

**Studies of fractional D-branes
in the gauge/gravity correspondence**

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**Flavored Chern-Simons quivers
for M2-branes**

Cyril Closset



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PART ONE is based on the following two papers:

- [1] R. Argurio, F. Benini, M. Bertolini, C. Closset, and S. Cremonesi, “**Gauge/gravity duality and the interplay of various fractional branes**” *Phys. Rev.* **D78** (2008) 046008, arXiv:0804.4470 [hep-th].
- [2] F. Benini, M. Bertolini, C. Closset, and S. Cremonesi, “**The N=2 cascade revisited and the enhancon bearings**” *Phys. Rev.* **D79** (2009) 066012, arXiv:0811.2207. [hep-th].

PART TWO is based on

- [3] F. Benini, C. Closset, and S. Cremonesi, “**Chiral flavors and M2-branes at toric CY4 singularities,**” *JHEP* **02** (2010) 036, arXiv:0911.4127 [hep-th].

The following lectures notes are reproduced in the APPENDIX:

- [4] C. Closset, “**Toric geometry and local Calabi-Yau varieties: An introduction to toric geometry (for physicists),**” arXiv:0901.3695 [hep-th].

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Chapter 1

Introduction, overview and summary

STRING theory is our best candidate for a theory of gravity [5, 6] which is coherent with our quantum theoretical understanding of Physics [7, 8]. To the outsider, string theory in its present formulation(s) might look rather Baroque. Nevertheless, the student who has spent some time amongst its wonders cannot help but suspect that string theory contains some of the deepest clues to a more comprehensive understanding of our Universe.

On a more down-to-earth tone, string theory contains the more familiar framework of quantum field theory, which is the main tool of XXth century Physics. More surprisingly, it turns out that some quantum fields theories are equivalent to string theories. Hence, quite independently from the considerations about quantum gravity, string theory can be seen as just another tool to study interesting field theories in regimes which were hitherto out of reach.

1.1 Motivations

Our current understanding of string theory relies crucially on the concept of *duality*. A duality between two theories is an exact physical equivalence, which means that any physical observable is the same in both theories. The two sides of the duality might or might not look alike, depending in part on whether we are dealing with perturbative or non-perturbative dualities.

The work of this thesis takes place in the context of the *AdS/CFT* correspondence and in the more general framework known as the gauge/gravity correspondence. This correspondence is a very surprising non-perturbative duality, in which the two dual theories seem utterly different. The revolutionary idea behind the *AdS/CFT* proposal [9] is that a quantum field theory in 3 or 4 dimensions can be dual to a gravitational theory in a higher dimensional space (10 or 11 dimensions for the best understood cases stemming from string theory). This relationship is best understood when the quantum field theory is a conformal field theory (CFT). The dual gravity theory involves gravity, and its ultraviolet completion in the form of a string theory, in Anti-de-Sitter (AdS) space-time.

In perturbative string theory, the basic object is the fundamental *string*. To under-

stand non-perturbative dualities in string theory, it is important to study higher dimensional objects known as *branes*. Branes are the underlying theme of this work. They are fascinating dual objects, in the sense that they can be interpreted very differently in different regimes. The best understood instance of a brane is the Dirichlet brane, or D-brane. In the weakly coupled string theory, D-branes support open strings, whose degrees of freedom contain a vector field. In the case of multiple coincident D-branes, these open string excitations possess non-Abelian gauge symmetries, similarly to the mathematical structure of the Standard Model. D-branes are also massive objects, which deform space-time according to the laws of General Relativity. In the appropriate supergravity limit of small space-time curvatures, they correspond to some kind of extremal black holes (or black branes) called p-branes.

The D3-brane is particularly important. Its worldvolume spans 3+1 dimensions, so that we could live on it. Moreover, its corresponding extremal 3-brane solution is smooth (the dilaton is constant), so that the supergravity approximation does not break down at the horizon. These two properties make D3-branes very interesting to study. By taking a near horizon limit on D3-branes, we obtain a string theory “derivation” of an AdS_5/CFT_4 correspondence. This duality has given us new tools to compute in four dimensional theories: one can “simply” do computations in the dual gravity background to extract observables in a strongly coupled field theory. This makes AdS/CFT an important tool on the road to an analytic understanding of low energy QCD, which is one of the greatest problems in theoretical physics. AdS/CFT has already been successfully used to study (at least at a qualitative level) strange properties of the quark-gluon plasma produced at RHIC. In Part One of the thesis we will study some models which have some resemblance with zero temperature supersymmetric QCD, although we should warn the reader that our main object of study will not be these low energy theories *per se*, but rather their exotic embedding into so-called “cascading field theories”.

Another instance of AdS/CFT correspondence is the duality stemming from considering the near horizon limit on M2-branes. It had been less well studied until recently, when interesting progress were made towards giving a Lagrangian description of the low energy theory living on multiple M2-branes. In that case, the field theory is three dimensional. The AdS/CFT correspondence could then be used as a tool to study non-perturbative properties of three dimensional field theories, which arise as descriptions of many condensed matter systems. For instance, there has been a lot of activity recently in trying to mimic high temperature superconductors from such a gravity construction. In Part Two of the thesis, we will extend the number of examples of AdS_4/CFT_3 dualities which have an explicit string theory “derivation”.

1.2 The thesis: an overview

The work presented in this thesis takes place in the context of the gauge/gravity correspondence. My two main points (theses) are:

1. The supergravity solution of Bertolini *et al.* [10] and Polchinski [11] describing backreacted fractional D3-branes on the $\mathbb{C}^2/\mathbb{Z}_2$ orbifold has a dual field theory interpretation as describing a particular vacuum on the Coulomb branch of the $SU(N+M) \times SU(N)$ $\mathcal{N} = 2$ quiver theory. This vacuum is the analog of the

baryonic root on the Coulomb branch of $\mathcal{N} = 2$ SQCD. One can also write a corrected supergravity solution which realizes explicitly the correct “enhancement mechanism”.

2. M2-branes on any non-compact eight dimensional toric Calabi-Yau cone with complex codimension two singularity have a low energy field theory description in term of a Chern-Simons quiver theory coupled to flavors (fields in the fundamental representation of some gauge groups of the quiver). The field theory description crucially relies on including the diagonal monopole operators (and related non-perturbative effects) in the discussion of the chiral ring.

These two points are carefully explained in **Part One** and **Part Two** of this thesis. In particular, the main arguments are contained in Chapter 6 and Chapter 13, respectively. The rest of this long text can be considered as a detailed explanation of the concepts involved, necessary for the full understanding of the above two points. In the rest of this section we give a non-technical overview of the contents of this thesis. In the two following sections we summarize our main results in more details, and we point out possible directions for future research.

This work takes for granted some standard knowledge about four dimensional supersymmetric field theories, and about various non-perturbative effects such as Seiberg duality. We however introduce the tools we use the most in two Appendices: $\mathcal{N} = 1$ supersymmetric theories and Seiberg duality are reviewed in Appendix C, while Appendix E offers a brief account of the Seiberg-Witten approach to $\mathcal{N} = 2$ supersymmetric theories in four dimensions. Three dimensional supersymmetric field theories, which tend to be less familiar, will be introduced thoroughly in Part Two. We have also attempted whenever possible to introduce all the string theory tools we use, at least at a superficial level.

In **Chapter 2**, we give a first look from above at our field of study. Our usual environment will be type II string theory, but we will make healthy walks into eleven dimensional M-theory as well. Our main concern lies with supersymmetric branes, which hide rich supersymmetric field theories in their bosom. In type II string theory, these are the D-branes and the NS5-branes. Of course, in this introductory chapter we have to take a lot for granted, but the properties we will mention are common knowledge among string theorists of all stripe.

Chapter 3 is an important review chapter which is relevant for both Parts of the thesis. It deals with D3-branes at Calabi-Yau (CY) threefold singularities, but it should be clear that the properties we discuss are valid for any D-branes (in particular for D2-branes on the same threefolds). The main point of this chapter is that all the holomorphic properties of a given CY threefold are encoded in a so-called *quiver*. The quiver is a particular field theory describing the dynamics of open strings on the D-branes, when the D-branes sit on top of the singularity. Particular focus is put on the toric case. Toric geometry is reviewed in Appendix B.

After these preliminaries, we enter **Part One**, which consists of four chapters:

Chapter 4 provides an introduction to the *AdS/CFT* correspondence, oriented towards later use. We introduce the important Klebanov-Witten theory in that chapter. The discussion focuses on the case of *AdS₅/CFT₄*, which stems from the study of D3-branes, but we also make general comments which apply to the setups discussed in Part

Two of the thesis.

Chapter 5 introduces the general concept of gauge/gravity correspondence. The motivation to consider such an extension of the duality conjecture are explained. We focus on one such model, called the Klebanov-Strassler model. It gives a supergravity description to a theory similar in many respects to $\mathcal{N} = 1$ $SU(N)$ Super-Yang-Mills in four dimensions. We will not enter into the study of the many nice properties of this model (which has become a huge field of research). Instead, we focus on a particular property of this model called a *duality cascade*. We will give a first non-technical definition of that term in the next section.

Chapter 6 discusses duality cascades in $\mathcal{N} = 2$ supersymmetric setup. We study in detail the Coulomb branch of the quiver for D3-branes at the $\mathbb{C} \times \mathbb{C}^2/\mathbb{Z}_2$ singularity, by a careful study of the associated Seiberg-Witten curve. In particular, we prove the Point 1 stated earlier. That chapter is based on [2]. We also derive a whole family of supergravity solutions corresponding to more generic Coulomb branch vacua.

Chapter 7, based on [1], considers fractional branes of various kinds on a \mathbb{Z}_2 orbifold of the conifold. It studies from the supergravity perspective the renormalization group flow in the dual quiver, resulting from generic brane charge assignments. This provides support for the conjecture that the discussion of [2] also applies to generic $\mathcal{N} = 1$ quiver theories corresponding to geometries with complex codimension one singularities. We also discuss the infrared behavior of the field theory, at the bottom of the cascade. This concludes Part One.

Part Two of the thesis is concerned with M2-branes and with the related AdS_4/CFT_3 correspondence. It consists of six chapters.

Chapter 8 explains the general problem. There is a natural Maldacena limit we can take on a stack of M2-branes, and consequently there should exist an AdS_4/CFT_3 correspondence. However, the explicit study of this correspondence had been impeded for a long time by our ignorance about the interacting CFT present at low energy on a stack of coincident M2-branes. We present a panorama of this issue prior to the recent M2-brane breakthrough.

Chapter 9 introduces supersymmetric field theories in 2+1 dimensions. In particular, it introduces the Chern-Simons interaction and its supersymmetric completion. It also discusses Chern-Simons matter theories, which can give us explicit weakly coupled examples of interacting CFTs in three dimensions.

Chapter 10 introduces monopole operators, which are of central importance to the discussion of Chapter 13. These are local operators which create magnetic flux at a point in Euclidian space-time. They generically play an important rôle in chiral ring of generic $\mathcal{N} = 2$ supersymmetric CFTs, and they give rise to large non-perturbative modifications of the moduli space of the theory.

Chapter 11 introduces the Aharony-Bergman-Jafferis-Maldacena (ABJM) theory, which is central to the recent progress in the study of CFTs for M2-branes. We focus on the study of its moduli space and we briefly discuss the importance of monopole operators in that context. In the second part of the chapter we present recent proposals for $\mathcal{N} = 2$ theories describing M2-branes at Calabi-Yau fourfold singularities, in the form of Chern-Simons quivers. We focus on the toric case, and explain a general algorithm which allows to find the classical moduli space of any $\mathcal{N} = 2$ toric Chern-Simons quiver.

Chapter 12 sets to explain the correspondence between Chern-Simons quivers and M2-branes at singularities thanks to string theory/M-theory dualities. We first explain the ABJM setup using fivebranes in type IIB, and its generalization by Tomasiello and Jafferis. This leads to the interesting possibility of adding extra “flavor” fields to the ABJM model, as we review, following a paper by Gaiotto and Jafferis. In the second part of the paper we review how one can derive the Chern-Simons quiver associated to any toric Calabi-Yau fourfold, through a simple type IIA reduction. We follow a proposal by Aganagic, which we slightly clarify (as already appeared in [3]).

The final **Chapter 13** contains all the original results of Part Two. It is based on [3]. We show that Chern-Simons quivers coupled to extra fields (flavors) naturally arise from M2-branes at non-isolated toric CY singularities (of complex codimension two). We use toric geometry to understand how the coupling of new flavors to a Chern-Simons quiver modifies the moduli space of the theory. We present a simple algorithm which relates the flavoring procedure to simple manipulations on the toric diagram, and we prove the equivalence between the geometric expectation and the non-perturbative moduli space of the flavored quiver. This last step uses crucially a non-perturbative chiral ring relation involving the monopole operators, which we conjecture to exist following similar results in earlier literature.

1.3 Cascading RG flow and $\mathcal{N} = 2$ fractional branes

The most studied generalization of the *AdS/CFT* correspondence to the non-conformal realm involves fractional D3-branes (wrapped D5-branes) at singularities. The most celebrated of such setups is the Klebanov-Strassler model, which considers fractional D3-branes at the conifold singularity. Fractional branes allow to brane-engineer generic $\mathcal{N} = 1$ supersymmetric field theories, which can be somewhat similar to the minimal supersymmetric extension of the Standard Model (MSSM), and their study is therefore of obvious interest. However, these MSSM-like theories always have some more exotic UV completion, in which the number of degrees of freedom grows continuously with the energy. In that respect they are very different from asymptotically free theories such as real world QCD (where the number of degrees of freedom goes to a constant in the UV). Such complicated theories are usually known as *cascading quiver theories*. It is possible in principle to decouple the exotic UV completion from the phenomenologically interesting IR dynamics, but not in the supergravity approximation. This squares well with the expectation that QCD in the large N limit can be described as a perturbative string theory [12]. Unfortunately we would have to deal with strings on curved space, which is too hard. The next best thing we can do is to consider the supergravity limit. In that limit, the dual field theory is strongly coupled at any scale.

The main focus of Part One of the thesis is on the study of these “cascading” UV completions. In the gravity description, which is very well understood, the unbounded growth in the number of degrees of freedom is encoded in “running fluxes”: the enclosed D3-brane charge increases logarithmically with the distance (large distance means high energy in the dual field theory). In setups which correspond to fractional D3-branes at conifold-like singularities ($\mathcal{N} = 1$ fractional branes), the dual field theory interpretation of these running fluxes is well understood too: the field theory can be described by a succession of Seiberg-dual theories, with the ranks of the gauge groups continuously

increasing as we go towards the UV. For instance, the UV limit of the Klebanov-Strassler theory is formally a $SU(\infty) \times SU(\infty)$ theory, quite different from the $SU(M)$ SYM theory present in the IR limit.

We will study a different kind of fractional branes, known as $\mathcal{N} = 2$ fractional branes. Such branes are localized at complex codimension one singularities, and they are therefore free to move along a complex line. This corresponds to an extra Coulomb branch in the dual field theory, associated to scalar fields in the adjoint representation. The supergravity solutions for such branes are very similar to the $\mathcal{N} = 1$ case, with analogous running fluxes¹. However, the dual field theory interpretation was not clear, because there does not exist an appropriate Seiberg duality with adjoint fields. Several partial and contradictory explanations were present in the literature, prior to our work. In Chapter 6 we will clarify the situation by studying the simplest $\mathcal{N} = 2$ example with the Seiberg-Witten technology. We show that the previously known supergravity solutions with running fluxes correspond to a particular vacuum on the Coulomb branch. This vacuum is very similar to the so-called baryonic root in $\mathcal{N} = 2$ SQCD: it is a point on the Coulomb branch where non-perturbative effects (instantons) break the non-Abelian part of the gauge group from $SU(N)$ to $SU(N_f - N)$. Hence the net effect is similar to Seiberg duality, although the mechanism is different. We call “*the $\mathcal{N} = 2$ cascade*” the RG flow described by successive transitions at baryonic roots. We will provide supergravity solutions for more general Coulomb branch vacua. We will also explain how the Klebanov-Strassler $\mathcal{N} = 1$ cascade can be recovered upon mass deformation from $\mathcal{N} = 2$ to $\mathcal{N} = 1$, in the field theory.

This complete understanding of the cascade associated with $\mathcal{N} = 2$ branes allows to consider theories where effects from both types of fractional branes are important. Particularly interesting for physical applications is the fact that generic fractional brane assignment can lead to supersymmetry breaking in the IR theory (an issue which we will not address in this work). In Chapter 7 we will analyze some generic cascade with mixed features, and we will also comment on the supersymmetric vacua. It would be interesting to study further to the issue of SUSY breaking at the bottom of generic cascades involving a choice of Coulomb vacuum at some steps. Another interesting (and related) direction of study would be to consider a flux compactification with a throat corresponding to a $\mathcal{N} = 2$ cascade. In a purely $\mathcal{N} = 2$ throat, we do not expect any metastable vacuum to be present (and hence no possibility of a de Sitter construction like in [13]) when putting a anti-D3 brane in the throat, because twisted flux can be transmuted into D3-branes at no cost, which would then annihilate the anti-D3-brane classically. However, in a cascade of mixed type the situation is less clear, and there might be interesting new possibilities.

1.4 Quivers for M2-branes and their generalizations

Fifteen years after its discovery, M-theory is still very mysterious. A fundamental object in M-theory is the M2-brane, which is the uplift of the fundamental string (and also of the D2-brane) from type IIA to M-theory. The *AdS/CFT* correspondence stemming from the

¹Although strictly speaking we cannot use the supergravity approximation because of the singularity line, we can still consider this approximation supplemented with extra fields, corresponding to the twisted sectors of the closed string theory.

near horizon limit on M2-branes was poorly understood until recently, due to the lack of explicit control over the CFT. The ABJM theory gave an explicit and (almost) maximally supersymmetric Lagrangian description for the low energy theory on M2-branes, either on flat space or on some orbifold $\mathbb{C}^4/\mathbb{Z}_k$. The ABJM theory has a weakly coupled limit at $k \rightarrow \infty$. In that limit the theory is dual to type IIA string theory on $AdS_4 \times \mathbb{CP}^3$. This correspondence is very similar to the well studied $AdS_5 \times S^5$ case; many similar avenues of research have then opened, for instance in the very active field of integrability.

This thesis is concerned with generalizations of the ABJM theory to theories with less supersymmetries. The overall aim is to understand the general relationship between M2-branes and the theories which describe their low energy dynamics. In particular, we would like to know whether these new instances of the AdS/CFT correspondence can shed new light on the “intrinsic” degrees of freedom of multiple M2-branes. By intrinsic, we mean some formulation that might be considered proper to M-theory itself. We will see in Part Two of the thesis that we give a rather pessimistic answer to that question: all the known theories which describe M2-branes are best understood as arising from a *dual* description of M-theory, either in type IIA or type IIB string theory². In fact, we stress that the intrinsically eleven-dimensional properties of M-theories are encoded in the dual CFTs through *non-perturbative effects*, which go beyond the information contained in the Lagrangian. These non-perturbative effects are related to the fact that ’t Hooft operators (monopole operators) are local in three dimensions. Further investigation of the monopole operators in these theories seems crucial to us. It is a complicated problem which goes beyond the scope of this work.

In Part Two, we will give an overview of these various issues, stressing what seemed the most important in order to make our point. The main results will be explained in the last chapter of the thesis. We show how M2-branes on Calabi-Yau with non-isolated singularities have a natural description in terms of Chern-Simons quivers with flavors.

There are numerous further directions one could take to extend the results presented here. Some of these questions are already under investigation. One obvious question, which might also be the hardest, is how one can extend the models presented in Chapter 13, which are purely Abelian, to non-Abelian quivers. The analysis of the non-Abelian case would be very hard (indeed it is already tricky in the maximally supersymmetric case). A crucial point, which should not be that hard to understand, is how the gauge charges of the monopole operators are modified in the non-Abelian case: indeed the presence of fundamental fermions should change the *representation* of the monopole under the gauge group, and would be very interesting to understand how this comes about.

A further point (which is work in progress) is to carefully study the Higgs branches of our models. In particular this is crucial to the study of $\mathcal{N} = 2$ mirror symmetry between different quivers which have the same geometric branch (and the same M-theory dual). It is also important to understand how the multiple-bound brane tilings square with our general picture.

Finally, on a more hypothetical note, it would be interesting to use some of our theories as toy models of condensed matter systems.

²There is a class of theories, called multiple-bound brane tiling, which we do not know how to describe using a string theory duality. More work needs to be done to clarify the status of such theories. We will discuss some of their problems in Part Two.

Chapter 2

Strings, branes and such

STRING theory is a vast subject. In this chapter, we give an patchy overlook of the framework taken for granted in the present thesis. Our main objects of study are *D-branes* in type II string theories. We work mainly in the supergravity limit. We also cast a first look on eleven-dimensional M-theory.

2.1 Elements of type II string theory

Let us consider type II superstring theories. In the limit of weak string coupling, the relevant degrees of freedom are closed superstrings, which live in 10 dimensional space-time. In the Ramond-Neveu-Schwarz (RNS) quantization of the superstring, requiring the absence of tachyons selects some sectors (by the GSO projection). There are two possibilities, called type IIA and type IIB:

$$\text{IIA : } (NS_+, NS_+), (R_+, NS_+), (NS_+, R_-), (R_+, R_-),$$

$$\text{IIB : } (NS_+, NS_+), (R_+, NS_+), (NS_+, R_+), (R_+, R_+).$$

Type II theories are called such because there are two gravitinos in the spectrum (one in each R-NS sector). In type IIA the gravitinos have opposite chiralities while in type IIB they have the same chirality (say positive). Hence type IIA string theory is non-chiral while type IIB is chiral.

Note also that the massless modes in the RR sector fill up a bispinor, which decomposes into various anti-symmetric tensors:

$$\text{IIA : } \mathbf{8}_+ \otimes \mathbf{8}_- = \mathbf{8}_v + \mathbf{56}_v,$$

$$\text{IIB : } \mathbf{8}_+ \otimes \mathbf{8}_+ = \mathbf{1} + \mathbf{28} + \mathbf{35}_{sd}$$

In type IIA we have a vector and a 3-form, while in type IIB we have a scalar, a 2-form and a 4-form with self-dual field strength. Moreover, the space-time theory is supersymmetric [14].

2.1.1 D-branes

In perturbative string theory, D-branes are defined as hypersurfaces on which open strings can end. Instead of Neumann boundary conditions $n^a \partial_a X^\mu = 0$, one can specify Dirichlet boundary conditions

$$X^{\mu_D} = \text{cst}$$

along some $9-p$ directions $\mu_D = p+1, \dots, 9$, at the cost of breaking translation invariance in space-time. The Dp-brane is the $(p+1)$ -dimensional hypersurface defined by these boundary conditions. Moreover, one can see already at the perturbative level that Dp-branes must be dynamical objects. The quantization of the open string on the Dp-brane gives, at the massless level, a $p+1$ dimensional vector field, $9-p$ scalars, and the additional fermions required by supersymmetry. The scalars correspond to the fluctuations of the D-brane in the $9-p$ transverse directions. From the space-time perspective they are interpreted as Goldstone bosons for the translation invariance spontaneously broken by the D-brane.

Indeed, the non-perturbative point of view is that D-branes are solitonic states of the type II theories. In that sense they are fully dynamical objects, albeit very heavy in the perturbative string limit. We are particularly interested in D-branes which preserve some supersymmetries. A stack of parallel Dp-branes of a single type preserve half of the supersymmetries [15], namely $Q + \Gamma^{p+1 \dots 9} \tilde{Q}$. Such BPS states are stable and carry conserved charges, which appear as central charges in the 10 dimensional super-Poincaré algebra [16, 14]. For a Dp-brane it appears as

$$\{Q_\alpha, \tilde{Q}_\beta\} = -2\tau_p Z_{\mu_1 \dots \mu_p}^R (\Gamma^{\mu_1} \dots \Gamma^{\mu_p})_{\alpha\beta} \quad (2.1)$$

in the commutator of two supercharges of opposite chirality in 10 dimensions. Since the charges $Z_{\mu_1 \dots \mu_p}^R$ carry Lorentz indices along the Dp-brane spatial directions, they are not central charges in the usual sense, but indeed this is because extended objects break rotational invariance. If we dimensionally reduce along the D-brane volume they become usual central charges associated to a charged particle in $10-p$ dimensions. The Dp-brane tension is

$$\tau_p = \frac{1}{(2\pi)^p \alpha'^{\frac{p+1}{2}} g_s}, \quad (2.2)$$

where g_s is the closed string coupling constant. Usually solitons have a tension which scales as $1/g^2$; D-branes are not real solitons of the closed string theory in that sense, but should rather be seen as additional fundamental objects in type II string theory.

A fundamental property of D-branes is that they carry non-Abelian gauge fields. In perturbative string theory this follows from the well-known argument using Chan-Paton factors [17, 18]: oriented open strings on a stack of N D-branes naturally carry $U(N)$ degrees of freedom. Many gauge field theories can be engineered using D-branes.

2.1.2 Supergravity limits and p-brane solutions

The low energy limit of type IIA/B string theory is type IIA/B supergravity. In this thesis we will mostly be concerned with the bosonic fields, so we write the bosonic actions only.

Common to both type IIA and type IIB is the low energy theory for the NS-NS sector, namely for the metric, the B-field and the dilaton. In Einstein frame, the action reads

$$S_{NS} = \frac{1}{2\kappa^2} \int d^{10}x \sqrt{-G} R - \frac{1}{4\kappa^2} \int [d\Phi \wedge *d\Phi + e^{-\Phi} H_3 \wedge *H_3] \quad (2.3)$$

where

$$2\kappa^2 = (2\pi)^7 \alpha'^4 g_s^2 \quad (2.4)$$

is basically the Newton coupling constant. The Einstein frame is defined from the string frame by rescaling the metric by the fluctuating part of the dilaton field, while its VEV $e^{\Phi_0} = g_s$ has been absorbed into κ . Φ is then the fluctuating part of the dilaton. $H_3 = dB_2$ is the NS-NS field strength. The full IIA bosonic action also contains kinetic terms for the RR potentials as well as various interactions, including a Chern-Simon term. Let us define the improved field strength for a p -form potential,

$$F_{p+1} = dC_p + C_{p-3} \wedge H_3. \quad (2.5)$$

The bosonic IIA action is

$$S_{IIA} = S_{NS} - \frac{1}{4\kappa^2} \int \left[e^{\frac{3\Phi}{2}} F_2 \wedge *F_2 + e^{\frac{\Phi}{2}} F_4 \wedge *F_4 + B_2 \wedge F_4 \wedge F_4 \right]. \quad (2.6)$$

The equations of motion are easily derived. We will not write them here because we will not really need them in this work.

The type IIB supergravity action is a bit more tricky, because we have to make sure that the field strength F_5 is self-dual. We can write the following action,

$$S_{IIB} = S_{NS} - \frac{1}{4\kappa^2} \int \left[e^{2\Phi} F_1 \wedge *F_1 + e^{\Phi} F_3 \wedge *F_3 + \frac{1}{2} F_5 \wedge *F_5 - C_4 \wedge H_3 \wedge F_3 \right] \quad (2.7)$$

but we have to supplement the equations of motion with the self-duality condition $F_5 = *F_5^1$. We review the type IIB equations of motions in Appendix A.

There exist an interesting class of solutions to type IIA/B SUGRA which were found by Horowitz and Strominger [20]. They are called black p -branes, and are a generalization of the Reissner-Nordstrom black hole to 10 dimensions. We will only be concerned with the extremal solutions, which are half-BPS. They are most easily written in *string frame*. Consider the action

$$S = \frac{1}{2\kappa^2} \int d^{10}x \sqrt{-G_s} \left(e^{-2\Phi} R + e^{-2\Phi} 4\partial_\mu \Phi \partial^\mu \Phi - \frac{1}{2} |F_{p+2}|^2 \right), \quad (2.8)$$

with $H_3 = 0$. The extremal p -brane metric reads

$$ds_s^2 = H^{-\frac{1}{2}} \eta_{\mu\nu} dx^\mu dx^\nu + H^{\frac{1}{2}} \delta_{ij} dy^i dy^j, \quad (2.9)$$

with $\mu = 0, \dots, p$, $i = p+1, \dots, 9$, and the H is any harmonic function in the transverse directions,

$$\Delta_y H = 0. \quad (2.10)$$

¹Note however that a formalism to write an action for a self-dual form has been developed in [19].

The dilaton and the RR gauge field are given by

$$e^\Phi = g_s H^{\frac{(3-p)}{4}}, \quad g_s C_{p+1} = H^{-1} dx^0 \wedge \cdots \wedge dx^p. \quad (2.11)$$

This is also a solution for $p = 3$, although the case of a 3-brane is more subtle since it is both an electric and a magnetic source for C_4 . A typical form for the warping function H is

$$H(y) = 1 + \sum_{i=1}^n \left(\frac{L_i}{|y - y_i|} \right)^{7-p}. \quad (2.12)$$

It corresponds to a multi-centered solution. We can interpret it as the gravity solution describing n stacks of Dp-branes, each stack carrying charge N_i . More precisely,

$$L_i^{7-p} = N_i \frac{2\kappa^2 \tau_p}{(7-p)\text{Vol}(S^{8-p})} = N_i \frac{(2\pi\sqrt{\alpha'})^{7-p}}{(7-p)\text{Vol}(S^{8-p})} g_s. \quad (2.13)$$

This second expression is valid in Einstein frame, in which case the brane tension τ_p is the one in (2.2). As expected for a BPS solution, we can superpose solutions at no cost in energy. One can also see from a probe brane analysis that for a Dp-brane the gravitational attraction is exactly canceled by the repulsion from the RR potential.

2.1.3 DBI action for the D-brane

A fundamental property of D-branes in type II string theory is that, as implied by the identification of the last subsection, they are charged under the RR fields. The central charge appearing in (2.1) can always be understood as related to the gauge potential coupling electrically to the extended object. In the case of a D-brane,

$$Z_{\mu_1 \cdots \mu_p} \sim \int d^{9-p} x (*d * dC_p)_{0\mu_1 \cdots \mu_p}, \quad (2.14)$$

where one integrates over the $9 - p$ spatial directions transverse to the D-brane. The usual coupling of a vector potential to a point particle generalizes to

$$S_{WZ} = \tau_p \int_{p+1} C_p. \quad (2.15)$$

We can also write an action describing the classical dynamics of a D-brane, similar to the Nambu-Goto action for the string. This low energy action also involves the B-field and the $U(1)$ vector field living on the D-brane. It can be determined, for instance, by asking for consistency with the various string dualities (to be discussed below). It is called the Dirac-Born-Infeld (DBI) action, being a supersymmetric generalization of the Born-Infeld action [21]. Its bosonic part is

$$S_{DBI} = -\tau_p \int_{D_p} d^{p+1} \xi \sqrt{-\det(G_s + B_2 + 2\pi\alpha' F_2)} \quad (2.16)$$

Here $F_2 = dA$ is the worldvolume field strength, and the background metric G_s and B-field B_2 are pulled-back quantities. The total bosonic action is the sum $S = S_{WZ} + S_{DBI}$. In this thesis we will rather work with the Einstein frame expression, which is given in Appendix A.

2.1.4 Half-BPS extended objects in type II string theory; D-branes and NS5-branes

Let us end this section by listing the simplest half-BPS objects existing in type II string theory.

	Type IIA	Type IIB
D-branes	D0, D2, D4, D6, D8	D(-1), D1, D3, D5, D7, D9
F-string	yes	yes
NS5-brane	yes	yes

We gave a “democratic” presentation of the D-branes [16], listing both the electric and magnetic branes. In general, if a D_p-brane couples electrically to the RR-potential C_{p+1} , there exists also a D(6-p)-brane which couples magnetically. In other words, the D(6-p)-brane is an electric source for the dual field strength $F_{8-p} = *F_{p+1}$, where $*$ is the Hodge duality operator in ten dimensions. One exception is a D9-brane, which is non-dynamical. Another exception is the D8-brane, which magnetically sources a non-dynamical field strength F_0 ; the D8-charge corresponds to the Romans mass of massive type IIA supergravity [22].

The fundamental string (F-string) is common to both string theories, of course. The F-string couples electrically to the NS-NS 2-form potential,

$$S \supset \tau_{F1} \int B_2, \quad \text{with} \quad \tau_{F1} = \frac{1}{2\pi\alpha'}. \quad (2.17)$$

There exist a corresponding extremal black string solution in supergravity (cfr. [23] for a pedagogical discussion).

There is also a magnetic dual of the F-string, which is called the NS5-brane. The corresponding extremal black brane background is (in string frame)

$$ds^2 = dx_\mu dx^\mu + H(y) dx^i dx^i, \quad e^\Phi = g_s H^{\frac{1}{2}}, \quad *H_3 = dx^0 \wedge \dots \wedge g_s dx^5 \wedge dH^{-1}, \quad (2.18)$$

with H an harmonic function in the 4-dimensional transverse space, such as $H = 1 + (nQ_5)/r^2$ for a bunch of n localized NS5-brane. The tension of a single NS5-brane is

$$\tau_{NS5} = \frac{1}{(2\pi)^5 \alpha'^3 g_s^2}. \quad (2.19)$$

The $1/g_s^2$ dependence allows us to view this object as a more conventional soliton of the closed string theory.

2.2 A few words about string dualities

Our modern understanding of string theory relies on a tight network of duality conjectures. Due to their non-perturbative nature, we cannot prove most of these conjectures for the moment, but they imply a host of non-trivial results, which we can check and which have been checked. See for instance the books [14, 24] for a modern overview.

At the level of pure principles, string theory has no parameters at all. Nevertheless, at a practical level, it is often said that string theory has two important parameters, namely α' and the string coupling g_s , which can both be used to define some useful “classical limits”.

- g_s is the string coupling, which measure the tendency of strings to split. The g_s expansion is an expansion in string loops. This expansion is very similar to the perturbative expansion of quantum field theories. It gives the S-matrix elements in term of the genus expansion of the string worldsheet. $g_s = 0$ is the classical string limit. It is well known that g_s is itself determined by the background value of the dilaton field, $g_s = e^\Phi$.
- The α' expansion is an expansion around the point particle limit; the length scale $\sqrt{\alpha'} = l_s$ is called the string length. The proper expansion parameter to consider is a dimensionless parameter such as $\alpha' p^2$, where p^2 is the characteristic scale of interest. l_s sets the mass of the massive string states, which are necessary for the spectacular UV finiteness of string theory. It can be taken as a unit of length in stead of the 10 dimensional Planck length $\kappa^{\frac{1}{4}}$ (see the relation (2.4)). In the low energy limit $p^2 \ll 1/\alpha'$ we obtain an effective theory of massless particles in the guise of a non-renormalizable supergravity theory.

Let us go for a brief review of string theory dualities.

2.2.1 T-duality

Historically the oldest, T-duality is a perturbative duality, which holds order by order in the g_s expansion. We refer to [25] for a comprehensive review. The simplest example of T-duality is for string theory on $\mathbb{R}^{8,1} \times S^1$, with S^1 a circle of radius R . The momentum along the periodic direction is quantized as n/R , $n \in \mathbb{Z}$. In the case of a string, there is also a conserved winding number w . T-duality states that any physical observable (spectrum and scattering amplitudes) is invariant under the exchange

$$n \leftrightarrow w \quad R \leftrightarrow \frac{\alpha'}{R}. \quad (2.20)$$

Momentum is exchanged with winding number, and a circle of radius R becomes a circle of radius α'/R . This duality famously lead to the discovery of D-branes: suppose we have open strings in 10 dimensions. The winding number of an open string is not conserved. In the T-dual picture, the momentum along S^1 should not be conserved either: we have a D8-brane, which breaks translational invariance. One can show explicitly that T-duality exchanges Neumann and Dirichlet boundary conditions for open strings [18].

We can perform T-duality on much more general backgrounds. T-duality relates two different backgrounds (the string target space) which are indistinguishable from the point of view of string theory, because the world sheet theory (or more precisely the Polyakov path integral) is invariant under the background field transformation. In type II superstring theory, T-duality exchanges type IIA and type IIB. In the presence of closed strings only, the change in the background is encoded in the Buscher's rules [26]. T-duality mixes together B-field and metric, and relates the string couplings as

$$g_s \leftrightarrow \frac{\sqrt{\alpha'}}{R} g_s. \quad (2.21)$$

Consistently, the Buscher rules transformation for the NS-NS fields are symmetries of the supergravity equations. The RR gauge potentials change as (schematically)

$$C_{p+1} \leftrightarrow C_p \text{ or } C_{p+2}, \quad (2.22)$$

according to whether C_p has or does not have a leg along the direction we T-dualize along. D-branes are T-dualized accordingly,

$$Dp\text{-brane} \quad \leftrightarrow \quad D(p-1)\text{-brane} \quad \text{or} \quad D(p+1)\text{-brane}. \quad (2.23)$$

2.2.2 T-duality between geometry and NS5-branes

For completeness, we should also discuss how T-duality acts on NS5-branes. That depends on the direction along which we T-dualize. If we T-dualize along the worldvolume of the NS5-brane, T-duality merely maps the type IIA NS5-brane to the type IIB NS5-brane. Things become more interesting when we want to T-dualize along a transverse direction. Because the B-field and the metric mix under T-duality, an NS5-brane can map to pure geometry [27, 28].

Let us consider the problem from the point of view of some geometry. The idea is to T-dualise along some S^1 fiber of the geometry. At points where the S^1 fiber shrinks to zero, there is a corresponding singularity in the T-dual B-field, which is interpreted as a NS5-brane source.

Let us work out part of the story at the level of supergravity, although the full story is more involved [27, 28]. We consider the simple case of the ALE (asymptotically locally euclidian) metric on the singularity $\mathbb{C}^2/\mathbb{Z}_n$, with coordinates

$$z_1 = r e^{i(\frac{\phi}{2} + \frac{\psi}{2n})} \cos \frac{\theta}{2}, \quad z_2 = r e^{i(\frac{\phi}{2} - \frac{\psi}{2n})} \sin \frac{\theta}{2}. \quad (2.24)$$

The range of ϕ and ψ is $[0, 2\pi)$ and $[0, 4\pi)$, respectively. The flat metric is simply a Hopf fibration,

$$ds_4^2 = dz_1 d\bar{z}_1 + dz_2 d\bar{z}_2 = dr^2 + \frac{r^2}{4} \left(d\theta^2 + \sin^2 \theta d\phi^2 + \frac{1}{n^2} (d\psi + n \cos \theta d\phi)^2 \right). \quad (2.25)$$

Considering this as a background of type IIA/B, we can T-dualize to type IIB/A in the supergravity limit, along the ψ direction, by using Buscher's rules [26]². The T-dual fields are

$$\hat{ds}_4^2 = dr^2 + \frac{r^2}{4} (d\theta^2 + \sin^2 \theta d\phi^2) + \frac{4n^2}{r^2} d\psi^2, \quad (2.26)$$

$$e^{2(\hat{\Phi} - \Phi_0)} = \frac{4n^2}{r^2}, \quad \hat{B} = n \cos \theta d\phi \wedge d\psi. \quad (2.27)$$

The linear dilaton profile and the $\hat{H}_3 = d\hat{B}$ flux should correspond to n NS5-branes. Indeed we have the magnetic charge

$$\frac{1}{16\pi^2} \int_{S^3} \hat{H}_3 = n. \quad (2.28)$$

²In this case the Buscher rules tell us that

$$\hat{G}_{\psi\psi} = \frac{1}{G_{\psi\psi}}, \quad e^{2\hat{\Phi}} = \frac{e^{2\Phi}}{G_{\psi\psi}}, \quad \hat{G}_{\mu\nu} = G_{\mu\nu} - \frac{G_{\mu\psi} G_{\nu\psi}}{G_{\psi\psi}}, \quad \hat{B}_{\mu\psi} = \frac{G_{\mu\psi}}{G_{\psi\psi}},$$

where G is the string frame metric, and μ, ν are any coordinate index different from ψ .

However, we should be more precise. The way the duality was first stated [27, 28] was not as a duality between NS5-branes and ALE space, but between NS5-branes and the Taub-NUT space, also called ALF space (asymptotically locally flat). The Taub-NUT metric reads

$$ds_{TN}^2 = h(d\rho^2 + \rho^2(d\theta^2 + \sin^2\theta d\phi^2)) + h^{-1}(d\psi + \omega)^2, \quad (2.29)$$

where h is an harmonic function on \mathbb{R}^3 and the one-form ω is such that $d\omega = *_3dh$. For the simple case of a single center geometry of charge n , we have

$$h(\rho) = 1 + \frac{n}{\rho}, \quad \omega = n \cos\theta. \quad (2.30)$$

At small r , we drop the factor of 1 and the metric (2.29) looks like

$$ds^2 = \frac{n}{\rho}d\rho^2 + n\rho \left(d\Omega_2^2 + \frac{1}{n^2}(d\psi + n \cos\theta d\phi)^2 \right). \quad (2.31)$$

Redefining $4n\rho = r^2$, we find the flat metric on $\mathbb{C}^2/\mathbb{Z}_n$ (2.25). Hence the ALE geometry actually describes the center of a Taub-NUT geometry. On the other hand, consider a stack of n NS5-branes transverse to $\mathbb{R}^3 \times S^1$. Its metric reads

$$ds_{10}^2 = dx_{5,1}^2 + H(\rho, \psi) (d\psi^2 + dy_3^2). \quad (2.32)$$

where ψ is the coordinate on the S^1 , and ρ of the radius of the \mathbb{R}^3 , and H an harmonic function on the transverse space. Smearing the n branes along the circle, we have $H = 1 + \frac{n}{\rho}$. Moreover, at small ρ ,

$$ds_{10}^2 = dx_{5,1}^2 + \frac{n}{\rho}(d\psi^2 + d\rho^2 + \rho^2 d\Omega_2^2), \quad (2.33)$$

where $d\Omega_2^2 = d\theta^2 + \sin^2\theta d\phi^2$ is the usual metric on S^2 . Upon the change of coordinates $r^2 = 4n\rho$, this is the metric (2.26) which we found from T-duality. So we have shown that the T-dual of an ALE space is the near-horizon limit on some (smeared) NS5-branes. The smearing is an artifact of the supergravity approximation, in string theory we really have NS5-branes localized on a circle [28]. Moreover, the distance between the NS5-branes on the circle is T-dual to the value of a flat B-field localized at the $\mathbb{C}^2/\mathbb{Z}_n$ singularity; see [29] for a recent discussion.

2.2.3 S-duality in type IIB

The second superstring revolution was triggered by the discovery of non-perturbative dualities, which exchange strong and weak coupling. Let us first consider type IIB string theory. It had been known for a long time that the type IIB supergravity equations are invariant under a continuous $Sl(2, \mathbb{R})$ symmetry. In Einstein frame, the metric G and the self-dual 5-form F_5 are $Sl(2, \mathbb{R})$ invariant, while the vector 2-forms and the *axio-dilaton*, defined as

$$\tau \equiv C_0 + ie^{-\Phi}, \quad (2.34)$$

transform as

$$\begin{pmatrix} B_2 \\ C_2 \end{pmatrix} \rightarrow \begin{pmatrix} a & b \\ c & d \end{pmatrix} \begin{pmatrix} B_2 \\ C_2 \end{pmatrix}, \quad \tau \rightarrow \frac{a\tau + b}{c\tau + d}, \quad \text{with } \begin{pmatrix} a & b \\ c & d \end{pmatrix} \in Sl(2, \mathbb{R}). \quad (2.35)$$

In the quantum theory (type IIB string theory), this symmetry cannot hold, because it would contradict the Dirac quantization of charge. Nevertheless, it is possible that the full type IIB string theory is invariant under the S-duality group $Sl(2, \mathbb{Z})$. In particular, the so-called S generator of $Sl(2, \mathbb{Z})$ acts as

$$\tau \rightarrow -\frac{1}{\tau}. \quad (2.36)$$

For $C_0 = 0$, this transformation sends the string coupling g_s to $1/g_s$ ³. This conjecture gives us a way to deal with type IIB string theory at strong coupling: we just have to consider the weakly coupled S-dual version of type IIB! Under S-duality, the F1-string is mapped to the D1-string, and the NS5-brane is mapped to the D5-brane. This is possible since, when $C_0 = 0$,

$$\tau_{F1} = g_s \tau_{D1} \quad \tau_{NS5} = \frac{1}{g_s} \tau_{D5}. \quad (2.37)$$

We also have more general (p, q) -strings and (p, q) -branes, with p being the NS-NS charge and q the RR charge. (p, q) objects with p and q coprime are all in the same $Sl(2, \mathbb{Z})$ orbit as $(1, 0)$. Each type of string gives a possible starting point for string perturbation theory, so that type IIB string theory has an infinite family of semi-classical limits.

2.2.4 Type IIA and M-theory

A natural question to ask is whether there is something similar to S-duality for the type IIA superstring. Type IIA supergravity has no similar S-duality, but another fact about supergravities turns out to be crucial.

In eleven dimension one can write a *unique* supergravity theory [30]. It contains a metric G_{11} , a gravitino and a 3-form gauge field A_3 , with field strength $G_4 = dA_3$. The bosonic part of the action is quite simple, consisting of a kinetic term and of a Chern-Simons term,

$$S = \frac{1}{2\kappa_{11}^2} \int d^{11}x \sqrt{-G_{11}} \left(R - \frac{1}{2} |G_4|^2 \right) - \frac{1}{12\kappa_{11}^2} \int A_3 \wedge G_4 \wedge G_4. \quad (2.38)$$

It is useful to define an eleven dimensional Planck length l_p by

$$(2\pi)^8 l_p^9 = 2\kappa_{11}^2. \quad (2.39)$$

To make contact with string theory, we should make the link with a 10 dimensional theory. This can be done by a Kaluza-Klein reduction. It is well known (see for instance [31, 32]) that the reduction of Einstein theory in D dimension along a circle leads to a theory for a $D - 1$ dimensional metric together with a $U(1)$ vector and a dilaton scalar field, which parametrizes the size of the circle. We must also reduce the three form A_3 , which gives both a 2-form and a 3-form in 10 dimensions. In total we have the bosonic content of type IIA supergravity, and the details (including the fermions) can be worked out too. At the end of the day the dimensional reduction of 11 dimensional supergravity precisely gives type IIA supergravity (2.6). The bosonic part of the story is easy to work out, and more details will be given in Chapter 8.

³We refer to the Appendix E for a discussion of a similar S-duality in the field theory context.

Type IIA supergravity is a consistent truncation of 11 dimensional supergravity, which means that any solution of the type IIA equations of motion will be a solution of the 11 dimensional theory, but not the other way around. The full eleven-dimensional theory can be accounted for by keeping the towers of Kaluza-Klein (KK) modes.

Such KK modes have masses $m = n/R_{10}$, with $n \in \mathbb{Z}$ and R_{10} the radius of the circle along which we compactify. The claim is that the KK modes corresponding to a graviton along the circle (together with its supersymmetric partners) precisely corresponds to the D0-brane of type IIA string theory. Since the mass of a D0-brane is $1/\sqrt{\alpha'}g_s$, this identification is possible only if

$$R_{10} = \sqrt{\alpha'}g_s. \quad (2.40)$$

According to this relation, at strong coupling $g_s \rightarrow \infty$ the eleventh dimension becomes large and essentially decompactifies. In general, the radius of the eleventh dimension might vary in space, in which case there is a non-trivial dilaton profile in type IIA. We also have the important relation

$$l_p = \sqrt{\alpha'}g_s^{1/3}. \quad (2.41)$$

While l_p and $l_s = \sqrt{\alpha'}$ provide natural units of length in 11 and 10 dimensions, respectively, we see that the conversion factor is given by $g_s^{1/3}$.

The conjecture is that type IIA at strong coupling is described by a eleven-dimensional quantum theory called *M-theory*, and that there is a duality at any coupling between type IIA and M-theory [33, 34, 35]. We usually call the circle of the eleventh direction the *M-theory circle*.

We do not know much about M-theory. What we know is that its low energy limit is 11 dimensional supergravity. We also know that it contains half-BPS extended objects of dimension $2 + 1$ and $5 + 1$, called M2- and M5-branes. Their tensions are

$$\tau_{M2} = \frac{1}{(2\pi)^2 l_p^3}, \quad \tau_{M5} = \frac{1}{(2\pi)^5 l_p^6}. \quad (2.42)$$

Using the relation (2.41), we see that the M2-brane tension is the same as the D2-brane tension. We will then identify these two objects: if the M-theory circle is transverse to the M2-brane, we obtain a D2-brane [36]. On the other hand, if the M-theory circle lies along the M2-brane worldvolume, the reduction will give a fundamental string [37], with the correct tension

$$\tau_{F1} = 2\pi R_{10}\tau_{M2} = \frac{1}{2\pi\alpha'}. \quad (2.43)$$

Similarly we may identify the M5-brane with the type IIA NS5-brane when the circle is transverse, or with a D4-brane when the circle is parallel. Indeed $\tau_{NS5} = \tau_{M5}$ and $\tau_{D4} = 2\pi R_{10}\tau_{M5}$.

Finally, we should discuss the case of D6-branes. They couple magnetically to C_1 . Hence their M-theory lift should be the “magnetic dual” of gravitons along the M-theory circle: the M-theory uplift of a D6-brane is a Taub-NUT space, also called KK monopole. In a KK monopole the M-theory circle S^1 is non-trivially fibered over \mathbb{R}^3 . While the total space is topologically \mathbb{R}^4 , the metric asymptotes to $S^1 \times S^2$ at infinity in \mathbb{R}^3 . In the case of n D6-branes, the first Chern class of the S^1 fibration is n , and the resulting Taub-NUT of charge n is topologically $\mathbb{R}^4/\mathbb{Z}_n$.

We can summarize the type IIA/M-theory duality in the following table:

M-theory	Type IIA
G_{11} A_3	G_{10} , C_1 , e^Φ C_3 , B_2
KK-mode	D0
M2	D2 , F1
M5	NS5 , D4
KK-monopole	D6
?	D8

Note that the D8-brane has no M-theory lift because there is no known way to obtain a Romans mass from a 11 dimensional theory.

We will discuss the M-theory/type IIA duality in a bit more detail in the second part of the thesis, which deals with the AdS_4/CFT_3 correspondence. We will also need this duality in Chapter 6, where we will take advantage of the fact that a system of NS5- and D4-branes lift to a single smooth M5-brane in M-theory (see also Appendix E).

Chapter 3

D3-branes at singularities

IN this Chapter we review the deep relationship which exists between D-branes at singularities and quiver gauge theories. The tools we introduce will be put to good use at various points in this thesis.

In section 3.1 we make our first encounter with $\mathcal{N} = 4$ super-Yang-Mills. In section 3.2, we explain how to find the field theory living on a stack of D3-branes at any flat orbifold singularity, which leads us to introduce quivers. In section 3.3 we briefly discuss the general case, for any Calabi-Yau singularity. In section 3.4 we introduce a simple Hanany-Witten brane construction, which is interesting to deal with a simple class of conifold singularities. In section 3.5 we consider the case of a general toric CY singularity, and explain what are brane tilings and why they are useful.

3.1 The D3-brane and $\mathcal{N} = 4$ SYM: a first encounter

D3-branes are special among the the zoo of Dp-branes. For instance, the p-brane metric is singular at $r = 0$ unless $p = 3$ [38]. Importantly, the extremal 3-brane solution (2.9)-(2.11) has constant dilaton, $e^\Phi = g_s$. In the open string picture, this corresponds to the fact that the Yang-Mills coupling, which appears in the action as

$$S_{YM} = -\frac{1}{4g_{YM}^2} \int d^4x \text{Tr} F_{\mu\nu} F^{\mu\nu}, \quad (3.1)$$

is classically marginal in four dimensions. We have the relation $g_{YM}^2 = 4\pi g_o^2 = 4\pi g_s$, where g_o is the open string coupling. It turns out that the quantum theory living on a stack of N D3-brane is a maximally supersymmetric $U(N)$ gauge theory which has an exactly marginal coupling g_{YM} . It is called $\mathcal{N} = 4$ super-Yang-Mills (SYM), and it contains a single $\mathcal{N} = 4$ multiplet in the adjoint of $U(N)$. In term of $\mathcal{N} = 1$ superfields, the $\mathcal{N} = 4$ multiplet splits into a vector multiplet V and three chiral multiplets Φ_i , $i = 1, 2, 3$. The chiral multiplets correspond to excitations along the three complex directions z_i transverse to the D3-brane. One has the relation $\Phi_i = 2\pi z_i / \alpha'$ between the VEVs of the scalar component of the chiral superfields and the positions z_i of the D3-branes. The $\mathcal{N} = 4$ theory written in $\mathcal{N} = 1$ form also has a superpotential

$$W = \Phi_1[\Phi_2, \Phi_3], \quad (3.2)$$

with a precise value for the coupling dictated by the extended supersymmetry. This is all we need to know about $\mathcal{N} = 4$ SYM for the moment.

3.2 D-branes at singularities and quivers

D-branes are interesting probes for singularities in string theory, because they can probe a geometry which is more local than the string length $\sqrt{\alpha'}$. This is obviously the case for a D0-brane, for instance, which is a point particle. Another possibility is to consider a D3-brane in $\mathbb{R}^{3,1} \times CY_3$, which is a point particle from the Calabi-Yau perspective. We want to know what happens to such a D3-brane probe when it goes to a singularity in the CY_3 .

We consider only algebraic singularities. Actually, we consider only Calabi-Yau singularities, so that there is still at least $\mathcal{N} = 1$ supersymmetry on the D3-brane worldvolume. An introduction to the relevant concepts of algebraic geometry, with particular focus on toric geometry, is provided in Appendix B.

The simplest local Calabi-Yau 3-fold to consider is \mathbb{C}^3 , which is just flat space. If we put a D3-brane on flat space, the low energy theory on its worldvolume is the $\mathcal{N} = 4$ SYM theory, as we just reviewed.

3.2.1 D3-branes at orbifold singularities

The simplest local algebraic singularity we can think of is an orbifold of flat space, \mathbb{C}^3/Γ , for Γ a discrete group. The low energy theory on probe D3-branes at the singularity was found in [39, 40]. The action of Γ should preserve the Calabi-Yau condition, which is equivalent to say that it preserves the Kähler and the holomorphic forms,

$$J = -i \sum_{i=1}^3 dz^i \wedge d\bar{z}^i, \quad \Omega = dz^1 \wedge dz^2 \wedge dz^3. \quad (3.3)$$

To preserve J , Γ we must preserve the norm $\sum_i |z^i|^2$ in \mathbb{C}^3 , while to preserve Ω it must be of unit determinant. Hence Γ must be a discrete subgroup of $SU(3)$. We denote the Γ action by

$$g \in \Gamma : z_i \mapsto \rho(g)_j^i z^j, \quad (3.4)$$

where $\rho(\Gamma)$ is some representation of Γ that we have to choose. We can understand the theory on a stack of N D3-brane by working on the covering space of \mathbb{C}^3/Γ , which is just \mathbb{C}^3 . There we have $|\Gamma| = \dim \Gamma$ images for each D3-brane. The theory for N D3-branes on the orbifold is then some projection of the $U(|\Gamma|N)$ $\mathcal{N} = 4$ theory: we only keep the modes which are invariant under Γ . The D3-brane images transform into the $|\Gamma|$ -dimensional representation $R(\Gamma)$, which is called the regular representation. It is a reducible representation which contains all the irreducible representations of $\Gamma_i(\Gamma)$,

$$R(g) = \oplus_i \dim(\Gamma_i) \Gamma_i(g), \quad \forall g \in \Gamma. \quad (3.5)$$

All fields in the $U(|\Gamma|N)$ $\mathcal{N} = 4$ theory are in the adjoint representation of the gauge group. Because of $\mathcal{N} = 1$ supersymmetry we can forget about the fermions and just state

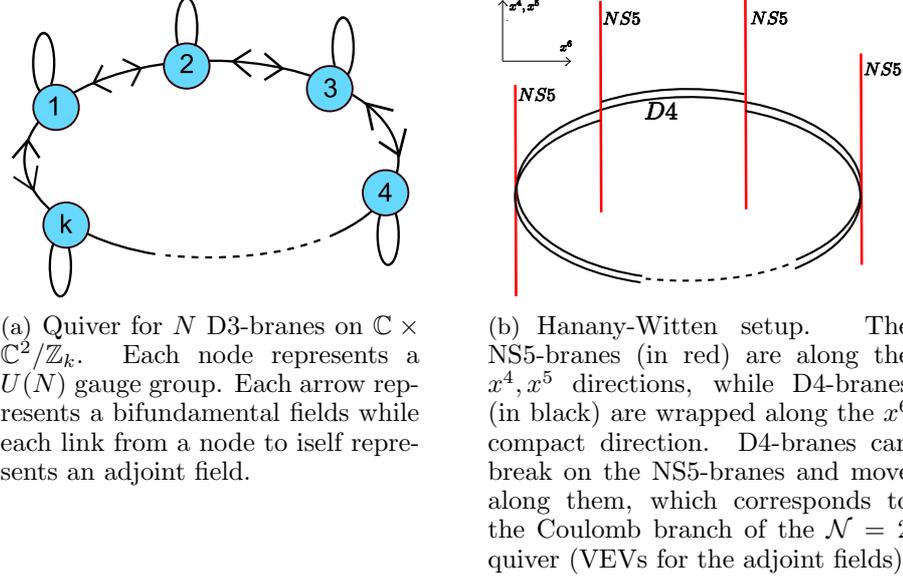


Figure 3.1: The quiver for D3-branes on $\mathbb{C} \times \mathbb{C}^2/\mathbb{Z}_k$ and the T-dual Hanany-Witten setup (explained in section 3.4 below).

the transformation laws of the bosonic fields under Γ . To have Γ invariance, we need

$$A_\mu = R(g)^{-1} A_\mu R(g), \quad (3.6)$$

$$\Phi_i = \rho(g)_i^j R(g)^{-1} \Phi_j R(g). \quad (3.7)$$

The representation ρ here is the same as in (3.4), since the complex field Φ_i corresponds to motion along z_i . The condition (3.6) means that the A_μ must be block diagonal. The gauge group of the orbifold theory is then a product gauge group

$$U(\dim \Gamma_1 N) \times U(\dim \Gamma_2 N) \times \dots, \quad (3.8)$$

according to the decomposition (3.5). The condition (3.7) tells us which chiral multiplets survive the projection. They will be either in an adjoint or in a bifundamental representation (fundamental of a gauge group and anti-fundamental of another). We also find a non-trivial superpotential by expanding the $\mathcal{N} = 4$ superpotential $W = \text{Tr} \Phi_1 [\Phi_2, \Phi_3]$ in term of the surviving fields.

Such a gauge theory can be usefully represented as a *quiver* diagram (together with the superpotential), as was advocated by Douglas and Moore [39]. One draws a node for each gauge group, and an arrow from node n to node m for each bifundamental field X_{mn} connecting those two gauge groups. See figure 3.1(a) for an example.

Let us work out an easy example. Consider $\Gamma = \mathbb{Z}_k$, acting as $(z_1, z_2, z_3) \mapsto (z_1, \omega^n z_2, \omega^{-n} z_3)$, for $\omega = e^{\frac{2\pi i}{k}}$. It is a subgroup of $SU(2)$, and so it actually preserves $\mathcal{N} = 2$ supersymmetry. Because \mathbb{Z}_k is Abelian each irreducible representation is of dimension one; this implies that the gauge group is $U(N)^k$. Moreover, the condition (3.7) is

$$(\Phi_1)_j^i = \omega^{j-i} (\Phi_1)_j^i, \quad (\Phi_2)_j^i = \omega^{j-i+1} (\Phi_2)_j^i, \quad (\Phi_3)_j^i = \omega^{j-i-1} (\Phi_3)_j^i. \quad (3.9)$$

We then have $\Phi_1 = \text{diag}(\phi_1, \dots, \phi_k)$ and

$$\Phi_2 = \begin{pmatrix} 0 & \cdots & 0 & X_{1k} \\ X_{21} & 0 & \cdots & 0 \\ 0 & X_{32} & 0 & \\ & & \ddots & 0 \end{pmatrix}, \quad \Phi_3 = \begin{pmatrix} 0 & X_{12} & 0 & \cdots \\ 0 & 0 & X_{23} & 0 \\ & & 0 & \ddots \\ X_{k1} & & & 0 \end{pmatrix}. \quad (3.10)$$

The quiver is shown in Fig.3.1(a). The superpotential is

$$W = \sum_{i=1}^k \phi_i (X_{ii-1}X_{i-1i} - X_{ii+1}X_{i+1i}). \quad (3.11)$$

This is known as the A_{k-1} quiver. Remark that in this case the form of the superpotential is actually fixed by the $\mathcal{N} = 2$ supersymmetry.

3.2.2 Brane fractionation at singularities

We have just learned that the low energy gauge theory on a D3-brane at an orbifold singularity is of the quiver type. This is because the D3-brane we considered had to transform in the regular representation of the orbifold group. For this reason it is often called a *regular D3-brane*. However, the reducibility of the regular representation hints at the fact that the regular D3-brane is a composite object: at the singularity, the brane can fractionate into a marginal bound state of some other objects. A D3-brane which transforms into an irrep Γ_i of Γ is called a *fractional D3-brane* of type i [41]. A fractional brane is stuck at the singularity, because it has no images in \mathbb{C}^3 . Its low energy gauge theory has a single $U(1)$.

In the case of $\mathcal{N} = 2$ singularities, which are of the form $\mathbb{C} \times \mathbb{C}^2/\Gamma$, however, we have a *non-isolated* singularity, along the line \mathbb{C} . The fractional branes can then move along the singularity line. This can also be seen in the quiver describing their low energy dynamics. For instance, consider the A_{k-1} quiver discussed above. The F-term relations following from the superpotential (3.11) (with $N = 1$, say) are

$$X_{ii+1}(\phi_i - \phi_{i+1}) = 0, \quad X_{i-1i}(\phi_{i-1} - \phi_i) = 0, \quad X_{ii-1}X_{i-1i} - X_{ii+1}X_{i+1i} = 0. \quad (3.12)$$

When $\phi_1 = \dots = \phi_k = \phi$, we have a branch which is $\mathbb{C} \times \mathbb{C}^2/\mathbb{Z}_k$, described by ϕ and $xy = t^k$, with $x = \prod_i X_{ii+1}$, $y = \prod_i X_{i-1i}$ and $t = X_{12}X_{21} = \dots = X_{k1}X_{1k}$. This branch is the moduli space for a regular D3-brane. There is also a branch of moduli space with $X_{ii-1} = X_{i+1i} = 0$ and the ϕ_i arbitrary, which corresponds to configurations of fractional branes along the singularity line.

Note that non-isolated singularities can also occur in $\mathcal{N} = 1$ orbifolds or in more general algebraic varieties.

3.2.3 Closed string perspective for D3-branes on orbifolds. Supergravity

Closed string theory on \mathbb{C}^3/Γ consist of $|\Gamma|$ sectors: one untwisted sector corresponding to the identity in Γ , and one twisted sector for each non-trivial element of Γ . Strings in

the twisted sector are stuck at the orbifold singularity. It is interesting that perturbative string theory (or more precisely a description of the closed string in term of a $\mathcal{N} = (2, 2)$ free CFT) imposes that the period $\int_{\mathcal{C}} B_2$ on any exceptional curve¹ \mathcal{C} is fixed [42]. In the presence of D-branes, we have that [43]

$$\frac{1}{4\pi^2\alpha'} \int_{\mathcal{C}_i} B_2 = \frac{\dim(\Gamma_i)}{|\Gamma|}, \quad \frac{1}{4\pi^2\alpha'} \int_{\mathcal{C}_0} B_2 = -\sum_{i \neq 0} \frac{\dim(\Gamma_i)}{|\Gamma|} \quad (3.13)$$

where the exceptional 2-cycles \mathcal{C}_i for $i \neq 0$ are in correspondence with the non-trivial irreducible representations of Γ appearing in (3.5). We have that $\mathcal{C}_0 = -\sum_{i \neq 0} \mathcal{C}_i$ in homology.

We are really interested in only the massless modes of type IIB string theory, so it is easier to consider just supergravity, taking good notice of the orbifold identification and of the additional twisted fields living on the singular submanifolds. In the supergravity limit it is certainly natural for $\int_{\mathcal{C}_i} B_2$ to take any value.

Moreover, we have the natural identification of fractional branes with *wrapped branes* [41]. A fractional D-brane is nothing but a wrapped D-brane on a vanishing cycle. Such a brane can still have non-zero tension because of a non-zero value for the B-field, as we now recall, following the review [44]. This is easily seen in the DBI approximation². Let us define the quantities

$$\tilde{b}_i = b_i + f_i \equiv \frac{1}{4\pi^2\alpha'} \int_{\mathcal{C}_i} (B_2 + 2\pi\alpha' F_2), \quad (3.14)$$

with F_2 the world-volume flux. The Einstein frame action for a probe D($p+2$)-brane wrapped on the *vanishing* 2-cycle \mathcal{C}_i is

$$S_i = -\tau_p \int d^{p+1} \xi e^{\frac{\Phi-3}{4}} \sqrt{-\det g} |\tilde{b}_i| + \tau_p \int_{p+1} (C_{p+1} \tilde{b}_i + A_{p+1}). \quad (3.15)$$

The twisted field A_{p+1} is defined such that

$$\tau_{p+2} \int_{p+3} C_{p+3} = \tau_p \int_{p+1} A_{p+1}. \quad (3.16)$$

Importantly, the wrapped D($p+2$)-brane couples to the RR potential C_{p+1} with a charge

$$Q_p^{(i)} = \tilde{b}_i \tau_p \quad \tilde{b}_i \in [0, 1). \quad (3.17)$$

It is BPS as long as \tilde{b}_i is positive, because $\tau_p^{(i)} = |Q_p^{(i)}|$. When $\tilde{b}_i = 0$ the fractional brane tension vanishes, which can lead to interesting phenomena, some of which are to be discussed in this thesis. One chooses the background value of $B_2 + 2\pi\alpha' F_2$ on each cycle \mathcal{C}_i , including \mathcal{C}_0 , such that $\sum_i \tilde{b}_i = 1$. This corresponds to the fact that since a regular D p -brane is a marginal bound state of all the possible fractional branes we should have $S_{Dp} = \sum_i S_i$.

¹An exceptional curve, or generally an exceptional cycle, is a cycle which would appear in some partial resolution preserving the CY condition (see e.g. section B.4.1 in the Appendix). In the orbifold limit the Kähler class of this cycle goes to zero. However, in string theory we have a complexified Kähler class $J + iB$ which doesn't need to vanish.

²The DBI action for a D-brane is technically valid only at large volume. However, for $\mathcal{N} = 2$ SUSY preserving orbifolds we can extrapolate the results from large volume to the orbifold fixed point. With $\mathcal{N} = 1$ SUSY only the information corresponding to the holomorphic data of the field theory on the D3-branes can be reliably extrapolated to small volume.

3.3 Branes at generic Calabi-Yau singularities

Having understood the case of an orbifold singularity, we now consider the general problem of finding the low energy field theory for D3-branes at any CY singularity. We can already anticipate that many of the ingredients of the orbifold case will carry over to the general case: the low energy field theory will be a quiver, and each node of the quiver will correspond to some brane wrapped on a vanishing cycle. Such wrapped branes are still called fractional branes. The question is how to find this quiver from the geometric data. In this section (which can be skipped) we consider the general case in a very sketchy manner; in the next sections we consider the case of toric singularities, where a solution can be given much more explicitly. The reason we have to be very sketchy in this section is that the way to phrase the problem rigorously uses quite a bit of extra mathematical machinery (sheaves and homological algebra) which is orthogonal to the tools used in this work. Nevertheless, for the sake of completeness, we will try to convey a rough idea of the deep relationship between CY singularities and quivers.

Our whole setup preserves $\mathcal{N} = 1$ supersymmetry in four dimensions. The question we are really asking is the following: Given a Calabi-Yau threefold singularity, what is the *holomorphic data* of the field theory living on a stack of D3-branes located at the singularity.

- What are the gauge groups and what is the matter content?
- What is the superpotential?

This is what we mean by finding the *quiver*. Only these pieces of information are encoded in the quiver-with-superpotential structure discussed in this chapter. In particular we are not asking any information about the Kähler potential of the field theory. The holomorphic structure of the field theory is fully encoded in the holomorphic data of the CY threefold. It is independent of the Kähler structure of the CY_3 , and it can thus be computed at large volume in the *resolved* singularity. This approach was first advocated by Wijnholt in [45]. See also [46, 47]. The appropriate language to describe D-branes wrapped on holomorphic cycles, at large volume, is the language of B-branes, which are branes in the topological B-model at $g_s = 0$; see [48, 49] for reviews. Mathematically, B-branes are *coherent sheaves*. A coherent sheaf is a mathematical object which makes well defined the notion of a “vector bundle over a holomorphic cycle”. At this level of discussion, coherent sheaves are just the branes wrapped around points, 2-cycles or 4-cycles (D3-, D5-, D7-branes, respectively).

The idea of the large volume perspective is that all the B-branes will correspond to some fractional branes at the singularity in the small volume limit³. One must find some kind of finite “basis” for the B-branes. Each coherent sheaf E_i in that basis should correspond to a fractional brane, and to a node in the quiver. Moreover, the B-branes “intersect” in a way which can be determined by algebraic geometry. At these intersection points we have massless open strings. These open string modes are counted by some generalization of cohomology groups called Ext groups. Between the coherent

³Actually it is not always true, one should usually also perform a so-called *stability* analysis, which goes beyond the B-model computation, to make sure that the B-brane is a good BPS D-brane in the small volume limit.

sheaves E and F , we have the groups

$$\mathrm{Ext}^k(E, F), \quad (3.18)$$

with $k = 0, \dots, n$ for a n -fold. Moreover, for Calabi-Yau n -folds we have the relations

$$\mathrm{Ext}^k(E, F) \cong \mathrm{Ext}^{n-k}(F, E), \quad (3.19)$$

which is called Serre duality; so we only need to compute half of the Ext groups. This topological information allows one to read the quiver, in principle. In particular one can sometimes argue (for instance as in [50]) that the Ext^0 groups correspond to a $\mathcal{N} = 1$ vector multiplets in the quiver, while the Ext^1 groups correspond to chiral multiplets. So $\dim \mathrm{Ext}^0(E, E)$ counts the rank of the gauge group at node E (we should have $\dim \mathrm{Ext}^0(E_i, E_j) = 0$ for $i \neq j$), and $\dim \mathrm{Ext}^1(E_i, E_j)$ counts the number of bifundamental fields between E_i and E_j . More complicated tricks also allow to compute the superpotential in some cases.

In practice any computation is horrendously complicated, and the above program has been carried out only in the case of complex cones over del Pezzo surfaces dP_n , $n = 1, \dots, 8$ (which are non-toric for $n > 3$); see also [51]. In that case there is a useful basis of B-branes called an exceptional collection which simplifies the problem [45, 47], and one can take advantage of existing mathematical results.

Note also that B-branes have the structure of a *category*: we have “objects”, the B-branes, and “morphism” between them, the Ext groups. It is in this category language that the comparison of B-branes on some geometry with the structure of a quiver becomes very natural, as we now explain. We are following [52, 53].

3.3.1 Homological algebra and the relation between quivers and singularities

Let us consider an abstract quiver with superpotential relations. By abstract quiver we mean that we only consider a *graph* made of nodes $i = 1, \dots, G$, and of arrows X_α between the nodes. A *path* from i to j is a string of subsequent arrows going from i to j . A quiver relation is a relation between several paths between the same two nodes. We call superpotential relation a relation which follows from a superpotential $W(X_\alpha)$, consisting of a formal sum of closed loops with signs ⁴, so that any relation F can be derived as

$$F_X = \partial_X W = 0, \quad (3.20)$$

for some arrow X . A quiver algebra is just the algebra of paths, $\mathbb{C}Q$, made of all the paths in the quiver (multiplication is the concatenation of paths), modulo the superpotential relations. We designate the relations by the ideal \mathcal{I} , so the quiver algebra is

$$\mathcal{A}(Q) = \mathbb{C}Q/\mathcal{I}. \quad (3.21)$$

Note that for any quiver with closed loops the path algebra $\mathbb{C}Q$ is infinite dimensional, although there are a finite number of generators, which are the arrows themselves.

⁴ W needs to consist of closed loops generically, so that $\partial_X W$ relates several paths between the two same nodes. It is the requirement of gauge invariance in the field theory.

Of course we could add more structure to this quiver. We could assign a vector space $E_i = \mathbb{C}^{n_i}$ to each node, and assign a linear map ($n_i \times n_j$ matrix)

$$X_{ij} : \mathbb{C}^{n_i} \rightarrow \mathbb{C}^{n_j}, \quad \text{i.e.} \quad X_{ij} \in \text{Hom}(E_i, E_j). \quad (3.22)$$

to each arrow. We could also assign a gauge symmetry $Gl(n_i, \mathbb{C})$ to each node i , acting in the obvious way. But this is some *extra* information, which is taken care of by the concept of *quiver representation*. A quiver representation is a representation of the quiver algebra, which can be generated by the assignments just made of vector spaces and linear maps. The set of all representations has the structure of a category: an object is a quiver representation, and a morphism is a set of linear maps between the vector spaces in the two quiver representations such that every square commutes⁵. This category is so-called Abelian, and we can go to the *derived category*. This is not as awful as it sounds, but we do not have space for definitions here. Suffice to say that the objects in the derived category, which we still denote E_i , are chain complex of the original objects. For instance we can have the trivial complex

$$\cdots \rightarrow 0 \rightarrow E_i \rightarrow 0 \rightarrow \cdots \quad (3.23)$$

The relevant morphisms are also called $\text{Ext}^n(E, F)$, for $n = 0, \dots, 3$ (they are “derived functors of the Hom functor”).

Physically speaking, the objects in this derived category represent the supersymmetric vacua of the quiver, for all possible ranks of the gauge groups. The group $\text{Ext}^0(E, F) = \text{Hom}(E, F)$ represent the gauge symmetries which would remain unbroken if you tried to combine objects E and F , while $\text{Ext}^1(E, F)$ is best thought as the physical modes (gauge equivalence class of modes satisfying the quiver relations) you can turn on between E and F . Lastly the elements of $\text{Ext}^2(E, F)$ are related to the quiver relations themselves. Now, for a quiver with *superpotential* relations it is possible to prove that [53]

$$\text{Ext}^1(E, F) \cong \text{Ext}^2(E, F), \quad (3.24)$$

which follows from the fact that X and F_X are somewhat “dual” because every relation follow from a superpotential⁶.

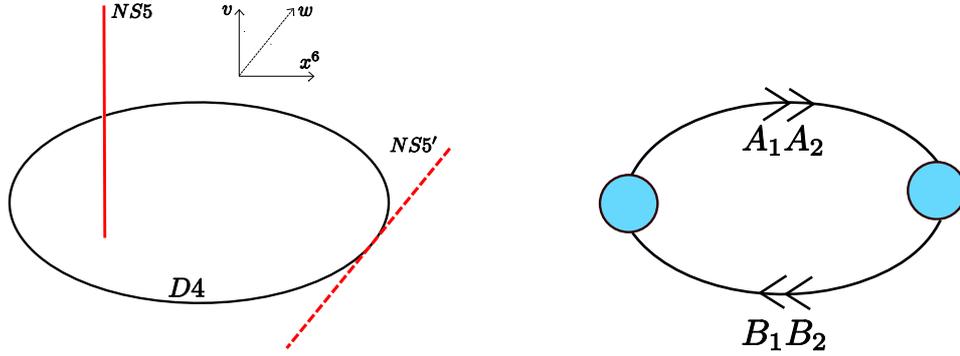
If we want to identify this quiver category structure with the B-brane category structure on a Calabi-Yau n -fold, we need that $n = 3$ because of (3.24) and (3.19). The precise conjecture is that the derived category of quiver representations is isomorphic to the B-brane category (which is actually also a derived category) on a Calaby-Yau *threefold*, at least in the case of a quiver with generic superpotential.

3.4 Hanany-Witten setups

In this section we succinctly discuss some systems of D4-branes suspended between NS5-branes, which are T-dual to systems of D3-branes on generalised conifolds in type IIB. In general, such Hanany-Witten setups [54] provide a very useful point of view on some

⁵The category of quiver representations is isomorphic to the category of $\mathcal{A}(Q)$ -modules.

⁶Actually this Serre relation holds only for certain superpotentials, such that every fractional brane representation has a projective resolution; for instance if $W = 0$ this is not the case [53].



(a) Orthogonal NS5-branes T-dual to the conifold.

(b) Conifold quiver (also known as Klebanov-Witten quiver). The superpotential is $W = A_1 B_1 A_2 B_2 - A_1 B_2 A_2 B_1$.

Figure 3.2: The Hanany-Witten setup T-dual to D3-branes on the conifold, and the associated quiver.

gauge theories, especially in the case of $\mathcal{N} = 2$ extended supersymmetry. For instance in Chapter 6 we will use such T-dual setup quite a lot.

Consider N D3-branes at the $\mathbb{C}^2/\mathbb{Z}_k$ singularity, as in section 3.2.1. T-dualising, we get k parallel NS5-branes localized around a circle in type IIA, as explained in section 2.2.2. Let x^6 be the direction of the circle. The spacing $(\Delta x^6)_i$ between the i -th and $(i + 1)$ -th NS5-brane corresponds to the the period of the B-field on the vanishing cycle \mathcal{C}_i in type IIB. There are also N D4-branes wrapped along the x^6 circle. This Hanany-Witten setup [54] is summarized in the following table:

	0	1	2	3	4	5	6	7	8	9	
NS5 _{<i>i</i>}	○	○	○	○	○	○					
D4	○	○	○	○			○				(3.25)

The D4-branes can split into k fractional branes, each fractional brane stretching between two adjacent NS5-branes. From this setup one can easily read the quiver field theory, which agrees with the one we computed in section 3.2.1. See the Figure 3.1(b). Because the NS5-branes are parallel, the “fractional” D4-branes stretched between them can move along them on the $v = x^4 + ix^5$ complex plane without breaking supersymmetry. This phenomena corresponds to fractional D3-branes moving along the singularity, or to the Coulomb branch in the low energy quiver theory.

Hanany-Witten setup of this type can also describe some $\mathcal{N} = 1$ geometries [55, 56]. The setup (3.25) preserves $\mathcal{N} = 2$ supersymmetry, and it also has a $U(1)_R \times SU(2)_R$ R-symmetry. The $U(1)_R$ corresponds to rotation in the (x^4, x^5) plane, while the $SU(2)_R$ rotates the vector (x^7, x^8, x^9) . If we rotate one NS5-brane with respect to the others, we break the R-symmetry to a diagonal $U(1)_R$, hence we must have broken the supersymmetry at least to $\mathcal{N} = 1$. This is indeed what happens [57]: if two NS5-branes are not parallel anymore, the adjoint field associated to motions of the D4-branes along the NS5-branes gets an $\mathcal{N} = 1$ mass term. When two NS5-branes are orthogonal, we can integrate out this adjoint field. In this way we obtain several new quivers from the

one of Figure 3.1(a) for $\mathbb{C} \times \mathbb{C}^2/\mathbb{Z}_k$: we simply integrate out any pair of adjoint fields for the two gauge groups located on each side of the NS5-brane which we rotate. Let us say we rotate several NS5-branes onto the $w = v^8 + ix^9$ direction, and we call these the NS5'-branes. We have

	0	1	2	3	4	5	6	7	8	9	
NS5 _{<i>i</i>}	○	○	○	○	○	○					
NS5' _{<i>j</i>}	○	○	○	○					○	○	
D4	○	○	○	○			○				

(3.26)

The simple example of the conifold geometry $xy - vw = 0$ is depicted in Figure 3.2, together with the associated quiver with superpotential. The superpotential can be deduced from the superpotential of the $\mathcal{N} = 2$ quiver by giving a mass to the adjoint fields and integrating them out. In general, this setup can describe D3-branes on any generalised conifold of the form

$$xy - v^m w^n = 0. \quad (3.27)$$

For $m = 0$ we have the $\mathcal{N} = 2$ case, $\mathbb{C}^2/\mathbb{Z}_n$. In general, we can see this geometry as a cylinder \mathbb{C}^* fibered over $\mathbb{C}^2 \cong \{v, w\}$ according to the equation $xy = f(v, w)$. When $f(v, w) = 0$ the S^1 of the cylinder pinches off. If we T-dualise along this S^1 the loci where the fiber degenerates corresponds to the locations of the T-dual NS5-branes [58]. For the generalized conifold, $f(v, w) = v^m w^n$ so that there are n NS5-branes along v (at $w^n = 0$) and m NS5-branes along w (at $v^m = 0$) in the T-dual.

Remark that from this T-duality argument we cannot specify the ordering of the various NS5 and NS5' branes along the x^6 circle. This is because such ordering depends on information about the Kähler structure (in particular about periods of the B-field).

3.5 Toric singularities and dimer models

Toric Calabi-Yau singularities form a very simple class of Calabi-Yau singularities (see Appendix B for an introduction). It is only natural to try to find the quiver for any toric CY singularity by taking advantage of the tools of toric geometry. This problem was first addressed in [59]; in this early approach, one embeds any toric singularity X into a larger $\mathbb{C}^3/(\mathbb{Z}_n \times \mathbb{Z}_m)$ singularity. The quiver of the parent orbifold is known, and one finds the quiver of X by Higgsing the orbifold quiver (corresponding to partial resolution in the geometry).

A few years ago a breakthrough was made thanks to the work of Hanany, Kennaway and collaborators [60, 61]. These papers found a beautiful underlying combinatoric structure to the so called toric quivers, which makes the translation between toric geometry and quivers very deep and surprisingly simple. The structure which made this simplification possible is known as brane tiling (and the associated dimer model). A lot of work followed, some of which will be reviewed in this section.

We are mainly concerned with the determination of the geometry associated to any toric quiver. For more comprehensive introduction to the subject, we refer to the beautiful review [62] by Kennaway.

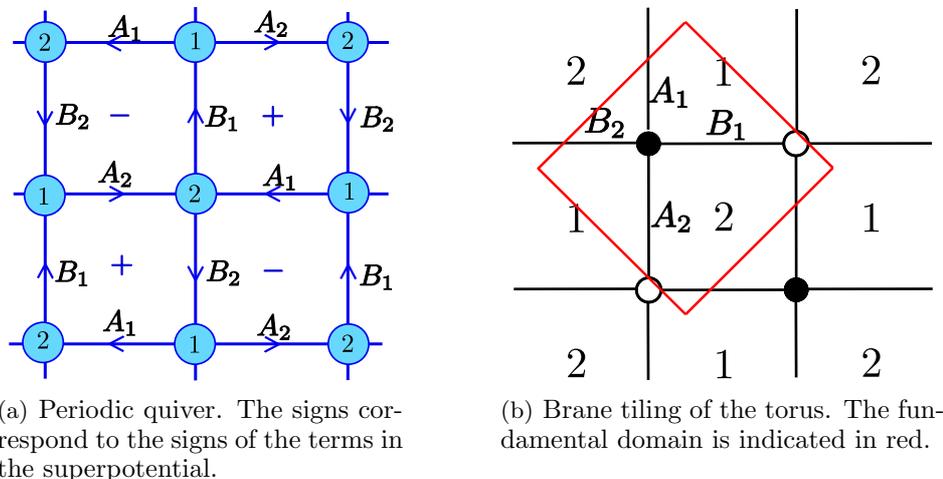


Figure 3.3: The periodic quiver and the brane tiling for the conifold quiver theory.

3.5.1 Toric quiver theories as dimer models

Toric quivers are a particular type of quiver which satisfy one additional condition: *each chiral field of the quiver appears only linearly in the superpotential, in exactly two terms with opposite signs*. This implies that all the F-term relations take the form “monomial = monomial”. For such a quiver, one can define a so-called planar quiver as follows [62]. Consider each superpotential term as a polygon. Give each polygon a positive or negative orientation according to the sign of the superpotential term, and glue the polygons together along common arrows. Since each field appears only twice, we then form an orientable Riemann surface, with the orientation inherited from the orientation of the polygons. We will assume that this Riemann surface is actually a *torus* (we can actually prove that by assuming conformality of the theory on the D3-branes, as will be discussed in the next chapter). This graph on the torus is the planar graph. We could also have drawn a bi-periodic tiling of the plane by repeating polygons; that graph is called the “periodic quiver”.

It is more interesting to consider the graph dual to the planar graph: each polygon with positive (resp. negative) orientation becomes a white (resp. black) vertex, and each quiver node becomes a face. This dual graph is bipartite graph on the torus called a *brane tiling* in the literature. An example is given in Figure 3.3. The dictionary between quiver and brane tiling is summarized in Insert 1. Given a brane tiling, a *dimer* is just a distinguished edge. A *perfect matching* is a configuration of dimers such that every vertex is touched exactly once. We define the perfect matching matrix $M_{\alpha k}$ as

$$M_{\alpha k} = \begin{cases} 1 & \text{if the perfect matching } p_k \text{ contains the fields } X_\alpha \\ 0 & \text{otherwise} \end{cases} \quad (3.28)$$

A dimer model is nothing but a brane tiling together with its perfect matchings.

Insert 1. Quiver from brane tiling

A brane tiling is a bipartite graph on the torus: it has white and black vertices in equal numbers, and non-intersecting edges connecting white to black vertices.

- Each face i corresponds to a gauge group.
- Each edge corresponds to a chiral field X_{ij} . Looking towards the white dot, the edge X_{ij} is a field in the fundamental of the gauge group i on its right and in the anti-fundamental of the gauge group j on its left.
- Each white (black) vertex corresponds to a single-trace superpotential term with the fields appearing in clockwise (counterclockwise) order, and with a plus (minus) sign in front.

3.5.2 From quiver to geometry: moduli space and the forward algorithm

There are many interesting applications of brane tilings and of the associated dimer models. Here we are mainly concerned with its use to solve for the classical moduli space of the Abelian theory [63]. This so-called *forward algorithm* [60, 61] built on the previous work [64, 65, 59].

There is a simple rule of thumb to see if a quiver can describe D3-branes at a CY_3 singularity: the vacuum moduli space should contain at least the CY threefold. This is because *regular* D3-branes are point-like probes of the geometry, and they can be displaced away from the singularity without breaking supersymmetry. Recall that the moduli space of a $\mathcal{N} = 1$ supersymmetric theory is a complex manifold of the form

$$\mathcal{M} = \{X_\alpha \mid dW = 0; D = 0\}/G. \quad (3.29)$$

We first impose the F-flatness conditions $dW = 0$ on $\mathbb{C}[X_\alpha]$, where X_α are the complex scalars in the chiral multiplets. The resulting complex manifold \mathcal{Z} was called the “master space” in recent literature [66]. Next we impose gauge invariance under the gauge group G together with the D-flatness conditions $D = 0$; equivalently one can just quotient by the complexified gauge group $G_{\mathbb{C}}$ [67] (in other words there is an equivalence between the Kähler quotient \mathcal{Z}/G and the holomorphic quotient $\mathcal{Z}/G_{\mathbb{C}}$).

The forward algorithm allows to compute efficiently the *Abelian* moduli space of any toric quiver. In the Abelian theory the gauge group is $U(1)^G$. Hence, if $W = 0$ (for instance in the Abelian conifold theory) the moduli space (3.29) has an obviously toric description as a GLSM (see Appendix B). For a non-trivial superpotential satisfying the toric condition, the master space itself should be a toric variety, because it has an affine description with relations of the form “monomial = monomial”. The master space can then be described as a GLSM as well. The non-trivial problem is to find the correct set of homogeneous coordinates for any quiver. As we will show, the perfect matchings actually do the job, although they usually give a redundant set of homogeneous coordinates.

Given a brane tiling, let us assign a complex field p_k (so called complex matching variable) to each perfect matching. Then assign to each quiver field a product of perfect matching variables, according to

$$X_\alpha = \prod_k p_k^{M_{\alpha k}}. \quad (3.30)$$

Due to the toricity condition on the superpotential, any F-term relation $\partial_{X_\alpha} W = 0$ is of the form

$$\prod'_{\beta \in V_m} X_\beta = \prod'_{\gamma \in V_n} X_\gamma \quad (3.31)$$

where \prod' means that the product does not include the field X_α linking the vertices V_m and V_n . Using the parametrization (3.30) solves the F-term relations because $\prod'_{\beta \in V_m} X_\beta = \prod_k p_k^{\sum'_{\beta \in V_m} M_{\beta k}}$, and $\sum'_{\beta \in V_m} M_{\beta k} = \sum'_{\gamma \in V_n} M_{\gamma k} = 1$ (or 0) $\forall k$, by the definition of perfect matching.

Then, we can easily construct a GLSM in term of the perfect matching variables. First there are the charges Q_D associated to the $U(1)^G$ gauge group of the quiver. Let E be the number of quiver fields (edges in the brane tiling) and m be the number of perfect matching variables. Then Q_D is a $(G-1) \times m$ matrix $(Q_D)_k^i$ which gives the charges of the p_k 's under the quiver gauge group (we really have only a $U(1)^{G-1}$ gauge group because the diagonal $U(1)$ does not couple to any field). It is defined as

$$M_k^\alpha (Q_D^t)^k_i = (Q^t)_i^\alpha \quad (3.32)$$

in term of the perfect matching matrix and of the charge matrix Q_α^i of the quiver. The definition of Q_D is only up to elements of $\ker M$. This ambiguity is fixed by introducing additional charges

$$Q_F^t = \ker M; \quad \text{i.e.} \quad M_k^\alpha (Q_F^t)^k_l = 0, \quad (3.33)$$

which are charges for the perfect matching variables which give zero charge to the quiver fields, according to the map (3.30). They are thus gauge symmetries (redundancies) of the parametrization (3.30). The GLSM is thus

$$\begin{array}{c|ccc|c} & p_1 & \cdots & p_m & \text{FI} \\ \hline U(1)_F^l & (Q_F)_1^l & \cdots & (Q_F)_m^l & 0 \\ U(1)_D^i & (Q_D)_1^i & \cdots & (Q_D)_m^i & \xi^i \end{array} \quad (3.34)$$

Note that we have also considered FI parameters here; however we do not have FI parameters for the $U(1)_F$ symmetries because they are not associated to dynamical gauge fields. It turns out that there are $m - G - 2$ charges Q_F^t . In total, the GLSM has $m - 3$ $U(1)$'s, so that the resulting space is a 3 dimensional toric variety. Actually it is a Calabi-Yau, since $\sum_{k=1}^m (Q_D)_k^i = 0$, and $\sum_{k=1}^m (Q_F)_k^l = 0$ too. The latter relation follows from the definition (3.33) and the fact that if we choose all the fields X_γ which end on the same vertex we have that $\sum_\gamma M_{\gamma k} = (1, \dots, 1)$ (all the perfect matchings are chosen once), which implies the result.

⁷It is an ‘‘experimental’’ observation [59, 66] that the master space of toric quivers has dimension $G + 2$. I am not aware of any proof in the literature (although see [63]). I thank Amihay Hanany for correspondence on this point.

Recall that, once given the GLSM charges $Q^t = (Q_F^t, Q_D^t)$, we can find the vectors of the toric cone $v = (v_x, v_y, 1)$,

$$G = \begin{pmatrix} v_x^{(1)} & v_x^{(2)} & \cdots & v_x^{(m)} \\ v_y^{(1)} & v_y^{(2)} & \cdots & v_y^{(m)} \\ 1 & 1 & \cdots & 1 \end{pmatrix}, \quad (3.35)$$

simply by taking the kernel of Q ,

$$QG^t = 0. \quad (3.36)$$

The third line of (3.35) can be chosen so due to the CY condition. Note that we have chosen m toric vectors corresponding to the m homogeneous coordinates (p_k) , but in general this is not a minimal presentation of the toric diagram (several toric vectors will be identical). For the physical meaning of these redundancies, see e.g. [62].

3.5.3 Kasteleyn matrix and fast forward algorithm

The remaining question is how to find the perfect matching variables for a given brane tiling. The answer is provided by the Kasteleyn matrix [68, 60], which we now review. Let us draw the bipartite graph on a fundamental domain, and denote γ_x and γ_y the two boundaries with orientation, which are particular representatives of the two primitive homology 1-cycles of the torus. Let us also denote by $m, n = 1, \dots, V$ the white and black vertices. Let X_{mn}^a be the edges linking the white node m to the black node n . Finally, let $\langle X, \gamma \rangle$ be the intersection number of the edge X with the 1-cycle γ . The improved *Kasteleyn matrix* is defined as

$$K_{mn} = \sum_{\gamma} X_{mn}^{\gamma} x^{\langle X_{mn}^a, \gamma_x \rangle} y^{\langle X_{mn}^a, \gamma_y \rangle}. \quad (3.37)$$

The perfect matching are enumerated by the permanent⁸ of K_{mn} , which is defined similarly to the determinant but with no minus signs,

$$\text{perm } K = \sum_{i_1, \dots, i_n} (\epsilon^{i_1 \dots i_n})^2 K_{i_1 1} \dots K_{i_n n}. \quad (3.38)$$

This permanent is known as the characteristic polynomial of the dimer model. It can be written as

$$\text{perm } K(x, y) = \sum_k p_k x^{\langle p_k, \gamma_x \rangle} y^{\langle p_k, \gamma_y \rangle}, \quad (3.39)$$

where each summand p_k is a perfect matching, written as

$$p_k = \prod_{\alpha} X_{\alpha}^{M_{\alpha k}}. \quad (3.40)$$

Note that one should not take this notation seriously, unlike in the relation (3.30); perfect matching are only a collection of edges, and would be more properly written as

$$p_k = \{ X_{\alpha} \mid M_{\alpha k} = 1 \}. \quad (3.41)$$

⁸We could also work with the determinant, but at the cost of introducing some extra signs in the definitions.

Insert 2. The fast forward algorithm

The fast forward algorithm [60, 61] is a very efficient way to find the toric diagram corresponding to the Abelian moduli space of a toric quiver. Given a brane tiling, we define the *Kasteleyn matrix*. Each row of this matrix represents a white node, each column a black node. Each entry is a sum of monomials

$$K_{mn} = \sum_{\gamma \in \{m \rightarrow n\}} X_\gamma x^{\langle X_\gamma, \gamma_x \rangle} y^{\langle X_\gamma, \gamma_y \rangle} \quad (3.44)$$

where we sum over the edges X_γ which go from the white node m to the black node n ; x, y are formal parameters and $\langle X_\gamma, \gamma_x \rangle, \langle X_\gamma, \gamma_y \rangle$ are the number of times, with sign, the field X_γ crosses the x and y boundaries of the fundamental domain. The perfect matching are found through the permanent (a determinant without the minus signs)

$$\text{perm } K = \sum_k p_k x^{h_x(k)} y^{h_y(k)}. \quad (3.45)$$

The set of points (h_x, h_z) in \mathbb{Z}^2 (sometimes called the Newton polygon of the polynomial $\text{perm } K$) is the toric diagram of the Calabi-Yau threefold we are looking for!

A change of fundamental domain (which changes the values of the $\langle X_\gamma, \gamma_{x,y} \rangle$) corresponds to a $Sl(3, \mathbb{Z})$ transformation of the toric diagram.

