CONDITIONAL SYMMETRIES AND CONDITIONAL INTEGRABILITY FOR NONLINEAR SYSTEMS

Pavel Winternitz
Centre de recherches mathématiques, Université de Montréal
C.P. 6128-A, Montréal, Québec, H3C 3J7, Canada

ABSTRACT.

Two methods of obtaining particular solutions of nonlinear differential equations are reviewed. The first makes use of "conditional symmetries", i.e. local Lie point transformations leaving a subset of solutions of an equation invariant. The second consists of adding further equations to the given one, so that the equation, together with the supplementary conditions, figures as compatibility conditions for an algebra of linear operators.

1. INTRODUCTION.

Virtually all of the fundamental equations of physics are nonlinear, as are most of the differential systems describing specific physical phenomena. It is hence both of conceptual and practical interest to develop techniques for solving nonlinear differential equations.

In this presentation we shall concentrate on methods of obtaining exact analytical solutions, rather than numerical ones. The motivation is that analytical solutions, even if they are particular, rather than general ones, very often provide a lot of insight into the qualitative behavior of a system. Analytical solutions can also serve as the starting point for further perturbative calculations. They often turn out to be particularly stable and may provide asymptotic formulas for solutions developing from wide classes of initial conditions.

Two different techniques are extremely useful in constructing solutions of nonlinear partial differential equations (PDE's). One is the method of *symmetry reduction.* It consists of a systematic application of Lie group theory to obtain solutions that are invariant under some subgroup of the symmetry group of the considered system. The invariance conditions are used to reduce the system of PDE's to a lower dimensional system, which may be easier to solve. The method is applicable if the differential system under consideration has a nontrivial symmetry group in the first place. The price we pay for reducing the number of independent variables is that initial conditions, or boundary conditions, can only be imposed on specific types of surfaces.

The method of symmetry reduction goes back to S. Lie and is reviewed in numerous books and articles [1]–[6]. A recent burst of activity in this area is due, among other reasons, to the possibility of performing otherwise tedious calculations on computers, using various symbolic languages [7], [8].

The other systematic technique for solving PDE's analytically is that of the spectral transform and its generalizations [9]–[11]. It is based on the possibility of finding a Lax pair, i.e. a pair of linear differential equations, for which the original nonlinear equations are compatibility conditions. The method, when applicable, provides large classes of solutions, among them solitons, multisolitons, periodic and quasiperiodic solutions. Nonlinear equations for which the spectral transform method is applicable are called "completely integrable", and they are relatively rare.

Our purpose here is to review some recent results extending the applicability of both of the above techniques for solving nonlinear PDE's.

2. CONDITIONAL SYMMETRIES AND THE DIMENSIONAL REDUCTION OF PARTIAL DIFFERENTIAL EQUATIONS.

The problem that we are addressing can be formulated as follows. We are given a system of partial differential equations (PDE's)

$$\begin{aligned} &\Delta_a(\vec{x},\vec{u},\vec{u}_{x_\mu},\vec{u}_{x_\mu x_\nu},...)=0 \\ &a=1,...,N,\ \vec{x}\in\mathbb{R}^p,\quad \vec{u}\in\mathbb{R}^q \end{aligned}\tag{2.1}$$

of any order M for q functions of p variables. How can we reduce it to a system of equations involving $k<p$ independent variables? We wish to do this by finding functions U_i and z_α such that the substitution

$$\begin{aligned} &u_i(\vec{x})=U_i(\vec{x},w_1(z_1,...z_k),...w_q(z_1,...z_k)) \\ &z_j=z_j(\vec{x}),\quad i=1,...,q \end{aligned}\tag{2.2}$$

will reduce the system (2.1) to a new system of differential equations of the form

$$\begin{aligned} &\tilde{\Delta}_a(\vec{z},\vec{w},\vec{w}_{z_\alpha},\vec{w}_{z_\alpha z_\beta},...)=0 \\ &a=1,...,N,\quad \vec{z}\in\mathbb{R}^k,\ \vec{w}\in\mathbb{R}^q,\ 0\le k<p. \end{aligned}\tag{2.3}$$

In eq. (2.2) U_i and z_j are known functions; the dimensional reduction is in the fact that eq. (2.3) involves only $w_1,...,w_q$ and $z_1,...z_k$ with $k<p$ ($k=0$ corresponds to an algebraic system, $k=1$ to a system of ordinary differential equations (ODE's)).

