4.4}$$

A good approximation to the equilibrium orbit scalloping can be found by choosing $a_o = 0$, so that r_o is the average radius of the equilibrium orbit. For $n \neq 0$,

$$\left\{ \begin{array}{l} a_n \cong \dfrac{\alpha g_n}{n^2 N^2} \\[2ex] b_n \cong \dfrac{\alpha f_n}{n^2 N^2} \end{array} \right. \tag{4.5}$$

and α satisfies the approximate equation

$$1 - g_o \alpha - \left(k + \frac{3}{2}\right) G^2 \alpha^2 = 0 \tag{4.6}$$

For a given momentum, α is a relation between radius and field. The circumference factor, the maximum of the ratio of the equilibrium orbit radius to its radius of curvature, is given by

$$C = \alpha \left(1 + x_e (\theta_o)\right)^{k+1} \left| \frac{B_z (\theta_o)}{B_o} \right|, \tag{4.7}$$

where θ_o is the azimuth at which the maximum occurs. The factors multiplying α in (4.7) simply scale the field from the average radius r_o to the radius $r_o \left(1 + x_e (\theta_o)\right)$ and replace B_o in α by $B_z (\theta_o)$.

The quadratic term of the α equation (4.6) introduces a second root which represents a second equilibrium orbit. This second orbit is of practical interest only in the two-way accelerator (where $g_o = 0$), where it represents the equilibrium motion of a particle traveling in the opposite direction. However, in all radial sector accelerators, the quadratic term makes an important contribution to the value of the principal root and therefore to the circumference factor. Physically, this is the contribution to the net guide field along the equilibrium orbit which arises because the orbit scallops out into stronger field in positive magnets and in into weaker field in negative magnets. In a one-way radial sector accelerator this contribution can lower the circumference factor by as much as 25%. In a two-way accelerator this effect gives the total net guide field, so that particles traveling in opposite directions simply interchange positive and negative magnets. Note from (4.6) that in a two-way accelerator we must have $k > -\frac{3}{2}$ in order that real α (and therefore real equilibrium orbits) exist.

V. LINEAR MOTION ABOUT THE EQUILIBRIUM ORBIT

As in the AG synchrotron,[4] the linear motion about the equilibrium orbit is a sine wave with ν waves per revolution with superimposed a wave (the "ripple" or "beat") having the period of the magnet structure (N waves per revolution). The phase change per magnet period is

$$\sigma = 2\pi \nu / N \tag{5.1}$$

In the theory of the AG synchrotron, matrices are used to calculate σ as a function of the magnet parameters. The matrix method is not as useful in FFAG accelerators, because the field gradients along the equilibrium orbit are not in general piecewise constant. However, the same general principle applies. When σ reaches an integral multiple of π, the edge of a stability zone is reached. Outside the stability zone, the motion is exponential rather than oscillatory. The familiar necktie of AG theory is just the folding together of the boundaries of the lowest stability zones for radial and vertical oscillations. Except where superperiods are present, FFAG accelerators operate in the lowest stability zones for both radial and vertical oscillations, because the ripple is very large in higher zones.

In the actual design of an accelerator one desires to find the betatron oscillation frequencies very accurately, and digital computation is ideally suited for this purpose. However, the simple results of the smooth approximation[5] are a valuable guide to the effects of changing various parameters. For $g_o \neq 0$ (spiral sector and one-way radial sector accelerators) the betatron oscillation frequencies are

$$\begin{cases} \nu_x^2 = k + 1 + \frac{1}{2} k^2 G^2 \\ \nu_y^2 = -k + \frac{1}{2} k^2 G^2 + F^2 \left(\frac{K^2}{N^2} + \frac{1}{2} \right) \end{cases} \qquad (5.2)$$

The terms in k alone are the constant gradient focusing due to the radial increase of the average fields; the terms $\frac{1}{2} k^2 G^2$ arise from alternating gradient focusing. The terms $F^2 K^2/N^2$ and $\frac{1}{2} F^2$ in ν_y^2 are the spiral focusing (alternating edge focusing) and Thomas focusing (constant edge focusing) respectively. Note that these terms do not enter the radial focusing. They cancel due to the scalloping of the equilibrium orbit. The alternating gradient terms are small in spiral sector accelerators, where $G \sim 1/N$. If these terms are neglected, we must have $k > -1$ in spiral sector accelerators to preserve radial focusing.