The coefficients

$$h_x(p_k) = \langle p_k, \gamma_x \rangle, \quad h_y(p_k) = \langle p_k, \gamma_y \rangle, \quad (3.42)$$

are called the slopes of the perfect matching p_k . They represent the flux that goes out of the fundamental domain through γ_x or γ_y , for a given perfect matching p_k :

$$h_x(p_k) = \sum_\alpha M_{k\alpha} \langle X_\alpha, \gamma_x \rangle, \quad h_y(p_k) = \sum_\alpha M_{k\alpha} \langle X_\alpha, \gamma_y \rangle. \quad (3.43)$$

This construction actually gives us more than just the perfect matchings. It directly gives us the *toric diagram*. This so-called fast forward algorithm is summarized in the Insert 2. The claim of [61], proven in [63], is that the points $(h_x, h_y) \in \mathbb{Z}^2$ are precisely the points of the toric diagram for the CY_3 associated to the brane tiling. To prove that, we need to prove that the matrix

$$G_h = \begin{pmatrix} h_x(p_1) & h_x(p_2) & \cdots & h_x(p_m) \\ h_y(p_1) & h_y(p_2) & \cdots & h_y(p_m) \\ 1 & 1 & \cdots & 1 \end{pmatrix} \quad (3.46)$$

is the same (or $Sl(2, \mathbb{Z})$ equivalent) to the matrix G defined in (3.35). In other words, we need to prove that

$$Q_F G_h^t = 0, \quad \text{and} \quad Q_D G_h^t = 0. \quad (3.47)$$

The first equality directly follows from (3.43) and (3.33). To prove the second equality, we note that, using (3.43) and (3.32),

$$(Q_D)^i_k (G_h^t)^k_x = (Q_D)^i_k M_{k\alpha} \langle X_\alpha, \gamma_x \rangle = \sum_\alpha Q_\alpha^i \langle X_\alpha, \gamma_x \rangle, \quad (3.48)$$

with \mathcal{Q}_α^i is the charge matrix of the quiver. We then easily show that the l.h.s. is zero $\forall i$ [63]. Suppose all the edges in the brane tiling are bifundamental fields (the incorporation of adjoint fields is straightforward). For each face i in the tiling, there are an even number of edges $\{X\}$ which cross γ_x . Consider two edges X and X' in $\{X\}$ which are consecutive as we go around the face i . From the definition of a brane tiling, we have that the charges $\mathcal{Q}^i(X)$ and $\mathcal{Q}^i(X')$ are the same (resp. of opposite sign) if the X and X' are separated by an odd (resp. even) number of edges. On the other hand $\langle X, \gamma_x \rangle$ and $\langle X', \gamma_x \rangle$ have opposite (resp. same) sign if the separation is odd (resp. even). Hence the contribution of X and X' cancels out in (3.48). The same reasoning holds for the y component. This completes the proof.

3.5.4 An example: the dP_1 quiver.

Let us work out an example, to see how the above concepts drastically simplify the computation of the moduli space. We consider the somewhat famous dP_1 quiver [59]. It is a quiver corresponding to D3-branes at the tip of a complex cone over the first del Pezzo surface⁹, $C_{\mathbb{C}}(dP_1)$ (which is also the real cone over $Y^{2,1}$). The quiver is shown in Fig. 3.4(a). The superpotential is

$$W = \epsilon^{ab} X_{12}^a X_{23}^b X_{31} + \epsilon^{ab} X_{41}^a X_{12}^b X_{24} + \epsilon^{ab} X_{12}^3 X_{23}^a X_{34} X_{41}^b \quad (3.49)$$

From the corresponding brane tiling shown in Fig. 3.4(b), we write down the Kasteleyn matrix,

$$K = \begin{pmatrix} X_{23}^2 & X_{31} & X_{12}^1 x^{-1} y \\ X_{34} + X_{12}^3 y^{-1} & X_{23}^1 & X_{41}^2 \\ X_{41}^1 & X_{12}^2 x & X_{24} \end{pmatrix} \quad (3.50)$$

and its permanent

$$\begin{aligned} \text{perm } K &= (X_{12}^1 X_{12}^2 X_{12}^3 + X_{23}^1 X_{23}^2 X_{24} + X_{24} X_{31} X_{34} + X_{31} X_{41}^1 X_{41}^2) + \\ &+ X_{12}^2 X_{23}^2 X_{41}^2 x + X_{12}^3 X_{24} X_{31} y^{-1} + X_{12}^1 X_{12}^2 X_{34} y + X_{12}^1 X_{23}^1 X_{41}^1 x^{-1} y. \end{aligned} \quad (3.51)$$

The resulting 2d toric diagram (made of 5 points) is shown in Figure 3.4(c). The perfect matchings¹⁰ and the corresponding points in the toric diagram are

$$\begin{aligned} a &= \{X_{12}^2, X_{23}^2, X_{41}^2\} = (1, 0, 0), & e &= \{X_{12}^1, X_{12}^2, X_{12}^3\} = (0, 0, 0), \\ b &= \{X_{12}^1, X_{12}^2, X_{34}\} = (0, 1, 0), & f &= \{X_{23}^1, X_{23}^2, X_{24}\} = (0, 0, 0), \\ c &= \{X_{12}^1, X_{23}^1, X_{41}^1\} = (-1, 1, 0), & g &= \{X_{24}, X_{31}, X_{34}\} = (0, 0, 0), \\ d &= \{X_{12}^3, X_{24}, X_{31}\} = (0, -1, 0), & h &= \{X_{31}, X_{41}^1, X_{41}^2\} = (0, 0, 0). \end{aligned} \quad (3.52)$$

The F-term relations of the theory are solved by

$$\begin{aligned} X_{12}^1 &= bce, & X_{41}^1 &= ch, & X_{23}^1 &= cf, & X_{34} &= bg, \\ X_{12}^2 &= abe, & X_{41}^2 &= ah, & X_{23}^2 &= af, \\ X_{12}^3 &= de, & X_{31} &= dgh, & X_{24} &= df, \end{aligned} \quad (3.53)$$

as one can easily check.

⁹The corresponding AdS/CFT duality was first discussed in [69, 70].

¹⁰We denote $(p_k) = (a, b, \dots, h)$ for later use in Chapter 13.

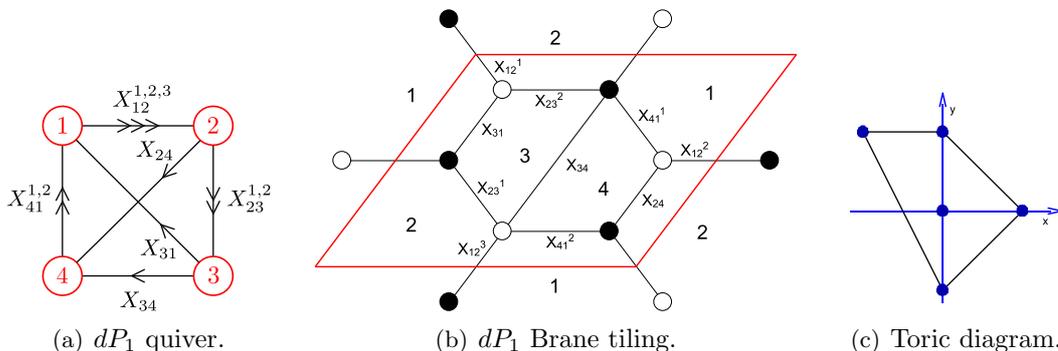


Figure 3.4: Quiver diagram, brane tiling and toric diagram of the dP_1 theory.

3.5.5 From geometry to quiver: the inverse algorithm

In the last subsections we discussed at some level of detail how to find the moduli space of any toric quiver. For completeness, we should also mention how we can recover the quiver theory from the geometric data. An efficient method (dubbed the fast inverse algorithm) was found by Hanany and Vegh in [71]. It relies on the identification of the external legs in the (p, q) -web (dual of the toric diagram) with the zig-zag paths in the brane tiling. A zig-zag path is a path along the edges of the tiling which turn maximally left at a node, maximally right at the next, and so on. Zig-zag paths form oriented loops (which do not self-intersect for a consistent quiver) which have homology charges (p, q) on the torus, and they are identified with (p, q) -legs. For more details on the algorithm, which involves a bit of drawing, we refer to the original paper [71]. An improved version of the algorithm was introduced by [72].

It is important to note that the result of the algorithm is not unique. In fact many quivers can have the same moduli space (or more precisely the same CY_3 as a branch of the moduli space). Such “toric dual” quivers are linked by some form of Seiberg duality [73, 74, 75]. This is a beautiful subject, which we will not study in this thesis.

To conclude this section, let us also note that there exists a string theory proof of the relation between D3-branes at toric singularities and brane tilings [76]. This proof uses mirror symmetry to map the set of fractional D3-branes on a cone to D6-branes intersecting in the type IIA mirror geometry. The brane tiling can then be read from this D6-brane system.

In the first Part of the thesis, we will consider systems of D3-branes on rather simple conifold geometries, in the context of the gauge/gravity correspondence. We will have to wait until the second Part, in particular in Chapter 13, to really use the power of the brane tiling techniques.

Part I

Gauge/gravity and cascades

Chapter 4

Conformal field theories and the AdS/CFT correspondence

THIS Chapter gives an introduction to the AdS/CFT correspondence. Particular focus is put on the case of the AdS_5/CFT_4 correspondence, stemming from properties of D3-branes. The general discussion of subsection 4.3.3 also holds for the case of the duality stemming from M2-branes, which we discuss in the second Part of the thesis.

4.1 Superconformal gauge theories in 3+1 dimensions

It is known since the work of Coleman and Mandula [77] that a non-trivial QFT cannot enjoy arbitrary space-time symmetries. In the presence of massive fields the most general bosonic symmetry is the Poincaré symmetry. If all the fields are massless, we can also have *conformal* symmetry. Conformal transformations are transformations which preserve the background metric up to an overall factor, $g_{\mu\nu}(x) \rightarrow e^{\omega(x)} g_{\mu\nu}(x)$.

The conformal group of d dimensional Minkowski space-time consists of the Poincaré group together with scale transformations and the special conformal transformations,

$$x^\mu \rightarrow \lambda x^\mu, \quad x^\mu \rightarrow \frac{x^\mu + a^\mu x^2}{1 + 2a_\nu x^\nu + a^2 x^2}. \quad (4.1)$$

Let us denote by $M_{\mu\nu}$ and P_μ the Poincaré generators (Lorentz transformations and translations), by D the scale transformation and by K_μ the special conformal transformations. The conformal algebra is actually isomorphic to the $SO(d, 2)$ algebra,

$$[J_{ab}, J_{cd}] = i\eta_{ac}J_{bd} + i\eta_{bd}J_{ac} - i\eta_{ad}J_{bc} - i\eta_{bc}J_{ad}. \quad (4.2)$$

Here $a = (\mu, d, d+1)$, and we use the metric $\eta_{ab} = \text{diag}(- + \dots + -)$. The $so(d, 2)$ generators are

$$(J_{ab}) = \begin{pmatrix} M_{\mu\nu} & \frac{1}{2}(K_\mu - P_\mu) & \frac{1}{2}(K_\mu + P_\mu) \\ -\frac{1}{2}(K_\nu - P_\nu) & 0 & -D \\ -\frac{1}{2}(K_\nu + P_\nu) & D & 0 \end{pmatrix} \quad (4.3)$$

4.1.1 $\mathcal{N} = 1$ superconformal algebra

Combining conformal transformations with \mathcal{N} -extended supersymmetry in four dimensions, one obtains the superconformal group $SU(2, 2|\mathcal{N})$. In the following we focus on the $\mathcal{N} = 1$ case. The generators of the $\mathcal{N} = 1$ superconformal algebra can be organised according to their scaling dimensions. We have

$$\Delta \begin{array}{cccc} & & P_\mu & \\ & Q_\alpha & & \bar{Q}_{\dot{\alpha}} \\ & & M_{\mu\nu} & R \\ & S_\alpha & & \bar{S}_{\dot{\alpha}} \\ & & K_\mu & \end{array}$$

The elements of the super-Poincaré subalgebra are the top ones: P_μ , Q , \bar{Q} and $M_{\mu\nu}$ have scaling dimension 1, $\frac{1}{2}$, $\frac{1}{2}$ and 0, respectively. Special conformal transformations K_μ have dimension -1 . We also have dilatation and R-symmetry generators Δ and R , both of dimension 0, and finally the conformal supersymmetry generators S and \bar{S} .

One can consider several real forms of this algebra. For doing quantum field theory we need a self-adjoint Hamiltonian. In Minkowsky space-time we then require that $Q^\dagger = \bar{Q}$, so that $P_\mu^\dagger = P_\mu$, and similarly for S , \bar{S} and K_μ . However this is not the most convenient way to deal with a CFT, because we cannot really define useful asymptotic states. Instead, we go with to the Euclidian theory on \mathbb{R}^4 and we consider correlation functions of operators, much like for a statistical mechanical system. Consider then \mathbb{R}^4 minus the origin. We can do a conformal transformation to the cylinder $\mathbb{R} \times S^3$,

$$dr^2 + r^2 d\Omega_3^2 \cong d\tau^2 + d\Omega_3^2, \quad (4.4)$$

with $r = e^\tau$, and then go to real time again, $\tau = it$. Then $\Delta = r\partial_r = -i\partial_t = H'$, namely the dilatation operator on \mathbb{R}^4 becomes an Hamiltonian on the cylinder. For this H' to be self-adjoint we must consider a real form of the algebra where $P_\mu^\dagger = K_\mu$ and $Q_\alpha^\dagger = S^\alpha$. This is called the radial quantization picture.

Any local operator in the Euclidean picture corresponds to a state on the cylinder. Indeed any operator insertion at the origin of \mathbb{R}^4 corresponds to a boundary condition at $t = -\infty$,

$$|\mathcal{O}\rangle = \lim_{r \rightarrow 0} \mathcal{O}|0\rangle. \quad (4.5)$$

Conversely, we can take any state on S^3 , consider it as a field configuration on some ball around the origin, and shrink the ball to zero size thanks to conformal invariance. This is the state-operator correspondence of conformal field theory.

In the radial quantization picture, one can derive interesting bounds from unitarity. In particular $\Delta \geq 1$ for scalar operators in any CFT. In any $\mathcal{N} = 1$ SCFT we also have the important bound [78]

$$\Delta \geq \frac{3}{2}|R| \quad (4.6)$$

for any scalar operator. The R-charge is normalized so that $[R, Q] = -Q$.

We now introduce the notion of primary and superprimary operators. Operators in the theory will appear in representations of the (super)conformal algebra. Notice then that K_μ lowers the conformal dimension of a field by -1 . Hence, because there is a lower

bound on Δ in a CFT, a conformal multiplet must have a lowest component, such that $[K_\mu, \mathcal{O}_p] = 0$. Such a \mathcal{O}_p is called a primary operator. Similarly, in a SCFT, S_α and $\bar{S}_{\dot{\alpha}}$ lower the dimensions by $-\frac{1}{2}$. A *superprimary* operator is the lowest component of a superconformal multiplet, i.e. it is such that

$$[S, \mathcal{O}_{sp}]_{\pm} = 0. \quad (4.7)$$

Any superprimary is also a primary. An operator which can be written as $[Q, \mathcal{O}]$ is called a superconformal *descendant*. So a superconformal multiplet consist of one superprimary operator and several descendants.

We are also interested in short representations of the superalgebra, which are called BPS or *chiral* multiplets, which are annihilated by some of the Q 's. The lowest component of a chiral multiplet is called a chiral superprimary, or chiral primary for short. It saturates the bound (4.6) as $\Delta = \frac{3}{2}R$. The dimension of a chiral primary is then protected from quantum corrections.

4.1.2 $\mathcal{N} = 4$ SYM

The $\mathcal{N} = 4$ Super-Yang-Mills theory is the maximally (rigid) supersymmetric theory in four dimension. Its Lagrangian can easily be worked out from dimensional reduction of maximally supersymmetric 10 dimensional SYM, for instance. In $\mathcal{N} = 1$ notation, it reads (traces are implicit)

$$\mathcal{L} = \int d^2\theta d^2\bar{\theta} \sum_{i=1}^3 \Phi_i e^V \Phi_i^\dagger + \int d^2\theta \left(\frac{\tau}{16\pi i} \mathcal{W}^2 + g\Phi_1[\Phi_2, \Phi_3] \right) + h.c. \quad (4.8)$$

All the fields are in the adjoint representation of $SU(N)$. It has a high degree of symmetry, the superconformal group $PSU(2, 2|4)$, whose maximal bosonic subgroup is $SO(4, 2) \times SU(4)$. The six real scalar fields ϕ_i ($\Phi_1 = \phi_1 + i\phi_2$, etc.) are in the $\mathbf{6}$ of the R-symmetry group $SU(4) \cong SO(6)$. According to the NSVZ beta function (cfr Appendix C), the gauge coupling is exactly marginal in perturbation theory, $\beta(g) = 0$. Indeed $b_0 = 0$, and the anomalous dimension for the chiral superfields must vanish by symmetry.

There is also strong evidence that this theory enjoys Montonen-Olive duality [79], also known as S-duality, exchanging strong and weak coupling, as $\tau \rightarrow -\frac{1}{\tau}$. Combining with the equivalence $\tau \rightarrow \tau + 1$ which shifts the θ angle by 2π , the conjecture is that $\mathcal{N} = 4$ SYM is invariant under any $Sl(2, \mathbb{Z})$ transformation of its gauge coupling,

$$\tau \rightarrow \frac{a\tau + b}{c\tau + d}, \quad \begin{pmatrix} a & b \\ c & d \end{pmatrix} \in Sl(2, \mathbb{Z}). \quad (4.9)$$

From the perspective of string theory, this is quite natural. $\mathcal{N} = 4$ SYM is the theory living on a bunch of flat D3-branes, and we have the identification $\tau = C_0 + i/g_s$ between the gauge coupling and the background value of the axio-dilaton. Montonen-Olive duality is then inherited from the S-duality of type IIB string theory (under which the D3-brane is self-dual).

As an important remark, note that although the gauge group on N D3-branes is $U(N)$, the diagonal $U(1)$ is free and can be decoupled in the IR. We will come back to this point later on.

4.1.3 An example of an $\mathcal{N} = 2$ SCFT

Generically theories with less supersymmetries are not conformal. However for particular field contents we can still tune all the beta functions to vanish, and hope (and hopefully check!) that the theory is really a superconformal field theory. An example which will be of particular interest to us in Chapter 6 is $\mathcal{N} = 2$ $SU(N)$ SQCD with $N_f = 2N$ fundamental hypermultiplets. In $\mathcal{N} = 1$ language, it contains one chiral superfield Φ in the adjoint representation, $2N$ chiral superfields Q in the fundamental and $2N$ chiral superfields \tilde{Q} in the anti-fundamental. Again, one can check that $b_0 = 0$ so the beta function vanishes at one loop. Actually, using the Seiberg-Witten theory one can argue that the theory is exactly superconformal [80]. It is also thought to enjoy Montonen-Olive duality.

This example of $N_f = 2N$ $\mathcal{N} = 2$ SQCD is actually an example of a continuous *family* of conformal field theories, indexed by the coupling τ which is exactly marginal. It is called a *line of fixed points* in coupling space. The SCFT we will consider in the first part of this work will also have this property of being part of a continuous line of fixed point. This is because we will consider theories which have an AdS_5 dual under the AdS/CFT map, and in all the cases the dual AdS_5 solution comes in a continuous family indexed by the VEV of the axio-dilaton.

In general, however, conformal field theories only exist as isolated fixed points in coupling space. The typical $\mathcal{N} = 2$ example is the Argyres-Douglas fixed point [81] (see also [82]). Anticipating, we note that this fact will be important in the second part of this work, which deals with SCFTs in three space-time dimensions. These theories will typically be isolated fixed points, and this is dual to the statement that the AdS_4 solutions of M-theory usually do not have marginal directions.

4.1.4 $\mathcal{N} = 1$ SCFT: an example of a strongly coupled fixed point

In four dimensional theories with extended supersymmetry, the line of fixed points always goes through the origin in coupling space, so that these theories can be made arbitrarily weakly coupled. In $\mathcal{N} = 1$ theories we can have more general families of CFTs which have no weak coupling limit. Consider for instance an $\mathcal{N} = 1$ $SU(N)$ theory with $N_f = 2N$ flavors, namely N_f fundamental fields Q_i and N_f antifundamental fields \tilde{Q}^j , with a superpotential of the form [83, 84]

$$W = h(Q_i \tilde{Q}^j)(Q_j \tilde{Q}^i). \quad (4.10)$$

Due to the flavor symmetry all the fields Q_i and \tilde{Q}^i must have the same anomalous dimension γ_0 . The beta functions are proportional to each others,

$$\beta(h) = (1 + 2\gamma_0)h, \quad \beta\left(\frac{8\pi^2}{g^2}\right) = N(1 + 2\gamma_0). \quad (4.11)$$

Hence the conditions for vanishing beta functions reduce to the single equation $\gamma_0(h, g) = -\frac{1}{2}$ in a two dimensional coupling space. Although we cannot compute $\gamma_0(h, g)$ in general, we can show that this line does not go through the origin by using Seiberg's conjecture about the existence of a non-trivial CFT in $\mathcal{N} = 1$ SQCD, namely at $h = 0$ [83]: since $\gamma_0(0, g) = -\frac{1}{2}$ has a solution for $g = g^* > 0$, the line of fixed point $\gamma_0(h, g) = -\frac{1}{2}$ never goes through the origin. Consequently, the theory is always strongly coupled.

Note that the superpotential (4.10) is not renormalizable. Consequently the theory needs some UV completion to be well defined. Actually there is the interesting option of embedding it into the finite $\mathcal{N} = 2$ $N_f = 2N$ SQCD [84]. This later theory can be deformed to $\mathcal{N} = 1$ by a mass term for the adjoint superfield Φ ,

$$W = gQ\Phi\tilde{Q} + \frac{1}{2}m\Phi^2. \quad (4.12)$$

Integrating out Φ , one recovers¹ the superpotential (4.10). Recalling that there is a S-duality symmetry in the $\mathcal{N} = 2$ theory, we can ask whether this property is inherited by the $\mathcal{N} = 1$ theory one obtains by mass deformation. The answer turns out to be positive: Seiberg duality maps the theory (4.10) to itself! These issues are discussed with great care in the original paper [84], as well as in [85], to which we refer for further details.

4.2 Anti-de-Sitter space and near horizon limit

Anti-de-Sitter (AdS) space is a maximally symmetric solution to Einstein equations with a negative cosmological constant Λ [86]. The metric for AdS space in $d + 1$ space-time dimensions reads

$$ds^2(\text{AdS}_{d+1}) = R^2 (-\cosh^2 \rho d\tau^2 + d\rho^2 + \sinh^2 \rho d\Omega_{d-1}^2), \quad (4.13)$$

with the AdS radius $R^2 = -d(d-1)/2\Lambda$. These are the so-called global coordinates. The time coordinate τ goes from $-\infty$ to $+\infty$, as part of our definition of the space (we are really considering the universal cover of AdS_{d+1}). The AdS boundary is at $\rho \rightarrow \infty$, and it has the form $\mathbb{R} \times S^{d-1}$. The isometry group of AdS is $SO(4,2)$, the same as the conformal group in d space-time dimensions². Instead of working in global AdS, we will only work on the so-called Poincaré patch, which covers only half of AdS. The Poincaré coordinates metric is

$$ds^2 = \frac{R^2}{r^2} dr^2 + \frac{r^2}{R^2} (\eta_{\mu\nu} dx^\mu dx^\nu). \quad (4.14)$$

The boundary at $r \rightarrow \infty$ is just Minkowski space-time $M^{d,1}$. Note that we took x and r to be dimensionful coordinates (x/R and r/R being dimensionless). For more details on the AdS space in our context, we refer to [38] and references therein.

4.2.1 Near horizon limit for D3-branes

Consider the $p = 3$ extremal p-brane solution (2.9), corresponding to N D3-branes in flat space. The string frame metric is

$$ds^2 = h^{-\frac{1}{2}} \eta_{\mu\nu} dx^\mu dx^\nu + h^{\frac{1}{2}} (dr^2 + r^2 d\Omega_5^2). \quad (4.15)$$

¹Actually one obtains $W = h \left((Q\tilde{Q})(Q\tilde{Q}) - \frac{1}{N} (Q\tilde{Q})^2 \right)$ because Φ is in the adjoint of $SU(N)$ and not of $U(N)$, but this subtlety will not concern us.

²Note that actually $SO(4,2)$ is the isometry of the AdS with compact time and not of its universal cover, so we really mean the algebra $so(4,2)$. The same comment applies when we consider the symmetry group of field theories: we actually never consider global issues, only the algebras.

There is a constant dilaton $e^\Phi = g_s$, and a 4-form potential $C_4 = g_s^{-1} h^{-1} dx^0 \wedge \dots \wedge dx^3$. The warp factor h is given by

$$h(r) = 1 + \frac{R^4}{r^4}, \quad \text{with} \quad R^4 = \frac{\pi^3}{\text{Vol}(S^5)} 4\pi \alpha'^2 g_s N. \quad (4.16)$$

This background preserves 16 supercharges and the metric is asymptotically flat, at $r \rightarrow \infty$. For small r , on the other hand, there is a *throat*, a steep gravitational well created by the massive D3-branes themselves. The supergravity description of the D3-branes is valid when $\sqrt{g_s N} \gg 1$, so that the curvature radius R^2 is large in units of α' . We also need $g_s \ll 1$ to suppress closed string loops. The near horizon geometry is the geometry obtained in the limit $r \ll R$,

$$ds^2 = \frac{r^2}{R^2} \eta_{\mu\nu} dx^\mu dx^\nu + \frac{R^2}{r^2} dr^2 + R^2 d\Omega_5^2. \quad (4.17)$$

It is a direct product of AdS_5 and a sphere S^5 , both of radius R . The near horizon geometry preserves twice as many supersymmetries (the new 16 supercharges are related to conformal supercharges of the dual SCFT).

4.3 The $\text{AdS}_5/\text{CFT}_4$ correspondence

The AdS/CFT conjecture stems from considering the dual description of D3-branes, in term of open or closed string degrees of freedom. We then apply a kind of “near horizon limit” to both descriptions. Here we follow the presentation of [38].

- The open string description of N D3-branes is valid when $g_s N \ll 1$. In that limit, D3-branes are rigid hypersurfaces in space-time and perturbative string theory is valid. At low energy $E < \alpha'^{-\frac{1}{2}}$, the system can be described by an effective action of the form

$$S = S_{\text{brane}} + S_{\text{bulk}} + S_{\text{int}}. \quad (4.18)$$

describing the dynamics of the massless modes of open and closed strings. In particular, S_{brane} should be the $\mathcal{N} = 4$ SYM Lagrangian plus α' corrections, for instance written in the form of a DBI action. We then take the limit $E\alpha'^{\frac{1}{2}} \rightarrow 0$, keeping the energy E fixed. In this limit, all interactions between the brane and the bulk fields vanish, and we are left with two decoupled system, $S_{\text{brane}} + S_{\text{bulk}}$: the $\mathcal{N} = 4$ SYM field theory, on the one hand, and free supergravity (a gas of non-interacting closed string modes) on the other hand.

- In the opposite limit $g_s N \gg 1$, the closed string description is valid. In the metric (4.15), there is a redshift

$$E = h^{-\frac{1}{4}} E_r \approx \frac{r}{R} E_r \quad (4.19)$$

between the energy E_r measured by an observer at constant r and the energy E measured by an asymptotic observer at infinity (the RHS being the approximation at small r). This means that small energies $E < \alpha'^{-\frac{1}{2}}$ can corresponds to arbitrarily high energies E_r at small r . We again take E to be small and fixed, and we take

the limit $E\alpha'^{\frac{1}{2}} \rightarrow 0$, while keeping $E_r\alpha'^{\frac{1}{2}}$ fixed so that we can excite arbitrarily heavy string modes in the throat. According to (4.19), we must then take $r \rightarrow 0$ with $\frac{r}{\alpha'}$ fixed. The resulting geometry is the near-horizon geometry (4.17) (with the understanding that the coordinate r is really r/α'). There is also a decoupled free supergravity system at large r , matching the open string description.

We can now state the Maldacena conjecture [9]: String theory on $AdS_5 \times S^5$ with N units of self-dual 5-form flux is equivalent to $SU(N)$ $\mathcal{N} = 4$ SYM in four dimensions. The former theory is a gravitational theory living in 10 dimensions, while the latter theory is a gauge theory in four dimensions. This duality is an example of an *holographic* duality.

Note that although the low energy gauge theory on N D3-branes is $U(N)$, the gauge group appearing in the conjecture is $SU(N)$. This is because the $U(1)$ factor is IR free and decouples in the IR. We briefly discuss this issue further in section 4.5.

As an obvious consistency check, note that both sides of the duality have the same global symmetries, the superconformal group $PSU(2, 2|4)$. In particular the R-symmetry $SU(4) \cong SO(6)$ of $\mathcal{N} = 4$ SYM is realized as the isometry group of the S^5 .

4.3.1 Various versions of the AdS/CFT conjecture

Strong form of the conjecture. In its strongest version, the AdS/CFT conjecture posits the exact equivalence of $\mathcal{N} = 4$ SYM with gauge group $SU(N)$ with type IIB string theory on $AdS_5 \times S^5$, with N units of F_5 flux through the S^5 . The relationship between the CFT and the string theory parameters are

$$g_{YM}^2 = 4\pi g_s, \quad \lambda = g_{YM}^2 N = \frac{R^4}{\alpha'^2}.$$

Here $\lambda = g_{YM}^2 N$ is the 't Hooft coupling. Since we don't really know how to deal with "string theory" at generic values of g_s and α' , this conjecture is often considered as a possible *definition* of non-perturbative string theory. In the strong form of the conjecture, what is really required is that the geometry asymptotes to $AdS_5 \times S^5$. In the bulk, we could have any quantum gravity effects, for instance bubbling geometries [87], and classical geometry should not be a valid approximation.

Classical limit. A milder version of the correspondence is obtained by turning off closed string interactions, $g_s \rightarrow 0$, while keeping fixed the curvature radius $R/\sqrt{\alpha'}$. It corresponds to the planar limit $N \rightarrow \infty$ of $\mathcal{N} = 4$ SYM,

$$g_{YM} \rightarrow 0, \lambda = g_{YM}^2 N \text{ fixed} \quad \Leftrightarrow \quad g_s \rightarrow 0, \frac{R^2}{\alpha'} \text{ fixed}.$$

Unfortunately, perturbative string theory on a curved background with RR fluxes is very hard. What has been checked however is that we can match string states to CFT states in the plane wave limit of $AdS_5 \times S^5$ [88].

Low energy limit. As a further simplification, we take the supergravity limit of low curvature. It is a strong coupling limit in the field theory,

$$g_{YM} \rightarrow 0, \lambda = g_{YM}^2 N \rightarrow \infty \quad \Leftrightarrow \quad g_s \rightarrow 0, \frac{R^2}{\alpha'} \rightarrow \infty.$$

This is the limit we will be interested in, in this work. In this limit we can use classical gravity to learn about a strongly coupled gauge theory.

4.3.2 The energy-radius relation

The *AdS/CFT* correspondence is a realization of the holographic principle [89, 90], in the sense that gravity in *AdS* is encoded in a lower dimensional theory on the boundary. It is tempting to identify the *AdS* boundary at $r \rightarrow \infty$ with the space-time on which the CFT lives (the “holographic screen”), but it would be a bit misleading. To understand better this point, we must understand the meaning of the non-compact radial direction from the point of view of the field theory. Recall that the CFT is invariant under space-time dilatations, $x^\mu \rightarrow \lambda x^\mu$. This is also a symmetry of the AdS metric (4.14), with

$$x^\mu \rightarrow \lambda x^\mu, \quad r \rightarrow \frac{1}{\lambda} r. \quad (4.20)$$

Note that r scales like the energy in the CFT. We are then lead to identify the radial coordinate r with the *energy scale* in the conformal theory,

$$E \propto \frac{r}{\alpha'}. \quad (4.21)$$

The proportionality constant does not really matter; what matters is that ratios of energies and ratios of radii are the same. The energy-radius relation is also called the UV/IR correspondence, because high energy in the field theory corresponds to large distance in the gravitational theory. It is a very important notion, especially when we want to discuss non-conformal versions of the correspondence.

4.3.3 The AdS/CFT map: general discussion

In order to flesh out the correspondence, we need a precise way to relate the observables in the two descriptions. The crucial steps to do so were taken in [91, 92]. The observables of a conformal field theory are the correlations functions, $\langle \mathcal{O} \rangle$ for any gauge invariant operator \mathcal{O} . Any such correlator should have a dual *state* in string theory. We cannot really be very specific about what are the “string theory states”, but we have come to learn that they can look very diverse: supergravity excitations, perturbative string states, but also D-branes states and even whole new geometries. In the supergravity limit of the correspondence, the available states are fewer and more manageable. From now on we will mainly work in that limit.

For later purposes, let us discuss the *AdS_{d+1}/CFT_d* map in any dimension d . It is practical to introduce a generating functional for the correlators of the CFT, of the form

$$\left\langle \exp \int_d \phi_0 \mathcal{O} \right\rangle_{CFT}, \quad (4.24)$$

where ϕ_0 is some source (one for each operator). The idea of the AdS/CFT map is that this generating function is given by the string partition function on *AdS*, with boundary conditions set at $r \rightarrow \infty$ for the excitation dual to \mathcal{O} . In supergravity, we can solve the classical equations for various fields in *AdS_{d+1}*, setting boundary conditions at $r \rightarrow \infty$. For instance, the solution for a massive scalar field in *AdS_{d+1}* has the following asymptotics at the boundary:

$$\lim_{r \rightarrow \infty} \phi(x, r) = r^{-\Delta_+} \phi_+(x) + r^{-\Delta_-} \phi_-(x), \quad (4.25)$$

Insert 3. The gist of the AdS/CFT map

To understand any AdS/CFT pair, we need to understand the dictionary between CFT operators and supergravity/string theory modes,

$$\phi(x, z) \Leftrightarrow \mathcal{O}. \quad (4.22)$$

In the expansion of a scalar mode near the boundary (at least when $m^2 R^2 > -\frac{d^2}{4} + 1$),

$$\lim_{r \rightarrow \infty} \phi(x, r) = r^{-\Delta} \phi_1(x) + r^{\Delta-d} \phi_0(x), \quad (4.23)$$

the non-normalizable mode $\phi_0(x)$ corresponds to a *source* in the CFT, while the normalizable mode $\phi_1(x)$ is proportional to the VEV $\langle \mathcal{O} \rangle$.

In other words, the normalizable modes *change the vacuum* of a given theory, while the non-normalizable modes *change the theory* by a perturbation of the UV Lagrangian. More generally the mode ϕ_0 is sometimes normalizable, in which cases it corresponds to a perturbation of the CFT by a relevant operator, which does not change the UV dynamics.

with

$$\Delta_{\pm} = \frac{1}{2} \pm \sqrt{\frac{d^2}{4} + m^2 R^2}. \quad (4.26)$$

In *AdS* a scalar field can have negative m^2 without being unstable as long as $m^2 R^2 \geq -\frac{d^2}{4}$ (the Breitenlohner-Freedman bound) [93, 94]. As long as $m^2 R^2 > -d^2/4 + 1$, the mode going like $r^{-\Delta_+}$ is normalizable, while the other one is not. If $-d^2/4 \leq m^2 R^2 < -d^2/4 + 1$ both modes are normalizable [38]. Suppose we are in the first (generic) situation, when the Δ_- is *non-normalizable*. The prescription of [92] is that this later mode is associated to the source of the dual operator, as $\phi_- = \phi_0$. Moreover, for the CFT coupling $\int \phi_0 \mathcal{O}$ to be scale invariant (recall the radius-energy relation discussed above), \mathcal{O} must scale as $r^{-\Delta_- + d}$. We then have that

$$d - \Delta_- = \Delta_+ \equiv \Delta = \dim \mathcal{O}. \quad (4.27)$$

This gives the relation between the mass of the scalar mode ϕ in supergravity and the conformal dimension of the dual gauge invariant scalar operator \mathcal{O} in the CFT. Similar relations can be worked out for modes of any spin; see [38] for more details. We can schematically summarize the AdS/CFT map as

$$\left\langle \exp \int d^d x \phi_0 \mathcal{O} \right\rangle_{CFT} = Z_{string} \left[\phi(x, r)|_{r \rightarrow \infty} \sim r^{\Delta-d} \phi_0 \right] \approx e^{-S_{SUGRA}^{\text{on-shell}}[\phi]}, \quad (4.28)$$

The RHS is the supergravity approximation, in which case we really know what we are talking about: to compute the CFT generating function in that limit, one must compute the on-shell supergravity action in term of the various fields ϕ , imposing the boundary condition we discussed above. Since such modes are non-normalizable, the on-shell action will generally diverge, and one will need to add counter-terms. This is actually the analog of the usual renormalization procedure in the CFT [95, 96].

We have just argued that non-normalizable modes correspond to boundary sources for the dual CFT operators. What about the normalizable modes, which go like $r^{-\Delta}$? Such modes decay fast enough at large r , but they become large at smaller radius. According to the UV/IR correspondence, it means that these modes have a strong effect on the IR dynamics of the CFT. They are identified as vacuum expectation values (VEV) for the CFT operator [97, 98], $\phi_- \propto \langle \mathcal{O} \rangle$.

More generally, as long as $\Delta < d$ the mode ϕ_0 is normalizable³. In that case it corresponds to a deformation of the CFT by a relevant operator, which is inconsequential in the UV. Let us note also that in the case when both ϕ_+ and ϕ_- are normalizable, we can choose either one of them to be the source, which allows to couple *AdS* scalars to operators of conformal dimensions between $\frac{d}{2} - 1$ and $\frac{d}{2} + 1$ in the CFT. In $\mathcal{N} = 4$ SYM the source always corresponds to ϕ_- , because all the gauge invariant operators have $\Delta > \frac{d}{2} = 2$, but in other CFTs it needs not be the case [98].

Let us also remark that the supergravity action in (4.28) is the $d + 1$ dimensional action on *AdS* after Kaluza-Klein (KK) decomposition of the modes on the compact part of the geometry X ($X = S^5$ in the $AdS_5 \times S^5$ case). Indeed near the boundary, $r \gg R$, the size of X gets much smaller than any finite length scale on the Minkowski boundary, so it makes sense to decompose the supergravity modes into partial waves on X .

4.3.4 The $AdS_5/\mathcal{N} = 4$ dictionary

We are not going to discuss in any details the *AdS/CFT* map between $\mathcal{N} = 4$ SYM and the dual supergravity solution. The KK spectrum from the reduction on S^5 with a self-dual F_5 flux was computed long ago in [99, 100]. All supergravity excitations fall into half-BPS representations of $SU(2, 2|4)$, which is not surprising since a long superconformal multiplet should contain states of spin higher than 2. There is a one-to-one correspondence between the supergravity excitations and the half-BPS superconformal multiplets in the $\mathcal{N} = 4$ CFT. The chiral primary operators in these multiplets are the single trace operators

$$\mathcal{O}_{i_1 \dots i_m} = \frac{1}{N} \text{Tr} \phi^{\{i_1} \dots \phi^{i_m\}}, \quad (4.29)$$

where the six scalar fields ϕ^i of the $\mathcal{N} = 4$ supermultiplet are fully symmetrized and trace-free (in term of Dynkin index, they are in the $(0, m, 0)$ of $SU(4)$). The supergravity mode dual to (4.29) is some mixture of the trace of the graviton and of fluctuations of the 5-form field [100]. For a complete list of the operators dual to supergravity modes, see the review [101] and references therein.

4.4 From $\mathcal{N} = 4$ to $\mathcal{N} = 1$. Non-spherical horizons

One might wonder whether the surprising Maldacena correspondence would not be due in large part to the very high degree of symmetry of the setup. In order to generalize the *AdS/CFT* correspondence, a first step is to consider conformal field theories with less supersymmetries. A way to do that is to replace the S^5 geometry by a more general 5-manifold which only preserves 4 supercharges. In the early days of the correspondence

³Actually the condition is $\Delta < \frac{d}{2} + 1$ for an operator dual to a scalar field in *AdS*.

this was investigated in [102, 70]. Such a “non-spherical horizon” will be a Sasaki-Einstein (SE) space (which just means that the associated cone is Calabi-Yau, as we review below).

In the same way as the maximally supersymmetric version of AdS/CFT stems from a scaling limit on D3-branes in flat space, we can consider the same limit on D3-branes in any geometry. If the D3-branes sits on a smooth point the near horizon limit will be $AdS_5 \times S^5$ as in flat space, since any smooth geometry is locally flat. We are then lead to consider D3-branes at a singularity. It should be Calabi-Yau so that the setup preserves $\mathcal{N} = 1$ supersymmetry in four dimensions (the CY itself preserves eight supercharges, and the parallel D3-branes break half of them).

On the one hand, we know the low energy field theory on such D3-branes from Chapter 3. On the other hand, there is a straightforward supergravity solution for N D3-branes on a Calabi-Yau cone. It is simply

$$ds^2 = h^{-\frac{1}{2}} \eta_{\mu\nu} dx^\mu dx^\nu + h^{\frac{1}{2}} (dr^2 + r^2 ds^2(X_5)) , \quad (4.30)$$

with X_5 the relevant SE space and

$$h(r) = 1 + \frac{R^4}{r^4} \quad \text{with} \quad R^4 = \frac{\pi^3}{\text{Vol}(X_5)} 4\pi\alpha'^2 g_s N . \quad (4.31)$$

We drop the 1 in the near-horizon limit, $h = (R/r)^4$. The $AdS_5 \times X_5$ background has an isometry group

$$SU(2, 2|1) \times G \quad \supset \quad SO(3, 1) \times U(1)_R \times G . \quad (4.32)$$

The superconformal group $SU(2, 2|1)$ has a $U(1)_R$ R-symmetry, which is an isometry of X_5 . The space X_5 might also have additional isometries G , which will correspond to additional global symmetries of the dual CFT. For instance, the X_5 base of a toric CY cone will have at least $U(1)^2 \subset G$.

4.4.1 Sasaki-Einstein manifolds

To define a Sasaki-Einstein manifold X_{2n-1} of dimension $2n - 1$, we first define its metric cone $C(X_5)$ as

$$ds(C(X_5)) = dr^2 + r^2 ds(X_5) . \quad (4.33)$$

The manifold X_5 is called Sasakian if $C(X_5)$ is Kähler. For any such Kähler metric we can define the so-called *Reeb vector*

$$\xi = I \left(r \frac{\partial}{\partial r} \right) , \quad (4.34)$$

with I the complex structure on $C(X_5)$ (it acts on the vector $r\partial_r$, giving another vector). ξ is a Killing vector (both on the cone and once pulled-back to X_5); see Appendix A of [103] for a proof. As explained in [103], a choice of metric on the cone corresponds to a choice of Reeb vector, and there exists a unique choice of ξ corresponding to a Ricci-flat metric. When the metric (4.33) is Ricci-flat, the metric on X_5 is Einstein, $R_{\alpha\beta} = (2n - 2)g_{\alpha\beta}$, as a short computation shows. Hence a Sasaki-Einstein metric on X_5 means that (4.33) is Calabi-Yau.

The Reeb vector of a SE space generates the $U(1)_R$ isometry dual to the R-symmetry of the CFT. If the orbits of ξ close, X_5 is called a regular SE manifold (or “quasi-regular” if ξ has some fixed points). Usually, the orbits of ξ do not close, and the SE space is called irregular.

The best known SE 5-manifold is the $T^{1,1}$ manifold at the base of the conifold singularity. It is reviewed in Appendix F.1. In that case $\xi = \partial_\psi$ and the space is regular. For a long time the only known Sasaki-Einstein metrics were the ones on S^5 and on $T^{1,1}$ [104]. In the last decade things changed with the construction of infinite families of irregular SE metrics on $S^2 \times S^3$ [105, 106, 107, 108]. These SE manifolds are known as $L^{a,b,c}$ (there is also the sub-family $Y^{p,q}$). The cones over them are CY toric singularities. There exists also a powerful technique, called Z-minimization, to extract the volumes of SE manifolds when the metric is not explicitly known, both in toric [109] and non-toric [103] cases.

4.4.2 Conformal $\mathcal{N} = 1$ toric quivers

In Chapter 3 we discussed the structure of the low energy field theory on D3-branes at singularities. In particular we explained how to find the quiver for any toric singularity. As we pointed out, this only extracted information about holomorphic data, which are protected by supersymmetry. On the other hand we could not learn anything about the running of the physical couplings, which depend on the renormalisation of the Kähler potential and which we cannot compute in general. The *AdS/CFT* correspondence allows us to extract more information on the *physical* quiver gauge theory. In particular, the correspondence predicts that there should be some line of fixed point in the quiver space of couplings, corresponding to the arbitrary constant value of the axio-dilaton in type IIB string theory.

In the case of toric quivers we can check this using a Leigh-Strassler type of argument [84]. Consider a toric quiver as defined in section 3.5.1. It has G gauge groups, which we suppose to be all $SU(N)$. It also has E bifundamental (or adjoint) fields and V superpotential terms. Because the corresponding brane tiling is drawn on the *torus* as a graph with G faces, E edges and V vertices, we have the Euler identity

$$G - E + V = 0. \quad (4.35)$$

The space of coupling of such a quiver is of dimension $G + V$, with G gauge couplings and V superpotential couplings. The conditions for the vanishing of all beta functions are

$$3 - \sum_{i \in v} (1 + \frac{1}{2} \gamma_i) = 0, \quad 3 - \frac{1}{2} \sum_{i \in g} (1 - \gamma_i) = 0, \quad (4.36)$$

with $v = 1, \dots, V$ and $g = 1, \dots, G$. Summing all these conditions, we have

$$\begin{aligned} \sum_v (3 - \sum_{i \in v} (1 + \frac{1}{2} \gamma_i)) + \sum_g (3 - \frac{1}{2} \sum_{i \in g} (1 - \gamma_i)) &= \\ 3V - 2E - \sum_i \gamma_i + 3G - E + \sum_i \gamma_i &= 3(G - E + V) = 0, \end{aligned} \quad (4.37)$$

where the last equality is (4.35). We then have at least one relation between the beta functions, so there should exist a *line of fixed points*. One can also reverse the argument

[61]; then conformality of the field theory implies that the planar graph of section 3.5.1 must be a graph on the torus.

4.4.3 Chiral ring of $\mathcal{N} = 1$ SCFTs

The chiral ring of any $\mathcal{N} = 1$ field theory is the cohomology of the supercharge $\bar{Q}_{\dot{\alpha}}$ (for a review, see for instance [110]). More precisely it is the set of gauge invariant operators \mathcal{O} which are chiral modulo the ones which are chiral exact,

$$[\bar{Q}_{\dot{\alpha}}, \mathcal{O}] = 0, \quad \mathcal{O} \sim \mathcal{O} + [\bar{Q}_{\dot{\alpha}}, \mathcal{O}']. \quad (4.38)$$

The classical chiral ring of a theory with superpotential W has the schematic form $\mathbb{C}[X]/\partial_X W$ in term of the fields X , and it is understood that we really consider gauge orbits only. The chiral ring relations are $\partial_X W = 0$ because the equations of motions for the chiral superfield X are of the form $\bar{D}\bar{D}X^\dagger = \partial_X W$, so that the lowest component of $\partial_X W$ is \bar{Q} -exact. In this discussion we do not include the gauge chiral superfields \mathcal{W}_α , but this can be done too (see [111] for instance). Ignoring that last point, the chiral ring is the coordinate ring of the classical moduli space seen as an algebraic variety⁴. It can be written in term of gauge invariant variables \mathcal{O} as

$$\mathbb{C}[\mathcal{O}_1, \dots, \mathcal{O}_n]/I, \quad (4.39)$$

where the ideal I contains the F-flatness relations following from $dW = 0$, while it also contains so-called *syzygies*, which are relations between gauge invariant fields following from their definition in term of gauge variant fields.

What we are really interested in is the quantum chiral ring. In general non-perturbative effects can change the chiral ring relations, with corrections involving the strong coupling scale Λ . However in the case of a conformal theory such corrections cannot arise, so that the classical and quantum chiral rings are isomorphic. Moreover, in a $\mathcal{N} = 1$ SCFT the chiral ring operators are the superconformal primaries, with their quantum dimension determined by their R-charge, $\Delta = \frac{3}{2}R$.

In the case of $\mathcal{N} = 1$ quiver SCFTs, the moduli space is a N -symmetric product of the Calabi-Yau variety associated to the quiver (we consistently ignore so-called baryonic operators in this simplified discussion), denoted $\text{Sym}^N(CY_3)$. The chiral primary operators can be identified with elements of the coordinate ring of that moduli space, as we said.

For instance, consider the case of the conifold quiver. The gauge invariant operators are of the form $\text{Tr}((AB)^k)$. To obtain a chiral primary the A 's and B 's involved must be symmetrized, due to the F-term relations. There will be many syzygies in that description. An alternative way to parametrize the chiral ring is to use the gauge invariance under the first gauge group to diagonalize the four commuting matrices $M_{ij} = A_i B_j$. Each eigenvalue of M_{ij} is identified with an affine coordinate z_{ij} for one of the N D3-branes on the conifold, and the permutation symmetry is nothing but the residual Weyl invariance of this gauge fixing. It is then clear that the eigenvalues of M_{ij} are coordinates on $\text{Sym}^N(CY_3)$, because the syzygy $M_{11}M_{22} = M_{12}M_{21}$ is precisely the conifold equation.

⁴This is possible because OPE's of chiral operators are never singular.

4.4.4 The Klebanov-Witten theory and remarks about the *AdS/CFT* map for $\mathcal{N} = 1$ quivers

The best studied (and historically the first) example of an explicit *AdS/CFT* pair which is not an orbifold of the $\mathcal{N} = 4$ case is the duality between string theory on $AdS_5 \times T^{1,1}$ and the so-called Klebanov-Witten (KW) theory [102]. The SE manifold $T^{1,1}$ is the homogeneous space $(SU(2) \times SU(2))/U(1)$ endowed with a SE metric reviewed in Appendix F. The CY cone $C(T^{1,1})$ is nothing but the conifold, whose associated quiver we already discussed.

The type IIB supergravity spectrum on AdS_5 from KK reduction over $T^{1,1}$ was analyzed in [112, 113, 114]. This spectrum can of course be matched precisely to operators in the $\mathcal{N} = 1$ CFT constructed by Klebanov and Witten. In the $\mathcal{N} = 1$ case not all the supergravity modes sit in short supermultiplets of $SU(2, 2|1)$. The particular subset of modes which do can be matched to $\mathcal{N} = 1$ chiral operators in the CFT. In particular, the chiral primary operators are scalar operators of the form

$$\text{Tr} A_{i_1} B_{j_1} \cdots A_{i_k} B_{j_k}, \quad (4.40)$$

which are completely symmetric in the i, j indices. They have dimension $\Delta = \frac{3k}{2}$ and R-charge k . They are mapped to the mode in the “chiral AdS multiplet” with the lowest *AdS* mass, as analyzed in [114].

For any SE manifold X_5 , the scalar KK modes with the lowest mass at fixed R-charge turn out to be modes dual to the chiral primary operators with this same R-charge. They are a mixing of the trace of the graviton with the F_5 fluctuations along X_5 , with masses [100, 98]

$$m^2 R^2 = 16 + E_r - 8\sqrt{E_r + 4}. \quad (4.41)$$

Here E_r is the eigenvalue of the scalar Laplacian on X_5 , which depends on the R-charge. The chiral primary operators are also classified by their representation under G in (4.32), of course. For instance $G = SU(2) \times SU(2)$ in the conifold theory, and the operators (4.40) are in the (k, k) of $SU(2) \times SU(2)$.

In the KW theory with $SU(N) \times SU(N)$ gauge group, we have three complex couplings, the gauge couplings τ_1, τ_2 and the superpotential coupling h , but the vanishing of the exact beta functions only requires that $\gamma_A = \gamma_B = -\frac{1}{2}$, similarly to what we wrote in section 4.1.4. Hence there is a surface of fixed points of two complex dimensions. One of these flat directions is related to the VEV of the axio-dilaton $\tau = C_0 + i/g_s$, according to the general argument of section 4.4.2, while the other is related to the period of the flat 2-form $C_2 + \tau B_2$ one can turn on over the 2-cycle of $T^{1,1}$. One can show⁵ that the precise dictionary is [102]

$$\tau_1 + \tau_2 + \frac{N}{\pi i} \log(h\mu) = \tau, \quad (4.42)$$

$$\tau_1 - \tau_2 = \frac{1}{2\pi^2 \alpha'} \int_{S^2} (C_2 + \tau B_2) - \tau. \quad (4.43)$$

The expressions on the LHS are RG invariant quantities on the space of coordinates, as

⁵I thank Stefano Cremonesi for interesting discussions about this point.

one can easily check. The imaginary part reads

$$\frac{8\pi^2}{g_1^2} + \frac{8\pi^2}{g_2^2} - 2N \log(|h\mu|) = \frac{2\pi}{g_s}, \quad (4.44)$$

$$\frac{8\pi^2}{g_1^2} - \frac{8\pi^2}{g_2^2} = \frac{1}{\pi\alpha' g_s} \int_{S^2} B_2 - \frac{2\pi}{g_s}. \quad (4.45)$$

Note that the relation (4.44) makes sense only for $|h\mu| > e^{-\frac{\pi}{g_s N}}$, in particular $|h\mu| \gtrsim 1$ in the supergravity limit of large $g_s N$. This means that we cannot probe the whole surface of fixed points in the supergravity limit, and in particular we have no hope to describe the fixed point which exists at $h = 0$. The later fixed point is very similar to the interacting fixed point in $N_f = 2N_c$ SQCD in the conformal window [83] so it would be very interesting to describe it holographically, but it seems that we would need string theory to do that.

4.5 Spontaneous breaking of scale invariance

To further generalize the AdS/CFT correspondence, the next step would be to break the conformal symmetry. This can be done *explicitly* in several way, and we will discuss a particularly interesting way to do this in the next chapters. In this section, we would like to discuss the *spontaneous* breaking of conformal invariance, due to a vacuum expectation value. We mainly follow [98] and some original computations, relegated to the Appendix D.

According to the strong form of the AdS/CFT correspondence, the dual string theory in asymptotically *AdS* space should not only reproduce the CFT, which sits at the origin of the moduli space of $\mathcal{N} = 4$, but also the full moduli space of vacua. This moduli space consists of a Coulomb branch only, corresponding to the configuration space of N D3-branes on \mathbb{R}^6 , with fixed center of mass. A natural thing to do is to try and take the Maldacena limit on some generic configuration of D3-branes, corresponding to a generic point on the Coulomb branch. For the supergravity approximation to be valid we should however consider big clumps of N_i D3-branes, with $\sum_i N_i = N$ and N_i large. We also assume that their separation is small in units of $\sqrt{\alpha'}$. Then the Maldacena limit just amounts to dropping the 1 in the expression (2.12) for the warp factor of a multi-centered solution.

At large r , the multi-centered solution looks like the $AdS_5 \times S^5$ solution. It is convenient to rewrite the warp factor in term of a multi-polar expansion, which makes it easy to compare with the CFT operators. For simplicity, consider a simple setup with two stacks of D3-branes separated by a distance s . A first stack of $N - N'$ D3-branes sits at $x = 0$ in \mathbb{R}^6 , and a second stack of N' D3-branes sits at $|x| = s$. In Appendix D we solve for the warp factor of the D3-brane metric, in some coordinates. The warp factor depends on a one angle θ_1 as well as on r . For $r > s$, we have

$$h(r, \theta_1) = \frac{R^4}{r^4} \left(1 + \frac{N'}{N} \sum_{m_0=1}^{\infty} \left(\frac{s}{r}\right)^{m_0} Y_{m_0}(\theta_1) \right). \quad (4.46)$$

We then need to interpret these subleading contribution as supergravity excitations around the *AdS* background. We consider the trace of the graviton along S^5 , which

corresponds to chiral primary operators as explained in section 4.4.4. Near the boundary of *AdS*₅ we have the expansion

$$h^\alpha{}_\alpha = 5 \frac{h^{1/2} - h_{AdS_5}^{1/2}}{h_{AdS_5}^{1/2}} = \frac{5N}{2N'} \sum_{m_0 > 0} \left(\frac{s}{r}\right)^{m_0} Y_{m_0} - \frac{5}{2} \left(\frac{N'}{N}\right)^2 \sum_{m_0, n_0 > 0} \left(\frac{s}{r}\right)^{m_0 + n_0} Y_{m_0} Y_{n_0} + \dots \quad (4.47)$$

The first terms in the RHS correspond to VEVs of single trace scalar operators in the CFT, the following terms corresponding to VEVs of multi-trace operators [98]. The operators which get VEVs in *SU(N)* $\mathcal{N} = 4$ SYM are of the form (4.29). These operators are in the $(0, m_0, 0)$ of *SO(6)*, and we can write them in term of some operators $\mathcal{O}^{(m)}$, defined tensorially by

$$\mathcal{O}_{i_1 \dots i_m} \hat{y}^{i_1} \dots \hat{y}^{i_m} = \mathcal{O}^{(m)} Y_m \quad (4.48)$$

(no summation on m). Here \hat{y}^i are orthonormal basis vectors of \mathbb{R}^6 , and Y_{m_0} stands for the scalar harmonics on S^5 . Of course in general m stands for the various quantum numbers (m_0, \dots, m_4) but in our simple example only the operators indexed by m_0 , $m_1 = \dots = m_4 = 0$, get a VEV, given by

$$\langle \mathcal{O}^{(m_0)} \rangle \propto \frac{N'}{(m_0 - 2)N} \left(\frac{s}{2\pi\alpha'}\right)^{m_0} \quad (4.49)$$

for $m_0 > 2$, and similarly for $m_0 = 2$, even though the analysis is more involved in that case [98]. Of course (4.49) it is simply a VEV for $\text{Tr}(\phi^1)^{m_0}$, involving only the scalar field ϕ^1 , but one can easily generalize the above analysis to any multi-centered solution.

Note that there is a $m_0 = 1$ term in (4.47). It would correspond to a VEV for a scalar operator of dimension one (i.e. a free field) :

$$\frac{1}{N} \langle \text{Tr} \phi^1 \rangle \propto \frac{N'}{N} \frac{s}{2\pi\alpha'}. \quad (4.50)$$

It corresponds to the diagonal *U(1)* of the *U(M)* theory living on the branes, and it is effectively decoupled from the *SU(M)* dynamics. It was argued in several ways that the AdS/CFT correspondence involves the gauge group *SU(M)* rather than *U(M)*, the main point being that this *U(1)* mode is free in the CFT, while everything couples at least to gravity in the bulk. In supergravity, this $m_0 = 1$ mode is related to the so-called doubleton mode that appears when dimensionally reducing type IIB on S^5 [100]. Actually, the $m_0 = 1$ term in (4.46) can be accounted for by a trivial coordinate translation to the center of mass coordinates [98], so that it is indeed non-physical (pure gauge).

The solution (4.46) is only asymptotically AdS. It can be shown [115] that it interpolates between an AdS solution at large r and another AdS throat around the $N - N'$ D3-branes sources at the origin. According to the energy-radius relation, going to small r corresponds to going towards the IR of the field theory, and one is lead to guess that the departure of the background from AdS as r decreases really describes a non-trivial *renormalization group flow* in the dual theory. In the present case, the claim is that the solution with the warp factor (4.46) describes a renormalization group flow interpolating between *SU(N)* $\mathcal{N} = 4$ in the UV and *SU(N - N')* $\mathcal{N} = 4$ in the IR, due to non-trivial VEVs. In the field theory there is a decrease in the number of degrees of freedom because

we should integrate out modes which become heavy due of the Higgs mechanism, while in the gravity description we see some continuous decrease in the F_5 flux,

$$-\frac{1}{(4\pi^2\alpha')^2 g_s} \int_{S^5} F_5 = N + N' \left(\frac{r^6}{(r^2 + s^2 - 2sr \cos \theta_1)^3} - 1 \right), \quad (4.51)$$

which interpolates between N at large r and $N - N'$ for $r \ll s$. However, it is difficult to make these statements very precise. For instance the θ_1 dependence seems to forbid any unambiguous identification of the radius r with the energy scale in the dual theory.

We will be able to make much more precise statements in the models we will study in the next chapters. In those setups some monotonic functions in the supergravity background will be matched exactly to the RG running of coupling constants as computed from the exact $\mathcal{N} = 1$ beta functions in the dual field theory.

Chapter 5

Fractional D-branes and gauge/gravity correspondence

5.1 Overview: the gauge gravity/correspondence

THE AdS/CFT correspondence can be explored and extended in many directions. One question of obvious physical interest is whether one could use the correspondence to achieve some better understanding of QCD. Let us be less ambitious, and ask whether we could construct models which are at least closer to QCD than superconformal field theories. In particular a reasonable goal is to construct some string theory dual to $\mathcal{N} = 1$ SQCD.

We will be even humbler, and investigate models dual to $\mathcal{N} = 1$ Super-Yang-Mills (SYM) theory. There are roughly three models on the market, which we will enumerate in historical order. The first is the Polchinski-Strassler model [116], in which the $\mathcal{N} = 4$ AdS/CFT duality is deformed by relevant operators, namely by mass terms breaking $\mathcal{N} = 4$ to $\mathcal{N} = 1$ SYM. The second model is the Maldacena-Nuñez (MN) solution [117], which is obtained by wrapping N D5- or NS5-branes around a 2-sphere, and finding an associated supergravity solution. The third is the Klebanov-Strassler (KS) solution [118], which we will discuss in detail in this chapter.

It is often the case that we only discuss the *supergravity limit* of some string theory models. However one must keep in mind that it is really their embedding into string theory which makes them so interesting. For instance a fundamental string can be dual to a flux tube, a D-brane can be dual to a domain wall, etc. The rule of the game is generally to find a consistent supergravity solution, and to interpret it in terms of the dual theory. The conjectured duality between non-conformal field theories on the one hand and supergravity (or string theory) models such as the MN or the KS solution on the other hand is called the *gauge/gravity correspondence*. In such models we usually know much less about the precise map between observables than in the AdS/CFT case. However the models we consider in this chapter are “close” to the AdS/CFT case in a precise sense, which allows to apply similar techniques. For instance in these models one can also apply the holographic renormalization program of AdS/CFT, as shown in [119].

We will consider models based on *fractional D3-branes*. Consider for instance the straightforward definition of fractional branes on $\mathcal{N} = 1$ or $\mathcal{N} = 2$ orbifolds given in

section 3.2.2. In Chapter 3 we discussed the quiver for regular D3-branes on \mathbb{C}^3/Γ but we could also discuss any subquiver associated to some other representation of Γ . In particular the field theory on M fractional branes of a single type is simply $SU(\dim\Gamma_i M)$ $\mathcal{N} = 1$ SYM. In the case of $\mathcal{N} = 2$ orbifolds the low energy theory on the fractional branes is $\mathcal{N} = 2$ SYM, and the adjoint scalar field in the $\mathcal{N} = 2$ vector multiplet corresponds to the fact that the singularity $\mathbb{C} \times \mathbb{C}^2/\Gamma$ is non-isolated. The story is similar for D3-branes on any Calabi-Yau singularity.

In reality, not any such fractional brane is physical, because typically there might be tadpoles. This happens if the compact holomorphic cycle wrapped by the fractional D3-brane is dual to another compact cycle in the Calabi-Yau threefold. Then the RR flux from the brane has nowhere to go and there is an inconsistency. Cancellation of RR tadpoles is dual to cancellation of anomalies in the quiver. The quiver for a regular D3-branes is typically a chiral theory, so there might be gauge anomalies. When considering a candidate fractional brane one must check that the quiver dual to N regular branes plus M fractional branes is not anomalous; this means that there must be as many incoming as outgoing chiral fermions at each gauge group. We will not need to discuss this point further because the models we will consider in the following are based on non-chiral quivers, such as the conifold quiver.

5.1.1 The issue of the UV completion

All the gauge/gravity dualities alluded to above have a common unavoidable problem: they might look very much like $\mathcal{N} = 1$ at small scale, but they always have some very different UV completion. In the Polchinski-Strassler model the UV is simply $\mathcal{N} = 4$, in the Maldacena-Nuñez model the UV is a six-dimensional theory. In these models one would like to decouple the SYM theory from its UV, but it is never possible in the supergravity limit. The generic reason is that the scale of the UV physics is set by a scale m (the mass for the adjoints in Polchinski-Strassler or the mass of the first KK mode in the MN solution), while the dynamically generated SYM scale is always of the form $\Lambda = me^{-\frac{c}{\lambda}}$, with λ the running 't Hooft coupling. In the supergravity limit λ is large and the SYM scale is of the same order as the scale of the UV physics. The story is similar in the Klebanov-Strassler model, although the UV physics is somewhat more exotic, as we will explain in this Chapter.

Actually we cannot expect to have a supergravity dual to any QCD-like theory. A general argument to that effect is that in Yang-Mills theories one expects to have a spectrum of glueballs of arbitrary spin, following a Regge trajectory, while in supergravity one cannot have excitations of spin higher than 2. We need string theory to reproduce a Regge trajectory; after all that was the original motivation to invent it.

Nevertheless, people have learned a great deal by studying supergravity solutions dual to SYM-like theories. Unfortunately we will not have the space to cover much of the many beautiful things people computed already at the level of supergravity.

5.2 Supersymmetry conditions

In this chapter we want to discuss supergravity solutions corresponding to fractional D3-branes, which are D5-branes wrapped on vanishing 2-cycles¹. Fractional branes couple magnetically to the RR potential C_2 , so the supergravity solutions we consider should have some F_3 flux turned on.

Let us consider the supersymmetry variations of the gravitini and dilatini in type IIB supergravity [120] (we write them as in the Appendix of [121]). We consider the case of a constant axio-dilaton $\tau = i$. Then, the conditions for supersymmetry are

$$\begin{aligned}\delta\psi_M &= \nabla_M\epsilon + \frac{i}{1920}F_{M_1\dots M_5}^{(5)}\Gamma^{M_1\dots M_5}\epsilon + \frac{i}{96}G_{M_1M_2M_3}^{(3)}(\Gamma_M^{M_1M_2M_3} - 9\delta_M^{M_1}\Gamma^{M_2M_3})\epsilon^* \quad (5.1) \\ \delta\lambda &= \frac{1}{24}G_{M_1M_2M_3}^{(3)}\Gamma^{M_1M_2M_3}\epsilon = 0 \quad (5.2)\end{aligned}$$

As is well known, in the absence of fluxes these conditions reduce to $\nabla_M\epsilon = 0$, in which case we need covariantly constant spinors to preserve supersymmetry. For an $\mathbb{R}^4 \times X_3$ ansatz this means we have at least one covariantly constant spinor on X_6 ², which reduces the holonomy from $Spin(6) \cong SU(4)$ to $SU(3)$, hence X_6 is a Calabi-Yau. In our case we do have fluxes, as well as warping functions in the metric. The general conditions for supersymmetry have been analysed in [122] in term of $SU(3)$ structures, with the upshot that in type IIB the space X_6 must still be a complex manifold, but need not be Kähler.

We will instead consider the simple case of a D3-brane-like ansatz,

$$ds^2 = h^{-\frac{1}{2}}\eta_{\mu\nu}dx^\mu dx^\nu + h^{\frac{1}{2}}ds^2(X_6), \quad (5.3)$$

$$F_5 = (1 + *_{10})dh^{-1} \wedge dx^0 \wedge \dots \wedge dx^3. \quad (5.4)$$

It corresponds to the so-called ‘‘type B’’ supersymmetric ansatz [123], for which we can split the 10 dimensional spinor ϵ as

$$\epsilon = \zeta \otimes \chi_1, \quad (5.5)$$

in which case the variations with respect to ϵ and ϵ^* are linearly independent. With this ansatz the variation $\delta\psi_\mu$ identically vanishes, while the ϵ part of $\delta\psi_m = 0$ (with m standing for the internal coordinates) implies that the metric on $ds^2(X_6)$ is Calabi-Yau [124, 123]. The remaining conditions imply that G_3 is a primitive $(2, 1)$ -form with respect to the Kähler and complex structures of the CY X_6 . This implies that it is imaginary self-dual (ISD) with respect to the CY metric on X_6 ,

$$*_6G_3 = iG_3. \quad (5.6)$$

This is all we need to know to understand the models discussed in this thesis. Remark however that this ansatz is very particular. For instance the KS solution is a particular vacuum, while the full family of vacua dual the baryonic branch of the KS field theory was found by solving a much more general ansatz [125] in term of $SU(3)$ structure conditions [126].

¹At the level of the B-model a fractional brane could be a wrapped 7-brane, but to have no tadpole such a brane should be dual to a non-compact 2-cycle. This would mean we are on a SE 5-manifold with a 1-cycle (and of course a 4-cycle) in homology, which is impossible for a CY_3 of strict $SU(3)$ holonomy.

²Then we have at least $\mathcal{N} = 2$ SUSY in 4 dimensions because we can vary independently both gravitini/dilatini along the $\mathbb{R}^{3,1}$.

5.3 Fractional branes at the conifold singularity

5.3.1 Backreacting fractional branes on the conifold: the KT solution

The “naive” solution for N regular D3-branes and M fractional D3-branes was first found by Klebanov and Tseytlin [127], and will be referred to as the KT solution. With the formalism explained so far, and assuming the ansatz of the previous section, the derivation of a supersymmetric solution becomes almost easy. We want a supersymmetric solution to the equations (A.5), with the N D3-branes replaced by N units of F_5 flux and an explicit source term for M wrapped D5-branes localised at $r = 0$. Imposing the ISD condition for $G_3 = F_3 + iH_3$, the only remaining equations are the Bianchi identities

$$dG_3 = -2\pi\alpha'g_sM\delta_{D5}, \quad dF_5 = -H_3 \wedge F_3. \quad (5.7)$$

The D5-brane is wrapped on a holomorphic 2-cycle of $T^{1,1}$, and its magnetic flux escapes through the dual 3-cycle³. As reviewed in Appendix F, there exist a 3-form (F.16) ω_3^{CF} which integrates to $8\pi^2$ over the conifold 3-cycle. We can solve the BI for F_3 by

$$F_3 = -\frac{\alpha'g_s}{2}M\omega_3^{CF}, \quad \text{so that} \quad -\frac{1}{4\pi^2\alpha'g_s} \int_{S^3} F_3 = M. \quad (5.8)$$

This gives the normalization for G_3 , which must otherwise be proportional to the primitive (2,1)-form (F.17) of the conifold,

$$G_3 = -\frac{\alpha'g_s}{2}M\omega^{(2,1)}. \quad (5.9)$$

This solution implies a $\log r$ dependence of the background B-field, as first noted in [128]:

$$H_3 = \frac{3}{2}\alpha'g_sM\frac{dr}{r} \wedge \omega_2^{CF}, \quad \Rightarrow \quad B_2 = \alpha' \left(\pi b_0 + \frac{3}{2}g_sM \ln \frac{r}{r_0} \right) \omega_2^{CF}. \quad (5.10)$$

The integration constants b_0 and r_0 are not independent, obviously, but it will turn out convenient to write B_2 in this way (then b_0 is the real integration constant while r_0 will be the integration constant in the warp factor solution (5.12) below). The remaining equation we have to solve is the BI for F_5 , which gives an equation for the warp factor h ; rotational symmetry (the branes are at the tip) implies that h depends only on r :

$$\frac{1}{r^5} \partial_r r^5 \partial_r h(r) = \frac{81}{2} (\alpha'g_sM)^2 \frac{1}{r}. \quad (5.11)$$

The solution is easily found to be $h \sim \ln r/r^4$. Remembering there must also be N units of flux through $T^{1,1}$ due to the regular D3-branes, the full solution reads

$$h(r) = \frac{R^4}{r^4} \left(1 + \frac{3}{2\pi} \alpha'g_s \frac{M^2}{N} \left(\ln \frac{r}{r_0} + \frac{1}{4} \right) \right). \quad (5.12)$$

³This can be seen for instance in toric geometry: the two-cycle is the intersection of two toric divisors corresponding to opposite points of the toric diagram, while the 3-cycle can be taken as the base of a third toric divisor.

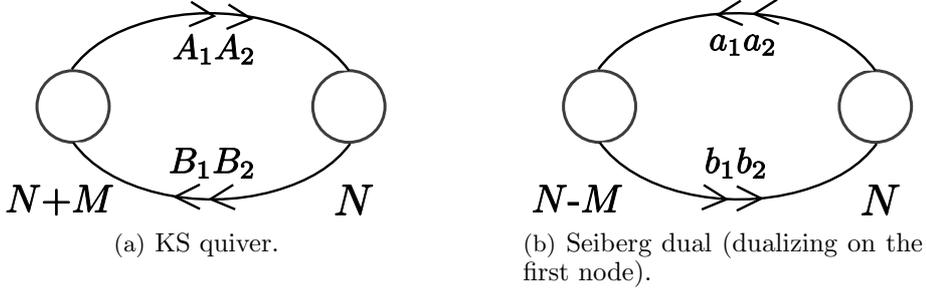


Figure 5.1: Quiver diagram of the $U(N+M) \times U(N)$ KT/KS theory (on the left) and its Seiberg dual (on the right). Remark that the dual quiver is self-similar with gauge group $U(N) \times U(N-M)$.

The constant R^4 is defined as in (4.31), with $X_5 = T^{1,1}$ (it has volume $\frac{16}{27}\pi^3$). The integration constant has been fixed in order to have N units of D3-brane charge at $r = r_0$. The D3-brane charge is actually running with r ,

$$Q_{D3} = -\frac{1}{(4\pi^2\alpha')^2 g_s} \int_{T^{1,1}} F_5 = N + \frac{3}{2\pi} g_s M^2 \ln \frac{r}{r_0}. \quad (5.13)$$

The KT solution we have just derived is perfectly smooth at large r . Differently from the solution discussed around equation (4.51), however, the metric is not asymptotically AdS but instead differs from it by logarithmic corrections. We will discuss this result and its interpretation in the next subsection, but we can already anticipate that these log corrections beautifully encode the renormalization group running of the coupling constants in the quiver theory.

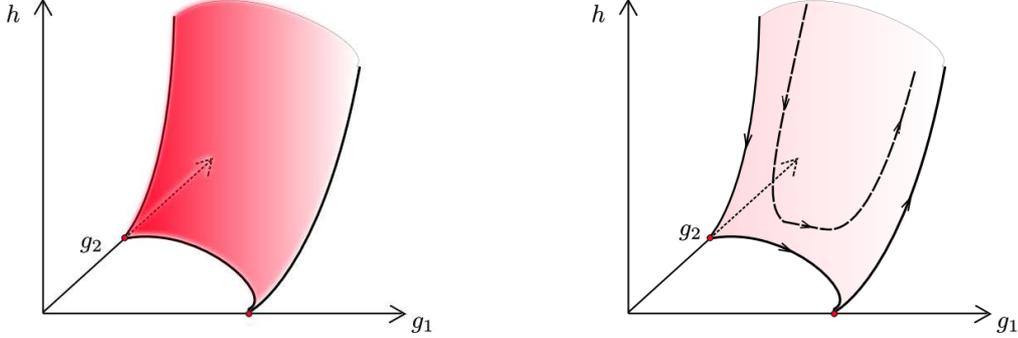
At small r , however, the metric has a naked singularity [127]. This could be expected from the fact that the D3-brane charge (5.13) goes negative for r small enough, which should not be allowed physically.

5.3.2 Cascade in the $\mathcal{N} = 1$ quiver

The naive interpretation of the KT solution is that it results from backreacting N regular branes and M fractional branes. In that case the dual quiver is the one given in Figure 5.1(a), with gauge group $SU(N+M) \times SU(N)$. However, the expression (5.13) for the D3-brane charge immediately leads to problems with this interpretation. Firstly it is continuous, and secondly it increases as r increases.

The first puzzle is not really a problem. The gauge invariant D3-brane charge $\int F_5$ does not count the number of D3-branes, because it is not quantized, as briefly reviewed in Appendix A. One can define the Page charge (A.9), which is quantized but not invariant under large gauge transformations. This Page charge is equal to $N + b_0 M$, although there is an ambiguity related to large gauge transformations, which is actually related to Seiberg dualities in the field theory. We will spell out this interpretation in more details in Chapter 7, section 7.3.3; see [129] for the original presentation.

In the following we want to interpret the logarithmic variation in F_5 and B_2 from the field theory perspective. Let us consider the theory in Figure 5.1(a), with gauge group



(a) Surface of fixed points in the KW model. The red dots on the axis of g_1 and g_2 are the SQCD fixed points.

(b) Renormalization group trajectories in the KT/KS model.

Figure 5.2: On the left we show the surface of fixed points in the Klebanov-Witten SCFT with $SU(N) \times SU(N)$ gauge group. On the right the gauge group is $SU(N+M) \times SU(N)$ and there is no surface of fixed points anymore, only the SQCD fixed points survive (the red dots). We show an RG trajectory which is first attracted to $g_2 = g_2^*$ while the superpotential coupling h decreases, until the perturbation due to the first gauge group takes over and g_1 begins to increase. As we go nearer to the $g_1 = g_1^*$ SQCD fixed point the coupling h becomes relevant and start increasing again. The area in pale red on the right is the surface where the RG running is very slow, as long as $N \gg M$, since this surface is inherited from the KW surface of fixed points. These figures are inspired from [85].

$SU(N+M) \times SU(N)$, and superpotential

$$W = h(A_1 B_1 A_2 B_2 - A_1 B_2 A_2 B_1) \quad (5.14)$$

We will analyze the RG flow in some detail, following [118, 85]. We have three distinct beta functions,

$$\beta\left(\frac{8\pi^2}{g_1^2}\right) = N + 3M + 2N\gamma_0, \quad \beta\left(\frac{8\pi^2}{g_2^2}\right) = N - 2M + 2(N+M)\gamma_0, \quad \beta(h) = (1 + 2\gamma_0)h. \quad (5.15)$$

If we set two of the coupling constants to zero, we can find a fixed point for the third coupling. Consider the SQCD fixed point at $g_2 = h = 0$, $g_1 = g_1^*$. At this fixed point we have $\gamma_0 = -\frac{1}{2} - \frac{3M}{2N}$, and we see that the coupling g_2 is irrelevant: for any small $g_2 > 0$, the RG flow will lead again to the SQCD fixed point of the first gauge group $SU(N+M)$, as long as $h = 0$. On the other hand, the coupling h is relevant there; this is completely analogous to the case of SQCD with $N_f < 2N_c$ reviewed in Appendix C. This behavior is shown in Figure 5.2(b).

The knowledge of SQCD with quartic superpotential explained in Appendix C allows us to follow the RG flow if we stay near the boundary of the surface of fixed points of the KW model. As the superpotential coupling h increases, what should we do? We can go to

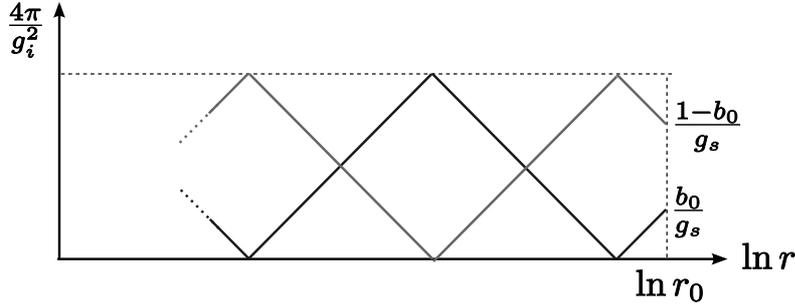


Figure 5.3: The KT/KS RG flow, a.k.a. Seiberg duality cascade. Each time one of the two gauge coupling diverges, one goes to another Seiberg dual frame; the coupling of the Seiberg dual gauge group decreases while the other coupling will now increase and eventually diverge.

a Seiberg dual description! The Seiberg dual theory has gauge group $SU(N-M) \times SU(N)$ shown in Figure 5.1(b) and a self-similar superpotential

$$W = h'(a_1 b_1 a_2 b_2 - a_1 b_2 a_2 b_1), \quad (5.16)$$

which arises after integrating out the dual mesons $M_{ij} = A_i B_j$. In this new description the coupling h' decreases and we flow towards the SQCD fixed point of $SU(N-M)$. But if g_2 is non-zero the $SU(N)$ gauge group is now in the same position as the first gauge group before: the SQCD fixed point of this $SU(N)$ with $N_f = 2N - 2M$ is now an attractor for the RG flow. As we approach the region of large g_2 the coupling h' becomes relevant, and the story unfolds as before. This is the *cascade*: we can follow the RG flow through multiple Seiberg dualities, and at each step the rank of one or the other gauge group decreases by $2M$, but the theory remains self-similar.

Going upstream in the cascade, we see that we have an $SU(N+M) \times SU(N)$ theory at any scale, but N gets bigger and bigger. It is a quite unconventional UV completion.

How does this RG flow match with the supergravity solution with “running fluxes”? To have a precise matching, we need a precise energy/radius relation, similarly to the AdS/CFT case, and this seems like a hard problem. However, let us see what happens if we keep the same relation as in the AdS case, $r/r_0 = E/E_0$. Suppose we define the dual theory at an energy E_0 corresponding to the integration constant r_0 in (5.13). There we identify $Q_{D3} = N$ with the parameter N in the ranks of the gauge groups of the quiver theory. We want to take $N \gg M$; this is just a choice of the integration constant r_0 to be large enough, and in the field theory it corresponds to going at very high energy in the cascade. In this UV region we can use the KW results, in particular $\gamma_0 = -\frac{1}{2}$ up to corrections of order $(M/N)^2$ ⁴. The field theory beta functions become

$$\beta\left(\frac{8\pi^2}{g_1^2}\right) = 3M + o\left(\frac{M}{N}\right)^2, \quad \beta\left(\frac{8\pi^2}{g_2^2}\right) = -3M + o\left(\frac{M}{N}\right)^2, \quad \beta(h) = o\left(\frac{M}{N}\right)^2. \quad (5.17)$$

We see that in this UV limit the superpotential coupling is at a quasi fixed point, as its running is much slower than the running of the gauge couplings. For simplicity we

⁴The first corrections are at this order because there is a \mathbb{Z}_2 symmetry which exchanges $M \rightarrow -M$ and $N \rightarrow N + M$ [118]. It is important that there is no $o(M/N)$ corrections, because that would be incompatible with the constancy of the dilaton in the gravity solution.

can take $h\mu \approx 1$. In this approximation the AdS/CFT relationships (4.44)-(4.45) can be used, and they simplify to

$$\frac{8\pi^2}{g_1^2} + \frac{8\pi^2}{g_2^2} \approx \frac{2\pi}{g_s}, \quad (5.18)$$

$$\frac{8\pi^2}{g_1^2} - \frac{8\pi^2}{g_2^2} = \frac{1}{\pi\alpha'g_s} \int_{S^2} B_2 - \frac{2\pi}{g_s}. \quad (5.19)$$

In the KT background, we have that

$$\frac{8\pi^2}{g_1^2} - \frac{8\pi^2}{g_2^2} = \frac{4\pi}{g_s} \left(b_0 - \frac{1}{2} \right) + 6M \ln \frac{r}{r_0}. \quad (5.20)$$

Using the energy/radius relation $r \sim E$, we can read the RG flow from supergravity. We have the running

$$\frac{8\pi^2}{g_1^2} = \frac{2\pi}{g_s} b_0 + 3M \ln \frac{r}{r_0}, \quad \frac{8\pi^2}{g_2^2} = \frac{2\pi}{g_s} (b_0 - 1) - 3M \ln \frac{r}{r_0}, \quad (5.21)$$

which exactly agrees with the field theory result (5.17), with initial conditions set by b_0 (when $b_0 = \frac{1}{2}$ both gauge couplings are equal at $r = r_0$). The gauge coupling g_1^2 increases while g_2^2 decreases. The puzzle is how can we follow the RG flow when g_1^2 blows up, which happens when $r = r_0 \exp(-\frac{2\pi b_0}{3Mg_s})$. At that scale the supergravity solution is perfectly well behaved, but the period $\int_{S^2} B_2$ becomes negative. Moreover, at that scale the Maxwell D3-brane charge (5.13) is shifted from N to $N - b_0 M$. This is the first step of the cascade: we can go to a Seiberg dual frame in the field theory, as we explained, and follow the RG flow. In supergravity the Seiberg duality corresponds to a shift $b_0 \rightarrow b_0 - 1$ of the B-field, which is a large gauge transformation [129]. From there we can go on. For each cascade step $r_k/r_{k-1} = \exp(-\frac{2\pi}{3Mg_s})$ the Maxwell charge shifts by $-M$. This duality cascade is shown in Figure 5.3.

5.3.3 The low energy theory and the deformed conifold

Suppose we start somewhere in the UV with an $SU(N+M) \times SU(N)$ gauge group, with $N = kM$ for k some large integer. As we go down along the cascade and flow to the IR, the gauge group will eventually reduce to $SU(2M) \times SU(M)$. The $SU(M)$ group is asymptotically free so it seems that the dynamics is dominated by the $SU(2M)$ gauge group, which has $2M$ flavors. The dynamics here is very close to what happens for SQCD with $N_f = N_c$, the only difference being that the flavor group is broken to $SU(M) \times SU(2) \times SU(2)$ by the superpotential (5.14). The low energy dynamics is equivalent to a system of gauge invariant (under $SU(2M)$) mesons and baryons [130, 131],

$$M_{ij} = B_i A_j, \quad \mathcal{B} \sim A^{2M}, \quad \bar{\mathcal{B}} \sim B^{2M}. \quad (5.22)$$

The baryons $\mathcal{B}, \bar{\mathcal{B}}$ are the fully antisymmetric combinations of the fields A_i and B_j , respectively, constructed by using the ϵ tensor of $SU(2M)$. They are $SU(M) \times SU(2) \times SU(2)$ singlets. It is well known that $N_f = N_c$ SQCD has a quantum deformed moduli space.

This deformation can be implemented through a Lagrange multiplier in the superpotential. In our case,

$$W = h(M_{11}M_{22} - M_{12}M_{21}) + \xi(\det \mathcal{M} - \mathcal{B}\bar{\mathcal{B}} - \Lambda_{2M}^{4M}). \quad (5.23)$$

Note that the matrix \mathcal{M} of mesons must be understood as a $2M \times 2M$ matrix. The scale Λ_{2M} is the dynamically generated scale of $SU(2M)$. The moduli space has two branches, as we will show. The F-terms relations are

$$h \begin{pmatrix} M_{22} & -M_{12} \\ -M_{21} & M_{11} \end{pmatrix} = \xi(\det \mathcal{M})\mathcal{M}^{-1}, \quad \xi\mathcal{B} = 0 = \xi\tilde{\mathcal{B}}, \quad (5.24)$$

together with the constraint $\det \mathcal{M} - \mathcal{B}\tilde{\mathcal{B}} = \Lambda_{2M}^{4M}$. If $\mathcal{B}, \tilde{\mathcal{B}} \neq 0$, we must have $\xi = 0$ and then $M_{ij} = 0$. It is the *baryonic branch*,

$$M_{ij} = 0, \quad \mathcal{B}\tilde{\mathcal{B}} = -\Lambda_{2M}^{4M}. \quad (5.25)$$

which is the merger of two classical baryonic branches into a single one. The second branch is the mesonic branch, with non-vanishing VEVs for M_{ij} . In that case we must have $\xi \neq 0$, which forces the baryons to vanish. We will not discuss this branch further here ⁵.

Let us concentrate on the baryonic branch. In this vacuum, we can integrate out all the modes because we expect a mass gap of order Λ_{2M} , as in any confining theory. At scales below Λ_{2M} , we are left with pure $SU(M)$ from the second gauge group, which has its own dynamical scale Λ_M . Actually, the baryonic branch vacuum breaks the baryonic symmetry $U(1)_B$ which gives charge $+1$ to A_i and -1 to B_i , hence we must also have a massless Goldstone boson [134] in the low energy theory. In summary, the theory in the IR must be $SU(M)$ SYM together with a massless Goldstone chiral multiplet.

What about the dual string theory solution? A natural expectation is that the singularity at small r in the KT solution signals the breakdown of the supergravity approximation near the fractional D3-brane sources, and that we need stringy effects to resolve this singularity. This expectation is correct, but it turns out, nevertheless, that the overall effect of these stringy corrections can be fully captured by a supergravity solution, as was shown in the seminal work of Klebanov and Strassler [118].

In a confining theory with a mass gap, one expects the holographic direction encoding the energy scale to terminate in the IR. One can see this using the area law criterion for Wilson loops, which are dual to open strings with their endpoints fixed at the boundary [135, 136]. We refer to the review [137] and references therein for more details.

A crucial clue as to how the warped conifold geometry could terminate comes from computing the moduli space of the $SU(M+1) \times SU(1)$ theory. This theory can be thought as dual to a single probe D3-brane on the background dual to $SU(M)$ (with the cascading UV completion discussed above), in which case the moduli space will be the space seen by this probe D3-brane. There will be an Affleck-Dine-Seiberg superpotential [138] which arises non-perturbatively for the gauge group $SU(N+1)$ with two flavors, and consequently one can show that the moduli space is of the form [118]

$$M_{11}M_{22} - M_{12}M_{21} \sim \left(h^{1-M} \Lambda_{M+1}^{3M+1} \right)^{\frac{1}{M}}. \quad (5.26)$$

⁵A thorough discussion of the moduli space of the cascading quiver can be found in [132]. A supergravity solution dual to some mesonic branch was discussed in [133].

This is the *deformed conifold*! So it must be that the fractional D3-branes at the apex of the cone trigger a complex deformation of it. This is called a *geometric transition*. Something very similar is known to exist in the context of topological strings [139, 140], in which case open strings on the resolved conifold are dual to closed strings on the deformed conifold. In our physical context, the understanding is that we have some stringy dynamical process (dynamical in the same sense as the scale Λ_{SYM} is dynamical) by which M D5-branes wrapped on the vanishing 2-cycle of the conifold are transmuted into pure 3-form flux on the 3-cycle of the deformed conifold,

$$-\frac{1}{4\pi^2\alpha'g_s}\int_{S^3}F_3=M. \quad (5.27)$$

The KS solution builds on this insight by proposing the supergravity ansatz (5.3)-(5.4) based on the *deformed conifold*, $X_6=C_{\text{def}}$. The Calabi-Yau metric on this space is known [104, 141]. To find a supersymmetric solution what we have to do is to construct a primitive (2,1)-form G_3 on C_{def} which asymptotes to the G_3 (5.9) of the KT solution at large r . Then one can integrate the warp factor equation (partly numerically), and one finds that the space is smooth at the tip of the cone. When g_sM is large the curvature is small everywhere, and supergravity is a good approximation.

We do not give the full detail of the KS solution here, because we will not need it much in this thesis ⁶. We refer the reader to the original papers [118, 142, 143, 144, 145, 146]. The Klebanov-Strassler solution is dual to the \mathbb{Z}_2 symmetric vacuum at $\mathcal{B}=\bar{\mathcal{B}}=i\Lambda_{2M}^{2M}$ on the baryonic branch (5.25), while a family of supergravity solutions covering the full baryonic branch was worked out in [126].

5.4 Fractional branes on various singularities

We have just seen that backreacting fractional branes at the cone $C(T^{1,1})$ leads to running fluxes at large radius and to a complex structure deformation (geometric transition) at small radius. How does this picture generalize to arbitrary singularities? The “naive” KT ansatz with explicit sources on a CY cone can be generalized easily whenever the conical CY metric is known, the only technical difficulty being the construction of the primitive (2,1)-form G_3 . This was done in [10, 11] for the case of the $\mathcal{N}=2$ singularity $\mathbb{C}\times\mathbb{C}^2/\mathbb{Z}_2$, while the KT-like solution for the full family of $C(Y^{p,q})$ metrics was worked out in [147]. In all the known cases the solution has running fluxes, $B_2\sim\ln r$, while the singularity in the warped factor leads to a naked singularity at small radius.

As for the conifold, there are two conceptually different issues to address. The first is the UV puzzle: the running fluxes indicate that the Maxwell D3-brane charge increases with r , and this must be interpreted in the dual quiver field theory, if possible. It turns out that there are two possible interpretation:

- The fractional brane may sit at an isolated singularity, so that it cannot move away from the tip. Then the RG flow as seen in supergravity can be followed in the quiver language by repeating a self-similar sequence of Seiberg dualities. These duality cascades can be quite intricate as the quiver grows more complex. These

⁶We will briefly introduce the deformed conifold metric in section 7.4.3 of Chapter 7 below.

Seiberg duality cascade were matched to the supergravity description for all the $Y^{p,q}$ quivers in [147].

- The fractional branes may sit at a non-isolated codimension four singularity. The simplest example is the one of a $\mathcal{N} = 2$ fractional brane on the orbifold $\mathbb{C} \times \mathbb{C}^2/\Gamma$. In general, fractional branes at non-isolated singularities are often simply called $\mathcal{N} = 2$ fractional branes as well, because their low energy dynamics is $\mathcal{N} = 2$ SYM away from the tip. In this case the running fluxes cannot be interpreted as a Seiberg duality, and there was some confusion in the literature concerning the correct field theory interpretation. We will discuss in detail the $\mathcal{N} = 2$ cascade in the $\mathbb{C} \times \mathbb{C}^2/\mathbb{Z}_2$ geometry in the next chapter.

The second issue is to understand the IR of the cascading quiver theory, and to implement the IR dynamical effects in the dual string theory background. This should cure the bad singularity of the KT-like solution. By direct analysis of the dual field theory quiver (using the known non-perturbative effects arising in $\mathcal{N} = 1$ theories), it was found that there are three kinds of fractional branes, depending on the IR dynamics of the low energy quiver [148]:

- *Deformation fractional branes.* These are the fractional branes which sit at a singularity which admits a complex structure deformation⁷. Such fractional branes trigger a geometric transition similarly to the conifold (it is actually always a conifold transition, locally). The low energy field theory is confining.
- *DSB fractional branes.* These are the fractional branes located at an isolated singularity which does not admit a complex structure deformation. Then the low energy quiver dynamically breaks supersymmetry (DSB) but there is no stable vacua, instead the theory has a runaway behavior [149, 148, 150, 151].
- *$\mathcal{N} = 2$ fractional branes,* at a non-isolated singularity. In this case the branes cannot undergo a geometric transition, because there exists no local complex deformation of a non-isolated singularity. An abelian gauge group survives at low energy in the quiver. The low energy dynamics is effectively $\mathcal{N} = 2$ and it can be understood thanks to Seiberg-Witten theory. We will discuss this case in the next chapter.

One can also consider fractional branes of various kinds simultaneously, which can lead to interesting physics. One such example with mixed features will be worked out at length in Chapter 7.

⁷We refer to subsection B.4.2 of Appendix B for details on this geometric condition.

Chapter 6

The $\mathcal{N} = 2$ cascade revisited and the enhançon bearings

IN the previous chapter we reviewed the prototypical example of a supergravity solution for so-called deformation fractional brane. There exists a rich literature which deals with the Klebanov-Strassler solution and its various generalizations.

The case of $\mathcal{N} = 2$ fractional branes, which involves twisted sector fields propagating along a complex line of singularity, was less understood so far. Here we aim at filling that gap. This chapter is based on [2], written in collaboration with Francesco Benini, Matteo Bertolini and Stefano Cremonesi.

6.1 Introduction and overview

The example we will consider is an $\mathcal{N} = 2$ model obtained by considering fractional branes at a $\mathbb{C} \times \mathbb{C}^2/\mathbb{Z}_2$ orbifold (also known as A_1 singularity) [10, 11]. The supergravity solution found in [10, 11] has been interpreted in various ways in the literature [11, 134, 152]. Consideration of probe fractional branes in the supergravity solutions [11] and recent methods based on the computation of Page charges [129, 1] suggest that the RG flow of the dual theories involves strong coupling transitions where the rank of the non-abelian factor in a gauge group with an adjoint chiral superfield drops according to the same numerology as in Seiberg duality, leading to a cascade. Since Seiberg-like dualities do not hold in this case, such strong coupling transitions cry for an explanation. It is worth stressing that such a phenomenon is not specific to $\mathcal{N} = 2$ models, but instead appears quite generically in any $\mathcal{N} = 1$ setup admitting non-isolated singularities together with isolated ones: the RG flow, as read from the gravity solution, is described by suitable combinations of Seiberg duality cascades and $\mathcal{N} = 2$ -like transitions. Therefore, clarifying which field theory dynamics governs these transitions is instrumental to understanding how string theory UV-completes field theories arising on systems of fractional branes at rather generic CY singularities.