For simplicity, we restrict ourselves to the case of one PDE, i.e. $N=1$ in (2.1) and (2.3).

The *classical answer* to the above question is provided by the method of symmetry reduction. The method consists of the following steps:

(1) Find the Lie group G of local point transformations leaving the system (2.1) invariant.

(2) Classify all subgroups $G_0 \subseteq G$ having generic orbits of codimension $k+1$ in the space $X \otimes U$ of independent and dependent variables, into conjugacy classes under the action of G and choose a representative of each class.

(3) For each representative subgroup find $k+1$ functionally independent invariants and choose a basis of invariants satisfying

$$I_i = \xi_i(x_1, ...x_p),\ i = 1, ..., k < p,\ I_{k+1} = F(x_1, ...x_p, u),\ \frac{\partial F}{\partial u} \neq 0. \qquad (2.4)$$

(4) Consider I_{k+1} as a function of $\xi_1, ...\xi_k$, solve for n and obtain an expression of the form (2.2), where the functions z_i are identified with the invariants ξ_i and I_{k+1} with $w(\xi_1, ...\xi_k)$.

(5) Substitute u into (2.1). The G invariance of (2.1) and the completeness of the set of G_0 invariants guarantees that (2.3) will involve only the invariants F, ξ_i and derivatives of F with respect to ξ_i.

The described procedure is entirely algorithmic; a classification of subgroups provides a classification of different reductions. The question is whether it provides all possible reductions.

The answer to the above question is, in general *no* and was provided by counterexample in a recent article by Clarkson and Kruskal [**12**]. The article was devoted to a reduction of the Boussinesq equation

$$u_{tt} + uu_{xx} + (u_x)^2 + u_{xxxx} = 0 \qquad (2.5)$$

to an ODE. Their approach was entirely straightforward, making the substitution (2.2) with $q = 1, k = 1, \vec{x} = (x, t)$ in (2.5) and requiring, by "brute force" that $w(z)$ should satisfy an ODE. In this manner they obtained known reductions, due to translational, or dilational invariance, but also several new reductions, not related to the symmetry group of the Boussinesq equation. The authors included a sentence to the effect: "We hope that a group theoretic explanation of the method will be possible in due course".

Such an explanation was indeed subsequently provided [**13**], moreover the explanation also yields an algorithm for performing the reduction. The framework for the group theoretical explanation has already existed for some time, namely the "nonclassical method" of Bluman and Cole [**14**].

The basic idea is to make use of "conditional symmetries " of an equation, that is transformations that only transform a subset of solutions into solutions, but take other solutions out of the solution set. The subset of solutions left invariant is characterized by a supplementary condition, i.e. an equation added to the one that we wish to solve. The point is to choose this supplementary condition in a way that will be as nonrestrictive as possible.

This is best done in infinitesimal language. Instead of looking for the symmetry group G of an equation, one looks for its Lie algebra. For a scalar equation with two independent variables, such as the Boussinesq equation (2.5), the Lie algebra L is realized by vector fields of the form

$$\hat{v} = \xi(x, t, u)\partial_x + \tau(x, t, u)\partial_t + \phi(x, t, u)\partial_u. \qquad (2.6)$$

The functions ξ, τ and ϕ must satisfy a set of determining equations, obtained by requiring that the n-th prolongation $pr^{(n)}v$ of $\hat{v}$ should annihilate the equation on its solution set:

$$pr^{(n)}\hat{v}\Delta^{(n)}\Big|_{\Delta^{(n)}=0} = 0. \tag{2.7}$$

Here $\Delta^{(n)} = 0$ is the considered differential equation and the prolongation of $\hat{v}$ is a differential operator acting on x, t, u and derivatives of u[1],[8]. Eq.(2.7) amounts to a system of PDE's for ξ, τ and ϕ. These equations are linear even if the equation $\Delta^{(n)} = 0$ is nonlinear.

To find "conditional symmetries", we add a first order equation, adapted to the vector field (2.6), namely

$$\Delta^{(1)} = \xi u_x + \tau u_t - \phi = 0, \tag{2.8}$$

which is automatically annihilated on its solution set by $pr^{(1)}\hat{v}$.