For $g_o = 0$, (the two-way accelerator)

$$\begin{cases} \nu_x^2 = 2\,k \\ \nu_y^2 = \dfrac{1 + \dfrac{2\,K^2}{N^2}}{k} \quad \dfrac{F^2}{G^2} \end{cases} \qquad (5.3)$$

The radial focusing is similar in appearance to a constant gradient term, but in fact is due solely to alternating gradient focusing. It looks strange because the equilibrium orbit depends on $k\,G^2$ through α (see eq. (4.6)). The vertical focusing has only spiral and Thomas focusing terms.

Amplitudes can be calculated in an analytical approximation equivalent to the smooth approximation. For small σ, the amplitude of the sine wave increases for a given initial angle with the equilibrium orbit just because its frequency decreases. For σ near π, the periodic ripple about the oscillation of frequency ν is very large, becoming infinite at $\sigma = \pi$. Even though the basic oscillation is small, the non-sinusoidal beating about it requires large apertures.

VI. NON-LINEAR FORCES

Scaling fields of the type (2.1) necessarily have terms which are not linear in x and y and these in turn necessarily give non-linear equations of motion. Non-scaling fields do not appear to help, though they have been investigated only in the smooth approximation.[6] It appears to be impossible to find a field linear in x which will keep ν_x and ν_y constant over a reasonable range of energies. Thus it seems that FFAG accelerators are forced to live with non-linearities.

There are profound unsolved questions about long range stability of motion which, like that in the FFAG accelerator, follows from a Hamiltonian which is periodic in time (in our case θ plays the role of time). But these questions are not really of great moment in practice, because gas scattering and radiation damping distort the Hamiltonian picture.

The relatively short range effects of non-linear forces are of much greater practical interest. These forces give rise to stability limits - amplitudes beyond which the motion is very unstable. For very small amplitudes, the frequency of oscillation is determined by the linear forces. As the initial amplitude is increased, the frequency changes until it coincides with a "resonance." For larger amplitudes the motion is unstable. These resonances are called "essential" because they occur in any accelerator with non-linear restoring forces, even though there are no field errors. The general essential resonance relation has the form

$$m_x \sigma_x + m_y \sigma_y = 2\pi m, \tag{6.1}$$

where m_x, m_y and m are integers. The largest contribution to instability comes from the term

$$x^{|m_x| - 1} y^{|m_y|} e^{i m \theta}$$

in the radial equation and

$$x^{|m_x|} y^{|m_y| - 1} e^{i m \theta}$$

in the vertical equation. Resonances with m_y odd do not occur because of symmetry about the median plane. Only resonances with $m_x \geqslant 0$, $m_y \geqslant 0$ give instability directly. Difference resonances, where $m_x m_y < 0$ can lead to instability indirectly by changing amplitudes (and therefore frequencies) until some other resonance is reached. Terms with $|m_x| + |m_y| > 4$ do not give instability for small amplitudes even when the resonance relation (6.1) is exactly satisfied.

Quite simple formulas can be derived for stability limits which are valid in the neighborhood of a resonance. Consider, for example, the one-dimensional equation