In this chapter we reconsider the cascading solution describing regular and fractional D3 branes at the $\mathbb{C} \times \mathbb{C}^2/\mathbb{Z}_2$ orbifold, as a prototype of the more general class of branes at non-isolated singularities, and we provide a solution for this problem. Our proposal

elaborates on previous ones [11, 134], and solves a number of problems raised there. The dual gauge theory is a $SU(N + M) \times SU(N)$ $\mathcal{N} = 2$ quiver with bifundamental matter, where N is the number of regular branes and M the number of fractional ones, and its dual supergravity solution is known [10]. The structure of such a gauge theory has many similarities with the conifold one, and the two are indeed related by a $\mathcal{N} = 1$ -preserving mass deformation [102]. In order to provide a precise interpretation of the cascading RG flow, we start approximating the dynamics around scales where one of the two gauge coupling diverges with an effective $\mathcal{N} = 2$ SQCD, treating the other group as global. This allows us to claim that the transition occurs at the baryonic root (*i.e.* the point of the quantum moduli space of $\mathcal{N} = 2$ SQCD where the baryonic branch meets the Coulomb branch), where the strongly coupled $SU(N + M)$ group is effectively broken to $SU(N - M)$ (plus abelian factors). As in the $\mathcal{N} = 1$ conifold model, this is an iterative process which has the effect of lowering the effective ranks of the two gauge groups as the energy decreases, in a way which is exactly matched by the dual supergravity solution. On the other hand, the power of the Seiberg-Witten (SW) curve technology allows us to check our claim exactly, in the full quiver theory. We provide a short introduction to Seiberg-Witten theory in Appendix E.

Models arising from branes at non-isolated singularities have the distinctive property of having, besides a Higgs branch, also a Coulomb branch. This allows for a rather mundane UV completion of the cascading quiver theory, starting with the conformal $SU(N + M) \times SU(N + M)$ theory engineered by $N + M$ D3 branes at the orbifold singularity, and Higgsing it at some scale z_0 [11]. This stops the cascade in the UV as the theory is in a superconformal phase at energies higher than z_0 (notice that such a simple SCFT completion is not possible for the $\mathcal{N} = 1$ conifold model; see [153] for alternative ways to UV-complete the $\mathcal{N} = 1$ cascade with a SCFT). We first discuss the case where the cutoff is at finite energy: by means of the relevant Seiberg-Witten curves [80, 154], we provide a detailed analysis of several vacua on the Coulomb branch, together with the corresponding supergravity duals. For vacua at the origin of the Coulomb branch, there is in fact no cascade at all [152], while we show that the smaller is the number of adjoints fields having vanishing VEV, the larger is the number of steps in the cascade.

We then consider the case where the cutoff is sent to infinity, corresponding to the infinite cascade limit. This setup is the one which makes contact with the conifold cascade, as the two are expected to be related by a mass deformation. Actually, only specific vacua of the $\mathcal{N} = 2$ theory survive such a mass deformation [155], and we provide the corresponding SW curve, with a parametrically high level of accuracy. To find the supergravity solution interpolating from the $\mathcal{N} = 2$ to the $\mathcal{N} = 1$ cascade is left to future research.

Our analysis also allows us to provide a description of an infinite class of new vacua along the Coulomb branch, where the RG flow alternates energy ranges where the theory runs, and others where the theory is in a superconformal phase. The borders between these subsequent regions are described by enhançon-like rings and we naturally dub the corresponding geometric structures enhançon bearings. We provide the corresponding supergravity duals and show, both from the gauge theory and supergravity points of view, how such vacua interpolate between the non-cascading and the cascading vacua.

The original supergravity solution of [10], which is the building block for all supergravity duals along the Coulomb branch that we analyse, presents an unphysical repulsive

region around the origin. Another distinctive property of $\mathcal{N} = 2$ models is the peculiar way in which such a singularity is cured. Models with $\mathcal{N} = 2$ supersymmetry are not confining, and the resolution of the IR singularity is associated to the enhançon mechanism [156] which excises the unphysical region giving back a singularity-free solution. The scale at which the excision occurs depends on the dual gauge theory vacuum one is studying [134, 152], and therefore the excised solutions will differ for different vacua. We work out the enhançon mechanism for all gauge theory vacua mentioned above, computing explicitly the warp factors of the excised solutions. It is worth noticing that the way the enhançon mechanism works here is qualitatively different from the original one discussed in [156], since in the present case the enhançon shell is not of real codimension one, *i.e.* it is not a domain wall: the modification of the solution corresponds to an actual excision for the twisted fields but not for the untwisted ones, most notably the metric and the RR 5-form field strength. In turn, the corrected warp factor and 5-form depend on the excised configuration of twisted fields and fractional branes dual to the field theory vacuum under consideration. We find that around the origin the metric is free of singularities and the new solutions we find perfectly match, within the supergravity approximation, the dual gauge theory expectations.

This chapter is organized as follows. In section 6.2 we briefly recall the $\mathcal{N} = 2$ quiver gauge theory at the A_1 singularity, the structure of its moduli space and that of the known supergravity duals, both for the conformal and non-conformal models. In section 3 we recall how the non-perturbative dynamics of the model can be studied through Seiberg-Witten curves, and review the enhançon mechanism. Section 4, which includes the main result of this chapter, is devoted to the analysis of the cascading vacua, while in section 5 we discuss a new class of vacua characterized by the presence of subsequent enhançon bearings. Finally, in section 6 we work out the excision procedure and the corresponding warp factors for all the gauge theory vacua previously discussed.

6.2 D3 branes on the $\mathbb{C}^2/\mathbb{Z}_2$ orbifold and a cascading solution

The low energy theory on N D3 branes placed at the origin of the $\mathbb{C} \times \mathbb{C}^2/\mathbb{Z}_2$ orbifold is a four-dimensional $U(N) \times U(N)$ $\mathcal{N} = 2$ gauge theory with two bifundamental hypermultiplets. The field content is summarized in the quiver diagram of figure 6.1. The beta functions of both $SU(N)$ factors vanish, the diagonal $U(1)$ is decoupled, while the anti-diagonal $U(1)$ becomes free in the IR and gives rise to a global symmetry, the baryonic symmetry $U(1)_B$.

The classical moduli space agrees precisely with the possible configurations of regular and fractional D3 branes on $\mathbb{C} \times \mathbb{C}^2/\mathbb{Z}_2$. In terms of $\mathcal{N} = 1$ superfields, the tree level superpotential (dictated by $\mathcal{N} = 2$ supersymmetry) reads

$$W = (B_1\Phi A_1 - B_2\Phi A_2) - (A_1\tilde{\Phi}B_1 - A_2\tilde{\Phi}B_2) , \quad (6.1)$$

where contractions over gauge indices are implied. The corresponding F-term equations are

$$\Phi A_i - A_i\tilde{\Phi} = 0 \quad , \quad B_i\Phi - \tilde{\Phi}B_i = 0 \quad , \quad A_1B_1 - A_2B_2 = B_1A_1 - B_2A_2 = 0 . \quad (6.2)$$

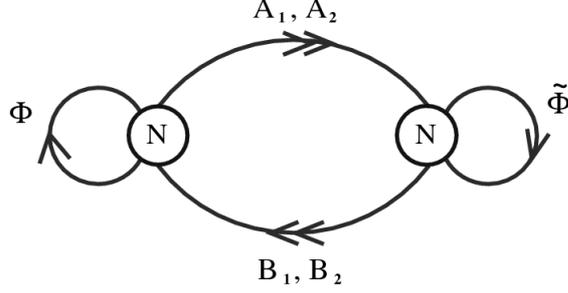


Figure 6.1: Quiver diagram of the $U(N)_L \times U(N)_R$ $\mathcal{N} = 2$ theory, in $\mathcal{N} = 1$ notation. Nodes correspond to gauge factors, arrows connecting different nodes represent bifundamental chiral superfields while arrows going from one node to itself represent adjoint chiral superfields.

The holomorphic gauge invariant operators, which descend to local coordinates on the moduli space, are given by traces of products of the operators $A_i B_j \equiv \varphi_{ij}$ and Φ for the first gauge group, and $B_i A_j \equiv \tilde{\varphi}_{ij}$ and $\tilde{\Phi}$ for the second one.

The moduli space consists of several branches. First we have the so-called Higgs branches, where the hypermultiplets obtain vacuum expectation values (VEV's). These VEV's result in the Higgsing of the quiver to a subgroup of the diagonal $U(N)$ gauge group, and the theory has an accidental $\mathcal{N} = 4$ supersymmetry in the IR. The Higgs branch has $(\mathbb{C} \times \mathbb{C}^2/\mathbb{Z}_2)^N/S_N$ geometry, corresponding to the displacement of regular D3 branes in the full transverse space, up to permutations. Because of $\mathcal{N} = 2$ supersymmetry, the Kähler metric on the Higgs branch is protected against any quantum corrections. Next we have the Coulomb branch, on which the hypermultiplet VEV's vanish while the VEV's for the two adjoint scalars can take arbitrary values: at a generic point on this branch, the surviving gauge group is $U(1)^{2N}$. The Coulomb branch has the form $\mathbb{C}^N/S_N \times \mathbb{C}^N/S_N$, which corresponds to the displacement of the two types of fractional D3 branes, each of them associated to one gauge factor, along the orbifold singularity line. The quantum corrected metric on the Coulomb branch is exactly calculable thanks to Seiberg-Witten theory [157]. Finally, there are mixed branches, where some hypermultiplet VEV's and some adjoint VEV's are turned on.

In the large N and large 't Hooft coupling limit, the low energy superconformal $SU(N) \times SU(N)$ sector is better described by its type IIB supergravity dual [158]. The full Higgs branch is dual to a family of supergravity solutions corresponding to D3 branes at arbitrary positions on the 6-dimensional transverse space,

$$ds^2 = Z^{-1/2} \eta_{\mu\nu} dx^\mu dx^\nu + Z^{1/2} \delta_{nm} dx^m dx^n \quad (6.3)$$

$$g_s F_5 = (1 + *) d\text{vol}_{3,1} \wedge dZ^{-1}, \quad (6.4)$$

where $\mu, \nu = 0, \dots, 3$, $m, n = 4, \dots, 9$ and the orbifold identification $\mathbf{x} = (x^m) \simeq (\tilde{\mathbf{x}}) \equiv (x^{4,5}, -x^{6,7,8,9})$ is understood. Z is a harmonic function of \mathbf{x} ,

$$Z = 4\pi g_s \alpha'^2 \sum_{j=1}^N \left(\frac{1}{|\mathbf{x} - \mathbf{x}_j|^4} + \frac{1}{|\mathbf{x} - \tilde{\mathbf{x}}_j|^4} \right). \quad (6.5)$$

The function contains the D3 branes and their images. Notice that the total 5-form flux on S^5/\mathbb{Z}_2 at infinity is N . The relation between the parameters x_j and the field theory

moduli is $x_j = 2\pi\alpha'\phi_j$, where ϕ_j is an eigenvalue of the VEV of some field. Φ and $\tilde{\Phi}$ are mapped to $x^4 + ix^5$, while φ_{ij} are mapped to algebraic coordinates z_{ij} on $\mathbb{C}^2/\mathbb{Z}_2$, such that $z_{12}z_{21} - z_{11}^2 = 0$ and $z_{22} = z_{11}$. The supergravity axio-dilaton $\tau = C_0 + i e^{-\Phi} = C_0 + \frac{i}{g_s}$ is constant,¹ as D3 branes do not couple to it. It is related to the field theory gauge couplings and theta angles by

$$\tau = \tau_1 + \tau_2 \quad \text{where} \quad \tau_j = \frac{\theta_j}{2\pi} + \frac{4\pi i}{g_j^2}, \quad j = 1, 2. \quad (6.6)$$

In the following we will take $\tau = i/g_s$ unless otherwise stated.

As noticed in [134], for a generic point on the Higgs branch (and more generally on any branch), the supergravity solution has large curvature. However, configurations where all the branes are in big clumps have a good supergravity description, and configurations where only a small number of branes are isolated are well described by probe branes in the background generated by the other branes.

The Coulomb branch of our $\mathcal{N} = 2$ quiver is described by fractional D3 branes along the orbifold singularity. In this case supergravity solutions include a non-trivial profile for the twisted field fluxes. Indeed, fractional D3 branes source magnetically the twisted scalar c and by supersymmetry they also source its NSNS partner, the twisted scalar b . This can be easily understood recalling [41] that fractional D3 branes are D5 branes wrapped on the exceptional 2-cycle \mathcal{C} which lives at the orbifold singularity. The twisted scalars are simply the reduction of the RR and NSNS 2-form potentials, C_2 and B_2 , on \mathcal{C} . They can be organized in a complex field as

$$\gamma \equiv c + \tau b = c + \frac{i}{g_s} b = \frac{1}{4\pi^2\alpha'} \int_{\mathcal{C}} \left(C_2 + \frac{i}{g_s} B_2 \right), \quad (6.7)$$

while

$$G_3 = F_3 + \frac{i}{g_s} H_3 = 4\pi^2\alpha' d\gamma \wedge \omega_2 \quad (6.8)$$

is the complexified 3-form field strength, where ω_2 is a closed anti-selfdual $(1, 1)$ -form with delta-function support at the orbifold plane, normalized as $\int_{\mathcal{C}} \omega_2 = 1$. Regular D3 branes do not couple to the twisted sector, hence the profile of γ is affected solely by fractional branes. The complex twisted scalar γ is then subject to a two-dimensional Laplace equation in \mathbb{C} with sources at the positions of the fractional branes. Supersymmetric solutions [123] have primitive, imaginary self-dual and $(2, 1)$ G_3 flux, which implies that $\gamma = \gamma(z)$ is a meromorphic function of $z = x^4 + ix^5$, such that $d\gamma(z)$ has simple poles at the locations of sources. For a bunch of N fractional and N anti-fractional² branes at positions z_j and \tilde{z}_j , respectively, we have

$$\gamma = \frac{i}{\pi} \left[\sum_{j=1}^N \log(z - z_j) - \sum_{j=1}^N \log(z - \tilde{z}_j) \right] + \gamma^{(0)}. \quad (6.9)$$

¹We work in the string frame. Here Φ is the full dilaton, which is constant in all the solutions under consideration, not to be confused with one of the adjoint chiral superfields. From now on we will rather use $g_s = e^{\Phi}$.

²With some abuse of language, following [152] we call ‘anti-fractional branes’ D5 branes wrapped on \mathcal{C} with the opposite orientation, with some worldvolume flux through \mathcal{C} in order to preserve the same supercharges as the fractional branes.

Here $\gamma^{(0)}$ is an integration constant: its imaginary part sets the value of b at large $|z|$ or in the theory at the origin of the moduli space, while the real part does not really have a physical meaning in the dual theory because of the presence of the axial anomaly, and we will set it to zero. The positions of the fractional branes z_j and \tilde{z}_j are classically identified with the eigenvalues $\Phi_j, \tilde{\Phi}_j$ of the field theory adjoint scalars. Corrections to this identification arise at quantum level and will be discussed in the next section.

The holographic relations between the Yang-Mills couplings and theta angles and the supergravity fields are

$$\tau_1 + \tau_2 = \tau \qquad \tau_1 - \tau_2 = 2\gamma - \tau = 2 \left[c + \tau \left(b - \frac{1}{2} \right) \right], \quad (6.10)$$

but we will often set $\tau = i/g_s$. In particular, when $b = 0$ the imaginary part of τ_1 vanishes and g_1 diverges, whereas for $b = 1$ it is g_2 which diverges.³ What we face in such cases is obviously a peculiar field theory, a SCFT with one divergent gauge coupling, in which instanton corrections dominate even in the large N limit [159], and about which not much is known. Although from the Seiberg-Witten curve analysis one does not expect extra massless fields in general, the supergravity description is a very incomplete description for this phase. When $c \in \mathbb{Z}$ as well, extra massless states do appear, and the theory enters a tensionless string phase, as originally suggested in [160] from consistency of T -duality with type IIA string theory.

So far, we have only discussed the superconformal $SU(N) \times SU(N)$ theory,⁴ which has a well behaved UV limit and whose stringy realization through AdS/CFT is unambiguous. However, what we are really interested in is the non-conformal $SU(N+M) \times SU(N)$ gauge theory. This can be easily obtained through Higgsing from the superconformal $SU(N+M) \times SU(N+M)$ theory, which can be engineered placing $N+M$ regular D3 branes at the origin of the orbifold: taking M VEV's of the second adjoint scalar to be at a scale $|z_0|/2\pi\alpha'$ produces an effective $SU(N+M) \times SU(N) \times U(1)^M$ theory below $|z_0|$,⁵ where the $U(1)$ factors are IR free and decouple. In the dual picture, this corresponds to placing M anti-fractional branes at, say, the roots of $\tilde{z}_j^M = -z_0^M$, while the other N anti-fractional branes and $N+M$ fractional branes sit classically at the origin. The twisted scalar in this configuration is then

$$\gamma = \frac{i}{\pi} \log \frac{z^M}{z^M + z_0^M} + \gamma^{(0)}. \quad (6.11)$$

For the sake of simplicity, unless differently specified, in the following we will set the orbifold point value $\gamma^{(0)} = \frac{i}{2g_s}$ [161, 162], so that in the UV $\tau_1 = \tau_2 = \frac{i}{2g_s}$. In the large M limit in which we work, (6.11) can be traded for its limiting behavior

$$\gamma = \begin{cases} i \frac{M}{\pi} \log \frac{z}{z_0} + \frac{i}{2g_s} \equiv i \frac{M}{\pi} \log \frac{z}{z_1} & \text{if } |z| < |z_0| \\ \gamma^{(0)} & \text{if } |z| > |z_0| \end{cases} \quad (6.12)$$

³Actually $b \in [0, 1]$ is the only range of validity of the formulas, because otherwise one would have negative square couplings. As noticed in [118] and extensively discussed in [129, 1], when b is outside this range one has to perform a large gauge transformation to shift it to the interval where (6.10) can be applied.

⁴From now on, we will often consciously forget the additional $U(1) \times U(1)$ factor which decouples at low energies.

⁵In the following, when speaking about scales we will often omit the $2\pi\alpha'$ factor.

where we set $z_1 = e^{i\frac{\pi}{M}\gamma^{(0)}} z_0 = e^{-\frac{\pi}{2g_s M}} z_0$. Note that the twisted fluxes break the $U(1)$ isometry corresponding to rotation in the z -plane to a discrete subgroup \mathbb{Z}_{2M} ⁶. This is dual to the breaking of the $U(1)$ R-symmetry because of anomalies in the gauge theory [163].

The gauge invariant D3 brane charge (Maxwell charge) carried by the fluxes of the solution is proportional to the 5-form flux; it is found by integrating the Bianchi identity in the absence of sources $dF_5 = -H_3 \wedge F_3$ on the angular S^5/\mathbb{Z}_2 of radius r and reads, for $r < \rho_0 = |z_0|$,

$$-\frac{1}{(4\pi^2\alpha')^2} \int F_5 = N + \frac{g_s M^2}{\pi} \log \frac{r}{\rho_1} \quad (6.13)$$

with $\rho_1 = |z_1|$.

We see from eqs.(6.12)-(6.13) that, similarly to the Klebanov-Tseytlin (KT) solution [127], the solution enjoys logarithmically varying B field and 5-form flux below the cutoff: this naturally suggests that the dual field theory might enjoy a cascading RG flow with subsequent infinite coupling transitions reducing the rank of the infinitely coupled non-abelian gauge group by $2M$ at scales $\rho_k = e^{-\frac{(2k-1)\pi}{2g_s M}} \rho_0$, $k = 1, \dots, l$, where $l \equiv [N/M]_-$ ⁷ [11]. This will be dealt with in section 4, where the $\mathcal{N} = 2$ cascading nature of the solution will be discussed in great detail.

Before attacking this problem, though, we have to deal with another phenomenon, which always arises in supergravity solutions dual to non-conformal supersymmetric gauge theories with eight supercharges. By analyzing the explicit form of the warp factor, it was shown in [10] that the ten-dimensional metric obtained using (6.11), besides the obvious singularity on the orbifold fixed plane, displays an unphysical repulsive region near the origin, at a scale of order $e^{-\pi N/g_s M^2} \rho_1$.⁸ One expects that, as suggested in [10], an enhançon-like mechanism [156] might be at work here, which excises the unphysical region rendering back a repulsion-free solution. We will show that this is indeed the case, discussing in the next section the specific way in which the enhançon mechanism manifests in this context, and providing in section 6.6 an excised and singularity-free solution.

6.3 The enhançon and the Seiberg-Witten curve

The quantum corrections to the Coulomb branch constrain the (anti)fractional D3 brane positions, z_j and \tilde{z}_j , in the gravity dual. The full quantum corrected moduli space is exactly encoded in the full family of Seiberg-Witten (SW) curves [80, 154]. The SW curves for the $\mathcal{N} = 2$ superconformal field theory at hand were found in [157]. At the classical level, the fractional brane positions z_j and \tilde{z}_j correspond to the eigenvalues of the VEV's of the adjoint scalars Φ and $\tilde{\Phi}$. In the quantum theory this identification cannot survive because the VEV's parametrize the moduli space and are unconstrained, whereas fractional brane positions are constrained. That is, in the large N limit one expects [134] quantum corrections and the consequent constraints on z_j and \tilde{z}_j to be bound, because

⁶Really, the cutoff fractional branes only preserve \mathbb{Z}_M , but this is irrelevant at large M or not very close to those branes.

⁷We denote by $[y]_{\pm}$ the ceiling and floor functions, namely the integers which better approximate y from above and below respectively.

⁸See appendix A of [164] for an analytic study of the warp factor found in [10].

of supersymmetry, to a non-negative 5-form flux (that means non-negative enclosed D3-charge) for all allowed configurations on the quantum moduli space, at least whenever the supergravity approximation is valid. This property is in fact at the core of the enhançon mechanism.

Let us detail this point by first considering a simplified example. Consider the theory discussed previously with $N = 0$: this is an $SU(M) \times SU(M)$ superconformal theory which can be engineered by M regular D3 branes. Below the UV scale $|z| = |z_0|$, the theory is effectively Higgsed to $SU(M)$ $\mathcal{N} = 2$ pure SYM (plus IR free $U(1)$ factors). The dual supergravity solution is the one in (6.12)-(6.13) with $N = 0$, and it corresponds to the M fractional branes classically at the origin. The quantum moduli space can be studied with a good approximation by means of the SW curves for $SU(M)$ [165, 166]

$$y^2 = \prod_{a=1}^M (v - \phi_a)^2 + 4\Lambda^{2M} , \quad (6.14)$$

where Λ is the strong coupling scale of $\mathcal{N} = 2$ $SU(M)$ SYM and ϕ_a are the eigenvalues of the adjoint scalar Φ parametrizing a family of hyperelliptic curves in $\mathbb{C}^2 = \{(v, y)\}$. The curves could also be written in terms of gauge invariant symmetric polynomials. Classically ($\Lambda = 0$) the eigenvalues ϕ_a coincide with the double branch points of (6.14), and correspond to the fractional brane positions on the z plane in the gravity description. An elegant way to see this is the following: type IIB string theory on the orbifold is T-dual to type IIA on a circle (with coordinate x^6) with two parallel NS5 branes along x^0, \dots, x^5 , separated in the compact direction x^6 (see [167] for a review). Fractional D3 branes are T-dual to D4 branes stretched along x^6 between the two NS5's. The classical Coulomb branch is then given by all the possible configurations of D4 branes on the plane $v = x^4 + ix^5$. The system can be further uplifted to M-theory, where the NS5's and the D4's are just part of a single M5 brane. The M5 brane seen as a Riemann surface is identified with the SW curve [157]. At the quantum level, the eigenvalues ϕ_a still parametrize the whole moduli space (up to Weyl gauge identifications), but they no longer correspond to double branch points nor fractional brane positions, strictly speaking. In the perturbative regime of the theory, $|\phi_a| \gg |\Lambda|$, the branch points still appear in pairs close to ϕ_a : in the M-theory picture the D4 branes are inflated into small tubes. As soon as the VEV's get into the non-perturbative region (at scales comparable with Λ), the branch points get well separated and it does not make much sense to talk about fractional brane positions anymore.

At the origin of the moduli space ($\Phi = 0$), the hyperelliptic curve (6.14) becomes $y^2 = v^{2M} + 4\Lambda^{2M}$, which has $2M$ separate branch points at $v^{2M} = -4\Lambda^{2M}$. In the large M limit, the branch points densely fill a ring of radius $2^{1/M}|\Lambda|$. It is also possible to see that, adding a probe fractional brane (in field theory terms, consider the $SU(M+1)$ theory with one additional VEV ϕ), in which case the SW curves are

$$y^2 = v^{2M}(v - \phi)^2 + 4\Lambda^{2M+2} , \quad (6.15)$$

the probe can freely move in the semi-classical region outside the ring, but it cannot penetrate it. For $|\phi| \gg |\Lambda|$, the two extra branch points are placed near ϕ , with a small separation of order $\Lambda(\Lambda/\phi)^M$, while the other $2M$ branch points are still on the ring. As $|\phi|$ approaches $|\Lambda|$ and then goes to zero, the branch points split and melt into the ring.

As anticipated, the dual string theory picture of this is the famous enhançon mechanism [156]. The tension of BPS fractional D3 branes is equal to their gauge invariant Maxwell D3-charge, which is γ

$$T_{n_f} = \frac{\mu_3}{g_s} |g_s \mathbb{I}m \gamma + n_f| = \frac{\mu_3}{g_s} |b + n_f|, \quad (6.16)$$

where n_f is the number of units of worldvolume flux on the exceptional 2-cycle \mathcal{C} (notice that neither b or n_f are gauge invariant, while their sum is). This turns out to be proportional to the perturbative moduli space metric on the Coulomb branch of the $SU(M)$ $\mathcal{N} = 2$ pure SYM theory⁹ [10]. At the scale $|\Lambda| = \rho_1$, b vanishes and fractional D3 branes, which are wrapped D5 branes with no worldvolume flux, become tensionless; below that scale they would be non-supersymmetric and they would feel a repulsive potential. Notice also that the enclosed D3 brane charge would become negative for smaller scales, which could hardly be the case if fractional D3 branes were at the origin. Moreover, a massive particle probe would experience an unphysical gravitational repulsion close to the origin. The resolution of this puzzle is that fractional branes cannot be brought all at the same place, but rather melt into a thin ring of radius ρ_1 : the *enhançon ring*. This changes the twisted fields distribution in the geometry: inside the ring, $b = 0$ (more generally it is integer), c is constant, and there is no D3 brane charge. The warp factor needs to be re-computed using the correct configuration of fractional branes and twisted field, and the result is that the suspicious repulsive region disappears, as will be shown in section 6.6.

In some sense, the whole region defined by $b = 0$ (more generally $b \in \mathbb{Z}$) behaves like a conductor: D5 charges (recall that the D3 charge vanishes along with the tension inside the enhançon) are pushed to the boundary and there is no field inside. We will call such a region the enhançon plasma. We already noticed in section 6.2 that the IR field theory dual to the interior region is quite peculiar: it is a conformal $SU(N) \times SU(N)$ theory with one divergent gauge coupling. However, in this particular case $N = 0$ and the dynamics is trivial inside the enhançon plasma: $SU(M)$ is simply broken by instantons to $U(1)^{M-1}$.

As discussed in [152], exactly the same kind of behavior can be found in the most generic situation, i.e. when $N \neq 0$ and the theory has product gauge group $SU(N+M) \times SU(N)$. Since the second gauge group is not asymptotically free, one should embed the theory into the $SU(N+M) \times SU(N+M)$ conformal one, properly Higgsed, as sketched at the end of Section 6.2. One can then exploit the power of the Seiberg-Witten technology. In order to write down the SW curve, let us define the complex coordinate

$$u = i \frac{x^6 + ix^{10}}{2\pi R_{10}}, \quad (6.17)$$

which parametrizes the M-theory torus defined by the identifications $u \simeq u + 1 \simeq u + \tau$. The complex structure τ is identified with the type IIB axio-dilaton. Let us also define the parameter $q = e^{2\pi i \tau}$ and the coordinate $t = e^{2\pi i u}$; note that $t \simeq qt$ on the torus.

⁹There is a matching with the perturbative result because in the large M limit instanton corrections are strongly suppressed, and abruptly show up at the scale Λ [159].

For concreteness, let us stick again to the case of equal gauge couplings in the UV CFT: $\tau_1 = \tau_2 = \tau/2$. In terms of the quasi-modular Jacobi θ -functions

$$\theta_2(2u|2\tau) = \sum_{n=-\infty}^{\infty} q^{(n-\frac{1}{2})^2} t^{2n-1} \quad (6.18)$$

$$\theta_3(2u|2\tau) = \sum_{n=-\infty}^{\infty} q^{n^2} t^{2n}, \quad \theta_4(2u|2\tau) = \sum_{n=-\infty}^{\infty} (-1)^n q^{n^2} t^{2n}, \quad (6.19)$$

the SW curve for the conformal theory can be written as [168]

$$\frac{S(v) + R(v)}{S(v) - R(v)} = f(u|\tau), \quad \text{with} \quad f(u|\tau) \equiv \frac{\theta_3(u|\tau/2)}{\theta_4(u|\tau/2)} = \frac{\theta_3(2u|2\tau) + \theta_2(2u|2\tau)}{\theta_3(2u|2\tau) - \theta_2(2u|2\tau)}, \quad (6.20)$$

or alternatively

$$\frac{R(v)}{S(v)} = g(u|\tau), \quad \text{with} \quad g(u|\tau) \equiv \frac{f-1}{f+1} = \frac{\theta_2(2u|2\tau)}{\theta_3(2u|2\tau)}. \quad (6.21)$$

Here $R(v) = \prod_{a=1}^{N+M} (v - \phi_a)$ and $S(v) = \prod_{a=1}^{N+M} (v - \tilde{\phi}_a)$ are degree $N + M$ polynomials whose zeros ϕ_a and $\tilde{\phi}_a$ are the eigenvalues for the adjoint scalars of the first and second gauge group, respectively.

Following [152], let us choose a \mathbb{Z}_M -invariant configuration for the anti-fractional branes Higgsing the CFT at large $|z|$ (i.e. large $|v|$ for the corresponding D4 branes), and consider the origin of the moduli space of the low energy $SU(N + M) \times SU(N)$ theory,

$$R(v) = v^{N+M} \quad S(v) = v^N (v^M - z_0^M). \quad (6.22)$$

The N common zeros of $R(v)$ and $S(v)$ factor out of the curve, without affecting the RG flow. They correspond to N D3 branes, whose moduli space is flat (apart from orbifold singularities when several branes coincide) and not quantum corrected. We are then left to consider an $SU(M) \times SU(M)$ theory, spontaneously broken to $SU(M) \times U(1)^{M-1}$ at the scale z_0 . Hence, if the IR dynamics is not much affected by the UV Higgsing, as it is natural to expect, the low energy physics should be similar to the enhançon mechanism previously discussed, but with N leftover regular D3 branes.

Let us give further evidence for the above claim. As explained in [157], we can extract the running of the gauge coupling from the bending of the two NS5 branes due to the unbalanced D4 branes tension. In the M-theory picture, the gauge couplings at a scale v can be extracted from the SW curve looking at the corresponding two values of u ; we have that

$$\Delta u = \tau_1, \quad \tau - \Delta u = \tau_2, \quad (6.23)$$

while the map between the type IIB twisted scalars (c, b) and the field theory couplings (τ_1, τ_2) was given in (6.10). In particular, the curve (6.20) at the point (6.22) on the Coulomb branch reads

$$1 - 2\left(\frac{v}{z_0}\right)^M = f(u|\tau). \quad (6.24)$$

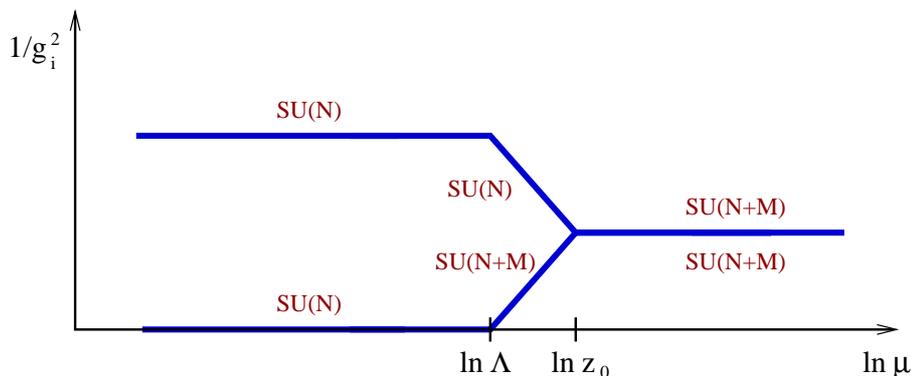


Figure 6.2: RG flow of the theory at the enhançon vacuum (origin of the moduli space). The low energy theory below Λ is a peculiar one, with one formally diverging coupling.

One can check [152] that in the UV regime $|v| > |z_0|$, the theory is conformal with equal gauge couplings. Comparing (6.24) with (6.14), one can see¹⁰ that the dynamically generated scale is at $\Lambda = q^{\frac{1}{4M}} z_0$. In the range $|\Lambda| < |v| < |z_0|$, the two gauge couplings are running with opposite β -functions

$$\beta = \frac{\partial}{\partial \log |v|} \frac{8\pi^2}{g_{1,2}^2(|v|)} = \pm 2M . \quad (6.25)$$

For $|v| < |\Lambda|$ the gauge couplings are constant with $8\pi^2/g_{1,2}^2 = 0, 2\pi/g_s$ respectively. The RG flow is sketched in figure 6.2. At the scale Λ , the gauge group is effectively broken by instantons from $SU(N+M) \times SU(N) \times U(1)^M$ to $SU(N) \times SU(N) \times U(1)^{2M}$, the latter being conformal up to an IR free abelian sector.

Further information is gained from the computation of branch points of the SW curve, which correspond to double points of the function $f(u|\tau)$: they are at $u_* = 0, 1/2, \tau/2, (1+\tau)/2$ where $f(u_*|\tau) = f_0, 1/f_0, -f_0, -1/f_0$ respectively, and $f_0 = 1 + 4q^{1/4} + \mathcal{O}(q^{1/2})$. The first set is located at

$$u = \frac{\tau}{2}, \frac{\tau+1}{2} : \quad v \simeq v_h^\pm = z_0 e^{2\pi i h/M} \left[1 \pm \frac{2}{M} \left(\frac{\Lambda}{z_0} \right)^M \right] \quad h = 1, \dots, M . \quad (6.26)$$

These are almost double branch points, which correspond to the M anti-fractional branes located near $|z_0|$, corresponding to the VEV's of $\tilde{\Phi}$ we used to Higgs the conformal theory. The second set is located at

$$u = 0, \frac{1}{2} : \quad v \simeq v_k = 2^{1/M} e^{2\pi i k/2M} \Lambda \quad k = 1, \dots, 2M . \quad (6.27)$$

These branch points correspond to M fractional branes melted into an enhançon ring at scale Λ .

¹⁰Notice that in the supergravity approximation, $g_s \rightarrow 0$ with $g_s N$ large, the parameter $q = e^{2\pi i \tau}$ has exponentially small modulus $|q| = e^{-2\pi/g_s}$, allowing for a series expansion of $f(t|q)$ in positive powers of q .

As in the pure SYM case, probe fractional branes can be studied on this background by means of the SW curves for the $SU(N + M + 1) \times SU(N + M + 1)$ theory

$$\frac{R(v)}{S(v)} = \frac{v^M(v - \phi)}{(v^M - z_0^M)(v - \tilde{\phi})} = g(u|\tau) , \quad (6.28)$$

where ϕ and $\tilde{\phi}$ parametrize the extra VEV for Φ and $\tilde{\Phi}$. The branch points corresponding to the eigenvalue ϕ (the fractional D3 probe) can freely move outside the enhançon ring, but as they approach it and ϕ goes to 0, the two branch points split and melt into the enhançon ring. The two branch points corresponding to the eigenvalue $\tilde{\phi}$ (the anti-fractional D3 probe) can instead penetrate the enhançon ring; when this happens, they unchain two branch points from the ring which follow them inside: an anti-fractional brane eats a melted fractional brane from the ring, forming a regular D3 brane free to move everywhere.

From this analysis, one concludes that, no matter the value of N , the fluxes in eqs. (6.12) and (6.13) do describe the physics of the $SU(N + M) \times SU(N)$ theory at the origin of its moduli space, provided that they are excised at radius $\rho_1 \simeq |\Lambda|$ by an enhançon mechanism. The solution should also be cut off at a radius $|z_0|$, or completed with M anti-fractional branes, providing a conformal AdS_5 UV completion. As already stressed, the warp factor needs to be recomputed in the presence of the correct configuration of fractional branes and excised twisted fields. This will be done in section 6.6.

Notice, however, that the supergravity solution of eqs. (6.12) and (6.13) does not seem to have any pathology below ρ_1 , at least down to a scale of order $e^{-\frac{\pi N}{g_s M^2}} \rho_1$, where the 5-form flux (6.13) vanishes and the problematic repulsive region starts. The question arises whether there is any field theory interpretation for such a solution, suitably excised only at a radius

$$\rho_{\min} = \rho_{l+1} \equiv e^{-\frac{\pi l}{g_s M}} \rho_1 \quad \text{with } l \equiv [N/M]_-, \quad (6.29)$$

the smallest infinite coupling scale outside the region of negative D3 brane charge. As already noticed, the presence of a constant 3-form flux and the logarithmic running of the 5-form flux strongly suggests a cascading behavior, as for the Klebanov-Tseytlin-Strassler $\mathcal{N} = 1$ model [118, 127], properly adapted to a $\mathcal{N} = 2$ setting. An interpretation of the would-be $\mathcal{N} = 2$ RG flow that can be extracted from the supergravity solution in terms of some sort of Seiberg duality cascade was in fact argued for in [11], but the existence of an appropriate $\mathcal{N} = 2$ duality had not been clarified, so far. On the other hand, in [134] the reduction of 5-form flux was interpreted as due to a distribution of D3 branes and/or wrapped D5 branes. It was further suggested that a suitable distribution of D3 branes only (Higgs branch) could perhaps account for it. However, the latter proposal encounters some problems in reproducing the running of gauge couplings and decrease of nonabelian gauge group ranks that is suggested by the supergravity solution.

Drawing on well established results about $\mathcal{N} = 2$ SQCD, we propose that there exist field theory vacua, not at the origin of the Coulomb branch, which display a *cascading* behavior. They are dual to the solution in (6.12) and (6.13), valid well below the first infinite coupling radius ρ_1 down to some much lower scale, at most until the so-called true enhançon scale $\Lambda_{\min} = \rho_{\min}$, where the twisted fields are excised. ρ_{\min} is named

the true enhançon radius since it is the scale at which the excision is performed. All the higher infinite coupling scales, ρ_j with $j = 1, \dots, l$, will be called generalized enhançon radii [134].

We provide a precise identification of these vacua in the next section. The excision of the twisted fields by means of the enhançon mechanism and the disappearance of the naive singularity will be discussed in section 6.6. Depending on the field theory vacua one is studying, the excision can take place at different scales, for instance at $\rho = \rho_1$, as in the vacua discussed in this section, or at the bottom of the cascade, at the scale $\rho = \rho_{\min}$, as for the cascading vacua to be discussed in section 4.

6.4 The cascading vacuum in field theory

The perturbative RG flow of the $SU(N + M) \times SU(N)$ theory, given in (6.25), is such that the largest group goes to strong coupling at a scale Λ . The supergravity solution we are considering suggests that, in the dual vacuum, a mechanism effectively reduces the gauge group to $SU(N - M) \times SU(N)$ below Λ , plus possible $U(1)$ factors. This statement can be supported by a computation of Page charges in supergravity, in the gauge that gives sensible field theory couplings (as extensively discussed in [129, 1]). The value of b , in the gauge in which $b \in [0, 1]$, is found from (6.12) to be

$$b = g_s \Im \gamma = \frac{g_s M}{\pi} \log \frac{\rho}{\rho_1} - \left[\frac{g_s M}{\pi} \log \frac{\rho}{\rho_1} \right]_-, \quad (6.30)$$

where $\rho = |z|$. The D5 and D3 brane Page charges at radius r are evaluated to be¹¹

$$Q_5^{\text{Page}} = -\frac{1}{4\pi^2 \alpha'} \int F_3 = 2M \quad (6.31)$$

$$Q_3^{\text{Page}} = -\frac{1}{(4\pi^2 \alpha')^2} \int (F_5 + B_2 \wedge F_3) = N + M \left[\frac{g_s M}{\pi} \log \frac{r}{\rho_e} \right]_-. \quad (6.32)$$

This shows that the non-abelian factors in the gauge group drop as $SU(N + M) \times SU(N) \rightarrow SU(N - M) \times SU(N)$ not only at the first strong coupling scale $\rho_1 = \Lambda_1 \equiv \Lambda$, but actually at each generalized enhançon, which occurs at a scale

$$\rho_k = \Lambda_k = e^{-\frac{\pi(k-1)}{g_s M}} \Lambda_1 = e^{-\frac{\pi(2k-1)}{2g_s M}} \rho_0 \quad k = 1, \dots, l, \quad (6.33)$$

where recall that $l = [N/M]_-$ and we also set $N = lM + p$. Finally, at $\Lambda_{l+1} \equiv \Lambda_{\min} \equiv e^{-\frac{\pi l}{g_s M}} \Lambda_1$ there is a true enhançon ring with M tensionless fractional branes, and the non-abelian factors in the gauge group reduce according to $SU(M + p) \times SU(p) \rightarrow SU(p) \times SU(p)$, with one infinite gauge coupling. Twisted fields have to be excised there so as to avoid negative D3-charge in the interior region.

In passing let us stress, as in [134], that even though their dynamics takes place at arbitrarily low energies, the possible additional $U(1)$ factors are described in the holographic setup by modes at a finite radius where the corresponding fractional D3 branes lie.

¹¹The 3-cycle where the D5 charge integration is performed is the product of the exceptional 2-cycle \mathcal{C} and an S^1 on the orbifold line. Since the intersection number is $(\mathcal{D}, \mathcal{C}) = -2$ where \mathcal{D} is the cone over the 3-cycle, the D5 charge is twice the number of wrapped D5 branes.

In order to have an intuition on the strong coupling dynamics at hand, let us first focus on the first such generalized enhançon, which occurs at the scale $\Lambda_1 = \Lambda$. This will clearly be a prototype for any generalized enhançons. As already stressed, at the scale Λ , the coupling of the largest gauge group diverges (and instantonic corrections dominate), while the other gauge coupling reaches the value $g_{\min}^2 = 4\pi g_s$. As a first step toward the understanding of the precise mechanism taking place, we can consider a corner of the parameter space of the gauge theory where $N g_{\min}^2 \rightarrow 0$. In this limit, the gauge dynamics of the second factor decouples and it effectively becomes a global symmetry: the theory around Λ is simply $SU(N + M)$ SQCD with $2N$ flavors. Moreover, possible VEV's for the smaller group adjoint scalar effectively behave as masses for the larger group hypermultiplets. In this case we are out of the supergravity approximation but this analysis will give us some good insight. Hence, let us quickly review some results about the moduli space of $\mathcal{N} = 2$ SQCD.

6.4.1 One cascade step: $\mathcal{N} = 2$ SQCD

The moduli space of $\mathcal{N} = 2$ SQCD [155] with N_c colors and N_f flavors consists of a Coulomb branch and of various Higgs branches. The Coulomb branch [169, 155] is parametrized by the vacuum expectation value of the adjoint scalar field Φ in the $\mathcal{N} = 2$ vector multiplet,

$$\Phi = \text{Diag}(\phi_1, \dots, \phi_{N_c}) \quad \sum_a \phi_a = 0, \quad (6.34)$$

and is thus given by the $N_c - 1$ dimensional complex space of ϕ_a 's modulo permutations (Weyl gauge transformations). The VEV's generically break the $SU(N_c)$ gauge group to its Cartan subgroup $U(1)^{N_c-1}$. However, at special submanifolds where the Higgs branches meet the Coulomb branch a non-abelian gauge symmetry survives. Higgs branches can be divided into a baryonic branch and various non-baryonic branches (according to whether baryonic operators acquire VEV's or not); the corresponding intersections with the Coulomb branch were dubbed roots.¹² Higgs branches are not quantum corrected, however their intersections among themselves and with the Coulomb branch are modified at quantum level.

The SW curve describing the Coulomb branch for vanishing masses is [169, 171]

$$y^2 = \prod_{a=1}^{N_c} (x - \phi_a)^2 + 4\Lambda^{2N_c - N_f} x^{N_f}. \quad (6.35)$$

Nonbaryonic branches are labeled by an integer $1 \leq r \leq \min([N_f/2]_-, N_c - 2)$. The low energy effective theory at the roots are the IR free or finite $SU(r) \times U(1)^{N_c-r}$ SQCD with N_f hypermultiplets in the fundamental representation and charged under one of the $U(1)$ factors. At special points along these submanifolds, the SW curve shows that $N_c - r - 1$ additional massless singlet hypermultiplets arise, each one charged under one of the remaining $U(1)$ factors. It is important that there are $2N_c - N_f$ such vacua, related by the broken $\mathbb{Z}_{2N_c - N_f}$ non-anomalous R-symmetry acting on the Coulomb branch.

¹²Issues related to the baryonic root of $\mathcal{N} = 2$ SQCD and the mass deformation to $\mathcal{N} = 1$ were recently discussed in [170].

The baryonic branch exists for $N_c \leq N_f$, and the baryonic root is a single point, invariant under the $\mathbb{Z}_{2N_c - N_f}$ R-symmetry. Thus its coordinates on the Coulomb branch are¹³

$$\Phi_{\text{bb}} = \underbrace{(0, \dots, 0)}_{N_f - N_c}, \phi\omega, \phi\omega^2, \dots, \phi\omega^{2N_c - N_f}, \quad (6.36)$$

where $\omega = \exp\{2\pi i/(2N_c - N_f)\}$, for some value of ϕ (and $\phi = 0$ classically). The gauge group is thus broken to $SU(N_f - N_c) \times U(1)^{2N_c - N_f}$, which is IR free.¹⁴ The requirement that a Higgs branch originates from this root implies the presence of $2N_c - N_f$ massless hypermultiplets charged only under the $U(1)$ factors; this singles out a point in the submanifold described by (6.36). The result is $\phi = \Lambda$, so that the SW curve takes the singular form

$$y^2 = x^{2(N_f - N_c)}(x^{2N_c - N_f} + \Lambda^{2N_c - N_f})^2. \quad (6.37)$$

The $x^{2(N_f - N_c)}$ factor corresponds to an unbroken $SU(N_f - N_c)$ gauge group. The remaining $2(2N_c - N_f)$ branch points show up in coincident pairs, located at $x_k = \Lambda\omega^{k - \frac{1}{2}}$ with $k = 1, \dots, 2N_c - N_f$, corresponding to the $2N_c - N_f$ mutually local massless hypermultiplets.

The reason for this detour should be clear by now: the non-perturbative dynamics at the baryonic root preserves the same $\mathbb{Z}_{2N_c - N_f} = \mathbb{Z}_{2M}$ R-symmetry as the supergravity solution we are discussing, and its low energy effective theory possesses an $SU(N_f - N_c) = SU(N - M)$ non-abelian gauge symmetry precisely matching the numerology of the cascading interpretation. Hence, iterating the above procedure at the subsequent generalized enhancements Λ_k (where the higher rank gauge group coupling diverges), it is natural to propose the supergravity solution in (6.12) and (6.13) (excised only down at the true enhancement ρ_{min}) to be dual to a cascading $SU(N + M) \times SU(N)$ quiver gauge theory at subsequent baryonic roots of the strongly coupled gauge groups.¹⁵ In what follows, we will provide several checks for the validity of our proposal.

6.4.2 The cascading vacuum in the quiver gauge theory

Let us now turn to the full quiver gauge theory $SU(N + M) \times SU(N)$. The vacuum we propose as the dual of the full cascading solution is a vacuum in which, at each step along the resulting cascade, the largest of the two gauge groups goes to strong coupling with a behavior analogous to the the baryonic root of SQCD. This vacuum is invariant under the same non-anomalous \mathbb{Z}_{2M} subgroup of the R-symmetry as the supergravity solution we started with. Moreover, not only has it the correct spontaneous symmetry breaking pattern but also the correct RG flow, including the beta functions and the separation of scales where the transitions occur, as can be extracted from supergravity.

It is worth stressing that our vacuum does not sit exactly at the baryonic roots, as there are no baryonic roots in the quiver theory (see section 6.4.4 for an exception).

¹³For $N_f > 3N_c/2$ there are other $\mathbb{Z}_{2N_c - N_f}$ -invariant submanifolds. However the baryonic root is just one point, and one can show that it in fact belongs to the submanifold (6.36) [155].

¹⁴We assume $N_f < 2N_c$ so that the microscopic theory is UV free. This bound is satisfied in the cascading quiver theory.

¹⁵We should mention that a proposal for an $\mathcal{N} = 2$ cascade at the baryonic root has been alluded to in [172], in the context of the M-theory realization of this elliptic model.

However, it does approximate them in the supergravity limit in which $q \rightarrow 0$, which is the limit of interest to us.

Let us start for concreteness with an $SU((2K+1)M) \times SU((2K+1)M)$ conformal theory in the UV and then break the gauge group to $SU((2K+1)M) \times SU(2KM)$ by giving VEV's of order z_0 in a \mathbb{Z}_M -invariant way to M eigenvalues of the adjoint scalar $\tilde{\Phi}$. We choose a vacuum in which, at each step of the RG flow, the most strongly coupled group is at its baryonic root (in the $q \rightarrow 0$ limit). Let us write the SW curve as $R(v)/S(v) = g(u|\tau)$ as in (6.21), where u is the coordinate on a torus of complex structure τ . We choose the polynomials $R(v)$ and $S(v)$ of degree $(2K+1)M$, as

$$\begin{aligned} R(v) &= v^M \prod_{j=0}^{K-1} (v^{2M} + q^{\frac{1}{2}+2j} z_0^{2M}) \\ S(v) &= (v^M - z_0^M) \prod_{j=0}^{K-1} (v^{2M} + q^{\frac{3}{2}+2j} z_0^{2M}) . \end{aligned} \tag{6.38}$$

The polynomial $R(v)$ is related to the $SU((2K+1)M)$ group that starts flowing toward strong coupling at the cutoff scale z_0 , whereas the polynomial $S(v)$ is related to the $SU((2K+1)M)$ group which is spontaneously broken to $SU(2KM)$ there.¹⁶ The eigenvalues of the two adjoint scalar fields are put, in an alternating manner, at energies corresponding to their subsequent strong coupling scales along the cascade: in the limit in which the dynamics of the weakly coupled group decouples at those scales, the vacua mimic the SQCD baryonic root. In agreement with the cascading RG flow of the supergravity solution, the hierarchy of strong coupling scales is controlled by $q = e^{2\pi i\tau}$. Because of the large M limit, the running is led by the perturbative beta functions except at the successive strong coupling scales, where instantonic corrections sharply appear. This field theory running can be explicitly checked either numerically using the exact SW curve we wrote, or analytically by expanding the polynomials energy range by energy range, in an effective field theory approach (see Appendix E.7). A plot of the resulting RG flow is shown in figure 6.3.

We now move on to the study of the branch points of the curve. Recall that branch points are double solutions in v at fixed u . In the dual type IIA construction, a pair of coincident branch points at v corresponds to a D4 brane stretched between the two NS5's, while in type IIB it corresponds to a fractional brane at position $z \simeq v$ on the orbifold singularity line. When the branch points are not in pairs, the full M-theory description is needed, fractional branes are no longer perturbative states in type IIB and their wavefunction is spread over the whole $b \in \mathbb{Z}$ region [156] (at least in the large M limit).

It turns out that the branch points for $u = 0, 1/2$, up to corrections of higher order in q , lie at

$$v^M \simeq \mp q^{n+1/4} z_0^M, \quad n = 0, \dots, K-1 \quad \text{and} \quad v^M \simeq \mp 2q^{K+1/4} z_0^M. \tag{6.39}$$

¹⁶Very similarly, we can also describe a cascade with an $SU(2KM) \times SU(2KM)$ UV completion: it amounts to putting the cutoff and the vanishing eigenvalues in the same adjoint field/polynomial in (6.38), otherwise preserving the structure of the polynomials. Finally, the generalization to the cascade with $N = lM + p$ can be achieved by multiplying R and S by the same degree p polynomial.

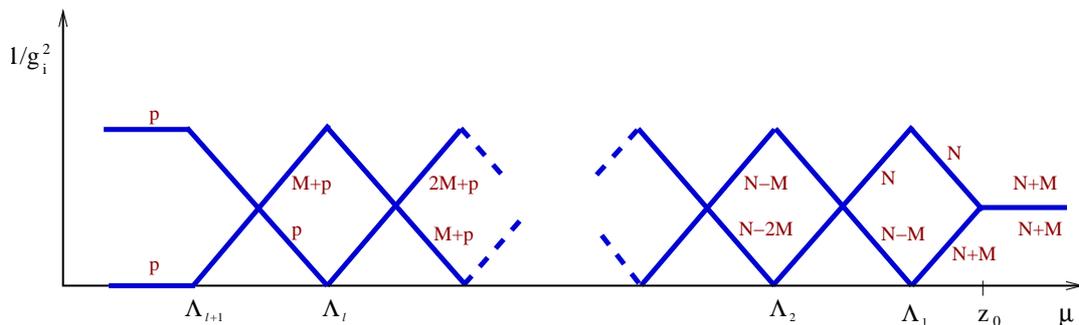


Figure 6.3: RG flow of the theory at the cascading vacuum (taking $p = 0$, for definiteness). Here, as well as in figures 6.4 and 6.6, the horizontal axis is logarithmic and we have omitted the SU factors for the gauge groups, to avoid clutter.

The former class of points consists of K sets of $2M$ double points (which are double up to an accuracy discussed at the end of the next subsection), corresponding to the K baryonic-root-like VEV's of the first gauge group, whereas the latter are $2M$ well separated branch points, corresponding to the true enhancement of the low energy $SU(M)$ theory. The branch points for $u = \tau/2, (\tau + 1)/2$ lie at

$$v^M = \mp q^{n+3/4} z_0^M, n = 0, \dots, K-1 \quad \text{and} \quad v^M = (1 \pm 2q^{1/4}) z_0^M. \quad (6.40)$$

The first class of points consists again of K sets of $2M$ (almost) double points, corresponding to the K baryonic-root-like VEV's of the second gauge group, while the second set of points are the almost paired branch points associated to semiclassical fractional branes at the cutoff scale z_0 .

6.4.3 The infinite cascade limit

In this subsection we analyse the case of an infinite cascade, created as the cutoff anti-fractional branes are sent to infinity. We are interested in this limit for two main reasons: first of all, this limit allows us to describe the field theory vacuum and the SW curve dual to the infinite cascade solution of [10], where there are no cutoff anti-fractional branes; secondly, this infinite cascade bears strong connections and similarities, that we will specify in the following, with the Klebanov-Tseytlin-Strassler $\mathcal{N} = 1$ cascade [127, 118], which is necessarily unbounded in the UV since fractional branes are stuck at an isolated conifold singularity.

In order to properly define this limit, we should keep fixed the IR enhancement scale Λ_{min} , as well as the generalised enhancement scales defined in (6.33). It is thus convenient to rewrite the two polynomials as

$$R_K(v) = v^M \prod_{j=1}^K (v^{2M} + q^{-2j} \Lambda_{min}^{2M})$$

$$S_K(v) = (v^M - q^{-\frac{1}{4}-K} \Lambda_{min}^M) \prod_{j=1}^K (v^{2M} + q^{1-2j} \Lambda_{min}^{2M}). \quad (6.41)$$

The limit of infinite cascade is formally $K \rightarrow \infty$. Let us define $x = (v/\Lambda_{min})^M$, obtaining the SW curve

$$T_K(x) \equiv \frac{R_K(v)}{S_K(v)} = \frac{x}{x - q^{-1/4-K}} \frac{\prod_{j=1}^K (x^2 + q^{-2j})}{\prod_{j=1}^K (x^2 + q^{1-2j})} = g(u|\tau). \quad (6.42)$$

Note that

$$T_K(x) = \frac{x}{x - q^{-1/4-K}} \frac{\prod_{j=1}^K (1 + q^{2j} x^2)}{q^K \prod_{j=1}^K (1 + q^{2j-1} x^2)} \quad (6.43)$$

converges pointwise as $K \rightarrow \infty$ for any fixed value of x (possibly with poles) since $|q| < 1$, even though it does not converge uniformly.

We can then show that the approximate double points become exact at any order in q at large enough $|v|$ (i.e. the monopoles become exactly massless in the upper reach of the cascade). We will make use of the following property of g at its double points: $g(0|\tau) = -g(1/2|\tau) = 1/g(\tau/2|\tau) = -1/g((1+\tau)/2|\tau)$. Moreover, the value of the periodic function at these points is given by

$$g_0(q) \equiv g(0|\tau) = \frac{\theta_2(0|2\tau)}{\theta_3(0|2\tau)} = 2q^{1/4} \prod_{j=1}^{\infty} \frac{(1 + q^{2j})^2}{(1 + q^{2j-1})^2}. \quad (6.44)$$

Let us start with the branch points at $u = 0, 1/2$ and $x = -\epsilon q^{-n}$, where $n = 1, \dots, K$ and $\epsilon = \pm 1$. After some manipulations one gets

$$T_K(-\epsilon q^{-n}) = \frac{2\epsilon q^{1/4}}{(1 + \epsilon q^{1/4+K-n})} \frac{\prod_{j=1}^{\min(n-1, K-n)} (1 + q^{2j})^2}{\prod_{j=1}^{\min(n, K-n)} (1 + q^{2j-1})^2} \frac{\prod_{j=\min(n-1, K-n)+1}^{\max(n-1, K-n)} (1 + q^{2j})}{\prod_{j=\min(n, K-n)+1}^{\max(n, K-n)} (1 + q^{2j-1})}. \quad (6.45)$$

Consequently, the equation $T_K(x) = \epsilon g_0(q)$ is solved up to corrections $\mathcal{O}(q^{2\min(n, K-n)+1})$, $\mathcal{O}(q^{2\min(n-1, K-n)+2})$ and $\mathcal{O}(q^{1/4+K-n})$. In particular, in the case $K \geq 3n$ which is the lower part of the cascade we get

$$\frac{T_K(-\epsilon q^{-n})}{\epsilon g_0(q)} = 1 + \mathcal{O}(q^{2n}), \quad (6.46)$$

and the branch points we found are correct up to $\mathcal{O}(q^{2n})$. Similarly, for the branch points at $u = \tau/2, (\tau+1)/2$ and $x = -\epsilon q^{-n+1/2}$ with $n = 1, \dots, K$, we get

$$T_K(-\epsilon q^{-n+1/2}) = \frac{1}{2\epsilon q^{1/4} (1 + \epsilon q^{3/4+K-n})} \times \frac{\prod_{j=1}^{\min(n-1, K-n+1)} (1 + q^{2j-1})^2}{\prod_{j=1}^{\min(n-1, K-n)} (1 + q^{2j})^2} \frac{\prod_{j=\min(n-1, K-n+1)+1}^{\max(n-1, K-n+1)} (1 + q^{2j-1})}{\prod_{j=\min(n-1, K-n)+1}^{\max(n-1, K-n)} (1 + q^{2j})}, \quad (6.47)$$

and in particular, for $K \geq 3n$

$$\frac{T_K(-\epsilon q^{-n+1/2})}{(\epsilon g_0(q))^{-1}} = 1 + \mathcal{O}(q^{2n-1}). \quad (6.48)$$

In order to show that these two sets of branch points are double, we compute

$$\frac{dT_K}{dx}(x) = T_K(x) \left\{ \frac{1}{x} + \sum_{j=1}^K \frac{2x}{x^2 + q^{-2j}} - \frac{1}{x - q^{-1/4-K}} - \sum_{j=1}^K \frac{2x}{x^2 + q^{1-2j}} \right\}. \quad (6.49)$$

One can show that $T'_K(-\epsilon q^{-n}) = \mathcal{O}(q^{n+1/2})$ and $T'_K(-\epsilon q^{-n+1/2}) = \mathcal{O}(q^n)$ so that the points are double, up to sub-leading corrections (from numerical studies it seems that the corrections actually appear at some much higher order).

In a similar way, one shows that the non-double branch points at $u = 0, 1/2$ and $x = -2\epsilon$ (enhancement) are correct up to $\mathcal{O}(q)$, whereas the almost double ones at $u = \tau/2, (\tau + 1)/2$ and $x = (1 + 2\epsilon q^{1/4})q^{-1/4-K}$ (cutoff) are correct up to $\mathcal{O}(q^{1/4})$.

Summarizing, our analysis shows that the SW curve (6.42) for the finite cascade has a well defined infinite cascade limit as we send $K \rightarrow \infty$. We also evaluated to which degree the approximate double points in the $q \rightarrow 0$ limit, appearing at all the strong coupling scales except the smallest one, depart from being exactly double; we find that in the infinite cascade limit the mass of the corresponding monopoles goes to 0 for any value of q as we consider higher and higher scales up in the cascade, that is large n . Finally, only at the bottom of the infinite cascade do we find equally separated double points (in the $q \rightarrow 0$ limit), filling a true enhancement ring in the large M limit.

6.4.4 Mass deformation

A not completely satisfactory feature of the cascading vacua we proposed is that, although they preserve the \mathbb{Z}_{2M} R-symmetry as the baryonic root of SQCD, the extra light monopoles are strictly massless only in the $q \rightarrow 0$ limit or for very large n . At finite q and n , our vacua are not really singled out as very special points in the moduli space. Surely this is enough to our purpose of finding the field theory vacua dual to the supergravity solutions in (6.12) and (6.13). However, it will be useful to argue for the existence of a cascading vacuum with exactly massless monopoles.

The task can be related to mass deformation of the $\mathcal{N} = 2$ theory to $\mathcal{N} = 1$, after the addition of a mass term for the adjoint scalars

$$W_{\text{mass}} = \frac{m}{2}(\Phi^2 - \tilde{\Phi}^2). \quad (6.50)$$

In the case of $\mathcal{N} = 2$ SQCD, a mass deformation lifts the moduli space and only the points on the Coulomb branch with $2N_c - N_f$ extra massless monopoles survive, that is the baryonic root and the $2N_f - N_c$ special points along the non-baryonic roots. The reason is that in the dual M-theory picture a mass deformation corresponds to a relative rotation of the two extended M5 branches (NS5-branes in IIA), and this is possible only if the curve has genus zero (because in the $\mathcal{N} = 1$ theory confinement breaks completely the gauge group, and the genus of the M-theory/SW curve equals the rank of the left over group). On the other hand, moduli space points with massless monopoles are singular

points where the genus of the curve reduces, and a maximal number of them is needed to reach zero genus.

This suggests that a special point on the moduli space of the quiver theory should be found after a mass deformation. There are two main problems however. The first is that the cascading theory is obtained from the conformal theory by spontaneous breaking at the cutoff z_0 ; this is no longer a solution after mass deformation. A possible solution is to consider an infinite cascade, as in the case of the conifold theory. From a more conservative point of view, one could consider an unstable time-dependent field configuration with a finite cascade (with a large number of steps) in which the VEV's for the spontaneous breaking are very large but collapsing to zero. In this case the dimensionless parameter controlling the time evolution of the field is $\ddot{\Phi}/\Phi^3 = -(m/\Phi)^2$, which is in fact very small for $\Phi \gg m$. This mechanism would “freeze” the cutoff in this limit. The other problem is that, unlike the SQCD case, after mass deformation the far IR is $SU(M)$ $\mathcal{N} = 1$ pure SYM, whose M vacua break \mathbb{Z}_{2M} to \mathbb{Z}_2 .

These observations suggest that we should look for a genus zero SW curve which breaks \mathbb{Z}_{2M} to \mathbb{Z}_2 , mimicking the curve for $SU(M)$, and which describes an infinite cascade. Let us start from one of the M genus zero curves of $\mathcal{N} = 2$ $SU(M)$ SYM: being of genus zero they are parametrized by a complex coordinate λ , from which one constructs two rational functions v and t [173, 174]

$$v = \lambda + \frac{\Lambda^2}{\lambda}, \quad t = \lambda^M \quad \Rightarrow \quad t^2 - P_M(v)t + \Lambda^{2M} = 0, \quad (6.51)$$

where $P_M(v)$ is a particular polynomial of degree M in v . In the following we will set $\Lambda = 1$; then $P_M(v)$ is a Chebishev polynomial [159]

$$P_M(v) = \left[\frac{v + \sqrt{v^2 - 4}}{2} \right]^M + \left[\frac{v - \sqrt{v^2 - 4}}{2} \right]^M. \quad (6.52)$$

The genus zero curve for the infinite cascade vacuum in the quiver theory is simply obtained by wrapping the SYM curve on the torus,

$$Q = \lim_{K \rightarrow \infty} Q_K = \lim_{K \rightarrow \infty} \prod_{j=-K}^K F(q^j t, v) = 0 \quad \text{with} \quad F(t, v) = t - P_M(v) + \frac{1}{t}, \quad (6.53)$$

where $t = e^{2\pi i u}$. This definition is mainly formal, as the infinite product above does not converge. However its zero locus in $T^2 \times \mathbb{C}$ (the curve itself) is well defined, and it consists of the SYM curve wrapped infinitely many times on the torus. It is clear that it has genus zero (being non-compact, we mean that it is parametrized by λ) and that it reproduces the correct IR behavior of $SU(M)$ SYM.

In order to make sense of it, and to check that it is the limit of a sequence of SW curves for longer and longer cascades, with the correct hierarchy of scales as expected from the RG flow at the baryonic roots, we consider finite K (eventually sent to ∞) and rewrite the curve as

$$\tilde{Q}_K = q^{K(K+1)} f(q) Q_K = f(q) \left(t - P + \frac{1}{t} \right) \prod_{j=1}^K \left(1 - P t q^j + t^2 q^{2j} \right) \left(1 - \frac{P}{t} q^j + \frac{q^{2j}}{t^2} \right) = 0, \quad (6.54)$$

where $f(q) = \prod_{j=1}^{\infty} (1 - q^{2j})(1 - q^{2j-1})^2$. The zero locus is the same as before, but now the product converges as $K \rightarrow \infty$. Then, we define a sequence of SW curves for $SU((2K+1)M) \times SU((2K+1)M)$ given by

$$\mathcal{Q}_K \equiv -\tilde{R}_K \theta_3(2u|2\tau) + \tilde{S}_K \theta_2(2u|2\tau) = 0, \quad (6.55)$$

with the polynomials \tilde{R}_K and \tilde{S}_K chosen as

$$\tilde{R}_K(v) = P(v) \prod_{j=1}^K (q^{2j} P(v)^2 + 1 - 2q^{2j} + q^{4j}) \quad (6.56)$$

$$\tilde{S}_K(v) = q^{-1/4} (1 - q^{K+1/4} P(v)) \prod_{j=1}^K (q^{2j-1} P(v)^2 + 1 - 2q^{2j-1} + q^{4j-2}). \quad (6.57)$$

Using the identities

$$\theta_3(2u|2\tau) = \prod_{j=1}^{\infty} (1 - q^{2j}) (1 + t^2 q^{2j-1}) (1 + t^{-2} q^{2j-1}) \quad (6.58)$$

$$\theta_2(2u|2\tau) = q^{1/4} (t + t^{-1}) \prod_{j=1}^{\infty} (1 - q^{2j}) (1 + t^2 q^{2j}) (1 + t^{-2} q^{2j}), \quad (6.59)$$

one can explicitly verify that

$$\tilde{\mathcal{Q}}_K = \mathcal{Q}_K \quad \text{up to orders } \mathcal{O}(q^{K+1/4}). \quad (6.60)$$

Moreover, since the polynomials $P_M(v)$ behave as v^M for $v \gg 1$, one can check that the hierarchy of scales of the cascading vacuum of subsection 6.4.3 is reproduced, up to IR corrections related to the different unbroken R-symmetries.

Let us comment on this result. Eq. (6.53)-(6.54) defines a genus zero curve with exactly double branch points for any value of q , which describes a theory with infinitely long cascade and exactly massless monopoles, dual to a specific type IIB supergravity solution with no *AdS* asymptotics. One could think of realizing the theory by wrapping an M5 brane along the curve, and then computing observables from it. However one could object that, unlike the $\mathcal{N} = 1$ infinite KS cascade which makes sense as a field theory through holographic renormalization [119], an infinite $\mathcal{N} = 2$ cascade probably does not. The reason is that as we cascade down the IR-free $U(1)$ factors accumulate, and an infinite cascade would require an infinite number of photons at finite energies, which does not make much sense. Thus in (6.55)-(6.56) we constructed a sequence of legitimate SW curves for any value of K , describing larger and larger field theories with cascade which, although not having genus zero because of the UV cutoff, approximate the genus zero curve (6.54) with arbitrary precision, for any value of q and M . We could compute observables in the sequence, getting in the limit the same answer as from (6.54). Therefore this procedure makes sense of the infinite cascade theory, in the sense that observables in finite sectors are insensible to the (possibly infinite number of) decoupled photons.

Eventually, notice that the sequence in (6.55)-(6.56) contains the finite q corrections to the $\mathcal{N} = 2$ cascade that are required to have exactly massless monopoles and that were missing in (6.41) because those were not visible in supergravity.

The mass deformation of this $\mathcal{N} = 2$ vacuum is particularly interesting because it induces a flow from the cascading $\mathcal{N} = 2$ theory to the $\mathcal{N} = 1$ Klebanov-Strassler (KS) cascade. This is expected on the field theory side because the adjoint fields have to be integrated out at the scale of the deformation mass parameter, leaving the Klebanov-Strassler field theory at smaller energies.

This is clear also in M-theory. The genus zero SW curve we proposed is the one of $\mathcal{N} = 2$ $SU(M)$ SYM, rewritten on the torus so as to create an elliptic model. Similarly to the M genus zero points on the Coulomb branch of $\mathcal{N} = 2$ SYM which survive mass deformation and flow to the M confining vacua of $\mathcal{N} = 1$ SYM, the M genus zero $\mathcal{N} = 2$ curves we proposed flow to the M cascading vacua of the $\mathcal{N} = 1$ KS theory, whose IR is in fact $\mathcal{N} = 1$ SYM.

The rotated $\mathcal{N} = 1$ curve in the limit $m \rightarrow \infty$ is easily written. As before, we start rotating the SW curve for $SU(M)$ SYM, exploiting the rational parametrization in terms of λ [174]

$$\begin{cases} v = \lambda \\ t = \lambda^M \\ w = \zeta \lambda^{-1} \end{cases} \quad \Rightarrow \quad \begin{cases} t = v^M \\ vw = \zeta \end{cases} \quad (6.61)$$

where the low energy strong coupling scale $\zeta = \Lambda_{\mathcal{N}=1}^3 = m \Lambda_{\mathcal{N}=2}^2$ is kept fixed in the limit, and a suitable rescaling of variables is performed [173]. The curve for the quiver theory is obtained by wrapping the curve on the M-theory torus: $0 = \prod_j (q^j t - v^M)$. After a rescaling to make the product converge, we get

$$0 = (t - v^M) \prod_{j=1}^{K \rightarrow \infty} \left(t v^M - q^j (t^2 + v^{2M}) + q^{2j} t v^M \right), \quad vw = \zeta. \quad (6.62)$$

Note however that while in the $\mathcal{N} = 2$ case the M5 brane embedding can be interpreted as the exact SW curve for the field theory, which encodes the prepotential and the full dynamics, after breaking to $\mathcal{N} = 1$ this is no longer the case. The theory on the M5 brane reduces to the field theory of interest only when, for particular choices of the parameters, the unwanted modes are decoupled, and we refer to [174, 175] for details.

It should be possible to reproduce this interpolating flow in supergravity, so as to gain insight also on the Kähler data of these $\mathcal{N} = 1$ vacua. In particular, if the mass deformation is much larger than the enhançon scale Λ , the solution should interpolate to the Klebanov-Tseytlin (KT) solution (before chiral symmetry breaking takes place in the IR). We leave the analysis of such an interpolating solution, which should be performed along the lines of [176], to the future.

6.5 More supergravity duals: enhançon bearings

In this section we study other vacua of the $SU(N+M) \times SU(N)$ theory, focusing on a class preserving the same \mathbb{Z}_{2M} R-symmetry as the supergravity solution of section 2. We will start from the non-cascading enhançon vacuum of section 6.3 and gradually construct the cascading vacuum discussed previously by pulling VEV's out of the origin. In this process, we will observe new nontrivial vacua, for which we will propose novel type IIB dual backgrounds.

Let us consider the following family of polynomials for the SW curves of the $SU(N+M) \times SU(N+M)$ theory, parametrized by ϕ

$$R(v) = v^{N-M} (v^{2M} - \phi^{2M}) \quad S(v) = v^N (v^M - z_0^M). \quad (6.63)$$

An overall v^{N-M} factor (interpreted as $N - M$ D3 branes at the origin) decouples from the SW curve (6.21), so that we will effectively reduce to the $SU(2M) \times SU(2M)$ case, with

$$R(v) = v^{2M} - \phi^{2M} \quad S(v) = v^M (v^M - z_0^M) . \quad (6.64)$$

For $\phi = 0$ we are at the origin of the moduli space of the $SU(2M) \times SU(2M)$ effective theory, where the enhançon mechanism takes place. We want to study the branch points of the SW curve as we vary ϕ continuously, in the supergravity approximation of small q , so that $g_0(q) = 2q^{1/4} + \mathcal{O}(q^{5/4})$. We will use the shorthand notation $\xi = v^M$ and define the enhançon scale $\Lambda = 2^{1/M} q^{1/4M} z_0$.

Let us first consider the branch points at $u = 0, 1/2$, related to the polynomial R . Depending on the value of $|\phi|$, we find:¹⁷

- $|\phi^M| < |q^{1/4} z_0^M|$

$$\xi \simeq \pm \Lambda^M , \quad \xi \simeq \pm \left(\frac{\phi^2}{\Lambda} \right)^M , \quad (6.65)$$

namely $2M$ equally separated branch points at the enhançon ring and $2M$ equally spaced branch points at a ring of radius $|\phi^2/\Lambda|$;

- $|\phi^M| > |q^{1/4} z_0^M|$

$$\xi \simeq \pm (1 + \epsilon q^{1/4}) \phi^M , \quad \epsilon = \pm 1 , \quad (6.66)$$

namely $2M$ pairs of branch points on a circle of radius $|\phi|$.

The branch points at $u = \tau/2, (1 + \tau)/2$ related to the polynomial S , as long as $|\phi^M| < |q^{-1/4} z_0^M|$ which will always be the case if $|\phi| < |z_0|$, are

$$\xi \simeq (1 \pm 2q^{1/4}) z_0^M , \quad \xi \simeq \pm 4q^{1/2} \left(\frac{\phi^2}{\Lambda} \right)^M , \quad (6.67)$$

namely M pairs of branch points along a circle of radius $|z_0|$ and $2M$ equally spaced branch points on a ring of radius $4^{1/M} q^{1/(2M)} |\phi^2/\Lambda|$.

In order to understand what the supergravity solutions dual to these vacua are, it will be useful to recall what are the BPS fractional branes at our disposal. They are obtained by wrapping D5 branes or anti-D5's ($\eta = \pm 1$ below, respectively) on the exceptional 2-cycle with n_f units of worldvolume flux. Their Wess-Zumino action reads

$$S_{WZ} = \eta \mu_3 \int_{M^{3,1}} \left[\tilde{c}_4 + (b + n_f) C_4 \right] , \quad (6.68)$$

where \tilde{c}_4 is a twisted potential dual to c . We will use the notation $D5_{n_f}$ and $\overline{D5}_{n_f}$ for the fractional branes with flux (recalling that n_f is gauge dependent while the D3-charge is gauge invariant). The BPS objects are those whose worldvolume flux ensures positive D3-charge $\eta(b + n_f)$, which then equals the tension (6.16); notice that when the D3 charge exceeds one, we simply have a marginally stable bound state of a fractional D3 brane with a number of regular D3 branes.

The picture which stems from the branch points of the curve and from the study of the RG flow is the following.

¹⁷We write the first corrections only when they are necessary to split double branch points.

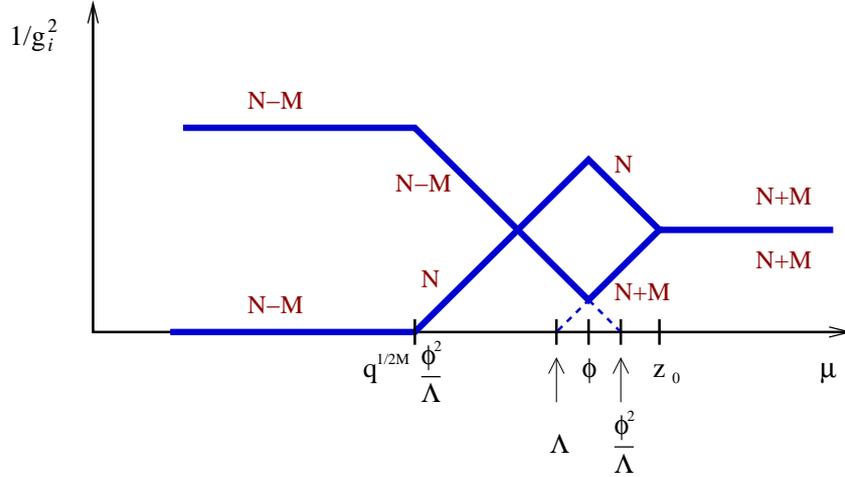


Figure 6.4: RG flow of the theory at a vacuum with a perturbative Higgsing at scale ϕ .

First, in the case $|\Lambda| < |\phi| < |z_0|$, whose corresponding RG flow is depicted in figure 6.4, the theory is conformal in the UV, down to z_0 where M eigenvalues of one adjoint scalar break the gauge group to $SU(N+M) \times SU(N) \times U(1)^M$, triggering the RG flow. They correspond to M semiclassical $\overline{D5}_{-1}$'s in the type IIB picture. At the scale ϕ there are $2M$ pairs of branch points at the positions of the $2M$ VEV's of the other adjoint scalar, which break further to $SU(N-M) \times SU(N) \times U(1)^{3M}$ and invert the RG flow. They correspond to $2M$ semiclassical D5's in the geometry, which invert the twisted fluxes; in particular b starts to grow as the radius decreases. At a lower energy scale $q^{1/(2M)}\phi^2/\Lambda$ the $SU(N)$ coupling diverges, instantons break the gauge group further to the conformal $SU(N-M) \times SU(N-M)$ theory with one divergent coupling (times the $U(1)^{4M}$ factor), and we find $2M$ branch points equally spaced along a ring. In type IIB, b reaches the value 1 at the ring and there leaves M tensionless $\overline{D5}_{-1}$'s smeared over the enhançon ring. It is possible to see by adding a $\overline{D5}_{-1}$ probe that it cannot penetrate into the interior, whereas a D5₀ can penetrate the enhançon ring, unchaining a $\overline{D5}_{-1}$ from it and making a D3 brane, which is free to move inside.

There is a more interesting behavior in the case of $|\phi| < |\Lambda|$. If $\phi = 0$ we are at the enhançon vacuum of Section 6.3. When ϕ does not vanish, the branch points follow the pattern of figure 6.5 whereas the RG flow is the one depicted in figure 6.6. As before, M $\overline{D5}_{-1}$'s are placed at the cutoff scale z_0 . From that scale downwards there is a flow with decreasing b towards smaller radii, and an enhançon ring with $2M$ equally spaced branch points at Λ , where b reaches 0 and M tensionless D5₀'s are melted on the ring. At lower energies the theory includes the conformal $SU(N) \times SU(N)$ factor with one divergent coupling: $b = 0$ in the dual supergravity solution, because of the M fractional branes at the enhançon ring. One could have expected that a new flow would start at a scale ϕ because of the VEV's, but it does not: it actually starts only at a lower scale ϕ^2/Λ , where there are $2M$ additional equally spaced branch points; below this energy scale, the gauge group with divergent coupling starts running towards weak coupling again, whereas the other one runs towards strong coupling. We enter a new perturbative regime, which ends

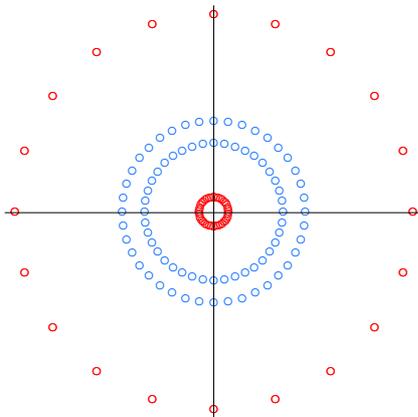


Figure 6.5: Branch points of the $U(20) \times U(20)$ theory at a vacuum with one enhançon bearing, a non-perturbative region between two enhançon rings. Red (blue) circles denote branch points related to the S (R) polynomial.

with a final ring of equally spaced branch points at scale $q^{1/(2M)} \phi^2/\Lambda$ where one gauge coupling diverges; in the interior we find a new conformal $SU(N - M) \times SU(N - M)$ sector, with one divergent coupling, down to the IR.

We will call the ring at scale ϕ^2/Λ an anti-enhançon. From the supergravity point of view it is indistinguishable from a usual enhançon. However from the field theory point of view it is quite peculiar: it represents instantonic effects that break the upper conformal theory to a running one. These effects at the scale ϕ^2/Λ are triggered by VEV's at the scale ϕ : they take some “affine RG time” to break the group; moreover this means that the effective conformal theory must have some remnant of the scale Λ . These issues deserve further investigations.

We dub the regions between enhançon and anti-enhançon rings, where $b \in \mathbb{Z}$ and the theory enjoys a superconformal phase, *enhançon bearings*.

It turns out that one can construct two different type IIB solutions that describe this RG flow. The first one, say Higgsing-inspired (H), by continuity with the case $|\phi| > |\Lambda|$ where a perturbative Higgs mechanism takes place, interprets the ring of branch points at ϕ^2/Λ as an anti-enhançon made of M tensionless $D5_0$'s (like the ones at Λ), which therefore force b to grow as the radius decreases, so that it remains bounded by 0 and 1. The innermost ring, placed where b reaches 1, is an enhançon ring made of smeared tensionless $\overline{D5}_{-1}$. In this picture the $D5_0$'s ($\overline{D5}_{-1}$'s) are always associated to the first (second) gauge group.

The second, say cascade-inspired (C), works by analogy with the Klebanov-Tseytlin-Strassler $\mathcal{N} = 1$ cascade and interprets the ring of branch points at ϕ^2/Λ as an anti-enhançon made of M tensionless $\overline{D5}_0$, and b becomes negative at smaller radii. Then b is monotonic, and the innermost ring at $b = -1$ is interpreted as an enhançon ring made of M tensionless $D5_1$. This is the picture that matches with the solution in (6.12)-(6.13) and which is usually considered in the literature. The association between fractional branes and gauge groups is such that wrapped (anti)D5 branes always correspond to the larger (smaller) gauge group.

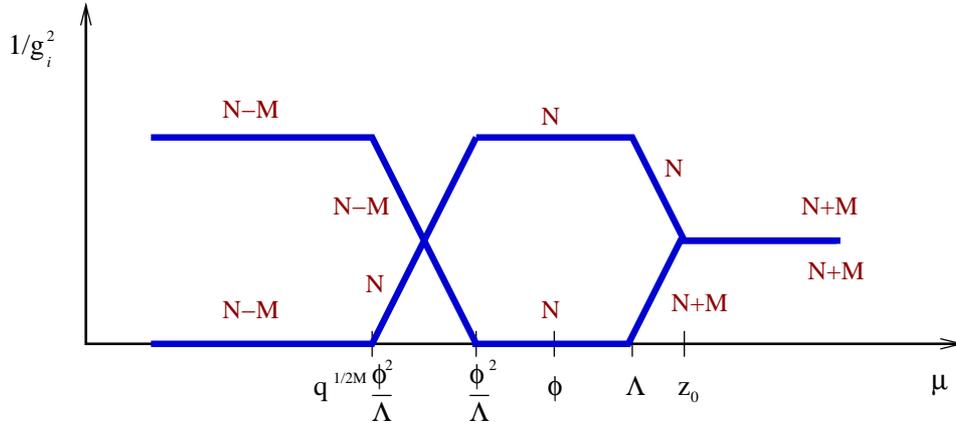


Figure 6.6: RG flow of the theory at a vacuum with one enhançon bearing. The theory is effectively the conformal $SU(N) \times SU(N)$ between the scales Λ and ϕ^2/Λ .

Type IIB solutions like the two we are discussing here can be explicitly constructed by excising and gluing twisted fields of the solution in (6.12)-(6.13) (possibly generated by one or the other kind of fractional branes) and of a fluxless solution, with suitable sources accounting for the discontinuities at the glued surfaces, along the lines of [177]. As already stressed in the case of the ordinary enhançon ring, this excision and gluing procedure works for twisted fields, which are constrained to the orbifold fixed plane. Instead, untwisted fields like the metric can propagate also in the four dimensions of the orbifold, and must be computed once the twisted fields and fractional brane configuration is specified; this will be done in section 6.6. It should be remarked that they turn out to be the same in the two pictures. One immediately realizes that all gauge invariant quantities one could compute from the two solutions will give the same answer, and in the field theory moduli space we have only one vacuum to match with the two solutions. This suggests that an ambiguity must be at work.

The ambiguity is particularly apparent in the T-dual type IIA/M-theory description. In type IIA, on each NS5-brane there is some worldvolume $G_1 = dA_0$ flux. Space-time filling I3 brane intersections of codimension two, where D4 branes end on an NS5 brane, are magnetic sources for A_0 ; the flux $\oint G_1$ through any closed path in the 2 dimensions of the NS5 worldvolume parametrized by v , in which I3 branes are points, jumps by one unit whenever the path crosses one of these points. In what follows we will consider circular paths centered in the origin of the v plane. One direction transverse to the NS5's, say x^6 , is compact of radius R and the distance between the two NS5-branes is $2\pi b R$. In figure 6.7(a) we plotted the local geometry around a ring where the perturbative Higgsing takes place as in the RG flow of figure 6.4: the NS5 on the left has a flux $\oint G_1 = -M$ (in suitable units) below the stretched D4 branes, that jumps to M above the D4's, while the opposite happens to the NS5 on the right whose flux jumps from M to $-M$. Along a generalized enhançon ring b is integer valued, so that the stretched D4's are degenerate and the NS5's touch, as in figure 6.7(b). This interpretation leads to the H-picture in IIB: b has a saw-shaped profile bounded by $[0, 1]$ and there are $2M$ fractional branes of one kind in the enhançon bearing, M on each boundary. But the same IIA configuration

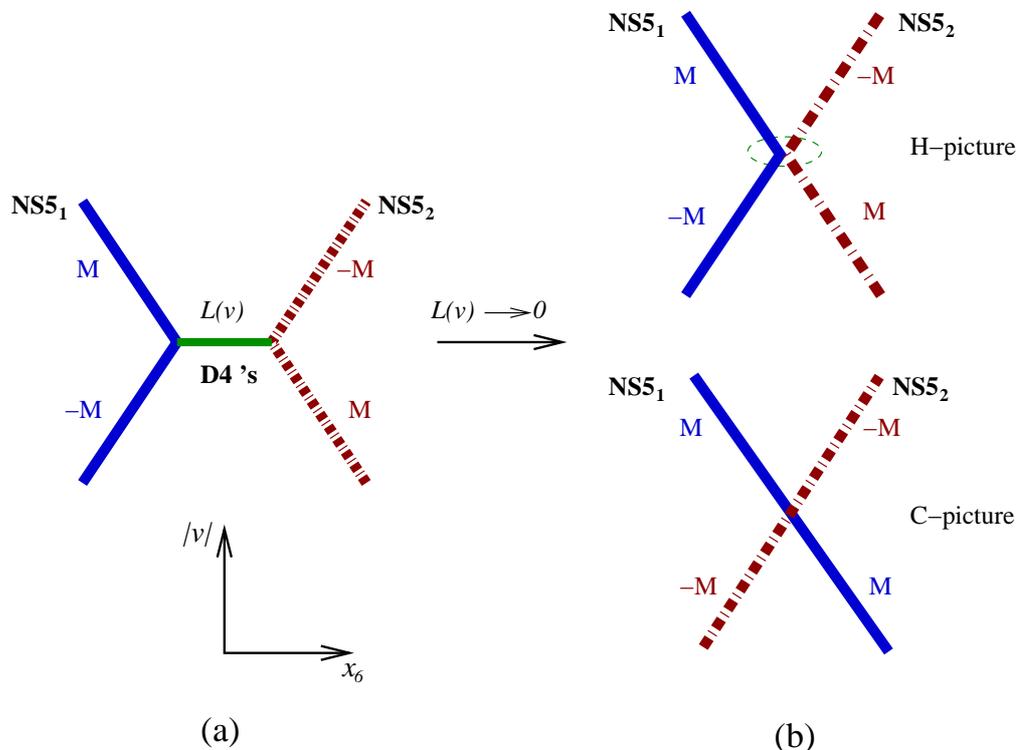


Figure 6.7: IIA description and ambiguity. (a) a point of the moduli space where $2M$ D4 branes are stretched between two NS5 branes. (b) another point where the D4 branes have collapsed to zero length. In the H-picture we interpret the D4 branes as still present, providing bending tension and flux jump; in the C-picture, the D4 branes are simply not there.

can be equally well interpreted as two NS5 branes that just cross, without any D4 branes between them and without any jump in the flux. This leads to the C-picture in IIB: b is monotonic, and the bearing has fractional branes on one side and anti-fractional on the other side, which cancel their charge. In the type IIA picture there is clearly a single configuration (dual to a single vacuum in field theory) which gives rise to two pictures in IIB.

In type IIB, the ambiguity is related to S-duality: the duality group $PSL(2, \mathbb{Z})$ acts covariantly on the parameter space, whilst the left over \mathbb{Z}_2 that acts as $(B_2, C_2) \rightarrow (-B_2, -C_2)$ and $(b, c) \rightarrow (-b, -c)$ on the twisted fields, is gauged. The novel feature here is that the enhançon bearings are domain walls on the \mathbb{C} orbifold line, and the \mathbb{Z}_2 can act on each domain separately. At the same time, as already stressed, the ambiguity does not affect the untwisted fields: F_5 and the warp factor are the same in the two pictures, since they depend on the twisted fields only quadratically in their field strengths; B_2 and C_2 are zero in the bulk.

We can keep playing the same game of adding suitable VEV's, explained so far in this section, to the newly found solutions, so as to generate longer and longer RG flows with more and more transitions and reductions of degrees of freedom. Of course the number of steps is at most $[N/M]$. In this way we produce a class of vacua with a sort of cascading behavior, with cascades of different lengths.

We conclude discussing the behavior of probes through the enhançon bearing, as extracted from the branch points of the SW curve with a pair of VEV's added in the perturbative regime outside the bearing, and interpreting it in the C-picture (the other one is equivalent). Consider first moving the VEV for the adjoint scalar of the gauge group related to the branch points of the bearing, keeping the VEV for the other adjoint fixed. As we decrease the VEV towards the outer enhançon scale, the two branch points reach the ring and there split and melt into it. Nothing happens until the VEV becomes smaller than the scale of the inner anti-enhançon scale, when two branch points escape from this ring, pair up and then continue their motion as almost double branch points. In the C-picture, this corresponds to a $D5_0$ which melts at the outer enhançon, and later comes out of the inner anti-enhançon as a $\overline{D5}_0$. Similarly, we can move the VEV for the adjoint scalar of the other gauge group. The corresponding two branch points cross the outer enhançon ring, unchaining two of its branch points. When they reach the inner ring, they leave two branch points there and move on. In the C-picture, this corresponds to a $\overline{D5}_{-1}$ that captures a $D5_0$ at the enhançon and becomes a D3-brane, free to move inside the bearing; then it leaves a $\overline{D5}_0$ at the anti-enhançon and becomes a $D5_1$ which is a minimal BPS object in the region $b \in [-1, 0]$ below the anti-enhançon ring. This behavior of probes through the enhançon bearings in the case of monotonic b precisely accounts for the non-trivial rearrangement of minimal objects in BPS bound states claimed in [11].

6.5.1 Reconstructing the cascading vacuum at the baryonic roots

We can now connect the enhançon bearing vacua discussed so far with the cascading vacuum at the baryonic roots of section 6.4. Such a cascading vacuum has the property that all the complexified strong coupling scales along the cascade are related by the same hierarchy $q^{1/2M}$, which ensures that, at least for $q \rightarrow 0$, the branch points pair up.

We start from a vacuum with an enhançon bearing and send the thickness of the bearing to zero sending $|\phi| \rightarrow |\Lambda|$ for the relevant strong coupling scale Λ . So doing, we end up with a single circle at scale Λ where $4M$ branch points lie, $2M$ coming from inside and $2M$ coming from outside. For generic phases of ϕ , these branch points do not pair up, and on the type IIB side we end up with a source term at the glued surface, accounting for a discontinuity of c . If instead the phase of ϕ is suitably tuned, branch points coming from the outer boundary and branch points coming from the inner boundary of the bearing collide, hence forming double branch points. Repeating the game with a vacuum with many enhançon bearings, we can obtain the cascading vacuum along the baryonic roots sending the thickness of each bearing to zero, see figure 6.8.

In type IIB, as we reduce the bearing to zero thickness we make the two smeared sources at the inner and outer boundaries of the bearing coincide. Following [134], we call the resulting shell a *generalized enhançon ring*. In the H-picture, this is made of $2M$ tensionless fractional branes, which account for the $U(1)^{2M}$ factor left over by the gauge breaking. The presence of the $2M$ massless hypermultiplets is more difficult to be claimed: one could think of them as arising at the $2M$ points along the ring where $\gamma \in \mathbb{Z} + \tau\mathbb{Z}$; however they should only be massless for the correct tuning of the phase of ϕ . Our belief is just that the IIB supergravity description is incomplete at the enhançon bearings. On the contrary, in the M-theory description the mass of BPS hypermultiplet states is given by the mass (proportional to the area) of M2 disks ending on the M5 brane

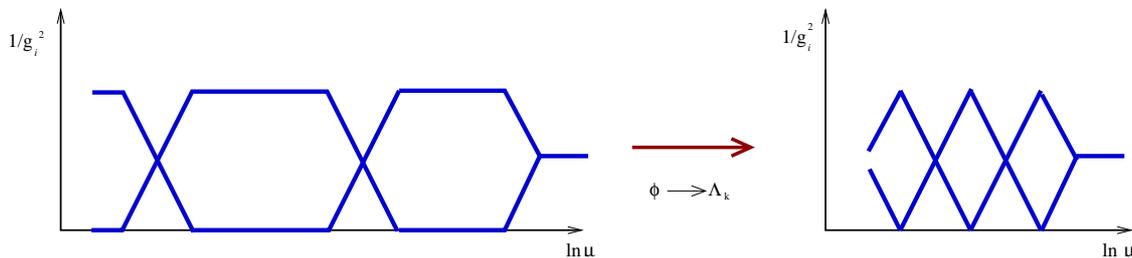


Figure 6.8: In the limit where the thickness of the bearings is sent to zero, one can reconstruct the cascading vacuum along subsequent baryonic roots.

[157, 178, 179, 180] which is the same as the SW curve; it is easy to see that the $2M$ double branch points corresponds to massless hypermultiplets.

In the C-picture the generalized enhançon is made of M fractional and M anti-fractional branes, both tensionless and D3-chargeless. When the phase of ϕ is suitably chosen and the inner and outer branch points coincide as we shrink the bearing, the D5-charges locally cancel leaving the continuous supergravity solution of Section 6.2; otherwise a source remains accounting for the discontinuity of c , and one might think of smeared dipoles of fractional/anti-fractional branes. In this picture the identification of the field theory modes is even subtler: even when a perfect annihilation seems to occur, this cannot be the case as the $U(1)^{2M}$ factor must still be there.