The "conditional symmetries" will now be vector fields of the form (1.6) satisfying

$$pr^{(n)}\hat{v}\cdot\Delta^{(n)}\Big|_{\Delta^{(n)}=0,\Delta^{(1)}=0} = 0, \quad pr^{(1)}\hat{v}\cdot\Delta^{(1)}\Big|_{\Delta^{(n)}=0,\Delta^{(1)}=0} = 0. \tag{2.9}$$

From (2.9) we obtain a system of determining equations for the coefficients ξ, τ and ϕ. Contrary to the case of ordinary symmetries, the determining equations will be nonlinear, since the same unknown functions ξ, τ and ϕ figure in the vector field and in the supplementary condition.

The set of vector fields satisfying (2.9) is hence larger than the set satisfying (2.7). It should be emphasized that these conditional symmetries do not form a vector space, still less a Lie algebra, since each vector field $\hat{v}$ has its own supplementary condition (2.8).

Each symmetry operator, be it an ordinary, or a conditional one, provides a reduction of the PDE to an ODE.

The idea of conditional symmetries, in different contexts and with different names, has been introduced by several authors [13]-[17]. As formulated here, conditional symmetries for a given equation $\Delta^{(n)} = 0$, are actually ordinary symmetries for a system of equations: $\Delta^{(n)} = 0, \Delta^{(1)} = 0$. Hence existing software can be used to construct these symmetries [8].

Let us now turn to the example of the Boussinesq equation (2.5). We restrict ourselves here to the case $\tau \neq 0$ in (2.6) and (2.8). With no loss of generality, we can then put $\tau = 1$. Using the MACSYMA program [8], we obtain 14 determining equations. They are coefficients of different terms of the type $u_x^k u_{xx}^\ell u_{xxx}^n, k, \ell, n \in \mathbb{Z}^+$ since u_t is eliminated using eq. (2.8) and u_{xxxx} using the Boussinesq equation. Solving the determining equations, we obtain

$$\begin{aligned} &\tau(x,t,u) = 1, \quad \xi(x,t,u) = \alpha(t)x + \beta(t) \\ &\phi(x,t,u) = -[2\alpha(t)u + 2\alpha(\alpha' + 2\alpha^2)x^2 + 2(\alpha\beta' + \alpha'\beta + 4\alpha^2\beta) + 2\beta(\beta' + 2\alpha\beta)] \end{aligned} \tag{2.10}$$

where

$$\alpha'' + 2\alpha\alpha' - 4\alpha^3 = 0, \quad \beta'' + 2\alpha\beta' - 4\alpha^2\beta = 0. \tag{2.11}$$

The invariants of the one-dimensional group corresponding to the obtained vector field are found by solving the characteristic system associated with $\hat{v}$.

Denoting the invariants w and z, viewing w as a function of z and solving for u, we obtain the reduction formulas

$$u(x,t) = K^2(t)w(z) - (\alpha x + \beta)^2$$
$$z(x,t) = xK(t) - \int_0^t \beta(s)K(s)ds, \quad K(t) = \exp\,[-\int_0^t \alpha(s)ds]. \tag{2.12}$$

Substituting (2.12) into the Boussinesq equation, we obtain the ODE

$$w'''' + ww'' + w'^2 + (Az+B)w' + 2Aw = 2(Az+B)^2, \tag{2.13}$$

where (2.11) implies that

$$A = \frac{\alpha^2 - \alpha'}{K^4}, \quad B = \frac{\alpha\beta - \beta'}{K^3} + \frac{\alpha^2 - \alpha'}{K^4}\int_0^t \beta(s)K(s)ds \tag{2.14}$$

are constants. To proceed further we must solve eq.(2.11) for $\alpha(t)$ and $\beta(t)$. We have

$$\alpha = \frac{H'}{2H}, \quad H'^2 = h_0H^3 + h_1, \; \beta = c_1\frac{H'}{H} + c_2\frac{H'}{H}\int_0^t \frac{H(s)}{[H'(s)]}ds, \tag{2.15}$$

where h_0, h_1, c_1 and c_2 are constants.

(1) $h_0 = h_1 = 0$. Then $\alpha = 0, \beta = \beta_0 + \beta_1 t$, $\beta_\mu =$ const. For $\beta_1 = 0$ we obtain travelling waves, due to (ordinary) translational invariance. For $\beta_1 \neq 0$ we can use translational invariance to set $\beta_0 = 0$. We obtain a new reduction

$$z = x - \frac{1}{2}t^2, \quad u = w(z) - t^2, \quad w''' + ww' - w = 2z + c_1 \tag{2.16}$$

and w is expressed in terms of the Painlevé transcendent P_{II}.