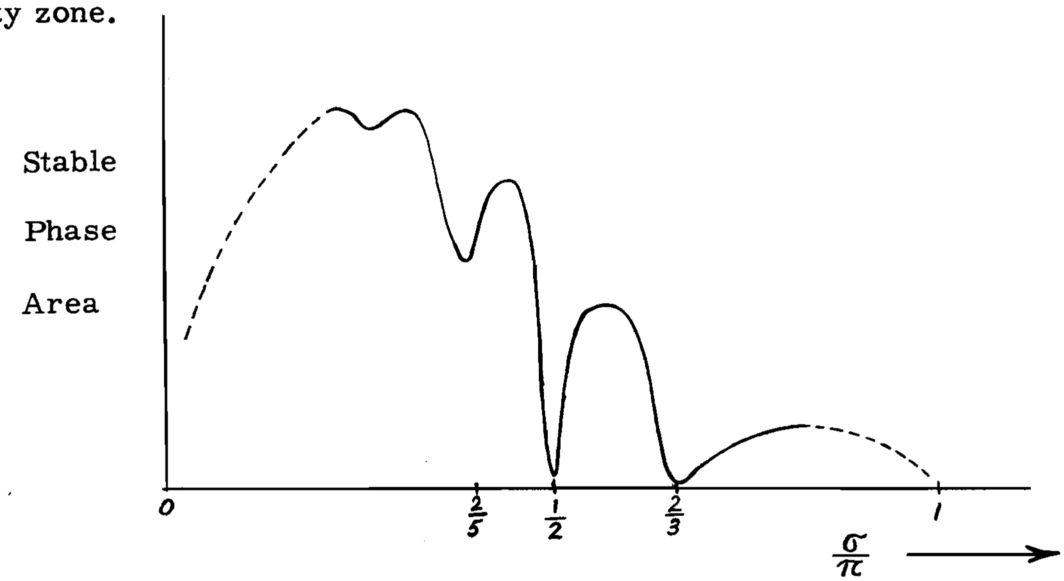
$$x'' + \nu_x^2 x = (B \sin N \theta) x^2 \tag{6.2}$$

The resonance $\sigma_x = \frac{2\pi}{3}$ ($\nu_x = \frac{1}{3} N$) which is excited by the quadratic term is in practice the most important one-dimensional resonance. The largest stable amplitude is[7]

$$A \cong \left| \frac{8}{3} \frac{N}{B} (\nu_x - \frac{1}{3} N) \right| . \tag{6.3}$$

For radial sector accelerators, $B \cong \frac{1}{2} g_1 k^2$, while for spiral sector accelerators $B \cong \frac{1}{2} g_1 K^2$.

Digital computer work gives a more extended picture. The sketch below is a plot of stable phase area (essentially $\nu_x A^2$) vs. σ for the first stability zone.



The region above $\sigma = 2\pi/3$ has much smaller stability limits than the region below. Above $\sigma \cong 0.8\pi$ the phase plane picture begins to break up into many small islands, which are not useful. The region near $\sigma = 0$ has a very complicated structure which is not useful on economic grounds.

Coupling resonances, where both m_x and m_y are different from zero, introduce thresholds. Near such a resonance, there is a certain radial amplitude below which the two-dimensional motion is stable, but above which the vertical amplitude increases exponentially.[8] For example, near the resonance $\sigma_x + 2\sigma_y = 2\pi$, the x-threshold for y-growth is

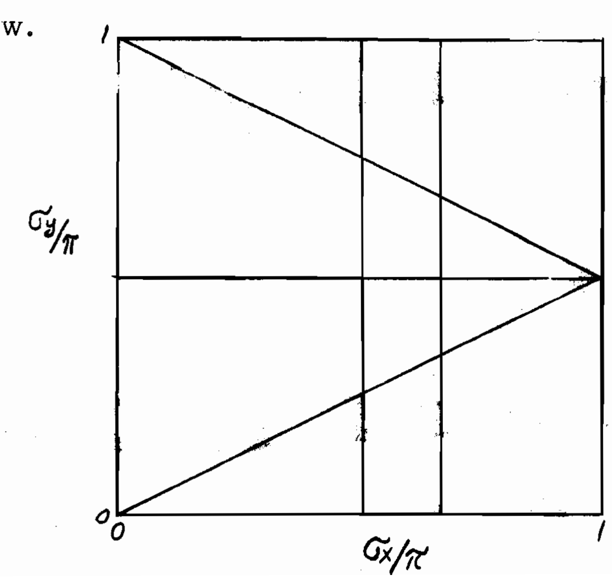
$$A_{x_o} = \frac{1}{2B}\left|\left((2\,\nu_y)^2 - (\nu_x + N)^2\right)\right| \qquad (6.4)$$