Let us conclude commenting on how the cascading vacuum at subsequent baryonic roots naturally arises as the dual of the supergravity solution of section 6.2. Such supergravity solution was constructed imposing rotational isometry on the \mathbb{C} orbifold line and without introducing any source. Rotational isometry translates to \mathbb{Z}_{2M} symmetry in field theory, whilst absence of sources requires all the VEV's to be at a strongly coupled scale. Among these vacua, only the cascading vacuum in the C-picture avoid seeming discontinuities of c (theta angles) and b .

6.5.2 More bearings: the enhançon plasma

So far we have described a class of \mathbb{Z}_{2M} -symmetric solutions of IIB supergravity, corresponding to vacua of the dual field theory with the same property, characterized by the presence of the enhançon plasma in the shape of fat rings (that we called enhançon bearings). From a simple numerical inspection of the field theory Coulomb branch, one discovers that the enhançon plasma can take quite different shapes (see for instance figure 6.9). We give here a general characterization of such vacua, in the large N limit.

We will show that from the point of view of IIB supergravity any choice of the enhançon plasma domains, with the only constraint of charge quantization, leads to an actual solution and represents a field theory vacuum. For definiteness, we will study the $SU(N) \times SU(N)$ conformal theory with $b = \frac{1}{2}$, spontaneously broken to non-conformal theories. Thus first of all we distribute some number of anti-fractional branes in a circular ring of radius ρ_0 in the \mathbb{C} -plane orbifold singularity. Then we will arbitrarily specify the enhançon plasma domains, without any restriction on the number of their holes and allowing nested domains.

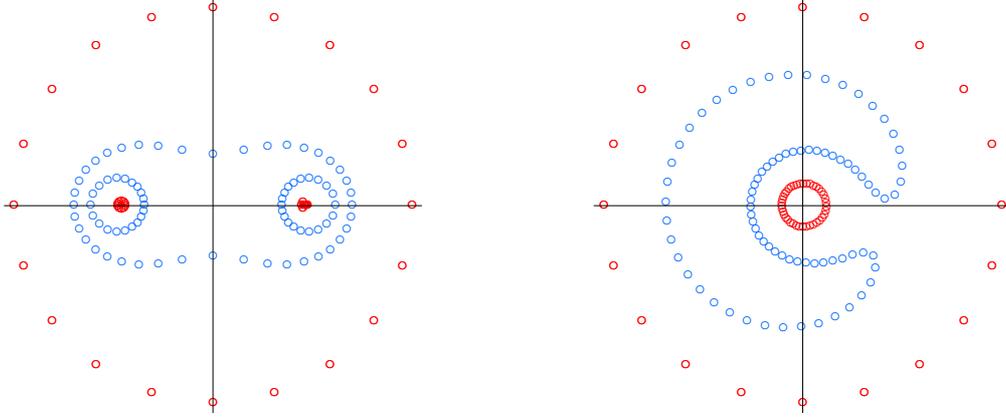


Figure 6.9: Some vacua of the $U(40) \times U(40)$ in which the enhançon plasma assumes exotic shapes. Left: the plasma has two holes. Right: the fat ring has broken into a horseshoe, disclosing the inner region.

The strategy to construct IIB supergravity solution is to solve for the twisted potentials b and c first, and then for F_5 and the warp factor.

The enhançon plasma domains behave as conductors for the objects carrying D5 charge, so that charges distribute themselves on the boundaries and inside there are no fields: b and c are constant with $b \in \mathbb{Z}$. Outside the plasma domains there are regions \mathcal{D}_i where b and c are non-trivial. Consider one of these regions, with its boundary given by a collection of curves $\mathcal{C}_{i,\alpha}$: there is one external curve $\mathcal{C}_{i,E}$ while we call the internal ones $\tilde{\mathcal{C}}_{i,\alpha}$. The boundary conditions in \mathcal{D}_i are that $b \in \mathbb{Z}$ on each curve $\mathcal{C}_{i,\alpha}$, and since we choose not to have generalized enhançon rings nor tensionful fractional branes around (they both can be obtained by sending to zero thickness an enhançon plasma with fat ring or circular shape), up to gauge transformations and picture ambiguity $b = 1$ on $\mathcal{C}_{i,E}$ and $b = 0, 1$ on $\tilde{\mathcal{C}}_{i,\alpha}$. The only exception is the outermost region \mathcal{D}_E where $b = \frac{1}{2}$ on the external ring at $|z| = \rho_0$ and the branes are tensionful, while $b = 0$ on $\mathcal{C}_{E,\alpha}$. Supersymmetry constrains $\gamma(z) = c + \frac{i}{g_s}b$ to be a local meromorphic function, and after our choice of boundary conditions actually holomorphic. To be precise, $e^{-i\gamma}$ must be a holomorphic section of a \mathbb{C}^* bundle. Rephrasing, we look for a harmonic real function b with fixed boundary conditions, and a local harmonic real function c which satisfies the Cauchy-Riemann relations.

The problem of finding a harmonic function b with prescribed values on the boundaries $\mathcal{C}_{i,\alpha}$ has one and only one solution. It can be found by minimizing the functional

$$D[u] = \int_{\mathcal{D}_i} |\partial u|^2 \quad (6.69)$$

amongst all $u \in C^{(1)}(\mathcal{D}_i \setminus \bigcup_{\alpha} \mathcal{C}_{i,\alpha}) \cap C^{(0)}(\mathcal{D}_i)$ with $u|_{\mathcal{C}_{i,\alpha}} = b(\mathcal{C}_{i,\alpha})$. A local harmonic function that satisfies the Cauchy-Riemann relation can be constructed as

$$c(z) = \frac{1}{g_s} \int_{p_0}^z (\partial_y b dx - \partial_x b dy), \quad g_s dc = \partial_y b dx - \partial_x b dy = - * db, \quad (6.70)$$

where $z = x + iy$, p_0 is an arbitrary reference point and $*$ is constructed with the flat metric on \mathbb{C} . Notice that $*d\gamma = -i d\gamma$. $c(z)$ turns out to be a multi-valued function with

monodromies which give a representation of the homology group of \mathcal{D}_i . However, as long as the twisted field-strength dc is concerned, this is globally defined.

As we will explain in section 6.6, the warp factor is obtained by solving a Poisson equation (6.75) on the orbifold $\mathbb{C}^2/\mathbb{Z}_2 \times \mathbb{C}$, with two kinds of D3-charge source terms, both localized along the orbifold line. One is proportional to $|\partial_z \gamma|^2$ and comes from the twisted fluxes. The other one is localized on the tensionful fractional branes in the external ring and represents their D3-charge. In general the brane density per unit length ω along the ring is not constant but rather given by

$$\omega = -\frac{1}{2} \mathbb{R}e \partial_t \gamma , \quad (6.71)$$

where the derivative is taken tangent to the boundary. This comes from the Bianchi identity $dF_3 \sim \delta_{D5}^{(4)}$. On the boundaries of the enhançon plasma domains there are fractional branes too with density (6.71), but they are tensionless as $b \in \mathbb{Z}$ inside. Thus the only contribution of the latter kind comes from the circular cutoff ring at $|z| = \rho_0$. We do not go into further details here, as the computation of the warp factor is fully explained in section 6.6. What matters is that there is always one and only one solution normalizable at infinity. The 5-form flux is then given by: $g_s F_5 = (1 + *)d\text{vol}_{3,1} \wedge dZ^{-1}$.

So far we showed that for any choice of the enhançon plasma domains, we can in principle solve the supergravity equations. The last constraint is the D5-charge quantization, which amounts to the monodromy of $c(z)$ being quantized

$$\oint dc \in 2\mathbb{Z} , \quad (6.72)$$

or in other terms $e^{-i\pi\gamma}$ being a single-valued function. A basis of monodromies is given by $\Delta c(\tilde{\mathcal{C}}_{i,\alpha})$ on the internal boundaries $\tilde{\mathcal{C}}_{i,\alpha}$, and the integral is the total number of fractional branes on them. As the solution only depends on the choice of the enhançon plasma boundaries (and the value of b on them), (6.72) descends to a constraint (in fact the only one) for them.

The total D3-charge of the system is then easily determined. The contribution from the fluxes in all the regions \mathcal{D}_i is

$$Q_3^{flux} = \frac{1}{2} \sum_i \int_{\mathcal{D}_i} dc \wedge db = -\frac{1}{2} \sum_i \sum_\alpha \int_{\mathcal{C}_{i,\alpha}} b dc = -\frac{1}{2} \sum_{i,\alpha} b(\mathcal{C}_{i,\alpha}) \Delta c(\mathcal{C}_{i,\alpha}) . \quad (6.73)$$

The contribution from the anti-fractional branes on the external cutoff ring can be read from (6.71) to be: $Q_3^{\text{cutoff}} = -\frac{1}{4} \oint_{\rho_0} dc$, because $b = \frac{1}{2}$ there. Since the external ring is the external boundary $\mathcal{C}_{E,E}$ of the outermost region \mathcal{D}_E , this contribution can be added to (6.73) by formally considering $b(\mathcal{C}_{E,E}) = 1$ instead of $1/2$. Notice that (6.73) is gauge and picture invariant. However, for our choice of gauge and picture the total charge is

$$Q_3^{total} = \sum_{i,\alpha} (1 - b(\tilde{\mathcal{C}}_{i,\alpha})) \Delta c(\tilde{\mathcal{C}}_{i,\alpha}) \equiv N , \quad (6.74)$$

where we used that $\mathcal{C}_{i,E} = -\sum_\alpha \tilde{\mathcal{C}}_{i,\alpha}$ in homology, and $b(\mathcal{C}_{i,E}) = 1$. This expression counts the number of fractional (as opposed to anti-fractional) branes. And in fact the

solution we constructed is dual to a vacuum of the $SU(N) \times SU(N)$ theory. It is clear that if we want to embed this vacuum in a larger theory, we can simply add regular D3 branes.

Summarizing, we have shown that any choice of enhançon plasma domains, up to the charge quantization constraint, gives rise to a solution of IIB supergravity with sources. Taking the limit of zero thickness, we can also include generalized enhançons and isolated bunches of fractional branes; bunches of regular branes are easily included as well. Each of these solutions is dual to a vacuum on the Coulomb branch of the $SU(N) \times SU(N)$ SCFT. Even though we cannot be more specific about the exact map (it should be worked out by computing operator VEV's holographically), this huge class of solutions helps in covering the moduli space of the dual field theory.

6.6 Excisions, warp factors and the cure of repulson singularities

In this section we take into account the excision of twisted fields inside the enhançon ring and bearings and work out the correct warp factor for a quite general rotationally symmetric configuration of fractional branes, which will be useful to describe the enhançon vacuum of section 6.3, the cut off cascading vacuum of section 6.2, the infinite cascade vacuum of section 6.4.3 and the vacua with rotationally symmetric bearings of section 6.5.

We stress once again that consistency of the configuration of fractional branes, in agreement with the dual field theory picture encoded in the SW curve, implies an excision of the naive twisted field solution at enhançon rings. Unlike the situation of [156], where there is an enhançon shell of codimension 1 in the non-compact part of the internal geometry, here we face enhançon rings having codimension 1 only for the twisted fields which are constrained to live on the orbifold plane, but not for the bulk fields which propagate also in the four additional dimensions of the orbifold. Consequently, the usual excision of [156, 177] works for twisted fields but not for untwisted fields; in particular, the warp factor has to be computed once and for all, once the correct configuration of fractional branes and twisted fields describing some gauge theory vacuum is specified.

The equation which determines the warp factor Z follows from the modified Bianchi identity for F_5 in the presence of sources at the locations of tensionful (anti-)fractional branes; it is a Poisson's equation which reads [10]

$$\Delta_6 Z + (4\pi^2 \alpha')^2 g_s^2 |\partial_z \gamma|^2 \delta^{(4)}(\vec{x}) + 2(4\pi^2 \alpha')^2 g_s \sum_i Q(\mathbf{x}_i) \delta^{(6)}(\mathbf{x} - \mathbf{x}_i) = 0, \quad (6.75)$$

where Δ_6 is the 6-dimensional Laplacian and $\mathbf{x} = (\vec{y}, \vec{x})$ a 6-dimensional vector, $\vec{y} \equiv (\text{Re } z, \text{Im } z) = (x^4, x^5)$ being a vector on the orbifold fixed plane \mathbb{R}^2 and $\vec{x} = (x^6, \dots, x^9)$ being a vector in the covering space \mathbb{R}^4 of the orbifold. In the previous formula, $Q(\mathbf{x}_i)$ is the gauge invariant D3 brane charge of a regular or (anti-)fractional D3 brane placed at \mathbf{x}_i , which depends on the object and on the value of fields at its position (in the case of fractional branes). The sum runs over all tensionful fractional D3 branes as well as regular D3 branes along with their images.

We will first consider M tensionless fractional branes melted in an enhançon ring of radius ρ_e in the fixed plane parametrized by z , together with M 'cutoff' anti-fractional

branes at the M roots of $z^M = -z_0^M$, which are used to Higgs the conformal UV theory at the scale $\rho_0 = |z_0|$. Here and in the following, ρ_e is the scale at which the excision should be performed and its actual value depends, case by case, on the vacuum one is actually considering. We will also impose that the total gauge invariant D3 brane charge of the configuration be $N + M$, adding regular D3 branes at the origin when needed, so that the dual gauge theory is $SU(N + M) \times SU(N + M)$ in the UV. Using the freedom of shifting the axion b by an integer via a large gauge transformation, we will also set $b(\rho) = 0$ for $\rho < \rho_e$. Finally, we will be general and place the cutoff anti-fractional branes at a scale ρ_0 such that $b(\rho_0)$ can acquire any positive value; the gauge invariant D3 brane charge supported by each of the anti-fractional branes is therefore

$$-n_f - b(\rho_0) = [b(\rho_0)]_+ - b(\rho_0) . \quad (6.76)$$

In other words, these anti-fractional branes are D5 branes wrapped on $-\mathcal{C}$, with $-[b(\rho_0)]_+$ units of worldvolume flux on it. Being in the large M limit, we can safely approximate the cutoff anti-fractional branes with a ring.

The warp factor gets different contributions. First of all, if there are some regular D3 branes at the origin, they source the usual term according to (6.5). Secondly, the M cutoff anti-fractional branes, because of their tension, contribute the following term in the ring approximation

$$Z_{ring, M}(\rho, \sigma; \rho_0) = 8\pi g_s M \alpha'^2 ([b(\rho_0)]_+ - b(\rho_0)) \frac{\sigma^2 + \rho^2 + \rho_0^2}{[(\sigma^2 + \rho^2 + \rho_0^2)^2 - 4\rho_0^2 \rho^2]^{3/2}} , \quad (6.77)$$

where $\rho = |\vec{y}|$ and $\sigma = |\vec{x}|$. Fractional branes at the enhançon ring, being tensionless, do not contribute directly to the warp factor. Finally, there is a term sourced by the twisted field strengths

$$d\gamma = \frac{iM}{\pi} \frac{dz}{z} \Theta(|z| - |z_e|) \Theta(|z_0| - |z|) . \quad (6.78)$$

In general it takes the form

$$Z_{fl}(\vec{y}, \vec{x}) = 4\pi \alpha'^2 g_s^2 \int d^2z |\partial_z \gamma|^2 \frac{1}{[|\vec{x}|^2 + |\vec{y} - \vec{z}|^2]^2} , \quad (6.79)$$

which in the case under consideration reduces to

$$\begin{aligned} Z_{fl, M}(\rho, r; \rho_e, \rho_0) = & \frac{2(g_s M \alpha')^2}{r^4} \left\{ 2 \log \frac{r^4 + \left(\rho_e^2 + \sqrt{(r^2 + \rho_e^2)^2 - 4\rho_e^2 \rho^2} \right) r^2 - 2\rho_e^2 \rho^2}{r^4 + \left(\rho_0^2 + \sqrt{(r^2 + \rho_0^2)^2 - 4\rho_0^2 \rho^2} \right) r^2 - 2\rho_0^2 \rho^2} + \right. \\ & \left. + 2 \log \frac{\rho_0^2}{\rho_e^2} + \frac{r^2}{r^2 - \rho^2} \left[\frac{3(r^2 - \rho^2) + \rho_0^2 - \rho^2}{\sqrt{(r^2 + \rho_0^2)^2 - 4\rho_0^2 \rho^2}} - \frac{3(r^2 - \rho^2) + \rho_e^2 - \rho^2}{\sqrt{(r^2 + \rho_e^2)^2 - 4\rho_e^2 \rho^2}} \right] \right\} , \end{aligned} \quad (6.80)$$

where $r^2 = \rho^2 + \sigma^2$.

Notice that the total D3 brane charge, which is $N + M$ if the UV theory has gauge group $SU(N + M) \times SU(N + M)$, gets sectioned in different pieces. The flux term carries a charge $M b(\rho_0)$, since $b(\rho_0) = \frac{g_s M}{\pi} \log \frac{\rho_0}{\rho_e}$; the cutoff anti-fractional branes carry a charge

$M ([b(\rho_0)]_+ - b(\rho_0))$; finally, there are $N - [b(\rho_0)]_- M$ regular D3 branes at the origin. This can be checked via the large r asymptotics of the different terms in the warp factor.

The vacuum considered in [152] and described in section 6.3 has N regular D3 branes at the origin, the enhançon ring at $\rho_1 = e^{-\frac{\pi}{2g_s M}} \rho_0$, and M cutoff anti-fractional branes at ρ_0 , where $b(\rho_0) = \frac{1}{2}$, carrying $M/2$ units of D3 charge; the twisted fluxes between fractional and anti-fractional branes carry other $M/2$ units of D3 charge.

The vacuum with a finite cascade starting at z_0 and reaching $SU(M)$ in the infrared has no regular D3 branes at the origin, M fractional branes with no D3 charge melted at an enhançon ring at $\rho_{\min} = \rho_{N/M} \equiv e^{-\frac{\pi N}{g_s M^2}} \rho_1$, and M cutoff anti-fractional branes at ρ_0 , where $b(\rho_0) = \frac{N}{M} + \frac{1}{2}$, carrying again $M/2$ units of D3 charge; this time the twisted fluxes between fractional and anti-fractional branes carry $N + M/2$ units of D3 charge. As we explained in detail in section 6.5, what happens is that at each generalized enhançon ring scale along the cascade (where $b \in \mathbb{Z}$) melted tensionless fractional and anti-fractional branes are left, naively annihilating if c is continuous crossing radially the generalized enhançon ring. In case $N = lM + p$ is not a multiple of M , then $\rho_{\min} = e^{-\frac{\pi l}{g_s M}} \rho_1$, $b(\rho_0) = l + \frac{1}{2}$ and there are p D3 branes at the origin: the IR theory below the enhançon scale is the $SU(p) \times SU(p)$ theory with one infinite coupling.

The infinite cascade limit can even be defined continuously: it is enough to send continuously the cutoff $\rho_0 \rightarrow \infty$ keeping ρ_{\min} fixed and $b(\rho_{\min}) = 0$. This can be achieved if $b(\rho_0) = \frac{g_s M}{\pi} \ln \frac{\rho_0}{\rho_{\min}}$: as we change the cutoff ρ_0 , we also change the value of the gauge couplings at the cutoff (and on the string side the tension of the cutoff branes) so that low energy physics is not modified. Notice that every time a $b(\rho_0) \in \mathbb{Z}$ threshold is crossed, the total D3 brane charge of the configuration (the ranks of the UV CFT) jumps by M units, and the cutoff anti-fractional branes change. The warp factor for the infinite cascade with no regular D3 branes is nothing but $Z_{fl, M}(\rho, r; \rho_{\min}, \infty)$, see eq. (6.80). If needed, the addition of p regular D3 branes is straightforward.

We can also find the warp factor for a configuration with any number of rotationally symmetric bearings. The total warp factor is sourced by twisted fluxes and possibly by cutoff anti-fractional branes, if there is no infinite cascade in the UV. Inside bearings fluxes vanish, whereas outside they take the usual form $|d\gamma| = \frac{M}{\pi} \frac{d\rho}{\rho}$. Therefore fluxes contribute to the warp factor by a sum of terms taking the schematic form $Z_{fl, M}(\rho, r; \rho_{>}^{(i+1)}, \rho_{<}^{(i)})$, where $\rho_{>}^{(i+1)}$ is the outer radius of the $(i+1)$ -th bearing and $\rho_{<}^i$ is the inner radius of the i -th bearing, if the ordering points inwards. The requirement that $\rho_{<}^{(i)}$ and $\rho_{>}^{(i+1)}$ be boundaries of subsequent bearings translates into $\rho_{<}^{(i)} = e^{\frac{\pi n_i}{g_s M}} \rho_{>}^{(i+1)}$, for some $n_i \in \mathbb{N}$.

Finally, by now it should also be clear how to write the warp factor in the case of perturbative Higgsings by backreacting rings of tensionful fractional and anti-fractional branes, adding terms like (6.77) sourced at suitable radii and with the suitable normalizations.

We end this section with some important remarks about the backreacted geometries. For concreteness, we concentrate on solutions without bearings nor perturbative Higgsing except at the cutoff, since the generalization of the statements we are about to make should be clear.

The warp factor diverges (and the gravitational potential felt by a massive particle has an absolute minimum) only at the locations of sources for it (fractional branes and

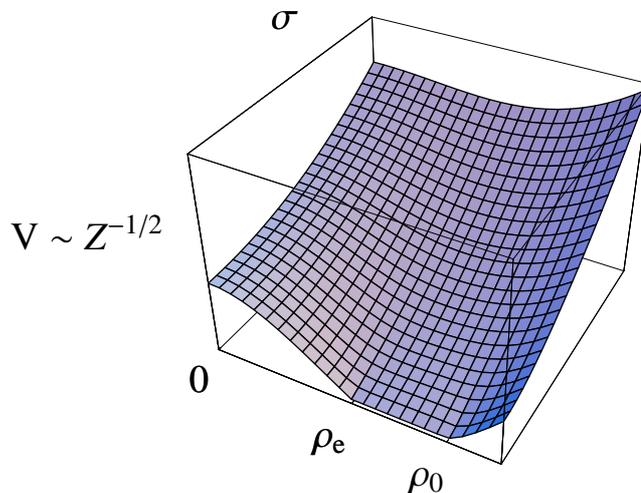


Figure 6.10: Potential $V = V(\rho, \sigma)$ felt by a massive particle in the background dual to a vacuum with finite cascade. The origin is a saddle point while the absolute minimum is on the $\sigma = 0$ axis all along the range where the dual field theory undergoes a RG flow, from the enhançon radius ρ_e up to the UV cut-off ρ_0 .

twisted field strengths), namely on the orbifold plane $\sigma = 0$ and for $\rho \in [\rho_e, \rho_0]$. There are no repulsive regions even when the D3 brane charge vanishes at some IR scale, as occurs at the enhançon scale in the vacuum of [152] with $N = 0$ and in the finite or infinite cascade solution with $p = 0$. Massive objects (but BPS ones) are always attracted by the sources of stress-energy: they want to go where twisted fluxes and fractional branes (and possibly regular D3 branes) lie. For concreteness, we report in figure 6.10 the shape of the effective potential $V(\rho, \sigma)$ felt by a massive particle: it is proportional to $Z^{-1/2}$, once the kinetic terms are normalized to be $(\frac{d\rho}{d\tau})^2 + (\frac{d\sigma}{d\tau})^2$, τ being the worldline proper time.

In these solutions the curvature diverges approaching the domain where twisted fluxes have support. Therefore, strictly speaking, the gravity solution cannot be trusted in that region and string theory is needed to resolve the curvature singularity. Still, the M-theory picture suggests that the form of the twisted fields will remain unchanged.

Finally, if there are no D3 branes at the origin the geometry smoothly approaches flat space at $r = 0$, where the warp factor approaches

$$Z(\mathbf{0}) = 2(g_s M \alpha')^2 \left(\frac{1}{\rho_e^4} - \frac{1}{\rho_0^4} \right) + 8\pi g_s M \alpha'^2 ([b(\rho_0)]_+ - b(\rho_0)) \frac{1}{\rho_0^4}, \quad (6.81)$$

signaling that excitations in the non-abelian sector have a minimal energy (consistently with the $SU(M)$ factor being broken to $U(1)^{M-1}$). If instead there are regular D3 branes at the origin, they dominate the IR asymptotics which is $AdS_5 \times S^5/\mathbb{Z}_2$, signaling a non-abelian fixed point.

6.7 Conclusions

In this chapter, we filled a gap in the understanding of the gauge theory dual interpretation of supergravity solutions with running fluxes, arising when considering fractional

branes at generic Calabi-Yau singularities. It has been known for some time that fractional branes at isolated singularities describe RG flows which can be described in terms of cascades of Seiberg dualities. A similar interpretation was not possible for branes at non-isolated singularities, since their effective dynamics is intrinsically $\mathcal{N} = 2$.

The basic outcome of our analysis is that, for branes at non-isolated singularities, the reduction of the gauge group ranks along the RG flow can be understood in terms of a sequence of strong coupling transitions reminiscent of the low energy description of the baryonic root of $\mathcal{N} = 2$ SQCD. The energy range spanned by the cascade depends on the point in the Coulomb branch one is sitting at; specifically, on the number of non-vanishing VEV's for the adjoint scalars.

We were also able to provide a gravity dual description of a new set of infinitely many vacua, characterized by new geometric structures, the enhançon bearings, where the dual gauge theory alternates energy ranges where it runs, with ranges in which it is in a strongly coupled superconformal phase.

For all these vacua, an enhançon mechanism takes place in the far IR. This changes the twisted fields configuration and ultimately the metric, whose correct repulson-free expression we provided for all vacua we have been studying.

We should also note that Stefano Cremonesi analysed further the IR of this model in [181], finding the exact twisted flux configuration corresponding to any Coulomb branch vacua (any Seiberg-Witten curve). The upshot of the analysis of [181] is that one should really think of the $\mathcal{N} = 2$ fractional branes as being transmuted into twisted flux. In that respect we see that the mechanism of resolution of the IR singularity at finite g_s is similar to the geometric transition for deformation fractional branes.

Our analysis focused, for definiteness, on the A_1 singularity, but our results have a much wider validity. First, they trivially extend to any $\mathcal{N} = 2$ singularity, as for instance the full ADE series. Second, any Calabi-Yau cone with non-isolated singularities, which upon the inclusion of branes generically gives rise to a $\mathcal{N} = 1$ theory, should present the same behavior. This is suggested from the supergravity solution and it is a rather non-trivial claim since SW techniques are not available in the $\mathcal{N} = 1$ context.

More complicated flows occur when fractional branes at isolated and non-isolated singularities are both present, which is in fact the most generic situation. In a generic cascade some cascade “steps” can be understood in terms of Seiberg duality, some of them cannot as the theory, due to the presence of adjoint fields, exhibits at some energies an effective $\mathcal{N} = 2$ behavior. We conjecture that in those cases too the rank reduction is due to the adjoint fields being at baryonic-root-like points of its moduli space. In the next chapter we will consider several examples of such cascades of mixed type, finding perfect agreement with our proposal.

Chapter 7

Cascades of mixed kind and interplay of various fractional branes

IN this Chapter we find the KT-like solution for arbitrary fractional branes on a non-chiral quiver corresponding to a \mathbb{Z}_2 orbifold of the conifold. The geometry has non-isolated singularities, so we can consider simultaneously the various kinds of fractional branes, to study their interplay. This geometry was studied in the earlier work [182] with the hope that one could brane-engineer dynamical SUSY breaking while bypassing the typical runaway instability (actually there is still a runaway direction along a baryonic branch, but the hope was to realize metastable DSB).

Our main reason to study this geometry here is because it is an interesting model to apply the ideas of the previous Chapter about the $\mathcal{N} = 2$ cascade in a more generic $\mathcal{N} = 1$ context. This Chapter is based on [1], written with Riccardo Argurio, Francesco Benini, Matteo Bertolini and Stefano Cremonesi.¹

7.1 Introduction

The correspondence between gauge theories with non-trivial low-energy dynamics and string theory backgrounds has an enormous potential. The string theory setup is usually established drawing uniquely on the holomorphic data of a supersymmetric gauge theory, including a specific choice of vacuum. Then, solving the classical equations of motion of supergravity one can in principle obtain, through the warp factor, all the dynamical informations on the gauge theory low-energy dynamics, that would instead usually imply precise knowledge of the Kähler sector. The limitation of this procedure to supergravity and not to full string theory corresponds in the gauge theory to taking some large N and strong 't Hooft coupling limit.

A fruitful arena where to address these issues has proven to be that of D3-branes at Calabi-Yau (CY) singularities. In this context, the most celebrated example where such

¹Actually the paper [1] was written before [2], and it gave the motivation to further study $\mathcal{N} = 2$ cascades, when we realized that they were poorly understood.

a program has been successfully completed is the warped deformed conifold [118], which describes a theory with confinement and chiral symmetry breaking.

It is of obvious interest to apply the above program to gauge theories with a varied low-energy behavior. D3-branes at CY singularities typically give rise to $\mathcal{N} = 1$ quiver gauge theories, which are supersymmetric theories characterized by product gauge groups, matter in the bifundamental representation and a tree level superpotential, all such data being dictated by the structure of the singularity. Most quiver gauge theories can have several different IR behaviors, depending on which branch of the moduli space one is sitting on. Already in the simple conifold theory, one has a baryonic branch displaying confinement and a mass gap in the gauge sector, and mesonic branches with a dynamics which is $\mathcal{N} = 4$ to a good approximation. In more general quivers, other kinds of low-energy behaviors are possible. Some quivers will actually have no vacua and display a runaway behavior [149, 148, 150, 151], but this leaves little hope of finding a regular gravity dual. Other quivers will on the other hand contain branches of the moduli space where the dynamics is approximately the one on the Coulomb branch of an $\mathcal{N} = 2$ theory. The latter can also be thought of as mesonic branches, albeit of complex dimension one instead of three as in the (generic) $\mathcal{N} = 4$ case.

In this chapter, we construct the gravity dual of the most generic gauge theory one can engineer using D3-branes at the tip of a \mathbb{Z}_k non-chiral orbifold of the conifold [56], focusing for simplicity, but with little loss of generality, on the case $k = 2$. This singularity admits different kinds of fractional branes, triggering confinement or enjoying an $\mathcal{N} = 2$ mesonic branch and known as deformation or $\mathcal{N} = 2$ fractional branes, respectively. We aim at describing the backreaction of the most general D3-brane bound state. The difficulty in doing so stems from the fact that the UV completion which corresponds to the supergravity solution is qualitatively different in the two cases. For deformation branes, the renormalization group (RG) flow is best described in terms of a cascade of Seiberg dualities which increases the overall rank of the quiver nodes towards the UV. For $\mathcal{N} = 2$ branes, the RG flow (which is indeed present and also increases the ranks towards the UV [10, 11]) seems to be better represented by some form of non-perturbative Higgsing, as argued in the previous chapter.

It should be clear that whenever there are $\mathcal{N} = 2$ branes around the IR of the gravity dual is bound to contain some singularity. This is because open string degrees of freedom cannot completely transmute into flux. Indeed, on the Coulomb branch we still have by definition some surviving abelian gauge group, which cannot be described in terms of closed string degrees of freedom. This situation is similar to the situation where one aims at describing theories with flavors. There too, flavor degrees of freedom must be described by open strings, and hence flavor branes must be present in the gravity dual as physical sources [183]. Thus in our set up we expect to have physical sources corresponding to $\mathcal{N} = 2$ fractional branes. The main difference with respect to the case of flavor branes is that $\mathcal{N} = 2$ fractional branes are not infinitely extended in the Calabi-Yau.

The main results of our analysis can be summarized as follows. We find an explicit supergravity solution describing a generic distribution of fractional branes, both of the deformation and $\mathcal{N} = 2$ kind, on the orbifolded conifold, and corresponding to the UV regime of the dual gauge theory. It describes holographically an RG flow which exactly matches the beta functions that one can compute in the dual field theory and the expected reduction of degrees of freedom towards the IR, which occurs through a cascade. We

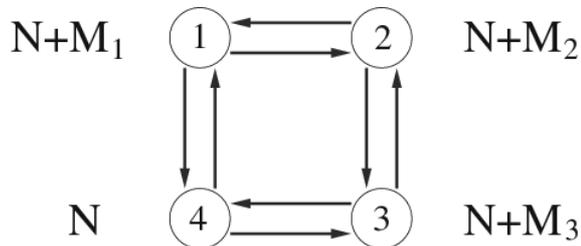


Figure 7.1: The quiver diagram of the gauge theory, for the most generic choice of ranks. Circles represent unitary gauge groups, arrows represent bifundamental chiral superfields. For later purposes we have parametrized the four independent ranks in terms of a common N .

develop an algorithm to follow the RG flow of each gauge coupling from the supergravity solution. An interesting feature is that in this general setting there are cascade steps that do not always have a simple interpretation in terms of Seiberg dualities. This is due to the presence of $\mathcal{N} = 2$ fractional branes, or more generally to the presence of twisted fluxes. Nevertheless, supergravity considerations and field theory expectations (based on the non-holomorphic beta function) exactly match. As far as the IR regime is concerned, we perform a non-trivial consistency check matching the field theory effective superpotential with that predicted from the geometric background. We also provide the solution for the 3-form fluxes and discuss the pattern of singularities resolution, while we only set the stage for computing the exact warp factor in this case.

This chapter is structured as follows. In section 7.2 we explain our set up and introduce the minimal geometrical data that is needed in the following. In section 7.3 we present the supergravity solution which is expected to reproduce the UV behavior of our quiver gauge theory. We take the CY base to be the orbifold of the singular conifold, but we take into account all the fluxes sourced by the fractional branes and compute their backreaction on the warp factor. We then check that the result is indeed compatible with the expected RG flow and perform a number of non-trivial gauge/gravity duality checks. In section 7.4 we discuss the extension of the previous solution towards the IR, discuss the singularity structure of our solution, their resolutions, and match the effective superpotential obtained on the two sides of the correspondence. The appendices contain many technical data which might help in better understanding the form of the supergravity ansatz that we solve in the main text and the geometric structure of the orbifolded conifold CY singularity we consider.

7.2 The orbifolded conifold

We consider in what follows an orbifolded avatar of the familiar conifold quiver. We focus on a non-chiral \mathbb{Z}_2 orbifold of the conifold and consider the corresponding $\mathcal{N} = 1$ supersymmetric quiver gauge theory obtained by placing a bound state of regular and fractional D3-branes at its tip. This theory has been analyzed at great length in [182], to which we refer for more details.

The quiver gauge theory is shown in Figure 7.1. The gauge theory has four gauge

factors and a tree level superpotential for the bifundamental fields

$$W = \lambda(X_{12}X_{21}X_{14}X_{41} - X_{23}X_{32}X_{21}X_{12} + X_{34}X_{43}X_{32}X_{23} - X_{41}X_{14}X_{43}X_{34}) , \quad (7.1)$$

where X_{ij} is a chiral superfield in the fundamental representation of the i -th gauge group and antifundamental representation of the j -th gauge group, and traces on the gauge degrees of freedom are understood.

We are interested in the dynamics of the gauge theory with the most generic rank assignment, as in Figure 7.1. Depending on the values of the M_i 's, various kinds of IR dynamics can occur: confinement, runaway behavior or a (locally $\mathcal{N} = 2$) quantum moduli space.

There is a relation between the ranks of the various gauge groups in the quiver and the number of fractional branes wrapping the different 2-cycles in the geometry. In turn, the fractional branes source the RR 3-form flux which is an important ingredient in order to determine the supergravity solution. In the following of this section we provide the link between these three sets of data (ranks, branes wrapping cycles, fluxes). For a more detailed discussion we refer to appendix F.2.

7.2.1 Regular and fractional branes

The superconformal theory ($N \neq 0$, $M_i = 0$) can be engineered by placing N regular D3-branes at the tip of the cone. Unbalanced ranks in the quiver of Figure 7.1 correspond instead to the presence of fractional D3-branes and the corresponding breaking of conformal invariance. From the gauge theory viewpoint, fractional branes correspond to independent anomaly free rank assignments in the quiver (modulo the superconformal one). Hence, in the present case, we have three types of fractional branes to play with.

As reviewed in Section 5.4, fractional branes can be classified in terms of the IR dynamics they trigger [148].

The deformation fractional branes are those associated to a single node in the quiver, or to several decoupled nodes, or else to several contiguous nodes whose corresponding closed loop operator appears in the tree level superpotential. This subsector of the quiver gauge theory undergoes confinement. Examples of this kind in our theory correspond to rank assignments $(1, 0, 0, 0)$, $(1, 0, 1, 0)$ or $(1, 1, 1, 0)$ and cyclic permutations.

The $\mathcal{N} = 2$ fractional branes are those associated to contiguous nodes in the quiver whose corresponding meson does not appear as an individual term in the superpotential. Rank assignments corresponding to this class of branes in our quiver are for instance $(1, 1, 0, 0)$ and cyclic permutations.

Finally, fractional branes of any other class (which is the most generic case, in fact) are DSB branes. They lead to ADS-like superpotential and runaway behavior. Geometrically, they are associated with geometries where the complex structure deformation is obstructed, this tension being the geometric counterpart of the runaway. In this case the occupied nodes have unbalanced ranks.

Obviously, combining different fractional branes of a given class, one can obtain fractional branes of another class. Hence one can choose different fractional brane bases to describe the gauge theory. In our present case, we will be able to choose a basis composed only of deformation and $\mathcal{N} = 2$ fractional branes. We have just seen to which rank

assignments the various branes should correspond, now we have to review which 2-cycles they are associated to.

7.2.2 Geometry, cycles and quiver ranks

There is a well established relation between quiver configurations, the primitive topologically non-trivial shrinking 2-cycles of a given CY singularity, and the possible existing fractional D3-branes, since the latter can be geometrically viewed as D5-branes wrapped on such cycles. Let us review such relation for our CY singularity (see appendix F.2 for a full analysis).

The conifold is a non-compact CY three-fold described by the following equation in \mathbb{C}^4 : $z_1 z_2 - z_3 z_4 = 0$. We consider a \mathbb{Z}_2 orbifold of such singularity defined by the symmetry

$$\Theta : (z_1, z_2, z_3, z_4) \rightarrow (z_1, z_2, -z_3, -z_4) . \quad (7.2)$$

The resulting orbifolded geometry is described by the following equation in \mathbb{C}^4

$$(z_1 z_2)^2 - xy = 0 , \quad (7.3)$$

where $x = z_3^2$ and $y = z_4^2$. There is a singular locus in this variety which consists of two complex lines, that we call the p and q lines, respectively. They meet at the tip $\{z_1 = z_2 = x = y = 0\}$ and correspond to the fixed point locus of the orbifold action Θ .

One can as well describe the variety as a real manifold. The coordinates we use are defined in appendix F.1. From this point of view the conifold is a real cone over $T^{1,1}$, which in turn is a $U(1)$ bundle over $S^2 \times S^2$. The orbifold action (7.2) reads in this case

$$\Theta : (\phi_1, \phi_2) \rightarrow (\phi_1 - \pi, \phi_2 + \pi) . \quad (7.4)$$

The two complex lines are defined, in complex and real coordinates respectively, as

$$\begin{aligned} p &= \{z_1 = x = y = 0, \forall z_2\} = \{\theta_1 = \theta_2 = 0, \forall r, \psi'\} \\ q &= \{z_2 = x = y = 0, \forall z_1\} = \{\theta_1 = \theta_2 = \pi, \forall r, \psi''\} , \end{aligned} \quad (7.5)$$

where $\psi' = \psi - \phi_1 - \phi_2$ and $\psi'' = \psi + \phi_1 + \phi_2$ are (well defined) angular coordinates along the singularity lines. In a neighborhood of the singular lines (and outside the tip) the geometry looks locally like the A_1 -singularity $\mathbb{C} \times \mathbb{C}^2/\mathbb{Z}_2$. The fixed point curve p sits at the north poles of both S^2 's while the curve q sits at the south poles. A sketch of the conifold geometry in these real coordinates and of the fixed points of Θ is given in Figure 7.2.

Our CY cone has three vanishing 2-cycles. Two of these three 2-cycles arise due to the orbifold action. Such exceptional 2-cycles are located all along the $\mathbb{C}^2/\mathbb{Z}_2$ singular lines p and q , and we call them \mathcal{C}_2 and \mathcal{C}_4 , respectively. The third relevant 2-cycle descends from the 2-cycle of the parent conifold geometry, whose base $T^{1,1}$ is topologically $S^2 \times S^3$. Correspondingly, we will have a basis consisting of three fractional branes.

In appendix F.2 we construct different fractional brane bases. However, the basis we will favor here is the one arising most naturally when viewing our singularity as a \mathbb{Z}_2 projection of the conifold, which as anticipated is given in terms of the two $\mathcal{N} = 2$ 2-cycles \mathcal{C}_2 and \mathcal{C}_4 and a deformation 2-cycle, \mathcal{C}_β . This basis of 2-cycles corresponds to a

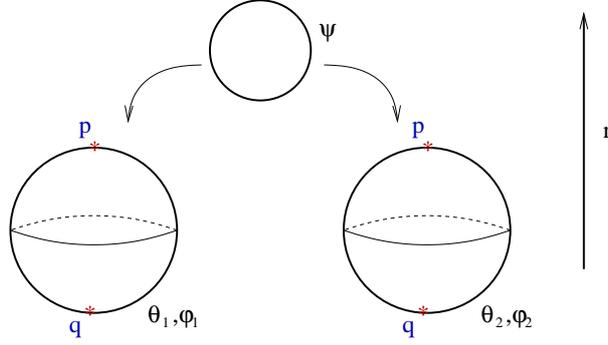


Figure 7.2: The singular conifold in real angular coordinates: it is a real cone in r over $T^{1,1}$, which in turn is a $U(1)$ fibration in ψ over the Kähler-Einstein space $\mathbb{C}P^1 \times \mathbb{C}P^1$ parameterized by θ_i and ϕ_i . The fixed point locus of the orbifold action Θ is given by two lines p and q , localized at antipodal points on the two S^2 's. At the tip the spheres shrink and p and q meet.

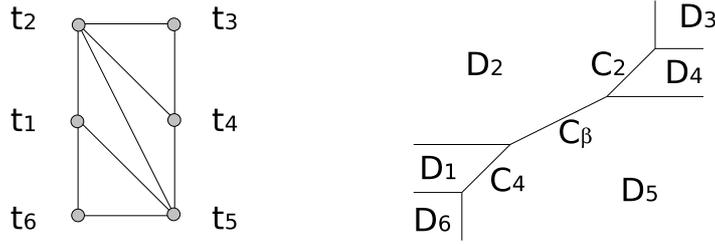


Figure 7.3: The (p, q) -web (right) associated to the specific triangulation (which corresponds to a specific resolution) of the toric diagram of the orbifolded conifold (left).

particular resolution of the singularity, which is encoded in the triangulation of the toric diagram (and the associated (p, q) -web) reported in Figure 7.3.

We now mention some results derived in appendix F.2. First, a linear combination of the three cycles above, $\mathcal{C}_{CF} \equiv 2\mathcal{C}_\beta + \mathcal{C}_2 + \mathcal{C}_4$, has a vanishing intersection with the exceptional 2-cycles \mathcal{C}_2 and \mathcal{C}_4 and it corresponds to the 2-cycle of the double covering conifold geometry. Hence, a brane wrapping it does not couple to closed string twisted sectors, which are those associated to exceptional cycles, and it gives rise to the orbifold of the configuration of a fractional brane at the singular conifold [127]. It thus corresponds to a quiver rank assignment $(1, 0, 1, 0)$. Given the obvious rank assignments $(0, 1, 1, 0)$ and $(1, 1, 0, 0)$ for branes wrapped on \mathcal{C}_2 and \mathcal{C}_4 respectively, it follows that the rank associated to a D5-brane wrapped on \mathcal{C}_β is $(0, -1, 0, 0)$. We will find it more convenient to use a D5-brane wrapped on $-\mathcal{C}_\beta \equiv \mathcal{C}_\alpha$, corresponding to the quiver $(0, 1, 0, 0)$.

Eventually, one needs to compute the RR 3-form fluxes sourced by each fractional brane. Our findings, which are derived in appendix F.2, are summarized in the Table

below:

	$-\int_{A_2} F_3$	$-\int_{A_4} F_3$	$-\int_{A_{CF}} F_3$	gauge theory
D5 on \mathcal{C}_2	2	0	0	(0, 1, 1, 0)
D5 on \mathcal{C}_4	0	2	0	(1, 1, 0, 0)
D5 on \mathcal{C}_α	1	1	-1	(0, 1, 0, 0)

(7.6)

where fluxes are understood in units of $4\pi^2\alpha'g_s$. The 3-cycle A_2 corresponds to the product of the exceptional 2-cycle \mathcal{C}_2 transverse to the p -line with the S^1 on p . Similarly, A_4 is the product of the exceptional \mathcal{C}_4 with the S^1 in the q -line. Finally, A_{CF} is the image of the compact 3-cycle of the double covering conifold under the orbifold projection.

The table above is all we need to translate directly a quiver with generic rank assignment to a supergravity solution with the corresponding 3-form flux.

7.3 Supergravity background for the UV regime

In this section we present the supergravity solution describing the most general D3-brane system one can consider on the orbifolded conifold. The solution is expected to be dual to the previously discussed gauge theory with the most general rank assignment: $(N + M_1, N + M_2, N + M_3, N)$.²

Fractional branes are magnetic sources for the RR 3-form flux. This typically results in some singularity of the backreacted supergravity solution. In some cases, namely when there are only deformation branes around, the singularity is smoothed out by the complex structure deformation the branes induce. One gets back a singularity-free solution where branes are replaced by fluxes [118, 140]. In more general situations it is more difficult to find a regular solution. As already noticed, in the case of $\mathcal{N} = 2$ fractional branes this is in fact not even expected to be possible, because there should always be some remaining open string modes corresponding to the left over $U(1)^{N-1}$ gauge degrees of freedom on the Coulomb branch. Hence, (a remnant of) the brane sources remains in the gravity dual.

This said, in order to take the leading effect of any such kind of fractional brane into account, it is enough to make an educated ansatz for the supergravity fields and to impose suitable boundary conditions on the system of differential equations. Therefore, in what follows, we will only consider the type IIB bulk action S_{IIB} , eq. (A.1), and implement the effects of each brane source by properly chosen boundary conditions.

7.3.1 The UV regime: running fluxes and singularity lines

The general solution we are looking for has constant axio-dilaton $\tau = C_0 + ie^{-\Phi} = i$, but non-trivial RR and NSNS 3-form fluxes (which are usually organized in a complex 3-form $G_3 = F_3 + ie^{-\Phi}H_3 = F_3 + iH_3$), RR 5-form field strength F_5 and warp factor. The ansatz reads

$$\begin{aligned}
 ds_{10}^2 &= h^{-1/2} dx_{3,1}^2 + h^{1/2} (dr^2 + r^2 ds_{T^{1,1}}^2) \\
 F_5 &= (1 + *_{10}) dh^{-1} \wedge d\text{vol}_{3,1} \\
 G_3 &= G_3^U + G_3^T
 \end{aligned}
 \tag{7.7}$$

²Our conventions for type IIB supergravity and D-brane actions, together with the equations of motion for the bulk fields, can be found in appendix A.

where the orbifold \mathbb{Z}_2 identification (7.2) acting on the internal coordinates is understood, h is the warp factor, while the superscripts U and T on the 3-form flux stand for untwisted and twisted sector fluxes, respectively. The above ansatz is the one of a warped singular cone. Any deformation of the singular geometry will still asymptote to this cone for large values of the radial coordinate, and it is in this sense that we will think of the solution as representing (at least) the UV regime of the dual gauge theory.

Recall that for the solution to be supersymmetric, the complex 3-form G_3 should be $(2, 1)$, primitive and imaginary-self-dual [123]

$$*_6 G_3 = i G_3 , \quad (7.8)$$

where $*_6$ is constructed with the unwarped metric. We will see that the warp factor depends on the radial coordinate as well as some of the angular coordinates, as typical for solutions with $\mathcal{N} = 2$ branes around [10].

The equations of motion we have to solve are written in appendix A, eqs. (A.5). The warp factor equation is given by the BI for F_5 . The Einstein equations are then automatically satisfied by our ansatz (7.7).

It is easy to check that, given all the geometrical data discussed in the previous section, and taking for simplicity all fractional branes sitting at the tip, the complex 3-form G_3 reads³

$$\begin{aligned} G_3 &= -\frac{\alpha'}{2} g_s (M_1 - M_2 + M_3) \left[\omega_3^{CF} - 3i \frac{dr}{r} \wedge \omega_2^{CF} \right] \\ &\quad + 2i\pi\alpha' g_s (-M_1 + M_2 + M_3) \frac{dz_2}{z_2} \wedge \omega_2^{(p)} + 2i\pi\alpha' g_s (M_1 + M_2 - M_3) \frac{dz_1}{z_1} \wedge \omega_2^{(q)} \\ &= -\frac{\alpha'}{2} g_s (M_1 - M_2 + M_3) \left[\omega_3^{CF} - 3i \frac{dr}{r} \wedge \omega_2^{CF} \right] \\ &\quad + i\pi\alpha' g_s (-M_1 + M_2 + M_3) \left(3 \frac{dr}{r} + i d\psi' \right) \wedge \omega_2^{(p)} \\ &\quad + i\pi\alpha' g_s (M_1 + M_2 - M_3) \left(3 \frac{dr}{r} + i d\psi'' \right) \wedge \omega_2^{(q)} , \end{aligned} \quad (7.9)$$

where ω_3^{CF} and ω_2^{CF} are defined in appendix F.1, and $\omega_2^{(p)}$ and $\omega_2^{(q)}$ are the two normalized exceptional 2-cocycles defined by the integrals below.

For the present purposes it suffices to recall that

$$\int_{\mathcal{C}_{CF}} \omega_2^{CF} = 4\pi , \quad \int_{\mathcal{C}_2} \omega_2^{(p)} = \int_{\mathcal{C}_4} \omega_2^{(q)} = 1 , \quad \text{and} \quad \int_{A_{CF}} \omega_3^{CF} = 8\pi^2 , \quad (7.10)$$

where A_{CF} is the image under the orbifold projection of the 3-sphere on the double covering conifold. The second equality in (7.9) can be easily obtained by using eqs. (F.2-

³The vielbein we use for the singular conifold can be found in (F.9). Appendix F.1 contains a review of the singular conifold geometry.

F.5). It is then easy to check that the RR 3-form fluxes on the A-cycles are

$$-\frac{1}{4\pi^2\alpha'g_s} \int_{A_{CF}} F_3 = M_1 - M_2 + M_3 \quad (7.11)$$

$$-\frac{1}{4\pi^2\alpha'g_s} \int_{A_2} F_3 = -M_1 + M_2 + M_3 \quad (7.12)$$

$$-\frac{1}{4\pi^2\alpha'g_s} \int_{A_4} F_3 = M_1 + M_2 - M_3 . \quad (7.13)$$

It is important to stress at this point that the above equations are really the input (i.e. the asymptotic conditions) in solving the equations. They are in one-to-one correspondence with a choice of ranks in the quiver. The real part of G_3 , that is F_3 , is thus essentially determined in this way. Then the imaginary self-dual condition (7.8) fixes also H_3 , the imaginary part of G_3 . The latter is thus the output of solving the supergravity equations. As we will see in the next subsection, this is a non-trivial output in the sense that it will contain information about the running of the gauge couplings. Further dynamical data on the dual gauge theory is contained in the warp factor.

From the ansatz (7.7), one sees that the warp factor should satisfy the following equation in the unwarped internal manifold

$$*_6 d *_6 dh \equiv \Delta h = - *_6 (H_3 \wedge F_3) , \quad (7.14)$$

with boundary conditions dictated by the D-brane sources. To compute $H_3 \wedge F_3$ from (7.9) and to solve for the warp factor h in (7.14), the first issue is whether there are mixed terms between twisted and untwisted sectors in the expansion of such 6-form in the cocycle basis. Let us consider a closed 2-form ω_2 , that represents the Poincaré dual of an exceptional cycle \mathcal{C} in any submanifold transverse to the singularity line, and α_2 a smooth 2-form with vanishing flux on the exceptional cycle. The 4-form $\omega_2 \wedge \alpha_2$, which would give mixed terms, vanishes at any point but the singular one. One can then write $\omega_2 \wedge \alpha_2 = C \delta_4$ and compute C as

$$C = \int \omega_2 \wedge \alpha_2 = \int_{\mathcal{C}} \alpha_2 = 0 . \quad (7.15)$$

This implies that there are no mixed terms between the twisted sector and the untwisted one. Then the 6-form $H_3 \wedge F_3$ is easily computed. From (7.9) for the 3-form fluxes, using

$$\begin{aligned} \frac{dr}{r} \wedge \omega_2^{CF} \wedge \omega_3^{CF} &= -\frac{54}{r} dr \wedge d\text{vol}_{T^{1,1}} \\ \omega_2^{(p)} \wedge \omega_2^{(p)} &= -\frac{1}{4\pi^2} \delta^{(2)}(1 - \cos \theta_1, 1 - \cos \theta_2) \sin \theta_1 d\theta_1 \wedge d\phi_1 \wedge \sin \theta_2 d\theta_2 \wedge d\phi_2 \\ \omega_2^{(q)} \wedge \omega_2^{(q)} &= -\frac{1}{4\pi^2} \delta^{(2)}(1 + \cos \theta_1, 1 + \cos \theta_2) \sin \theta_1 d\theta_1 \wedge d\phi_1 \wedge \sin \theta_2 d\theta_2 \wedge d\phi_2 , \end{aligned} \quad (7.16)$$

we get

$$\begin{aligned} H_3 \wedge F_3 &= 81 \alpha'^2 g_s^2 \frac{1}{r^6} \left\{ \frac{1}{2} (M_1 - M_2 + M_3)^2 + (M_1 - M_2 - M_3)^2 \delta^{(2)}(1 - \cos \theta_1, 1 - \cos \theta_2) \right. \\ &\quad \left. + (M_1 + M_2 - M_3)^2 \delta^{(2)}(1 + \cos \theta_1, 1 + \cos \theta_2) \right\} dr \wedge r^5 d\text{vol}_{T^{1,1}} . \end{aligned} \quad (7.17)$$

The equation we have to solve for the warp factor is then

$$\Delta h = -81 \alpha'^2 g_s^2 \frac{1}{r^6} \left\{ \frac{1}{2} (M_1 - M_2 + M_3)^2 + (M_1 - M_2 - M_3)^2 \delta^{(2)}(1 - \cos \theta_1, 1 - \cos \theta_2) \right. \\ \left. + (M_1 + M_2 - M_3)^2 \delta^{(2)}(1 + \cos \theta_1, 1 + \cos \theta_2) \right\}. \quad (7.18)$$

Defining the angular function

$$f(x, y) = \frac{1}{24} \sum_{(n,m) \neq (0,0)}^{\infty} \frac{(2n+1)(2m+1)}{n(n+1) + m(m+1)} P_n(x) P_m(y), \quad (7.19)$$

where $P_n(t)$ are Legendre polynomials, and which satisfies the differential equation

$$\Delta_{ang} f(\cos \theta_1, \cos \theta_2) = -\delta^{(2)}(1 - \cos \theta_1, 1 - \cos \theta_2) + \frac{1}{4}, \quad (7.20)$$

the solution finally reads (see appendix F.3 for details)

$$h = \frac{27\pi\alpha'^2}{2} \frac{1}{r^4} \left\{ g_s N + \frac{3g_s^2}{4\pi} \left[(M_1 - M_2 + M_3)^2 + (M_1 - M_3)^2 + M_2^2 \right] \left(\log \frac{r}{r_0} + \frac{1}{4} \right) \right. \\ \left. + \frac{6g_s^2}{\pi} \left[(M_1 - M_2 + M_3)^2 f(\cos \theta_1, \cos \theta_2) + (M_1 + M_2 - M_3)^2 f(-\cos \theta_1, -\cos \theta_2) \right] \right\}. \quad (7.21)$$

The constant terms inside the $\{\dots\}$ in eq. (7.21) have been fixed in such a way that the effective D3-charge at $r = r_0$ is N . This is a choice for the physical meaning one wants to give to r_0 , as any such constant term can be absorbed into a redefinition of r_0 .

The above solution is not smooth, as the warp factor displays singularities at small r . Moreover, as already anticipated, we expect an enhancement behavior to be at work whenever there are $\mathcal{N} = 2$ branes in the original bound state. Similarly to [10, 11], the enhancement radius can be defined by the minimal surface below which the effective D3-charge changes sign. The resolution of the singularities has to do with the IR dynamics of the dual gauge theory. The structure of the vacua, as well as the phases the gauge theory can enjoy, depend crucially on the classes of fractional branes present and on the hierarchy of the scales Λ_i associated to each quiver node. Hence, the way the singularity is dealt with will change accordingly. These issues will be discussed in detail in section 7.4. Here we just want to stress that no matter the hierarchy between the dynamically generated scales Λ_i and the specific fractional branes content, the above solution is a good description of the UV regime of the dual gauge theory. In the following we will then present a number of non-trivial checks of the duality which apply in this regime.

7.3.2 Checks of the duality: beta functions and Maxwell charges

In this subsection we perform some non-trivial checks of the proposed gauge/gravity duality: we discuss the computation of gauge coupling beta functions and analyze the RG flow of our solutions using standard techniques. In the following subsection we adopt

a new perspective proposed in [129], which is based on Page charges [184] and enables us to get stronger predictions from supergravity.

Typically, given a supergravity background dual to a quiver gauge theory, the knowledge of the various brane charges at any value of the radial coordinate r allows one, in principle, to extract the gauge ranks of the dual theory at the scale μ holographically dual to r . Furthermore, from the value of closed string fields, one can learn about parameters and running couplings appearing in the dual field theory. In theories like IIB supergravity, whose action contains Chern-Simons terms leading to modified Bianchi identities for the gauge invariant field strengths, different notions of charges carried by the same fields may be introduced [184]. Following standard techniques, we will start using the so-called Maxwell charges, which are integrals of gauge invariant RR field strengths.

In order to specify the dictionary between the string and the gauge sides, one needs to understand the details of the microscopic D-brane configuration that realizes the field theory. As explained in [11], the idea is to match the brane charges of the supergravity solution at some value of r with the charges of a system of fractional branes that, in the presence of the same closed string fields as those of the supergravity solution, engineers the field theory: in this way one reads the effective theory at the scale μ . A complication arises because the meaningful brane configuration changes along the radial direction: when certain radial thresholds are crossed the D3-charge of one of the effective constituents of the system changes sign, and the system is no longer BPS. One has then to rearrange the charges into different BPS constituents. The field theory counterpart is that, when one of the gauge couplings diverges, one has to resort to a different description.

When the theory admits only deformation fractional branes, the link between different field theory descriptions is established by Seiberg duality. This was originally proposed and checked in the conifold theory [118], then applied to other singularities [147, 185] and even to theories with non-compact D7-branes [186, 129]. In $\mathcal{N} = 2$ solutions like the one of [10] the procedure works also well [11]. In this latter case, however, one expects the cascade not to be triggered by subsequent Seiberg dualities: the correct interpretation is through a non-perturbative duality as we explained in Chapter 6 (which formally looks like a mundane Higgsing phenomenon).

The supergravity solution presented in Section 7.3.1 is the first example of a solution describing the backreaction of a bound state containing both deformation and $\mathcal{N} = 2$ fractional branes, and hence represents an excellent opportunity to study their interplay. One expects $\mathcal{N} = 2$ fractional branes to behave as their cousins in pure $\mathcal{N} = 2$ setups, and we will find good evidence that this is the case. The novelty is that even deformation fractional branes, when probing a geometry admitting $\mathcal{N} = 2$ branes, may have that kind of behavior, sometimes.

Let us first compare the gauge theory beta functions with the supergravity prediction. The anomalous dimensions of matter fields in the UV are to leading order the same as in the conformal theory, $\gamma = -1/2$. Defining $\chi_a = 8\pi^2/g_a^2$, the four one-loop beta functions $b_a \equiv \partial/\partial(\log \mu) \chi_a$ are then

$$\begin{aligned} b_1 &= \frac{3}{2}(2M_1 - M_2) & b_2 &= \frac{3}{2}(-M_1 + 2M_2 - M_3) \\ b_4 &= \frac{3}{2}(-M_1 - M_3) & b_3 &= \frac{3}{2}(-M_2 + 2M_3) . \end{aligned} \tag{7.22}$$

On the other hand, inspection of the action of probe fractional D3-branes allows one to

find the dictionary between the gauge couplings and the integrals of B_2 on the corresponding shrinking 2-cycles [158, 70, 102, 128].⁴ With the conventions laid out in appendix A, the dictionary is easily found to be

$$\begin{aligned} \chi_2 + \chi_3 &= \frac{1}{2\pi\alpha'g_s} \int_{\mathcal{C}_2} B_2 & \chi_1 + \chi_3 &= \frac{1}{2\pi\alpha'g_s} \int_{\mathcal{C}_{CF}} B_2 \\ \chi_1 + \chi_2 &= \frac{1}{2\pi\alpha'g_s} \int_{\mathcal{C}_4} B_2 & \chi_1 + \chi_2 + \chi_3 + \chi_4 &= \frac{2\pi}{g_s}, \end{aligned} \quad (7.23)$$

with a radius-energy relation in the UV region $r/\alpha' = \mu$, like in the conformal case. Recall that $\mathcal{C}_{CF} = \mathcal{C}_2 + \mathcal{C}_4 - 2\mathcal{C}_\alpha$.

Integrating the NSNS 3-form given in eq. (7.9) one gets for the B_2 field

$$\begin{aligned} B_2 &= \frac{3}{2} \alpha' g_s \log \frac{r}{r_0} \left[(M_1 - M_2 + M_3) \omega_2^{CF} + 2\pi(-M_1 + M_2 + M_3) \omega_2^{(p)} \right. \\ &\quad \left. + 2\pi(M_1 + M_2 - M_3) \omega_2^{(q)} \right] + \pi\alpha' \left[a_{CF} \omega_2^{CF} + 4\pi(a_2 \omega_2^{(p)} + a_4 \omega_4^{(p)}) \right], \end{aligned} \quad (7.24)$$

where a_{CF} , a_2 , a_4 are integration constants. This implies that

$$\begin{aligned} \frac{1}{2\pi\alpha'g_s} \int_{\mathcal{C}_{CF}} B_2 &= 3(M_1 - M_2 + M_3) \log \frac{r}{r_0} + \frac{2\pi}{g_s} a_{CF} \\ \frac{1}{2\pi\alpha'g_s} \int_{\mathcal{C}_2} B_2 &= \frac{3}{2}(-M_1 + M_2 + M_3) \log \frac{r}{r_0} + \frac{2\pi}{g_s} a_2 \\ \frac{1}{2\pi\alpha'g_s} \int_{\mathcal{C}_4} B_2 &= \frac{3}{2}(M_1 + M_2 - M_3) \log \frac{r}{r_0} + \frac{2\pi}{g_s} a_4. \end{aligned} \quad (7.25)$$

The three integration constants a_{CF} , a_2 , a_4 correspond to the periods of B_2 at $r = r_0$, the latter having been chosen to be the value of the holographic coordinate where the effective D3-brane charge is N , see the discussion after eq. (7.20). We can think of it as a UV cut-off for the dual gauge theory, i.e. the scale where the dual UV bare Lagrangian is defined. Then the integration constants fix, through eqs. (7.23), the bare couplings of the dual non-conformal gauge theory. It is easy to check that the logarithmic derivatives of (7.25) give exactly the same beta functions as the field theory computation in (7.22).

As generically happens in supergravity solutions dual to non-conformal theories, the Maxwell D3-charge runs. It is easily computed from eq. (A.4) and (7.21) to be in our case

$$Q_{D3}(r) = N + \frac{3g_s}{2\pi} [M_1^2 + M_2^2 + M_3^2 - M_1M_2 - M_2M_3] \log \frac{r}{r_0}. \quad (7.26)$$

As in [118], the periods of B_2 are no more periodic variables in the non-conformal supergravity solutions. One should then investigate what the shift in $Q_{D3}(r)$ is once we move in the radial direction from r down to r' , where $\Delta r = r - r' > 0$ is the minimal radius shift for which all the periods of B_2 on \mathcal{C}_α , \mathcal{C}_2 , \mathcal{C}_4 change by an integer (in units of $4\pi^2\alpha'$). The shift in $Q_{D3}(r)$ should then be compared against the gauge theory expectation for

⁴Such formula apply in the UV of the supergravity solution because the superpotential coupling is at a quasi-fixed point, as explained in Chapter 5.

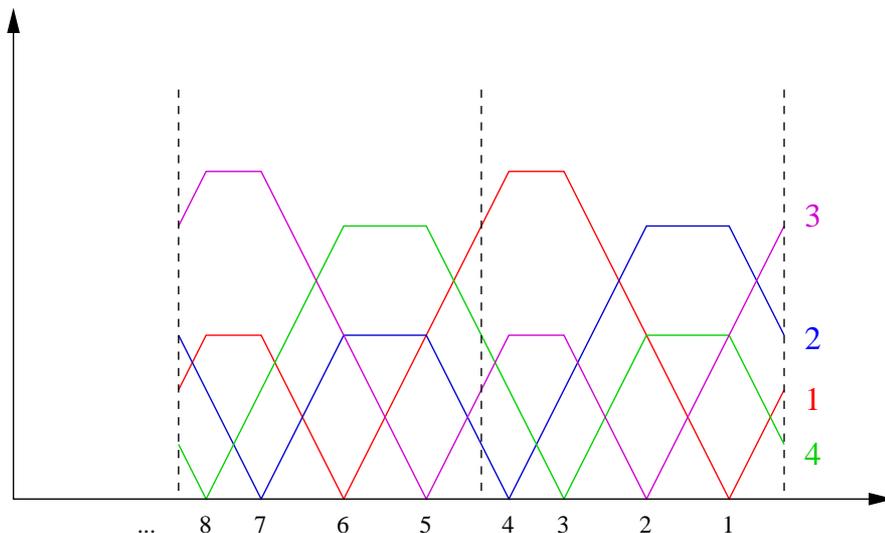


Figure 7.4: Example of the pattern of the cascade of Seiberg dualities for ranks $(N + P, N, N + P, N)$ as derived from the field theory. Black numbers indicate Seiberg dualities, performed on gauge groups with diverging couplings. Inverse squared gauge couplings are plotted versus the logarithm of the energy scale.

the decrease of the ranks under a specific sequence of cascade steps. What changes after such a sequence are the ranks of the gauge groups, all decreasing by the same integer number, the theory being otherwise self-similar, and with the initial values of the couplings. Sometimes a cyclic permutation of the gauge group factors is also needed, as in [118]. We will call such a sequence of cascade steps a quasi-period.

We are now ready to check the supergravity predictions against the field theory cascade in some simple cases. We consider three examples with deformation fractional branes only, where the RG flow can be followed by performing successive Seiberg dualities, while in the fourth example below we need an $\mathcal{N} = 2$ duality at some steps of the cascade.

1. $(N + P, N, N + P, N)$

This theory is the daughter of the duality cascade discussed in [118]. There are P deformation branes of type $(1, 0, 1, 0)$ (corresponding to D5-branes wrapped over \mathcal{C}_{CF}). We get for the charge and the periods

$$\begin{aligned}
 Q_{D3}(r) &= N + \frac{3g_s}{4\pi} 4P^2 \log \frac{r}{r_0} \\
 b_{\mathcal{C}_\alpha} &= -\frac{3g_s}{4\pi} 2P \log \frac{r}{r_0} + a_\alpha, & b_{\mathcal{C}_2} &= a_2, & b_{\mathcal{C}_4} &= a_4,
 \end{aligned}
 \tag{7.27}$$

where $a_{CF} = a_2 + a_4 - 2a_\alpha$ and $b_{\mathcal{C}_i}$ are the periods of B_2 along the cycle \mathcal{C}_i in units of $4\pi^2\alpha'$. From the above equation we see that $r' = r \exp[-4\pi/(6g_sP)]$, and under this radial shift $Q_{D3}(r') = Q_{D3}(r) - 2P$. This matches with the gauge theory expectations since the theory is quasi-periodic with a shift $N \rightarrow N - 2P$, which is obtained after four subsequent Seiberg dualities on the different gauge groups. See Figure 7.4 for an explicit example of the RG flow computed in field theory, for some values of the bare couplings.

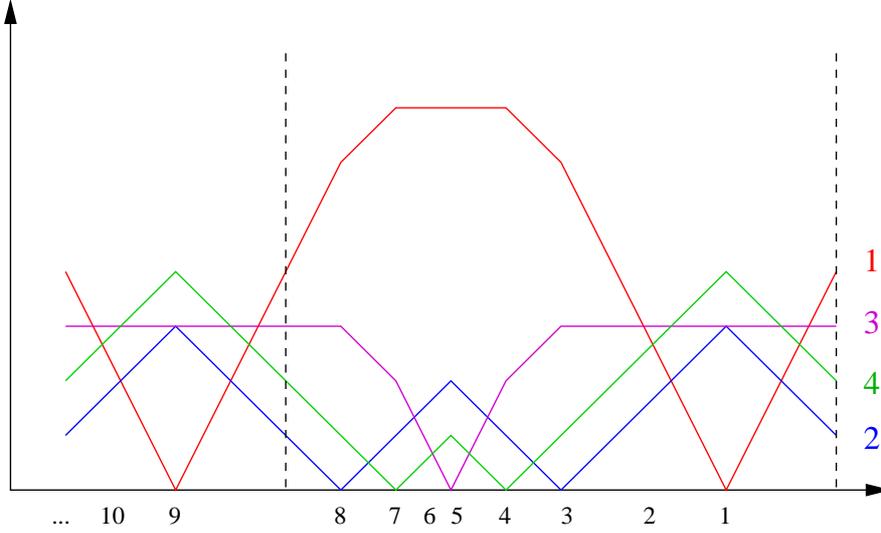


Figure 7.5: Example of the pattern of the cascade of Seiberg dualities for ranks $(N + P, N, N, N)$ as derived from the field theory.

Obviously, for any cyclic permutation of the above rank assignment we have the same story.

2. $(N + P, N, N, N)$

$$Q_{D3}(r) = N + \frac{3g_s}{4\pi} 2P^2 \log \frac{r}{r_0}$$

$$b_{C_\alpha} = -\frac{3g_s}{4\pi} P \log \frac{r}{r_0} + a_\alpha, \quad b_{C_2} = -\frac{3g_s}{4\pi} P \log \frac{r}{r_0} + a_2, \quad b_{C_4} = \frac{3g_s}{4\pi} P \log \frac{r}{r_0} + a_4. \quad (7.28)$$

From the above equation we see that $r' = r \exp[-4\pi/(3g_s P)]$ and consequently $Q_{D3}(r') = Q_{D3}(r) - 2P$. This matches again with gauge theory expectations. Although the quiver looks self-similar after four Seiberg dualities, the theory is not: the gauge couplings return to their original values only after eight Seiberg dualities, as shown in Figure 7.5. Hence in this case a quasi-period needs eight dualities and the shift in the ranks is indeed $N \rightarrow N - 2P$. Again, similar conclusions hold for any cyclic permutations of the above rank assignment.

3. $(N + Q, N + Q, N + Q, N)$

$$Q_{D3}(r) = N + \frac{3g_s}{4\pi} 2Q^2 \log \frac{r}{r_0}$$

$$b_{C_\alpha} = a_\alpha, \quad b_{C_2} = \frac{3g_s}{4\pi} Q \log \frac{r}{r_0} + a_2, \quad b_{C_4} = \frac{3g_s}{4\pi} Q \log \frac{r}{r_0} + a_4. \quad (7.29)$$

Here, $r' = r \exp[-4\pi/(3g_s Q)]$ and $Q_{D3}(r') = Q_{D3}(r) - 2Q$. A quasi-period requires eight Seiberg dualities and again agreement with field theory expectations is found. Notice that this theory appears along the RG flow of the theory $(N', N', N', N' + Q)$.

4. $(N + P, N + P, N, N)$

$$\begin{aligned}
Q_{D3}(r) &= N + \frac{3g_s}{4\pi} 2P^2 \log \frac{r}{r_0} \\
b_{C_\alpha} &= \frac{3g_s}{4\pi} P \log \frac{r}{r_0} + a_\alpha, \quad b_{C_2} = a_2, \quad b_{C_4} = \frac{3g_s}{4\pi} 2P \log \frac{r}{r_0} + a_4.
\end{aligned} \tag{7.30}$$

Here $r' = r \exp[-4\pi/(3g_s P)]$ and $Q_{D3}(r') = Q_{D3}(r) - 2P$. For such $\mathcal{N} = 2$ fractional branes we have to perform some duality at nodes with adjoints (cfr. the left quiver on Figure 7.7 below), at some step along the cascade. We conjecture that it can be understood as some $\mathcal{N} = 1$ generalization of the baryonic root transition found in the $\mathcal{N} = 2$ cascade. Formally its only effect is to change the rank of the gauge group with an adjoint from $N + P$ to $N - P$. For this reason we will loosely refer to this non-perturbative transition as ‘‘Higgsing’’. The RG flow is shown in Figure 7.10 (we will explain how this can be obtained from supergravity below). The quiver looks self-similar after three Seiberg dualities, which consist of one Seiberg duality on node 1, one Higgsing on node 2 and again one Seiberg duality on node 1. However a quasi-period needs six dualities, as apparent in Figure 7.10, and the shift in the ranks is $N \rightarrow N - 2P$, as expected.

7.3.3 Page charges and the RG flow from supergravity

There is another way of matching our running supergravity solutions (and more generally type IIB solutions constructed from fractional branes at conical singularities) with cascading field theories. The method was originally proposed in [129], working on ideas in [184]. Instead of using Maxwell charges, which are conserved and gauge invariant but not quantized nor localized, the method is based on Page charges [187] which are conserved and quantized, and therefore more suitable to be identified with gauge ranks, even though they shift under large gauge transformations.

Let C be a formal sum (polyform) of RR potentials $C = \sum C_p$, and $F = (d + H_3 \wedge) C$ the field strength polyform. Suppose we have a D_p -brane, whose dual current (loosely speaking its Poincaré dual) is a $(9 - p)$ -form Ω_{9-p} , with world-volume flux F_2 . Then the EOM/BI for the fluxes read

$$\begin{aligned}
(d + H_3 \wedge) F &= e^{\mathcal{F}} \wedge \sum_p \sigma_p 2\kappa^2 \tau_p \Omega_{9-p} \\
\Rightarrow dF^{Page} &\equiv d(e^{B_2} \wedge F) = e^{2\pi\alpha' F_2} \wedge \sum_p \sigma_p 2\kappa^2 \tau_p \Omega_{9-p},
\end{aligned} \tag{7.31}$$

where $\sigma_1 = \sigma_7 = 1$ and $\sigma_{-1} = \sigma_3 = \sigma_5 = -1$. In particular F^{Page} is a closed polyform outside the branes. Then Maxwell and Page charges are defined as

$$\text{Maxwell: } Q_p = \frac{\sigma_p}{2\kappa^2 \tau_p} \int_{S^{8-p}} F \quad \text{Page: } Q_p^{Page} = \frac{\sigma_p}{2\kappa^2 \tau_p} \int_{S^{8-p}} e^{B_2} \wedge F. \tag{7.32}$$

The idea is that it is possible to read the field theory RG flow from supergravity pointwise. At fixed radial coordinate r dual to some scale μ , standard formulæ allow us to compute the gauge couplings from the dilaton and the integrals of B_2 . Such formulæ

do not give real couplings in general, but need particular integer shifts of B_2 , which are large gauge transformations. Consequently, Page charges get shifted by some integer values. Having at hand a dictionary, they are readily mapped to the ranks of the gauge theory at that scale.

At some specific radii, in order to keep the couplings real, one has to perform a further large gauge transformation, shifting B_2 and therefore ending up with different ranks. These points connect different steps of the cascade and can usually be interpreted in the field theory as Seiberg dualities [118] or non-perturbative Higgsings. In particular, ranks are not continuously varying functions but rather integer discontinuous ones. This is not the end of the story: in general the shifts of B_2 are not enough to save us from imaginary couplings, and one is forced to introduce multiple dictionaries. We will see how everything beautifully merges.

Let us make the point clear using the Klebanov-Strassler cascade [127, 118]. The first step is to identify a dictionary between the field theory ranks and Page charges. An $SU(N + M) \times SU(N)$ theory is microscopically engineered with N regular and M fractional D3-branes at the tip of the conifold, thus from eq. (7.31) $Q_3^{Page} = N$, $Q_5^{Page} = M$. The formulæ for the gauge couplings are

$$\chi_1 = \frac{2\pi}{g_s} b \quad \chi_2 = \frac{2\pi}{g_s} (1 - b) , \quad (7.33)$$

where $\chi_a = 8\pi^2/g_a^2$ and $a = 1$ refers to the larger group, while $4\pi^2\alpha' b = \int_{S^2} B_2$. From the actual UV solution [127], we have (for B_2 in some gauge)

$$b = \frac{1}{4\pi^2\alpha'} \int_{S^2} B_2 = \frac{3g_s M}{2\pi} \log \frac{r}{r_0} \quad Q_3 = -\frac{1}{2\kappa^2\tau_3} \int_{T^{1,1}} F_5 = N + \frac{3g_s M^2}{2\pi} \log \frac{r}{r_0} . \quad (7.34)$$

At any radius/energy scale $x \equiv \log r/r_0$ one should perform a large gauge transformation and shift b by some integer Δb such that $\chi_a \geq 0$, compute the Page charges in such a gauge, and finally use the dictionary to evaluate the ranks at that scale.

It is easy to evaluate Δb and Q_3^{Page} in this example. They read

$$\Delta b = -\left[\frac{3g_s M}{2\pi} x \right]_- \quad Q_3^{Page} = N - \Delta b M = N + \left[\frac{3g_s M}{2\pi} x \right]_- M , \quad (7.35)$$

where the floor function $[y]_-$ is the greatest integer less than or equal to y . Applying the algorithm at any x , we can plot the RG flow of the gauge couplings and the ranks along it. The result (the famous KS cascade) is depicted in Figure 7.6. Notice that we never imposed continuity of the gauge couplings (even though it is a well motivated physical requirement), nevertheless the supergravity solution predicts it. Moreover it also suggests a reduction in the gauge group ranks without explaining the corresponding field theory mechanism. It turns out that in this case Seiberg duality can beautifully account for it [118, 85].

We want to apply the same procedure to our class of solutions. In order to do that, however, we need some more machinery. Given a basis of 2-cycles C_i and 3-cycles A_j on radial sections, one defines an intersection matrix

$$C_i \cdot A_j = \mathcal{I}_{ij} \quad i, j = 1 \dots p , \quad (7.36)$$

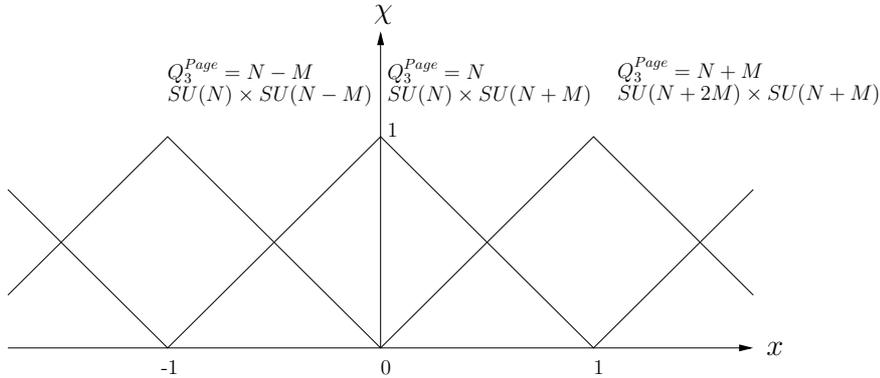


Figure 7.6: Flow in the KS theory as computed with the algorithm. x is in units of $2\pi/3g_sM$ while χ in units of $2\pi/g_s$. At integer values of x a large gauge transformation is required. At each step the Page D3-charge and the field theory is indicated.

where p is the number of fractional branes. Let $(n_I) = (\#D5_i, \#D3)$, $I = 1 \dots p + 1$ be the occupation vector, that is the numbers of D5-branes wrapped on \mathcal{C}_i and of D3-branes. A *dictionary* $F_{(m)}$ relates this system to the ranks r_a , $a = 1 \dots P$ of the dual gauge theory

$$r_a = [F_{(m)}]_{aI} n_I . \quad (7.37)$$

In general $P \geq p + 1$, but for our non-chiral theory $P = p + 1$ and $F_{(m)}$ is invertible. In the following $i, j = 1 \dots p$ while $I, J, a, b = 1 \dots p + 1$. Let (Q_I) be the vector of Page charges

$$(Q_I) = \left(-\frac{1}{2\kappa^2\tau_5} \int_{A_j} F_3, -\frac{1}{2\kappa^2\tau_3} \int F_5^{Page} \right) , \quad (7.38)$$

then the Bianchi identity eq. (7.31) implies that $Q_j = -\mathcal{I}_{ji}^t n_i$. Introducing the matrix $\tilde{\mathcal{I}} = \text{diag}(-\mathcal{I}^t, 1)$ we can write: $Q_I = \tilde{\mathcal{I}}_{IJ} n_J$. It follows that (suppressing indices)

$$r = \left(F_{(m)} \tilde{\mathcal{I}}^{-1} \right) Q . \quad (7.39)$$

The formulæ relating the gauge couplings to the supergravity solution can be derived by considering the worldvolume action of probe D3- and wrapped D5-branes [185]. Let $\chi_a = 8\pi^2/g_a^2$ as before. Considering D3-branes one concludes that $\sum \chi_a = 2\pi/g_s$; then the integral of B_2 on some 2-cycle \mathcal{C}_j is related to the gauge coupling on the probe D5-brane, which is itself related to the sum of the χ 's corresponding to the ranks increased by the D5, as in (7.23). Defining the vector

$$(B_I) = \left(\frac{1}{4\pi^2\alpha'} \int_{\mathcal{C}_i} B_2, 1 \right) \quad (7.40)$$

one can summarize the relations by

$$\frac{2\pi}{g_s} B = F_{(m)}^t \chi \quad \Rightarrow \quad \chi = \frac{2\pi}{g_s} F_{(m)}^{-1t} B . \quad (7.41)$$

Under large gauge transformations the integrals of B_2 change by integer amounts, thus the first p components of the vector B undergo a particular shift $B_i \rightarrow B_i + Z_i$, for some $Z_i \in \mathbb{Z}$. As a result the Page D3-charge is shifted by

$$\Delta Q_3^{Page} = -\frac{1}{2\kappa^2\tau_3} \int \Delta B_2 \wedge F_3 = Q_j (\mathcal{I}^{-1})_{jk} Z_k, \quad (7.42)$$

while the inferred gauge couplings change according to eq. (7.41).

We now apply the algorithm to our solutions (7.9), where the integrals of B_2 are (7.25), for some values of the charges (equivalently for some M_i 's). Using the basis $\{\mathcal{C}_2, \mathcal{C}_4, \mathcal{C}_\alpha\}$ for the 2-cycles and $\{A_2, A_4, A_{CF}\}$ for the 3-cycles, the intersection matrix \mathcal{I}_{ij} is given by

$$\mathcal{I}_{ij} = \begin{pmatrix} -2 & 0 & 0 \\ 0 & -2 & 0 \\ -1 & -1 & 1 \end{pmatrix} \quad (7.43)$$

as in (F.39), while the dictionary $[F_{(1)}]_{aI}$ derived in section 7.2.2 (see Table (7.6)), referring to the central quiver in Figure 7.7, is reported in Figure 7.8. One quickly discovers that, for generic values of the integration constants a_i and of the radial coordinate r , there is no gauge transformation that produces positive χ_a in eq. (7.41).

One is led to the conclusion that *multiple dictionaries* are needed. This had to be expected since performing any Seiberg duality on the central quiver in Figure 7.7 one obtains the lateral quivers (depending on the node chosen), which are substantially different and cannot be described by the same dictionary, even up to reshuffling of the nodes.

It turns out that even two dictionaries are not enough in our case. We provide a set of six dictionaries such that, at any energy, for one and only one dictionary there is one large gauge transformation that gives non-negative χ_a , see Figure 7.8.

The dictionaries besides $F_{(1)}$ are obtained from it through formal Seiberg dualities. Consider a system with occupation vector $n = (n_1, n_2, n_3, N)$. Start with the central quiver where the ranks are given by eq. (7.37) using $F_{(1)}$. Then a formal Seiberg duality on one node gives a new quiver with new ranks (and superpotential), from which a new dictionary $F_{(m)}$ is directly read. Actually there is an ambiguity because the number of D3-branes N could have changed in the process (but not the other charges) and then one is free to add lines of 1's to any of the first three columns. One can show that the physical result, that is the gauge couplings and ranks in the correct gauge of B_2 , is not affected. In our case, a Seiberg duality on node 1 gives $F_{(4)}$, on node 2 $F_{(6)}$, on node 3 $F_{(3)}$, on node 4 $F_{(5)}$ and on two opposite nodes $F_{(2)}$.

We can finally apply the algorithm at any radius $x \equiv \log r/r_0$, that is:

- find a dictionary in the set $\{F_{(m)}\}$ and a large gauge transformation $B_i(x) \rightarrow B_i(x) + Z_i$ such that, according to eq. (7.41), $\chi_I \geq 0 \quad \forall I$. It turns out that there is always one and only one solution;⁵
- compute the D3-brane Page charge in this gauge, using eq. (7.42) (D5-brane charges are invariant);

⁵To be precise, when one of the χ_I vanishes there are two dictionaries (with their gauges) that do the job. At these radii there is the transition between the validity domains of two different field theory duals.

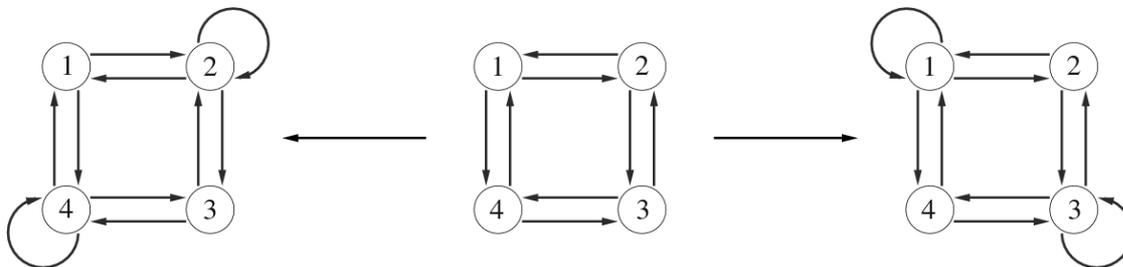


Figure 7.7: Seiberg dual quivers. The central quiver is the most extensively discussed one in this chapter. The left quiver is obtained with a Seiberg duality on node 1 or 3, while the right one on node 2 or 4.

$$\begin{array}{ccc}
 F_{(3)} = \begin{pmatrix} 0 & 1 & 0 & 1 \\ 1 & 1 & 1 & 1 \\ 0 & 1 & 1 & 1 \\ 0 & 0 & 0 & 1 \end{pmatrix} & F_{(1)} = \begin{pmatrix} 0 & 1 & 0 & 1 \\ 1 & 1 & 1 & 1 \\ 1 & 0 & 0 & 1 \\ 0 & 0 & 0 & 1 \end{pmatrix} & F_{(5)} = \begin{pmatrix} 0 & 1 & 0 & 1 \\ 1 & 1 & 1 & 1 \\ 1 & 0 & 0 & 1 \\ 1 & 1 & 0 & 1 \end{pmatrix} \\
 F_{(4)} = \begin{pmatrix} 1 & 0 & 1 & 1 \\ 1 & 1 & 1 & 1 \\ 1 & 0 & 0 & 1 \\ 0 & 0 & 0 & 1 \end{pmatrix} & F_{(2)} = \begin{pmatrix} 1 & 0 & 1 & 1 \\ 1 & 1 & 1 & 1 \\ 0 & 1 & 1 & 1 \\ 0 & 0 & 0 & 1 \end{pmatrix} & F_{(6)} = \begin{pmatrix} 0 & 1 & 1 & 1 \\ 0 & 0 & 0 & 1 \\ 1 & 0 & 1 & 1 \\ 0 & 0 & 1 & 1 \end{pmatrix}
 \end{array}$$

Figure 7.8: A set of six dictionaries for the orbifolded conifold theory. $F_{(3)}$, $F_{(4)}$ refer to the left quiver, with adjoints on nodes 2-4; $F_{(1)}$, $F_{(2)}$ refer to the central quiver, without adjoints; $F_{(5)}$, $F_{(6)}$ refer to the right quiver, with adjoints on nodes 1-3. The four columns represent the nodes activated by a D5-brane on \mathcal{C}_2 , \mathcal{C}_4 , \mathcal{C}_α and a D3-brane respectively.

- use the dictionary and the charges in eq. (7.39) to evaluate the ranks at that scale in the corresponding quiver.

As a result, one can plot the gauge couplings along the flow and keep track of the various field theory descriptions.

It is clear that the transition radii between two different descriptions (dictionaries) occur when one of the χ_I vanishes. But in principle there is no reason why one should expect, from the procedure above, continuous couplings at the transition points. Surprisingly enough, it turns out that the resulting couplings are indeed continuous. Some plots with explanation are in Figures 7.9, 7.10, 7.11, 7.12 (obtained via a mathematica code). In the following, we comment on interesting examples.

1. $(N + P, N, N + P, N)$

The RG flow, as computed from supergravity with the algorithm above, is plotted in Figure 7.9 (for $P = 1$ and some typical choice of the integration constants a_2 , a_4 , a_α and the starting radius $x = \log r/r_0$). It precisely matches with the field theory expectations, with respect to both gauge couplings and ranks at any step. All transition points can be interpreted by means of a single Seiberg duality, as the prototypical example in [118]. Notice that the integral of B_2 on \mathcal{C}_2 and \mathcal{C}_4 is constant and generically not integer.

Figures: the following figures represent the RG flow as computed from SUGRA with the algorithm, for typical values of the integration constants a_2, a_4, a_α and the initial radius $x = \log r/r_0$. The gauge couplings are in units of $2\pi/g_s$. On the right side we report, for each step, the dictionary used and the ranks in the quiver; the addition of N is understood. Underlined ranks signal an adjoint chiral superfield at the corresponding node. The red line represents the first group, the orange the second one, the light green the third one, the dark green the fourth one.

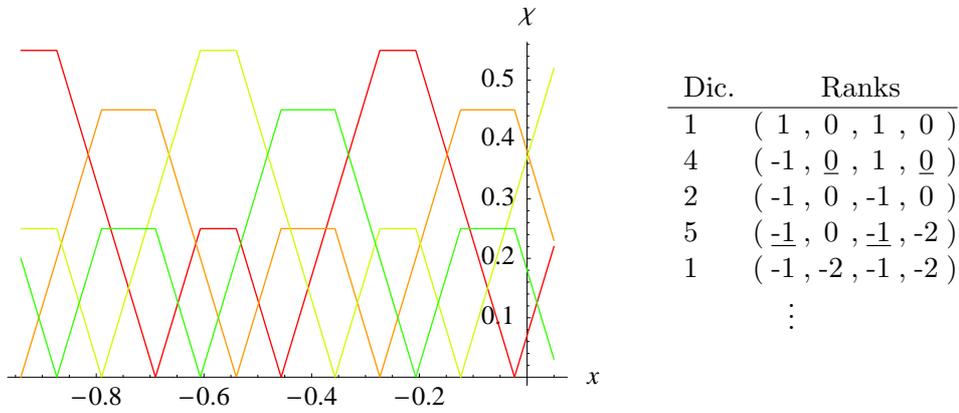


Figure 7.9: RG flow for the $(N + 1, N, N + 1, N)$ theory from SUGRA.

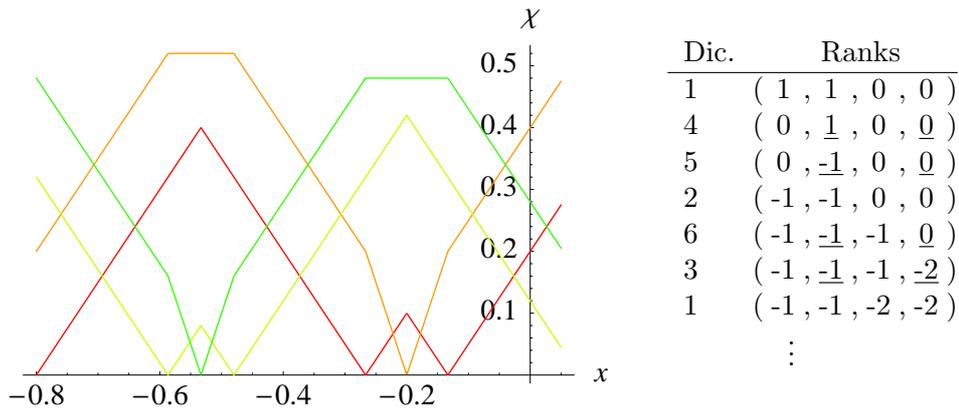
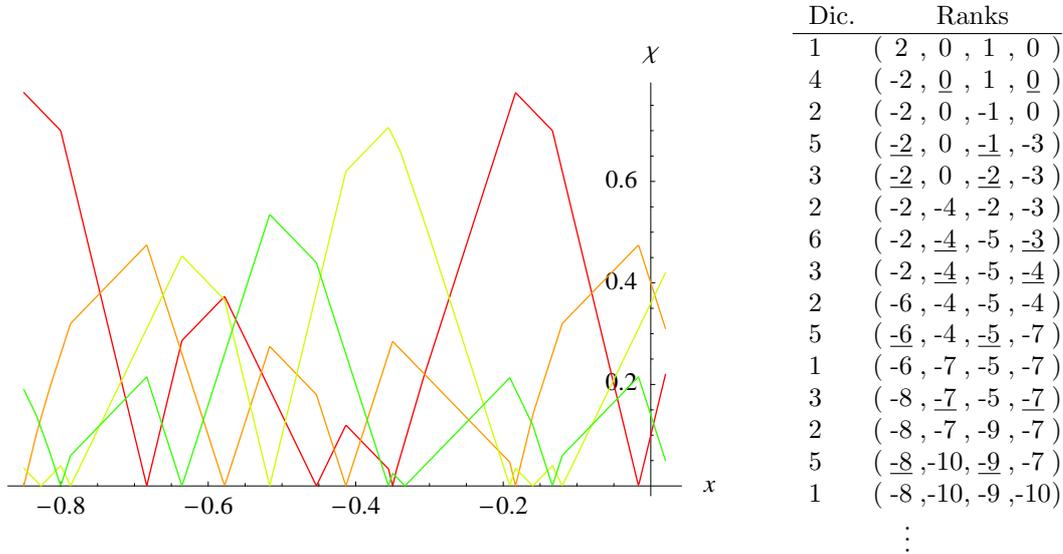
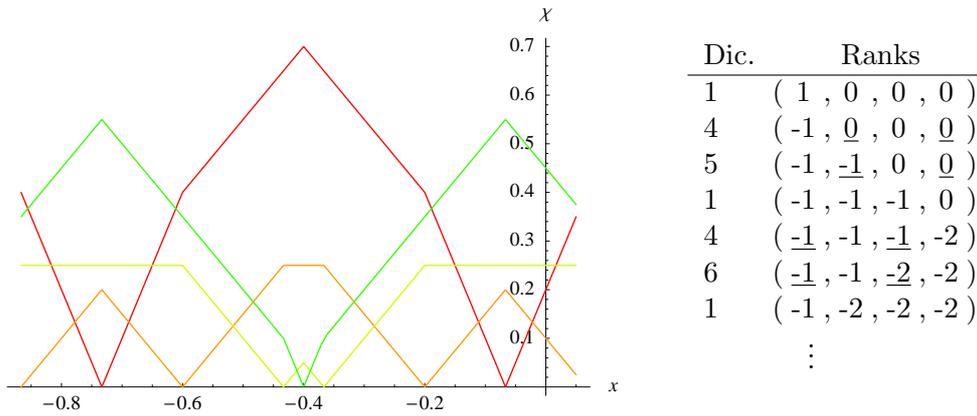


Figure 7.10: RG flow for the $(N + 1, N + 1, N, N)$ theory from supergravity.