(2) $h_0 \neq 0$, $h_1 = 0$. We obtain $\alpha = -1/t$, $\beta = \beta_1 t^4 + \beta_2/t$. Using ordinary symmetries (translations and dilations), we set $(\beta_1, \beta_2) = (1,0)$ or $(0,0)$. The reduction is

$$z = xt - \frac{1}{6}\beta_1 t^6, \; u = w(z)t^2 - \left(\frac{x}{t} - \beta_1 t^4\right)^2$$
$$w''' + ww' - 5\beta_1 w = 50\beta_1 z + c_1, c_1 = \text{const.} \tag{2.17}$$

Depending on the values of β_1 and c_1 we obtain solutions in terms of elliptic functions, or the Painlevé transcendents P_I or P_{II}.

(3) $h_0 = 0, h_1 \neq 0$. Simplifying by translations and dilations, we obtain $\alpha = 1/2t, \beta = \beta_1 t, \beta_1 = 1$, or $\beta_1 = 0$. For $\beta_1 = 0$ we obtain a known reduction (due to dilational invariance). For $\beta_1 = 1$ we obtain a new reduction

$$z = \frac{x}{\sqrt{t}} - \frac{2}{3}t^{3/2}, \quad u = \frac{1}{t}w(z) - \left(\frac{x}{2t} + t\right)^2$$
$$w'''' + ww'' + w'^2 + \frac{3}{4}zw' + \frac{3}{2}w = \frac{9}{8}z^2 \tag{2.18}$$

(4) $h_0 \neq 0, h_1 \neq 0$. This is the generic case and we obtain

$$\alpha = \frac{1}{2}\frac{\mathcal{P}'}{\mathcal{P}}, \; \beta = \frac{\beta_1}{4}\frac{\mathcal{P}'}{\mathcal{P}} + \beta_2 \frac{\mathcal{P}'}{\mathcal{P}} \int \frac{\mathcal{P}(t)dt}{[\mathcal{P}'(t)]^2}$$
$$(\mathcal{P}')^2 = 4\mathcal{P}^3 - g_3 \tag{2.19}$$

where $\mathcal{P}(t)$ is the Weierstrass elliptic function. The reduction is new and can be written as

$$z(x,t) = x[\mathcal{P}(t)]^{-1/2} + \frac{1}{3}\beta_2 g_3^{-1}[\mathcal{P}(t)]^{-1/2} \int_0^t \mathcal{P}(s)ds$$
$$u(x,t) = w(z)\mathcal{P}^{-1} - \frac{1}{4}\left[\frac{\mathcal{P}'}{\mathcal{P}}\right]^2 \left(x + \beta_2 \int_0^t \frac{\mathcal{P}(s)ds}{[\mathcal{P}'(s)]^2}\right)^2, \tag{2.20a}$$

where $w(z)$ satisfies an equation having the Painlevé property

$$w'''' + ww'' + w'^2 - \frac{3}{4}g_3 w' - \frac{3}{2}g_3 w = \frac{9}{8}g_3^2 z^2. \tag{2.20b}$$

At this stage we can draw some conclusions.

(1) Conditional symmetries for the Boussinesq equation are very important; they give more reductions than ordinary symmetries.

(2) Conditional symmetries for a PDE are ordinary symmetries for the PDE plus the supplementary conditions. Hence, existing software can be used to derive the determining equations for the conditional symmetries.

(3) For the Boussinesq equation, the conditional and ordinary symmetries, taken together, give all reductions of the type (2.2). The final result coincides with that obtained by Clarkson and Kruskal using their direct method [**12**].

(4) The procedure we advocate is to first find all ordinary symmetries and then to use them to simplify the equations for the conditional symmetries.

We mention that both the direct procedure and the conditional symmetry one have recently been applied to the Kadomtsev-Petviashvili equation [**18**]. New reductions to PDE's in two variables and to ODE's were obtained, yielding new solutions. The two different methods again give the same results.

Open questions that remain are:

(1) Can one tell, without going through all the calculations, when does an equation or system of equations, allow conditional symmetries[2]. For some equations, such as the Korteweq-deVries equation, the modified KdV equation, the Burgers equation and quite a few others, all conditional symmetries coincide with ordinary ones.

(2) Is our conjecture, that conditional symmetries, together with ordinary ones, provide all possible reductions of a PDE, correct?