and the y-growth rate is

$$\mu \cong \frac{B}{\nu_y\,(\nu_x + N)^2}\sqrt{A_x^2 - A_{x_o}^2}\;, \qquad (6.5)$$

where B is the same coefficient as in the $\sigma_x = 2\pi/3$ case because the field satisfies Maxwell's equations.

The essential resonances of greatest importance in the first stability zone are sketched below.



The working point (σ_x, σ_y) of an accelerator must be chosen far enough from any given resonance so that the stable phase volume available is sufficient for the intensity desired, since the density in phase space is determined by the injector, according to Liouville's theorem.

We remark in closing that betatron oscillations damp adiabatically as $B^{-\frac{1}{2}}$. The forced oscillation of the equilibrium orbit increases proportionally with radius, as do the coordinate dimensions of the betatron oscillation phase space, while the momentum dimensions, which are the sines of the angles which a trajectory makes with the equilibrium orbit, do not change with radius. Thus the region of betatron oscillation phase space which is filled with particles shrinks relative to the stability limits during acceleration. Liouville's theorem is not dissatisfied because betatron oscillation phase space is not the total phase space.

VII. MISALIGNMENTS AND FIELD ERRORS

FFAG accelerators have the same integral and half-integral resonances as AG synchrotrons. In addition, there are effects due to the non-linear forces which are as yet largely unexplored.

Just as in the AG synchrotron, there is an equivalence between field errors and misalignments. Moving a magnet a distance Δr produces a change in field

$$\Delta B = \frac{k B}{r} \Delta r \tag{7.1}$$

in a radial sector accelerator. In a spiral sector accelerator k is roughly replaced by K for some purposes, but not for others. As k and K increase, misalignments increase in importance relative to field errors which arise from inhomogeneities in the magnets themselves.

One may make a distinction between static and dynamic effects in the theory of misalignments. The static effects, where ν and the field errors are taken as time-independent, has been worked out quite thoroughly in the linear approximation. Dynamic effects, where ν passes through a resonance in a finite number of revolutions, have received very little attention. One might well hope that such effects can be eliminated for all practical purposes by trimming the D. C. fields of FFAG accelerators. It would be of interest to know how far such corrections should be carried. Note also from (7. 1) that the field error due to a misalignment depends on radius, so that dynamic effects can result.

Adapting the integral resonance calculations from AG synchrotrons,[4] one can say that with 98% probability the displacement of the equilibrium orbit will be less than

$$P = \frac{2\nu}{|\sin \pi \nu|} \quad \frac{k}{\nu M^{1/2}} \; F^{1/2} \tag{7.2}$$

times the rms displacement of individual magnets. M is the total number of magnets and F is the form factor (the square of the ratio of the amplitude of the AG oscillation to the amplitude of a sine wave of the same frequency). For given σ, F is constant, M varies as N, ν varies as N and k varies as N^2 (from (5. 2)), so that P varies as $N^{1/2}$ or $k^{1/4}$. Even in a spiral sector accelerator, k should not be replaced by K, because the equilibrium orbit is relatively independent of spiraling. P, the factor by which magnet displacements multiply to produce orbit displacements, is, as in AG synchrotrons, of order 50.