Figure 7.11: RG flow for the $(N + 2, N, N + 1, N)$ theory from supergravity.Figure 7.12: RG flow for the $(N + 1, N, N, N)$ theory from supergravity.

2. $(N + P, N + P, N, N)$

The supergravity RG flow is shown in Figure 7.10 (for $P = 1$ and typical integration constants). This theory is realized with $\mathcal{N} = 2$ fractional branes only, and one expects a behavior quite similar to the $\mathcal{N} = 2$ setup of [10]. The algorithm confirms that there are steps of the cascade where the node with divergent coupling has an adjoint chiral field and $\mathcal{N} = 2$ superpotential. In the example of Figure 7.10, after a Seiberg duality on node 1, one is left with the left hand side quiver of Figure 7.7, and superpotential

$$W = -X_{12}X_{21}X_{14}X_{41} + M_{22}(X_{21}X_{12} - X_{23}X_{32}) + X_{32}X_{23}X_{34}X_{43} - M_{44}(X_{43}X_{34} - X_{41}X_{14}). \quad (7.44)$$

The next node with diverging coupling is node 2. Notice that if one neglects the gauge dynamics on the other nodes and possible subtleties related to a non-trivial Kähler potential and anomalous dimensions of node 2, the theory is effectively $\mathcal{N} = 2$ massless SQCD with $N + P$ colors and $2N$ flavors. We conjecture that it can be described along

the lines of the $\mathcal{N} = 2$ cascade [2] discussed in the previous chapter: we can go to a dual description at the analog of the baryonic root. We refer to this as (non-perturbative) Higgsing.

On the gravity side this step in the cascade occurs when

$$\frac{1}{4\pi^2\alpha'} \int_{\mathcal{C}_2} B_2 \in \mathbb{Z} \quad \text{or} \quad \frac{1}{4\pi^2\alpha'} \int_{\mathcal{C}_4} B_2 \in \mathbb{Z} \quad (7.45)$$

(in this case only \mathcal{C}_4). Since \mathcal{C}_2 and \mathcal{C}_4 are shrunk 2-cycles along the $\mathcal{N} = 2$ singularity lines, at these radii (the generalized enhançons) there are extra massless fields and tensionless objects in supergravity.

3. $(N + P, N, N + Q, N)$

The supergravity RG flow for the case $(N + 2, N, N + 1, N)$ is shown in Figure 7.11. This theory is realized with deformation fractional branes only. Nevertheless, the fact that the geometry admits $\mathcal{N} = 2$ fractional branes causes that, at some steps, there is a reduction of rank in a node with adjoint; as before, this is interpreted in term of Higgsing. Shells where such transitions occur are precisely at radii where one of the periods of B_2 on \mathcal{C}_2 or \mathcal{C}_4 vanishes.

This rather intriguing fact can be understood by noticing that in some intermediate steps, i.e. when there are nodes with adjoints, the relevant dictionary forces us to reinterpret the configuration as if it were composed of deformation fractional branes together with a number of $\mathcal{N} = 2$ fractional branes.

For generic P and Q things can be analysed in a similar way. Notice that for P and Q large and coprime, the flow becomes quickly very complicated.

4. $(N + P, N, N, N)$

The supergravity RG flow for the case $(N + 1, N, N, N)$ is shown in Figure 7.12. As in the previous examples, when one of the periods of B_2 on \mathcal{C}_2 or \mathcal{C}_4 vanishes supergravity predicts some transition that cannot be interpreted as a Seiberg duality in the FT. This flow is anyway peculiar because performing a Seiberg duality on a conformal node it is possible to provide a dual FT interpretation of the RG flow using only Seiberg dualities, as was done in the previous subsection. However, supergravity seems to predict a different pattern of dualities which nevertheless leads to the same evolution of the gauge couplings.

Let us summarize what we found. There exists a well-defined algorithm that, given a minimal set of dictionaries, allows one to derive the field theory RG flow from a supergravity solution. For toric singularities, as the one we are describing, the dictionaries can be derived using standard techniques (see for instance [188]) and, given the first, the other ones follow applying formal Seiberg dualities. It is not clear to us how to determine the minimal number of dictionaries, and we have obtained them by hand. Moreover, it would be interesting to understand how to extend the algorithm to supergravity solutions dual to chiral gauge theories, as those in [147].

Our geometry admits both deformation and $\mathcal{N} = 2$ fractional branes. We saw examples of cascades from deformation branes that can be interpreted in term of Seiberg dualities only, examples with $\mathcal{N} = 2$ branes that are very close to pure $\mathcal{N} = 2$ theories and whose interpretation should be similar to the $\mathcal{N} = 2$ cascade, but also examples

which one would say are realized with deformation branes only that nevertheless require something like non-perturbative Higgsing, at some steps.

7.4 The IR regime of the theory

As already noticed, the solution presented in the previous section is singular. In this section we discuss how to extend it towards the IR (i.e. at small radii on the gravity side). It is not difficult to see that the warp factor (7.21) becomes singular at short distances, so that the metric has a repulson type singularity.

This is of course expected, since our solution is similar to the ones of [127] and [10]: we are considering the backreaction of the branes in the supergravity limit, but supergravity cannot be the full story near the branes themselves, where the stringy dynamics should be dominant. Resolving the singularity then amounts to a clever guess of what these stringy effects would lead to. Deformation fractional branes and $\mathcal{N} = 2$ fractional branes are very different in that respect.

In the case of deformation branes at conifold points, the singularity can be smoothed out in supergravity by considering the warping of the deformed conifold instead of the singular conifold. This is what has been done in [118], and the procedure introduces a dimensionful parameter ϵ , related to the dynamical scale of a confining gauge group.

In the case of $\mathcal{N} = 2$ fractional branes, one does not expect that the repulson singularity can be smoothed in a similar way. Indeed, the $U(1)^N$ abelian degrees of freedom on the Coulomb branch can only appear through the presence of left-over open string modes in the gravity dual. This means that physical branes are still present, although they are expected to form a ring that effectively cloaks the singularity [10]. This is the enhançon mechanism first discussed in [156]. The enhançon radius (where probe fractional branes become tensionless) then provides a dimensionful parameter, which basically corresponds to the dynamically generated scale of the $\mathcal{N} = 2$ gauge theory.

Note that in addition to the repulson singularity, the presence of twisted flux makes the warp factor singular all along the Coulomb branch, which coincides with the line of orbifold singularities. It then signals that one should include new massless modes in the low energy effective theory also at large values of r . This is what happens in our $\mathcal{N} = 1$ orbifolded conifold setup as well. Still, the supergravity solution can already give us some important insight into the dynamics, particularly about the RG flow trajectory of the gauge theory dual, as we saw in the previous section.

We now turn to the IR effective theory at the bottom of the cascade. In our solutions, it is clear that the IR behaviour can be quite different depending on which dynamics dominates, i.e. which nodes in the low-energy quiver have the largest dynamical scale. As was argued in the previous section, the RG flow will, in a way or another, reduce the ranks of the gauge groups by a common additive factor. In other words, the effective number of regular branes will diminish as we go inwards to the IR, and we assume that we eventually reach a point where the quiver has only three nodes.

In the following, we will first analyze the low-energy dynamics from the gauge theory point of view. We perform the analysis in two different regimes: either the $\mathcal{N} = 2$ effective dynamics is the most important effect, or else the $\mathcal{N} = 1$ confining behavior dominates. As a consistency check of the candidate gravity dual, we reproduce the effective superpotential from the holomorphic data of the geometry in that latter limit.

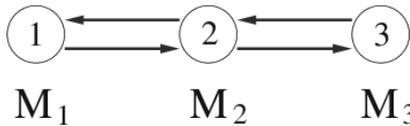


Figure 7.13: The 3-node quiver that corresponds to the IR bottom of the cascade.

We eventually consider the equations determining the warp factor. The latter is related to data encoded in the full Kähler potential of the gauge theory. Hence, computing the warp factor would be the main challenge in order to gather new dynamical information on the low-energy theory. To do that, the two limits in which the dynamics is predominantly confining or $\mathcal{N} = 2$ are quite different. In the latter case, we will argue that the enhancement is so large that a possible local deformation of the geometry would be irrelevant, and so the UV solution presented in the previous section is basically the correct gravity dual up to the enhancement radius. When instead the confining dynamics is the strongest, one expects to have a gravity dual consisting of the orbifold of the deformed conifold, with singularities along the orbifold fixed line. We must anticipate that we will stop short of actually computing the warp factor in that case.

7.4.1 Gauge theory IR dynamics

In this subsection we perform the gauge theory analysis for the low-energy behavior of a generic 3-node quiver, see Figure 7.13. It will often prove useful to actually think of moduli spaces in terms of mobile (fractional) branes, so we will freely make reference to this interpretation even in the course of the purely gauge theoretic analysis.

Let us call Λ_i the dynamically generated scale of the i -th node of the quiver, with $i = 1, 2, 3$. We consider two qualitatively different regimes.

First we analyze the regime $\Lambda_2 \gg \Lambda_{1,3}$, where the dominant quantum effects come from the second node. As we will see, for $M_2 < M_1 + M_3$, there is no deformation of the (mesonic) moduli space, which itself corresponds to having a stack of $\mathcal{N} = 2$ fractional branes on their Coulomb branch. For $M_2 > M_1 + M_3$, we find a runaway behavior on the Coulomb branch. This is interpreted in the gravity dual as a fully regular deformation of the geometry in the presence of $\mathcal{N} = 2$ fractional branes. Indeed, in this case the exceptional cycle the branes wrap is blown-up and minimizes its volume at infinity: the $\mathcal{N} = 2$ branes are pushed away.

Secondly, the regime $\Lambda_2 \ll \Lambda_{1,3}$ is analyzed (a similar analysis was performed in the appendix of [182]). One finds gaugino condensation for both nodes one and three, with $S_1 = S_3$. On the dual geometric side, the deformation branes trigger a geometric transition that still preserves an orbifold singularity line in the resulting deformed geometry. The singularity line can accommodate some left over $\mathcal{N} = 2$ branes which explore their moduli space.

Regime $\Lambda_2 \gg \Lambda_{1,3}$

In this regime, the only gauge dynamics we take into account is the one of the second node. The quiver configuration is $(M_1, M_2, M_3, 0)$ with $3M_2 > M_1 + M_3$, so that node 2

has a strongly coupled IR dynamics and it makes sense to neglect the scales of the other nodes as a first approximation.

The tree level superpotential is

$$W_{tree} = \lambda X_{12} X_{23} X_{32} X_{21} \quad (7.46)$$

and the quantum corrected one is

$$W = W_{tree} - (M_1 - M_2 + M_3) \left(\frac{\det \mathcal{K}}{\Lambda_2^{3M_2 - M_1 - M_3}} \right)^{\frac{1}{M_1 - M_2 + M_3}}. \quad (7.47)$$

If $M_2 > M_1 + M_3$, this is the familiar Affleck-Dine-Seiberg (ADS) superpotential [138], while if $M_2 < M_1 + M_3$, it is the effective superpotential for the free Seiberg dual mesons and vanishing dual quark VEVs.⁶

The meson matrix for the second node is

$$\mathcal{K} \equiv \begin{pmatrix} X_{12} X_{21} & X_{12} X_{23} \\ X_{32} X_{21} & X_{32} X_{23} \end{pmatrix} \equiv \begin{pmatrix} K_{11} & K_{13} \\ K_{31} & K_{33} \end{pmatrix}. \quad (7.48)$$

Let us denote

$$S_2 \equiv \left(\frac{\det \mathcal{K}}{\Lambda_2^{3M_2 - M_1 - M_3}} \right)^{\frac{1}{M_1 - M_2 + M_3}}. \quad (7.49)$$

We want to determine the moduli space of such a theory. Considering the effective superpotential in terms of the mesons, one has the following F-flatness conditions

$$\begin{aligned} S_2(\mathcal{K}^{-1})_{11} = 0 &= S_2(\mathcal{K}^{-1})_{33} \\ \lambda K_{31} - S_2(\mathcal{K}^{-1})_{31} = 0 &= \lambda K_{13} - S_2(\mathcal{K}^{-1})_{13} \end{aligned} \quad (7.50)$$

This implies

$$M_1 S_2 = M_3 S_2. \quad (7.51)$$

We must then have⁷ that $S_2 = 0$, which implies that K_{13} and K_{31} must vanish, and $\det \mathcal{K} = \det K_{11} \det K_{33}$. When $M_2 < M_1 + M_3$ the constraint $S_2 = 0$ means that $\det \mathcal{K} = 0$. Using the gauge freedom of the first and third nodes, the general solution consists of \mathcal{K} diagonal with M_2 non-vanishing eigenvalues. There are as many distinct such solutions as there are possibilities of choosing M_2 out of the $M_1 + M_3$ $\mathcal{N} = 2$ subquiver configurations $(1, 1, 0, 0)$ or $(0, 1, 1, 0)$.

When we have instead $M_2 > M_1 + M_3$, there is an ADS superpotential, and the constraint on the mesons become

$$\det \mathcal{K} = \det K_{11} \det K_{33} \rightarrow \infty. \quad (7.52)$$

⁶In principle, we should worry about additional baryonic directions in the effective dynamics. Their fully quantum analysis is beyond the scope of the present analysis, however both the classical gauge theory analysis of the higgsing patterns and their interpretation in terms of brane motions hint that the statements concerning the mesonic VEVs should not be modified.

⁷Unless $M_1 = M_3$, where we have another possible solution: $K_{11} = K_{33} = 0$ and $S_2 = \Lambda_2^3 (\lambda \Lambda_2)^{\frac{M_1}{M_2 - M_1}}$, presumably related to a non-Coulomb branch.

This corresponds to a runaway behavior of the $\mathcal{N} = 2$ brane configuration (the same phenomenon was observed in [189, 190]). Indeed, after all the $\mathcal{N} = 2$ configurations have been accounted for (by moving on the Coulomb branch), there remains the configuration $(0, M_2 - M_1 - M_3, 0, 0)$, that confines, and we know that this should correspond to the following deformation of the geometry seen by D3-branes

$$(z_1 z_2 - S_2) z_1 z_2 = xy . \quad (7.53)$$

This space only has a singularity at the origin, so that the Coulomb branch (which corresponds to a singularity line in the orbifolded conifold) is lifted, the supersymmetric vacua being preserved only at infinity. Geometrically what happens is that the $\mathcal{N} = 2$ branes become non-BPS, as they wrap a blown-up cycle, and they can only minimize their tension by moving off to infinity.

Regime $\Lambda_2 \ll \Lambda_{1,3}$

In the regime $\Lambda_{1,3} \gg \Lambda_2$, gaugino condensation at the first and third nodes is the dominant effect in the IR. This corresponds to a complex structure deformation of the geometry, induced by the deformation fractional branes. We again consider the quiver configuration $(M_1, M_2, M_3, 0)$ with tree level superpotential (7.46).

Let us restrict to the case where $M_2 < M_1, M_3$.⁸ The first and the third gauge groups develop an ADS superpotential at the quantum level, while the second gauge group can be considered classical. In term of the mesons $\mathcal{M} = X_{21} X_{12}$ and $\mathcal{N} = X_{23} X_{32}$ of the first and third nodes respectively (which are both $M_2 \times M_2$ matrices in the adjoint plus singlet of the second node), the full effective superpotential reads

$$W = \lambda \mathcal{M} \mathcal{N} + (M_1 - M_2) \left(\frac{\Lambda_1^{3M_1 - M_2}}{\det \mathcal{M}} \right)^{\frac{1}{M_1 - M_2}} + (M_3 - M_2) \left(\frac{\Lambda_3^{3M_3 - M_2}}{\det \mathcal{N}} \right)^{\frac{1}{M_3 - M_2}} . \quad (7.54)$$

Instead of solving for the extrema of the above superpotential, we find it useful to first integrate in the glueball superfields for the two confining gauge groups. We are also motivated in doing this by the approach which uses the Gukov-Vafa-Witten (GVW) [191] superpotential to make the link between the gauge theory and the geometrical quantities, and which will be pursued in section 7.4.2. We thus obtain

$$W = \lambda \mathcal{M} \mathcal{N} + (M_1 - M_2) S_1 - S_1 \log \frac{S_1^{M_1 - M_2} \det \mathcal{M}}{\Lambda_1^{3M_1 - M_2}} + (M_3 - M_2) S_3 - S_3 \log \frac{S_3^{M_3 - M_2} \det \mathcal{N}}{\Lambda_3^{3M_3 - M_2}} , \quad (7.55)$$

which is a Taylor-Veneziano-Yankielowicz (TVY) [192] kind of superpotential. Of course, extremizing with respect to S_1 and S_3 will lead us back to the previous ADS-like superpotential. However let us extremize with respect to all fields together

$$\lambda \mathcal{N} = S_1 \mathcal{M}^{-1}, \quad \lambda \mathcal{M} = S_3 \mathcal{N}^{-1}, \quad (7.56)$$

⁸If $M_2 > M_1, M_3$, two Seiberg dualities on nodes one and three bring us back to the case analyzed previously because we can assume that the dual scales are such that $\tilde{\Lambda}_{1,3} \ll \Lambda_2$. If $M_1 > M_2 > M_3$, it is possible to show that the system has a runaway behavior.

$$\log \frac{S_1^{M_1-M_2} \det \mathcal{M}}{\Lambda_1^{3M_1-M_2}} = 0, \quad \log \frac{S_3^{M_3-M_2} \det \mathcal{N}}{\Lambda_3^{3M_3-M_2}} = 0. \quad (7.57)$$

The above equations imply that \mathcal{M} is proportional to the inverse of \mathcal{N} , and that

$$S_1 = S_3 \equiv S = \left(\lambda^{M_2} \Lambda_1^{3M_1-M_2} \Lambda_3^{3M_3-M_2} \right)^{\frac{1}{M_1-M_2+M_3}}. \quad (7.58)$$

This of course implies that also $\det \mathcal{M}$ is fixed, while the moduli space is spanned by the values of \mathcal{M} subject to this constraint. Once the effective $\mathcal{N} = 2$ dynamics of the $SU(M_2)$ gauge group is taken into account, the moduli space reduces to the $M_2 - 1$ directions in the Cartan subalgebra.

Let us also consider two limiting cases. If $M_1 = M_2 = M_3 \equiv M$, one can check that the mesonic and the baryonic branches decouple. On the mesonic branch, the superpotential (7.55) is correct and the solution to its extremization is

$$\det \mathcal{M} = \Lambda_1^{2M}, \quad \det \mathcal{N} = \Lambda_3^{2M}, \quad S_1 = S_3 = \left(\lambda^M \Lambda_1^{2M} \Lambda_3^{2M} \right)^{\frac{1}{M}}. \quad (7.59)$$

The dynamics is essentially the same as before. Note that the S_i act effectively as Lagrange multipliers, and their being non zero is a signal of the decoupling of the mesonic from the baryonic branch. This was the case of most interest in [182].

The other limiting case is $M_2 = 0$. Here there are no mesons \mathcal{M} and \mathcal{N} , and hence no coupling between nodes one and three. We just have a sum of two Veneziano-Yankielowicz superpotentials [193] for two decoupled SYM theories. Consistently, we obtain upon extremization

$$S_1 = \left(\Lambda_1^{3M_1} \right)^{\frac{1}{M_1}}, \quad S_3 = \left(\Lambda_3^{3M_3} \right)^{\frac{1}{M_3}}. \quad (7.60)$$

In this case, the two VEVs S_i are independent. It corresponds to a generic deformation of the geometry, as reviewed in appendix F.2.

7.4.2 The Gukov-Vafa-Witten superpotential

In this subsection, we make an important consistency check of our gauge/gravity set up by matching the GVW superpotential [191] to the gauge theory effective superpotential considered in the previous subsection.

It is well known that Calabi-Yau compactification of type IIB in the presence of fluxes helps to restrict the allowed values of the complex structure moduli. The dynamics of these moduli can be encoded in an effective superpotential W_{GVW} for the resulting four dimensional supergravity. In the gauge/gravity correspondence setup, W_{GVW} can also be computed, provided we fix some boundary conditions at infinity on the non-compact CY we are using. It can be written as

$$W_{GVW} = \frac{i}{2\pi g_s \alpha'^4} \int_{\mathcal{M}_6} G_3 \wedge \Omega, \quad (7.61)$$

where Ω is the holomorphic 3-form. One can then compare this W_{GVW} superpotential to the dual gauge theory superpotential, since they are expected to agree on-shell.⁹

⁹Remark that W_{GVW} is a supergravity superpotential, in particular $dW_{GVW} = 0 = W_{GVW}$ on supersymmetric compactifications. To decouple gravity we must consider non-compact manifolds and accordingly on the dual gauge theory side we only have $dW = 0$.

In the absence of brane sources, the G_3 flux is closed and depends only on the cohomology class of G_3 . Adding some D5-brane sources for G_3 , however, one must keep track of the position of these branes [174, 194, 195]. Separating G_3 into a bulk contribution (i.e. closed part) G_3^b and a contribution from the sources G_3^s , and using Riemann relations for the closed part, one has

$$W_{GVW} = \frac{i}{2\pi g_s \alpha'^4} \sum_j \left(\int_{A_j} G_3^b \int_{B_j} \Omega - \int_{B_j} (G_3^b + G_3^s) \int_{A_j} \Omega \right) - \frac{2\pi i}{\alpha'^3} \sum_{\mathcal{N}=2 \text{ branes}} \int_{\Xi_3} \Omega, \quad (7.62)$$

where Ξ_3 is a 3-chain that extends from the 2-cycle wrapped by the D5-brane to some reference 2-cycle near infinity.¹⁰

Let us now compute W_{GVW} in our orbifolded conifold geometry. We consider a generic smooth deformation, with the two complex structure parameters ϵ_1, ϵ_3 arbitrary, see (F.36), and we take the limit where the wrapped D5-branes are far from the deformation near the tip. With an obvious linear change of coordinates, the geometry is defined by

$$xy - (u^2 - v^2 + \epsilon_1)(u^2 - v^2 + \epsilon_3) = 0 \quad (7.63)$$

in $\mathbb{C}^4 \cong \{x, y, u, v\}$. The holomorphic 3-form Ω is given by

$$\Omega = \frac{1}{2\pi^2} \frac{du \wedge dv \wedge dx}{x}. \quad (7.64)$$

We obtain the usual results for the periods of Ω on the A and B cycles (see appendices F.2 and F.4 for more details)

$$\int_{A_j} \Omega = \epsilon_j, \quad \text{and} \quad \int_{B_j} \Omega = \frac{\epsilon_j}{2\pi i} \log \left(\frac{\epsilon_i}{4ev_0^2} \right) + \text{regular}, \quad (7.65)$$

where $v = v_0$ is a cut-off for the non-compact B -cycles. The contribution to (7.62) coming from D5-branes wrapped on \mathcal{C}_2 is computed in appendix F.4 : for a D5-brane located at $v = \xi$, in the limit $|v_0|^2, |\xi|^2 \gg |\epsilon_k|$, we have the simple result

$$\int_{\Xi_3} \Omega = -\frac{1}{2\pi i} (\epsilon_1 - \epsilon_3) \log \frac{\xi}{v_0} + \mathcal{O}\left(\frac{\epsilon^2}{\xi^4}\right). \quad (7.66)$$

Let us now consider the following F_3 fluxes

$$-\frac{1}{4\pi^2 g_s \alpha'} \int_{A_1} G_3^b = M_1, \quad -\frac{1}{4\pi^2 g_s \alpha'} \int_{A_3} G_3^b = M_3 - M_2. \quad (7.67)$$

This means we assume that M_1 and $M_3 - M_2$ D5-branes that were wrapped on the 2-cycles \mathcal{C}_1 and \mathcal{C}_3 , see eq. (F.35), have undergone geometric transition independently.¹¹

¹⁰For an intuitive feel for the meaning of that formula, one can think of a one-dimensional analogy: $dF^{(s)} = \delta_{\text{source}}$ means that $F^{(s)}$ is a step function that begins at the location of the source. It is easy to generalise the argument to 6 dimensions, at least formally by integration by part.

¹¹There is thus an arbitrariness in choosing these fluxes, and we actually wrote the flux assignment that makes the following arguments the simplest. The identifications (7.74) and (7.76) below consistently reflect this choice.

We also have M_2 D5-branes wrapped on \mathcal{C}_2 , at positions $|\xi_i|^2 \gg |\epsilon_{1,2}|$. Let us finally denote the B -periods of G_3 by the complex numbers

$$\mathcal{B}_k \equiv -\frac{1}{4\pi^2 g_s \alpha'} \int_{B_k} G_3, \quad k = 1, 3. \quad (7.68)$$

Plugging all this into (7.62) and denoting the product of the positions ξ_i by ξ^{M_2} , we get

$$\alpha'^3 W_{GVW} = -\epsilon_1 \ln \left(\frac{\epsilon_1^{M_1}}{e^{M_1} (2v_0)^{2M_1 - M_2} (2\xi)^{M_2}} e^{-2\pi i \mathcal{B}_1} \right) - \epsilon_3 \ln \left(\frac{\epsilon_3^{M_3 - M_2} (2\xi)^{M_2}}{e^{M_3 - M_2} (2v_0)^{2M_3 - M_2}} e^{-2\pi i \mathcal{B}_3} \right). \quad (7.69)$$

This flux plus branes configuration should correspond to the mesonic branch of the gauge theory $(M_1, M_2, M_3, 0)$ in the regime of section 7.4.1. In order to compare this superpotential to the gauge theory result, we need to find the correct gauge/gravity dictionary. Let us identify as usual the cutoff of the B-cycle with the UV cutoff in the field theory, so that we have

$$\frac{1}{\alpha'^3} (2v_0)^2 = \mu_0^3 \quad \frac{1}{\alpha'^3} \epsilon_{1,3} = S_{1,3}, \quad (7.70)$$

Naturally, μ_0 is the UV scale at which we define the gauge theory, while S_1 and S_3 are the gaugino condensates of the first and third node of the quiver. We also know from the gauge theory analysis that the eigenvalues n_i of the meson matrix \mathcal{N} are to be identified with the coordinates $z_2^{(i)}$ on the p -line of singularities. More precisely

$$n_i \propto z_2^{(i)} = \xi_i + \sqrt{\xi_i^2 + \epsilon} \approx 2\xi_i, \quad \text{for } |\xi_i|^2 \gg |\epsilon_{1,3}|, \quad (7.71)$$

taking the root close to ξ_i . Equating the dimensionless ratios $\xi/v_0 = n_i/\mu_0^2$ on both sides of the correspondence, we find the relation

$$\frac{1}{\alpha'^{3/2}} 2\xi_i = \frac{n_i}{\mu_0^{1/2}}. \quad (7.72)$$

We still have to relate the B-periods of G_3 (7.68) to gauge theory quantities. This is the most subtle part, since these periods are not topological, but instead depend crucially on the boundary conditions at infinity (and hence on the bare Lagrangian of the field theory). By the non-renormalisation theorem, we know that W should not depend on the cut-off. Imposing $\mu_0 \frac{\partial W}{\partial \mu_0} = 0$ gives us the following two conditions

$$-2\pi i \frac{\partial \mathcal{B}_1}{\partial \ln \mu_0} = 3M_1 - 2M_2, \quad -2\pi i \frac{\partial \mathcal{B}_3}{\partial \ln \mu_0} = 3M_3 - M_2. \quad (7.73)$$

In the particular case $M_1 = 0$, $S_1 = 0$, only the second condition has to be imposed. Then, it is easy to see that

$$\mathcal{B}_3 \equiv \tau_0^{(3)} \quad (7.74)$$

should be identified with the UV value of the holomorphic coupling of the third node, which provides the correct beta function. We have then reproduced the effective superpotential for the $(0, M_2, M_3, 0)$ quiver, where the second node is treated as a flavor group

$$W = -S_3 \ln \left(\frac{S_3^{M_3 - M_2} \det \mathcal{N}}{e^{M_3 - M_2} \mu_0^{3M_3 - M_2} e^{2\pi i \tau_0^{(3)}}} \right). \quad (7.75)$$

In the general case, $M_1 \neq 0$, in order to satisfy the relations (7.73) we get for \mathcal{B}_1 the identification

$$\mathcal{B}_1 = \tau_0^{(1)} + \frac{M_2}{2\pi i} \ln(\mu_0 \lambda), \quad (7.76)$$

where $1/\lambda$ is some scale, independent of μ_0 , that we will identify with the inverse of the tree level quartic coupling in the gauge theory.

Defining the usual holomorphic SQCD scales

$$\Lambda_1^{3M_1-M_2} = \mu_0^{3M_1-M_2} e^{2\pi i \tau_0^{(1)}}, \quad \Lambda_3^{3M_3-M_2} = \mu_0^{3M_3-M_2} e^{2\pi i \tau_0^{(3)}}, \quad (7.77)$$

we then find the following superpotential

$$W = M_1 S_1 - S_1 \ln \left(\frac{S_1^{M_1-M_2}}{\Lambda_1^{3M_1-M_2}} \frac{S_1^{M_2}}{\lambda^{M_2} \det \mathcal{N}} \right) + (M_3 - M_2) S_3 - S_3 \ln \left(\frac{S_3^{M_3-M_2} \det \mathcal{N}}{\Lambda_3^{3M_3-M_2}} \right). \quad (7.78)$$

This superpotential is precisely equal to the gauge theory result (7.55), provided the first F-flatness condition of (7.56) is imposed. This field theory constraint has a technical counterpart in our analysis: in supergravity we need to assume that a geometric transition has taken place, so that we have a smooth geometry. Hence, $\mathcal{C}_2 = \mathcal{C}_4$ and the p - and q -lines meet smoothly, so there is only one type of wrapped D5-brane to consider. This is why we only dealt with one single brane position ξ while there are two different mesons \mathcal{M} and \mathcal{N} in the gauge theory.

7.4.3 IR regime and singularities resolution

Let us now investigate how the backgrounds discussed in section 3 must be modified at small radii in order to take into account the non-trivial IR dynamics of the full physical quiver gauge theory. As already mentioned previously, the dynamical scales at low energies correspond to different dimensionful quantities in the supergravity solution, depending on the qualitative dynamics of the relevant node. For nodes 1 and 3, whose low-energy dynamics is $\mathcal{N} = 1$, the scales Λ_1, Λ_3 are related to the deformation parameters of the geometry ϵ_1, ϵ_3 . For node 2, which leads essentially to $\mathcal{N} = 2$ dynamics, the scale Λ_2 is related to the enhançon radius ρ_c at which a probe $\mathcal{N} = 2$ fractional brane becomes tensionless; ρ_c is related to the twisted flux terms in (7.9).

Let us first briefly consider the regime where the dominant IR dynamics is $\mathcal{N} = 2$, that is when $\Lambda_2 \gg \Lambda_1, \Lambda_3$. This translates in supergravity in a hierarchy where the length scale defined by ρ_c is much larger than the length scales defined by $\epsilon_1 = \epsilon_3$ (recall that the two deformation parameters must be equal if there are BPS $\mathcal{N} = 2$ fractional branes around). Since the enhançon radius effectively cloaks the singularity, length scales smaller than ρ_c are not accessible any more. Hence, the geometry which can be probed is always at length scales for which the deformation is negligible. We thus conclude that in this regime the UV solution of section 3 is a very good approximation even as far as the IR behavior is concerned. Of course, the low-energy dynamics is $\mathcal{N} = 2$ in this case and the gravity dual description of it has the usual drawback of being essentially singular.

We now consider the richer case of the opposite regime, when $\Lambda_1, \Lambda_3 \gg \Lambda_2$ and the dominant IR dynamics is confining. Here we expect to be able to probe length scales where the deformation drastically changes the underlying geometry.

There is actually a simple way to approach this problem. One can have BPS $\mathcal{N} = 2$ fractional branes in the deformed geometry only when the two deformation parameters are equal and the geometry is given by

$$(z_1 z_2 - \epsilon)^2 = xy . \quad (7.79)$$

As remarked in [182], this can obviously be seen as the orbifold of the deformed conifold

$$z_1 z_2 - \epsilon = z_3 z_4 \quad (7.80)$$

under $\Theta : (z_1, z_2, z_3, z_4) \rightarrow (z_1, z_2, -z_3, -z_4)$. There is a single singularity line along $z_3 = z_4 = 0$, $z_1 z_2 = \epsilon$.

We can relate this complex form of the embedding to the real coordinates on the deformed conifold as follows

$$\begin{aligned} z_1 &= \sqrt{\epsilon} e^{\frac{i}{2}(\phi_1 + \phi_2)} \left\{ \sin \frac{\theta_1}{2} \sin \frac{\theta_2}{2} e^{(\tau + i\psi)/2} + \cos \frac{\theta_1}{2} \cos \frac{\theta_2}{2} e^{-(\tau + i\psi)/2} \right\} \\ z_2 &= \sqrt{\epsilon} e^{-\frac{i}{2}(\phi_1 + \phi_2)} \left\{ \cos \frac{\theta_1}{2} \cos \frac{\theta_2}{2} e^{(\tau + i\psi)/2} + \sin \frac{\theta_1}{2} \sin \frac{\theta_2}{2} e^{-(\tau + i\psi)/2} \right\} \\ z_3 &= \sqrt{\epsilon} e^{-\frac{i}{2}(\phi_1 - \phi_2)} \left\{ \cos \frac{\theta_1}{2} \sin \frac{\theta_2}{2} e^{(\tau + i\psi)/2} - \sin \frac{\theta_1}{2} \cos \frac{\theta_2}{2} e^{-(\tau + i\psi)/2} \right\} \\ z_4 &= \sqrt{\epsilon} e^{\frac{i}{2}(\phi_1 - \phi_2)} \left\{ \sin \frac{\theta_1}{2} \cos \frac{\theta_2}{2} e^{(\tau + i\psi)/2} - \cos \frac{\theta_1}{2} \sin \frac{\theta_2}{2} e^{-(\tau + i\psi)/2} \right\} . \end{aligned} \quad (7.81)$$

Note that τ is a dimensionless radial coordinate, and that for τ large $\epsilon e^\tau \rightarrow r^3$, we asymptote to the singular conifold described in (F.2-F.5). We refer to appendix F.1 for the notation used hereafter.

The Calabi-Yau metric on the deformed conifold reads

$$\begin{aligned} ds_6^2 &= \frac{2^{\frac{2}{3}}}{3} \epsilon^{2/3} K(\tau) \left[\frac{1}{3K^3(\tau)} (d\tau^2 + \zeta^2) + \frac{1}{2} \sinh^2 \frac{\tau}{2} ((\sigma_1 - \Sigma_1)^2 + (\sigma_2 - \Sigma_2)^2) \right. \\ &\quad \left. + \frac{1}{2} \cosh^2 \frac{\tau}{2} ((\sigma_1 + \Sigma_1)^2 + (\sigma_2 + \Sigma_2)^2) \right] , \end{aligned} \quad (7.82)$$

with

$$K(\tau) = \frac{(\sinh \tau \cosh \tau - \tau)^{1/3}}{\sinh \tau} . \quad (7.83)$$

The orbifold action is $(\phi_1, \phi_2) \rightarrow (\phi_1 - \pi, \phi_2 + \pi)$, like in the singular case. The fixed line at $z_3 = z_4 = 0$ is described by two halves: $p = \{\theta_1 = \theta_2 = 0\}$ and $q = \{\theta_1 = \theta_2 = \pi\}$. We have

$$\begin{aligned} z_1 \Big|_p &= \sqrt{\epsilon} e^{-(\tau + i\psi')/2} & z_2 \Big|_p &= \sqrt{\epsilon} e^{(\tau + i\psi')/2} \\ z_1 \Big|_q &= \sqrt{\epsilon} e^{(\tau + i\psi'')/2} & z_2 \Big|_q &= \sqrt{\epsilon} e^{-(\tau + i\psi'')/2} \end{aligned} \quad (7.84)$$

with $\psi' = \psi - \phi_1 - \phi_2$ and $\psi'' = \psi + \phi_1 + \phi_2$. This line is completely smooth now: the p - and q -lines are glued together at $\tau = 0$, with the identification $\psi' = -\psi''$. The full submanifold can alternatively be described with a single patch, by extending the domain

of τ to $-\infty < \tau < +\infty$ and using, say, only ψ' . With this observation in mind, the metric on the singularity line is

$$ds^2 = \frac{2^{\frac{2}{3}} \epsilon^{2/3}}{9K^2(\tau)} (d\tau^2 + d\psi'^2) . \quad (7.85)$$

It is a cylinder, on which we can introduce the complex coordinate $w = \tau + i\psi'$. We can construct the following 1-form on the line

$$\gamma = \frac{dz_2}{z_2} \Big|_p \equiv d \log z_2 = \frac{1}{2}(d\tau + i d\psi') = \frac{1}{2}dw . \quad (7.86)$$

Consider now a SUSY preserving ansatz similar to (7.7), but with a warped deformed conifold metric

$$ds_{10}^2 = h^{-1/2} dx_{3,1}^2 + h^{1/2} ds_6^2 . \quad (7.87)$$

The untwisted G_3 will be as in [118], and the twisted part will get contribution by $\mathcal{N} = 2$ branes and by deformation branes, generically. It can be written as

$$\begin{aligned} G_3 = & \frac{\alpha'}{2} g_s (-M_1 + M_2 - M_3) \left[\omega_3^{KS} - \frac{i}{g_s} dB_2^{KS} \right] \\ & - 2\pi i \alpha' g_s (M_1 + M_2 - M_3) d \log z_2 \wedge \omega_2 + 4\pi i \alpha' g_s \sum_{j=0}^{M_2} d \log(z_2 - z_2^{(j)}) \wedge \omega_2 \end{aligned} \quad (7.88)$$

where ω_3^{KS} and dB_2^{KS} are the ones of [118]. In particular $d\omega_3^{KS} = 0$ and $\int_{A_{CF}} \omega_3^{KS} = 8\pi^2$. Instead ω_2 is the anti-self-dual form at the orbifold point, normalised such that $\int_{\mathcal{C}_2} \omega_2 = 1$. Moreover, $z_2^{(j)}$ are the positions of the M fractional branes on the z_2 plane. We get

$$\begin{aligned} -\frac{1}{4\pi^2 \alpha' g_s} \int_{A_{CF}} F_3 &= M_1 - M_2 + M_3 \\ -\frac{1}{4\pi^2 \alpha' g_s} \int_{A_2} F_3 &= -M_1 + M_2 + M_3 \\ -\frac{1}{4\pi^2 \alpha' g_s} \int_{A_4} F_3 &= M_1 + M_2 - M_3 , \end{aligned} \quad (7.89)$$

which exactly match those of the UV solution. These integrals are easily performed by noticing that, in A_2 , the circle on the p line at infinity is around $z_2 = \infty$, while in $A_4 \equiv -A_2$ the circle on the q line at infinity is around $z_2 = 0$.¹² The M_2 sources provide for the difference between $\int_{A_2} F_3$ and $-\int_{A_4} F_3$.

We can consider a simpler configuration, where the $\mathcal{N} = 2$ fractional branes are located at $\tau = \tau_0$ and are smeared on the circle parametrized by ψ' . We then consider

$$\sum_{j=0}^{M_2} d \log(z_2 - z_2^{(j)}) \rightarrow \frac{M_2}{2\pi i} \oint \frac{dz_0}{z_0} d \log(z_2 - z_0) , \quad (7.90)$$

with $z_0 = \sqrt{\epsilon} e^{\frac{1}{2}(\tau_0 + \psi'_0)}$, and the integrand is a differential in z_2 . The integral is thus performed at fixed τ_0 . It is easy to see that

$$\frac{1}{2\pi i} \oint \frac{dz_0}{z_0} d \log(z_2 - z_0) = \frac{dz_2}{z_2} \frac{1}{2\pi i} \oint dz_0 \left(\frac{1}{z_0} - \frac{1}{z_0 - z_2} \right) . \quad (7.91)$$

¹²Notice also that $A_2 \cong \mathcal{C}_2 \times \psi'$ while $A_4 \cong \mathcal{C}_4 \times \psi'' = -\mathcal{C}_2 \times \psi'$.

The integral is vanishing if $|z_2| < |z_0|$ (that is $\tau < \tau_0$), while it is unity if $|z_2| > |z_0|$, which is $\tau > \tau_0$. Hence, if we take the branes to be smeared along the $\tau = 0$ circle, the 3-form flux reads

$$G_3 = \frac{\alpha'}{2} g_s (-M_1 + M_2 - M_3) \left[\omega_3^{KS} - \frac{i}{g_s} dB_2^{KS} \right] - \pi i \alpha' g_s \left[(M_1 + M_2 - M_3) - 2M_2 \Theta(\tau) \right] dw \wedge \omega_2, \quad (7.92)$$

where Θ is the Heaviside step function. It is straightforward to see that the twisted part of the 3-form flux we get here is exactly equal to the one of the singular conifold case (7.9).

The warp factor equation reads

$$\Delta h = - *_6 (H_3 \wedge F_3) - \frac{M_2}{2} (4\pi^2 \alpha')^2 g_s *_6 \delta_6. \quad (7.93)$$

We have included an explicit source term because in this case the source branes are located at an otherwise smooth point of the geometry. As in the singular case, the twisted and untwisted 3-form terms do not mix, and we can write the above equation in a way much similar to the one appearing in (7.18). There will be a first, completely smooth term on the r.h.s. coming from $*_6(H_3^{KS} \wedge F_3^{KS})$. The terms coming from the twisted flux will be similar to the ones in (7.18), with a τ -dependent prefactor. Eventually, the term coming from the explicit source term will contain a $\delta(\tau)$. Of course, the warp factor will be a sum of the particular inhomogeneous solutions of the Laplace equation with the various source terms. For instance, there will be a first piece which will be given by $h^{KS}(\tau)$. The other pieces will necessarily involve a dependence on the other coordinates. Because of the smearing, we can consider an ansatz for h which does not depend on ϕ_i . However as we will see instantly, we will have to keep explicit ψ dependence in h .¹³

The Laplacian on the deformed conifold for $h(\tau, \psi, \theta_1, \theta_2)$ reads (see also the appendix of [133])

$$\begin{aligned} \frac{2^{\frac{2}{3}} \epsilon^{\frac{2}{3}}}{3} \Delta h = & \frac{3}{\sinh^2 \tau} \partial_\tau (K^2 \sinh^2 \tau \partial_\tau h) + 6K^2 \partial_\psi^2 h \\ & + \frac{2 \cosh \tau}{K \sinh^2 \tau} \left(\partial_1^2 h + \cot \theta_1 \partial_1 h + \cot^2 \theta_1 \partial_\psi^2 h + \partial_2^2 h + \cot \theta_2 \partial_2 h + \cot^2 \theta_2 \partial_\psi^2 h \right) \\ & + \frac{4}{K \sinh^2 \tau} \left[\cos \psi (\cot \theta_1 \cot \theta_2 \partial_\psi^2 h - \partial_1 \partial_2 h) + \sin \psi (\cot \theta_1 \partial_1 \partial_\psi h + \cot \theta_2 \partial_\psi \partial_2 h) \right]. \end{aligned} \quad (7.94)$$

We see that the angular operator on the third line has explicit dependence on ψ . A solution of the Laplace equation independent on ψ must then be also independent of θ_1 and θ_2 , which is not consistent with the functional dependence of the source terms. Hence we are forced to consider a ψ dependent warp function.

We can now view the Laplace operator on h as a sum (weighted by functions of τ) of angular operators, which can be thought of as acting on the variables defining the 5-dimensional space $T^{1,1}$. The angular operators appearing in the first two lines

¹³This is because ∂_ψ does not generate an isometry of the deformed conifold. Hence smearing the sources along ψ does not help.

are actually the three angular operators which define the Laplacian on $T^{1,1}$, ∂_ψ^2 and $(\partial_i^2 + \cot \theta_i \partial_i + \cot^2 \theta_i \partial_\psi^2)$ for $i = 1, 2$, when they act on functions which do not depend on the ϕ_i angles. We can thus find a complete basis of functions on $T^{1,1}$ which are simultaneously eigenfunctions of these three operators.

In the deformed conifold however, we also have the additional angular operator on the third line of eq. (7.94). This operator will inevitably mix eigenfunctions of the previous three operators, hence making the problem of finding solutions to the Laplace equation a problem of solving an (infinite) system of ordinary differential equations.

Going over this analysis, even qualitatively or numerically, is obviously beyond the scope of the present work. The main reason is that locally, the solution for the warp factor will again look like the one for $\mathcal{N} = 2$ fractional branes at a $\mathbb{C}^2/\mathbb{Z}_2$ singularity, with its enhançon-like singular behavior. Hence the deep IR region has the difficulties common to the other $\mathcal{N} = 2$ gravity duals. Nevertheless, it could be interesting to go further along the analysis of the IR region of this configuration.

Let us now end this section with a very short remark on a particular case, which is the one occurring when $M_2 = 0$. From the gauge theory point of view, we expect a completely regular geometry

$$(z_1 z_2 - \epsilon_1)(z_1 z_2 - \epsilon_3) = xy . \quad (7.95)$$

In particular, this geometry no longer possesses lines of A_1 -singularities. However, from the UV expression for the 3-form fluxes (7.9) or (7.88), it seems that when $M_1 \neq M_3$ there is still a twisted piece. This cannot be completely correct of course. The $\epsilon_1 = \epsilon_3$ geometry is locally a $\mathbb{C}^2/\mathbb{Z}_2$ fibration over the fixed line (topologically a cylinder). When turning on different deformations $\epsilon_1 \neq \epsilon_3$, the $\mathbb{C}^2/\mathbb{Z}_2$ singularity is blown-up fiberwise, with a base-dependent volume of the blown-up 2-cycle. In particular its volume is a τ -dependent parameter $a(\tau)$ such that $a \rightarrow 0$ when $\tau \rightarrow \pm\infty$, while it reaches a maximum around $\tau = 0$. The 3-form can be constructed from the ASD 2-form on the ALE space which is the blow-up of $\mathbb{C}^2/\mathbb{Z}_2$, and is therefore completely smooth in the bulk of the geometry. However it asymptotes a δ -function behaviour for large radii, i.e. in the UV region. Hence, there is no contradiction in the fact that the UV solution displays twisted flux also when there is no real orbifold fixed line.

7.5 Conclusions

In this chapter we presented a supergravity solution which describes fractional branes at the orbifolded conifold. The input is essentially given by the geometry probed by the branes and its possible deformations, together with the RR 3-form fluxes sourced by the fractional branes. The output can be summarized in the NSNS 3-form flux and the warp factor, which should thus shed light on the characteristics of the dual gauge theory which are not directly related to the holomorphic sector.

We have performed some non-trivial checks both on the UV behavior of the NSNS flux, matching with a cascading interpretation of the RG-flow of the gauge theory, and on the IR low-energy theory by matching the effective superpotentials. The latter check of course only concerns the holomorphic sector, but clarifies the IR effects that the fractional branes have on the geometry.

Part II

Chern-Simons quivers and M-theory

Chapter 8

AdS_4/CFT_3 and the quest for a theory of multiple M2-branes

M-THEORY is the strong coupling limit of type IIA string theory, and it is also linked to the other string theories by various dualities. Since its discovery [33, 34], it has remained rather clouded in mystery. We know that it has 11 dimensional supergravity as its low energy limit, and we know that it contains extended objects preserving half of the supersymmetry, as we recalled in Chapter 2.

As we will review, there is a natural Maldacena limit that we can take on a stack of M2-branes [9]. The gravity dual is an AdS_4 background in M-theory. Until recently, almost nothing was known about the dual CFT, which should arise as the low energy theory on M2-branes. In the supergravity limit, we can compute correlators in the dual conformal theory (and make prediction about the dual theory when it is not known), using the general AdS/CFT techniques reviewed in Chapter 4, section 4.3.3. The quest for an explicit description of the dual CFT will be the subject of this second part of the thesis.

8.1 M2-brane solution in eleven dimensional supergravity

The bosonic action for eleven-dimensional supergravity is

$$S = \frac{1}{(2\pi)^8 l_p^9} \left\{ \int d^{11}x \sqrt{-G_{11}} \left(R - \frac{1}{2} |G_4|^2 \right) - \frac{1}{2} \int A_3 \wedge G_4 \wedge G_4 \right\}, \quad (8.1)$$

with l_p the 11d Planck length. The bosonic equations of motion are the Einstein equations and the equation for the 4-form,

$$d * G_4 + \frac{1}{2} G_4 \wedge G_4 = -(2\pi)^8 l_p^9 \frac{\delta S_{loc}}{\delta A_3}, \quad (8.2)$$

where we allowed for the possibility of localized sources coupling electrically to A_3 , which are M2-branes with minimal coupling

$$S_{M2} \supset \tau_{M2} \int A_3. \quad (8.3)$$

We are mainly interested in solutions corresponding to M2-branes at singularities [196]. We would like to place M2-branes at the tip of a 8-dimensional cone $C(X_7)$, with X_7 the 7-dimensional base. The ansatz is

$$ds^2 = h^{-\frac{2}{3}} dx^\mu dx_\mu + h^{\frac{1}{3}} (dr^2 + r^2 ds^2(X_7)), \quad (8.4)$$

$$G_4 = dx^0 \wedge dx^1 \wedge dx^3 \wedge dh^{-1}, \quad (8.5)$$

where x^μ are the 2+1 coordinates along the flat M2-brane worldvolume, and the warp factor h is a function on $C(X_7)$. The equation (8.2) reduces to a Poisson equation on the cone,

$$\Delta h = -(2\pi l_p)^6 \sum_i^N \delta_{M2}^{(i)}, \quad (8.6)$$

where the RHS corresponds to the M2-brane sources. Let us consider a stack of N M2-branes at the tip. We then have

$$h(r) = 1 + \frac{R^6}{r^6} \quad \text{with} \quad R^6 = N \frac{(2\pi l_p)^6}{6\text{Vol}(X_7)}. \quad (8.7)$$

This solution is an extremal black brane solution similar to the ones which exist for D-branes, with a (non-singular) horizon at $r = 0$. The number of preserved supersymmetries will depend on whether the cone $C(X_7)$ admits globally defined spinors, implying reduced holonomy [197, 196]. To preserve \mathcal{N} supersymmetries in 2+1 dimensions¹, $C(X_7)$ should have a reduced holonomy group

$$Spin(8 - \mathcal{N}) \subset Spin(8). \quad (8.8)$$

We will be interested in the case of $\mathcal{N} \geq 2$ SUSY. For $\mathcal{N} = 2$, the cone is Calabi-Yau ($Spin(6) \cong SU(4)$ holonomy) and the base X_7 is a Sasaki-Einstein manifold². For $\mathcal{N} = 3$ the cone is hyper-Kähler, and the base is 3-Sasakian. The maximally supersymmetric case, $\mathcal{N} = 8$, is obviously flat space, \mathbb{R}^8 , which is a cone over the 7-sphere S^7 . Solutions with $\mathcal{N} \geq 4$ always involve quotients of S^7 .

In the near horizon limit, we obtain a source-less solution (often called Freund-Rubin [198] solution)

$$ds^2 = \left(\frac{r}{R}\right)^4 dx^\mu dx_\mu + R^2 \frac{dr^2}{r^2} + R^2 ds^2(X_7), \quad (8.9)$$

$$G_4 = \frac{6r^5}{R^6} dx^0 \wedge dx^1 \wedge dx^3 \wedge dr, \quad (8.10)$$

The geometry is $AdS_4 \times X_7$, with N units of flux through X_7 ,

$$\frac{1}{(2\pi l_p)^6} \int_{X_7} *G_4 = N. \quad (8.11)$$

¹We will review 3d SUSY in the next chapter. \mathcal{N} SUSY means that there are $2\mathcal{N}$ supercharges. The formula (8.8) is just a convenient way to summarize results which must be derived for each \mathcal{N} .

²Cfr. section 4.4 in Chapter 4.

It is often convenient to work with different coordinates. If we define the dimensionless coordinates $z = (R/r)^2$ and $y = 2x/R$, we can express the solution in term of the “unit AdS_4 ” space

$$ds^2(AdS_4) = \frac{dy^\mu dy_\mu + dz^2}{z^2} \quad \text{with} \quad d\text{vol}(AdS_4) = \frac{dy^0 \wedge dy^1 \wedge dy^2 \wedge dz}{z^2}. \quad (8.12)$$

We have

$$ds^2 = R^2 \left(\frac{1}{4} ds^2(AdS_4) + ds^2(X_7) \right), \quad (8.13)$$

$$G_4 = -\frac{3}{8} R^3 d\text{vol}(AdS_4), \quad (8.14)$$

Remark that the radius of the internal manifold X_7 is twice the radius of AdS_4 .

Since the near horizon geometry has an AdS factor, we have a doubling of supercharges, which should correspond to the superconformal charges in a dual three dimensional SCFT. The full group of isometries of the solution contains a supergroup called $OSp(\mathcal{N}|4)$. We have

$$OSp(\mathcal{N}|4) \times G \supset SO(3,2) \times SO(\mathcal{N})_R \times G, \quad (8.15)$$

where $SO(\mathcal{N})_R \times G$ is the isometry group of X_7 .

8.1.1 Energy/radius relation

To discuss the energy/radius relation in the AdS_4/CFT_3 correspondence, we find it convenient to consider the metric (8.9), because the radial direction r of the cone $C(X_7)$ is the most natural coordinate in the field theory context. In these coordinate, the dilatation symmetry acts as

$$x^\mu \rightarrow \lambda x^\mu, \quad r \rightarrow \frac{1}{\lambda^{\frac{1}{2}}} r. \quad (8.16)$$

We see that r scales like the square root of the energy. We also have that $\sqrt{g_{00}} = (r/R)^2$, so that energies are redshifted as r^2 . We must therefore have an energy/radius relation of the form

$$E_{CFT} \sim \frac{r^2}{l_p^3}. \quad (8.17)$$

This also matches with the fact that scalar fields have classical dimension $\frac{1}{2}$ in 2+1 dimensions.

8.1.2 Type IIA reduction

It will be important later on to have some basic understanding of the type IIA reduction. Let us choose a direction parametrized by an angle $\psi \sim \psi + 2\pi$, along which to reduce the eleven dimensional metric to ten dimension. The reduction ansatz is

$$ds_{11d}^2 = e^{-\frac{2\Phi}{3}} ds_{IIA}^2 + e^{\frac{4\Phi}{3}} \left(C_1 + l_p g_s^{\frac{2}{3}} d\psi \right)^2, \quad (8.18)$$

$$G_4 = F_4 + g_s^{\frac{2}{3}} l_p H_3 \wedge d\psi, \quad (8.19)$$

Here Φ is identified with the fluctuating part of the type IIA dilaton, and we have included a factor of $g_s = e^{\Phi_0}$, for later convenience. The metric decomposition (8.18) describes the 11d space-time as a circle fibration over a 10d space-time. The coordinate radius of the circle is

$$R_{10} = g_s^{\frac{2}{3}} l_p. \quad (8.20)$$

The connection C_1 is the RR 1-form potential of type IIA supergravity, with curvature (field strength) $F_2 = dC_1$. Plugging this reduction ansatz into the action (8.1) we obtain the type IIA bosonic action [14],

$$S_{IIA} = \frac{1}{2\kappa^2} \int d^{10}x \sqrt{-G_s} e^{-2\Phi} R + \frac{1}{2\kappa^2} \int \left\{ e^{-2\Phi} 4d\Phi \wedge *d\Phi - \frac{1}{2} F_2 \wedge *F_2 \right. \\ \left. - e^{-2\Phi} \frac{1}{2} H_3 \wedge *H_3 - \frac{1}{2} \tilde{F}_4 \wedge *\tilde{F}_4 \right. \\ \left. - \frac{1}{2} B_2 \wedge F_4 \wedge F_4 \right\}, \quad (8.21)$$

where the three lines correspond to the three terms in (8.1). We have defined $\tilde{F}_4 = F_4 - H_3 \wedge C_1$, and $2\kappa^2 = (2\pi)^7 l_p^8 / g_s^{2/3}$. Note that although the action (8.21) is written in the string frame, the 10d Newton constant is the physical one. Comparing with (2.4), we find that

$$l_p = \sqrt{\alpha'} g_s^{\frac{1}{3}}. \quad (8.22)$$

For an $AdS_4 \times X_7$ background, we choose the M-theory circle to correspond to some circle fiber in X_7 , which mean that we see X_7 as a fiber bundle with base M_6 ,

$$S^1 \rightarrow X_7 \xrightarrow{\pi} M_6. \quad (8.23)$$

The fibration might be singular (if the circle degenerates somewhere), in which case there might be additional objects in the type IIA reduction. We will discuss this possibility later. In general, let us suppose we can write the metric on X_7 as ³

$$ds^2(X_7) = ds^2(M_6) + \frac{w}{k^2} (kP + d\psi)^2 \quad (8.24)$$

with ψ an angle of period 2π , w some function of the coordinates on M_6 , and k a parameter which we can tune in order to make the M-theory circle arbitrarily small. It follows from (8.18)-(8.19) that the type IIA background in the string frame is (recall that $l_p = \sqrt{\alpha'} g_s^{1/3}$)

$$ds_{IIA}^2 = \frac{\sqrt{w}}{kg_s \sqrt{\alpha'}} R^3 \left(\frac{1}{4} ds^2(AdS_4) + ds^2(M_6) \right), \quad (8.25)$$

$$e^\Phi = w^{\frac{3}{4}} \left(\frac{R}{kg_s \sqrt{\alpha'}} \right)^{\frac{3}{2}}, \quad F_2 = kg_s \sqrt{\alpha'} dP, \quad F_4 = -\frac{3}{8} R^3 d\text{vol}(AdS_4) \quad (8.26)$$

Remark that when w is non-trivial, we have both a dilaton profile and a non-trivial warping of the AdS_4 space.

³Here I follow the notation of [199].

8.2 AdS_4/CFT_3 : the AdS side

Having understood the general case of M2-branes at singularities from the supergravity point of view, we would like to know whether we can find the dual superconformal field theories. Such theories are quite elusive. One obvious difference from the AdS_5/CFT_4 case is that the theory dual to $AdS_4 \times X_7$ has typically no weakly coupled limit, because there is no dimensionless parameter in M-theory. *A priori*, the only clutch we have is the relation

$$\frac{R}{l_p} \sim N^{1/6}, \quad (8.27)$$

which means that the 11 dimensional supergravity approximation is valid in the large N limit, N being the number of M2-branes.

Let us consider the maximally supersymmetric case of M2-branes on flat space. The associated $AdS_4 \times S^7$ background has been studied in detail quite a while ago [200, 201]. The supergravity fluctuations organize into short superconformal multiplets, very similarly to the $AdS_5 \times S^5$ case. Such BPS excitations should correspond to chiral primary operators in the dual SCFT.

Some further insight can be uncovered by considering the M-theory/type IIA relationship⁴. In particular, it is instructive to consider the sphere S^7 as a Hopf fibration over CP^3 . For $M_6 = CP^3$, $w = 1$ and $k \in \mathbb{Z}$, the metric (8.24) describes a smooth quotient space S^7/\mathbb{Z}_k ,

$$S^1 \rightarrow S^7/\mathbb{Z}_k \xrightarrow{\pi} CP^3. \quad (8.28)$$

In the type IIA reduction F_2 carries k units of flux along the $CP^1 \subset CP^3$ [202]. We also have $R = (2\pi k N g_s^2)^{1/6} \sqrt{\alpha'}$, and

$$e^{2(\Phi+\Phi_0)} = \sqrt{2}\pi \frac{N^{1/2}}{k^{5/2}}. \quad (8.29)$$

Note that we have reabsorbed the constant $g_s = e^{\Phi_0}$ defined in (8.18) into the full dilaton, which is the actual type IIA string coupling. We see that perturbative string theory is valid when

$$k \gg N^{1/5}, \quad (8.30)$$

The metric is

$$ds_{IIA}^2 = R_s^2 \left(\frac{1}{4} ds^2(AdS_4) + ds^2(CP^3) \right), \quad \text{with} \quad R_s^2 = \sqrt{2}\pi \sqrt{\frac{N}{k}} \alpha', \quad (8.31)$$

R_s is the curvature radius in 10d, and type IIA supergravity is a good approximation as long as

$$k \ll N \ll k^5. \quad (8.32)$$

We will see in the next chapters examples of dual conformal field theories where the parameter N/k is the analog of a 't Hooft coupling, which can be perturbative when $k \gg N$.

⁴This is a key point to understand the ABJM theory [202], which we will review in Chapter 11.

8.3 SCFT on M2-branes?

We would like to shed some light on the low energy theory living on a stack of M2-branes, which we expect to be an interacting superconformal theory. At first sight we have no good tools to tackle the construction of this theory beyond $N = 1$. For a single M2-brane, the M2-brane action can be written as a standard Nambu-Goto action (for the bosonic part),

$$S_{M2} = -\tau_{M2} \int d^3\xi \sqrt{G_{11}} + \tau_{M2} \int A_3, \quad (8.33)$$

where the background fields are pulled-back. The worldvolume bosonic fields are just 8 scalars ϕ^a , corresponding to the 8 transverse directions. We usually write the action in the classically equivalent sigma-model form,

$$S_{M2} = -\frac{\tau_{M2}}{2} \int d^3\xi \sqrt{\gamma} \left(\gamma^{ij} \partial_i \phi^a \partial_j \phi^b G_{ab} - 1 \right) + \tau_{M2} \int A_3, \quad (8.34)$$

where γ^{ij} is an auxiliary worldvolume metric. The supersymmetric M2-brane action was constructed in the late 80's in [203, 204], and it explicitly preserves space-time supersymmetry, similarly to the Green-Schwarz superstring action. In flat space, we have a free theory with $\mathcal{N} = 8$ rigid supersymmetry [203], and the ϕ^a transform in the vector representation of a global $SO(8)$, corresponding to rotation in the transverse direction.

We are not interested in the M2-brane theory by itself, but rather in the low energy theory. For a single M2-brane at a smooth point in space, the low energy theory should just be a free theory. For $N > 1$, we are at a loss. In the case of D-branes in string theory, we know from perturbative string theory that the correct generalization from $N = 1$ to $N > 1$ involves taking into account the non-abelian degrees of freedom in the vector excitation of the open string. In the case of M2-branes, we do not have such understanding. Fundamental strings uplift to M2-brane tubes ending on open M2-branes and there is no perturbative way to deal directly with such an interacting system. What we can do, and what we will do, is to avoid these difficulties by using the various M-theory/string theory dualities.

8.3.1 M2- from D2-brane: dual photon

A first way to think about the problem is to use the simplest duality between M-theory and type IIA, obtained by compactifying flat space on a circle of radius $R_{10} = \sqrt{\alpha} g_s$, $\mathbb{R}^{1,9} \times S^1$. A stack of N M2-branes transverse to this circle (say along (x^0, x^1, x^2)) is dual to a stack of N D2-branes in flat space. We certainly know how to deal with such a system when g_s is small. The low energy theory is just a Yang-Mills theory with $U(N)$ gauge group, and with 7 scalars in the adjoint representation (and the associated fermions required by supersymmetry),

$$-\frac{1}{g_{YM}^2} \int d^3x \text{Tr} \left(\frac{1}{4} F_{\mu\nu} F^{\mu\nu} + \frac{1}{2} \sum_{i=1}^7 D_\mu \phi^i D^\mu \phi^i \right). \quad (8.35)$$

The problem is that the coupling constant g^2 has dimension of mass. This means that in three dimensions the Yang-Mills gauge coupling is classically *relevant*, because it runs

already at tree level. Simply, for the dimensionless coupling $\alpha \equiv g^2/\mu$

$$\frac{\partial \alpha}{\partial \ln \mu} = -\alpha + o(\alpha^2) \quad (8.36)$$

and the theory is asymptotically free. In the infrared, the theory is strongly coupled and we lose control. Note that we have

$$g_{YM}^2 = \frac{g_s}{\sqrt{\alpha'}} \quad (8.37)$$

so the theory becomes strongly coupled at scales $\mu \leq g_s/\sqrt{\alpha'}$. Taking into account the backreaction of the D2-branes, we have a curved background with a dilaton profile

$$e^\Phi \sim \frac{N^{\frac{1}{4}}}{r^{\frac{5}{4}}}, \quad (8.38)$$

where $r \rightarrow 0$ roughly corresponds to the IR limit in the field theory. We did not gain anything, since the string theory description breaks down near the branes, and one should really consider the M-theory description instead. Indeed, from the last relation it seems that the M-theory circle decompactifies as we go near the D2-branes.

In the case of a single D2-brane, there is an interesting way to see how the M-theory description can arise, in field theory terms. The bosonic content of the D2-brane theory consists of a photon and 7 free scalars. It is an interesting fact that a $U(1)$ connection is dual to a scalar in $2+1$ dimensions. Consider the action

$$S = -\frac{1}{2g^2} \int F \wedge *F - \frac{1}{2\pi} \int \varphi \wedge dF, \quad (8.39)$$

which consists of the Maxwell action together with a dimensionless auxiliary field φ which imposes the Bianchi identity $dF = 0$. In the absence of source, this theory is equivalent to the Maxwell theory formulated in terms of the vector potential. We just need to path integrate over φ and F instead of A ,

$$Z = \int \mathcal{D}F \mathcal{D}\varphi e^{iS[F,\varphi]}. \quad (8.40)$$

Integrating out F only, we find the dual action

$$S_{dual} = -\frac{1}{2} \frac{g^2}{(2\pi)^2} \int d\varphi \wedge *d\varphi = -\frac{1}{2g^2} \int d\phi \wedge *d\phi, \quad (8.41)$$

describing a free scalar field $\phi = g^2\varphi/2\pi$ of dimension one. Note that, because of its definition $d\phi = *F$, this scalar is only defined up to a constant shift. Moreover, this scalar is periodic of period g^2 ,

$$\phi \sim \phi + g^2 \quad (8.42)$$

or $\varphi \sim \varphi + 2\pi$ because of the flux quantization condition, $\int F \in 2\pi\mathbb{Z}$ around any 2-sphere. Hence an abelian gauge field is dual to a periodic free scalar with a shift symmetry. This scalar is often called the *dual photon*. At low energy, $\mu \ll g^2$, we can ignore the periodicity condition on ϕ . This is the field theory analog of the fact that the M-theory circle decompactifies near the D2-brane horizon. We can also easily perform the inverse duality from scalar to vector. If one does this carefully for the M2-brane action, one can recover the full D2-brane action [36].

8.3.2 Flavors and large N_f limit

As an aside, let us remark that for a generic Yang-Mills theory coupled to matter, we will have the following one-loop correction to the beta function,

$$\frac{\partial \alpha}{\partial \ln \mu} = -\alpha \left(1 + \frac{b_0}{4\pi} \alpha \right), \quad (8.43)$$

with b_0 a one-loop coefficient given by [205]

$$b_0 = \frac{23}{12} T(adj) - \frac{1}{6} \sum_{\text{fermion } i} T(r_i) - \frac{1}{12} \sum_{\text{scalar } j} T(r_j). \quad (8.44)$$

Here $T(r)$ are representation indices ($T(fund) = 1$ and $T(adj) = 2N$ for $SU(N)$). We see that b_0 is negative when there is a lot of matter (or in the abelian case), and there exists the interesting possibility of having a Wilson-Fisher fixed point [206] at one-loop, with $\alpha^* = 4\pi/|b_0|$. This fixed point is weakly coupled if we choose the field content appropriately.

For instance, consider a $\mathcal{N} = 2$ supersymmetric $U(N)$ theory with N_f chiral superfields in the fundamental and antifundamental representations (so called flavors). In that case, the one-loop coefficient is the same as the one loop coefficient in $\mathcal{N} = 1$ SQCD in 4d,

$$b_0 = 3N - N_f. \quad (8.45)$$

Such a system can be engineered by adding N_f D6-branes on top of the D2-branes, and one can then find a supergravity description as in [207]. It would be interesting to see whether one can find some explicit string theory description of such a fixed point. In this part of the thesis we will introduce field theory models which have similar properties, being expected to flow to a non-trivial fixed point which can be made weakly coupled when the number of flavors is large. For those models we will know which is the correct string theory dual.

In the case of $\mathcal{N} = 2$ supersymmetric $U(1)$ theories coupled to matter, we expect the existence of a SCFT for any number of matter fields. These theories can be investigated using 3d mirror symmetry [208]. This is a beautiful subject in its own, to which we will not do justice.

In the case of the $\mathcal{N} = 6$ theory on a stack of D2-branes, the one-loop coupling vanishes, $b_0 = 0$: the quantum effects do not alter the *classical* running of g_{YM} ⁵.

⁵At least at one-loop, but it is tempting to guess that this property survives at all order, similarly to the case of $\mathcal{N} = 4$ in 4d, which is T-dual. It would be something interesting to investigate, since in 3d there is no analog to the “exact” NSVZ beta function [209].

Chapter 9

Superconformal theories in three dimensions

IN the hope of giving a Lagrangian description of the low energy field theory living on a stack of M2-branes, we would like to construct supersymmetric theories which are classically conformal. A step in that direction was taken by Schwarz [210], building on earlier work [211, 212, 213]. The idea is to use supersymmetric Chern-Simons theories coupled to matter.

In this Chapter we review the field theory background of this second Part of the thesis. We review particle states and supersymmetry in 2+1 dimensions; in particular we explain in detail and fix our notation for the $\mathcal{N} = 2$ superspace formalism. Next we explain the construction of Chern-Simons (CS) theories and CS-matter theories, and comment on some quantum properties of such theories in the $\mathcal{N} = 2$ supersymmetric case.

9.1 Spinors in three dimensions and supersymmetry

We begin by fixing our conventions for spinors (for this we follow the Appendix of [214]), then we explain in some details the representation theory of the Poincaré and super-Poincaré algebra. In three dimensions with Minkowski metric $\eta^{\mu\nu} = \text{diag}(-++)$, the Lorentz group is $SO(2, 1)$ and the spin group is $SU(1, 1)$. The associated Dirac matrices are taken to satisfy

$$\{\gamma^\mu, \gamma^\nu\} = 2\eta^{\mu\nu}. \quad (9.1)$$

They can be chosen as $(\gamma^\mu)_\alpha^\beta = (i\sigma^2, \sigma^1, \sigma^3)$, with the spin index $\alpha, \beta = 1, 2$:

$$\gamma^0 = \begin{pmatrix} 0 & 1 \\ -1 & 0 \end{pmatrix}, \quad \gamma^1 = \begin{pmatrix} 0 & 1 \\ 1 & 0 \end{pmatrix}, \quad \gamma^2 = \begin{pmatrix} 1 & 0 \\ 0 & -1 \end{pmatrix}. \quad (9.2)$$

They satisfy

$$\gamma^\mu \gamma^\nu = \eta^{\mu\nu} + \epsilon^{\mu\nu\rho} \gamma_\rho, \quad \text{tr}(\gamma^\mu \gamma^\nu \gamma^\rho) = 2\epsilon^{\mu\nu\rho}. \quad (9.3)$$

The generators of the Dirac representation of the Lorentz group are obtained as

$$\Sigma^{\mu\nu} = \frac{i}{4}[\gamma^\mu, \gamma^\nu] = \frac{i}{2}\epsilon^{\mu\nu\rho} \gamma_\rho. \quad (9.4)$$

Dirac spinors (ψ_α) are then vector with two complex components transforming in the fundamental representation of $SU(1, 1)$ ¹. We can raise and lower spinor indices with $\epsilon^{\alpha\beta}$ and $\epsilon_{\alpha\beta}$ ($\epsilon^{12} = -\epsilon_{12} = 1$); in particular $(\gamma^\mu)_{\alpha\beta} = (-1, -\sigma^3, \sigma^1)$.

Remark that in three dimensions there is no equivalent of γ^5 (a matrix which anti-commutes with all the Dirac matrices), and consequently there is no notion of chirality and no projection to Weyl spinors. As in any dimension, there is a charge conjugation matrix C [215],

$$C = \sigma^2 = \begin{pmatrix} 0 & -i \\ i & 0 \end{pmatrix}, \quad C^t = -C, \quad (\gamma^\mu)^t = -C\gamma^\mu C^{-1}. \quad (9.5)$$

Note that $C = C^{-1} = C^\dagger$ too. We can impose a reality condition $\psi^* = -i\gamma^0 C \psi$, which is simply

$$\psi^* = \psi \quad (9.6)$$

in this representation with real Dirac matrices (9.2). The minimal spinor in 3 dimensions is this Majorana spinor, which has two real components. Let us also define the Dirac conjugate and the Majorana conjugate

$$\bar{\psi} = \psi^\dagger \gamma_0, \quad \bar{\psi}^M = -i\psi^t C. \quad (9.7)$$

Note that $\bar{\psi} = \bar{\psi}^M$ for a Majorana spinor. The free action for a fermion (whether Majorana or not) is then²

$$\mathcal{L} = -i\bar{\psi}\gamma^\mu\partial_\mu\psi - im\bar{\psi}\psi. \quad (9.8)$$

9.1.1 The parity symmetry in 2 + 1 dimensions

The discrete parity transformation P acts as a change of sign on one of the spatial coordinates, say $(x^0, x^1, x^2) \rightarrow (x^0, -x^1, x^2)$. Requiring that the Dirac kinetic term be invariant under P implies that fermions transform as

$$\psi \rightarrow \gamma^1 \psi. \quad (9.9)$$

This implies that a fermion mass term $im\bar{\psi}\psi$ is *odd* under parity. A gauge field transforms as

$$(A_0, A_1, A_2) \rightarrow (A_0, -A_1, A_2) \quad (9.10)$$

The standard Yang-Mills kinetic term is P -even, but the Chern-Simons term (to be reviewed below) is P -odd.

¹The group $SU(1, 1)$ is the group of 2×2 matrices U of unit determinant and such that $U^\dagger \omega U = \omega$, where we can take $\omega = \sigma^2 = \begin{pmatrix} 0 & -i \\ i & 0 \end{pmatrix}$. Since $U = e^{iT}$, the corresponding generators T are traceless and such that $T^\dagger = \omega T \omega$. This is the case of the $i\gamma^\mu$ appearing in (9.4).

²The i in the mass term is necessary to make it real, due to our conventions for γ -matrices. Cfr. the Insert 4 below.

9.1.2 Poincaré algebra

The representation theory of the Poincaré algebra is slightly unusual, so it might be a good idea to pause and consider it in some detail. In this subsection we follow [216].

In $2 + 1$ dimensions, we can dualize the Lorentz generators $M_{\mu\nu}$ to a pseudo-vector

$$L^\mu = \frac{1}{2}\epsilon^{\mu\nu\rho}M_{\nu\rho}, \quad (9.11)$$

in term of which the Poincaré algebra reads

$$[L_\mu, L_\nu] = i\epsilon_{\mu\nu\rho}L^\rho, \quad [L_\mu, P_\mu] = i\epsilon_{\mu\nu\rho}P^\rho \quad [P_\mu, P_\nu] = 0. \quad (9.12)$$

The irreducible representations are characterized by two Casimirs, as in four dimensions:

$$P^2 = P_\mu P^\mu \quad W = P_\mu L^\mu. \quad (9.13)$$

The mass m and spin s of a single particle state are defined by the Wigner conditions

$$P^2|\psi\rangle = -m^2|\psi\rangle, \quad W|\psi\rangle = -sm|\psi\rangle. \quad (9.14)$$

As in four dimensions, one can study how the algebra can be represented on fields, with the eigenvalue conditions understood as a physical requirement (generally imposed by the equations of motion, for local fields). P_μ is represented on any field $\phi(p)$ by p_μ in momentum space (or $p_\mu = -i\partial_\mu$ in position space), but the non-trivial information is in the representation of the Lorentz group. For scalar fields and Dirac spinors, we have

$$\text{Scalar: } L^\mu = -i\epsilon^{\mu\nu\rho}p_\nu \frac{\partial}{\partial p^\rho}, \quad \text{Dirac spinor: } L^\mu = -i\epsilon^{\mu\nu\rho}p_\nu \frac{\partial}{\partial p^\rho} \mathbf{1} + \frac{i}{2}\gamma^\mu \quad (9.15)$$

The conditions (9.14) are solved with $s = 0$, $(p^2 + m^2)\phi = 0$ (the Klein-Gordon equation) for a scalar field ϕ , and with $s = \pm\frac{1}{2}$ and $(\pm i\gamma^\mu p_\mu + m)\psi = 0$ (the Dirac equation) for a fermion ψ . For a gauge field A_μ , let us consider the gauge invariant vector $\tilde{F}^\mu = \frac{1}{2}\epsilon^{\mu\nu\rho}F_{\nu\rho}$. We can represent the algebra with

$$(L^\mu)^\nu{}_\rho = -i\epsilon^{\mu\alpha\beta}p_\alpha \frac{\partial}{\partial p^\beta} \delta_\rho^\nu + i\epsilon^{\mu\nu\sigma}\eta_{\sigma\rho}. \quad (9.16)$$

The condition $(PL + sm)\tilde{F} = 0$ reads

$$(ip_\mu\epsilon^{\mu\rho\sigma} + sm\eta^{\rho\sigma})\tilde{F}_\sigma = 0. \quad (9.17)$$

This can be realised in term of equations of motions for a gauge field with a purely Chern-Simons term, as we will see later. In that case we have $s = 1$.

As a curiosity, remark that in $2+1$ dimensions it is possible to solve for the constraints (9.14) for any *real* spin s ; such states are the so-called ‘‘anions’’. Such anions can be realised in theories with Chern-Simons gauge field interaction. This interesting possibility will not concern us directly in this thesis, however, because we will consider conformal fields theories, which do not have any particle interpretation.

9.1.3 \mathcal{N} -extended supersymmetry

The supersymmetry algebra in $2 + 1$ dimensions is similar to the one in four dimensions. We consider \mathcal{N} -extended supersymmetry, where the supercharges Q_α^i , $i = 1, \dots, \mathcal{N}$ are Majorana spinors. In addition to (9.12), we have

$$[P_\mu, Q_\alpha^i] = 0, \quad [L^\mu, Q_\alpha^i] = \frac{i}{2}(\gamma^\mu)_\alpha{}^\beta Q_\beta^i, \quad (9.18)$$

while the supercharges anti-commute between themselves as

$$\{Q_\alpha^i, Q_\beta^j\} = -\gamma_{\alpha\beta}^\mu P_\mu \delta^{ij} = \begin{pmatrix} P_0 + P_1 & -P_2 \\ -P_2 & P_0 - P_1 \end{pmatrix} \delta^{ij} \quad (9.19)$$

This relation implies that $P_0 = (Q_1)^2 + (Q_2)^2$, so the energy is positive definite (recall that the two components Q are real in our conventions). The \mathcal{N} -extended SUSY algebra has an automorphism group, also known as R-symmetry group, which is $SO(\mathcal{N})$; see [215] and references therein. The maximal rigid supersymmetry allowed in three dimensions in $\mathcal{N} = 8$. The real spinors Q^i transform in the vector representation of $SO(\mathcal{N})$. Since we are going to consider superconformal theories, we will not be interested in the possibility of adding a central charge Z^{ij} on the RHS of (9.19)³.

Massive representations. To discuss the massive representations of the SUSY algebra, it is useful to define complex supercharges

$$\mathcal{Q}^i = Q_1^i - iQ_2^i, \quad \bar{\mathcal{Q}}^i = Q_1^i + iQ_2^i \quad (9.20)$$

which are eigenvectors of the rotation generator $L^0 = M_{12}$,

$$[L^0, \mathcal{Q}^i] = -\frac{1}{2}\mathcal{Q}^i, \quad [L^0, \bar{\mathcal{Q}}^i] = +\frac{1}{2}\bar{\mathcal{Q}}^i, \quad (9.21)$$

In the center of mass reference frame $P_\mu = (m, 0, 0)$, the SUSY algebra (9.19) takes the familiar form

$$\{\mathcal{Q}^i, \bar{\mathcal{Q}}^j\} = 2m\delta^{ij}, \quad \{\mathcal{Q}^i, \mathcal{Q}^j\} = 0, \quad \{\bar{\mathcal{Q}}^i, \bar{\mathcal{Q}}^j\} = 0. \quad (9.22)$$

The massive supermultiplet then has $2^\mathcal{N}$ complex components, by the usual construction of a fermionic Fock space. For $\mathcal{N} = 1$, we have a multiplet of two complex components,

$$|m, s\rangle, \quad \text{and} \quad \mathcal{Q}|m, s\rangle = |m, s + \frac{1}{2}\rangle. \quad (9.23)$$

The shift in the spin follows from (9.21) and the definition (9.14). For rigid supersymmetry, we would have either a complex scalar and Dirac spinor, or a Dirac spinor and a massive vector. Note that in term of the real supercharges the four real-valued states you can build from a reference state Ω are $\{\Omega, Q_1\Omega, Q_2\Omega, Q_1Q_2\Omega\}$; from this representation, however, it is not directly obvious how the states organize into Poincaré representations of different spins.

³When $\mathcal{N} = 2$ the central charge Z corresponds to a real mass for some fields [217], which we will briefly discuss in the context of Chapter 13.

Massless representations. The “helicity” group of a massless particle is $O(1) \cong \mathbb{Z}_2$, corresponding to the even or odd statistic of the fields. The two irreducible representations of \mathbb{Z}_2 correspond to scalar and Majorana spinor fields [218]. The case of massless vector fields is covered too because they can be dualized to scalars⁴, as we saw in the last chapter. It is often convenient to consider the massless representations as the $m = 0$ limit of the massive ones, which will help us to retain some distinction between scalar and vector fields in physical contexts.

Fixing a reference frame $P_\mu = (E, 0, E)$, the SUSY algebra reads

$$\{Q_\alpha^i, Q_\beta^j\} = \begin{pmatrix} 2E & 0 \\ 0 & 0 \end{pmatrix} \delta^{ij} \quad (9.24)$$

This means $Q_2^i = 0$, and we are left with the charges Q_1^i , such that $\{Q_1^i, Q_1^j\} = 2E\delta^{ij}$. This gives supermultiplets of $2^{\mathcal{N}}$ real components. In the case of $\mathcal{N} = 1$, we have just two real components, Ω and $Q_1\Omega$, which correspond to a real scalar and a Majorana fermion.

9.2 $\mathcal{N} = 2$ supersymmetry, superspace and superfields

We now turn to the $\mathcal{N} = 2$ case, which will be the typical amount of SUSY discussed in this thesis. For $\mathcal{N} = 2$, we can define the complex charges

$$Q_\alpha = Q_\alpha^1 + iQ_\alpha^2, \quad \bar{Q}_\alpha = Q_\alpha^1 - iQ_\alpha^2, \quad (9.25)$$

corresponding to Dirac spinors. This will define a complex structure on field space in any $\mathcal{N} = 2$ field theory. The SUSY algebra reads

$$\{Q_\alpha, \bar{Q}_\beta\} = -2\gamma_{\alpha\beta}^\mu P_\mu, \quad \{Q_\alpha, Q_\beta\} = 0, \quad \{\bar{Q}_\alpha, \bar{Q}_\beta\} = 0. \quad (9.26)$$

This is the same as the $\mathcal{N} = 1$ SUSY algebra in four dimensions (with 4d Weyl spinors corresponding to 3d Dirac spinors), and consequently we can borrow our knowledge from this much more familiar setup. The two irreducible massless representations are the chiral multiplet and the vector multiplet. Notice that the R-symmetry is $SO(2)_R \cong U(1)_R$.

We can introduce a superspace $(x^\mu, \theta^\alpha, \bar{\theta}^\alpha)$. We refer to the Insert 4 for more details and useful formulas. The supersymmetry generators are represented on superspace as

$$Q_\alpha = \partial_\alpha - i(\gamma^\mu \bar{\theta})_\alpha \partial_\mu, \quad \bar{Q}_\alpha = -\bar{\partial}_\alpha + i(\theta \gamma^\mu)_\alpha \partial_\mu, \quad (9.27)$$

while the supercovariant derivative are

$$D_\alpha = \partial_\alpha + i(\gamma^\mu \bar{\theta})_\alpha \partial_\mu, \quad \bar{D}_\alpha = -\bar{\partial}_\alpha - i(\theta \gamma^\mu)_\alpha \partial_\mu. \quad (9.28)$$

Let us construct the off-shell representations of the $\mathcal{N} = 2$ SUSY algebra in term of superfields. The two basic supermultiplets are the chiral multiplet and the vector multiplet. The chiral multiplet consists of a complex scalar ϕ and a Dirac fermion ψ . The corresponding chiral superfield contains an additional auxiliary complex field F ,

$$\Phi(y, \theta) = \phi(y) + \sqrt{2}\theta \psi(y) + \theta^2 F(y), \quad (9.29)$$

⁴Remark also that, consistently, massless gravity is non-propagating in 3d.

Insert 4. 3d spinors and $\mathcal{N} = 2$ superspace formulas

We follow the conventions of [214] for spinors, as spelled out in section 9.1. The superspace coordinates are organized into a Dirac spinor

$$\theta_\alpha, \quad \bar{\theta}^\alpha = (\theta^\dagger)^\beta (\gamma^0)_\beta{}^\alpha,$$

We define the following notation (both for θ and for any Dirac spinor)

$$\chi\psi = \chi^\alpha\psi_\alpha = \epsilon^{\alpha\beta}\chi_\beta\psi_\alpha, \quad \theta^2 = \theta^\alpha\theta_\alpha, \quad \bar{\theta}^2 = \bar{\theta}^\alpha\bar{\theta}_\alpha, \quad \text{etc.}$$

Useful superspace identities are

$$\begin{aligned} \theta^\alpha\theta^\beta &= \frac{1}{2}\epsilon^{\alpha\beta}\theta^2, \quad \theta_\alpha\theta_\beta = \frac{1}{2}\epsilon_{\alpha\beta}\theta^2, \quad \text{etc.}, \\ (\theta\bar{\theta})^2 &= -\frac{1}{2}\theta^2\bar{\theta}^2, \quad (\theta\bar{\theta})(\theta\gamma^\nu\bar{\theta}) = 0, \quad (\theta\gamma^\mu\bar{\theta})(\theta\gamma^\nu\bar{\theta}) = \frac{1}{2}g^{\mu\nu}\theta^2\bar{\theta}^2. \end{aligned}$$

Care must also be taken when taking hermitian conjugates (the rules are different from 4d, due to our convention for γ^0). We have

$$(\chi\psi)^\dagger = -\bar{\chi}\bar{\psi}, \quad (\chi\bar{\psi})^\dagger = -\chi\bar{\psi}$$

and in particular

$$(\theta^2)^\dagger = -\bar{\theta}^2, \quad (\theta\bar{\theta})^\dagger = -\theta\bar{\theta}.$$

On the other hand, the combination $\theta\gamma^\mu\bar{\theta}$ is real.

in term of the chiral coordinate $y^\mu = x^\mu + i\theta\gamma^\mu\bar{\theta}$. Of course we have that $\bar{D}_\alpha\Phi = 0$. Similarly, we have anti-chiral superfields $\bar{\Phi} = \Phi^\dagger$, such that $D_\alpha\bar{\Phi} = 0$,

$$\bar{\Phi}(\bar{y}, \bar{\theta}) = \bar{\phi}(\bar{y}) - \sqrt{2}\bar{\theta}\bar{\psi}(\bar{y}) - \bar{\theta}^2\bar{F}(\bar{y}). \quad (9.30)$$

The vector multiplet V contains a real scalar σ and a vector A_μ (which can be seen as the dimensional reduction of a 4d vector), as well as a Dirac fermion χ . In the standard Wess-Zumino gauge, the corresponding vector superfield contains only one additional auxiliary field D ,

$$V(x, \theta, \bar{\theta}) = 2i\theta\bar{\theta}\sigma(x) + 2\theta\gamma^\mu\bar{\theta}A_\mu(x) + \sqrt{2}i\theta^2\bar{\theta}\bar{\chi}(x) - \sqrt{2}i\bar{\theta}^2\theta\chi(x) + \theta^2\bar{\theta}^2D(x). \quad (9.31)$$

One can easily check that $V^\dagger = V$, using Insert 4. Under (abelian) gauge transformation, the vector multiplet transforms as

$$V \rightarrow V + \Lambda + \Lambda^\dagger, \quad (9.32)$$

for Λ an arbitrary chiral superfield. In Wess-Zumino gauge, we have that $V^n = 0$ for $n \geq 3$, and

$$\frac{1}{2}V^2 = (\eta^{\mu\nu}A_\mu A_\nu + \sigma^2)\theta^2\bar{\theta}^2 \quad (9.33)$$

It is also interesting to define so-called linear multiplets Σ , which are real multiplets satisfying the conditions

$$D^2\Sigma = 0, \quad \bar{D}^2\Sigma = 0. \quad (9.34)$$

Such a multiplet contains a conserved current J^μ which appears as $(\theta\gamma^\mu\bar{\theta})J_\mu \subset \Sigma$.

9.2.1 Abelian gauge field, conserved current and $\mathcal{N} = 2$ Lagrangian

In $2 + 1$ dimensions, there is conserved current associated to any abelian gauge field, as long as the gauge field does not couple to a magnetic current. This current is simply the dual field strength $*F = \tilde{F}$,

$$*F = \frac{1}{2}\epsilon_{\mu\nu\rho}F^{\nu\rho} dx^\mu = \tilde{F}_\mu dx^\mu, \quad (9.35)$$

which is conserved due to the Bianchi Identity. In $\mathcal{N} = 2$ supersymmetric theories, this current is part of a linear multiplet

$$\Sigma \equiv \frac{i}{4}\bar{D}^\alpha D_\alpha V, \quad (9.36)$$

which reads

$$\begin{aligned} \Sigma = & \sigma + \frac{1}{\sqrt{2}}\theta\bar{\chi} - \frac{1}{\sqrt{2}}\bar{\theta}\chi + i\theta\bar{\theta}D + \frac{1}{2}\theta\gamma^\rho\bar{\theta}\epsilon_{\rho\mu\nu}F^{\mu\nu} \\ & - \frac{i}{2\sqrt{2}}\bar{\theta}^2\theta\gamma^\mu\partial_\mu\chi + \frac{i}{2\sqrt{2}}\theta^2(\partial_\mu\bar{\chi}\gamma^\mu\bar{\theta}) + \frac{1}{4}\partial_\mu\partial^\mu\sigma\theta^2\bar{\theta}^2. \end{aligned} \quad (9.37)$$

This Σ is gauge invariant. The $\mathcal{N} = 2$ Lagrangian is simply the square of this linear multiplet

$$\mathcal{L}_{\text{gauge}} = \frac{1}{g^2} \int d^2\theta d^2\bar{\theta} \Sigma^2 = \frac{1}{g^2} \left\{ -\frac{1}{2}\partial_\mu\sigma\partial^\mu\sigma - \frac{i}{2}\bar{\chi}\gamma^\mu\partial_\mu\chi - \frac{1}{4}F_{\mu\nu}F^{\mu\nu} + \frac{1}{2}D^2 \right\}. \quad (9.38)$$

As we already noted in the last chapter, the coupling constant g^2 has mass dimension one, making the theory strongly coupled in the infrared.

For later purposes, it will be interesting to look at the SUSY variation of the gaugino χ , which is easily obtained by computing $[\epsilon Q + +\bar{\epsilon}\bar{Q}, \Sigma]$. We have

$$\delta_\epsilon\chi = \frac{1}{\sqrt{2}}\epsilon^\alpha Q_\alpha\chi^\beta = i\epsilon^\alpha \left((\gamma^\mu)_\alpha{}^\beta (\partial_\mu\sigma + i\tilde{F}_\mu) - D\delta_\alpha^\beta \right), \quad (9.39)$$

and similarly for $\delta_\epsilon\bar{\chi}$.

The coupling of matter in $\mathcal{N} = 2$ theories is done as in $\mathcal{N} = 1$ theories in 4d, with a standard kinetic term (mind an extra minus sign) and the possibility of turning on a superpotential,

$$\mathcal{L}_{\text{matter}} = \int d^2\theta d^2\bar{\theta} \left(-\Phi^\dagger e^V \Phi \right) + \int d^2\theta W(\Phi) + \int d^2\bar{\theta} \bar{W}(\Phi^\dagger). \quad (9.40)$$

9.2.2 Vector/scalar duality

We can easily generalize the considerations of section 8.3.1 to the $\mathcal{N} = 2$ formalism, showing that a vector superfield is dual to a chiral superfield. Consider the action

$$\int d^3x \left\{ \int d^4\theta \frac{1}{g^2} \Sigma^2 + \frac{1}{8\pi} \int d^2\theta \Phi \bar{D}^2 \Sigma + \frac{1}{8\pi} \int d^2\bar{\theta} \Phi^\dagger D^2 \Sigma \right\}, \quad (9.41)$$

with Φ a dimensionless chiral superfield, imposing the Bianchi Identity $D^2\Sigma = 0$, $\bar{D}^2\Sigma = 0$. Integrating out Σ , we find the dual action

$$\int d^3x \int d^4\theta \left(-\frac{g^2}{8\pi^2} \Phi^\dagger \Phi \right). \quad (9.42)$$

We have that $\Sigma = -g^2(\Phi + \Phi^\dagger)/4\pi$. In particular, the scalar components of Φ are related to the real scalar σ and to the dual photon φ (defined as $g^2\partial_\mu\varphi = 2\pi\tilde{F}_\mu$) according to

$$\phi = -\frac{2\pi}{g^2}\sigma + i\varphi. \quad (9.43)$$

The dual photon φ is only defined up to a constant shift. If exact, this shift symmetry forbids the chiral superfield Φ from entering in the superpotential [217], but in general the shift symmetry can be broken by non-perturbative effects [219].

9.2.3 Non-abelian generalization

We could also construct the $\mathcal{N} = 2$ generalization of the Yang-Mills Lagrangian, for any simple gauge group G , simply by replacing derivatives by covariant derivatives in (9.38). The non-abelian generalization of (9.32) is

$$e^V \rightarrow e^{\Lambda^\dagger} e^V e^\Lambda. \quad (9.44)$$

The field strength superfield is

$$\Sigma = \frac{i}{4} \bar{D}^\alpha e^{-V} D_\alpha e^V, \quad (9.45)$$

and the super-Yang-Mills Lagrangian remains of the simple form

$$\frac{1}{g^2} \int d^2\theta d^2\bar{\theta} \Sigma^2. \quad (9.46)$$

9.3 Chern-Simon term and topologically massive photon

Our purpose in this part of the thesis is to discuss superconformal field theories. Since the Yang-Mills coupling is a dimensionful parameter in three dimensions, it seems that we cannot use SYM theories for our purposes. Matter seems even worse because the gauge coupling runs to strong coupling in the infrared, and we lose any control on our theory. Nevertheless we saw in the last chapter that we can have a weakly theory at large N_f . There exists another way to get a weak coupling limit, through the addition of a Chern-Simons term. We will review that construction [220] in this section.