3. CONDITIONAL INTEGRABILITY.

We shall, somewhat loosely, call a nonlinear PDE "integrable" if large classes of solutions of this equation can be obtained, using linear techniques. Recognising integrable equations has always been a problem, but there are some heuristic criteria. Typically, integrable equations have the Painlevé property. For a PDE we say that it has the Painlevé property if all of its solutions are single valued in the neighbourhood of any noncharacteristic singularity surface [19]. An ODE has the Painlevé property if its solutions have no movable critical points, i.e. no singularities depending on the initial conditions, other than poles [20], [21]. Another indication of integrability is the existence of three-soliton solutions. Finally, integrable equations in more than two space-time dimensions tend to have infinite-dimensional Lie point symmetry groups and their Lie algebras have a Kac-Moody-Virasoro structure [6], [22], [23].

The term "conditional integrability" has been coined to describe a situation when a PDE is not integrable, but becomes integrable if a further condition, for instance a further PDE is added [24].

As an example, let us consider the PDE

$$w_{xxxy} + 3w_{xy}w_x + 3w_y w_{xx} + 2w_{yt} - 3w_{xz} = 0. \tag{3.1}$$

This is the second equation of the Kadomtsev-Petviashvili hierarchy, as introduced by Jimbo and Miwa [25]. If this equation is taken on its own, then it does not pass the Painlevé test, its solitary wave solutions do not combine into three-soliton solutions and its symmetry algebra, while infinite dimensional, is not of the Kac-Moody-Virasoro type [24].

Now let us impose a further equation on $w(z, y, z, t)$, namely the potential Kadomtsev-Petviashvili (PKP) equation itself (for z fixed):

$$w_{xxxx} + 6w_x w_{xx} + 3w_{yy} - 4w_{xt} = 0. \tag{3.2}$$

Simultaneous solutions of (3.1) and (3.2) do pass the Painlevé test and the two equations together determine three soliton solutions. The Lie point symmetry group of the pair of equations (3.1) and (3.2), has recently been calculated and analyzed [26]. A general element of its Lie algebra can be written as

$$\hat{v} = Z(f) + T(g) + Y(h) + X(k) + W(G), \tag{3.3}$$

where $f(z), g(z), h(z), k(z)$ and $G(z,t)$ are arbitrary C^∞ functions of the indicated variables.

We have

$$\begin{aligned}
&W(G) = G(z,t)\partial_w, \quad X(k) = k\partial_x - yk'\partial_w, \\
&Y(h) = h\partial_y + \frac{3}{4}th'\partial_x - \frac{1}{4}(2xh' + 3tyh'')\partial_w \\
&T(g) = g\partial_t + \frac{1}{32}(16yg' + 9t^2g''|\partial_x + \frac{3}{4}tg'\partial_y - \frac{1}{32}[4(3tx + 2y^2)g'' + 9yt^2g''']\partial_w \\
&Z(f) = f(z)\partial_z + \frac{1}{4}[xf' + \frac{3}{2}ytf'' + \frac{9}{32}t^3f''']\partial_x + \frac{1}{2}(yf' + \frac{9}{16}t^2f'')\partial_y \\
&\quad + \frac{3}{4}tf'\partial_t - \frac{1}{4}[wf' + xyf'' + \frac{3}{4}t(y^2 + \frac{3}{4}tx)f''' + \frac{9}{32}yt^3f'''']\partial_w.
\end{aligned} \tag{3.4}$$

This Lie algebra, the corresponding group transformations and their applications are analyzed in detail in Ref. 26. Here let us just note that T, X, Y and W form a subalgebra of a Kac-Moody algebra, (without a central extension). The vector fields $Z(f)$ form a Virasoro algebra which is also centerless.

To sum up, the pair of equations (3.1) and (3.2) taken together, as opposed to eq.(3.1) alone, passes all the tests indicating integrability.

This example of conditional integrability gives rise to several questions, such as: given a nonlinear PDE, how does one recognize that it is conditionally integrable? How does one determine the conditions to be imposed? How does one proceed to find solutions and what sort of solution set can one expect to obtain? How does one generate families of conditionally integrable equations?