At half-integral resonance, there is a frequency shift

$$\Delta \nu = \frac{k}{\nu M^{1/2}} \left\langle \frac{\Delta k}{k} \right\rangle_{rms} \tag{7.3}$$

and a stopband of width

$$\delta \nu = \frac{2k}{\nu M^{1/2}} \left\langle \frac{\Delta k}{k} \right\rangle_{rms} \tag{7.4}$$

centered on the resonance is opened up. K should replace k for spiral sector accelerators in (7.3) and (7.4). An additional beating of the oscillations is engendered, of amount

$$G = 1 + \sum_p \frac{(\delta \nu)_p}{2(\nu - p/2)} , \tag{7.5}$$

where $(\delta \nu)_p$ is the width of the stopband at p/2. If ν is halfway between two resonances, the beating is enlarged by a factor of perhaps 1.2. The coefficient of $\left\langle \frac{\Delta k}{k} \right\rangle_{rms}$ is of order unity, so that $\left\langle \frac{\Delta k}{k} \right\rangle_{rms}$ must be held to values of order 0.05 in practice.

At linear coupling resonances ($\nu_x + \nu_y = m$) a stopband of width

$$\delta \nu = \frac{4k}{\nu M^{1/2}} \langle \theta \rangle_{rms} \tag{7.6}$$

is engendered, where $\langle \theta \rangle_{rms}$ is the rms tilt of the magnetic median plane with respect to the geometric median plane. As in AG synchrotrons, this angle must be controlled to about 10^{-3} radians.

VIII. RF

There is remarkably little interaction between the design of the rest of an FFAG accelerator and that of the rf accelerating system. The radial gradient k enters the synchrotron oscillation equations because of the scaling of momentum with radius given by (2.4). A short calculation gives for the change of revolution frequency with energy,

$$\frac{E}{f}\ \frac{df}{dE} = \frac{(k+1) - \gamma^2}{(k+1)(\gamma^2 - 1)} \tag{8.1}$$

From (8.1), the transition energy E_t, where the phase oscillation frequency is zero, is

$$E_t = (k+1)^{1/2} E_o \tag{8.2}$$

where E_o is the rest energy of the particle. For the values of k found practical for betatron oscillations, the transition energy is usually of the order of several Bev. Acceleration over the transition energy is not too difficult, but stacking close to the transition energy is not at all easy. The rf voltage must be reduced drastically in the neighborhood of E_t. In the 15 Bev two-way accelerator proposed by MURA in 1958, it was found to be completely impractical to stack within $\pm$ 200 Mev of the transition energy. The transition energy is avoided completely in an accelerator with $k < 0$. But we saw above that two-way and spiral sector accelerators cannot be built with large negative k. One of the most interesting possibilities inherent in non-scaling accelerators is the elimination of the transition energy. One can envisage ways of doing this, but they have not been looked at in any detail.

There are also problems of rf knockout, where betatron oscillations can be induced in the stacked beam by the rf accelerating voltage bringing up further bunches to the stack. These problems can be surmounted in practice by reducing the rf voltage as the rf frequency passes through harmful resonant values.

A problem which is trivial in principle but difficult in practice is that of providing enough straight section room for the rf cavities. The return windings connecting the back with the forward wound magnet currents require a great deal of room at the sides of the magnets. In the 40 Mev electron accelerator, this problem was met by providing shelves above the pole face, so that the magnet cross section perpendicular to a radius is a T with the pole face the cross bar.

If it is desired to operate on the h^{th} harmonic of the revolution frequency, it is very desirable that N be a multiple of h, so that the total voltage per turn can be split up among h cavities which are placed in straight sections. This is more of a problem in intra-group communication than in accelerator physics.

IX. CONCLUSIONS

Different considerations are of prime importance in radial and spiral sector accelerators, so that we shall discuss them separately.

In radial sector accelerators the overriding consideration is magnet weight. The design of radial sector machines is pushed hard in the direction of high k by three factors: the magnet weight varies as $1/k^2$ from (3.6), the scalloping contribution to the net guide field increases with k, from (4.6), and the radial aperture required for a given momentum spread decreases rapidly as k increases, from (2.4). As k increases and the scalloping contribution becomes larger, the average field (- k) and AG ($\frac{1}{2} k^2 G^2$) terms of ν_y^2 in (5. 2) approach each other in magnitude. The vertical focusing is more and more due only to the Thomas focusing term, as in the two-way accelerator. Thus, if a large radial sector accelerator is to be built, it might as well be a two-way accelerator. Its focusing will approximate two-way focusing, so that the second beam costs nothing.