9.3.1 The pure Chern-Simons action

Given a gauge group G and a gauge field A with field strength $F = dA + [A, A]$, we can always write a Chern-Simons (CS) action

$$S_{CS} = \frac{k}{4\pi} \int_M \text{Tr} \left(A \wedge dA + \frac{2}{3} A \wedge A \wedge A \right) \quad (9.47)$$

which is defined on any 3-manifold M , and does not depend on the metric. Under a finite gauge transformation

$$A \rightarrow A' = U^{-1}AU + U^{-1}dU, \quad (9.48)$$

the CS action changes by

$$\delta S_{CS} = \frac{k}{4\pi} \int_M d(A \wedge dUU^{-1}) - \frac{k}{12\pi} \int_M U^{-1}dU \wedge U^{-1}dU \wedge U^{-1}dU. \quad (9.49)$$

The first term is a total derivative, which we assume to vanish. The second term is actually the *winding number* [220] of the gauge transformation around the 3-manifold M ,

$$w = \frac{1}{24\pi^2} \int_M (U^{-1}dU)^3 \in \mathbb{Z}. \quad (9.50)$$

This winding number is an integer, which is non-zero for so-called large gauge transformations (the ones not connected to the identity). For flat space compactified to S^3 , the possible winding numbers are classified by the third homotopy group of the gauge group, $\pi_3(G)$, similarly to instantons in four dimensions. Invariance of the path integral

$$Z = \int \mathcal{D}A e^{iS_{CS}} \quad (9.51)$$

implies the condition

$$k \in \mathbb{Z} \quad (9.52)$$

for non-abelian simple gauge groups, for which $\pi_3(G) = \mathbb{Z}$. We will take k to be quantized also in the abelian case, for simplicity. k is often called the *Chern-Simons level*. The CS equations of motions are simply

$$F = 0, \quad (9.53)$$

so the pure CS theory describes flat connections. It is an interesting theory nevertheless for non-trivial manifolds M , for which (9.51) and other quantum observables are related to topological invariants [221]. In this work we are interested in flat space, and the reason why the Chern-Simons term will be interesting nevertheless is that the gauge field can be coupled to dynamical matter fields, as we will soon discuss. In components and in Minkowski space-time, the CS Lagrangian reads

$$\mathcal{L}_{CS} = \frac{k}{4\pi} \text{Tr} \epsilon^{\mu\nu\rho} \left(A_\mu \partial_\nu A_\rho + \frac{2}{3} A_\mu A_\nu A_\rho \right). \quad (9.54)$$

9.3.2 Topologically massive gauge field

Consider an abelian theory with a photon A_μ , and an action which consists of both a Maxwell term (with electric coupling g of mass dimension $\frac{1}{2}$) and a Chern-Simons term at level k ,

$$S = \int d^3x \text{Tr} \left\{ -\frac{1}{4g^2} F_{\mu\nu} F^{\mu\nu} + \frac{k}{4\pi} \epsilon^{\mu\nu\rho} A_\mu \partial_\nu A_\rho \right\} \quad (9.55)$$

The equations of motion are

$$\partial_\mu F^{\mu\nu} + \frac{g^2 k}{4\pi} \epsilon^{\mu\rho\nu} F_{\mu\rho} = 0. \quad (9.56)$$

We can also write them in term of the dual field strength \tilde{F}^μ (9.35),

$$\epsilon^{\mu\nu\rho}\partial_\mu\tilde{F}_\rho + \frac{g^2k}{2\pi}\tilde{F}^\nu = 0. \quad (9.57)$$

This is exactly the equation (9.17), with

$$s = 1, \quad m = m_{CS} = \frac{g^2k}{2\pi} \quad (9.58)$$

Hence the Lagrangian (9.55) describes a massive excitation of spin one (if k is negative we take $s = -1$ and $m = -g^2k/2\pi$). Such a gauge field is called *topologically massive*, because its mass comes entirely from the topological Chern-Simons term. This is a novel phenomenon which is particular to $2 + 1$ dimensions. In addition and independently, we could also have a Brout-Englert-Higgs mechanism as in $3 + 1$ dimensions.

To check that we are indeed dealing with a massive spin-one single particle state, we can compute the propagator [220]. In Landau gauge,

$$\Delta_{\mu\nu}(p) = \frac{1}{p^2 + m_{CS}^2} \left(\eta_{\mu\nu} + \frac{p_\mu p_\nu + m_{CS} \epsilon_{\mu\nu\rho} p^\rho}{p^2} \right). \quad (9.59)$$

It indeed has a pole at $p^2 = -m_{CS}^2$. Remark that if we also include a Higgs mechanism, the propagator would have two independent poles [216] (and the massive vector would have two independent physical degrees of freedom).

As we flow to the IR, the Maxwell-Chern-Simons theory becomes trivial. The Wilsonian effective action would only contain the Chern-Simons term, which by itself has no dynamics.

9.4 $\mathcal{N} = 2$ Chern-Simons theories

One can easily write a $\mathcal{N} = 2$ supersymmetric CS term, using the superfields (9.31) and (9.37). It must be a classically marginal operator (dimension 3), which is either a F-term of dimension 2 or a D-term of dimension 1. Since the lowest dimensional F-term we can construct is the Yang-Mills superfield \mathcal{W}^2 (as in 4d), which has dimension 3, the CS term must be a D-term. In the abelian case, the correct answer is

$$S_{CS} = \int d^3x \frac{k}{4\pi} \int d^2\theta d^2\bar{\theta} V \Sigma. \quad (9.60)$$

In components,

$$S_{CS} = \int d^3x \frac{k}{4\pi} \{ \epsilon^{\mu\nu\rho} A_\mu \partial_\nu A_\rho - i \bar{\chi} \chi + 2\sigma D \}. \quad (9.61)$$

Note that the Chern-Simons action (9.60) is invariant under the generalized gauge transformation (9.32), since

$$\delta \int d^4\theta V \Sigma = \int d^4\theta (\Lambda + \Lambda^\dagger) \Sigma = \frac{1}{16} D^2 \bar{D}^2 (\Lambda + \Lambda^\dagger) \Sigma = 0.$$

The non-abelian generalization for the $\mathcal{N} = 2$ CS term is simply

$$\int d^3x \frac{k}{4\pi} \left\{ \epsilon^{\mu\nu\rho} \left(A_\mu \partial_\nu A_\rho + \frac{2}{3} A_\mu A_\nu A_\rho \right) - i \bar{\chi} \chi + 2\sigma D \right\}, \quad (9.62)$$

with the trace over gauge indices left implicit. The superspace expression is a bit particular, however:

$$S_{CS} = \int d^3x \frac{k}{4\pi} \int d^2\theta d^2\bar{\theta} \int_0^1 dt \frac{i}{2} \text{Tr} \{ V \bar{D}^\alpha e^{-tV} D_\alpha e^{tV} \}. \quad (9.63)$$

9.4.1 Topologically massive vector multiplet

Consider the $\mathcal{N} = 2$ Maxwell-Chern-Simons system,

$$\mathcal{L} = \int d^4\theta \frac{1}{g^2} \left\{ \Sigma^2 + \frac{m}{2} V \Sigma \right\} \quad (9.64)$$

with $m = \frac{g^2 k}{2\pi}$. In addition to (9.56), the equations of motion are

$$i(\gamma^\mu \partial_\mu + m)\chi = 0, \quad \text{and} \quad (\partial_\mu \partial^\mu - m^2)\sigma = 0, \quad (9.65)$$

as expected by supersymmetry (we have integrated over the auxiliary field D). From now on we will focus on theories possessing a Chern-Simons term and no Yang-Mills term, but one can always think of a Maxwell/Yang-Mills UV completion, which is irrelevant in the IR.

Let us also remark that in a $SU(N)$ Yang-Mills-Chern-Simons (YM-CS) theory with no SUSY, the CS coefficient is renormalized at one-loop, to $k \rightarrow k + N$ [222]. In the pure CS theory there seems to be no such correction, or rather it seems that this correction is strongly dependent on the regularization procedure [223]. This confusing situation greatly improves with supersymmetry. It was shown in [224] that in $\mathcal{N} = 1$ YM-CS theory the CS shift is of $N/2$ at one loop, while in $\mathcal{N} = 2$ theories the CS level is not renormalized at all.

9.4.2 Chern-Simons-matter superconformal theories

We will now consider coupling the Chern-Simons action to matter fields ϕ ,

$$S = S_{CS}[A] + \int d^3x A_\mu J^\mu[\phi] \quad (9.66)$$

with J^μ the matter current. The equation of motion for the gauge field is a constraint

$$\frac{k}{2\pi} \tilde{F}^\mu = -J^\mu \quad (9.67)$$

which equates the matter current J with the ‘‘topological’’ current \tilde{F} (note that both currents are conserved).

In a non-abelian theory (and in the unbroken phase), it is clear that k cannot be renormalized beyond one-loop, because higher loop contributions would go like $1/k^{L-1}$ and k could not remain an integer, leading to a non-perturbative inconsistency. When coupling the Chern-Simons action to matter fields, there is strong evidence⁵ that the CS

⁵In [225] it is proven that the beta function of the coupling $1/k$ vanishes in any CS-matter theory. We could nevertheless have a one-loop shift, as in YM-CS-matter theories, although the result [222], valid in the YM-CS case, suggests that also in the pure CS-matter case there is no one-loop shift when we have $\mathcal{N} \geq 2$ SUSY.

level is not renormalized at all, whether in the abelian or non-abelian case [225]. We will assume this to be the case in the following.

Let us consider a $\mathcal{N} = 2$ theory with generic matter Φ_i coupled to some gauge group, together with a Chern-Simons term (9.62) for the vector multiplet,

$$S = S_{CS}^{\mathcal{N}=2} + \int d^3x \left\{ \int d^4\theta \sum_i (-\Phi_i^\dagger e^V \Phi_i) + \int d^2\theta W(\Phi) + c.c. \right\} \quad (9.68)$$

where all representation indices and traces are left implicit. Consider first the case when there is no superpotential. In components, the matter Lagrangian reads

$$\mathcal{L}_{\text{matter}} = -D_\mu \phi_i^\dagger D^\mu \phi_i - i\bar{\psi}_i \gamma^\mu D_\mu \psi_i + F_i^\dagger F_i - \phi_i^\dagger D \phi_i - \phi_i^\dagger \sigma^2 \phi_i + i\bar{\psi}_i \sigma \psi_i + i\phi_i^\dagger \chi \psi_i + i\bar{\psi}_i \bar{\chi} \phi_i, \quad (9.69)$$

where D_μ is the gauge covariant derivative, while the fields σ and D also act on the fields as appropriate representations of the gauge group. The sum over i (the matter fields) is implicit. The auxiliary field D appears only linearly in (9.68). Integrating it out determines the non-dynamical scalar field σ in term of the matter fields,

$$\sigma = \frac{4\pi}{k} \sum_i (\phi_i^\dagger T_{R_i}^a \phi_i) t^a, \quad (9.70)$$

with t^a the adjoint representation generators of the gauge group, normalized so that $\text{tr}(t^a t^b) = \frac{1}{2} \delta^{ab}$, and T_{R_i} the generator of the representation under which ϕ_i transforms. We can also integrate χ and $\bar{\chi}$ (and F , trivially), which gives a Lagrangian of the form

$$\begin{aligned} \mathcal{L} = & \mathcal{L}_{CS}(A) - D_\mu \phi_i^\dagger D^\mu \phi_i - \frac{16\pi^2}{k^2} (\phi_i^\dagger T^a \phi_i) (\phi_j^\dagger T^b \phi_j) (\phi_k^\dagger T^a T^b \phi_k) \\ & - i\bar{\psi}_i \gamma^\mu D_\mu \psi_i - \frac{4\pi}{k} i(\bar{\psi}_i T^a \psi_i) (\phi_j^\dagger T^a \phi_j) - \frac{8\pi}{k} i(\phi_i^\dagger T^a \psi_i) (\bar{\psi}_j T^a \phi_j) \end{aligned} \quad (9.71)$$

The first term is (9.54). This whole Lagrangian is classically marginal. In [226] it was argued that this theory is actually exactly marginal, giving us a Lagrangian description of a $\mathcal{N} = 2$ SCFT. The only coupling constant here is $1/k$, which we have seen cannot be renormalized. Also, no superpotential can be dynamically generated: for each chiral superfield Φ_i there is a $U(1)$ global symmetry which changes its phase; this symmetry also holds in the quantum theory, because there is no chiral anomaly in 3d. Consequently no holomorphic superpotential $W(\Phi)$ can be generated. The only non-trivial renormalization of (9.71) which may occur is by wave function renormalization, affecting the Kähler potential. Gaiotto and Yin [226] argued that such corrections do occur but can only lead to irrelevant terms in the Wilsonian effective action, or else can be reabsorbed by a rescaling of the fields. These general arguments should hold at least at weak coupling (k large), as was also checked by explicit perturbative computations.

There is an important difference here from the case of conformal theories with a continuous coupling, such as $\mathcal{N} = 4$ SYM theories in 4d. When there is a continuous coupling we can deform the theory by adding the corresponding marginal operator to the Lagrangian. In $\mathcal{N} = 4$ SYM we can add the Lagrangian itself, $\delta\mathcal{L} = \alpha\mathcal{L}$, with α infinitesimal corresponding to a small shift in the gauge coupling. Here the coupling

is quantized, so that such a continuous shift of \mathcal{L} is not allowed⁶. Each CS level k corresponds to an *isolated* conformal field theory.

In the absence of superpotential, any $\mathcal{N} = 2$ chiral operator (the lowest component of a chiral superfield) is also a chiral primary in the SCFT. The quantum dimension of the simplest gauge invariant chiral operator $\text{Tr}\phi_i\phi_j$ can be computed perturbatively at small 't Hooft coupling. Consider for instance the case of an $SU(N)$ theory with M adjoint chiral superfields Φ_i , $i = 1, \dots, M$. Perturbative computations involve the 't Hooft coupling $\lambda = N/k$. The scalar potential is

$$V = \frac{16\pi^2}{k^2} \text{Tr}[\phi_i, \phi_i^\dagger][\phi_j, \phi_j^\dagger][\phi_k, \phi_k^\dagger]. \quad (9.72)$$

In this example the conformal dimension of $\text{Tr}\phi_i\phi_j$ was computed in the large N limit and at two loops in [226],

$$\Delta(\text{Tr}\phi^2) = 1 - 2\lambda^2(M - 1). \quad (9.73)$$

Since $\Delta = R$ for chiral primaries, this results gives the R-charge of ϕ at two loops,

$$R(\phi) = \frac{1}{2} - \lambda^2(M - 1). \quad (9.74)$$

which then determines the quantum dimension of any chiral primary operator $\text{Tr}(\phi^k)$ as $\Delta = kR(\phi)$.

9.4.3 $\mathcal{N} = 2$ SCFT with superpotential and weak non-renormalization theorem

It seems a general rule that the chiral superfields have a lower R-charge in the interacting $\mathcal{N} = 2$ CS-matter theory (when $W = 0$). This means that the classically marginal superpotential of the form

$$W = \alpha_{ijkl} \text{Tr}(\Phi_i\Phi_j\Phi_k\Phi_l) \quad (9.75)$$

is actually *relevant* for $\alpha \ll 1/k$, so α must grow towards the IR (the theory at $W = 0$ is an unstable fixed point). On the other hand, if $\alpha \gg 1/k$ we can approximate the theory by a Wess-Zumino model, in which case α decreases along the RG flow [226]. We can conclude that there exist a RG fixed point at finite α , with superconformal R-charge $R(\Phi_i) = \frac{1}{2}$ (all R-charges are equal due to the flavor symmetry). We would expect such fixed points to be isolated in the space of couplings α_{ijkl} . It was argued in [226] that there exists a manifold of fixed point at two loops, but one would suspect that only isolated fixed points survive at higher order. Somewhat surprisingly, it was recently argued (and explicitly computed at 4 loops) that the continuous manifold of fixed points survives at all orders [227].

In 3d the non-renormalization ‘‘theorems’’ for the superpotential couplings in a CS-matter theory are less strong than in 4d Yang-Mills theories. If a particular superpotential couplings α_{ijkl} is zero, then it stays zero in perturbation theory, because it would otherwise break a $U(1)$ symmetry. This is the only useful non-renormalization result we have.

⁶However, since the quantization of k is non-perturbative, it is not clear to me whether the Lagrangian \mathcal{L} seen as an operator is renormalized or not (whether it is chiral primary or not).

It is true that the spurious R-symmetry which assigns charge 2 to α_{ijkl} implies that α_{ijkl} only appears linearly (by holomorphy). In 4d Yang-Mills theory one can argue that there is no perturbative renormalization of the holomorphic coupling α through the gauge coupling either. In the Chern-Simons theory corrections by λ do occur. In any case, we are interested in the physical coupling, which is also renormalized through the wave function renormalization of the chiral fields, and in that respect it is not so different from the 4d case: the anomalous dimension of the operators entering in (9.75), at any finite α , will be some function of both λ and the α 's.

9.4.4 $\mathcal{N} = 3$ CS-matter theory

We can obtain a $\mathcal{N} = 3$ supersymmetric theory by starting with the matter content of an $\mathcal{N} = 4$ theory (same as $\mathcal{N} = 2$ in 4d). In particular, the vector superfield now contains an additional complex scalar Φ . Let us consider a theory with N_f hypermultiplets Q, \tilde{Q} (two chiral superfields in $\mathcal{N} = 2$ notation, in conjugate representations of the gauge group). The $\mathcal{N} = 3$ theory reads

$$\mathcal{L}^{\mathcal{N}=3} = \mathcal{L}_{CS}^{\mathcal{N}=2} + \int d^4\theta \left(-Q_i^\dagger e^V Q_i - \tilde{Q}_i^\dagger e^{-V} \tilde{Q}_i \right) + \int d^2\theta \left(-\frac{k}{8\pi} \Phi^2 + \tilde{Q} \Phi Q \right) + c.c. \quad (9.76)$$

The field Φ is auxiliary in the same way as the other fields in the vector multiplet. We can integrate it out, giving us the quartic superpotential

$$W = \frac{2\pi}{k} (\tilde{Q} T^a Q) (\tilde{Q} T^a Q). \quad (9.77)$$

This superpotential has the form (9.75), but this particular fixed point at $\alpha = \frac{2\pi}{k}$ has an enhanced $\mathcal{N} = 3$ supersymmetry. This theory is again an isolated fixed point (at least when we only allow for this coupling, which preserves an $SU(N_f)$ flavor symmetry). From the argument of the last subsection, we know that the operator $(\tilde{Q}Q)^2$ is actually irrelevant as a deformation of the $\mathcal{N} = 3$ fixed point, since the fixed point is IR stable. In $\mathcal{N} = 3$ theories the superconformal R-charge is part of a non-abelian $SU(2)_R$ group, and it cannot be renormalized. In particular the chiral primary fields retain their classical dimension. For instance the mesons $\tilde{Q}Q$ have dimension 1. However, there are also non-trivial chiral ring relations due to (9.77); in particular $(\tilde{Q}Q)^2$ is a chiral descendant, which can acquire a (positive) anomalous dimension. The fact that the dimension of $(\tilde{Q}Q)^2$ is unprotected by the superconformal algebra means that this operator is not a good coordinate on a putative larger manifold of fixed points. In our case the fixed point is isolated and there does not exist such coordinates.

Chapter 10

Monopole operators in three dimensions

WE introduce in this chapter some important players in three dimensional CFTs, the monopole operators. They are local operators which insert some magnetic source at a point in \mathbb{R}^3 .

In the first section we review the Dirac monopole [228] and the related Goddard-Nuyts-Olive (GNO) monopoles [229] in $U(N)$ gauge theory. We describe in some detail such monopole configurations and explain how they relate to particular chiral operators parametrizing the Coulomb branch. We then explain in the free abelian example how this relates to the concept of monopole operator as a local operator inserting a magnetic source at a point.

In the second section we introduce monopole operators in any CFT, as defined by Borokhov, Kapustin and Wu in [230, 231]. In the weakly coupled limit one can compute the charge induced by quantum fluctuations of matter fields. The only contribution comes from some zero modes of a Dirac operator on S^2 . It has been conjectured in some cases that this result is “semi-topological” and therefore holds at any coupling [231].

We emphasize that the monopole operators will play a crucial work in our work [3]. It is somewhat disappointing that we will have to make some conjecture concerning the non-perturbative validity of the formulas for the induced charges. A complete field theory derivation of the assumptions of [3] is beyond the scope of this thesis, and is left for future work.

Interesting operators which we will *not* consider are Wilson lines, which are non-local operators which insert an electric particle along a worldline. We just remark that in abelian Chern-Simons theories Wilson lines can be local and equivalent to monopole operators [232, 233].

In this chapter we work exclusively in Euclidean space, either in \mathbb{R}^3 with the metric $ds^2 = dr^2 + r^2 d\Omega_2$, or in $\mathbb{R} \times S^2$ with the metric $ds^2 = d\tau^2 + d\Omega_2$.

10.1 Monopoles in three dimensional SYM theories

The bosonic part of the Euclidean $\mathcal{N} = 2$ super-Yang-Mills action can be written in term of differential forms as

$$S = \frac{1}{2g^2} \int (F \wedge *F + \mathcal{D}\sigma \wedge *\mathcal{D}\sigma), \quad (10.1)$$

where $F = dA + A \wedge A$, while $\mathcal{D} = d + A$ is the covariant derivative in the adjoint representation, and an overall trace is implied. In the above action and in the following we have set the auxilliary field D to zero. We can equivalently write this action as

$$S = \frac{1}{4g^2} \int ((F + *\mathcal{D}\sigma) \wedge *(F + *\mathcal{D}\sigma) + (F - *\mathcal{D}\sigma) \wedge *(F - *\mathcal{D}\sigma)), \quad (10.2)$$

from which one derives the bound

$$S \geq \mp \frac{1}{g^2} \int F \wedge \mathcal{D}\sigma = \mp \frac{1}{g^2} \int_{S^2_\infty} F\sigma. \quad (10.3)$$

This inequality is saturated if and only if the fields satisfy the Bogomolny equations [234]

$$F = \mp *\mathcal{D}\sigma \quad (10.4)$$

This BPS bound is of course linked to supersymmetry. The Euclidian version of the gaugino variation (9.39) is

$$\delta_\epsilon \chi = -i\epsilon(\gamma^\mu(\mathcal{D}_\mu\sigma - \tilde{F}_\mu) - D), \quad \delta_{\tilde{\epsilon}} \bar{\chi} = -i\epsilon(\gamma^\mu(\mathcal{D}_\mu\sigma + \tilde{F}_\mu) - D), \quad (10.5)$$

with γ^μ the Euclidean γ -matrices, defined such that $\gamma^\mu\gamma^\nu = \delta^{\mu\nu} + i\epsilon^{\mu\nu\rho}\gamma^\rho$ (cfr. [235] for more detailed conventions). We will call a field configurations BPS when $\delta_{\tilde{\chi}} = 0$, and anti-BPS when $\delta\chi = 0$. Henceforth we will focus on the BPS case, for which (recall that $D = 0$)

$$*F = -\mathcal{D}\sigma, \quad \Leftrightarrow \quad F = -*\mathcal{D}\sigma. \quad (10.6)$$

It is well known that the field configurations satisfying this equation are the 't Hooft-Polyakov monopoles [236, 237], arising in the non-abelian theory spontaneously broken to its Cartan subgroup,

$$G \rightarrow U(1)^r, \quad (10.7)$$

due to a VEV for σ . While monopoles are solitons in four dimensions (finite energy solutions), they are instead instantons in three dimensions, in the sense that they minimize the Euclidean action. We are not particularly interested in the solutions to the non-linear Bogomolny equations, but only in the generic properties of the low energy theory in the presence of magnetic flux.¹ Let us consider the gauge group $U(N)$. A generic GNO monopole has a magnetic charge along the $U(1)^N$ Cartan subgroup. It can be described by a non-trivial $U(1)^N$ gauge connection on $\mathbb{R}^3 \setminus \{0\}$, defined on two patches surrounding the monopole localized at $r = 0$,

$$A_\pm = \frac{H}{2}(\pm 1 - \cos\theta)d\phi. \quad (10.8)$$

¹Smooth monopole solutions are classified by $\pi_2(G/U(1)^r)$ and there is no smooth monopole for $G = U(1)$; the GNO monopoles we consider are broader in that we do allow for singularities in the fields.

The two connections are related by a $U(1)^N$ gauge transformation $A_+ = A_- + Hd\phi$. Requiring single-valuedness of this gauge transformation as we go around the equator, $\phi \rightarrow \phi + 2\pi$, we are lead to the Dirac quantization condition of magnetic flux,

$$H = \text{diag}(n_1, \dots, n_N), \quad n_i \in \mathbb{Z}. \quad (10.9)$$

Note that permutations of the n_i 's is a gauge symmetry. For generic gauge group G the magnetic fluxes define the roots of a dual group G^\vee [229], but in our case $U(N)^\vee = U(N)$ and H is a element of the root space of $U(N)$ itself. To preserve $\mathcal{N} = 2$ supersymmetry, we need to solve (10.6). We are interested in the abelian solution only (which approximates any complete non-linear solution at large distances), so that we simply have $F = - * d\sigma$. This gives the diagonal solution

$$F = \frac{H}{2} d\Omega_2, \quad \sigma = \sigma_0 + \frac{H}{2r}. \quad (10.10)$$

The diagonal matrix $\sigma_0 = \text{diag}(\sigma_1, \dots, \sigma_N)$ is an integration constant. The action of the monopole configuration is

$$S_{\text{on-shell}} = -\frac{1}{g^2} \int_{S_\infty^2} \text{Tr}(F\sigma) = -\frac{2\pi}{g^2} \text{Tr}(H\sigma_0) = -\frac{2\pi}{g^2} \sum_i n_i \sigma_i. \quad (10.11)$$

Monopole (3d instantons) constitute saddle points which will contribute to the path integral of 3d Yang-Mills theory, giving rise to non-perturbative corrections. Since the action is positive-definite by construction, we must have that $\sum_i n_i \sigma_i \leq 0$, which restricts the allowed choice of σ_0 . Each possible magnetic flux along some $U(1)_i$ corresponds to a distinct topological sector, leading to a new perturbative expansion around the monopole saddle point. Such contributions are weighted by an overall factor

$$e^{-2\pi|n_i \sigma_i|/g^2} \quad (10.12)$$

(no sum implied), and they are therefore negligible as long as $\sigma_i \gg g^2$. On the other hand, near the origin of the Coulomb branch, $\sigma_i < g^2$, we expect the contributions from every topological sector to be equally important, but we lack any direct perturbative control.

10.1.1 The Coulomb branch

The VEV $\sigma_0 = \text{diag}(\sigma_1, \dots, \sigma_N)$ parametrizes the Coulomb branch of the $U(N)$ theory. When the eigenvalues σ_i as well as their differences $\sigma_i - \sigma_j$ are all large, the gauge group is broken to $U(1)^N$ at a correspondingly high scale. In a theory with $\mathcal{N} = 2$ supersymmetry, we expect the moduli space to be a complex manifold. Indeed we can also give VEVs to the dual photons of the low energy $U(1)^N$ group, and parametrize the resulting complexified Coulomb branch by the scalar components of the chiral superfield Φ defined in (9.43). Remark that in Euclidean space we have some extra factor of i in the definition of the dual photon,

$$\partial_\mu \varphi = -\frac{2\pi i}{g^2} \tilde{F}_\mu. \quad (10.13)$$

The dual photon φ being a periodic variable, it is natural to define a single valued chiral superfield as

$$T = e^\Phi = \exp\left(-2\pi\frac{\sigma}{g^2} + i\varphi\right). \quad (10.14)$$

We define a T_i for each $U(1)_i$ in $U(1)^N$. These fields have charge one under the shift symmetry of the dual photons. These are important coordinates on the Coulomb branch, which however are good only away from the origin. As shown in [217, 219], quantum effects will change the topology of the Coulomb branch when additional matter fields are present, splitting it into several branches which meet at the origin. There we expect an interacting SCFT to exist.

Note that the shift symmetry of the dual photon is nothing but the magnetic symmetry associated to the conserved current \tilde{F}^μ . Hence the field T has unit magnetic charge, and represents a supersymmetric “monopole” in some very concrete sense. We can define such a monopole with generic flux H by

$$T^{(H)} = \exp\left(\sum_i n_i \Phi_i\right). \quad (10.15)$$

This is our first encounter with a so-called monopole operator. One should think as this T as surviving as a special operator in the conformal field theory at the origin of the Coulomb branch.

10.1.2 Monopole operator in the 3d Maxwell theory

The monopole operator $T(x)$ inserts a Dirac monopole/instanton at the point x . In the case of a free Maxwell theory, this can be easily understood. The insertion of the operator

$$T(x)^{(n)} = e^{in\varphi(x)} \quad (10.16)$$

in the path integral changes the allowed boundary conditions for the fields, allowing for n units of magnetic flux around the point x . This is seen for instance in the dual photon formulation of the Maxwell theory (8.41). We have

$$\langle T(0)^{(n)} \rangle = \int D\varphi \exp\left\{-\frac{1}{2} \int d^3x \frac{g^2}{4\pi^2} \partial_\mu \varphi \partial^\mu \varphi + in \int d^3x \varphi \delta^3(x)\right\}. \quad (10.17)$$

This insertion changes the perturbative saddle point from $\varphi = 0$ to

$$\varphi = \frac{n}{2} \frac{2\pi i}{g^2} \frac{1}{r}, \quad (10.18)$$

Such a singularity in the dual photon field corresponds to n units of magnetic flux at $r = 0$, as we can see by using the relation (10.13). This reasoning is directly extended to the $\mathcal{N} = 2$ BPS operator

$$T(x)^{(n)} = e^{n\left(-\frac{2\pi}{g^2}\sigma(x) + i\varphi(x)\right)}. \quad (10.19)$$

The operator $e^{\lambda\sigma}$ is a “point operator” which likewise inserts a singularity for the real scalar field σ , and the particular value of λ in (10.19) is fixed by supersymmetry.

This discussion is isomorphic to the discussion of line operators given by Kapustin in [238]. In particular the operator T is nothing but the ’t Hooft operator dimensionally

reduced from four to three dimensions. This means that T stands for 't Hooft, although we will call it a “monopole operator” to conform to fashion.

Once we turn on interactions, the dual photon trick is not applicable anymore, but nevertheless the monopole operator is still a well defined concept. At least this is so in any conformal field theory, which we will now discuss.

10.2 Monopole operators in 3d CFT

In [230, 231], monopole operators were defined in 3d conformal field theories with abelian gauge fields. Consider for instance an abelian Chern-Simons theory coupled to matter. It is classically conformal, and we assume for the moment that this property is not broken by quantum effects. A monopole operator, also sometimes called vortex operator, is simply one which inserts n units of magnetic charge at a single point in \mathbb{R}^3 . This means that the OPE of such an operator \mathcal{O} with the conserved current \tilde{F}^μ has a singularity

$$\tilde{F}^\mu(x)\mathcal{O}(y) \sim \frac{n}{2} \frac{1}{|x-y|^2} \mathcal{O}(y). \quad (10.20)$$

In the path integral language, such an operator inserts a new boundary condition on the gauge field, requiring that the magnetic flux around the point $p = \{y^\mu\}$ be n . To have a complete definition, we need to specify also the new boundary conditions for all the other fields in the theory, which must now live on the punctured space $\mathbb{R}^3 \setminus \{p\}$. In general there might be many allowed operators with the same magnetic charge.

A more manageable definition is obtained by using radial quantization instead of the operator language. In the radial quantization picture a monopole operator is nothing but a state on $\mathbb{R} \times S^2$ with n units of magnetic flux through S^2 . This makes also clear why *local* monopole operators are specific to three dimensions: In four dimensions there is no such local operators because vector bundles on S^3 do not carry any interesting topological numbers (4d 't Hooft operator are extended operators corresponding to the insertion of a monopole worldline) [238]. To go from the theory on \mathbb{R}^3 to the theory on $\mathbb{R} \times S^2$, we change coordinates to $\tau = \ln r$, and we redefine all the fields as

$$\phi = e^{-\Delta(\phi)\tau} \tilde{\phi}, \quad (10.21)$$

where $\Delta(\phi)$ is the scaling dimension of ϕ , and $\tilde{\phi}$ are the new fields we use in the radial quantization picture.

We must also discuss the issue of gauge invariance. Gauge invariance of a state on $\mathbb{R} \times S^2$ is simply Gauss law, which can be written

$$\frac{\delta S}{\delta A_\tau} |\psi\rangle = 0, \quad (10.22)$$

for any physical state $|\psi\rangle$ (A_τ is the τ component of the gauge field). In the case of a Chern-Simons matter theory of the generic form (9.66), we have

$$\left(\frac{k}{2\pi} \tilde{F}^\tau + \rho \right) |\psi\rangle = 0, \quad (10.23)$$

where $\rho = J^\tau$ is the charge density operator. This means that, for gauge invariance of a state carrying electric charge Q together with n units of magnetic flux, we need

$$kn + Q = 0. \quad (10.24)$$

In particular, monopole operator must always be “dressed” by appropriate matter fields in a Chern-Simons theory. Indeed, a “bare” monopole operator of magnetic charge n corresponds to a state with an electric tadpole, since its action contains a term

$$S \supset \frac{k}{4\pi} \int A \wedge dA = kn \int A_0 dt. \quad (10.25)$$

This tadpole can only be cancelled by adding additional matter fields for a total charge of $Q = -kn$. In the following chapters we will keep referring to the bare monopole operator $T^{(n)}$ as “the” monopole operators, mainly for notational convenience. We will write gauge invariant operators schematically as

$$\mathcal{O}^{(n)} \sim T^{(n)} \phi_1 \cdots \phi_{kn}. \quad (10.26)$$

Nevertheless it should be kept in mind that only the gauge invariant operators have a physical meaning. This is true in particular for such operators which do not have any description in term of Lagrangian fields, in general. We will not discuss in any detail the non-abelian extension of these results, although they are of much interest in relation with the AdS_4/CFT_3 correspondence.

10.2.1 $\mathcal{N} = 2$ BPS monopole operators

Consistently with the discussion of section (10.1), $\mathcal{N} = 2$ BPS monopole operators which create some flux at $r = 0$ in \mathbb{R}^3 must also set a boundary condition

$$\lim_{r \rightarrow 0} \sigma = \frac{n}{2r} \quad (10.27)$$

for the scalar field σ in the vector supermultiplet. To discuss the corresponding state in $\mathbb{R} \times S^2$, we need to consider the curved space version of the gaugino variation (10.5). This was studied in [235], where it is found that (setting to zero the VEV of all the other fields)

$$\delta_{\bar{\epsilon}} \bar{\chi} \sim \gamma^\mu \left(\tilde{F}_\mu \bar{\epsilon} + (\partial_\mu \sigma) \bar{\epsilon} + 2\sigma \nabla_\mu \bar{\epsilon} \right), \quad (10.28)$$

where ∇ is the spin connection, and $\bar{\epsilon}$ must be a Killing spinor. The solution to $\delta_{\bar{\epsilon}} \bar{\chi} = 0$ is more easily found by simply rescaling the solution (10.10), which gives

$$F = \frac{H}{2} d\Omega_2, \quad \tilde{\sigma} = \frac{n}{2}. \quad (10.29)$$

This means that the field σ must have a constant VEV $n/2$ at $\tau = -\infty$.

10.2.2 Induced charges from quantum effects

So far we have discussed the concept of monopole operator assuming that the CFT has a weakly coupled Lagrangian description. For most of the models with $\mathcal{N} = 2$ we will consider this is not the case. One way to circumvent this difficulty is to work in a weakly coupled UV completion, for instance in the UV of a YM-CS-matter theory which UV completes the CS-matter theory, as done in [235]. Another weakly coupled crutch is obtained by taking the CS level k to be large, or one can also take the number of flavor fields N_f to be large [230, 231].

We now discuss some quantum properties of monopole operator in such a weakly coupled limit, for $\mathcal{N} = 2$ theories. In particular, we would like to compute the exact R-charge of the bare monopole operator in the SCFT. With that knowledge we can then compute the exact dimension of any *chiral primaries* monopole operator, in principle, if we know the R-charge of all the Lagrangian fields. Note that in general we do not know which is the exact superconformal R-charge, but we can still express the correction to the R-charge of a given monopole operator in term of the unknown R-charges of the matter fields. Actually we can also compute the quantum correction to any $U(1)$ charge Q .

The conformal dimension of the bare monopole operator is the energy of the corresponding state in radial quantization. This energy vanishes classically, but in the quantum theory it is usually strongly renormalized. In the weakly coupled theory the main effect will come from fluctuations of free quantum fields charged under a classical monopole background. In [230] it was shown that only the fermions contribute to this zero point energy, so we will discuss the fermions only.

Consider a particular $U(1)$ gauge group (it could be part of a larger gauge group). In the presence of a $\mathcal{N} = 2$ monopole background for this vector multiplet, any fermion ψ charged under this $U(1)$ will satisfy the wave equation

$$(\gamma^\mu \mathcal{D}_\mu \pm \frac{n}{2})\psi(\tau, \theta, \phi) = 0. \quad (10.30)$$

The \pm is for BPS or anti-BPS background. The first term is the Dirac operator in this monopole background,

$$\gamma^\mu \mathcal{D}_\mu = \gamma^\tau \partial_\tau + \mathcal{D}_{S^2}, \quad \text{with} \quad \mathcal{D}_{S^2} = \gamma^a (\nabla_a + iA_a). \quad (10.31)$$

The second term in (10.30) comes from the coupling $\bar{\psi}\sigma\psi$. The eigenvalues and eigenfunctions of \mathcal{D}_{S^2} were worked out in [239], for instance. Let J be the total angular momentum operator, with eigenvalue j , and let n be the magnetic flux. We have

$$\mathcal{D}_{S^2} \Psi_{nm}^0 = 0 \quad \text{for} \quad j = \frac{|n| - 1}{2} \quad (10.32)$$

$$\mathcal{D}_{S^2} \Psi_{njm}^\pm = i\Delta_{jn}^\pm \Psi_{njm}^\pm \quad \text{for} \quad j = \frac{|n| + 1}{2}, \frac{|n| + 3}{2}, \dots \quad (10.33)$$

with $m = -j, \dots, j$, and Δ_{jn}^\pm some eigenvalues which can be found e.g. in [235]. Hence there are $|n|$ zero modes of this Dirac operator on S^2 . On the other hand, the equation (10.30) has no zero energy solution: each mode Ψ_{njm}^\pm corresponds to a state of energy $\pm|j + \frac{1}{2}|$, while the zero modes Ψ_{nm}^0 correspond to states of energy $\frac{|n|}{2}$. This means that the vacuum state of the bare monopole is unique. From there one can compute any charge

by normal ordering, with some subtleties well explained in [231, 235]. It turns out that only the $|n|$ modes Ψ_{nm}^0 contribute to the normal ordering constant.

The final answer is that for any abelian symmetry Q under which the fermion ψ is charged, with charge $Q(\psi)$, there is an induced charge for the bare BPS monopole operator

$$\delta Q(T^{(n)}) = -\frac{|n|}{2}Q(\psi), \quad (10.34)$$

and minus that result for an anti-BPS monopole. In particular we can compute the R-charge, using the fact that in the semi-classical theory $R(\psi) = -\frac{1}{2}$ for any matter fermion. Assuming $T^{(n)}$ is a chiral primary operator, we deduce that it has conformal dimension $\frac{|n|}{2}N_f$ if there are N_f fermions coupled to the relevant $U(1)$ vector.

There is some circumstantial evidence that the above result for the quantum charges of monopole operators is actually valid for any coupling, in the case of $\mathcal{N} = 2$ supersymmetry (both by direct arguments and using 3d mirror symmetry) [231]. This certainly *seems* true for $\mathcal{N} > 2$, and there is some proof that it is so in some interesting cases [235], such as in the ABJM theory we will discuss in the next Chapter.

10.2.3 OPE of monopole operators

As a final remark, note that in radial quantization we can in principle compute the OPE between $T^{(n)}$ and $T^{(-n)}$ [231]. In general we can have

$$T^{(n)}T^{(-n)} \sim \sum_i c_i \mathcal{O}_i, \quad (10.35)$$

where the \mathcal{O}_i are all the operators with no monopole charge and twice the quantum numbers of $T^{(n)}$ (since $Q(T^{(n)}) = Q(T^{(-n)})$ according to (10.34)). We have also taken advantage of the fact that these monopole operators are chiral, so that the OPE is non-singular. In particular, if the conformal dimension of T vanishes (as well as all other quantum numbers), the only possible OPE is

$$T^{(n)}T^{(-n)} \sim 1 \quad \text{or} \quad 0. \quad (10.36)$$

The coefficients c_i in (10.35) can be computed in radial quantization as the matrix element

$$\langle T^{(-n)\dagger} | \mathcal{O}_i | T^{(n)} \rangle. \quad (10.37)$$

The actual computation of such elements in any non-trivial theory is however quite beyond the scope of this thesis. Note also that in general the bare monopole operators carry electric charges (from the CS interaction, and/or from quantum effects), in which case the above relation should be understood to hold only for gauge invariant chiral primary operators; schematically,

$$\mathcal{O}^{(n)}\mathcal{O}^{(-n)} \sim \sum_i c_i \mathcal{O}_i^{(0)}. \quad (10.38)$$

Chapter 11

The ABJM theory and Chern-Simons quivers

RECENT progress in writing down theories corresponding to M2-branes originates in the work of Bagger-Lambert [240, 241] and Gustavsson [242] (BLG), who wrote down a Lagrangian with explicit $\mathcal{N} = 8$ supersymmetry. This earlier work did not use Chern-Simons theories but rather an exotic structure called a three-algebra, which we will not discuss. It was soon shown [243] that the BLG theory was equivalent to a more familiar Chern-Simons theory. One shortcoming of the BLG theory is that it apparently describes the low energy theory on only two coincident M2-branes (with additional subtleties we will not enter into).

Building on this pioneering work, Aharony, Bergman, Jafferis and Maldacena (ABJM) proposed a Chern-Simons theory which has a $U(N) \times U(N)$ gauge group, with Chern-Simons levels $(k, -k)$. It should correspond to the low energy theory on a stack of N M2-branes. They gave a string theory derivation of their proposal by using a duality between a particular Hanany-Witten setup in type IIB and M2-branes on some orbifold singularity (this construction will be reviewed in the next Chapter).

A general lesson taught by the ABJM proposal is that in order to give a Lagrangian description of multiple M2-branes, we might have to abandon the hope of an explicit realization of all the global symmetries. Some symmetries might only arise as accidental symmetries at the superconformal fixed point. For instance, in the ABJM theory for N M2-branes in flat space only a subset $\mathcal{N} = 6$ of the $\mathcal{N} = 8$ supersymmetry is realized at the level of the Lagrangian, while the remaining supersymmetries are realized thanks to non-perturbative effects involving monopole operators. We should also note that in that case the field theory is strongly coupled, so the Lagrangian description is not really useful anyway.

This might seem like a step back from the ambition of a fully explicit Lagrangian theory with $\mathcal{N} = 8$ SUSY. It might be that this dream is not realizable, and that M2-branes theory in flat space cannot have a weakly coupled Lagrangian description, as suggested by the fact that the M-theory setup has no tunable parameter.

The truly great achievement of the ABJM model is that it has a weakly coupled limit, for k large and $\lambda = N/k$ small. On the other hand for $\lambda \gg 1$ we can also use either 11 dimensional supergravity or 10 dimensional type IIA supergravity, as described already

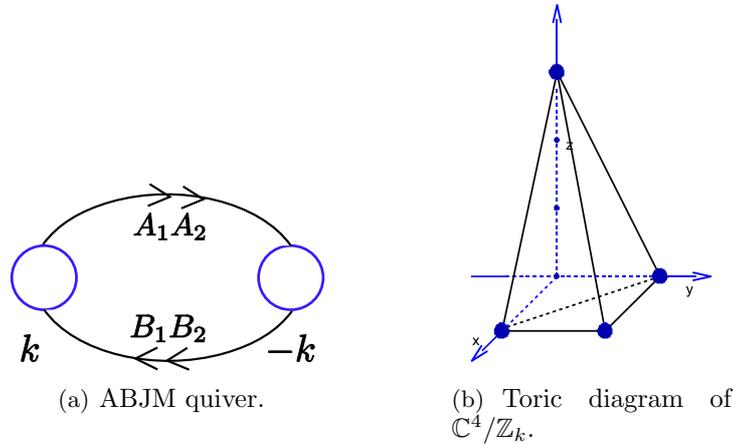


Figure 11.1: The ABJM quiver and its moduli space. The gauge group is $U(N) \times U(N)$ with Chern-Simons levels $(k, -k)$. The toric diagram of $\mathbb{C}^4/\mathbb{Z}_k$ contains four points, which can be taken as $(1, 0, 0)$, $(0, 1, 0)$, $(1, 1, 0)$ and $(0, 0, k)$ (here $k = 3$).

in Chapter 8. In that context all the heavy machinery of the AdS/CFT correspondence can be applied fruitfully. For instance we can extrapolate results for BPS quantities from weak to strong coupling¹.

This Part of the thesis is concerned with the generalization of the ABJM construction to instances of AdS_4/CFT_3 dualities with less supersymmetries. In the next chapter we will review an interesting $\mathcal{N} = 3$ construction, which directly generalizes ABJM in several directions. In this chapter we are particularly interested in $\mathcal{N} = 2$ Chern-Simons *quiver* theories [244, 245]. They are candidates to describe M2-branes at Calabi-Yau fourfold singularities. Remark that examples with minimal $\mathcal{N} = 1$ supersymmetry have also been proposed [246, 247, 248, 249].

In the first two sections of this Chapter we review the ABJM proposal and explain the crucial role played by monopole operators. In the remaining of the Chapter we briefly present some attempts in the literature towards building Chern-Simons quiver for M2-branes on any Calabi-Yau cone. We will focus on the toric case, which is more manageable². In particular we explain a generalization of the fast forward algorithm [251, 252, 253] which will be useful in Chapter 13.

11.1 The ABJM theory

In this section we introduce the famous ABJM theory [202], which is a Chern-Simons theory with $\mathcal{N} = 6$ supersymmetry. The matter content is the same as the famous conifold quiver, as shown in Fig. 11.1(a). It is a Chern-Simons matter theory with gauge

¹This assumes that we can treat λ as continuous in some regimes.

²The non-toric case is of interest too, mainly because toric Calabi-Yau fourfolds do not admit complex deformations [250]. A correspondence between a Chern-Simons quiver and a non-toric 8 dimensional conifold was initiated in [199], where it is shown that a complex deformation of the cone corresponds to a relevant deformation of the superpotential.

group $U(N)_k \times U(N)_{-k}$, where the subscripts $(k, -k)$ are the Chern-Simons levels of each $U(N)$ factor. In $\mathcal{N} = 2$ notation, we have four chiral matter fields (A_1, A_2, B_1, B_2) , where A_1, A_2 transform in the (N, \bar{N}) of the gauge group, and B_1, B_2 in the (\bar{N}, N) . Let us denote the $\mathcal{N} = 2$ vector superfields by V_1 and V_2 , for the first and second gauge group. The ABJM Lagrangian reads

$$\begin{aligned} \mathcal{L}_{ABJM} = & \mathcal{L}_{CS}[V_1; k] + \mathcal{L}_{CS}[V_2, -k] + \int d^4\theta \sum_{i=1}^2 \text{Tr} \left(-A_i^\dagger e^{V_1} A_i e^{-V_2} - B_i^\dagger e^{-V_1} B_i e^{V_2} \right) \\ & + \int d^2\theta \frac{4\pi}{k} \text{Tr} \left(\epsilon^{ij} \epsilon^{kl} A_i B_k A_j B_l \right) + c.c. \end{aligned} \quad (11.1)$$

The $\mathcal{N} = 2$ Chern-Simons terms \mathcal{L}_{CS} are given by (9.62)-(9.63). The superpotential coupling $4\pi/k$ is fixed by the requirement of $\mathcal{N} = 3$ supersymmetry, as in Section 9.4.4. We can derive it by writing the action in term of an auxiliary complex scalar for each vector multiplet (in which case the auxiliary fields fill a $\mathcal{N} = 4$ vector multiplet, which contains three real scalars). Consider the superpotential

$$W = -\frac{k}{8\pi} \text{Tr}(\Phi_1^2 - \Phi_2^2) + \sum_i \text{Tr}(B_i \Phi_1 A_i + A_i \Phi_2 B_i) \quad (11.2)$$

where the coupling between the $\mathcal{N} = 4$ hypermultiplets $H_i = (A_i, B_i)$ and the scalars from the vector multiplets is fixed by supersymmetry, while the Chern-Simons mass term for Φ_1 and Φ_2 breaks $\mathcal{N} = 4$ to $\mathcal{N} = 3$. Integrating out these fields, we obtain

$$W = \frac{4\pi}{k} \text{Tr}(A_1 B_1 A_2 B_2 - A_1 B_2 A_2 B_1). \quad (11.3)$$

One can check that it can be written as in (11.1), which makes explicit that the theory preserves a larger $SU(2)_A \times SU(2)_B$ flavor symmetry, under which (A_1, A_2) and (B_1, B_2) transform independently. In addition, the theory has a $SU(2)_R$ R-symmetry, under which (A_1, B_1^\dagger) and (A_2, B_2^\dagger) are doublets. In the $\mathcal{N} = 2$ notations we only see explicitly the Abelian subgroup $U(1)_R$, under which all chiral superfields have charge $\frac{1}{2}$. The $SU(2)_R$ does not commute with the $SU(2)_A \times SU(2)_B$ flavor symmetry. It is easy to see that the three $SU(2)$ factors generate a larger $SU(4)_R$ R-symmetry, under which the fields

$$(C_1, C_2, C_3, C_4) = (A_1, A_2, B_1^\dagger, B_2^\dagger) \quad (11.4)$$

transform in the fundamental representation. Since this $SU(4) \cong Spin(6)$ is a R-symmetry, this is the first hint that the ABJM action in fact preserves $\mathcal{N} = 6$ supersymmetry. By integrating out all the auxiliary fields and writing the Lagrangian in its full glory, in term of component fields, one can check that the action is indeed invariant under this $SU(4)_R$ [202, 214]. The full global symmetry of the ABJM model is

$$SU(4)_R \times U(1)_b, \quad (11.5)$$

where the so-called baryonic symmetry $U(1)_b$ is the symmetry which assigns charge $+1$ to A_i and -1 to B_i .

The proposal of ABJM is that this Chern-Simons SCFT is dual to M-theory on $AdS_4 \times S^7/\mathbb{Z}_k$, for a particular \mathbb{Z}_k quotient which we will discuss momentarily. At large

k the theory is weakly coupled if the 't Hooft coupling $\lambda = N/k$ is small, while for λ large we can use either M-theory or type IIA string theory, as described in Chapter 8, section 8.2.

This duality is very interesting because of the large amount of supersymmetry preserved. In the last two years a great deal of work has lent credence to this proposal. Basically any of the checks that could be done in the case of the duality between $AdS_5 \times S^5$ and $\mathcal{N} = 4$ SYM has some analog in the ABJM context, and much of it has been done already.

11.1.1 The ABJM moduli space

Let us compute the moduli space of the ABJM theory. Consider first the Abelian case. The gauge group is $U(1)_k \times U(1)_{-k}$. Integrating out the auxiliary fields D_1, D_2 fixes the value of σ_1, σ_2 to

$$\sigma_1 = \sigma_2 = \frac{2\pi}{k} \sum_i (|A_i|^2 - |B_i|^2). \quad (11.6)$$

Since $W = 0$ in the Abelian case, we have the four variables A_i, B_i , which remain unconstrained³. We still have to take care of gauge invariance, however. Naively we would have to impose the $U(1)_1 - U(1)_2$ gauge symmetry, which is the only one which couples to the matter fields. However this would not give a Kähler moduli space (because due to the CS terms there is no associated moment map to make a proper Kähler quotient). A more careful consideration of the CS interactions give the resolution of this puzzle. Let us define

$$A_D = A_1 + A_2, \quad A_b = A_1 - A_2. \quad (11.7)$$

The Abelian Chern-Simons term is then written

$$S_{CS} = \frac{k}{4\pi} \int A_b \wedge dA_D, \quad (11.8)$$

Only A_b couples to the matter fields, while A_D only appears here, in that combination. We can then use the dual photon trick, writing the action in term of $F_D = dA_D$ and imposing the Bianchi identity through a Lagrange multiplier φ

$$S_{CS} = \frac{k}{4\pi} \int A_b \wedge F_D + \frac{1}{2\pi} \int \varphi \wedge dF_D. \quad (11.9)$$

Integrating out F_D , we see that the dual photon is related to the gauge field A_b , as

$$A_b = \frac{2}{k} d\varphi, \quad (11.10)$$

so we can trade one for the other, and φ is not gauge invariant. Although it is invariant under $U(1)_D$, under a $U(1)_b$ gauge transformation $A_b \rightarrow A_b + d\Lambda$ we have

$$\varphi \rightarrow \varphi + \frac{k}{2} \Lambda. \quad (11.11)$$

³The reader should contrast this with the situation in the $\mathcal{N} = 1$ conifold theory, where we have a D-term constraint $|A_i|^2 - |B_i|^2 = 0$. Here the ‘‘D-term’’ can be arbitrary, and its value only determines the value of the auxiliary fields σ_1, σ_2 .

Flux quantization implies that φ has period π (see below around equ. (11.41) for a discussion in the general case of $\mathcal{N} = 2$ CS quivers). If we gauge fix $\varphi = 0$, we still have a residual \mathbb{Z}_k gauge symmetry, $\Lambda = 2\pi n/k$, $n \in \mathbb{Z}$, which acts on the matter fields as

$$(A_i, B_i) \rightarrow (e^{\frac{2\pi i n}{k}} A_i, e^{-\frac{2\pi i n}{k}} B_i). \quad (11.12)$$

Let us summarize the logic of this computation. The naive thing to do would be to quotient by $U(1)_b$ from the start, but this would not give a complex moduli space. The resolution of this puzzle is that we can dualize the photon of $U(1)_D$, giving us an extra real moduli. We then need to quotient by $U(1)_b$, but the dual photon φ transforms under the gauge symmetry $U(1)_b$. It is convenient to get rid of φ by gauge fixing it to zero, which however leaves a discrete \mathbb{Z}_k gauge symmetry. We then have the moduli space $\mathbb{C}^4/\mathbb{Z}_k$, with \mathbb{Z}_k acting on

$$(z_1, z_2, z_3, z_4) \rightarrow (\omega z_1, \omega z_2, \omega^{-1} z_3, \omega^{-1} z_4), \quad \omega = e^{\frac{2\pi i n}{k}}. \quad (11.13)$$

We could also choose a different complex structure, for instance the one of (11.4), which makes the $SU(4)_R$ and the associated $\mathcal{N} = 6$ SUSY more manifest. We prefer to focus on the description in term of the complex structure (11.13) since it is the one induced by the particular $\mathcal{N} = 2$ notation we chose, and we are interested in generalizing the discussion to arbitrary $\mathcal{N} = 2$ Chern-Simons quivers. Note that $\mathbb{C}^4/\mathbb{Z}_k$ is toric, with the toric diagram given in Figure 11.1(b).

The non-Abelian generalization is straightforward in this case, because we can diagonalize all the matrices involved (due to the F-term relations). As expected, the moduli space of the $U(N) \times U(N)$ theory corresponds to the moduli space of N M2-branes at a $\mathbb{C}^4/\mathbb{Z}_k$ singularity,

$$\mathcal{M} = \text{Sym}^N (\mathbb{C}^4/\mathbb{Z}_k). \quad (11.14)$$

11.2 Chiral ring and monopole operators

An important check of this *AdS/CFT* duality proposal is whether the chiral ring of the large N theory matches the supergravity spectrum. Consider first the Abelian case, $N = 1$, and focus on $\mathcal{N} = 2$ chiral fields for simplicity. If we only consider the fields appearing in the Lagrangian, the gauge invariant operators are of the form $A_i B_j$, which gives an algebraic description of the conifold, of complex dimension 3, instead of the orbifold $\mathbb{C}^4/\mathbb{Z}_k$. The chiral ring also contains monopole operators, which makes the study of chiral rings a much more complicated problem than in 4d SCFT, where the classical and quantum chiral rings are isomorphic. We can have either so-called diagonal or off-diagonal fluxes along $U(1)_k \times U(1)_{-k}$. A bare monopole operator with off-diagonal fluxes $(n, -n)$ would have an electric charge (kn, kn) . Such an operator cannot be made invariant using the field content of the theory, hence it cannot appear in the spectrum. Another way to see this is that Gauss law for A_D in (11.8) would read

$$\int_{S^2} F_b = 0 \quad (11.15)$$

for any state on $\mathbb{R} \times S^2$. Hence we only have to consider the diagonal operator $T^{(n)}$, which has magnetic flux (n, n) along $U(1)_k \times U(1)_{-k}$. It has electric charge $(nk, -nk)$,

according to the discussion of the previous Chapter. It can be shown that the conformal dimension of this bare monopole operator is precisely zero [214]. We can write chiral operators of the form (take $n \geq 0$)

$$\mathcal{O}^{(n)} \sim T^{(n)} B^{kn} (AB)^l, \quad \mathcal{O}^{(-n)} \sim T^{(-n)} A^{kn} (AB)^l, \quad (11.16)$$

of conformal dimension $\frac{1}{2}kn + l$. Recall that this way of writing the monopole operators is mainly a notational convenience. Their proper definition is through the corresponding state in $\mathbb{R} \times S^2$. It can be checked that the basic monopole operators

$$\mathcal{O}_{i_1 \dots i_k}^{(-1)} = T^{(-1)} A_{i_1} \dots A_{i_k}, \quad \mathcal{O}_{i_1 \dots i_k}^{(1)} = T^{(1)} B_{i_1} \dots B_{i_k}, \quad (11.17)$$

generate the full chiral ring

$$\mathbb{C}[\mathcal{O}_{i_1 \dots i_k}^{(-1)}, \mathcal{O}_{i_1 \dots i_k}^{(1)}] \cong \text{Spec}(\mathbb{C}^4/\mathbb{Z}_k). \quad (11.18)$$

Monopole operators correspond to states which have momentum n/R along the M-theory circle (D0-branes in type IIA). The isometry $U(1)_M$ associated to the M-theory fibration corresponds to the conserved current $*F_D$ in the ABJM theory. Indeed, we have

$$\frac{k}{4\pi} *F_D = J_b, \quad (11.19)$$

where J_b is the current of the baryonic symmetry $U(1)_b$. The isometry $U(1)_M \cong U(1)_b$ corresponds to a shift of the dual photon φ or equivalently to a rotation of the phase of the bifundamental fields. This global symmetry is to be distinguished from the gauge symmetry which corresponds to a *simultaneous* shift of the dual photon and of the phase of the bifundamental fields.

11.2.1 Non-Abelian case

In the non-Abelian case, a generic diagonal monopole has fluxes $H = (n_1, \dots, n_N)$, and H lies in the root lattice of the diagonal $U(N)$. The space dual to the root lattice is the weight lattice. Any vector in the weight lattice of $U(N)$, $W = (e_1, \dots, e_N)$, determines a corresponding representation of highest weights given by W [254, 255]. Consider an ordering of W with $e_1 \geq e_2 \geq \dots \geq e_N$. Then, the corresponding Young tableau has e_1 boxes on the first row, e_2 boxes on the second row, etc. It is a result of [254] that in a $U(N)$ theory with CS level k , the monopoles H transform in the $U(N)$ representation

$$W = kH = (kn_1, \dots, kn_N). \quad (11.20)$$

It was shown in [235] that all of these bare monopole operators have vanishing R-charge at the quantum level. Of particular interest are the monopoles with flux $H = (n, 0, \dots, 0)$, which transforms in the kn -symmetric product of the fundamental representation. The gauge invariant operators one can build from this particular $T^{(n)}$ are the analog of (11.16). They should be chiral primary operators, corresponding to a chiral ring isomorphic to

$$\text{Spec}(\mathbb{C}^4/\mathbb{Z}_k), \quad (11.21)$$

and they should match to the supergravity excitations along $AdS_4 \times S^7/\mathbb{Z}_k$. On the other hand, all the other possible monopole operators do not have the right symmetries to be matched to supergravity states. Hence we expect that they are not BPS and that they acquire large anomalous dimensions. Anomalous dimensions of operators with no magnetic charge can be computed in the large k limit, for instance in [256], matching expectations. But it is not clear (to the best of my knowledge) whether it is possible to do similar perturbative computations in the presence of monopole operators. If we work in the large k limit, monopole operators decouple from the spectrum, while in the gravity dual we can consider type IIA string theory on $AdS_4 \times \mathbb{CP}^3$. It is in this context that most direct checks of the ABJM proposal have been performed.

The punchline is that monopole operators are precisely very important if one would like to understand the M-theory limit rather than the type IIA limit.

11.2.2 Enhanced SUSY

In the ABJM theory, the monopole operators have one additional virtue. They enhance the supersymmetry from $\mathcal{N} = 6$ to $\mathcal{N} = 8$ when $k = 1, 2$. Indeed, \mathbb{C}^4 and $\mathbb{C}^4/\mathbb{Z}_2$ both preserve a larger $SO(8)_R$ symmetry, and the corresponding $\mathcal{N} = 8$ SUSY, but such enhanced supersymmetry is not seen in the Lagrangian. It was shown explicitly in [235, 257] how this enhancement comes about. There are 16 conserved currents associated to the $SU(4)_R \times U(1)_b$ symmetry, which take the form

$$j_{IJ}^\mu = i \left(C_I D_\mu C_J^\dagger - (D_\mu C_I) C_J^\dagger + \text{fermions} \right), \quad (11.22)$$

where the C_I 's are as defined in (11.4). The traceless part give the currents of $SU(4)_R$, while the trace $\sum_I j_{II}$ corresponds to $U(1)_b$. When $k = 1, 2$, we can write down 12 additional gauge invariant currents, by using monopole operators with $H = (n, \dots, 0)$:

$$T^{(-n)} i (C_I D_\mu C_J - C_J D_\mu C_I + \text{fermions}), \quad (11.23)$$

written here as 6 complex currents, and $n = 1, 2$ for $k = 2, 1$, respectively. In total we have 28 conserved currents, which generate a full $SO(8)_R$ symmetry [235]. The explicit form of the on-shell $\mathcal{N} = 8$ SUSY transformations were written down in [257].

11.3 $\mathcal{N} = 2$ Abelian quivers and their classical moduli space

The ABJM theory has the structure of a quiver. At first sight this is somewhat unexpected. As we saw in Chapter 3 quivers are tightly linked with D-branes at Calabi-Yau threefold singularities. We will see in the next Chapter that this is the case here too; for instance the fact that the ABJM quiver is the conifold quiver is not a coincidence [258]. One of the themes of this second Part of the thesis is to understand this relationship better.

In this section we review the constructions of $\mathcal{N} = 2$ Chern-Simons quivers initiated in [244, 245], followed by a large number of works [251, 252, 253, 259, 260, 261, 262, 263, 258, 264, 199, 265, 266, 267]. The rules of the game are to work out the moduli space of any such Chern-Simons quiver and to check whether it can correspond to M2-branes on a CY fourfold. In the toric case we can use brane tiling methods, and an interesting algorithm

has been developed to compute the relevant branch of the moduli space [251, 252, 253], directly generalizing the forward algorithm reviewed in section 3.5.

A Chern-Simons quiver theory is a quiver in which each node correspond to a 3d vector multiplet with $U(N)$ gauge group, and with Chern-Simons interactions only (no Yang-Mills term). The Chern-Simons level of the gauge group $U(N)_i$ is written k_i , and it is a quantized integer. Apart from that the quiver has the same structure as the ones we use for D-branes: there are matter fields in bifundamental and adjoint representations only, and a superpotential. We restrict to quivers with gauge group

$$U(N)_1 \times U(N)_2 \times \cdots \times U(N)_G. \quad (11.24)$$

Different ranks for the gauge groups are possible and this is linked with either fractional M2-branes [268] or cascading solutions [269, 270]; we will not discuss such developments⁴.

From now on we will focus on the Abelian case. Consider a quiver with gauge group $U(1)^G$ and CS levels (k_1, \dots, k_G) . The matter fields are denoted either by X_α , $\alpha = 1, \dots, M$ or by X_{ij} when we want to emphasize that the field X goes from node i to node j ; we also denote $i = h(\alpha)$ and $j = t(\alpha)$ for the node at the head or tail of the arrow X . The definition of the theory also includes some superpotential $W(X)$. It should be of classical dimension 2 if we want to have a weakly coupled description at large CS levels, but in general we can consider W some arbitrary polynomial in the fields, in the hope that the theory flows to a strongly coupled fixed point for any CS levels. The Lagrangian reads

$$\mathcal{L} = \sum_i \mathcal{L}_{CS}[V_i; k_i] + \sum_\alpha \int d^4\theta \left(-X_\alpha^\dagger e^{V_{t(\alpha)}} X_\alpha e^{-V_{h(\alpha)}} \right) + \int d^2\theta W(X) + c.c. \quad (11.25)$$

The scalar superpotential is

$$V = V_F + V_D, \quad (11.26)$$

with the F-term part familiar from $\mathcal{N} = 1$ theories in 4d,

$$V_F = \sum_\alpha \left| \frac{\partial W}{\partial X_\alpha} \right|^2 \quad (11.27)$$

and

$$V_D = - \sum_i \frac{k_i}{2\pi} \sigma_i D_i + \sum_\alpha |X_\alpha|^2 (D_{t(\alpha)} - D_{h(\alpha)}) + \sum_\alpha |X_\alpha|^2 (\sigma_{t(\alpha)} - \sigma_{h(\alpha)})^2 \quad (11.28)$$

Let us define the quantity,

$$D_i = \sum_{\alpha|t(\alpha)=i} |X_\alpha|^2 - \sum_{\alpha|h(\alpha)=i} |X_\alpha|^2 = \sum_\alpha g_i[X_\alpha] |X_\alpha|^2, \quad (11.29)$$

which in four dimensions would be the $\mathcal{N} = 1$ D-term for the gauge group $U(1)_i$ (here $g_i[X_\alpha]$ is the electric charge of X_α under $U(1)_i$, which is ± 1 or 0 in our quivers). Integrating out the fields D_i , we have a simple scalar potential

$$V_D = \sum_\alpha |X_\alpha|^2 (\sigma_{t(\alpha)} - \sigma_{h(\alpha)})^2, \quad (11.30)$$

⁴In the non-toric case we also expect that some theories describing “regular” M2-branes can have gauge groups with different ranks, similarly to the case of D-branes on non-toric CY threefolds.

together with a set of constraints

$$\frac{k_i}{2\pi}\sigma_i = \mathcal{D}_i, \quad (11.31)$$

which imply $\sum_i k_i \sigma_i = 0$. To compute the moduli space of such a quiver, we impose the conditions $V_F = 0$ and $V_D = 0$. The F-flatness conditions $\partial W / \partial X_\alpha = 0$ define an algebraic variety (sometimes called the master space)

$$\mathcal{Z} = \{X_\alpha \mid dW = 0\} \subset \mathbb{C}^M, \quad (11.32)$$

like for a quiver theory in 3+1 dimensions. In addition we need to solve $V_D = 0$ and the constraints (11.31). We are interested in the branch of the moduli space for which

$$\sigma_1 = \cdots = \sigma_G \equiv \sigma, \quad (11.33)$$

in which case V_D automatically vanishes. Such a branch exists only if

$$\sum_i k_i = 0. \quad (11.34)$$

This later condition will be part of our definition of a Chern-Simons quiver, because it is necessary in order to have a Calabi-Yau fourfold ⁵ [244], for this particular branch of the moduli space. The case $\sum_i k_i \neq 0$ is associated to a non-zero Romans mass in the type IIA gravity dual [247], a situation we will not consider. On the branch (11.33), σ is determined by the quantities \mathcal{D}_i ,

$$\frac{\sigma}{2\pi} = \sum_i \frac{k_i}{|k|^2} \mathcal{D}_i. \quad (11.35)$$

The constraints (11.31) reduce to $G - 2$ constraints

$$\sum_i v_i \mathcal{D}_i = 0 \quad \forall \{v_i\} \text{ such that } \sum_i v_i k_i = 0. \quad (11.36)$$

In other words, in contrast to quivers in 3+1 dimensions, we only impose $G - 2$ D-flatness conditions, corresponding to the subgroup of $U(1)^G$ orthogonal to both $\sum_i U(1)_i$ and $\sum_i k_i U(1)_i$. The moduli space is then a Kähler quotient $\mathcal{Z} // U(1)^{G-2}$. However we must be careful with our gauge fixing. Let us define the photons

$$A_D = \sum_{i=1}^G A_i, \quad A_b = \frac{1}{q} \sum_{i=1}^G k_i A_i \quad (11.37)$$

where $q = \gcd\{k_i\}$. One can show that the only way A_D enters the action is through a term [244]

$$\frac{q}{2\pi G} \int A_b \wedge dA_D. \quad (11.38)$$

Dualizing A_D , we have

$$d\varphi = \frac{q}{G} A_b, \quad (11.39)$$

⁵For a non-toric quiver with different ranks N_i the condition to have a CY₄ is $\sum_i k_i N_i = 0$.

with gauge transformation

$$A_b \rightarrow A_b + d\Lambda, \quad \varphi \rightarrow \varphi + \frac{q}{G}\Lambda. \quad (11.40)$$

The periodicity of φ depends on the allowed magnetic fluxes for all the $F_i = dA_i$. On this particular branch (11.33), we need $F_1 = \dots = F_G \equiv F$ to preserve supersymmetry [244]; when $\varphi \rightarrow \varphi + c$ we then have

$$\delta S = \frac{c}{2\pi} \int_{S^2} F_D = \frac{cG}{2\pi} \int_{S^2} F = cGn, \quad (11.41)$$

and since the quantum theory is invariant if $\delta S = 2\pi n$, φ has periodicity $2\pi/G$. Hence the gauge fixing of φ leaves a residual symmetry

$$\mathbb{Z}_q \subset U(1)_b, \quad (11.42)$$

which we must impose. As a result, the moduli space is a \mathbb{Z}_q quotient of the Kähler quotient of the master space \mathcal{Z} by $U(1)^{G-2}$:

$$\mathcal{M} = (\mathcal{Z}/U(1)^{G-2})/\mathbb{Z}_q. \quad (11.43)$$

We call this branch of the moduli space the geometric branch; it can also be called the Coulomb branch, since there is an unbroken $U(1)_D$ gauge symmetry. If this Coulomb branch is a CY_4 , it is tempting to guess that the quiver describes the low energy theory of a single M2-brane at the tip of the CY fourfold. Further checks in that direction would be the matching of the other branches of the moduli space to partial resolutions of the CY_4 singularity; work in that direction was performed e.g. in [262, 264, 271], but much remains to be understood.

11.3.1 Toric Chern-Simons quivers and the Kasteleyn matrix algorithm

In the case of so called toric quivers, as defined in Chapter 3, section 3.5, we can use a generalized fast forward algorithm to find the moduli space [251, 252, 253]. The output of this algorithm is the 3d toric diagram for a Calabi-Yau fourfold. The extra information in the 2+1 dimensional case are the CS levels k_i . One assigns an integer n_{ij} to each bifundamental field X_{ij} . The CS levels are then defined to be [251, 252]

$$k_i = \sum_j (n_{ij} - n_{ji}), \quad (11.44)$$

where the sum is over all the edges X_{ij} around the face i in the brane tiling; each edge contributes $+n_{ij}$ ($-n_{ij}$) to the group on its right (left), looking towards the white node. With this parametrization, the constraint $\sum_i k_i = 0$ is automatically satisfied. In Figure 11.2 we show the refined brane tiling for the dP_1 quiver of section 3.5.4.

The Kasteleyn matrix algorithm proceeds as in the 3+1 dimensional case, except that we add one more formal variable z in the definition of the Kasteleyn matrix,

$$K_{mn} = \sum_{\gamma \in \{m \rightarrow n\}} X_\gamma x^{\langle X_\gamma, \gamma_x \rangle} y^{\langle X_\gamma, \gamma_y \rangle} z^{n_\gamma}. \quad (11.45)$$

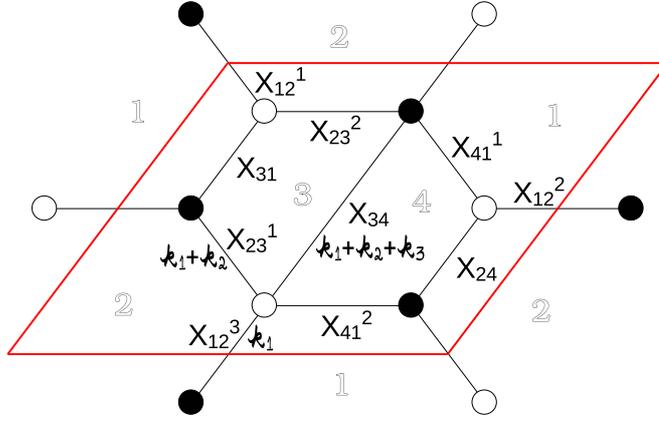


Figure 11.2: Brane tiling for the dP_1 quiver, refined with weights n_α . The corresponding Chern-Simons quiver has CS levels $(k_1, k_2, k_3, -k_1 - k_2 - k_3)$.

Computing the permanent, we have⁶

$$\text{perm } K = \sum_k t_k x^{h_x(k)} y^{h_y(k)} z^{h_z(k)}. \quad (11.46)$$

The slopes h_x, h_y are defined as in (3.43), while h_z is

$$h_z(t_k) = \sum_\alpha M_{k\alpha} n_\alpha. \quad (11.47)$$

We easily see that the points

$$(h_x, h_y, h_z) \in \mathbb{Z}^3 \quad (11.48)$$

describe the toric diagram of the CY_4 (the geometric branch) of the last section. Simply, we have that (in the notation of section 3.5)

$$\sum_k (Q_F)^l_k h_z(k) = 0, \quad \sum_k (G_D)^i_k h_z(k) = \sum_\alpha \mathcal{Q}_\alpha^i n_\alpha = k_i. \quad (11.49)$$

The first relation follows from (11.47) and (3.33), while the second follows from (11.47), (3.32) and the definition of the weights n_α for the fields X_α . It means that we only impose a $U(1)^{G-2}$ subgroup of the quiver gauge symmetry, corresponding to the electric charges orthogonal to $U(1)_b$ (and $U(1)_D$).

For our dP_1 example, the improved perm K is

$$\begin{aligned} \text{perm } K = & X_{31} X_{41}^1 X_{41}^2 + X_{12}^2 X_{23}^2 X_{41}^2 x + X_{12}^1 X_{12}^2 X_{41}^1 x^{-1} y + \\ & + X_{12}^1 X_{12}^2 X_{12}^3 z^{k_1} + X_{12}^3 X_{24} X_{31} y^{-1} z^{k_1} + X_{23}^1 X_{23}^2 X_{24} z^{k_1+k_2} + \\ & + X_{24} X_{31} X_{34} z^{k_1+k_2+k_3} + X_{12}^1 X_{12}^2 X_{34} y z^{k_1+k_2+k_3}. \end{aligned} \quad (11.50)$$

Notice that setting $z = 1$ we reproduce the 3+1 dimensional algorithm. Thus the projection of the 3d toric diagram of the 2+1 dimensional theory on the plane $h_z = 0$ is the 2d toric diagram of the 3+1 dimensional theory.

⁶Note that we change notation with respect to Chapter 3: from now on we will denote the perfect matching variables by t_k instead of p_k .

If $q = \gcd\{k_i\} > 1$, some internal points are not represented by any perfect matching, and the result of the GLSM has to be quotiented by \mathbb{Z}_q . Alternatively we can include all points of the toric diagram in the GLSM, at the price of adding new fields and gauge symmetries.

According to (11.49), the coordinates h_z determine the particular $U(1)_b \subset U(1)^G$, whose photon A_b is related to the dual photon φ of the diagonal gauge group. As we discussed already in the ABJM case (and we will come back to the point at length in Chapter 13), this $U(1)_b$ corresponds to the M-theory circle in the geometric setup.

11.4 A look at proposals for M2-brane theories

In the discussion of the previous section, we did not assume that the CS quiver had a well defined “parent quiver” in 3+1 dimension. In 3+1 dimensions, there are various consistency conditions that a quiver has to satisfy in order to describe D3-branes at a singularity, see *e.g.* [72]. One such condition is cancellation of gauge anomalies, which restricts the allowed matter content. Another condition is that the quiver theory should flow to a superconformal fixed point in the IR; basically this restricts the field content so that the number of color and flavor in each node is in the Seiberg conformal window (see Appendix C). In 2+1 dimensions, none of these restrictions are necessary: there is no continuous anomaly and almost any matter content is believed to lead to a SCFT in the infrared.

In the absence of a clear understanding of the link between M2-branes and quivers, this led to the study of many proposals for M2-brane theories [244, 245, 251, 252, 253, 259, 260, 261, 262, 263, 258, 264, 199, 265, 266, 267, 272]. The main tool to construct such theories is that their moduli space reproduces the expected CY_4 geometry, and that one can understand the partial resolution from one singularity to another one as Higgsing in the quiver, much like in 3+1 dimensions [262, 264, 271].

It is fair to say that most of the studies of such quivers in the literature consider the gauge theory at the classical level only. This often implies taking a large k (CS levels) limit, even though in general the anomalous dimensions are large and there really does not exist any weak coupling limit.

11.4.1 Brane tilings with multiple bounds

Of particular interest to us is a new kind of brane tiling which is not obviously inconsistent in 2+1 dimensions: the so-called multiple bounds brane tilings. These are tilings in which a white and a black node are connected by two or more edges (an example with double bounds is shown in Figure 11.3(a)).

The associated quiver has gauge groups with only one arrow entering and one arrow leaving; in 3+1 dimensions this corresponds to $\mathcal{N} = 1$ SQCD with $N_f = N_c$, which is certainly not conformal, and this is why such tilings never appear when considering D3-branes at CY_3 singularities.

A particular property of such tilings is that there are several perfect matchings which have the same slopes (h_x, h_y) , but different values of h_z . Indeed, consider a tiling with no multiple bound, and choose some edge X , which is contained in some number of perfect matchings $\{t_X\}$. We obtain a n -tuple bound tiling simply by replacing this edge

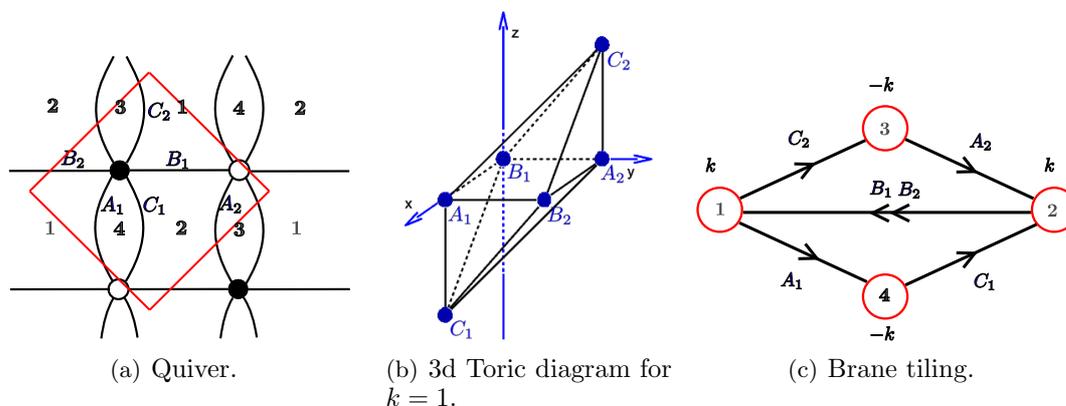


Figure 11.3: Brane tiling, quiver and toric diagram of the $Q^{1,1,1}$ model of [259]. Notice the two double-bounds in the brane tiling. The superpotential is $W = -\epsilon^{ij} A_i C_1 B_i C_2 A_2 B_j$.

X by n edges C_1, \dots, C_n . For any perfect matching p_X of the single bound quiver there are n perfect matchings in this new tiling. For a generic assignment of the weights n_α to the new edges C_α , we have $n - 1$ new points in the toric diagram, with coordinates (h_x, h_y, h_z) , but with the particular property that they all project to the *same* point (h_x, h_y) when $z = 0$. This means that in general the toric diagram contains a line of $h + 1$ external points (where h depends on the choice of n_α). This corresponds to a *complex codimension 2 singularity*, which is locally $\mathbb{C}^2 \times \mathbb{C}^2 / \mathbb{Z}_h$. A difficulty with the interpretation of the multiple bound theories as M2-brane theories is that they do not seem to account for such a non-isolated singularity: there are extra modes with $SU(h)$ gauge symmetry localized at the singularity in M-theory (or h D6-branes in a type IIA reduction), which are not seen in the field theory.

In Chapter 13 we will present an alternate description of the same geometries with codimension 2 singularities. It might be that the multiple bound theories are somewhat dual to the theories of Chapter 13. Another possibility is that the multiple bound theories are actually inconsistent (they do not describe M2-branes).

Example: a double-bound tiling for $Q^{1,1,1}$.

An interesting example is the CS quiver proposed in [259] as a candidate for the description of M2-branes on the $C(Q^{1,1,1})$ singularity. This CS quiver theory was further studied in [262, 273]. The Sasaski-Einstein seven manifold $Q^{1,1,1}$ is a homogenous space $(SU(2) \times SU(2) \times SU(2))/U(1)^2$, and it can also be seen as a regular circle fibration of $\mathbb{C}P^1 \times \mathbb{C}P^1 \times \mathbb{C}P^1$. For these reasons it is a nice space to study, much like the five dimensional $T^{1,1}$ in type IIB.

The brane tiling is shown in Figure 11.3(a). It can be obtained from the conifold tiling by doubling both edges A_1 and A_2 (compare to Figure 3.3(b) in Chapter 3), as

$$A_1 \rightarrow A_1, C_1, \quad A_2 \rightarrow C_2, A_2. \quad (11.51)$$

The quiver is the one of Figure 11.3(c), with CS levels $(k, k, -k, -k)$ and sextic superpotential

$$W = A_1 C_1 B_2 C_2 A_2 B_1 - A_1 C_1 B_1 C_2 A_2 B_2. \quad (11.52)$$

The permanent of the Kasteleyn matrix is

$$\text{perm } K = A_x + A_2 y + B_1 + B_2 x y + C_1 x z^{-k} + C_2 y z^k. \quad (11.53)$$

From this we can read the toric diagram of $Q^{1,1,1}/\mathbb{Z}_k$, shown in Fig. 11.3(b). In this simple case, each perfect matching corresponds to a single field since the Abelian superpotential vanishes. The geometric branch of the moduli space is described by the GLSM

	A_1	A_2	B_1	B_2	C_1	C_2	
$U(1)_{B_1}$	1	1	-1	-1	0	0	
$U(1)_{B_2}$	0	0	1	1	-1	-1	
\mathbb{Z}_k	-1	0	0	0	1	0	(11.54)

The last line keeps track of the charge of the homogeneous coordinates under the $\mathbb{Z}_k \subset U(1)_b$ by which we have to quotient $C(Q^{1,1,1})$:

$$(A_1, A_2, B_1, B_2, C_1, C_2) \sim (e^{-\frac{2\pi i}{k}} A_1, A_2, B_1, B_2, e^{\frac{2\pi i}{k}} C_1, C_2). \quad (11.55)$$

We see that the quotient breaks the $SU(2)^3$ isometry of $Q^{1,1,1}$ to $SU(2)$. At $k = 1$ we expect an enhanced symmetry due to the monopole operators. Indeed, by using the diagonal monopoles operators T, \tilde{T} which have magnetic flux ± 1 along $U(1)_D$, we can form the $SU(2)$ doublets

$$(A_1, T A_2), \quad (B_1, B_2), \quad (C_1, \tilde{T} C_2), \quad (11.56)$$

which allows to restore the larger $SU(2) \times SU(2) \times SU(2)$ symmetry. For $k > 1$, there is a column of k points below A_1 , and similarly above A_2 . This corresponds to the fact that the \mathbb{Z}_k action (11.55) has a locus of fixed points, consisting of two branches $\{A_1 = C_1 = 0\}$ and $\{A_2 = C_2 = 0\}$. The topology of each of these branches is \mathbb{C}^2 .

In the next Chapter we will obtain a better string theory understanding of $\mathcal{N} = 2$ Chern-Simons quivers related to M2-brane theories, at least in the toric case. At the time of writing, the status of the multiple-bounds brane tilings in that overall picture is still problematic, since it is not known how or whether one can obtain them by some stringy duality from better established theories.

Chapter 12

Chern-Simons quivers from stringy dualities

GIVEN the construction of Chern-Simons quivers reviewed in the last chapter, it is natural to wonder why does the quiver structure appear at all. After all, quivers are useful tools to describe D-branes, but it is not *a priori* clear that M2-branes should behave similarly. For instance, it is known that M2-branes theories do not behave like D-brane theories upon orbifolding [214, 274]: the Douglas-Moore [39] orbifolding procedure (see Chapter 3) on a M2-brane quiver for a space X does not lead to a quiver for M2-branes on X/Γ .

In this Chapter we explain the present understanding of the relation between M2-branes and quivers. This understanding is always through some string theory duality. In the first section we explain how the ABJM theory and some $\mathcal{N} = 3$ supersymmetric generalizations can be obtained from a type IIB brane setup. In the second section, we explain an interesting generalization of the ABJM quiver which naturally arises from this type IIB understanding: we consider adding flavors (fields in the fundamental/anti-fundamental representation) to the ABJM quiver. This prepares the ground for the understanding of the next Chapter.

In the last section we will consider a different understanding of Chern-Simons quiver, which comes about through the type IIA/M-theory duality. Part of that section is based on our paper [3], written in collaboration with Stefano Cremonesi and Francesco Benini.

12.1 Fivebrane systems, M-theory/type IIB duality and ABJM

In the original paper [202] the relation between the CS quiver theory and M2-branes on $\mathbb{C}^4/\mathbb{Z}_k$ was given a nice string theory derivation through an interesting Hanany-Witten setup in type IIB. This brane setup engineers a YM-CS-matter theory which flows to the CS-matter ABJM theory in the IR.

Consider type IIB string theory. Take two NS5-branes on a circle with coordinate x^6 and N D3-branes wrapping this circle. Let us put the NS5-brane at $x^6 = 0$ and the NS5' at $x^6 = \pi R_6$ (assuming $x^6 \sim x^6 + 2\pi R_6$). This is T-dual to D2-branes on $\mathbb{C} \times \mathbb{C}^2/\mathbb{Z}_2 \times \mathbb{R}$

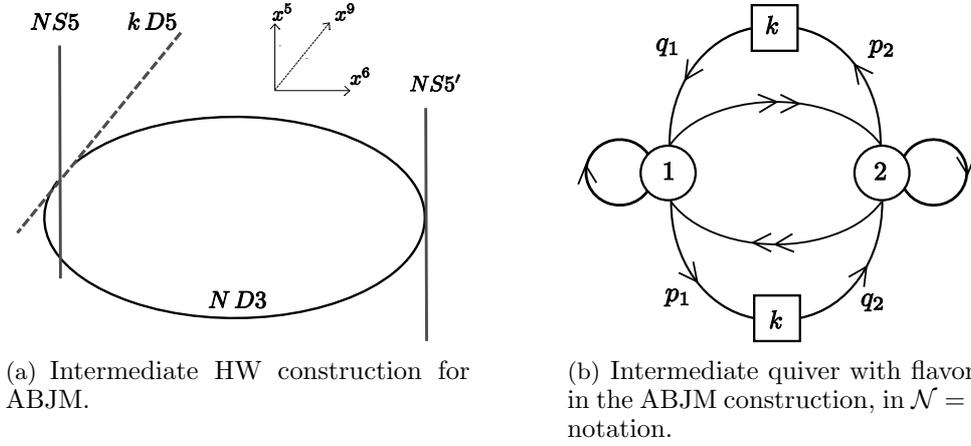


Figure 12.1: Intermediate step in the fivebrane construction of the ABJM theory. We show both the branes and the low energy quiver which lives on the D3-branes. The boxes represent the $U(k)$ flavor groups living on the stack of D5-branes. There are two independent flavor groups because the D5-branes can break in two halves on the NS5-brane.

in type IIA, similarly to the discussion of Section 3.4 in Chapter 3. The low energy theory is the $\mathbb{C} \times \mathbb{C}^2/\mathbb{Z}_2$ quiver with Yang-Mills terms and no CS levels. It has $\mathcal{N} = 4$ SUSY, as well as an explicit R-symmetry $SO(4)_R \cong SO(3) \times SO(3)$ which corresponds to rotations in (x^3, x^4, x^5) and (x^7, x^8, x^9) (see (12.1 below)). To introduce Chern-Simons levels, we have to replace one of the NS5-branes by a $(1, k)$ -fivebrane [275, 276]. An interesting way to see this is given in [276, 202]. Let us add k D5-branes to the setup, as follows:

	0	1	2	3	4	5	6	7	8	9
N D3	○	○	○				○			
NS5, NS5'	○	○	○	○	○	○				
k D5	○	○	○	○	○					○

(12.1)

and such that the D5- and the NS5-brane intersect at $x^6 = x^7 = x^8 = 0$. This breaks the supersymmetry to $\mathcal{N} = 2$ in the $2 + 1$ dimensional low energy theory on the D3-branes. The effect of the D5-branes is to add $\mathcal{N} = 2$ massless flavors in the theory, as depicted in Fig. 12.1(b). The flavor group is $U(k) \times U(k)$, corresponding to the fact that the D5-branes might break on the NS5-brane. These flavors give us a way to generate a Chern-Simons term, because in $2 + 1$ dimensions integrating out charged fermions can generate such a term in the effective action [277, 278]. This is due to a one-loop contribution of the form¹

$$\frac{N_f}{2} \int \frac{d^3q}{(2\pi)^3} A_\mu(-p) \text{tr} \left(\gamma^\mu \frac{\gamma^\rho (p_\rho + q_\rho) + im}{(p+q)^2 + m^2} \gamma^\nu \frac{\gamma^\sigma p_\sigma + im}{(p+q)^2 + m^2} \right) A_\nu(p), \quad (12.2)$$

with N_f the number of Dirac fermions (all of mass m), and the trace is over fermionic indices. The Chern-Simons term originates from the contribution proportional to $\text{tr}(\gamma^\mu \gamma^\nu \gamma^\rho) = 2\epsilon^{\mu\nu\rho}$. For N_f massive Dirac fermions in the fundamental or anti-fundamental represen-

¹We write the formula for the Abelian case, for simplicity.

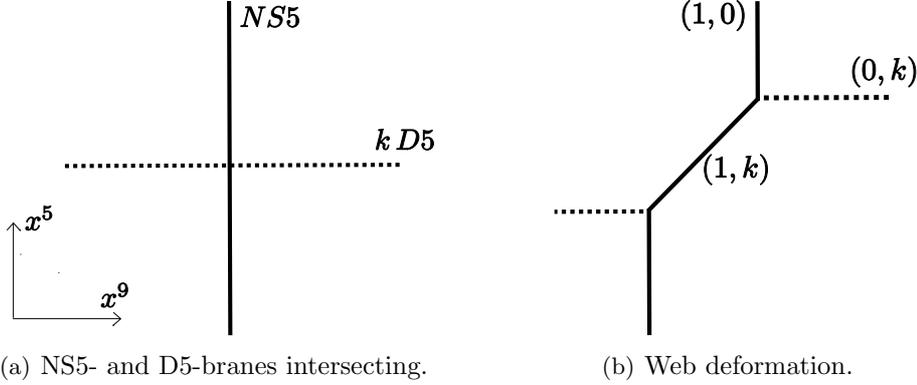


Figure 12.2: Web deformation of the NS5-/D5-brane intersection. The D3 branes intersect the $(1, k)$ -fivebrane in the middle.

tation of $U(N)$, the CS level of the gauge group is shifted by

$$k \rightarrow \frac{N_f}{2} \frac{m}{|m|}, \quad \text{when } |m| \rightarrow \infty. \quad (12.3)$$

This effects is possible in a $\mathcal{N} = 2$ supersymmetric theory because in 2+1 dimensions one can give a real *real* mass to a chiral multiplet X through the Lagrangian term [219, 217]

$$\int d^4\theta X^\dagger e^{2im\theta\bar{\theta}} X. \quad (12.4)$$

This term contains a real mass term $im\bar{\psi}\psi$ for the Dirac fermion in the chiral multiplet. It also breaks parity, like the Chern-Simons term generated in the IR effective theory. In our case, we can give real masses to ψ_q and ψ_p . Let us choose

$$\int d^4\theta (-p_1^\dagger e^{2im\theta\bar{\theta}} p_1 - q_2^\dagger e^{-2im\theta\bar{\theta}} q_2 - q_1^\dagger e^{2i\tilde{m}\theta\bar{\theta}} q_1 - p_1^\dagger e^{-2i\tilde{m}\theta\bar{\theta}} p_2). \quad (12.5)$$

Taking $m = -\tilde{m}$ (vector mass) corresponds to moving the D5-brane along the x^5 direction (equivalently one could give a VEV $\sigma = m = -\tilde{m}$ to the scalar in the $\mathcal{N} = 2$ vector multiplet). We are rather interested in the mass deformation with $m = \tilde{m} = m_A$, the so-called axial mass. In that case the fermion mass terms are

$$im_A(\bar{\psi}_{p_1}\psi_{p_1} + \bar{\psi}_{q_1}\psi_{q_1} - \bar{\psi}_{p_2}\psi_{p_2} - \bar{\psi}_{q_2}\psi_{q_2}). \quad (12.6)$$

Integrating out all the fermions, we obtain Chern-Simons levels k and $-k$ for the two gauge groups. In the brane picture, the axial mass corresponds to a web-deformation of the fivebranes [279], as shown in Figure 12.2. This effect is possible because the D5-brane can break into two halves on the NS5, each half being a localized source for the scalar $\phi^9 \sim x^9$ on the NS5-brane. The profile of the NS5-brane in (x^5, x^9) is given by the solution to a 1-dimensional Laplace equation [279]

$$\partial_5^2 \phi^7(x^5) \sim \frac{g_s}{2} \{k\delta(x^5 - 2\pi\alpha' m_A) - k\delta(x^5 + 2\pi\alpha' m_A)\}, \quad (12.7)$$

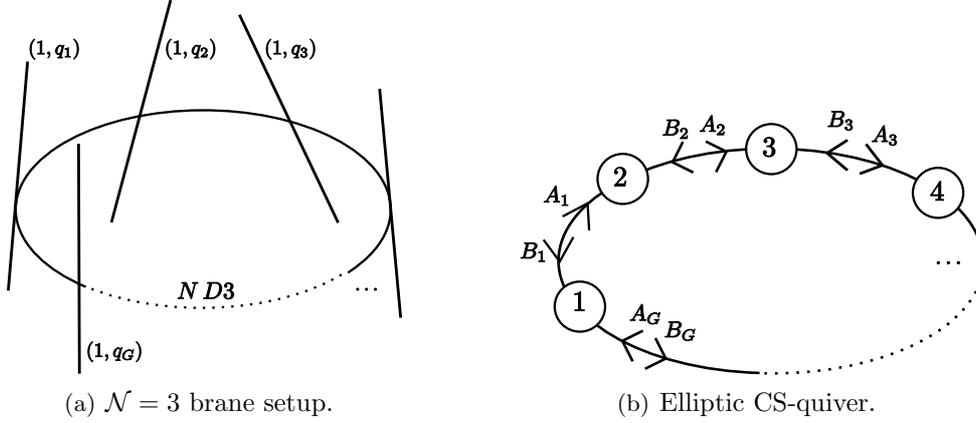


Figure 12.3: $\mathcal{N} = 3$ brane setup and the corresponding elliptic CS-quiver.

which gives Figure 12.2(b). The brane stretching at angle must be a $(1, k)$ -fivebrane bound state, because of flux conservation. The angle θ between the NS5- and the $(1, k)$ is fixed by $\mathcal{N} = 2$ supersymmetry in term of the type IIB axio-dilaton; when $C_0 = 0$, we have $\tan \theta = g_s k$, as follows from (12.7). Sending $m_A \rightarrow \infty$, we are left with only the $(1, k)$ -fivebrane, together with the remaining NS5'-brane. The low energy theory is the $\mathbb{C} \times \mathbb{C}^2/\mathbb{Z}_2$ quiver with $\mathcal{N} = 2$ Chern-Simons terms. We can obtain a $\mathcal{N} = 3$ theory by adding a mass term $m = \pm k/8\pi$ for the complex adjoint chiral superfields Φ_1, Φ_2 , which is the $\mathcal{N} = 3$ completion of the Chern-Simons interaction (we already encountered it at the end of Chapter 9). This corresponds to a rotation of the NS5'-brane by a angle θ in both the planes (x^3, x^7) and (x^4, x^8) . Thus we have reconstructed the Yang-Mills-Chern-Simons UV completion of the ABJM theory from a type IIB brane setup.