Some preliminary results in this direction have been obtained [27] and point towards an algebra of linear operators. Indeed, consider the PKP equation (3.2) for a function $w(x, y, t, z)$ and 3 linear operators L_1, L_2 and L_3 governing the evolution of an auxiliary function $\psi(x, y, z, t)$, in $(x, y), (t, x, y)$ and (z, x, y), respectively. The commutation relations $[L_1, L_2] = 0$ and $[L_1, L_3] = 0$ are respectively equivalent (on solutions of the linear equations $L_i\psi = 0$) to the PKP equation and a new equation in (x, y, z) integrable in the usual sense. The commutation relation

$$[L_2, L_3] = \alpha L_2 + \beta L_2 + \gamma L_3 + \Delta_{123} = 0$$

for $L_2\psi = 0, L_3\psi = 0$, where α, β and γ are constants, implies a new equation, $\Delta_{123} = 0$, in all four variables (x, y, z, t) under the condition that we also put $L_1\psi = 0$. In the case under consideration $\Delta_{123} = 0$ is equivalent to the Jimbo-Miwa equation (3.1). The implication is that we can hope to obtain, by linear techniques, families of solutions that depend on functions of two variables, rather than three, as would be required to satisfy general Cauchy conditions.

4. ACKNOWLEDGEMENTS

The author's research is partially supported by research grants from NSERC of Canada and FCAR du Gouvernement du Québec.

REFERENCES

1. P.J. Olver, "Applications of Lie Groups to Differential Equations," Springer, New York, 1986.
2. G.W. Bluman and J.D. Cole, "Similarity Methods for Differential Equations," Springer, New York, 1974.
3. L.V. Ovsiannikov, "Group Analysis of Differential Equations," Academic, New York, 1982.
4. N.H. Ibragimov, "Transformation Groups Applied to Mathematical Physics," Reidel, Boston, 1985.
5. C. Rogers and W. F. Ames, "Nonlinear Boundary Value Problems in Science and Engineering," Academic, New York, 1989.
6. P. Winternitz, in "Partially Integrable Evolution Equations in Physics," p. 515-567, Kluwer Dordrecht, 1990, Editors R. Conte and N. Boccara.
7. F. Schwarz, Computing, 34, 91, (1985).
8. B. Champagne and P. Winternitz, Preprint CRM-1278, Montréal ,1985 ; B. Champagne, W. Hereman and P. Winternitz, CRM-1689, Montréal, 1990.
9. M.J. Ablowitz and H. Segur, "Solitons and the Inverse Scattering Transform," SIAM, Philadelphia, 1981.
10. S. Novikov, S.V. Manakov, L.P. Pitaevskii and V.E. Zakharov, "Theory of Solitons. The Inverse Method," Plenum, New York, (1984).

11. F. Calogero and A. Degasperis, "Spectral Transform and Solitons," North Holland, Amsterdam, (1982).
12. P.A. Clarkson and M.D. Kruskal, J. Math. Phys., **30**, 2201 (1989).
13. D. Levi and P. Winternitz, J. Phys. A., **22**, 2915 (1989).
14. G.W. Bluman and J.D. Cole, J. Math. Mech., **18**, 1025 (1969).
15. A.S. Fokas and R.L. Anderson, Lett. Math. Phys., **3**, 117 (1979).
16. V.I. Fushchich and N.I. Serov, in "Simmetriya i resheniya uravnenii matematicheskoi fiziki," Kiev, 1989.
17. P.J. Olver and Ph. Rosenau, SIAM J. Appl. Math., **47**, 263 (1987).
18. P.A. Clarkson and P. Winternitz, Preprint CRM-1701, Montréal, 1990.
19. J. Weiss, M. Tabor and G. Carnavale, J. Math. Phys. (1983), p. 522, **24**, 522 (1983).
20. M.J. Ablowitz, A. Ramani, and H. Segur, J. Math. Phys., **21**, 715 (1980).
21. D. Rand and P. Winternitz, Comp. Phys. Commun., **42**, 359 (1986).
22. D. David, N. Kamran, D. Levi and P. Winternitz, Phys. Rev. Lett., **55**, 2111 (1985).
23. L. Martina and P. Winternitz, Ann. Phys. (NY), **196**, 231 (1989).
24. B. Dorizzi, B. Grammaticos, A. Ramani and P. Winternitz, J. Math. Phys., **27**, 2848 (1987).
25. M. Jimbo and T. Miwa, Publ. Res. Inst. Math. Sci. Kyoto Univ., **19**, 943 (1983).
26. J. Rubin and P. Winternitz, J. Math. Phys., **31**, 2085 (1990).
27. A.V. Mikhailov and P. Winternitz, work in progress.