We see from (5. 2) or (5. 3) that σ_x varies as k/N^2. If we increase k without increasing N, we soon run into the stability boundary at $\sigma = \pi$. If k is increased keeping k/N^2 roughly constant, the misalignment tolerances become more serious. At k = 250 the rms misalignments allowable are of order 0.002 inches, so that this appears to be a practical upper limit. A note of caution must be given. k = 250 gives a transition energy very close to 15 Bev, which would make stacking in a 15 Bev accelerator uncomfortable. The 1958 MURA proposal chose a $k \cong 210$ for these reasons.

A rough set of parameters for a radial sector accelerator might well be the following:

$k \cong 200$	$\sigma_x \cong \frac{3}{4}\pi$
$N \cong 70$	$\nu_x \cong 27$
$C \cong 6$	$\sigma_y \cong \pi/7$
	$\nu_y \cong 5$

There are two difficulties with this design. First, σ_x is so large that the stability limits are very small. Economic considerations force the radial sector accelerator to a working point where non-linear difficulties are just as severe as in the inherently more non-linear spiral sector accelerator. These tight stability limits plague all multiple-turn injection and particle handling schemes.

The second difficulty is that ν_y is so low. This makes misalignment tolerances quite severe for this dimension and in addition makes the space charge limit small. ν_y can be improved by spiraling, as one can see from (5.3), at the cost of increased magnet complication.

Let us turn now to the spiral sector accelerator. Here the non-linear forces are at least as important a factor as magnet weight, because the circumference factor and magnet weight are considerably smaller than in the radial machine. The non-linear forces in the spiral sector accelerator are almost completely determined by K. The value of K is fixed fairly definitely by the linear vertical motion. For example, for $N \cong 30$, $F \cong 1$ σ_y moves through the whole first stability zone from 0 to π as K changes from about 220 to about 300. K is much larger than k and (2.1) shows that the term x^n in an expansion of B_z has a term $K^n/n!$ in its coefficient. Since the non-linear forces are large, one would not think of trying to operate a spiral sector accelerator with $\sigma_x > 2\pi/3$, as one does in a radial sector accelerator.

If k is increased (to decrease magnet weight and radial aperture), N must increase as $\sqrt{k}$ to keep $\sigma_x < 2\pi/3$. K must vary as N^2 to keep σ_y from decreasing, so that k and K vary together. If N increases more rapidly than $\sqrt{k}$, so that σ_x decreases, K must increase faster than k to keep σ_y from decreasing.

A rough set of parameters for a spiral sector accelerator might well be:

k	$\cong$	50	σ_x	$\cong$	0.55π
N	$\cong$	30	ν_x	$\cong$	8
K	$\cong$	250	σ_y	$\cong$	0.4π
C	$\cong$	2	ν_y	$\cong$	6

In such a design, non-linear stability limits are still somewhat troublesome for multi-turn injection and similar particle handling. There is a possible method of avoiding these troubles in a non-scaling accelerator.* The flutter F, spiral angle K and gradient k can be made functions of radius, so that σ_x and σ_y are still independent of energy, but at the injection radius, where F is larger and K smaller (so that the accelerator is more like a radial sector accelerator), the stability limits are larger. The adiabatic damping is then used to keep particles within the decreasing stability limits as energy is increased, when the flutter decreases and K increases.

This "transition" accelerator has the advantage that its circumference factor, while large at injection, is small at high energy and high field,

*This suggestion has been made by many people in MURA and is not put forward here as being in any way original with the writer.

where circumference factor is expensive. It also raises the transition energy and might keep the whole accelerator below it. It appears feasible to construct magnets with flutter and spiral angle varying with radius.

The main factor which has not been explored in this machine is the influence of small stability limits on the stacked beam, where the many-particle effects not discussed here become important. The MURA 40 Mev electron accelerator should provide data on this subject.

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