This $\mathcal{N} = 3$ brane setup has a simple M-theory dual, which is obtained by T-dualizing along x^6 and further uplifting to M-theory. In fact the hyper-Kähler metric of the dual M-theory geometry is precisely known [280], in term of a $U(1)^2$ fibration over \mathbb{R}^6 . The D3-branes are simply M2-branes on that geometry. These hyper-Kähler spaces are interesting 8 dimensional generalizations of a multi-Taub-NUT. The IR limit on the M2-branes amounts to zooming on the origin of the geometry, which has the effect of washing out the effect of the Taub-NUT-like structure (the topology $\mathbb{R}^6 \times S^1 \times S^1$ at infinity) and leaving us with a conical singularity at the origin.

Here we are simply interested in the complex structure of such geometries, with the implicit understanding that we perform the above near-horizon limit, so that we obtain a conical hyper-Kähler metric. In the case of ABJM, the complex structure is $\mathbb{C}^4/\mathbb{Z}_k$. In the low energy theory only the Chern-Simons interactions remain.

12.1.1 $\mathcal{N} = 3$ generalizations

This fivebrane realization of ABJM is easily generalized to a family of $\mathcal{N} = 3$ quiver-Chern-Simons theories [281, 282], the so-called elliptic quivers. The brane configuration consists of G fivebranes along a circle, as shown in Figure 12.3(a). The i -th brane has

charge $(1, q_i)$, and it spans the directions:

	0	1	2	6	[3,7]	[4,8]	[5,9]	
$(1, q_i)$ -brane	○	○	○		$\theta_i/$	$\theta_i/$	$\theta_i/$	(12.8)
N D3	○	○	○	○				

In this table we also included N D3-branes along x^6 . The orientation angle θ_i in each plane (x^i, x^{i+4}) is such that $\tan \theta_i = q_i$ (for $\tau = i$). The quiver (Fig. 12.3(b)) is made of G gauge groups connected through a single loop. The Lagrangian is similar to the one of (11.1), with the $\mathcal{N} = 3$ superpotential

$$W = \sum_{i=1}^G \frac{2\pi}{k_i} (A_i B_i - B_{i-1} A_{i-1})^2. \quad (12.9)$$

The Chern-Simons levels (k_1, \dots, k_G) are determined in term of the fivebrane charges,

$$k_i = q_i - q_{i-1}, \quad \Leftrightarrow \quad q_i = \sum_{j=1}^i k_j. \quad (12.10)$$

Note that a transformation which shifts all the q 's by a constant does not affect the CS levels. We can choose the last fivebrane ($i = G$) to be a NS5-brane (by performing a $Sl(2, \mathbb{Z})$ transformation $\tau \rightarrow 1/(1 - q_G \tau)$).

12.1.2 Moduli space and hyper-toric geometry

In [282], it was shown that the moduli space of this $\mathcal{N} = 3$ quiver is a hyper-toric variety. Let us briefly explain this geometric construction. Hypertoric geometry is a generalization of the Kähler approach to toric geometry, jumping from the field \mathbb{C} to \mathbb{H} (the quaternions). A nice account is provided in [283]. A hypertoric variety is a $4n$ -real-dimensional variety admitting a $(\mathbb{C}^*)^n$ torus action (much like a $2n$ -real-dimensional toric variety does), which is hyper-Kähler (i.e. admits 3 complex structures satisfying the quaternion algebra).²

Let us define a set of d vectors (u_i) in a lattice $N \cong \mathbb{Z}^n$, defining a linear map

$$\beta \equiv (u_1, \dots, u_d) : \mathbb{R}^d \rightarrow \mathbb{R}^n \quad (12.11)$$

and define the charge covectors (Q^1, \dots, Q^{d-n}) as spanning the kernel of β . Then the variety $M(u, \lambda)$ is defined as the hyper-Kähler quotient

$$M(u, \lambda) = \mathbb{H}^d // G, \quad (12.12)$$

with $G = \ker \beta$. Here λ stands for hyper-Kähler parameters. Choosing one of the 3 complex structures, one can split each quaternion variable h as $h = (z, w) \in \mathbb{C}^2$. The torus \mathbb{C}^* acts on h as

$$t \cdot h = (tz, t^{-1}w) \quad (12.13)$$

²We see that, confusingly, a hyper-toric variety is generally *not* toric, since the typical isometry of a hyper-toric variety of complex dimension $2n$ is $U(1)^n$ instead of $U(1)^{2n}$.

Hence, we have $\mathbb{H}^n \cong (z_i, w_i)$. The quotient is taken by defining a triplet of moment maps

$$\mu_a^{\mathbb{R}} = \frac{1}{2} \sum_i Q_a^i (|z_i|^2 - |w_i|^2), \quad (12.14)$$

$$\mu_a^{\mathbb{C}} = \sum_i Q_a^i z_i w_i. \quad (12.15)$$

for each $U(1)_a$, $a = 1, \dots, d - n$ and setting

$$\mu_a^{\mathbb{R}} = \lambda_a^{\mathbb{R}}, \quad \mu_a^{\mathbb{C}} = \lambda_a^{\mathbb{C}}. \quad (12.16)$$

where the constants λ are the hyper-Kähler parameters. To obtain a cone, we need to set $\lambda = 0$. Then, we quotient by the $U(1)^{d-n}$ gauge symmetry. There is an important subtlety, much like for the toric case, in defining $\ker \beta$. If (u) only generate a sublattice N' of N , then we have to quotient also by the discrete group $\Gamma = N/N'$:

$$G = U(1)^{d-n} \times \Gamma. \quad (12.17)$$

In our case, we set $n = 2$, so that the space is eight dimensional. The hyper-toric variety $M(u, \lambda)$ is then defined by a set of $d = G$ integer vectors on the *plane*. In that case, the Theorem (4.1) of [283] says that $M(u, 0)$ is the Riemannian cone over a compact 3-Sasakian manifold if and only if: (i) every subset of two vectors in (u) is linearly independent and (ii) every single vector in (u) is a part of a \mathbb{Z} -basis of \mathbb{Z}^2 .

If condition (ii) of the theorem does not hold, but instead one generates a sublattice of \mathbb{Z}^2 , the hyper-Kähler cone has non-isolated quotient singularities.

The crucial observation of [282] is that the metric β of vectors in the hyper-toric fan corresponds to the charges of the (p, q) branes in the type IIB setup. For the setup of Fig. 12.3(a), we have

$$\beta = (u_1, \dots, u_G) = \begin{pmatrix} 1 & 1 & \cdots & 1 & 1 \\ q_1 & q_2 & \cdots & q_{G-1} & 0 \end{pmatrix}. \quad (12.18)$$

This characterization of the moduli space in term of the hyper-Kähler quotient (12.12) allows to use many known mathematical results to study the associated *AdS/CFT* correspondence. It also suggests interesting generalizations, to which we now turn.

12.2 $\mathcal{N} = 3$ Flavors in ABJM: the Gaiotto-Jafferis construction

The generic hyper-toric geometry characterized by

$$\beta = \begin{pmatrix} p_1 & p_2 & \cdots & p_d \\ q_1 & q_2 & \cdots & q_d \end{pmatrix} \quad (12.19)$$

can be realized as a type IIB setup involving (p, q) branes [280]. However, we have almost no understanding of the field theory on D3-branes suspended between arbitrary (p, q) -branes, so we cannot use that construction to study explicit instances of the *AdS₄/CFT₃* correspondence.

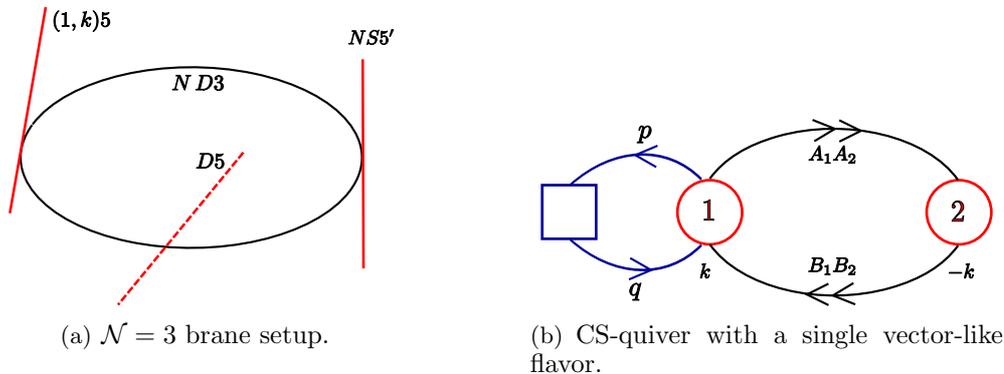


Figure 12.4: The Gaiotto-Jafferis $\mathcal{N} = 3$ brane setup and the corresponding CS-quiver with flavors.

There is however one case which we understand very well, which is simply the case of D3-branes intersecting (0, 1)-fivebranes (D5-branes) [54]. A D5-brane intersecting a stack of D3-branes introduces fields in the fundamental and antifundamental representation of the $U(N)$ gauge group on the D3-branes, also called *flavors*. Let us consider in detail an example with

$$\beta = \begin{pmatrix} 1 & 1 & 0 \\ k & 0 & 1 \end{pmatrix}. \tag{12.20}$$

This corresponds to adding a single D5-brane to the ABJM setup, in between the NS5'- and the $(1, k)5$ -brane, for instance. The brane construction and the $\mathcal{N} = 3$ quiver with flavors are shown in Fig. 12.4. The base of the cone characterized by (12.20) is a smooth \mathbb{Z}_k orbifold of a space called $N^{0,1,0}$, which was well studied in the string theory context, for instance in [284] in the early days of the AdS/CFT correspondance. It is a homogenous space of the form $SU(3)/U(1)$, with isometry group $SU(3) \times SU(2)$.

In the following, we will construct this space as a affine variety, and show how this geometry is reproduced by the moduli space of the flavored quiver (when $k = 1$, although the generalization is straightforward), following Gaiotto and Jafferis [285]. We use $\mathcal{N} = 2$ supersymmetric notations because we will generalize this kind of analysis to $\mathcal{N} = 2$ models in the next chapter. Note that $\mathcal{N} = 3$ flavors were also studied in [286, 287].

12.2.1 The $N^{0,1,0}$ space as an algebraic variety

Let us consider the geometry characterized by the hyper-toric fan (12.20) with $k = 1$. It is the cone over $N^{0,1,0}$, whose resolution is the cotangent bundle T^*CP^2 [283]. To the hyper-toric fan, we associate the GLSM

$$\frac{}{U(1)} \left| \begin{array}{cccccc} z_1 & z_2 & z_3 & w_1 & w_2 & w_3 \\ 1 & -1 & k & -1 & 1 & -k \end{array} \right. \tag{12.21}$$

A useful point of view is that the GLSM of a hyper-toric variety defines a toric variety, and that the hyper-toric variety is obtained by imposing additional ‘‘F-terms relations’’

(12.15). In our example, we have the following 9 gauge invariant complex variables:

$$\begin{aligned} x_0 &= z_1 z_2, & x_1 &= z_1 w_1, & x_2 &= z_1 w_3, \\ x_3 &= z_3 z_2, & x_4 &= z_3 w_1, & x_5 &= z_3 w_3, \\ x_6 &= w_2 z_2, & x_7 &= w_2 w_1, & x_8 &= w_2 w_3, \end{aligned} \quad (12.22)$$

and the “intermediate” 5-complex-dimensional toric variety is defined by

$$\begin{aligned} x_0 x_4 - x_1 x_3 &= 0, & x_0 x_5 - x_2 x_3 &= 0, & x_0 x_7 - x_1 x_6 &= 0, \\ x_0 x_8 - x_2 x_6 &= 0, & x_1 x_5 - x_2 x_4 &= 0, & x_1 x_8 - x_2 x_7 &= 0, \\ x_3 x_7 - x_4 x_6 &= 0, & x_3 x_8 - x_5 x_6 &= 0, & x_4 x_8 - x_5 x_7 &= 0, \end{aligned} \quad (12.23)$$

Moreover, we have an additional “F-term” condition $\mu^{\mathbb{C}} = 0$:

$$x_1 + x_5 - x_6 = 0, \quad (12.24)$$

which allows to eliminate x^5 , say, in term of the other variables. For $k > 1$, the story is similar, although more tedious to write down.

12.2.2 Recovering $N^{0,1,0}$ from the quantum chiral ring

Consider now the theory of Figure 12.4. We have the chiral superfields fields p and q , in the fundamental and anti-fundamental of the first gauge group, and the bifundamental fields A_i, B_i as in ABJM. The superpotential is fixed by $\mathcal{N} = 3$ supersymmetry,

$$W = \frac{2\pi}{k} (A_1 B_1 - A_2 B_2 + pq)^2 - \frac{2\pi}{k} (B_2 A_2 - B_1 A_1)^2. \quad (12.25)$$

For $p = q = 0$, it reduces to the ABJM superpotential (11.3). The explicitly realized global symmetry is

$$SU(2)_R \times SU(2)_F \times U(1)_B. \quad (12.26)$$

The “baryonic” symmetry $U(1)_B$ rotates p and q with opposite charges. When $k = 1$, $SU(2)_F$ is enhanced to $SU(3)$ due to non-perturbative effects, as we will show. We focus on the Coulomb branch, for which $p = q = 0$. We also consider only the abelian theory, in which case there is no F-term relations. At first sight it is not clear how to recover the algebraic construction of the last subsection from the quiver theory with a single flavor, since it would seem that the Coulomb branch is the same as in ABJM. However, we need to consider more carefully the monopole operators. They have the same electric charges as in ABJM. However, they now have a non-vanishing R-charge. This follows directly from the discussion of Chapter 10, equation (10.34). Since the fermions ψ_p, ψ_q in the chiral multiplets p, q have R-charge $-\frac{1}{2}$ (in $\mathcal{N} = 3$ theories the R-charges are the classical ones), the induced R-charge of the monopole operators is

$$R(T^{(n)}) = -\frac{|n|}{2} (R(\psi_p) + R(\psi_q)) = \frac{|n|}{2}. \quad (12.27)$$

This means that the bare monopole operators operators T, \tilde{T} carrying ± 1 units of magnetic flux along $U(1)_D$ have dimension $\frac{1}{2}$, the same dimension as the chiral field A_i, B_i . This allows to construct an enhanced $S\tilde{U}(3)$ global symmetry, by forming $SU(3)$ triplets

$$(A_1, A_2, T), \quad (B_1, B_2, \tilde{T}). \quad (12.28)$$

The abelian chiral ring is generated by the following gauge invariant operators:

$$\begin{aligned} X_0 &= A_1 B_1, & X_1 &= A_1 B_2, & X_2 &= A_1 \tilde{T}, \\ X_3 &= T B_1, & X_4 &= T B_2, & X_5 &= T \tilde{T}, \\ X_6 &= A_2 B_1, & X_7 &= A_2 B_2, & X_8 &= A_2 \tilde{T}, \end{aligned} \quad (12.29)$$

The correspondence with the hyper-toric description is clear:

$$\frac{z_1}{A_1} \quad \frac{z_2}{B_1} \quad \frac{z_3}{T} \quad \frac{w_1}{B_2} \quad \frac{w_2}{A_2} \quad \frac{w_3}{\tilde{T}} \quad (12.30)$$

The $U(1)$ of the gauge linear model (12.21) is the $U(1)_b$ of the quiver³. To obtain $C(N^{0,1,0})$, the crucial relation is the F-term relation (12.24). In the field theory language, it reads

$$T \tilde{T} = A_2 B_1 - A_1 B_2. \quad (12.31)$$

This is a completely quantum relation. As explained in Chapter 10, the OPE $T \tilde{T}$ can a priori contain any operator with the same quantum numbers. Since $T \tilde{T}$ has dimension 1 and is a singlet under $SU(2)_F$, the only possible operator which could have appeared was $\epsilon^{ij} A_i B_j$. That it indeed appears was argued in [285]. In the next Chapter we will conjecture a similar OPE in a $\mathcal{N} = 2$ setup, allowing us to consider generic flavors in $\mathcal{N} = 2$ quivers. Before we do that, we will now introduce another string theory derivation of a larger class of AdS_4/CFT_3 dualities.

12.3 Stringy derivation of $\mathcal{N} = 2$ Chern-Simons quivers

For some of the $\mathcal{N} = 2$ quivers proposed in the literature, such as the ones reviewed in the last chapter, it is not known how or whether they can be constructed from string theory. This puts the associated AdS_4/CFT_3 duality conjectures on a less firm ground than in the case of the $\mathcal{N} = 3$ quivers of this chapter, which do have a clear string theory derivation in term of D3- and fivebranes.

In this section we review and slightly clarify a type IIA derivation of the $\mathcal{N} = 2$ Chern-Simons quiver associated to M2-branes on any toric Calabi-Yau fourfold singularity [258]. It will provide use with a better understanding as to why M2-brane quivers look like D-brane quivers. This derivation was first explicitly considered by Aganagic [258]. The relation between the CY_4 geometric moduli space of a 3d quiver CS theory and the $CY_3 = CY_4//U(1)_M$ mesonic moduli space of a 4d gauge theory with the same quiver and superpotential was pointed out before in [282, 244, 245].

We consider toric Calabi-Yau singularities only. The basic idea of [258] is to write a toric CY_4 cone as line fibration over a toric Calabi-Yau threefold,

$$\mathbb{C} \rightarrow CY_4 \rightarrow CY_3. \quad (12.32)$$

The fibration structure of the $S^1 \subset \mathbb{C}$ corresponds to the fibration structure of the M-theory circle. Reducing along this circle, the type IIA setup is the CY_3 fibered over a real line. M2-branes become D2-branes on the CY_3 in type IIA, in which case we know their

³Indeed for Chern-Simons level k , the $U(1)$ charge of the monopole operator is $\pm k$, as in (12.21).

associated quiver very well. Moreover the non-trivial fibration leads to Chern-Simons interaction in the field theory, as we will review.

We stress again that there is generally several ways to choose the M-theory circle, corresponding to several different weak string coupling limits of M-theory.

12.3.1 CY_4 as a $U(1)$ fibration

Let us explain how to rewrite a toric CY d -fold as a (possibly singular) $U(1)$ fibration over a manifold, which in turn is the fibration of a toric CY $(d-1)$ -fold along a real line, with Kähler moduli that vary linearly along the line. We begin by following the exposition in [258], but we further stress the regularity conditions one has to impose on the choice of fibration structure.

Consider a toric CY d -fold, realized as the moduli space of a gauged linear sigma-model (GLSM). There are $R+d$ chiral superfields ϕ_s with $s=1, \dots, R+d$, and R $U(1)$ gauge groups with integer charges Q_s^a (of maximal rank) with $a=1, \dots, R$. The CY condition is $\sum_s Q_s^a = 0$ for all a . The number $R+d$ of fields can be taken to be equal to the number of dots in the $(d-1)$ -dimensional toric diagram. Then the charge matrix Q_s^a encodes the R linear relations $\sum_s Q_s^a \vec{v}_s = 0$ between the vectors $\{\vec{v}_s\}$ in the toric fan. The CY Y_d is simply the Kähler quotient $\mathbb{C}^{R+d}/U(1)^R$, which corresponds to imposing the moment map (D-term) equations

$$\sum_s Q_s^a |\phi_s|^2 = r^a \quad (12.33)$$

and quotienting by the gauge group

$$\phi_s \rightarrow e^{i \sum_a \lambda_a Q_s^a} \phi_s . \quad (12.34)$$

The moment map (or FI) parameters r^a are the resolution parameters of Y_d . We will be mainly interested in the conical case $r^a = 0$. Moreover, for each a the charges Q_s^a can be taken coprime without loss of generality.

To exhibit the fibered structure, we add the complex variable $r_0 + i\theta_0$ and choose a set of charges Q_s^0 satisfying the CY condition $\sum_s Q_s^0 = 0$. Then we impose one more equation and divide by one more gauge symmetry:

$$\sum_s Q_s^0 |\phi_s|^2 = r_0 ; \quad \theta_0 \rightarrow \theta_0 + \lambda , \quad \phi_s \rightarrow e^{i\lambda Q_s^0} \phi_s . \quad (12.35)$$

It is easy to check that the manifold is the same as before: using (12.35) r_0 can be eliminated while θ_0 can be gauged away (without leaving any residual gauge transformation).

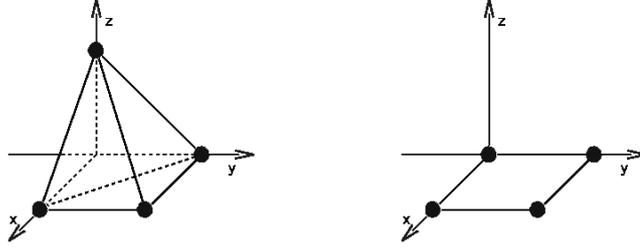
On the other hand, we can fix r_0 and think of θ_0 as a $U(1)$ fibration. The base manifold Y_{d-1} is then the Kähler quotient $\mathbb{C}^{R+d}/U(1)^{R+1}$:

$$\sum_s Q_s^0 |\phi_s|^2 = r_0 , \quad \sum_s Q_s^a |\phi_s|^2 = r^a \quad \forall a = 1, \dots, R , \quad (12.36)$$

modded out by

$$\phi_s \rightarrow e^{i\lambda Q_s^0 + i \sum_a \lambda_a Q_s^a} \phi_s . \quad (12.37)$$

Y_{d-1} is toric and Calabi-Yau. Moreover, Y_{d-1} is fibered over the real line r_0 , with a particular combination of the resolution parameters (set by Q_s^0) varying linearly with r_0 . The tip of Y_d is at $r_0 = 0$.


 Figure 12.5: Projection of the toric diagram of \mathbb{C}^4 -left- to one of the conifold -right-.

Projecting the toric diagram. Given the set of charges Q_s^a , the toric fan of Y_d is given by $R + d$ primitive vectors $\{\vec{v}_s\}$ in \mathbb{Z}^d which solve the R linear conditions $\sum_s Q_s^a \vec{v}_s = 0$ for all $a = 1, \dots, R$. We can collect the vectors as columns of a matrix $(G_K)^i_s$, with $i = 1, \dots, d$, of maximal rank. Then

$$Q_s^a (G_K^T)_i^s = 0 \quad (12.38)$$

and the rows of G_K span the kernel of Q_s^a as a map from \mathbb{R}^{R+d} to \mathbb{R}^R . We can use a transformation of $SL(d, \mathbb{Z})$ to map the vectors to $\vec{v}_s = (1, \vec{w}_s)$. The same equation can be used to obtain the charges of a GLSM, given the matrix G_K of all vectors in the toric fan.

The toric diagram of Y_{d-1} can be obtained in the same way. We add the extra condition $\sum_s Q_s^0 \vec{v}_s = 0$. The vectors \vec{v}_s do not satisfy it, because the rows of $(G_K)^i_s$ are linearly independent. In order to satisfy the extra relation, we must project the vectors on a hyperplane in such a way that the linear combination⁴

$$\vec{v}_M \equiv \text{primitive} \sum_s Q_s^0 \vec{v}_s \quad (12.39)$$

vanishes, that is a hyperplane orthogonal to \vec{v}_M . Notice that the CY condition on Q_s^0 plus the particular chosen frame $\vec{v}_s = (1, \vec{w}_s)$ assures that $\vec{v}_M = (0, \vec{w}_M)$. To make the projection clearer, we can perform an $SL(d-1, \mathbb{Z})$ transformation that maps \vec{v}_M to $(0, \dots, 0, 1)$, and changes the toric diagram of Y_d accordingly. Then the toric diagram of Y_{d-1} is obtained from the one of Y_d with the “vertical” projection that forgets the last component, as in Figure 12.5.

Fixed points. The reduction $Y_{d-1} = Y_d // U(1)_M$ can always be done. However, whenever the $U(1)_M$ fiber degenerates, we should expect some extra object or singularity in the type IIA background, on top of any possible geometric toric singularity (even non-isolated) of Y_{d-1} .

A first class of singularities arises from loci where the fiber $U(1)_M$ shrinks:

- each strictly external dot \vec{v}_s in the $(d-1)$ -dimensional toric diagram of Y_d is a conical toric divisor (complex codimension one) where the circle \vec{v}_s shrinks;
- each external edge v_{sr} connecting two adjacent dots \vec{v}_s and \vec{v}_r is a conical codimension 2 surface where the span in $U(1)^d$ of the two circles shrinks;

⁴We mean that \vec{v}_M is the primitive vector in \mathbb{Z}^d which is parallel to $\sum_s Q_s^0 \vec{v}_s$.

- each external polyhedron $v_{s_1 \dots s_n}$ of dimension $n - 1$ constructed between the strictly external dots $\vec{v}_{s_1}, \dots, \vec{v}_{s_n}$ is a conical codimension n surface where the span in $U(1)^d$ of the n circles shrinks.

In order to have a non-singular Kähler quotient for the projection, we should make sure that the circle \vec{v}_M is not contained in any of the spans above (the first case is automatically excluded). Practically, we require \vec{w}_M not to be parallel to any external sub-object in the convex polyhedron of the $(d - 1)$ -dimensional toric diagram. We stress that we are not worried about singularities in the quotient Y_{d-1} , but rather about degenerations of the fiber.

There is a second class of possible singularities, where the $U(1)_M$ fiber degenerates to $U(1)/\mathbb{Z}_p$ for some p . This happens if some of the charges in Q_s^0 have modulus larger than 1. In this case, there could be a conical surface where the fiber $U(1)_M$ degenerates: we have to make sure that the only point where this happens is the tip of Y_d .

The case of \mathbf{CY}_4 . Specializing to the case of interest – Y_4 and $Y_3 = Y_4//U(1)_M$ –, whenever none of the singularities above arises in the quotienting, we are sure that the reduction of M-theory on Y_4 along $U(1)_M$ gives a pure IIA background (to which the arguments in [258] can be applied), without extra objects on top of it.

In particular, we should make sure that: 1) there are no external edges in the 3d toric diagram parallel to \vec{w}_M ; 2) there are no external faces parallel to \vec{w}_M ; 3) once \vec{w}_M is expressed as an integer sum of the \vec{w}_s in the 3d toric diagram, if some coefficients have modulus larger than 1, the fiber does not degenerate outside the tip of Y_4 .

On the contrary, whenever the fiber degenerates, we should expect some extra objects in type IIA that have to be taken into account. In the next Chapter, we will study what happens if the fiber shrinks on a complex codimension-two submanifold of the four-fold (giving rise to D6-branes). The other cases deserve a separate study.

12.3.2 Chern-Simons quivers from type IIA

By the above construction, we have a duality between M2-branes at the tip of a \mathbf{CY}_4 and D2-branes on a \mathbf{CY}_3 fibered over \mathbb{R} (the D2-branes sit at $r_0 = 0$ and at the tip of the threefold). According to (12.35), the coordinate r_0 is a Kähler modulus of the Calabi-Yau threefold. More generally, we can consider

$$\sum_s Q_s^0 |\phi_s|^2 = hr_0; \quad \theta_0 \rightarrow \theta_0 + h\lambda, \quad \phi_s \rightarrow e^{i\lambda Q_s^0} \phi_s, \quad (12.40)$$

for h some positive integer. This changes Y_d to Y_d/\mathbb{Z}_h , corresponding to shrinking the θ_0 circle by a factor of h . This is usually by such a further orbifold that a weakly coupled type IIA background is obtained, as in the ABJM case. The gauge invariant vielbein characterizing the M-theory fibration is

$$d\theta_0 + i \sum_s q_s d\phi_s / \phi_s, \quad (12.41)$$

with q_s such that

$$\sum_s q_s Q_s^a = 0, \quad \sum_s q_s Q_s^0 = h. \quad (12.42)$$

This identifies the connection $\sum_s q_s d\phi_s / \phi_s$ as the RR 1-form C_1 . The RR 2-form flux $F_2 = dC_1$ is [258]

$$F_2 = \sum_s q_s \omega_s, \quad \text{with } \omega_s = \delta^2(\phi_s) d\phi_s \wedge d\bar{\phi}_s \quad (12.43)$$

where ω_s is a two-form which localizes on the toric divisor $D_s = \{\phi_s = 0\}$. In term of cohomology classes,

$$[F_2] = \sum_s q_s [D_s] \quad (12.44)$$

We can then compute the flux of F_2 through any curve by standard toric geometry methods [288].

If the fibration were trivial, the $\mathcal{N} = 2$ field theory on the D2-branes would be the dimensional reduction of the $\mathcal{N} = 1$ quiver theory on D3-branes, for the same CY_3 . Such quivers were discussed at length in this thesis. Each node of the quiver corresponds to a fractional D2-brane, which is some linear combination of D2-branes, D4-branes on 2-cycles and D6-branes on 4-cycles. For any holomorphic two-cycle \mathcal{C} in Y_{d-1} there is a corresponding $U(1)$ in the GLSM, corresponding to a D-term

$$\sum_s Q_s^{\mathcal{C}} |\phi_s|^2 = r_{\mathcal{C}}. \quad (12.45)$$

The flux of F_2 through \mathcal{C} is

$$\int F_2 = \sum_s q_s Q_s^{\mathcal{C}} = n_{\mathcal{C}} h. \quad (12.46)$$

Due to (12.42), the only contribution will come from Q^0 , with some integer coefficient $n_{\mathcal{C}}$ depending on \mathcal{C} .

Consider N fractional D2-branes wrapping \mathcal{C} in Y_{d-1} . The Wess-Zumino term in the D2-brane action will contain a term

$$\int_{\mathbb{R}^{2,1} \times \mathcal{C}} C_1 \wedge \text{Tr}(F \wedge F) = \int_{\mathcal{C}} F_2 \int_{\mathbb{R}^{2,1}} \text{Tr}(A \wedge dA + \frac{2}{3} A \wedge A \wedge A), \quad (12.47)$$

where we integrated by part. Hence the flux F_2 due to the non-trivial fibration of the M-theory circle induces a Chern-Simons level

$$k_{\mathcal{C}} = n_{\mathcal{C}} h \quad (12.48)$$

for the gauge group corresponding to \mathcal{C} . Similar considerations can be made for D6-branes on 4-cycles [258]. Hence, with some knowledge of the algebraic geometry of the Calabi-Yau threefold, we can in principle reconstruct the Chern-Simons levels of the CS-quiver theory. In total we obtain a Yang-Mills-Chern-Simons theory on the D2-branes.

The ABJM quiver. As an example, let us see how this works for $\mathbb{C}^4/\mathbb{Z}_k$. We go from this space to the conifold by a Kähler quotient. Let us call the homogenous coordinates A_1, A_2, B_1, B_2 . Due to the D-term relation

$$|A_1|^2 + |A_2|^2 - |B_1|^2 - |B_2|^2 = k r_0, \quad (12.49)$$

the type IIA geometry is the resolved conifold fibered over $\mathbb{R} \cong \{r_0\}$. There are two fractional branes on the conifold, corresponding to $[\mathbb{C}P^1]$ and $[1] - [\mathbb{C}P^1]$. The RR flux through CP^1 is simply

$$\int_{\mathbb{C}P^1} F_2 = k. \quad (12.50)$$

Hence the Chern-Simons theory is the conifold quiver with Chern-Simons levels $(k, -k)$, as expected.

The construction of [258] gives a neat explanation of the string theory origin of Chern-Simons quiver. The explanation of the quiver structure of ABJM and its generalizations boils down to well known properties of D-branes. This is worth stressing, because it means that the quiver, by itself, does not contain any properties which would be “intrinsic” to the M2-brane. In particular, many different quivers will be dual (generalizing 3d mirror symmetry [208]) if they correspond to the same M-theory geometry. The situation is different from the case of Seiberg duality for D3-brane quivers, because in that latter case the dualities are well understood as a reshuffling of the fractional branes. Here there is no such “mechanical” understanding of the M-theory duality, because we still do not have any direct control over the M2-brane degrees of freedom. Instead we have to rely on a dual picture, either in type IIB like in the first part of this chapter, or in type IIA like in this section.

In the next chapter we will generalize the type IIA derivation of $\mathcal{N} = 2$ M2-brane theories to the case where the M-theory Calabi-Yau fourfold contains non-isolated singularities in complex codimension two.

Chapter 13

Flavors in $\mathcal{N} = 2$ toric Chern-Simons quivers

IN this second Part of the thesis we have discussed in some detail various theories which are conjectured to describe the low energy theory on M2-branes sitting on some eight dimensional cone. Each such proposal leads to a new instance of a AdS_4/CFT_3 duality, although in the $\mathcal{N} = 2$ case there is generally no weak coupling limit in the field theory (similarly to the case of $\mathcal{N} = 1$ quiver SCFTs in 4d). Almost all of the field theory which we reviewed involve gauge groups with only adjoint and bifundamental matter, like conformal quiver gauge theories in 3+1 dimensions. In the previous Chapter we have seen that this is not a coincidence.

The string theory understanding of the correspondence leads to natural generalizations. It has been proposed recently that the dynamics of M2-branes on some hyper-Kähler cones ($\mathcal{N} = 3$ SUSY) is described by flavored quiver CS theories, including matter in the fundamental and antifundamental representation of the gauge groups [285, 286, 287]; we reviewed such a proposal in section 12.2 in the last chapter. Flavors were further studied in [289, 290, 291].

We aim to extend this program to M2-branes probing toric CY_4 singularities. This Chapter is entirely based on [3], written in collaboration with Stefano Cremonesi and Francesco Benini. A similar approach was advocated by Daniel L. Jafferis [292], with whom we coordinated the release of the arXiv preprints. The later work [292] also gives a discussion of some non-toric examples.

13.1 Motivation and overview

One of the problems we want to address is what happens when the fourfold has conical complex codimension-two singularities, which means that the base H_7 itself has codimension-two singularities: this is related to the addition of *flavors* – fields in the fundamental representation of the gauge groups. An A_{h-1} complex codimension-two singularity locally looks like $\mathbb{C}^2 \times \mathbb{C}^2/\mathbb{Z}_h$. M-theory on such a background develops $SU(h)$ gauge fields living along the singularity, and by the AdS/CFT map there must be an $SU(h)$ global symmetry in the boundary theory. Many models in the literature have

such singularities, however the large non-Abelian symmetry is not manifest. It is natural to look for a description in terms of flavors in the quiver theory.

Another way to understand the issue is to select a $U(1)$ isometry of the CY_4 that preserves the holomorphic 4-form Ω_4 , quotient the geometry by $\mathbb{Z}_k \subset U(1)$ and reduce along the circle to type IIA. The resulting background is a warped product $AdS_4 \times_w H_6$, with RR fluxes and varying dilaton. If $1 \ll k \ll N^{1/5}$ type IIA is weakly coupled, whereas for $k \gg N$ one expects a Lagrangian description for the 2-brane theory with weakly coupled gauge groups. If the $U(1)$ circle shrinks on a complex codimension-two surface in the CY_4 , we get D6-branes in the type IIA background, filling AdS_4 and wrapping a 3-cycle in H_6 . In fact, $\mathbb{C}^2 \times \mathbb{C}^2/\mathbb{Z}_h$ is the complex structure of a multi-Taub-NUT which, if reduced along its $U(1)$ isometry, gives rise to h D6-branes. It is known that the D2-D6 system introduces flavors in the theory living on D2-branes, as happens in the $\mathcal{N} = 3$ case [285, 286, 287].

A more systematic tool to derive the theory on M2-branes probing a CY_4 geometry is the Kähler quotient approach [258] which we reviewed in the last Chapter. This approach is powerful because it does not need metric details of the four-fold, but only algebraic geometric data. When the $U(1)$ fiber shrinks on codimension-two submanifolds of the CY_4 , we get D6-branes wrapping divisors of the CY_3 , and the theory on M2-branes has the same quiver and superpotential as the theory on D3-branes on the CY_3 in the presence of D7-branes wrapping the same divisors. For h D7-branes wrapping an irreducible divisor, the effect is that of introducing h pairs of quarks (p, q) coupled via the superpotential term

$$\delta W = p (\text{divisor equation}) q ,$$

where the divisor equation is written in terms of the bifundamental matter fields in the theory.

Led by these considerations, we can study what happens if we start with an $\mathcal{N} = 2$ quiver Chern-Simons theory, dual to a toric CY_4 geometry, and we flavor it. We mean that we select a subset $\{X_\alpha\}$ of bifundamental fields in the quiver, and for each of them we introduce h_α pairs of chiral multiplets (p_α, q_α) in the (anti)fundamental representation of the gauge groups, coupled by the superpotential term

$$W = W_0 + \sum_\alpha p_\alpha X_\alpha q_\alpha ,$$

W_0 being the “unflavored” superpotential. Because of the parity anomaly, this has to be accompanied by a shift of Chern-Simons levels. A concept of “chirality” is induced by $\mathcal{N} = 2$ supersymmetry, and inherited from four dimensions.

To study the chiral ring and moduli space of this theory, a crucial rôle is played by BPS diagonal monopole operators $T^{(n)}$ [293, 294, 232, 230, 231, 295, 233, 235]. Due to quantum corrections, they acquire global and gauge charges in the presence of flavors. Generically there is only one possible non-trivial OPE compatible with all the symmetries, that in the Abelian case reads

$$T\tilde{T} = \prod_\alpha (X_\alpha)^{h_\alpha} .$$

We conjecture that this quantum F-term relation holds, since our results strongly support this claim from the AdS/CFT point of view. The moduli space has Higgs and Coulomb

branches. We show that the geometric branch, in which $p_\alpha = q_\alpha = 0$, is described by the matter fields X_a plus the two monopole operators T and \tilde{T} , subject to the classical F-term relations from W plus this quantum F-term relation, modded out by the full gauge group $U(1)^G$. The geometric moduli space is still a toric CY_4 , that we precisely identify. Similar ideas appeared in [296].

The Chapter is organized as follows. In Section 13.2 we start with a top-down perspective, and analyze the Kähler quotient reduction of M-theory in the presence of KK monopoles. In Section 13.3 we turn to a bottom-up approach and flavor quiver Chern-Simons theories; their moduli space is studied in Section 13.4. Section 13.5 is devoted to deformations by real and complex masses. In Section 13.6 we work out many examples.

13.2 M-theory reduction and D6-branes : A top-down perspective

Let us consider M2-branes probing a toric conical Calabi-Yau four-fold Y_4 in M-theory.¹ We are interested in the type IIA string theory background that one obtains by reducing along a $U(1)$ isometry, in particular in the case that the four-fold contains KK monopoles and the $U(1)$ shrinks along them. The isometry group of a toric four-fold contains $U(1)^4$. A specific $U(1)$ or more generally \mathbb{R} subgroup is the superconformal R-symmetry, while the remaining commuting $U(1)_F^3$ leaves the holomorphic 4-form invariant. Reduction along a circle in $U(1)_F^3$ manifestly preserves eight supercharges in type IIA.

The toric data of the four-fold are specified by a Lagrangian $U(1)^4$ fibration over a strictly convex rational polyhedral cone. Each facet of the cone represents a toric divisor. In fact the normal vector to a facet, normalized to have integer components, represents the $U(1)$ cycle that shrinks on the facet. The collection $\{\vec{v}_s\}$ in \mathbb{Z}^4 of the normal vectors to all facets is called the toric fan. The Calabi-Yau condition is equivalent to the end-points of all vectors in the toric fan being coplanar; one can then use an $SL(4, \mathbb{Z})$ transformation to rewrite them as $\vec{v}_s = (1, \vec{w}_s)$, with $\{\vec{w}_s\}$ vectors in \mathbb{Z}^3 . The information encoded in the toric fan can be summarized by the 3d toric diagram: a 3d convex polyhedron whose strictly external points are \vec{w}_s . We will call *strictly* external, among the external points, a point which does not lie along a line connecting two external points – this means that strictly external points are not inside an edge nor a face of the toric diagram. Each strictly external point represents a conical toric divisor. The elements $(0, 1, 0, 0)$, $(0, 0, 1, 0)$, $(0, 0, 0, 1)$ in the \mathbb{Z}^4 ambient space of the toric fan generate the flavor $U(1)_F^3$ symmetry group that commutes with the R-symmetry.

Over the intersection of two adjacent facets, two $U(1)$ cycles in the fiber shrink. Suppose that the shrinking cycles are $(1, x, y, z)$ and $(1, x, y, z + 1)$ (two points vertically aligned) in \mathbb{Z}^4 : at the intersection of the two facets the $U(1)_M$ cycle $(0, 0, 0, 1)$, linear difference of the previous ones, shrinks as well. This happens along a complex codimension-two conical submanifold of the four-fold, and one can locally view the M-theory background as a KK monopole for that $U(1)_M$ action. Reducing along $U(1)_M$, one gets a D6-brane on some type IIA background.

As reviewed in section 12.3, the type IIA background can be written as the fibration

¹We refer the reader to [297, 62, 298, 4] for a simple introduction to basic facts about toric geometry and its relevance for quiver gauge theories.

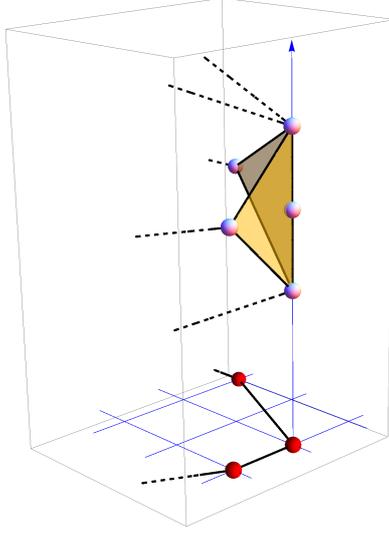


Figure 13.1: Vertical projection from the 3d toric diagram of Y_4 to the 2d toric diagram of Y_3 . The three aligned points give rise to two D6-branes.

of a CY_3 cone Y_3 over a real line $r_0 \in \mathbb{R}$ [258]. Degeneration loci of the $U(1)_M$ fiber result in various “objects” in the type IIA background. The toric threefold Y_3 is defined by a 2d toric diagram which is the projection of the 3d toric diagram to a plane orthogonal to the primitive vector \vec{v}_M that represents the cycle $U(1)_M$ used for the reduction. We can always perform an $SL(3, \mathbb{Z})$ transformation of the 3d toric diagram and map \vec{v}_M to $(0, 0, 0, 1)$; then the 2d toric diagram of Y_3 is the “vertical” projection of the 3d diagram to the plane $z = 0$.

In our example, the fact that two adjacent external points² in the 3d toric diagram project to the same point (which is then necessarily strictly external) in the 2d toric diagram, implies the presence of a D6-brane wrapping a toric divisor of the CY_3 in type IIA (spanning the spacetime $\mathbb{R}^{2,1}$ and localized at $r_0 = 0$). The toric divisor is specified by the projected point. More generally, if the 3d toric diagram has a collection of $h + 1$ aligned adjacent external points $(1, x, y, z_j)$ with $z_j = z, z + 1, \dots, z + h$, and we reduce along the $U(1)_M$ cycle $(0, 0, 0, 1)$, all $h + 1$ points project down to the same strictly external point in the 2d toric diagram and give rise to h coincident D6-branes wrapping a toric divisor of the CY_3 (see Figure 13.1).

Along the D6-branes lives a $U(h)$ gauge theory, which by the AdS/CFT map corresponds to a $U(h)$ global symmetry on the boundary. We could then expect the boundary theory to admit a description in which such a symmetry is manifest. The same conclusion can be reached in M-theory: $h + 1$ adjacent external points in the 3d toric diagram indicate h KK monopoles in Y_4 (a multi-Taub-NUT geometry), whose complex structure is locally $\mathbb{C}^2 \times \mathbb{C}^2/\mathbb{Z}_h$; the A_{h-1} singularity carries $SU(h)$ gauge fields, besides the $U(1)$ gauge field from the KK reduction of the M-theory potential C_3 . More precisely, in the near core limit the latter $U(1)$ KK mode is non-normalizable – correspondingly in the dual theory the diagonal $U(1)$ in $U(h)$ is actually gauged.

²Points are adjacent if the line connecting them is strictly external, not contained in a face.

13.2.1 IIA background as a CY_3 fibration with D6-branes

The symplectic reduction of Y_4 to a CY_3 is useful because it allows to exploit all the powerful techniques available for D3-branes probing toric singularities, to get information about the field theory. Given a toric CY_3 singularity in type IIB and N D3-branes probing it, the dual SCFT in 3+1 dimensions can be generically found with the fast inverse algorithm [71] briefly discussed in Chapter 3.³

We want to extend the correspondence of [258] to cases in which the Kähler quotient has complex dimension-two degeneration loci. To begin with, let us understand what the toric divisors of Y_3 correspond to. Each strictly external point \vec{p} in the 2d toric diagram corresponds to a toric divisor, to which is associated a collection of Q bifundamental fields $\{X_\eta\}_{\eta=1,\dots,Q}$ in the quiver theory, that have the same charges under all global (but not gauge) symmetries. The number Q is given by [299]

$$Q = \left| \det \begin{pmatrix} \Delta x & \Delta y \\ \Delta x' & \Delta y' \end{pmatrix} \right|, \quad (13.1)$$

where $(\Delta x, \Delta y)$ is the vector connecting the strictly external point \vec{p} to the next strictly external point along the perimeter, while $(\Delta x', \Delta y')$ is the vector connecting \vec{p} to the previous strictly external point.⁴ A time-filling D3-brane wrapped on the 3-cycle which is the radial section of the toric divisor (such embedding is supersymmetric) corresponds to a dibaryonic operator X_η^N [300, 297, 301]. Since the 3-cycle has the topology of a Lens space with fundamental group \mathbb{Z}_Q [297], the D3-branes admit a \mathbb{Z}_Q flat connection resulting in Q degenerate vacua. They correspond to the Q different dibaryonic operators $\{X_\eta^N\}$. An easy way to identify the set of fields is through perfect matchings in the brane tiling construction [68, 60, 61, 76].

Instead of wrapping a D3-brane on a radial section, one can wrap h spacetime-filling D7-branes on the whole toric divisor (this problem has been considered, *e.g.*, in [302, 303, 304, 129, 186]). They introduce a $U(h)$ global symmetry in the field theory, and h flavors of chiral fields p_η, q_η coupled to one of the bifundamental fields X_η through the superpotential term $W = p_\eta X_\eta q_\eta$. A \mathbb{Z}_Q connection, flat everywhere but at the tip, can be specified on the D7-branes to distinguish which bifundamental is flavored.

The same discussion holds in type IIA: D6-branes wrapping toric divisors of the CY_3 provide chiral flavors to the quiver gauge theory on D2-branes at the tip. Each stack of h D6-branes introduces a $U(h)$ flavor group (this is not *always* the case: we will be more precise in Section 13.5) and flavor chiral multiplets $p_{\hat{k}i}, q_{j\hat{k}}$ coupled to a bifundamental X_{ij} through a superpotential term⁵

$$W = \text{Tr} p_{\hat{k}i} X_{ij} q_{j\hat{k}}. \quad (13.2)$$

Here \hat{k} stands for a flavor group, i, j for gauge groups and fields are in the fundamental (anti-fundamental) of the first (second) index; all indices are contracted. We will jump between the notations X_a and X_{ij} for bifundamental fields. The field X_{ij} is determined as explained above.

³A huge number of examples has been explicitly worked out, see [62, 298] and references therein.

⁴ Q is more conveniently defined as the modulus of the cross product of two consecutive legs in the (p, q) -web that is dual to the 2d toric diagram.

⁵We absorb superpotential couplings inside chiral superfields.

The D6-branes are localized along \mathbb{R} : reducing the cone Y_4 their position is $r_0 = 0$. More generally, the position along r_0 corresponds to a real mass for the quarks in field theory, and to a partial resolution (or Kähler) parameter in M-theory (see Section 13.5.1). Since D6-branes, possibly with worldvolume flux, are sources for RR fields, the 2- and 4-form fluxes on 2- and 4-cycles vanishing at the CY_3 singularity jump at r_0 :

$$\delta \int_{\mathcal{C}_2} F_2 = \#(\mathcal{C}_2, D6), \quad \delta \int_{\mathcal{C}_4} F_4 = \#(\mathcal{C}_4, D6, \mathcal{C}_4^{(F_{\text{wv}})}) , \quad (13.3)$$

where the jump depends on the intersection on Y_3 between the cycles, the divisor and the cycle representing the worldvolume flux. This means that moving the D6's to the left or to the right of the D2-branes, the CS levels must jump as well. We will study this in detail.

Summarizing, whenever the $U(1)_M$ action has codimension-two fixed loci which descend in type IIA to D6-branes wrapping divisors of the CY_3 , the field theory derived using the CY_3 singularity is actually flavored.

We conclude this section with some comments. Two important differences between chiral flavors in AdS_5/CFT_4 and in AdS_4/CFT_3 must be borne in mind. Firstly, in 4d gauge theories chiral flavors are constrained by gauge anomaly cancelation, whereas in 3d such a constraint does not exist. The dual statement is that D7-branes wrapping divisors are constrained by RR C_0 tadpole cancelation, whilst D6-branes are not because the RR F_2 flux can escape to infinity along the transverse non-compact real line. The number of fundamental minus anti-fundamental fields for a gauge group in 3d need not vanish: if it is odd, the parity anomaly requires the presence of half-integral CS levels [305, 277, 278]. Secondly, in general the addition of flavors to an AdS_5/CFT_4 pair breaks conformal invariance and the RG flows leads the theory to a fixed point which is outside the validity of supergravity [304] (the dual statement is that D7-branes force the dilaton to run towards $-\infty$ at the tip). Flavoring AdS_4/CFT_3 pairs, the theory still flows to an interacting fixed point which however in many examples [285, 286, 287, 207, 306] (and in the ones of this work too) is still described by type IIA/M-theory.

In the following, we will focus on the Abelian case: we will consider a single M2/D2-brane and the corresponding quiver theory will have $U(1)$ gauge groups. One expects the low energy field theory on a stack of N M2/D2-branes to be described by the same quiver with $U(N)$ gauge groups, and the geometric moduli space to be the symmetric product of N copies of Y_4 . We leave the non-Abelian extension for the future.

13.3 Flavoring Chern-Simons-matter theories : A bottom-up perspective

In the rest of the chapter we turn to a bottom-up perspective. We start with a generic toric CY_4 geometry and a regular (as described in Section 12.3.1 in the last Chapter) IIA reduction along $U(1)_M$, such that the Chern-Simons-matter theory dual to M2-branes probing Y_4 can be read off [258]. Then we study the effect of chirally flavoring such a theory in a very general way, and in particular we study how the flavoring deforms the moduli space of the quiver theory. Alternatively, we can start with a toric CY_3 geometry

and its dual quiver theory (which in 3+1 dimensions is the theory dual to D3-branes probing Y_3), add to it generic $\mathcal{N} = 2$ Chern-Simons couplings (which corresponds to fibering Y_3 over \mathbb{R} and adding RR fluxes) and flavors (D6-branes), and study what is the resulting CY_4 geometry seen by M2-branes.

To begin with, let us specify the flavoring procedure. The starting point is an $\mathcal{N} = 2$ quiver Chern-Simons theory in 2+1 dimensions. The matter fields are chiral multiplets X_α in the adjoint or bifundamental representation, and we restrict ourselves to the Abelian case. Then we introduce B families of flavor chiral multiplets (p_α, q_α) , each coupled to a matter field X_α via the superpotential

$$W = W_0 + \sum_{\alpha} p_{\alpha} X_{\alpha} q_{\alpha} . \quad (13.4)$$

Here p_{α} (q_{α}) transform in the anti-fundamental (fundamental) of the gauge group under which X_{α} is in the fundamental (anti-fundamental). Each pair (p_{α}, q_{α}) really represents h_{α} fields, and introduces a $U(h_{\alpha})$ flavor symmetry.

We could be more general and couple a flavor pair to a bifundamental operator $\mathcal{O}_{\alpha} = \prod_{\beta=1}^n X_{\beta}$ constructed from a string of matter fields X_{β} . This is equivalent to coupling each of the X_{β} to its own flavor pair (p_{β}, q_{β}) , and then introducing complex masses

$$W = W_0 + \sum_{\beta=1}^n p_{\beta} X_{\beta} q_{\beta} + \sum_{\beta=1}^{n-1} m_{\beta} p_{\beta+1} q_{\beta} . \quad (13.5)$$

Integrating out the massive fields, we flavor the operator \mathcal{O}_{α} (see Section 13.5.2).

Every time we introduce two new flavor fields (p, q) coupled to X_{α} , the parity anomaly [305, 277, 278] requires to shift two CS levels as

$$\delta k_i = \pm \frac{1}{2} g_i [X_{\alpha}] , \quad (13.6)$$

g_i being the gauge charges. The sign is a choice of theory. If we add h_{α} flavors (p_{α}, q_{α}) , we choose sign h_{α} times, so that the shift

$$\delta k_i = \left(\frac{h_{\alpha}}{2} - \gamma_{\alpha} \right) g_i [X_{\alpha}] \quad (13.7)$$

is parametrized by an integer γ_{α} with $0 \leq \gamma_{\alpha} \leq h_{\alpha}$.

The reason for this is that gauge invariance requires [305, 277, 278]

$$k_i + \frac{1}{2} \sum_{\psi} (g_i[\psi])^2 \in \mathbb{Z} , \quad (13.8)$$

where the sum runs over all fermions charged under the i -th gauge group. When the second term is half-integral, the fermion determinant is multiplied by (-1) under certain gauge transformations, and the lack of gauge invariance of the CS terms cures it. In our setup the gauge charges of flavors are $g_i = \pm 1$, so consistency requires that each addition of two flavor fields is accompanied by a half-integral opposite shift of two CS levels (unless X_{α} is in the adjoint).

We can proceed in the opposite way and integrate the flavors out, by giving them a real mass. To do this, we promote the $U(h)$ flavor symmetry to a background gauge symmetry;

then a VEV $\langle \sigma_F \rangle$ for the real adjoint background scalar field σ_F in the $U(h)$ vector multiplet provides real masses to all flavors charged under $U(h)$. After diagonalization of $\langle \sigma_F \rangle$ by a flavor rotation, each flavor of charge q_ψ acquires a real mass $M_\psi = q_\psi \langle \sigma_F \rangle$. When integrating out massive fermions $\{\psi\}$, the CS levels k_i are shifted by one-loop diagrams as

$$k_i \rightarrow k_i + \frac{1}{2} \sum_\psi (g_i[\psi])^2 \text{Sgn}(M_\psi). \quad (13.9)$$

Integrating out just two flavor fields (p, q) , $M_p = \langle \sigma_F \rangle$ and $M_q = -\langle \sigma_F \rangle$; we can then write $\delta k_i = \frac{1}{2} g_i [X_\alpha] \text{Sgn}(\langle \sigma_F \rangle)$. The choice of $\text{Sgn}(\langle \sigma_F \rangle)$ corresponds to the choice of sign in (13.6): a choice of positive (negative) sign in (13.6) is undone by $\langle \sigma_F \rangle < 0$ ($\langle \sigma_F \rangle > 0$).

In the next subsection we compute the effect of flavors on monopole operators, while in Section 13.4 we study the moduli space of the flavored theories.

13.3.1 Monopole operators and flavors

A fundamental rôle in the study of the quantum moduli space of the flavored theories is played by monopole operators [293, 294, 232, 230, 231, 295, 233, 235]. We discussed these operators in Chapter 10.

In this section we discuss the effect of the fundamental matter fields on the “diagonal” monopole operators, that we will denote $T^{(n)}$. They have the same magnetic flux n along each $U(1)$ gauge group in the quiver⁶. They pick up electric charges (nk_1, \dots, nk_G) under $U(1)^G$, where G is the number of gauge factors and k_i are the CS levels. They were studied in detail in [235] and shown to be BPS (after having been dressed by scalar modes) in the ABJM theory [202] and its $\mathcal{N} = 3$ elliptic generalizations [281, 282]; we expect them to be BPS in generic $\mathcal{N} = 2$ theories describing M2-branes on CY_4 , since they correspond to modes of eleven-dimensional supergravity in short multiplets.

The monopole operators $T^{(n)}$ can acquire a charge under any $U(1)$ symmetry of the theory, both global and gauged, from quantum corrections [230, 231, 295, 235]. In the case of global symmetries the charge comes entirely from fermionic modes, while in the case of gauge symmetries the quantum contribution adds to the contribution of the CS levels. The quantum correction (in the Abelian case) to the charge Q from fermionic modes is

$$\delta Q[T^{(n)}] = -\frac{|n|}{2} \sum_\psi Q[\psi], \quad (13.10)$$

where we sum over all fermions ψ in the theory. Notice that only fermions in chiral representations contribute. The result is proportional to the mixed Q -gravitational anomaly that the same theory would have in 3+1 dimensions.

Formula (13.10) implies that in Chern-Simons quiver theories satisfying the toric condition, diagonal monopole charges do not receive any quantum correction. Quiver theories have matter chiral multiplets in the adjoint and bifundamental representation only; the toric condition is that all gauge ranks are the same (here 1), each matter field appears in the superpotential in exactly two monomials, and the number G of gauge groups plus the number P of monomials in the superpotential equals the number E of matter fields. Let Q be a non-R global or gauge symmetry: each monomial W_μ in the

⁶A detailed study of the full spectrum of monopole operators would be of great interest, but is beyond the scope of this work.

superpotential must have vanishing charge. Summing over all monomials: $2 \sum_{\psi} Q[\psi] = 2 \sum_a Q[X_a] = \sum_{\mu} \sum_{a \in \mu} Q[X_a] = 0$, where X_a are all matter fields, and gaugini must be chargeless. In the case of the R-symmetry, each monomial W_{μ} must have R-charge 2, so that: $2 \sum_{\psi} R[\psi] = 2G + 2 \sum_a R[\psi_a] = 2G - 2E + 2 \sum_a R[X_a] = 2G - 2E + 2P = 0$, where we used the fact that gaugini have R-charge 1.

Therefore, let us start with a quiver theory in which the monopole fields $T^{(n)}$ have only gauge charges (nk_1, \dots, nk_G) . Then we flavor the theory as in (13.4): we couple a set of flavor pairs (p_{α}, q_{α}) , each in number h_{α} , to some bifundamental operators X_{α} in the quiver, constructed as products of bifundamental fields,⁷ via

$$W = W_0 + \sum_{\alpha} p_{\alpha} X_{\alpha} q_{\alpha} . \quad (13.11)$$

We are interested in the charges induced on the monopole operators by flavors. Let us start with non-R symmetries. First, there are the new flavor symmetries $U(h_{\alpha})$ of which p_{α} and q_{α} are in conjugate representations, so that the diagonal monopole operators cannot get a charge under $U(h_{\alpha})$.⁸ Next, for any $U(1)$ flavor symmetry of W_0 under which X_{α} has charge Q_{α} , $(qp)_{\alpha}$ must have charge $-Q_{\alpha}$. Then, according to (13.10), the diagonal monopoles pick up a charge

$$Q[T^{(n)}] = \frac{|n|}{2} \sum_{\alpha} h_{\alpha} Q[X_{\alpha}] \quad (13.12)$$

in the flavored quiver. In the case of *gauge* charges, the contribution from fermions has to be summed with the contribution from Chern-Simons couplings:

$$g_i[T^{(n)}] = nk_i + \frac{|n|}{2} \sum_{\alpha} h_{\alpha} g_i[X_{\alpha}] , \quad (13.13)$$

where g_i are the gauge charges under $U(1)^G$. Eventually, consider the R-symmetry: $R[p_{\alpha}] + R[q_{\alpha}] = 2 - R[X_{\alpha}]$ at the IR fixed point, so that the monopoles get an R-charge

$$R[T^{(n)}] = -\frac{|n|}{2} \sum_{\alpha} h_{\alpha} (R[\psi_{p_{\alpha}}] + R[\psi_{q_{\alpha}}]) = \frac{|n|}{2} \sum_{\alpha} h_{\alpha} R[X_{\alpha}] . \quad (13.14)$$

These charges allow us to conjecture the following holomorphic quantum relation:

$$T^{(n)} T^{(-n)} = \left(\prod_{\alpha} X_{\alpha}^{h_{\alpha}} \right)^{|n|} , \quad (13.15)$$

which is consistent with all manifest symmetries in the action. This is understood as an operator statement: the equation must be multiplied on both sides by the necessary fields to form gauge-invariant operators. In Section 13.4 we show that in the usual unflavored case (where quantum corrections seem not to play a rôle) the relation $T^{(n)} T^{(-n)} = 1$ can be inferred from the form of the moduli space. Moreover, (13.15) is analogous to the quantum relation which appeared in the $\mathcal{N} = 3$ setup of [285] (see also [231]), and we will show that it reproduces the CY_4 moduli spaces as expected from the M-theory reduction, as we also check in several examples in Section 13.6. In the following we will use the notation

$$T^{(1)} \equiv T , \quad T^{(-1)} \equiv \tilde{T} , \quad (13.16)$$

for the simplest diagonal monopole operators.

⁷We are mainly interested in the case that X_{α} are pure bifundamental fields, but the arguments that follow apply as well to composite bifundamental fields, *i.e.* connected open paths in the quiver.

⁸To apply (13.10), take any generator of $U(h_{\alpha})$ and consider the $U(1)$ subgroup it generates.

13.4 Moduli space of flavored quivers

In this section we compute the geometric branch of the moduli space of any $\mathcal{N} = 2$ flavored Chern-Simons quiver, using the conjectured OPE (13.15) as a crucial ingredient. We first give an alternative derivation of the moduli space of *unflavored* CS quivers, emphasizing the rôle played by the monopoles operators. Next we discuss the moduli space of flavored CS quiver. We will use brane tilings with multiple bounds as an intermediate tool, which will allow us to use the Kasteleyn matrix algorithm of section 11.3.1 to easily extract the toric diagram.

13.4.1 Unflavored quivers and monopoles

The moduli space of any (unflavored) $\mathcal{N} = 2$ Chern-Simons quiver theory was worked out in [244], whose construction we reviewed in Chapter 11. In that analysis we gauge fixed the $U(1)_b$ symmetry, and there was a remaining discrete gauge identification which lead to the conclusion that the moduli space was a particular \mathbb{Z}_q orbifold of the “naive” moduli space. Here we present an alternative analysis of the moduli space, which shows how the monopole operators T, \tilde{T} can be included at the classical level. Instead of gauge fixing the dual photon φ , we keep it in the description of the moduli space. Given the periodicity of φ , we can construct the two complex fields

$$T = \rho e^{iG\varphi}, \quad \tilde{T} = \tilde{\rho} e^{-iG\varphi}, \quad (13.17)$$

where their dimensionless moduli ρ and $\tilde{\rho}$ are not specified yet. The gauge transformations are

$$T \rightarrow e^{i\sum k_i\theta_i} T, \quad \tilde{T} \rightarrow e^{-i\sum k_i\theta_i} \tilde{T}, \quad (13.18)$$

so that their gauge charges are $\pm(k_1, \dots, k_G)$ respectively. Keeping T, \tilde{T} in the description, we will have to divide by the full gauge group $U(1)^G$ (still nothing is charged under $U(1)_{\text{diag}}$). We can rewrite the D-term equations (11.36) and (11.35) as

$$0 = \tilde{\mathcal{D}}_i = k_i g^2 |T|^2 - k_i g^2 |\tilde{T}|^2 + \mathcal{D}_i \quad \forall i \quad (13.19)$$

with the extra complex constraint

$$T\tilde{T} = 1, \quad (13.20)$$

where $\tilde{\mathcal{D}}_i$ are “improved D-terms”. Here g^2 is some mass scale, discussed below. The improved D-term equations can be thought of as arising in the presence of extra chiral fields T, \tilde{T} with charges $\pm(k_1, \dots, k_G)$.

The equivalence works as follows:

$$\begin{aligned} 0 &= \sum_i c_i \tilde{\mathcal{D}}_i = \sum_i c_i \mathcal{D}_i && \forall \{c_i\} \text{ s.t. } \sum_i c_i k_i = 0 \\ 0 &= \sum_i \frac{k_i}{|k|^2} \tilde{\mathcal{D}}_i = g^2 |T|^2 - g^2 |\tilde{T}|^2 + \sum_i \frac{k_i}{|k|^2} \mathcal{D}_i. \end{aligned} \quad (13.21)$$

The first set is exactly (11.36). The second equation is equivalent to (11.35) if we express ρ and $\tilde{\rho}$ in terms of σ through the equations (13.20) and

$$|T|^2 - |\tilde{T}|^2 + \frac{\sigma}{2\pi g^2} = 0. \quad (13.22)$$

These two equations have one and only one solution in terms of σ .

As a result, the *same* moduli space can be obtained by adding T, \tilde{T} to the set $\{X_a\}$ of chiral fields, adding (13.20) to the set of classical F-term relations derived from the superpotential, and dividing by the full gauge group $U(1)^G$. Rephrasing, we start with a larger algebraic variety

$$\tilde{\mathcal{Z}} = \{X_a, T, \tilde{T} \mid dW = 0, T\tilde{T} = 1\} \subset \mathbb{C}^{M+2}, \quad (13.23)$$

and construct the geometric moduli space as the Kähler quotient

$$\mathcal{M} = \tilde{\mathcal{Z}} // U(1)^G. \quad (13.24)$$

It is natural to associate T and \tilde{T} with the monopole operators. In fact, following [217], it is natural to combine the vector multiplet scalar σ and the scalar dual to the photon in a chiral multiplet. The mass scale g^2 does not affect the moduli space, and we can use the coupling of the diagonal photon A_{diag} in a YM-CS UV completion of the theory. The relation (13.20) is a particular case of the quantum relation (13.15): we see that in the unflavored case it appears at the classical level, in the parametrization of the moduli space. Moreover T and \tilde{T} are necessary to parametrize the moduli space with operators invariant under the full $U(1)^G$ gauge group, as we saw in our discussion of the ABJM theory in Chapter 11.

13.4.2 Flavored quivers

Let us study the geometric moduli space of a quiver theory, flavored along the lines of Section 13.3. Let $\{X_\alpha\}$ be the set of bifundamental fields which are flavored, with superpotential $W = W_0 + \sum_\alpha p_\alpha X_\alpha q_\alpha$, and h_α the number of flavors in each family.

The F-term equations $dW = 0$ are clearly modified. In particular there could be Higgs branches where p_α, q_α get a VEV. This can happen when $X_\alpha = 0$, which, in the dual gravitational theory, corresponds either in IIA to the D2-brane ending on the D6-branes and turning on instanton field-strength configurations on their worldvolume, or in M-theory to the M2-brane ending on the local $\mathbb{C}^2 \times \mathbb{C}^2 / \mathbb{Z}_{h_\alpha}$ singularity. However we will not study Higgs branches. Therefore on the branch where

$$p_\alpha = 0, \quad q_\alpha = 0 \quad \forall \alpha \quad (13.25)$$

the F-term equations $dW = 0$ are the same as in the unflavored case. To those, we add the conjectured quantum relation (13.15):

$$T\tilde{T} = \prod_\alpha X_\alpha^{h_\alpha}. \quad (13.26)$$

We get an algebraic variety

$$\tilde{\mathcal{Z}} = \{X_a, T, \tilde{T} \mid dW = 0, T\tilde{T} = \prod_\alpha X_\alpha^{h_\alpha}\} \subset \mathbb{C}^{M+2}, \quad (13.27)$$

where M is the total number of bifundamental chiral fields. $\tilde{\mathcal{Z}}$ has to be divided by the complexified gauge group $U(1)^G$, so that the moduli space of the flavored quiver is

$$\mathcal{M}_{\text{flav}} = \tilde{\mathcal{Z}} // U(1)^G. \quad (13.28)$$

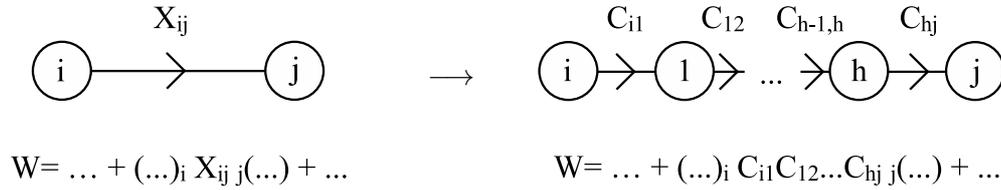


Figure 13.2: Deformation of the unflavored theory to construct the A-theory.

The gauge charges of T and \tilde{T} are in (13.13), and recall that, generically, in the flavoring process the Chern-Simons levels have to be shifted as explained in Section 13.3. Notice that, even though not discussed in this chapter, the same construction goes through if we couple a flavor group not to a bifundamental field X_α but to a bifundamental operator $\mathcal{O}_\alpha = \prod_\beta X_\beta$ built out of a connected open path in the quiver.

13.4.3 Toric flavored CS quivers

In case the CS-matter quiver theory is a brane tiling, and thus its geometric moduli space is a toric CY_4 , the flavoring of Section 13.3 produces a new theory whose geometric moduli space is still a toric CY_4 , and we can explicitly provide its toric diagram.

Toricity is easy to understand: if we interpret the tiling as a quiver theory in $3+1$ dimensions, its mesonic moduli space is $\mathcal{M}_{3+1} = \{X_\alpha | dW = 0\} // U(1)^G$, which is a toric threefold and thus has (at least) $U(1)^3$ symmetry. The space $\mathcal{M}_{\text{flav}}$ in (13.28) is then a fourfold, has an extra $U(1)$ symmetry acting on T, \tilde{T} and is then toric.

The strategy is to consider a *different theory* – that we call the A(auxiliary)-theory, as opposed to the flavored theory under consideration⁹ – of which we can easily construct the toric diagram, and then show that its geometric moduli space is the same as $\mathcal{M}_{\text{flav}}$ in (13.28). The A-theory is a usual CS-matter brane tiling theory, and its geometric moduli space can be computed with the Kasteleyn matrix algorithm. It is constructed as follows.

We start with the brane tiling of the unflavored theory, refined by numbers n_{ij} that encode the CS levels as $k_i = \sum_j (n_{ij} - n_{ji})$. Every time in the flavored theory we add h_α flavors (p_α, q_α) coupled to a bifundamental $X_\alpha \equiv X_{ij}$, in the A-theory we introduce h_α new gauge groups $U(1)_1^{(l)}$ with $l = 1, \dots, h_\alpha$ and substitute X_{ij} by $h_\alpha + 1$ bifundamental fields $C_{i1}, C_{12}, \dots, C_{h_\alpha j}$ coupled to the new groups in a chain as in Figure 13.2. The new superpotential of the A-theory is equal to the old one, but with the substitution $X_{ij} \rightarrow C_{i1} C_{12} \dots C_{h_\alpha j}$. In the tiling this corresponds to substituting the edge X_{ij} by $h_\alpha + 1$ nearby edges $C_{i1}, C_{12}, \dots, C_{h_\alpha j}$, connecting the same two superpotential nodes as X_{ij} , and enclosing h_α new faces $U(1)_1^{(l)}$ between them.¹⁰

Then we assign integers to the C fields: going from C_{i1} to $C_{h_\alpha j}$, they must be a sequence of increasing consecutive integers including n_{ij} (the old integer of X_{ij}). This

⁹We call the A-theory “auxiliary” because it is not our primary object of study, but rather a tool to compute the toric diagram of $\mathcal{M}_{\text{flav}}$.

¹⁰Such a feature of the tiling has been dubbed “multi-bond” and studied in [253, 260, 263]. See the discussion in Chapter 11.

means that we can choose an integer γ_α , with $0 \leq \gamma_\alpha \leq h_\alpha$, and then the numbers n are:

$$(C_{i1}, C_{12}, \dots, C_{h_\alpha j}) \rightarrow (n_{ij} - \gamma_\alpha, n_{ij} - \gamma_\alpha + 1, \dots, n_{ij} - \gamma_\alpha + h_\alpha). \quad (13.29)$$

The parameter γ_α , that represents the choice of theory, must be taken equal to the one in (13.7). The CS levels of the new gauge groups $U(1)_1^{(l)}$ are all 1; the CS levels of $U(1)^{(i)}$ and $U(1)^{(j)}$ are shifted as $k_i \rightarrow k_i - \gamma_\alpha$ and $k_j \rightarrow k_j + \gamma_\alpha - h_\alpha$.

We claim that the moduli space $\mathcal{M}_{\text{flav}}$ in (13.28) is a CY_4 , and its 3d toric diagram is the toric diagram obtained from the A-theory, for instance by the Kasteleyn matrix algorithm. The proof is given in the following subsection.

The deformation of the unflavored moduli space at the level of toric diagram is readily understood. The perfect matchings t_k that define the unflavored 3d toric diagram, obtained by the Kasteleyn matrix algorithm, have ‘‘horizontal’’ coordinates (x, y) and ‘‘height’’ z . For each perfect matching t_k , we add a number of consecutive points above and below t_k , with the same horizontal coordinates (x, y) as t_k . The points are added to the perfect matchings which appear in the parametrization (3.30) of flavored fields X_α . To be precise, the number of consecutive points above and below t_k is:

$$t_k \rightarrow \sum_\alpha M_{k\alpha}(h_\alpha - \gamma_\alpha) \text{ above and } \sum_\alpha M_{k\alpha}\gamma_\alpha \text{ below,}$$

where $M_{k\alpha}$ is the perfect matching matrix. A rich zoology of examples is provided in Section 13.6.

The reason for the addition of points goes as follows. In constructing the tiling of the A-theory, we substitute the edges X_α with $h_\alpha + 1$ new edges connecting the same two superpotential nodes, and assign them the integers in (13.29). Therefore, for each perfect matching that was constructed using X_α , we get h_α new perfect matchings with the same horizontal coordinates and consecutive heights determined by their integers. It is easy to check that the net result on the toric diagram is the one claimed above.

Finally, since each field X_a appears in at least one strictly external perfect matching, the deformed 3d toric diagram of the flavored theory has external ‘‘columns of vertically aligned points’’, which correspond to local KK monopoles in the CY_4 that is local $\mathbb{C}^2 \times \mathbb{C}^2/\mathbb{Z}_{h_\alpha}$ singularities. Thus the bottom-up approach gives results in perfect agreement with the top-down analysis of Section 13.2.

13.4.4 Moduli space of flavored quivers from the A-theory: a proof

In this subsection we prove that the geometric moduli space of the A-theory is the same as $\mathcal{M}_{\text{flav}}$ in (13.28).

Consider a single bifundamental $X_\alpha \equiv X_{ij}$ flavored by h_α quarks (p_α, q_α) in the flavored theory. In the A-theory X_{ij} has been substituted by $h_\alpha + 1$ bifundamentals $C_{i1}, C_{12}, \dots, C_{h_\alpha j}$, h_α new gauge groups $U(1)_1^{(l)}$ with $l = 1, \dots, h_\alpha$ have been added, and the other two gauge groups involved have CS levels $k_i - \gamma_\alpha$ and $k_j + \gamma_\alpha - h_\alpha$, with $0 \leq \gamma_\alpha \leq h_\alpha$, in terms of the levels k_i and k_j before flavoring. As we showed in Section 13.4.1, the geometric moduli space of the A-theory is the Kähler quotient

$$\mathcal{M}_{\text{A-theory}} = \{X_a, R, \tilde{R} \mid dW_A = 0, R\tilde{R} = 1\} // U(1)^{\tilde{G}}, \quad (13.30)$$

where R, \tilde{R} are the monopoles in the A-theory, W_A its superpotential and $\tilde{G} = G + h_\alpha$ is the total number of gauge groups.

The only fields charged under the h_α new groups $U(1)_1^{(l)}$ are $C_{i1}, C_{12}, \dots, C_{h_\alpha j}, R$ and \tilde{R} . Their charges, including $U(1)_{k_i - \gamma_\alpha}$ and $U(1)_{k_j + \gamma_\alpha - h_\alpha}$, are:

	C_{i1}	C_{12}	\dots	$C_{h_\alpha - 1, h_\alpha}$	$C_{h_\alpha j}$	R	\tilde{R}
$U(1)_{k_i - \gamma_\alpha}$	1	0	\dots	0	0	$k_i - \gamma_\alpha$	$-k_i + \gamma_\alpha$
$U(1)_1^{(1)}$	-1	1		0	0	1	-1
$U(1)_1^{(2)}$	0	-1		0	0	1	-1
\vdots	\vdots	\vdots		\vdots	\vdots	\vdots	\vdots
$U(1)_1^{(h_\alpha - 1)}$	0	0		1	0	1	-1
$U(1)_1^{(h_\alpha)}$	0	0		-1	1	1	-1
$U(1)_{k_j + \gamma_\alpha - h_\alpha}$	0	0	\dots	0	-1	$k_j + \gamma_\alpha - h_\alpha$	$-k_j - \gamma_\alpha + h_\alpha$

Let us perform the Kähler quotient by the complexified gauge group $\prod_{l=1}^{h_\alpha} U(1)_1^{(l)}$ only: it is done by introducing gauge invariants and relations between them. The independent gauge invariants are:

$$\begin{aligned}
T &\equiv R(C_{i1})^{h_\alpha} (C_{12})^{h_\alpha - 1} \dots (C_{h_\alpha - 1, h_\alpha})^1 & R\tilde{R} &= 1 \\
\tilde{T} &\equiv \tilde{R}(C_{12})^1 (C_{23})^2 \dots (C_{h_\alpha j})^{h_\alpha} & X_{ij} &\equiv C_{i1} C_{12} \dots C_{h_\alpha j},
\end{aligned} \tag{13.31}$$

where we dubbed one of them as the old field X_{ij} . The only relation is

$$T\tilde{T} = (X_{ij})^{h_\alpha}. \tag{13.32}$$

We see that, after quotienting, the new monopole operators are T, \tilde{T} and obey a ‘‘quantum’’ F-term relation. The charges of X_{ij}, T, \tilde{T} under the remaining two groups are:

	X_{ij}	T	\tilde{T}
$U(1)_{k_i - \gamma_\alpha}$	1	$k_i - \gamma_\alpha + h_\alpha$	$-k_i + \gamma_\alpha$
$U(1)_{k_j + \gamma_\alpha - h_\alpha}$	-1	$k_j + \gamma_\alpha - h_\alpha$	$-k_j - \gamma_\alpha$

We want to compare these charges with those in the flavored theory. When the flavored theory is being flavored by h_α quarks, its CS levels have to be shifted by $\delta k_l = (\frac{h_\alpha}{2} - \gamma_\alpha) g_l[X_{ij}]$ (13.7), where γ_α is a choice of theory. Plugging into (13.13) we get:

$$g_l[T] = k_l + (h_\alpha - \gamma_\alpha) g_l[X_{ij}] \quad g_l[\tilde{T}] = -k_l + \gamma_\alpha g_l[X_{ij}]. \tag{13.33}$$

This precisely agrees with the table above if we identify the choice of $0 \leq \gamma_\alpha \leq h_\alpha$ between the flavored and A-theory. So the quotient by $\prod_{l=1}^{h_\alpha} U(1)_1^{(l)}$ gives the A-theory monopoles the same quantum charges as in the flavored theory.¹¹

Let us now consider the classical F-term relations. In the A-theory, the F-terms are of two sorts: differentiating W_A by a field which is not $C_{i1}, \dots, C_{h_\alpha j}$ we get the same

¹¹The CS levels are different in the flavored and A-theory, but this does not matter. What matters for the moduli space are the charges of chiral fields.

equation as in the flavored theory, but with $X_{ij} \rightarrow C_{i1} \dots C_{h_\alpha j}$; differentiating W_A by one of $C_{i1}, \dots, C_{h_\alpha j}$ we get the same equation as in the flavored theory, but multiplied by the other C fields:

$$\left(\prod C\right) (\text{flavored theory relation}) = 0. \quad (13.34)$$

As long as no more than one of the C fields vanishes, we exactly reproduce the same F-terms as in the flavored theory. When more than one C field vanishes, all equations become trivial and the A-theory could develop a branch which is not contained in the geometric moduli space of the flavored theory. However, the geometric moduli space of the A-theory (which is the CY₄) is the one where the F-terms are solved by the parametrization $X_\alpha = \prod_k t_k^{M_{k\alpha}}$, thus if the flavored theory relations are satisfied at $C \neq 0$, they are satisfied also at $C = 0$.

We have thus shown that:

$$\begin{aligned} \mathcal{M}_{\text{A-theory}} &= \{X_a^A \mid dW_A = 0\} // U(1)^{\tilde{G}-2} = \\ &= \{X_a^A, R, \tilde{R} \mid dW_A = 0, R\tilde{R} = 1\} // U(1)^{\tilde{G}} = \\ &= \{X_a^F, T, \tilde{T} \mid dW_{\text{flav}} = 0, T\tilde{T} = \prod X_\alpha^{h_\alpha}\} // U(1)^G = \mathcal{M}_{\text{flav}}. \end{aligned} \quad (13.35)$$

The argument is straightforwardly generalized to the case that we flavor multiple fields X_α , each with its own h_α quarks. This concludes the proof.

Let us conclude with a remark. Suppose that the theory before flavoring has some global Abelian symmetry, under which X_{ij} has charge Q . Then also the A-theory has such a symmetry, if we assign charges $Q/(h_\alpha + 1)$ to $C_{i1}, \dots, C_{h_\alpha j}$. It is easy to compute that, after modding out by $\prod_l U(1)_1^{(l)}$, both T and \tilde{T} have charge $h_\alpha Q/2$. This reproduces the quantum formulæ (13.12) and (13.14).

13.5 Back to geometry: real and complex masses

Each non-compact toric divisor of a toric CY₃ is a strictly external point of its 2d toric diagram. In the field theory it corresponds to a set of fields $\{X_\eta\}_{\eta=1, \dots, Q}$ (with the same global charges), where Q is determined by (13.1): the equation $X_\eta = 0$, for any of the Q fields, defines the divisor as a submanifold of the mesonic moduli space. Placing a stack of h D6-branes on the divisor introduces h flavors coupled to one of the fields $\{X_\eta\}$ via the superpotential $\delta W = pX_\eta q$. This follows from the fact that the modes from 2-6 strings described by (p, q) become massless when some D2-branes are on top of the D6-branes. Moreover the D6-branes carry $U(h)$ gauge fields, which by the AdS/CFT map give rise to $U(h)$ global symmetry in the boundary theory.

There are Q fields such that the equation $X_\eta = 0$ describes the same irreducible divisor. The reason is that the radial section of the divisor can have non-trivial fundamental group (in the toric case $\pi_1(S^3/\mathbb{Z}_Q) = \mathbb{Z}_Q$); therefore a flat connection can be specified as boundary condition on the D6 worldvolume, distinguishing which of the Q fields it is coupled to. The connection is then flat everywhere but at the tip, where its flux can affect the shift of CS levels via (13.3). Indeed, flavoring different fields in the set $\{X_\eta\}$ implies shifting different CS levels (13.6). Clearly we can pile up D6-branes with different flat connection.

The converse is not true: a generic field X_a corresponds – via the equation $X_a = 0$ – to a collection of pairwise intersecting toric divisors, rather than to a single irreducible divisor. More precisely, each field is part of a set $\{X_\eta\}_{1,\dots,Q}$ which corresponds to a collection of consecutive strictly external points along the perimeter of the toric diagram. The number Q of fields in the set is still given by the formula in footnote 4, but taking the cross product between two non-consecutive legs (in the (p, q) -web) that enclose the sequence of points [299]. Flavoring one of the fields X_η via $\delta W = pX_\eta q$ is accomplished by placing a stack of D6-branes on the collection of intersecting divisors, described by $X_\eta = 0$. The map is easily worked out with perfect matchings and the Kasteleyn matrix algorithm; we gave an example in Section 3.5.4.

All these statements translate to M-theory. A stack of h D6-branes on the fibered CY_3 uplift to a CY_4 with h KK monopoles, which locally have complex structure $\mathbb{C}^2 \times \mathbb{C}^2/\mathbb{Z}_h$ and the geometry of a multi-Taub-NUT. The equation $X_\alpha = 0$ describes the location of the core of the multi-Taub-NUT. Such a singularity in M-theory carries $SU(h)$ gauge fields, while the extra $U(1)$ comes from the KK reduction of the bulk potential C_3 . In fact the geometry of h coincident KK monopoles is

$$ds_{KK}^2 = U d\vec{x} \cdot d\vec{x} + \frac{1}{U} (d\theta + A_\omega)^2 \quad \text{with} \quad U = \frac{1}{|\vec{x}|} + \frac{1}{\lambda^2}, \quad (13.36)$$

where $\vec{x} \in \mathbb{R}^3$, U is a harmonic function on \mathbb{R}^3 , $A_\omega = \vec{\omega} \cdot d\vec{x}$ is a $U(1)$ connection on \mathbb{R}^3 such that $dU = *_3 dA_\omega$, θ has period $4\pi/h$ and λ is the asymptotic radius of the circle. For $h = 1$ the metric is smooth, otherwise it has an A_{h-1} singularity. The 2-form

$$B = d\Lambda = d \left[\frac{|\vec{x}|}{|\vec{x}| + \lambda^2} (d\theta + A_\omega) \right] \quad (13.37)$$

is closed, anti-self-dual, regular and integrable. Thus a local KK reduction $C_3 = A \wedge B$ gives an extra $U(1)$ gauge field propagating around the core of the multi-Taub-NUT.

The flat boundary condition for the connection on the D6-branes uplifts to a flat boundary condition for C_3 (and possibly the gauge fields at the singularity). However, since in type IIA the connection is not flat at the tip and its flux can affect the CS levels which ultimately determine the fibration of the CY_3 along \mathbb{R} , in M-theory different boundary conditions can uplift to different geometries. An example will be given in subsection 13.6.3.

13.5.1 Real masses and partial resolutions

We can introduce real masses for chiral fields with the term

$$\int d^4\theta Z^\dagger e^{2i\tilde{m}\theta\bar{\theta}} Z. \quad (13.38)$$

As in Section 13.3, we can think of the real mass as a VEV for a background scalar σ_F , in the $\mathcal{N} = 2$ vector multiplet of $U(h)$. In this way we give opposite mass to the flavors p and q . The VEV of σ_F corresponds to the position of the D6-branes along the real line \mathbb{R} transverse to the CY_3 . When the D6-branes at r_0 are displaced from the D2-branes at the tip, the flavors can be integrated out at low energy. We showed in (13.3) that

opposite signs for σ_F affect the CS levels, consistently with the field theory discussion in Section 13.3.

Real masses, like Fayet-Iliopoulos parameters, do not affect the superpotential [217]. Uplifting to M-theory, real masses do not affect the complex structure of the CY₄ but rather its Kähler parameters: they correspond to blowing up a 2-cycle. In simple examples, integrating out a flavor pair corresponds to removing a single strictly external point from the 3d toric diagram: the local $\mathbb{C}^2 \times \mathbb{C}^2/\mathbb{Z}_h$ singularity manifests itself as a column of $h + 1$ external points, and integrating out a quark pair with negative (positive) $\langle \sigma_F \rangle$ corresponds to a partial resolution of the upmost (lowest) point in the column. Only in this limit of infinite mass/resolution parameter, the effective complex structure changes, as the removal of the point in the toric diagram shows. In more complicated situations, the partial resolution corresponding to giving infinite real mass to a flavor pair could correspond to removing more than one point: the precise map is via perfect matchings, as analyzed in Section 13.4.2.

13.5.2 Complex masses

Complex masses for the flavors can correspond to geometric deformations of the D6-brane embeddings, but not always. Suppose we want to flavor a bifundamental operator $\mathcal{O}_\alpha = \prod_\beta X_\beta$, made of an open chain of bifundamental fields. We can proceed in the following way: we flavor each field X_β separately, and then introduce complex masses for each chiral pair:

$$W = W_0 + \sum_{\beta=1}^n p_\beta X_\beta q_\beta + \sum_{\beta=1}^{n-1} m_\beta p_{\beta+1} q_\beta . \quad (13.39)$$

After integrating out the massive flavors, we get

$$W = W_0 + \frac{(-1)^{n-1}}{\prod_\beta m_\beta} p_1 \left(\prod_\beta X_\beta \right) q_n \equiv W_0 + p_\alpha \mathcal{O}_\alpha q_\alpha , \quad (13.40)$$

with suitable redefinition of fields. Since fermions in vector-like representations do not contribute to the monopole charges, the quantum F-term relation is unmodified:

$$T\tilde{T} = \prod_\alpha (X_\alpha)^{h_\alpha} = (\mathcal{O}_\alpha)^{h_\alpha} . \quad (13.41)$$

Therefore the two theories where we flavor \mathcal{O}_α or each X_β separately have the same geometric moduli space, and can only differ in their Higgs branches.

The complex masses m_β do *not* correspond to deformations of the D6-brane embeddings. In fact we can probe the embedding with D2-branes: the quarks become massless on $\bigcup_\beta \{X_\beta = 0\}$, which does not depend on m_β . Such masses correspond to VEVs for higher dimensional fields localized at the intersection of the D6-branes, which do not change the embedding.

This leads to the following natural generalization. Consider starting with a conical CY₃, not necessarily toric, and its dual quiver theory defined by D3-branes probing it. We can always include RR fluxes and fiber it along \mathbb{R} , that is add $\mathcal{N} = 2$ Chern-Simons terms in field theory (the geometry then uplifts to a CY₄ in M-theory). Then consider a

collection of divisors of the CY_3 , defined by a set of “bifundamental equations” written in terms of bifundamental fields in the quiver theory:

$$\bigcup_{\alpha} \{\text{equation}_{\alpha} = 0\}. \quad (13.42)$$

Each equation is a bifundamental operator and, if it is an adjoint, a mass term $\mu\mathcal{K}$ can be included. We place h_{α} D6-branes on the divisor $\{\text{equation}_{\alpha} = 0\}$. For each equation, this corresponds to introducing a pair of h_{α} flavor fields, with the correct gauge charges to couple to the bifundamental operator. They contribute to the charges of monopole operators precisely such that the only non-trivial possible quantum relation is

$$T\tilde{T} = \prod_{\alpha} (\text{equation}_{\alpha})^{h_{\alpha}}. \quad (13.43)$$

It then follows that the moduli space is the CY_4

$$\mathcal{M}_{\text{flav}} = \{X_{\alpha}, T, \tilde{T} \mid dW = 0, T\tilde{T} = \prod_{\alpha} (\text{equation}_{\alpha})^{h_{\alpha}}\} // U(1)^G. \quad (13.44)$$

It would be nice to check or prove this statement.

13.6 Examples – various flavored quiver gauge theories

In this section we discuss various examples of three-dimensional toric quiver gauge theories with flavors. Some of the flavored quivers have Chern-Simons terms, others do not. However, even when there are no CS terms, the models have a large N_f expansion (N_f being generically the number of flavors) and in the large N and large N_f limit they are expected to be dual to type IIA string theory on a weakly curved background with D6-branes. When the CS levels do not vanish and there are flavors, two independent expansion parameters k and N_f may be taken large and allow a reduction to type IIA string theory.

All the YM-CS quivers we consider are expected to flow to an interacting fixed point. Using the conjectured OPE of monopole operators explained in Section 13.3.1, we discuss the quantum chiral ring at this fixed point. Given any toric flavored Chern-Simons quiver, we can use the Kasteleyn matrix algorithm in the A-theory to find the toric diagram of the geometric moduli space. We will see in various examples how this works in detail. Practically, we solve the moduli space equations of the flavored theory by introducing new perfect matching variables as suggested by the A-theory. The associated GLSM corresponds to the toric CY_4 of the geometric moduli space.

Recall that the gauge invariant functions of the GLSM are the affine coordinates of the toric variety, and that they satisfy an algebra which defines the geometry as an algebraic variety. It follows from our construction that the quantum chiral ring of the quiver corresponds to the ring of affine coordinates on the toric variety. This is an important point, since this equivalence is a necessary condition for the existence of an AdS/CFT correspondence.

For each example we can consider the charges $Q^0 \equiv Q_M$ of the GLSM fields under $U(1)_M$. In our convention the charges are such that $\sum_s Q_s^M \vec{v}_s = (0, 0, 0, 1)$, see Section 13.2.¹² Then, one can work out in each case what is the locus of fixed points of the $U(1)_M$

¹²This only defines Q_M modulo the baryonic symmetries (the other $U(1)$ s in the GLSM). However the $U(1)_M$ charges of the affine coordinates are unambiguous.

action, and to which divisors it corresponds to in the type IIA reduction, making the link with the top-down approach of Section 13.2.

Let us fix the notation. The perfect matching variables t_i of the unflavored quiver are denoted a_z, b_z, c_z, \dots , with z the vertical coordinate of the corresponding point in the toric diagram. The toric diagram of the flavored theory is obtained by adding columns of points above and below some of the original points, as explained in Section 13.4.2. By an $SL(4, \mathbb{Z})$ transformation, we can always set the base of three of the columns of points to $z = 0$. We will always choose such a convenient frame. Although we consider quivers with Abelian gauge groups only, we nevertheless write the non-Abelian superpotentials, in order to make the link with well-known quivers more explicit.

13.6.1 Flavoring the \mathbb{C}^3 quiver

Our first example is the flavoring of $\mathcal{N} = 8$ SYM, the low energy field theory on a D2-brane on flat $\mathbb{C}^3 \times \mathbb{R}$. The quiver is simply that of $\mathcal{N} = 4$ SYM in 3+1 dimensions. In $\mathcal{N} = 2$ notation, we have a single vector superfield and three adjoint chiral superfields Φ_1, Φ_2, Φ_3 , with superpotential $W = \Phi_1[\Phi_2, \Phi_3]$.

We can add one, two or three flavor groups by coupling flavors to the appropriate chiral superfields, as shown in Figure 13.3. We denote by p_i and q_i the fundamental and antifundamental fields in the i -th flavor group coupled to the field Φ_i . The flavoring of a Φ_i corresponds to introducing D6-branes at $z_i = 0, x_9 = 0$, and D2/D6-brane intersections induce the superpotential

$$W = \Phi_1[\Phi_2, \Phi_3] + \sum_{i=1}^{h_1} p_{1,i} \Phi_1 q_{1,i} + \sum_{j=1}^{h_2} p_{2,j} \Phi_2 q_{2,j} + \sum_{l=1}^{h_3} p_{3,l} \Phi_3 q_{3,l} . \tag{13.45}$$

In the general case, the flavor group is $G_F = U(h_1) \times U(h_2) \times U(h_3) / U(1)$. The charges of the fields under the various gauge and global symmetries are summarized in the following table:

	Φ_i	p_1	q_1	p_2	q_2	p_3	q_3	\tilde{T}	T
$U(1)$	0	-1	1	-1	1	-1	1	0	0
$U(h_1)$	(1)	(h_1)	$(\overline{h_1})$	(1)	(1)	(1)	(1)	(1)	(1)
$U(h_2)$	(1)	(1)	(1)	(h_2)	$(\overline{h_2})$	(1)	(1)	(1)	(1)
$U(h_3)$	(1)	(1)	(1)	(1)	(1)	(h_3)	$(\overline{h_3})$	(1)	(1)

(13.46)

In this simple case, flavor groups are non-chirally coupled and so the monopole operators T, \tilde{T} do not acquire any gauge charge. Nevertheless, they do acquire some R-charge,

$$R(T) = R(\tilde{T}) = \frac{1}{2} (h_1 R(\Phi_1) + h_2 R(\Phi_2) + h_3 R(\Phi_3)) . \tag{13.47}$$

The quantum holomorphic relation (13.15) is

$$T\tilde{T} = \Phi_1^{h_1} \Phi_2^{h_2} \Phi_3^{h_3} . \tag{13.48}$$

It describes an affine variety whose affine coordinates are the five gauge invariant operators T, \tilde{T} and Φ_i (in the case of a $U(N)$ gauge group one should consider the eigenvalues). Let us discuss a few particular cases related to known models in the literature [253, 263, 290].

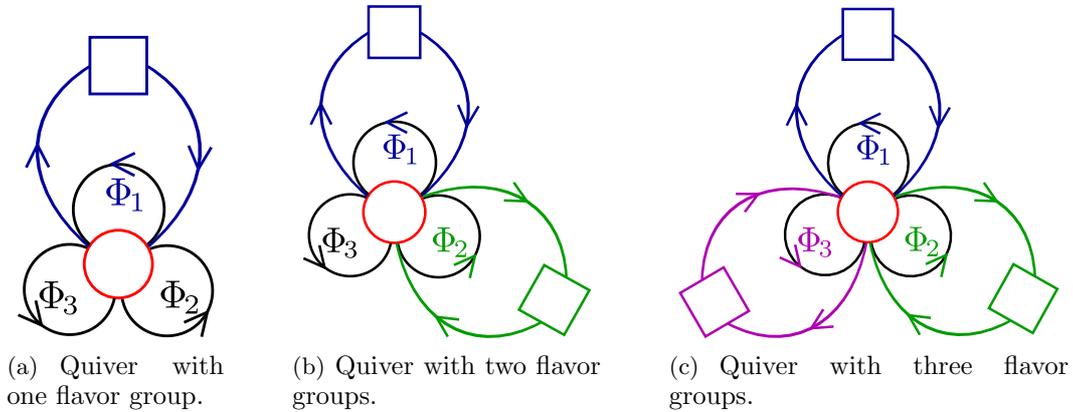


Figure 13.3: Quivers for flavored SQED. Circles are gauge groups, squares are flavor groups. Colored arrows indicate bifundamental fields coupled to flavors via a superpotential term.

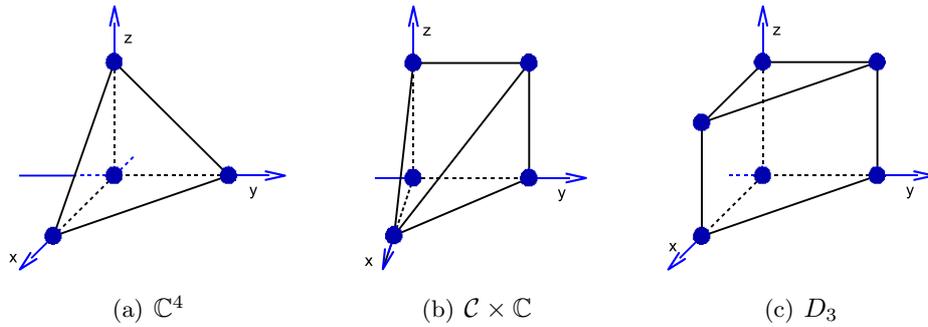


Figure 13.4: Toric diagrams corresponding to some flavors for the \mathbb{C}^3 quiver.

- In the case $h_2 = h_3 = 0$, The chiral ring relation is

$$T\tilde{T} = \Phi_1^{h_1}, \quad (13.49)$$

and the geometric branch of the moduli space is $\mathbb{C}^2 \times \mathbb{C}^2 / \mathbb{Z}_{h_1}$. For $h_1 = 1$ we have \mathbb{C}^4 , see Figure 13.4(a). This model is related to the dual ABJM model of [253, 263]. We discuss it in a bit more details in Section 13.6.1.

- For $h_1 = h_2 = 1$, $h_3 = 0$, we have $\mathbb{C} \times \mathcal{C}$ (\mathcal{C} the conifold), see Fig. 13.4(b). The A-theory for this model is the so-called Phase III of $\mathbb{C} \times \mathcal{C}$ discussed in [263].
- For $h_1 = 2$, $h_2 = 1$, $h_3 = 0$, we have \mathbb{C} times the suspended pinch point (SPP). This was also noticed in [290]. In general, for $h_1 = a$, $h_2 = b$, the geometry is $\mathbb{C} \times C(L^{aba})$.
- For $h_1 = h_2 = h_3 = 1$, the geometry is D_3 , see Fig. 13.4(c). The A-theory for this model is the Phase III of D_3 discussed in [263].

When some $h_i > 1$, these geometries have non-isolated singularities. Remark that we have considered the most general toric flavoring of the \mathbb{C}^3 quiver. The GLSM for the

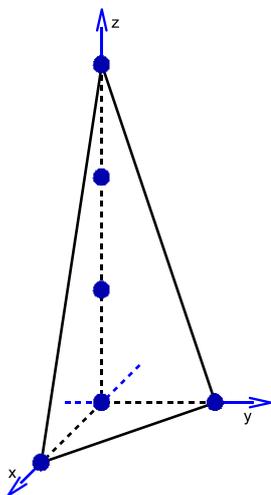


Figure 13.5: Toric diagrams of $\mathbb{C}^2 \times \mathbb{C}^2 / \mathbb{Z}_h$, for $h = 3$.

strictly external points is

	t_1	t_2	t_3	t_4	t_5	t_6	
$U(1)_{B_1}$	h_2	$-h_1$	0	$-h_2$	h_1	0	
$U(1)_{B_2}$	0	h_3	$-h_2$	0	$-h_3$	h_2	
$U(1)_{B_3}$	$-h_3$	0	h_1	h_3	0	$-h_1$	(13.50)

This GLSM does not encode various orbifold identifications which might in general arise: for a full description of the geometry one should consider the full GLSM, encoding all the relations in the toric diagram, with $h_1 + h_2 + h_3 + 3$ homogeneous coordinates.

Flavoring Φ_1 : the dual ABJM geometry

Let us discuss a bit more in detail the case $\mathbb{C}^2 \times \mathbb{C}^2 / \mathbb{Z}_h$. This geometry has the toric diagram shown in Figure 13.5,

$$a_0 = (0, 0, 0), \quad \dots, \quad a_h = (0, 0, h), \quad b_0 = (0, 1, 0), \quad c_0 = (1, 0, 0) \tag{13.51}$$

There are $h + 3$ homogeneous coordinates, and GLSM

	a_0	b_0	c_0	a_1	a_2	a_3	\dots	a_{h-1}	a_h	
$U(1)_{B_1}$	1	0	0	-2	1	0	\dots	0	0	
$U(1)_{B_2}$	0	0	0	1	-2	1	\dots	0	0	
$U(1)_{B_3}$	0	0	0	0	1	-2	\dots	0	0	
\vdots							\ddots			
$U(1)_{B_{h-1}}$	0	0	0	0	0	0	\dots	-2	1	
$U(1)_M$	1	0	0	-1	0	0	\dots	0	0	(13.52)

The five affine coordinates are

$$\begin{aligned} x_1 = \Phi_1 &= a_0 a_1 \dots a_{h-1} a_h, & x_2 = T &= a_0^h a_1^{h-1} \dots a_{h-2}^2 a_{h-1}, \\ x_3 = \tilde{T} &= a_1 a_2^2 \dots a_{h-1}^{h-1} a_h^h, & x_4 = \Phi_2 &= b_0, & x_5 = \Phi_3 &= c_0, \end{aligned} \quad (13.53)$$

and of course they satisfy

$$x_2 x_3 = x_1^h. \quad (13.54)$$

Also, the Q_M charges of (x_1, \dots, x_5) are $(0, 1, -1, 0, 0)$, so that $U(1)_M$ has fixed points at $x_2 = x_3 = 0$. Gauging $U(1)_M$, we get the type IIA geometry, which is \mathbb{C}^3 spanned by $(z_1, z_2, z_3) = (x_1, x_4, x_5)$ since the gauge invariant coordinate $x_2 x_3$ can be eliminated by (13.54). The locus of fixed points of $U(1)_M$ in the CY_4 descends to the divisor $x_1 = 0$ in \mathbb{C}^3 , where we must have a stack of h D6-branes. This was the argument of section 13.2, which motivates the field theory we presented.

Note that the same geometry is obtained as the moduli space of the so-called dual ABJM model of [253], at CS level h . This model was also studied in [263, 307, 272], and some puzzles were found. At $h = 1$, the dual ABJM model corresponds to the A-theory for our flavored theory with a single flavor. For h flavors, our A-theory is a tiling with an $(h + 1)$ -ple bond. It would be interesting to compare in more details our proposal to the one of [253].

For some specific values of the superpotential couplings, the supersymmetry of our flavored quiver gets enhanced to $\mathcal{N} = 4$, since the geometry $\mathbb{C}^2 \times \mathbb{C}^2/\mathbb{Z}_h$ is hyper-Kähler. Indeed, our setup is a $\mathcal{N} = 2$ version of the setup considered in [207].

13.6.2 Flavoring the conifold quiver

Consider the quiver of the ABJM theory, equal to the Klebanov-Witten (KW) quiver for D-branes on the conifold \mathcal{C} . It has two nodes, four bifundamental fields, A_1, A_2, B_1, B_2 , and superpotential $W = A_1 B_1 A_2 B_2 - A_1 B_2 A_2 B_1$. There are four points in the toric diagram of $\mathbb{C}^4/\mathbb{Z}_k$, corresponding to the four perfect matchings in the brane tiling of the conifold theory and to the bifundamental fields: because the F-term relations are trivial in the Abelian theory, we can write (with abuse of notation)

$$\begin{aligned} a_k = A_1 &= (0, 0, k), & b_0 = B_1 &= (0, 1, 0), \\ c_0 = A_2 &= (1, 1, 0), & d_0 = B_2 &= (1, 0, 0). \end{aligned} \quad (13.55)$$

We then consider the toric diagram obtained by adding four columns of points of heights h_a, h_b, h_c, h_d above the four base points (any other choice of adding the points above or below, is $SL(4, \mathbb{Z})$ equivalent to this up to a change in k):

$$a_{k+i} = (0, 0, k + i), \quad b_j = (0, 1, j), \quad c_l = (1, 1, l), \quad d_m = (1, 0, m), \quad (13.56)$$

where $i = 0, \dots, h_a, j = 0, \dots, h_b, l = 0, \dots, h_c, m = 0, \dots, h_d$. See Figure 13.6(b).

This toric geometry corresponds to a generic flavoring of the ABJM theory at level k , with flavor group $G_F = U(h_a) \times U(h_b) \times U(h_c) \times U(h_d)/U(1)$. The quiver is shown in Figure 13.6(a), and the superpotential is

$$\begin{aligned} W &= A_1 B_1 A_2 B_2 - A_1 B_2 A_2 B_1 + \\ &+ \sum_{i=1}^{h_a} p_{1,i} A_1 q_{1,i} + \sum_{j=1}^{h_b} p_{2,j} B_1 q_{2,j} + \sum_{l=1}^{h_c} p_{3,l} A_2 q_{3,l} + \sum_{r=1}^{h_d} p_{4,r} B_2 q_{4,r}. \end{aligned} \quad (13.57)$$

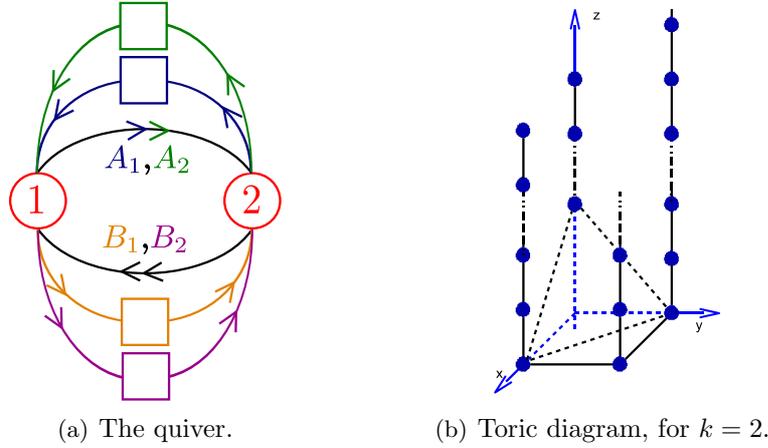


Figure 13.6: Quiver for a generic flavoring of ABJM, and the corresponding toric diagram, with four columns of heights h_a, \dots, h_d .

Before studying several interesting cases, let us discuss the general solution for the geometric moduli space in this family of models. We have the quantum relation (13.15),

$$T\tilde{T} = A_1^{h_a} B_1^{h_b} A_2^{h_c} B_2^{h_d}, \quad (13.58)$$

and the CS levels are $(k + f, -k - f)$, with $f = \frac{1}{2}(h_a - h_b + h_c - h_d)$. The gauge charges of bifundamental fields and monopole operators are (schematically)

	A_i	B_j	T	\tilde{T}	
$U(1)_{k+f}$	1	-1	$k + 2f$	$-k$	(13.59)
$U(1)_{-(k+f)}$	-1	1	$-k - 2f$	k	

The relation (13.58) can be solved by the perfect matching variables, as

$$A_1 = \prod_{i=0}^{h_a} a_{k+i}, \quad B_1 = \prod_{j=0}^{h_b} b_j, \quad A_2 = \prod_{l=0}^{h_c} c_l, \quad B_2 = \prod_{m=0}^{h_d} d_m \quad (13.60)$$

and

$$T = \left(\prod_{i=0}^{h_a} a_{k+i}^{h_a-i} \right) \left(\prod_{j=0}^{h_b} b_j^{h_b-j} \right) \left(\prod_{l=0}^{h_c} c_l^{h_c-l} \right) \left(\prod_{m=0}^{h_d} d_m^{h_d-m} \right) \quad (13.61)$$

$$\tilde{T} = \left(\prod_{i=0}^{h_a} a_{k+i}^i \right) \left(\prod_{j=0}^{h_b} b_j^j \right) \left(\prod_{l=0}^{h_c} c_l^l \right) \left(\prod_{m=0}^{h_d} d_m^m \right),$$

Notice that each perfect matching variable (13.55) of the ABJM theory is replaced by the product of all GLSM fields associated to the relevant column of points in the toric diagram. Monopole operators are instead products of fields along the four columns, with increasing or decreasing powers as we move vertically. This is to be compared to (13.31). It is easy to show that the $U(1)$ ambiguities of this parametrization reproduce the GLSM associated to the toric diagram (13.56).

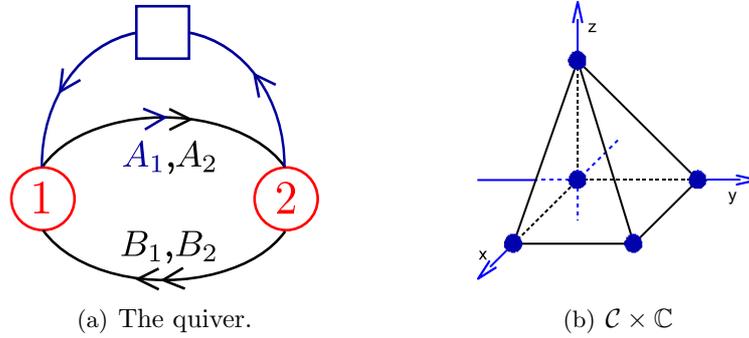


Figure 13.7: ABJM quiver with one chiral flavor, and its dual geometry.

Flavoring the field A_1 : the $\mathcal{C} \times \mathbb{C}$ geometry

Let us add a $U(1)$ flavor group to the 3d KW theory ($k = 0$), coupled to the bifundamental field A_1 as in Figure 13.7(a). The superpotential is $W = A_1 B_1 A_2 B_2 - A_1 B_2 A_2 B_1 + p A_1 q$, and the CS levels are $(\frac{1}{2}, -\frac{1}{2})$. The charges of the fields under the gauge and flavor groups are

	A_i	B_i	p	q	T	\tilde{T}	
$U(1)_{\frac{1}{2}}$	1	-1	0	-1	1	0	
$U(1)_{-\frac{1}{2}}$	-1	1	1	0	-1	0	(13.62)
$U(1)_F$	0	0	-1	1	0	0	

There are seven gauge invariant operators, namely $A_i B_j$, $T B_i$ and \tilde{T} . Using the quantum relation $T \tilde{T} = A_1$, we can however express $A_1 B_i$ as $\tilde{T} T B_i$, so that we actually have only 5 generators of the chiral ring,

$$x_1 = T B_1, \quad x_2 = A_2 B_2, \quad x_3 = T B_2, \quad x_4 = A_2 B_1, \quad x_5 = \tilde{T}, \quad (13.63)$$

subject to the relation

$$x_1 x_2 - x_3 x_4 = 0. \quad (13.64)$$

Hence, the moduli space is $\mathcal{C} \times \mathbb{C}$. Indeed, the quantum relation can be solved by $T = a_0$, $\tilde{T} = a_1$ and $A_1 = a_0 a_1$. The GLSM is

	a_0	b_0	c_0	d_0	a_1	
$U(1)_B$	1	-1	1	-1	0	
$U(1)_M$	1	0	0	0	-1	(13.65)

where we also specified the $U(1)_M$ charges. The toric diagram is shown in Figure 13.7(b). The locus of fixed points of the $U(1)_M$ action descends to the toric divisor $\{a_0 = 0\}$ in the conifold, where the D6-brane sits.

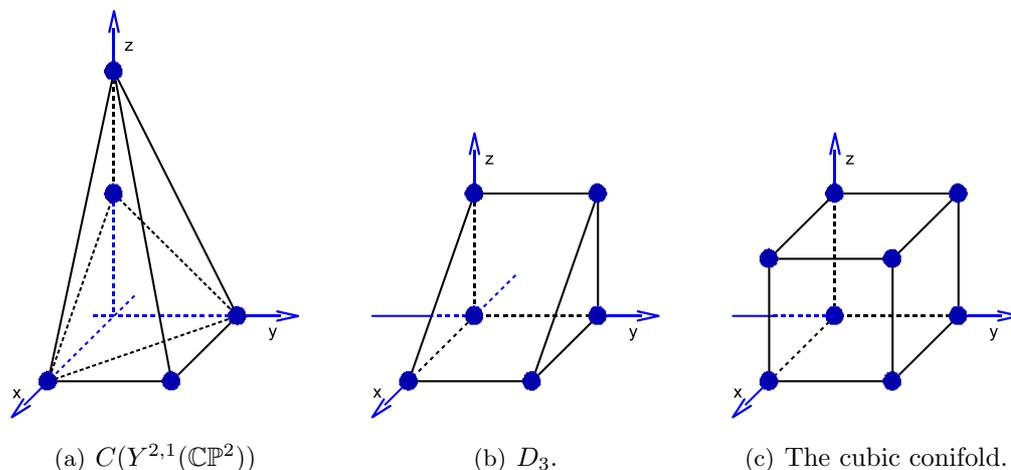


Figure 13.8: Toric diagrams corresponding to some flavors for the ABJM quiver.

Flavoring the field A_1 : the $C(Y^{2,1}(\mathbb{CP}^2))$ geometry

Let us then couple a $U(1)$ flavor group to A_1 in the ABJM theory at level $k = 1$. Now the CS levels are $(\frac{3}{2}, -\frac{3}{2})$ and the fields have gauge charges

	A_i	B_i	p	q	T	\tilde{T}	
$U(1)_{\frac{3}{2}}$	1	-1	0	-1	2	-1	(13.66)
$U(1)_{-\frac{3}{2}}$	-1	1	1	0	-2	1	

The quantum relation is solved by $T = a_1$, $\tilde{T} = a_2$, $A_1 = a_1 a_2$. The GLSM is

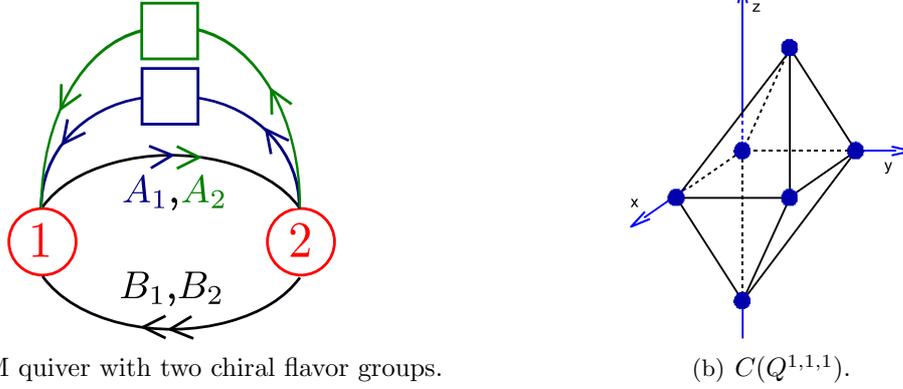
	a_1	b_0	c_0	d_0	a_2	
$U(1)_B$	2	-1	1	-1	-1	(13.67)
$U(1)_M$	1	0	0	0	-1	

The corresponding toric diagram is shown in Fig. 13.8(a), and it corresponds to the cone over $Y^{2,1}(\mathbb{CP}^2)$ [308]. This geometry and a related theory (actually the A-theory for our flavored theory) was discussed in [271]. There are nine gauge invariant operators for this quiver, matching the nine affine coordinates of the $C(Y^{2,1}(\mathbb{CP}^2))$ singularity:

$$\begin{aligned}
 x_1 &= TB_1 B_1 = a_1 b_0^2, & x_2 &= TB_2 B_2 = a_1 d_0^2, & x_3 &= \tilde{T} A_1 = a_1 a_2^2, \\
 x_4 &= TB_1 B_2 = a_1 b_0 d_0, & x_5 &= A_1 B_1 = a_1 b_0 a_2, & x_6 &= A_1 B_2 = a_1 d_0 a_2, \\
 x_7 &= A_2 B_1 = b_0 c_0, & x_8 &= A_2 B_2 = c_0 d_0, & x_9 &= \tilde{T} A_2 = c_0 a_2
 \end{aligned}$$

The chiral ring relations are:

$$\begin{aligned}
 x_1 x_8 &= x_4 x_7, & x_2 x_9 &= x_6 x_8, & x_3 x_7 &= x_5 x_9, & x_4 x_9 &= x_5 x_8, \\
 x_1 x_9 &= x_5 x_7, & x_2 x_7 &= x_4 x_8, & x_3 x_8 &= x_6 x_9, & x_4 x_9 &= x_6 x_7, \\
 x_1 x_2 &= x_4^2, & x_1 x_3 &= x_5^2, & x_2 x_3 &= x_6^2, & & \\
 x_1 x_6 &= x_4 x_5, & x_2 x_5 &= x_4 x_6, & x_3 x_4 &= x_5 x_6. & &
 \end{aligned} \tag{13.68}$$



(a) ABJM quiver with two chiral flavor groups.

(b) $C(Q^{1,1,1})$.

Figure 13.9: ABJM quiver with two chiral flavor groups, and a dual geometry.

Flavoring the fields A_1 and A_2 : the $C(Q^{1,1,1})$ geometry

Consider the conifold quiver with two $U(1)$ flavor groups coupled to A_1 and A_2 respectively, as in Fig. 13.9(a). The superpotential is

$$W = A_1 B_1 A_2 B_2 - A_1 B_2 A_2 B_1 + p_1 A_1 q_1 + p_2 A_2 q_2, \tag{13.69}$$

and we choose vanishing CS levels. In the toric diagram, this corresponds to adding one point below a_0 and one point above c_0 , see Fig. 13.9(b). The gauge charges of the fields and monopole operators are

	A_i	B_i	p_i	q_i	T	\tilde{T}	
$U(1)_0$	1	-1	0	-1	1	1	(13.70)
$U(1)_0$	-1	1	1	0	-1	-1	

The monopole operators satisfy the relation

$$T\tilde{T} = A_1 A_2. \tag{13.71}$$

We can solve it by introducing two new perfect matching variables a_{-1} and c_1 :

$$\begin{aligned} A_1 &= a_{-1} a_0, & B_1 &= b_0, & T &= a_{-1} c_0, \\ A_2 &= c_0 c_1, & B_2 &= d_0, & \tilde{T} &= a_0 c_1. \end{aligned} \tag{13.72}$$

The associated GLSM is a minimal presentation of the one for the real cone over $Q^{1,1,1}$:

	a_0	b_0	c_0	d_0	a_{-1}	c_1	
$U(1)_{B_1}$	1	-1	1	-1	0	0	(13.73)
$U(1)_{B_2}$	1	0	1	0	-1	-1	
$U(1)_M$	0	0	1	0	0	-1	

The gauge invariant operators generating the chiral ring are:

$$\begin{aligned} x_1 &= A_1 B_1 = a_{-1} a_0 b_0, & x_2 &= A_2 B_2 = c_0 c_1 d_0, & x_3 &= A_2 B_1 = b_0 c_0 c_1, \\ x_4 &= A_1 B_2 = a_{-1} a_0 d_0, & x_5 &= \tilde{T} B_1 = a_0 b_0 c_1, & x_6 &= \tilde{T} B_2 = a_0 c_1 d_0, \\ x_7 &= T B_1 = a_{-1} b_0 c_0, & x_8 &= T B_2 = a_{-1} c_0 d_0. \end{aligned}$$

They of course correspond to the affine coordinates on $C(Q^{1,1,1})$, whose algebra is

$$\begin{aligned} x_1x_2 - x_3x_4 = 0, & & x_1x_2 - x_5x_8 = 0, & & x_1x_2 - x_6x_7 = 0, \\ x_1x_3 - x_5x_7 = 0, & & x_1x_6 - x_4x_5 = 0, & & x_1x_8 - x_4x_7 = 0, \\ x_2x_4 - x_6x_8 = 0, & & x_2x_5 - x_3x_6 = 0, & & x_2x_7 - x_3x_8 = 0. \end{aligned} \quad (13.74)$$

Remark that the affine coordinates have $U(1)_M$ charges

$$\begin{array}{c|cccccccc} & x_1 & x_2 & x_3 & x_4 & x_5 & x_6 & x_7 & x_8 \\ \hline U(1)_M & 0 & 0 & 0 & 0 & -1 & -1 & 1 & 1 \end{array} \quad (13.75)$$

so the $U(1)_M$ fixed point locus is at $x_5 = x_6 = x_7 = x_8 = 0$, $x_1x_2 = x_3x_4 = x_1x_3 = x_2x_4 = 0$. This locus of fixed points has two branches:

$$\begin{aligned} 1) & \quad x_1 = x_4 = 0, \quad x_5 = x_6 = x_7 = x_8 = 0, \quad \forall x_2, x_3 & \iff & \quad a_0 = a_{-1} = 0 \\ 2) & \quad x_2 = x_3 = 0, \quad x_5 = x_6 = x_7 = x_8 = 0, \quad \forall x_1, x_4 & \iff & \quad c_0 = c_1 = 0. \end{aligned} \quad (13.76)$$

It is easy to see that they descend to the toric divisors $\{a_0 = 0\}$ and $\{c_0 = 0\}$ in the conifold \mathcal{C} . The D6-branes wrapping these divisors provide us with the chiral flavors in the quiver field theory.

Another quiver for the low energy field theory on M2-branes on $C(Q^{1,1,1})$ was proposed in [259], and further studied in [262]. The quiver of [259], which has two double-bonds, is precisely the A-theory of our chirally flavored conifold theory.

Flavoring the fields A_1 and B_1 : the D_3 geometry

Let us now couple a $U(1)$ flavor group to A_1 and a $U(1)$ flavor group to B_1 , with $\delta W = p_1 A_1 q_1 + \tilde{p}_1 B_1 \tilde{q}_1$ and vanishing CS levels. In this case there is no induced gauge charge for the monopole operators, because there are as many incoming as outgoing arrows in each gauge group. We have the quantum relation $T\tilde{T} = A_1 B_1$, which is solved by $A_1 = a_0 a_1$, $B_1 = b_0 b_1$, $T = a_0 b_0$ and $\tilde{T} = a_1 b_1$. The associated GLSM is

$$\begin{array}{c|cccccc} & a_0 & b_0 & c_0 & d_0 & a_1 & b_1 \\ \hline U(1)_{B_1} & 1 & -1 & 1 & -1 & 0 & 0 \\ U(1)_{B_2} & 1 & -1 & 0 & 0 & -1 & 1 \\ \hline U(1)_M & 1 & 0 & 0 & 0 & -1 & 0 \end{array} \quad (13.77)$$

The toric diagram, shown in Fig. 13.8(b), is the one of the D_3 geometry. The generators of the chiral ring are

$$x_1 = \tilde{T}, \quad x_2 = A_2 B_2, \quad x_3 = T, \quad x_4 = A_1 B_2, \quad x_5 = A_2 B_1. \quad (13.78)$$

As expected, they satisfy the defining equation of the D_3 singularity:

$$x_1 x_2 x_3 - x_4 x_5 = 0. \quad (13.79)$$

The locus of fixed points of $U(1)_M$ has two branches which descend to the two divisors $\{a_0 = 0\}$ and $\{b_0 = 0\}$ in the conifold.

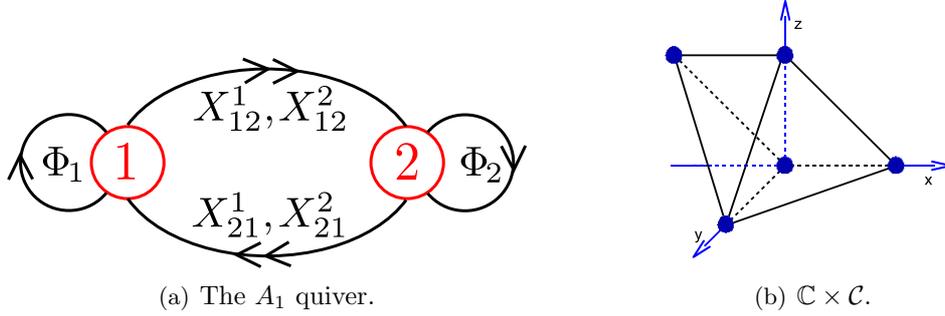


Figure 13.10: Quiver of the modified $\mathbb{C} \times \mathbb{C}^2/\mathbb{Z}_2$ model (CS levels $(1, -1)$), and moduli space.

Flavoring A_1, A_2, B_1, B_2 : the cubic conifold

Consider coupling a $U(1)$ flavor group to each bifundamental field, with vanishing CS levels. The quantum relation is

$$T\tilde{T} = A_1B_1A_2B_2 . \tag{13.80}$$

One can check that the moduli space is described by the following GLSM:

	a_0	b_0	c_0	d_0	a_1	b_1	c_1	d_1
$U(1)_{B_1}$	1	-1	1	-1	0	0	0	0
$U(1)_{B_2}$	0	0	0	0	1	-1	1	-1
$U(1)_{B_3}$	1	0	0	-1	-1	0	0	1
$U(1)_{B_4}$	1	-1	0	0	-1	1	0	0
$U(1)_M$	1	0	0	0	-1	0	0	0

(13.81)

The toric diagram is shown in Fig. 13.8(c), and we will call this geometry the *cubic conifold*. The gauge invariant operators are

$$\begin{aligned} x_1 &= a_0b_0c_0d_0 = T, & x_2 &= a_1b_1c_1d_1 = \tilde{T}, & x_3 &= a_0b_0a_1b_1 = A_1B_1, \\ x_4 &= c_0d_0c_1d_1 = A_2B_2, & x_5 &= a_0d_0a_1d_1 = A_1B_2, & x_6 &= b_0c_0b_1c_1 = A_2B_1, \end{aligned}$$

satisfying the equations

$$x_1x_2 - x_3x_4 = 0, \quad x_1x_2 - x_5x_6 = 0 . \tag{13.82}$$

This is a complete intersection. The $U(1)_M$ charges of (x_1, \dots, x_6) are $(1, -1, 0, 0, 0, 0)$. The locus of fixed point is at $x_1 = x_2 = 0, x_3x_4 = x_5x_6 = 0$, which has four branches and descend to the four toric divisors of the conifold.

13.6.3 Flavoring the modified $\mathbb{C} \times \mathbb{C}^2/\mathbb{Z}_2$ theory

In this section we add flavors to the so-called modified $\mathbb{C} \times \mathbb{C}^2/\mathbb{Z}_2$ theory of [245]. The quiver of the unflavored theory, Fig. 13.10(a), is the one for D-branes at a $\mathbb{C} \times \mathbb{C}^2/\mathbb{Z}_2$ singularity; we choose the height numbers n_{ij} equal to 1 for the bifundamental X_{12}^1 and 0 otherwise, so that the two gauge groups have CS levels $(1, -1)$. The superpotential is

$$W = \Phi_1(X_{12}^1X_{21}^2 - X_{12}^2X_{21}^1) - \Phi_2(X_{21}^2X_{12}^1 - X_{21}^1X_{12}^2) . \tag{13.83}$$

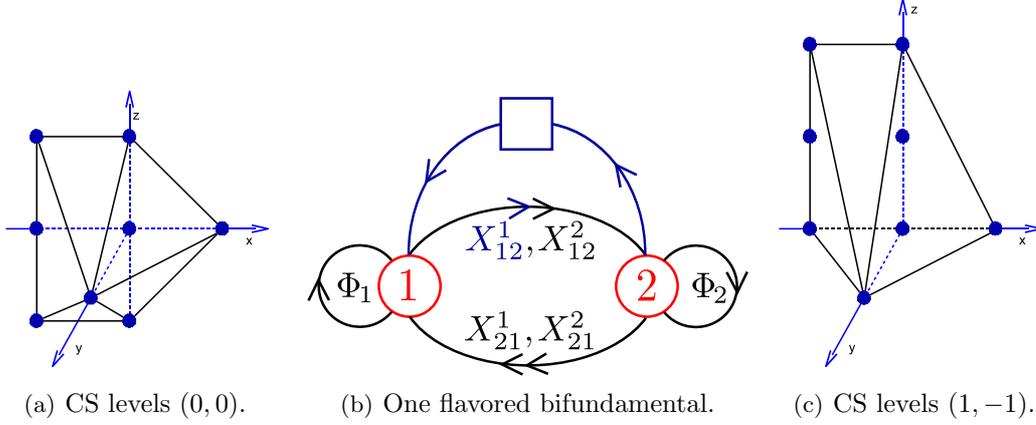


Figure 13.11: The $\mathbb{C} \times \mathbb{C}^2/\mathbb{Z}_2$ quiver with a $U(2)$ flavor group coupled to X_{12}^1 . Those two toric quivers are obtained by fixing the CS levels as indicated.

From the permanent of the Kasteleyn matrix,

$$\text{Perm } K = X_{21}^1 X_{21}^2 + X_{12}^2 X_{21}^2 x + X_{21}^1 X_{12}^1 x^{-1} z + X_{12}^1 X_{12}^2 z + \Phi_1 \Phi_2 y, \quad (13.84)$$

we see that the perfect matchings are

$$\begin{aligned} a_0 &= \{X_{21}^1, X_{21}^2\} = (0, 0, 0), & d_1 &= \{X_{12}^1, X_{12}^2\} = (0, 0, 1), \\ b_0 &= \{X_{12}^2, X_{21}^2\} = (1, 0, 0), & e_0 &= \{\Phi_1, \Phi_2\} = (0, 1, 0), \\ c_1 &= \{X_{21}^1, X_{12}^1\} = (-1, 0, 1). \end{aligned} \quad (13.85)$$

The 3d toric diagram, Fig. 13.10(b), is the one of $\mathbb{C} \times \mathcal{C}$. The F-term equations imply $X_{12}^1 X_{21}^2 = X_{12}^2 X_{21}^1$ and $\Phi_1 = \Phi_2$ along the mesonic branch. They are solved by

$$X_{12}^1 = c_1 d_1, \quad X_{12}^2 = b_0 d_1, \quad X_{21}^1 = a_0 c_1, \quad X_{21}^2 = a_0 b_0, \quad \Phi_1 = \Phi_2 = e_0.$$

The face in the 3d toric diagram whose vertices are $\{a_0, c_1, d_1, b_0\}$ is vertical, therefore additional objects may appear in the type IIA background. Nevertheless, encouraged by the results of [245] where the geometric moduli space was successfully matched with $\mathbb{C} \times \mathcal{C}$, we will trust the duality and add flavors to this model.

We will study three illustrative examples where two flavor pairs are added to this theory.

$U(2)$ flavor group coupled to X_{12}^1 : levels $(0, 0)$

We study two cases where we couple a $U(2)$ flavor group to X_{12}^1 , as in Fig. 13.11(b). Consider first the case where the CS levels vanish. The bifundamental fields and monopole operators of the quiver theory have gauge charges

	X_{12}	X_{21}	Φ	T	\tilde{T}
$U(1)_0$	1	-1	0	1	1
$U(1)_0$	-1	1	0	-1	-1

(13.86)

The gauge invariant operators in the geometric branch are Φ , $X_{12}X_{21}$, TX_{21} , $\tilde{T}X_{21}$.

In the A-theory, this flavoring corresponds to replacing the edge X_{12}^1 with $n_{X_{12}^1} = 1$ in the original brane tiling with a triple-bond with $n = -1, 0, 1$. It amounts to considering a 3d toric diagram with the points $\{a_0, b_0, c_{-1}, c_0, c_1, d_{-1}, d_0, d_1, e_0\}$ as in Fig. 13.11(a). We solve for the F-term relation and the quantum relation $T\tilde{T} = X_{12}^1$ by

$$\begin{aligned} X_{12}^1 &= c_{-1} c_0 c_1 d_{-1} d_0 d_1, & X_{12}^2 &= b_0 d_{-1} d_0 d_1, \\ X_{21}^1 &= a_0 c_{-1} c_0 c_1, & X_{21}^2 &= a_0 b_0, & \Phi_1 &= \Phi_2 = e_0, \\ \tilde{T} &= c_0 c_1^2 d_0 d_1^2, & T &= c_{-1}^2 c_0 d_{-1}^2 d_0. \end{aligned} \quad (13.87)$$

The charges of the homogeneous coordinates of the four-fold and of the quiver theory fields under the associated $U(1)^5$ GLSM are

	a_0	b_0	c_1	d_1	e_0	c_0	d_0	c_{-1}	d_{-1}	X_{12}	X_{21}	Φ	T	\tilde{T}
$U(1)_{B1}$	1	-1	-1	1	0	0	0	0	0	0	0	0	0	0
$U(1)_{B2}$	1	0	1	-1	0	-1	0	0	0	-1	1	0	-1	-1
$U(1)_{B3}$	1	0	0	0	0	0	-1	0	0	-1	1	0	-1	-1
$U(1)_{B4}$	1	0	0	0	0	-1	0	1	-1	-1	1	0	-1	-1
$U(1)_{B5}$	-2	0	0	1	0	0	0	0	1	2	-2	0	2	2

matching the gauge charges (13.86). The affine coordinates of the fourfold match the gauge invariant operators of the flavored quiver theory:

$$\begin{aligned} x_1 &= e_0 = \Phi_1 = \Phi_2, & x_2 &= a_0 b_0^2 d_{-1} d_0 d_1 = X_{12}^2 X_{21}^2, \\ x_3 &= a_0 b_0 c_0 c_1^2 d_0 d_1^2 = \tilde{T} X_{21}^2, & x_4 &= a_0 c_{-1} c_0^2 c_1^3 d_0 d_1^2 = \tilde{T} X_{21}^1, \\ x_5 &= a_0 b_0 c_{-1} c_0 c_1 d_{-1} d_0 d_1 = X_{12}^1 X_{21}^2 = X_{12}^2 X_{21}^1, & & \\ x_6 &= a_0 c_{-1}^2 c_0^2 c_1^2 d_{-1} d_0 d_1 = X_{12}^1 X_{21}^1, & & \\ x_7 &= a_0 b_0 c_{-1}^2 c_0 d_{-1}^2 d_0 = T X_{21}^2, & x_8 &= a_0 c_{-1}^3 c_0^2 c_1 d_{-1}^2 d_0 = T X_{21}^1. \end{aligned} \quad (13.88)$$

$U(2)$ flavor group coupled to X_{12}^1 : levels $(1, -1)$

Consider now the case of CS levels $(1, -1)$. The gauge charges are:

	X_{12}	X_{21}	Φ	T	\tilde{T}
$U(1)_1$	1	-1	0	2	0
$U(1)_{-1}$	-1	1	0	2	0

(13.89)

The gauge invariant operators are Φ , $X_{12}X_{21}$, $T(X_{21})^2$, \tilde{T} .

In the A-theory, this flavoring corresponds to replacing the edge X_{12}^1 with $n_{X_{12}^1} = 1$ in the original brane tiling by a triple-bond with $n = 0, 1, 2$. The GLSM field appearing in the 3d toric diagram, Fig. 13.11(c), are $\{a_0, b_0, c_0, c_1, c_2, d_0, d_1, d_2, e_0\}$. We solve for the geometric moduli space by setting

$$\begin{aligned} X_{12}^1 &= c_0 c_1 c_2 d_0 d_1 d_2, & X_{12}^2 &= b_0 d_0 d_1 d_2, \\ X_{21}^1 &= a_0 c_0 c_1 c_2, & X_{21}^2 &= a_0 b_0, & \Phi_1 &= \Phi_2 = e_0, \\ \tilde{T} &= c_1 c_2^2 d_1 d_2^2, & T &= c_0^2 c_1 d_0^2 d_1. \end{aligned} \quad (13.90)$$

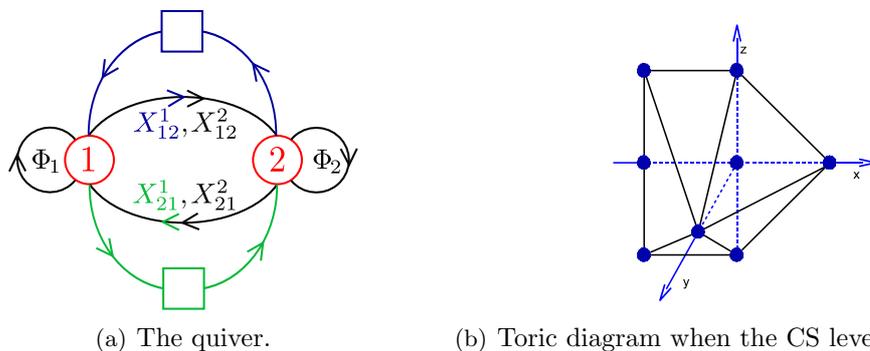


Figure 13.12: Modified $\mathbb{C} \times \mathbb{C}^2/\mathbb{Z}_2$ model with two flavored bifundamentals, and dual geometry.

The charges of the homogeneous coordinates of the fourfold and of the quiver theory fields under the $U(1)^5$ GLSM are

	a_0	b_0	c_1	d_1	e_0	c_0	d_0	c_2	d_2	X_{12}	X_{21}	Φ	T	\tilde{T}
$U(1)_{B1}$	1	-1	-1	1	0	0	0	0	0	0	0	0	0	0
$U(1)_{B2}$	1	0	1	-1	0	-1	0	0	0	-1	1	0	-2	0
$U(1)_{B3}$	1	0	0	0	0	0	-1	0	0	-1	1	0	-2	0
$U(1)_{B4}$	1	0	0	-2	0	0	0	0	1	-1	1	0	-2	0
$U(1)_{B5}$	0	0	-2	0	0	1	0	1	0	0	0	0	0	0

matching the gauge charges (13.89). The affine coordinates of the four-fold match the holomorphic gauge invariants of the flavored quiver theory:

$$\begin{aligned}
 x_1 &= a_0 b_0 c_0 c_1 c_2 d_0 d_1 d_2 = X_{12}^1 X_{21}^2 = X_{12}^2 X_{21}^1, & x_2 &= e_0 = \Phi_1 = \Phi_2, \\
 x_3 &= a_0^2 b_0^2 c_0^2 c_1 d_0^2 d_1 = T(X_{21}^2)^2, & x_4 &= a_0 c_0^2 c_1^2 c_2^2 d_0 d_1 d_2 = X_{12}^1 X_{21}^1, \\
 x_5 &= a_0^2 b_0 c_0^3 c_1^2 c_2 d_0^2 d_1 = T X_{21}^1 X_{21}^2, & x_6 &= a_0 b_0^2 d_0 d_1 d_2 = X_{12}^2 X_{21}^2, \\
 x_7 &= a_0^2 c_0^4 c_1^3 c_2^2 d_0^2 d_1 = T(X_{21}^1)^2, & x_8 &= c_1 c_2^2 d_1 d_2^2 = \tilde{T}.
 \end{aligned}$$

$U(1)^2$ flavor groups coupled to X_{12}^1 and X_{21}^1 : levels $(1, -1)$

Let us study a case where we couple a $U(1)$ flavor group to X_{12}^1 and a $U(1)$ flavor group to X_{21}^1 , as in Fig. 13.12(a). The quantum relation reads $T\tilde{T} = X_{12}^1 X_{21}^2$. We consider the case with CS levels $(1, -1)$: bifundamentals and monopole operators charges are

$$\begin{array}{c|ccccc}
 & X_{12} & X_{21} & \Phi & T & \tilde{T} \\
 \hline
 U(1)_1 & 1 & -1 & 0 & 1 & -1 \\
 U(1)_{-1} & -1 & 1 & 0 & -1 & 1
 \end{array} \tag{13.91}$$

The gauge invariant operators are $\Phi, X_{12} X_{21}, T X_{21}, \tilde{T} X_{12}$.

In the A-theory, this flavoring corresponds to replacing the edge X_{12}^1 with $n_{X_{12}^1} = 1$ in the brane tiling by a double-bond with $n = 0, 1$, and the edge X_{21}^1 with $n_{X_{21}^1} = 0$ by another double-bond, with $n = -1, 0$. All the other n_{ij} vanish. This gives a 3d toric diagram with points $\{a_{-1}, a_0, b_0, c_{-1}, c_0, c_1, d_0, d_1, e_0\}$, Fig. 13.12(b). This is not a

minimal presentation of the toric diagram. In particular, unlike for the other multiplicities, the distinction between c_0 and c'_0 is not needed to express the bifundamentals and monopole operators in terms of GLSM fields solving the F-term equations. It is possible to replace the two of them by a single field \tilde{c}_0 (setting $c_0 c'_0 = \tilde{c}_0$ in the formulæ below), getting rid of a $U(1)$ in the GLSM. We will do that in the following. Keeping instead all the perfect matching fields of the A-theory may be useful in the study of partial resolutions dual to real mass terms.

We solve for the geometric moduli space by setting

$$\begin{aligned} X_{12}^1 &= c_{-1} \tilde{c}_0 c_1 d_0 d_1, & X_{12}^2 &= b_0 d_0 d_1, \\ X_{21}^1 &= a_{-1} a_0 c_{-1} \tilde{c}_0 c_1, & X_{21}^2 &= a_{-1} a_0 b_0, & \Phi_1 &= \Phi_2 = e_0, \\ \tilde{T} &= a_0 \tilde{c}_0 c_1^2 d_1, & T &= a_{-1} c_{-1}^2 \tilde{c}_0 d_0. \end{aligned} \quad (13.92)$$

The charges of the homogeneous coordinates of the four-fold and of the quiver theory fields under the resulting $U(1)^5$ GLSM are

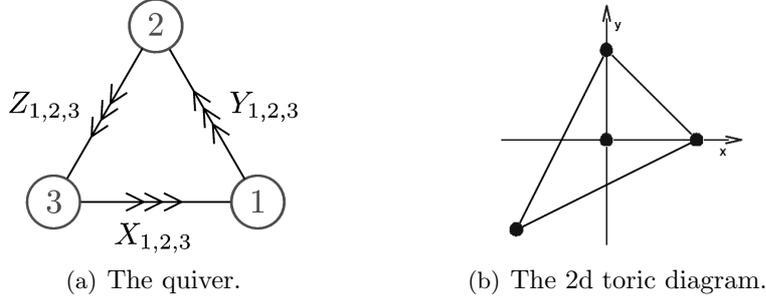
	a_0	b_0	\tilde{c}_0	d_0	e_0	a_{-1}	c_1	c_{-1}	d_1	X_{12}	X_{21}	Φ	T	\tilde{T}
$U(1)_{B1}$	1	0	0	-1	0	0	0	0	0	-1	1	0	-1	1
$U(1)_{B2}$	-2	1	1	0	0	0	0	0	0	1	-1	0	1	-1
$U(1)_{B3}$	1	0	-1	0	0	0	1	0	-1	-1	1	0	-1	1
$U(1)_{B4}$	1	0	-1	0	0	-1	0	1	0	0	0	0	0	0
$U(1)_{B5}$	0	0	2	0	0	0	-1	-1	0	0	0	0	0	0

matching the gauge charges (13.91). The affine coordinates of the four-fold match the holomorphic gauge invariants of the flavored quiver theory:

$$\begin{aligned} x_1 &= e_0 = \Phi_1 = \Phi_2, & x_2 &= a_{-1} a_0 b_0^2 d_0 d_1 = X_{12}^2 X_{21}^2, \\ x_3 &= a_{-1}^2 a_0 c_{-1}^3 c_0^2 c_0'^2 c_1 d_0 = T X_{21}^1, & x_4 &= a_0 c_{-1} c_0^2 c_0'^2 c_1^3 d_0 d_1^2 = \tilde{T} X_{12}^1, \\ x_5 &= a_{-1}^2 a_0 b_0 c_{-1}^2 c_0 c_0' d_0 = T X_{21}^2, & x_6 &= a_0 b_0 c_0 c_0' c_1^2 d_0 d_1^2 = \tilde{T} X_{12}^2, \\ x_7 &= a_{-1} a_0 c_{-1}^2 c_0^2 c_0'^2 c_1^2 d_0 d_1 = X_{12}^1 X_{21}^1 = \tilde{T} T, & & \\ x_8 &= a_{-1} a_0 b_0 c_{-1} c_0 c_0' c_1 d_0 d_1 = X_{12}^1 X_{21}^2 = X_{12}^2 X_{21}^1. & & \end{aligned} \quad (13.93)$$

The toric diagram of the CY_4 is the same as in the double-flavored X_{12}^1 model with CS levels $(0, 0)$ studied in subsection 13.6.3: thus the geometric branches of the moduli spaces of these two theories are the same, although the manifest flavor symmetries of the quivers are different. Presumably, the M-theory backgrounds will differ in monodromies of the 3-form potential C_3 .

The three double flavored models analyzed here for the modified $\mathbb{C} \times \mathbb{C}^2/\mathbb{Z}_2$ model lead to D6-branes along the same toric divisor inside the CY_3 . However there are different gauge connections on the flavor branes, everywhere flat but at the tip, and gauge fluxes on the 2-cycles at the singularity. In spite of the D6-branes being identically embedded at the level of the complex structure, the type IIA/M-theory backgrounds differ, because the different gauge fluxes at the singularity generate RR fluxes that backreact onto the metric.

Figure 13.13: The dP_0 quiver and the 2d toric diagram.

13.6.4 Flavoring the dP_0 quiver

The dP_0 quiver, Fig. 13.13(a), is the quiver for D-branes at a $\mathbb{C}^3/\mathbb{Z}_3$ singularity. It has three nodes and nine bifundamental fields, $X_i, Y_i, Z_i, i = 1, 2, 3$. We choose to parametrize the CS levels by $(k_1, k_2, k_3) = (q - p, q, p - 2q)$. The charges under the $U(1)^3$ gauge group are

	X_i	Y_i	Z_i	T	\tilde{T}
$U(1)_{q-p}$	-1	1	0	$q - p$	$-q + p$
$U(1)_q$	0	-1	1	q	$-q$
$U(1)_{p-2q}$	1	0	-1	$p - 2q$	$-p + 2q$

(13.94)

The superpotential is $W = X_i Y_j Z_k \epsilon^{ijk}$, so the indices ijk are fully symmetric in the chiral ring. From the permanent of the Kasteleyn matrix,

$$\text{Perm } K = X_1 Y_1 Z_1 x z^p + X_2 Y_2 Z_2 x^{-1} y^{-1} + X_3 Y_3 Z_3 y + X_1 X_2 X_3 z^{p-q} + Y_1 Y_2 Y_3 z^q + Z_1 Z_2 Z_3,$$

we read off the perfect matchings and the coordinates of the points in the toric diagram:

$$\begin{aligned} a_p &= \{X_1, Y_1, Z_1\} = (1, 0, p), & e_{p-q} &= \{X_1, X_2, X_3\} = (0, 0, p - q), \\ b_0 &= \{X_2, Y_2, Z_2\} = (0, 1, 0), & f_q &= \{Y_1, Y_2, Y_3\} = (0, 0, q), \\ c_0 &= \{X_3, Y_3, Z_3\} = (-1, -1, 0), & g_0 &= \{Z_1, Z_2, Z_3\} = (0, 0, 0). \end{aligned} \quad (13.95)$$

The choice of $SL(4, \mathbb{Z})$ frame is such that for $p, q > 0$ we have the geometry $Y^{p,q}(\mathbb{CP}^2)$ as presented in [308]. In particular, this family includes the geometry $M^{3,2} = Y^{3,2}(\mathbb{CP}^2)$. The perfect matching variables allow to solve the F-term relations as

$$\begin{aligned} X_1 &= a_p e_{p-q}, & Y_1 &= a_p f_q, & Z_1 &= a_p g_0, \\ X_2 &= b_0 e_{p-q}, & Y_2 &= b_0 f_q, & Z_2 &= b_0 g_0, \\ X_3 &= c_0 e_{p-q}, & Y_3 &= c_0 f_q, & Z_3 &= c_0 g_0, \end{aligned} \quad (13.96)$$

and the redundancies in this parametrization correspond to a non-minimal GLSM for the toric geometry. We couple chiral flavors to bifundamental fields in the dP_0 quiver, and consider a few simple but interesting examples, flavoring the theory with vanishing CS levels $p = q = 0$. The 2d diagram is shown in Fig. 13.13(b).

$U(1)$ flavor group coupled to X_1

Let us couple one flavor to the field X_1 in the quiver with vanishing CS levels, which induces CS levels $(-\frac{1}{2}, 0, \frac{1}{2})$. The quantum relation is $T\tilde{T} = X_1$, and the gauge charges of the fields and monopole operators are:

$$\begin{array}{c|ccccc} & X_i & Y_i & Z_i & T & \tilde{T} \\ \hline U(1)_{-\frac{1}{2}} & -1 & 1 & 0 & -1 & 0 \\ U(1)_0 & 0 & -1 & 1 & 0 & 0 \\ U(1)_{\frac{1}{2}} & 1 & 0 & -1 & 1 & 0 \end{array} \quad (13.97)$$

To find the geometric branch of the moduli space, we solve both the F-terms and the quantum relation by adding two new variables a_1 and e_1 to the solution (13.96):

$$\begin{array}{llll} X_1 = a_0 a_1 e_0 e_1, & Y_1 = a_0 a_1 f_0, & Z_1 = a_0 a_1 g_0, & T = a_0 e_0, \\ X_2 = b_0 e_0 e_1, & Y_2 = b_0 f_0, & Z_2 = b_0 g_0, & \tilde{T} = a_1 e_1, \\ X_3 = c_0 e_0 e_1, & Y_3 = c_0 f_0, & Z_3 = c_0 g_0. & \end{array} \quad (13.98)$$

The associated GLSM is

$$\begin{array}{c|cccccccc} & a_0 & b_0 & c_0 & e_0 & f_0 & g_0 & a_1 & e_1 \\ \hline U(1)_{B_1} & 1 & 1 & 1 & -2 & 0 & -1 & 0 & 0 \\ U(1)_{B_2} & 1 & 1 & 1 & -1 & -2 & 0 & 0 & 0 \\ U(1)_{B_3} & 1 & 1 & 1 & 0 & -1 & -2 & 0 & 0 \\ U(1)_{B_4} & 1 & 0 & 0 & -1 & 0 & 0 & -1 & 1 \\ \hline U(1)_M & 1 & 0 & 0 & 0 & 0 & 0 & -1 & 0 \end{array} \quad (13.99)$$

The three first rows correspond to the gauge group $U(1)^3$ of the quiver. This GLSM is a non-minimal presentation of the toric geometry of Fig. 13.14(a), corresponding to adding two points a_1 and e_1 as suggested by the A-theory. We have also specified the Q_M charges. Gauging $U(1)_M$ leads to the $CY_3 \mathbb{C}^3/\mathbb{Z}_3$, and the locus of fixed points projects to the non-compact divisor $\{a_0 = 0\}$. Let us check that the gauge invariant operators match the affine coordinates of the toric variety. There are 10 operators of the form XYZ , 6 of the form TYZ , and \tilde{T} , but the quantum relation makes $X_1YZ = \tilde{T}TYZ$ redundant, so that we are left with 11 generators of the chiral ring. We can check that they match all the gauge invariant functions of the GLSM:

$$\begin{array}{lll} x_1 = TY_1Z_1 = a_0^3 e_0 f_0 g_0 a_1^2, & x_6 = TY_2Z_2 = a_0 b_0^2 e_0 f_0 g_0, & x_{11} = \tilde{T} = a_1 e_1. \\ x_2 = X_2 Y_2 Z_2 = b_0^3 e_0 f_0 g_0 e_1, & x_7 = X_2 Y_2 Z_3 = b_0^2 c_0 e_0 f_0 g_0 e_1, & \\ x_3 = X_3 Y_3 Z_3 = c_0^3 e_0 f_0 g_0 e_1, & x_8 = TY_3 Z_3 = a_0 c_0^2 e_0 f_0 g_0, & \\ x_4 = TY_1 Z_2 = a_0^2 b_0 e_0 f_0 g_0 a_1, & x_9 = X_2 Y_3 Z_3 = b_0 c_0^2 e_0 f_0 g_0 e_1, & \\ x_5 = TY_1 Z_3 = a_0^2 c_0 e_0 f_0 g_0 a_1, & x_{10} = TY_2 Z_3 = a_0 b_0 c_0 e_0 f_0 g_0, & \end{array}$$

 $U(1)^2$ flavor groups coupled to X_1 and Y_1

Consider flavoring $X_1 = ae$ and $Y_1 = ef$. There are two possible CS levels, but let us consider the case $(0, -\frac{1}{2}, \frac{1}{2})$ corresponding to adding four perfect matching variables a_1 ,

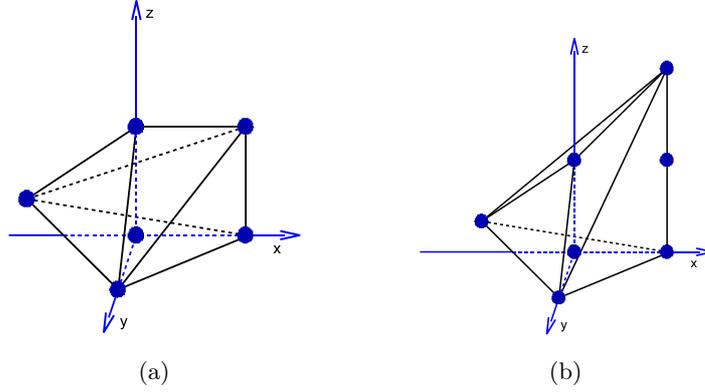


Figure 13.14: Toric diagram obtained by flavoring one or two fields in the dP_0 quiver.

a_2, e_1, f_1 . The toric diagram is in Fig. 13.14(b). The field theory gauge charges are

	X_i	Y_i	Z_i	T	\tilde{T}	
$U(1)_0$	-1	1	0	0	0	
$U(1)_{-\frac{1}{2}}$	0	-1	1	-1	0	(13.100)
$U(1)_{\frac{1}{2}}$	1	0	-1	1	0	

The quantum relation is $T\tilde{T} = X_1Y_1$. There are again 11 gauge invariant operators: $X_iY_jZ_k, TZ_i$ and \tilde{T} , but the three operators $X_1Y_1Z_i$ are redundant due to the quantum relation. We can solve the moduli space equations by

$$\begin{aligned}
 X_1 &= a_0a_1a_2e_0e_1, & Y_1 &= a_0a_1a_2f_0f_1, & Z_1 &= a_0a_1a_2g_0, & T &= a_0^2a_1e_0f_0, \\
 X_2 &= b_0e_0e_1, & Y_2 &= b_0f_0f_1, & Z_2 &= b_0g_0, & \tilde{T} &= a_1a_2^2e_1f_1. \\
 X_3 &= c_0e_0e_1, & Y_3 &= c_0f_0f_1, & Z_3 &= c_0g_0, & &
 \end{aligned}$$

and the associated GLSM is

	a_0	b_0	c_0	e_0	f_0	g_0	a_1	e_1	a_2	f_1	
$U(1)_{B_1}$	1	1	1	-2	0	-1	0	0	0	0	
$U(1)_{B_2}$	1	1	1	-1	-2	0	0	0	0	0	
$U(1)_{B_3}$	1	1	1	0	-1	-2	0	0	0	0	(13.101)
$U(1)_{B_4}$	1	0	0	0	0	0	-2	0	1	0	
$U(1)_{B_5}$	1	0	0	-1	0	0	-1	1	1	0	
$U(1)_{B_6}$	1	0	0	0	-1	0	-1	0	0	1	

The map between affine coordinates and gauge invariant operators is

$$\begin{aligned}
 x_1 &= TZ_1 = a_0^3a_1^2a_2e_0f_0g_0, & x_6 &= X_1Y_2Z_2 = a_0a_1a_2b_0^2e_0e_1f_0f_1g_0, \\
 x_2 &= X_2Y_2Z_2 = b_0^3e_0e_1f_0f_1g_0, & x_7 &= X_2Y_2Z_3 = b_0^2c_0e_0e_1f_0f_1g_0, \\
 x_3 &= X_3Y_3Z_3 = c_0^3e_0e_1f_0f_1g_0, & x_8 &= X_1Y_3Z_3 = a_0a_1a_2c_0^2e_0e_1f_0f_1g_0, \\
 x_4 &= TZ_2 = a_0^2a_1b_0e_0f_0g_0, & x_9 &= X_2Y_3Z_3 = b_0c_0^2e_0e_1f_0f_1g_0, \\
 x_5 &= TZ_3 = a_0^2a_1ce_0f_0g_0, & x_{10} &= X_1Y_2Z_3 = a_0a_1a_2b_0c_0e_0e_1f_0f_1g_0, \\
 & & x_{11} &= \tilde{T} = a_1a_2^2e_1f_1.
 \end{aligned}$$

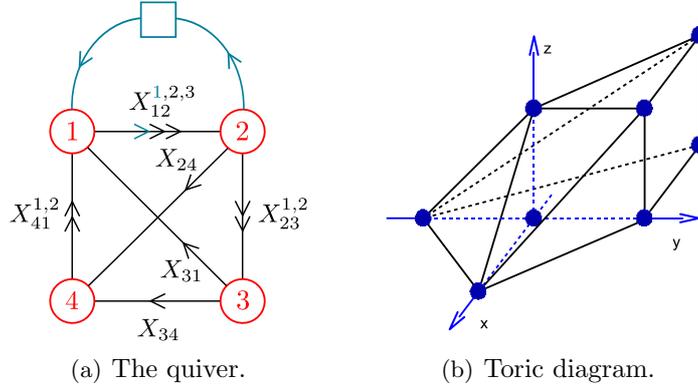


Figure 13.15: The dP_1 quiver with one flavor. Flavoring X_{12}^1 has the effect of adding a new point above two external points and one internal point in the toric diagram.

13.6.5 Flavoring the dP_1 quiver

The dP_1 quiver describes D-branes at the $C(Y^{2,1})$ CY_3 singularity. The quiver has 4 nodes and 10 bifundamental fields, as section 3.5.4. The brane tiling and quiver are shown in Fig. 3.4 and its perfect matchings are given in (3.52). Consider coupling a single flavor to the field X_{12}^1 , as in Figure 13.15(a). This time the field we flavor corresponds to two external points b_0 and c_0 , as well as an internal point e_0 , in the toric diagram of dP_1 . The Chern-Simons levels are $(\frac{1}{2}, -\frac{1}{2}, 0, 0)$, which corresponds to adding three points b_1 , c_1 and e_1 in the toric diagram, as shown in Figure 13.15(b).

	X_{12}^j	X_{23}^i	X_{41}^i	X_{31}	X_{24}	X_{34}	T	\tilde{T}
$U(1)_{\frac{1}{2}}$	1	0	-1	-1	0	0	1	0
$U(1)_{-\frac{1}{2}}$	-1	1	0	0	1	0	-1	0
$U(1)_0$	0	-1	0	1	0	1	0	0
$U(1)_0$	0	1	1	0	-1	-1	0	0

(13.102)

The quantum relation is $T\tilde{T} = X_{12}^1$. The F-term equations are solved by

$$\begin{aligned}
 X_{12}^1 &= b_0 b_1 c_0 e_0 e_1, & X_{41}^1 &= c_0 c_1 h_0, & X_{23}^1 &= c_0 c_1 f_0, & X_{34} &= b_0 b_1 g_0, \\
 X_{12}^2 &= a_0 b_0 b_1 e_0 e_1, & X_{41}^2 &= a_0 h_0, & X_{23}^2 &= a_0 f_0, & T &= b_0 c_0 e_0, \\
 X_{12}^3 &= d_0 e_0 e_1, & X_{31} &= d_0 g_0 h_0, & X_{24} &= d_0 f_0 g_0, & \tilde{T} &= b_1 c_1 e_1.
 \end{aligned}$$

The GLSM is

	a_0	b_0	c_0	d_0	e_0	f_0	g_0	h_0	b_1	c_1	e_1
$U(1)_{B_1}$	0	0	0	0	1	0	0	-1	0	0	0
$U(1)_{B_2}$	0	0	0	0	-1	1	0	0	0	0	0
$U(1)_{B_3}$	0	0	0	0	0	-1	1	0	0	0	0
$U(1)_{B_4}$	1	0	1	1	-1	0	-1	-1	0	0	0
$U(1)_{B_5}$	0	1	0	1	-1	-1	0	0	0	0	0
$U(1)_{B_6}$	0	1	-1	0	0	0	0	0	-1	1	0
$U(1)_{B_7}$	0	1	0	0	-1	0	0	0	-1	0	1

(13.103)

The three first lines correspond to the gauge charges under the first three gauge groups. Using the F-term relations together with $T\tilde{T} = X_{12}^1$, one can show that there are only 10 independent generators of the chiral ring,

$$\begin{aligned} x_1 &= X_{12}^3 X_{24} X_{41}^1, & x_4 &= T X_{24} X_{41}^2, & x_7 &= T X_{23}^1 X_{34} X_{41}^2, & x_{10} &= \tilde{T}. \\ x_2 &= X_{12}^3 X_{24} X_{41}^2, & x_5 &= X_{12}^3 X_{23}^2 X_{34} X_{41}^2, & x_8 &= T X_{23}^2 X_{34} X_{41}^2, \\ x_3 &= T X_{24} X_{41}^1, & x_6 &= T X_{23}^1 X_{34} X_{41}^1, & x_9 &= X_{12}^2 X_{23}^2 X_{34} X_{41}^2. \end{aligned}$$

and that they match the 10 affine coordinates of the toric geometry of Figure 13.15(b).

13.7 Conclusions

In this chapter, we studied the chiral ring of (conjectured) CFTs describing the IR fixed point of general 3d $\mathcal{N} = 2$ supersymmetric quiver gauge theories with chiral flavors, with or without CS terms, focusing on the toric case. These CFTs are conjectured to be holographically dual to M-theory on $AdS_4 \times H_7$ backgrounds.

We have generalized the stringy derivation of the quiver theories [258] to cases where the M-theory circle degenerates at complex codimension-two loci in the toric CY_4 cone, leading to flavor D6-branes wrapping toric divisors of the fibered CY_3 in type IIA string theory. The holomorphic embedding of flavor branes determines the superpotential couplings between the (anti)fundamental flavor superfields and bifundamental matter in the dual theory, whereas the RR F_2 fluxes contributed by D6-branes shift the CS levels.

Conversely, we have studied the addition of flavors coupled to bifundamental fields in toric 3d Abelian quiver theories. Flavoring is accompanied by shifts of some CS levels in order to balance the parity anomaly. We proved that the geometric branch of the moduli space (the one where flavor fields do not acquire a VEV) of the chirally flavored quiver theories is a toric conical CY_4 , and provided a recipe for deriving the toric diagram, exploiting an auxiliary quiver theory whose brane tiling has multi-bonds instead of flavors. The derivation of the moduli space relies on the existence of a non-trivial holomorphic OPE between BPS diagonal monopole operators, that we conjecture to appear at the quantum level since it is consistent with all global and gauge symmetries of the theory. Applying the reduction of [258] to the CY_4 branch, we can provide a stringy derivation of the proposed flavored gauge theories, closing the circle.

Firstly, it would be interesting to explore the Higgs branches of the flavored theories. In the presence of intersecting D6-branes, it will be crucial to understand whether new superpotential interactions arising from flavor branes intersections can appear and be marginal at the IR fixed point. The issue may be addressed using orbifold techniques and following the result of partial resolutions, as suggested in [303].

Secondly, it would be nice to understand whether the auxiliary multi-bond brane tilings are dual to the flavored quiver theories we studied. This issue requires the study of the full flavored theory and A-theory moduli spaces. Partial resolutions, interpreted as Higgsings (removal of one edge in a multi-bond) in the A-theory, correspond to explicit breaking of the flavor groups due to real mass terms in the flavored theory. Even though this is reminiscent of mirror symmetry, the P- and A-theory are not geometric dual in the sense of [290]: they correspond to the same M-theory reduction. The stringy derivation naturally leads to the flavored theory. Moreover, adding multi-bonds or flavorings are

local operations in the brane tiling/quiver, therefore any duality between the two theories must be a local operation as well. Finally, notice that giving a VEV to a bifundamental field in the flavored theory not only Higgses the gauge groups but also gives mass to all flavors coupled to it. In the brane tiling of the A-theory, all the edges between two vertices are removed.

It would also be interesting to extend our analysis to the full class of *ADE* singularities in M-theory, which goes beyond the toric case: *D*-type singularities descend to orientifolds in type IIA. One could also consider the addition of a Romans mass to the type IIA gravity duals with D6-branes, contributing a CS term to the diagonal gauge group [247, 309]: this would be particularly interesting for models with no CS terms, since it would provide a manifestly conformal action in the sense of ABJM [202]. Finally, one could apply the projection of [249] to identify $\mathcal{N} = 1$ dual pairs with flavors.

Appendix A

Type IIB SUGRA action, charges and equations of motion

We follow conventions according to which the action of Type IIB supergravity reads, in Einstein frame

$$S_{IIB} = \frac{1}{2\kappa^2} \left\{ \int d^{10}x \sqrt{-g} R - \frac{1}{2} \int \left[d\Phi \wedge *d\Phi + e^{2\Phi} F_1 \wedge *F_1 + \frac{1}{2} F_5 \wedge *F_5 + e^{-\Phi} H_3 \wedge *H_3 + e^\Phi F_3 \wedge *F_3 - C_4 \wedge H_3 \wedge F_3 \right] \right\}, \quad (\text{A.1})$$

where $\kappa^2 = \pi(2\pi)^6 \alpha'^4 g_s^2$ is the Newton coupling constant and the gauge invariant field strengths are defined as

$$F_1 = dC_0, \quad F_3 = dC_2 + C_0 H_3, \quad F_5 = dC_4 + C_2 \wedge H_3, \quad H_3 = dB_2. \quad (\text{A.2})$$

In our conventions the Einstein frame is defined from the string frame by rescaling the metric by the *fluctuating* part of the dilaton field. Moreover, our RR fields are normalized so as to appear in the action in a democratic way with respect to the NSNS fields, that is the Newton coupling constant κ enters as an overall factor in front of the Einstein frame supergravity action. As a consequence, the dilaton field Φ appearing in the action (A.1) is its fluctuating part, only, as its VEV has been absorbed into κ . With these conventions, the world-volume action for a Dp-brane is

$$S_{loc}^{Dp} = -\tau_p \int_{Dp} d^{p+1}\xi e^{\frac{p-3}{4}\Phi} \sqrt{-\det(\hat{g} + e^{-\Phi/2} \mathcal{F})} + \tau_p \int C \wedge e^{\mathcal{F}} \wedge \Omega_{9-p}, \quad (\text{A.3})$$

where $\mathcal{F} = \hat{B}_2 + 2\pi\alpha' F_2$ (the hat on the NSNS 2-form means that the form is pulled-back on the D-brane world-volume) and $\tau_p = 1/[(2\pi)^p \alpha'^{\frac{p+1}{2}} g_s]$. Finally, C is a polyform $C = \sum C_p$, with C_p being all possible RR potentials (electric and magnetic), and Ω_{9-p} is a form localized on the Dp-brane worldvolume (the Poincaré dual to the cycle) and closed.

With these conventions, the D3-brane and D5-brane charges (so called Maxwell charges [184], see below) are, respectively

$$Q_{D3} = -\frac{1}{(4\pi^2 \alpha')^2 g_s} \int F_5, \quad Q_{D5} = -\frac{1}{4\pi^2 \alpha' g_s} \int F_3. \quad (\text{A.4})$$

The equations of motion for the fields relevant to the solutions discussed in this thesis are

$$\begin{aligned} d e^\phi * F_3 &= H_3 \wedge F_5 - 2\kappa^2 \frac{\delta S_{loc}}{\delta C_2} \\ dF_5 &= -H_3 \wedge F_3 - 2\kappa^2 \frac{\delta S_{loc}}{\delta C_4} \\ d e^{-\phi} * H_3 &= -F_3 \wedge F_5 - 2\kappa^2 \frac{\delta S_{loc}}{\delta B_2}, \end{aligned} \quad (\text{A.5})$$

where we have imposed self-duality of F_5 on shell. By comparing the equations with the Bianchi identities of the dual field strengths we get the relation

$$F_7 = - e^\phi * F_3. \quad (\text{A.6})$$

The BIs modified by D-brane sources are

$$dF_3 = -2\kappa^2 \frac{\delta S_{loc}}{\delta C_6} \quad dH_3 = 0. \quad (\text{A.7})$$

Remark that in our conventions, the complex 3-form $G_3 = dC_2 + \tau H_3$ is simply

$$G_3 = F_3 + iH_3 \quad (\text{A.8})$$

when the axio-dilation is constant.

A.1 Maxwell and Page charges for D3-branes

The charges defined in (A.4) are gauge invariant quantities, but they are not quantized in general, because the Dirac quantization condition only applies to dC_4 and dC_2 , which are closed, while F_5 and F_3 are not closed in the presence of generic RR flux. Such charges were called Maxwell charges in [184].

Let us focus on the case of F_5 , with only F_3 flux and no F_1 flux, which is the case of interest in this thesis (we refer to [129] for a general discussion in the presence of D3-, D5- and D7-brane charges). The Page charge for D3-branes is defined as [184]

$$Q_{D3}^{\text{Page}} = -\frac{1}{(4\pi^2\alpha')^2 g_s} \int (F_5 + B_2 \wedge F_3). \quad (\text{A.9})$$

The form $F_5 + B_2 \wedge F_3$ is closed in the absence of localized sources, since $dF_5 = -H_3 \wedge F_3$. Locally, $F_5 + B_2 \wedge F_3$ can be written as $d(C_4 + B_2 \wedge C_2)$, so the Page charge will be non-zero only if there is some magnetic-monopole-like configuration. The Page charge is then a topological quantity, and it is quantized.

However, the Page charge is not gauge invariant under large gauge transformations of the B-field, $B_2 \rightarrow B_2 + \Lambda_2$ for Λ_2 some closed 2-form in a non-trivial cohomology class.

Appendix B

Algebraic geometry and toric geometry

In this Appendix we will review the relevant mathematical concepts we need in algebraic geometry, and particularly in toric geometry. The goal is to present toric geometry in its natural conceptual environment, which is algebraic geometry. All the relevant concepts are introduced along the way. This introduction to toric geometry is based on lectures given by the author at the Modave summer school in september 2008 [4].

Section B.1 is an introduction to the basic concepts of algebraic geometry. In section B.2 we will discuss the Calabi-Yau condition. In section B.3 we will delve into the core of the subject, defining toric varieties as particular holomorphic quotients, and showing how to introduce local coordinates in term of affine varieties (affine patches). Remark that we will mainly be interested in *local* properties, and so we will mostly concentrate on non-compact toric varieties. In particular we will consider Calabi-Yau toric varieties, which are always non-compact. In section B.4 we will introduce the notion of singularity in algebraic geometry, and we will show how we can deal with singular points in the toric case. In section B.5 we introduce a second way to define toric varieties, the Kähler quotient, also known as gauged linear sigma-model.

The presentation aims to be as self-contained as possible, but general knowledge of complex geometry is assumed at some point, especially in section B.2. Good introductions to complex geometry and Calabi-Yau manifolds can be found for instance in [310, 311, 312]. A very interesting mathematical reference is [313]. The standard reference on toric geometry is [314]. Posterior developments as explained in [315] are also important, as they actually simplify matters. See [316] for a nice account from a physicist perspective.

B.1 Algebraic geometry: the gist of it

We know that in geometry we always deal with some bunch of “points” that has more or less structure to it. A set of points together with a topology is called a topological space. Recall that a topology is what you define to be the open sets in your space, hence it provides a notion of locality. A topological space that locally looks like the euclidian space \mathbb{R}^n is called a manifold. If moreover the transition functions are differentiable (C^∞ for instance), it is called a differentiable manifold.

Smooth algebraic varieties can be seen as particular kind of manifolds which are simpler in some sense. Roughly speaking, they can be thought of as manifolds with rational transition functions¹. On the other hand, generic algebraic varieties are *not* manifolds, since they allow for various singularities; in that sense they are more general.

Remark that it is possible to define algebraic varieties *intrinsically*, in a way similar to what one does in differential geometry, but for doing so we would need to introduce the language of sheaves, and that would carry us too far afield. We will follow the more down to earth route, which defines algebraic varieties *extrinsically* as the algebraic set of zeros of some polynomials. Given a function $f : \mathbb{R}^n \rightarrow \mathbb{R}$, we can define a subset of \mathbb{R}^n ,

$$\mathbb{R}^n \supset \Sigma = \{f^{-1}(0)\} = \{x \in \mathbb{R}^n \mid f(x) = 0\},$$

which locally inherits its manifold structure from \mathbb{R}^n . However, this Σ is badly singular in general. If we restrict f to be a polynomial, things become much more tractable. It is one of the great advantages of the algebraic side of algebraic-geometry that singularities become easier to deal with.

Therefore we are now considering algebraic equations only. Hence it is very convenient to work with polynomials valued in \mathbb{C} , because \mathbb{C} is algebraically complete. From now on, unless otherwise stated, all variables are \mathbb{C} -valued, and by dimension we always mean complex dimension (half the real dimension).

Some algebraic definitions and propositions are reviewed in Insert 5, without demonstrations. For more details, see any algebraic geometry textbook, such as [317].

In this section we will first define affine varieties, which are the basic objects of algebraic geometry. Next we define the projective space $\mathbb{C}P^n$, which provides us with a particular example of the holomorphic quotient construction that we will encounter in detail when we define toric varieties in section B.3. For completeness we also define projective varieties, which are subvarieties of $\mathbb{C}P^n$.

B.1.1 Affine varieties

Varieties defined as algebraic subset of \mathbb{C}^n lead to the concept of *affine varieties*. Consider $\mathbb{C}^n = \{(x_1, \dots, x_n)\}$. Associated to it, we have the ring of polynomials in n variables, which is denoted by

$$\mathcal{R}_n \equiv \mathbb{C}[x_1, \dots, x_n]. \quad (\text{B.2})$$

It is obviously a ring (it is an additive group together with an associative product, distributive with respect to the addition); moreover it is a commutative ring. An *algebraic subset* $Z(\mathcal{T})$ of \mathbb{C}^n is defined as the zero locus of a set of polynomials $\mathcal{T} \subset \mathcal{R}_n$:

$$Z(\mathcal{T}) = \{(x_1, \dots, x_n) \in \mathbb{C}^n \mid p_i(x_1, \dots, x_n) = 0, \forall p_i \in \mathcal{T}\}. \quad (\text{B.3})$$

On the other hand, for any subset $Y \subset \mathbb{C}^n$, we denote the set of all polynomials that vanish on Y by $\mathcal{J}(Y)$. A natural question to ask is what is the relation between $\mathcal{J}(Z(\mathcal{T}))$ and \mathcal{T} . This is the content of the famous Hilbert's Nullstellensatz.

The whole idea of algebraic geometry is that you can define a space by the algebra of functions defined on it². Let us look at the polynomials which give rise to well defined

¹For toric varieties we will see that it is precisely that.

²Note that the ring of polynomials is naturally an *algebra* too.

Insert 5. A few notions of algebra

Ring. A ring is a set R equipped with two binary operations, $+$ and \cdot , such that

- (i) $(R, +)$ is a commutative group,
- (ii) \cdot is associative and there exist a neutral element (called unity). If moreover \cdot is commutative we talk of a commutative ring (it is the case in these lectures).
- (iii) \cdot is distributive over $+$.

Examples: The set of all integers \mathbb{Z} is a ring. Another example is the ring of polynomials in n variables, denoted $\mathbb{C}[x_1, \dots, x_n]$.

Ideal. An ideal I of a ring R is a subset $I \subset R$ such that

- (i) $i, j \in I \Rightarrow i - j \in I$,
- (ii) $i \in I, r \in R \Rightarrow ir \in I$.

Proposition: I an ideal of R implies that the quotient R/I is a ring too.

Notation: Given a set of elements $\{r_1, \dots, r_k\} \subset R$, we denote (r_1, \dots, r_k) the ideal generated by this set, which is the smallest ideal of R containing $\{r_1, \dots, r_k\}$.

Prime ideal. An ideal $P \subset R$ is a prime ideal if for any ideals $I, J \subset R$,

$$I \cdot J = \{ij \in R \mid i \in I, j \in J\} \subset P \Rightarrow I \subset P \text{ or } J \subset P.$$

Example: In the ring $\mathbb{C}[x, y]$, the ideal (xy) is not prime. It has a primary decomposition into (x) and (y) .

Radical of an ideal. Let I be an ideal of R . The radical of I , denoted $\text{rad}(I)$, is the intersection of all the prime ideals containing I . ($\text{rad}(I)$ is itself an ideal.)

Example: In $\mathbb{C}[x, y]$, $\text{rad}((x^n y^m)) = (xy)$.

An ideal $I \subset R$ is said to be radical if $\text{rad}(I) = I$.

Height of a prime ideal. The height $h(P)$ of a prime ideal $P \subset R$ is the largest integer h such that there exist a chain of strict inclusions of prime ideals P_i

$$P_0 \subset P_1 \subset \dots \subset P_h = P. \tag{B.1}$$

It gives a notion of the dimension of an ideal. Moreover it can be shown that the dimension of the affine variety corresponding to the quotient ring $\mathbb{C}[x_1, \dots, x_n]/P$ is $n - h(P)$.

Zero divisor and integral domain. An element $r \in R$, $r \neq 0$, is called a zero divisor if there exists $s \in R$, $s \neq 0$, such that $rs = 0$. A commutative ring without zero divisor is called an integral domain.

Proposition: Given R an integral domain, and $I \subset R$ an ideal, then R/I is an integral domain if and only if I is prime.

functions on the algebraic set (B.3). Two polynomials p_1 and p_2 will take the same value on $Z(\mathcal{T})$ if $p_1 - p_2 = t$, with some $t \in \mathcal{T}$, since t vanishes on $Z(\mathcal{T})$ by definition. We then only need to consider the equivalence classes of polynomials in \mathcal{R}_n that are linearly equivalent up to elements of \mathcal{T} . This is denoted by

$$\mathcal{A}(Z(\mathcal{T})) = \mathbb{C}[x_1, \dots, x_n]/\mathcal{T}. \quad (\text{B.4})$$

We want this quotient to define a proper ring of functions on $Z(\mathcal{T})$. This happens if \mathcal{T} is an *ideal* of the ring $\mathbb{C}[x_1, \dots, x_n]$. An ideal of a ring R is a subset $I \subset R$ such that I is a subgroup for the addition and is invariant under multiplication by any element in R . Given any set of polynomials, it is not difficult to extend it into a full-fledged ideal, as one can see in the examples below. One usually denote the ideal generated this way by (p_1, \dots, p_k) .

Examples:

- Take the ring $\mathbb{C}[x]$ of polynomials in x . The set $\{x\}$ is not an ideal (for instance it is not even a subgroup), but we can generate one simply by multiplying with every element of $\mathbb{C}[x]$. The ideal, denoted (x) , is simply the set of all polynomials without constant term. The quotient by the ideal simply gives the constants:

$$\mathbb{C}[x]/(x) = \mathbb{C}. \quad (\text{B.5})$$

- Consider the ideal (x^2) instead. The quotient $\mathbb{C}[x]/(x^2)$ is a ring generated by the two elements $\{1, x\}$ such that $x \cdot x = 0$. Such a x is called a zero divisor.
- On the ring $\mathbb{C}[x, y]$, consider the ideal (xy) . The quotient ring $\mathbb{C}[x, y]/(xy)$ has two zero divisors (x and y).

This last example corresponds to the surface $xy = 0$ in \mathbb{C}^2 . It consists of two branches which meet at the origin. In general, any algebraic set will consist of several “branches”,

$$Z(\mathcal{T}) = \Sigma_1 \cup \dots \cup \Sigma_m, \quad (\text{B.6})$$

and correspondingly the quotient ring (B.4) will have zero divisors. To avoid zero divisors, one must ask that the ideal be *prime* (see the Appendix 5 for the definition). In our example, (xy) is not prime, but it has a decomposition in two prime factors (x) and (y) . These two ideals correspond to the two “branches” $x = 0$ and $y = 0$.

Each component in the decomposition (B.6) is called *irreducible* if it cannot be decomposed further.

Definition: An *affine variety* is an irreducible algebraic subset of \mathbb{C}^n .

It is called “affine” simply because it is defined in \mathbb{C}^n , which is an affine space (i.e. a vector space where you can shift the origin anywhere). The very important thing to remember is that there is a one-to-one correspondence between affine varieties and prime ideals:

$$\Sigma = Z(P) \quad \xleftrightarrow{1-1} \quad \mathcal{A}(\Sigma) = \mathbb{C}[x_1, \dots, x_n]/P. \quad (\text{B.7})$$

Insert 6. Hilbert's Nullstellensatz

Consider an ideal I of $\mathbb{C}[x_1, \dots, x_n]$. Given the algebraic subset $Z(I) \subset \mathbb{C}^n$, as defined in section B.1.1, is the knowledge of $Z(I)$ enough to reconstruct the ideal I ? The answer is that you can only find $\text{rad}(I)$. This is the content of the famous Hilbert's Nullstellensatz. More precisely: **Theorem.** For any ideal I of $\mathbb{C}[x_1, \dots, x_n]$,

$$\mathcal{J}(Z(I)) = \text{rad}(I),$$

where $\mathcal{J}(Z(I))$ is the set of all polynomials vanishing on $Z(I)$.

We then have a one-to-one correspondence between algebraic sets and radical ideals. Remark that in dimension one, it implies that a polynomial with isolated zeros is fully determined by its roots; the Nullstellensatz is a generalisation of the fundamental theorem of algebra to higher dimensions.

This is a consequence of the Hilbert's Nullstellensatz (see Insert 6), which implies that if $\mathcal{T} = P$ is a prime ideal then the set of polynomials vanishing on $Z(P)$ is P itself, $\mathcal{J}(Z(P)) = P$.

Definition: The ring $\mathcal{A}(\Sigma)$ defined as in (B.4),

$$\mathcal{A}(\Sigma) = \mathbb{C}[x_1, \dots, x_n]/P, \tag{B.8}$$

is called the *coordinate ring*, or *structure ring*, of the affine variety Σ . This construction is familiar from supersymmetric theories: there the x_i are the gauge invariant operators, and P is generated by the F-terms. The structure ring in that case is called the *chiral ring*.

Example: the conifold. The ubiquitous conifold, \mathcal{C}_0 , which has been such a central tool in recent developments in string theory, is an affine variety defined by a single equation in \mathbb{C}^4 ,

$$x_1x_2 - x_3x_4 = 0. \tag{B.9}$$

Mathematicians call it a “threefold ordinary double point”, or *node*. Its coordinate ring is

$$\mathcal{A}(\mathcal{C}_0) = \mathbb{C}[x_1, x_2, x_3, x_4]/(x_1x_2 - x_3x_4). \tag{B.10}$$

B.1.2 Projective varieties

Affine varieties, being defined by polynomial equations in \mathbb{C}^n , are not compact. The projective space \mathbb{CP}^n is the simplest example of a compact algebraic variety (actually it is toric too). The standard way to define it is as the set of complex lines in \mathbb{C}^{n+1} ,

$$\mathbb{CP}^n = \frac{(\mathbb{C}^{n+1} \setminus \{0\})}{\mathbb{C}^*}. \tag{B.11}$$

The action of $\mathbb{C}^* = \mathbb{C} \setminus \{0\}$ is to multiply all coordinates in \mathbb{C}^{n+1} by $\lambda \in \mathbb{C}^*$, which defines the equivalence relation

$$[x_0, \dots, x_n] \sim [\lambda x_0, \dots, \lambda x_n]. \tag{B.12}$$

The origin $\{0\}$ was removed before taking the quotient so that \mathbb{C}^* may act freely. The resulting space is fully regular. The x_i are called homogeneous coordinates, and a point in \mathbb{CP}^n is represented by the equivalence class $[x_0, \dots, x_n]$. We can cover \mathbb{CP}^n with $n+1$ affine patches, one for each $x_i \neq 0$. The local coordinates on the i -patch are $z_k^{(i)} = x_k/x_i$, and the transition functions are the rational functions

$$z_k^{(i)}(z^{(j)}) = \frac{z_k^{(j)}}{z_i^{(j)}}. \quad (\text{B.13})$$

The Riemann sphere \mathbb{CP}^1 is the best known example. It has two patches, and the transition function on the equator is $z_N = 1/z_S$.

We can define subvarieties of \mathbb{CP}^n by taking the vanishing locus of a set of polynomials $p_i \in \mathcal{R}_{n+1}$. For the equations $p_i = 0$ to make sense, they should be constant on any equivalence class $[x_0, \dots, x_i]$, which means the p 's are *homogeneous* (i.e. they are sums of monomials of fixed degree):

$$p_i(x_0, \dots, x_n) \sim \lambda^d p_i(x_0, \dots, x_n). \quad (\text{B.14})$$

Definition: Given a homogeneous prime ideal P_h in \mathcal{R}_{n+1} , the associated *projective variety* is defined as

$$\Sigma(P_h) = \{[x_0, \dots, x_n] \mid p_i = 0 \quad \forall p_i \in P_h \subset \mathcal{R}_{n+1}\}. \quad (\text{B.15})$$

It is easy to check that if the p_i 's are homogenous of degree d , so is the ideal (p_i) .

The *homogeneous coordinate ring* is denoted by

$$\mathcal{S}(\Sigma) = \mathcal{R}_{n+1}/P_h. \quad (\text{B.16})$$

Projective plane curves. In \mathbb{CP}^2 , consider a hypersurface defined by a single polynomial p of degree d . If moreover

$$\frac{\partial p(x)}{\partial x_i} = 0 \quad \forall i, \quad \forall x \text{ s.t. } p(x) = 0, \quad (\text{B.17})$$

the curve is regular; it is a Riemann surface. Such Riemann surfaces are classified by their genus. There exists a theorem stating that

$$g = \frac{(d-1)(d-2)}{2}. \quad (\text{B.18})$$

In particular, for $d=3$, we have a *torus*, or elliptic curve ($g=1$). The general equation reads

$$\sum_{i+j+k=3} c_{ijk} x_0^i x_1^j x_2^k = 0. \quad (\text{B.19})$$

We have 10 parameters here. However 9 of them can be removed by a $Gl(3, \mathbb{C})$ transformation on the homogeneous coordinates. This leaves us with one parameter, which is basically the complex structure modulus of the torus. We will come back to the important issue of complex structure moduli later on in these lectures.

Remark that there are many more algebraic varieties than just affine and projective ones. In general, one can patch together affine varieties to obtain any algebraic variety, similarly to the idea of patching together open sets to form manifolds in differential geometry. We will see this explicitly in the simpler context of toric varieties.

B.1.3 Spectrum and scheme, in two words

Let us introduce the notion of spectrum of a ring. This is done only to set a useful notation that you might often encounter in the literature. The concepts of spectrum and scheme stem from taking seriously the idea that it is really the algebra of functions on it which defines a space. One starts with a purely algebraic object : given *any* ring A , one defines its *spectrum*

$$\text{Spec}(A) \equiv \{P \subset A\}, \quad (\text{B.20})$$

to be the set of all prime ideals of A (except A itself). This set can be given a natural *topology*, and it is then shown that, in the particular case of the coordinate ring of an affine variety,

$$\text{Spec}(\mathcal{A}(\Sigma)) \cong \Sigma, \quad (\text{B.21})$$

up to important subtleties that we shall willfully skip (in particular we are really talking about the maximal ideals here). The scheme structure is then obtained by introducing local coordinates by means of a so-called *structure sheaf* (for interesting introductions to sheaf concepts in physics, see for instance [48, 49]).

B.2 The Calabi-Yau condition

In this section we consider algebraic *manifolds*, i.e. *non-singular* algebraic varieties. It is fair to warn the reader that we will be applying results of this section in singular cases in the next section, although we will not discuss how this can be rigorously done.

An algebraic manifold is obviously a complex manifold: all the quantities we are dealing with are holomorphic by construction, and the variety inherits its complex structure from the embedding space \mathbb{C}^n or $\mathbb{C}P^n$. In this section, since we deal with manifolds, we can take a more direct, “intrinsic”, differential-geometric standpoint. This will simplify matter, since differential geometry is bound to be more familiar to the reader.

B.2.1 Holomorphic vector bundles and line bundles

Consider a complex manifold X of dimension m . On every open set we have local coordinate functions z_1, \dots, z_m , and we can define the exterior algebra of these coordinate functions, generated by *one-forms*

$$dz_1, \dots, dz_m. \quad (\text{B.22})$$

At any point p in the open set, $\{dz_i(p)\}$ form a basis for the *holomorphic cotangent space* T_p^*X at p . The multiplication operation on forms is the exterior product. All in all we have 2^m linearly independent elements

$$1, \quad dz_i, \quad dz_{i_1} \wedge dz_{i_2}, \quad \dots, \quad dz_{i_1} \wedge \dots \wedge dz_{i_m}, \quad (\text{B.23})$$

which form a graded algebra. At each degree, p -forms at any particular point span a vector space of dimension $\frac{n!}{p!(n-p)!}$.

Using holomorphic $Gl(m, \mathbb{C})$ -valued transition functions, we can patch all cotangent spaces together into the *holomorphic cotangent bundle* T^*X :

$$\mathbb{C}^m \quad \longrightarrow \quad T^*X \quad \xrightarrow{\pi} \quad X, \quad (\text{B.24})$$

which is itself a manifold of dimension $2m$. This is a particular case of an holomorphic vector bundle E ,

$$\mathbb{C}^k \longrightarrow E \xrightarrow{\pi} X, \quad (\text{B.25})$$

with \mathbb{C}^k the fiber, and π the natural projection, which is an holomorphic map. k is called the rank of the bundle.

Definition: An *holomorphic line bundle* (or *line bundle* for short) is an holomorphic vector bundle of rank one.

A very important line bundle is the *canonical bundle* K_X . It is defined as the m^{th} exterior product of T^*X ,

$$\mathbb{C} \longrightarrow K_X \equiv \Lambda^{(m,0)}T^*X \xrightarrow{\pi} X. \quad (\text{B.26})$$

Sections of the canonical bundle are holomorphic m -forms, that we can write (on each coordinate patch)

$$\Omega = f(z)dz_1 \wedge \cdots \wedge dz_m, \quad (\text{B.27})$$

for $f(z)$ some holomorphic function.

B.2.2 Calabi-Yau manifolds. Kähler and complex moduli

The Calabi-Yau condition is that the canonical bundle be *trivial*, i.e.

$$\Lambda^{(m,0)}T^*X \cong \mathbb{C} \times X. \quad (\text{B.28})$$

This implies the existence of a never vanishing *global section*. Standard arguments then imply that the function $f(z)$ in (B.27) must be a constant. This unique (up to rescaling by a constant) Ω is usually called the holomorphic m -form of the Calabi-Yau manifold X .

Kähler structure. A complex manifold can be endowed with a *Kähler structure*. There is no room here to explain in detail what this is, see [310, 311]. In two words though, a Kähler structure is a symplectic structure compatible with the complex structure: you need a closed and non-degenerate $(1, 1)$ -form ω . The nice thing is that complex structure plus Kähler structure implies there is a compatible Riemannian structure, i.e. a *hermitian metric*. This metric is defined by

$$g(\partial_z, \bar{\partial}_{\bar{z}}) = \omega(\partial_z, i\bar{\partial}_{\bar{z}}) \quad (\text{B.29})$$

for any two vectors $\partial_z, \bar{\partial}_{\bar{z}}$ in the tangent space (holomorphic and anti-holomorphic). The Kahler form ω is a representative of a Dolbeault cohomology class

$$[\omega] \in H^{1,1}(X). \quad (\text{B.30})$$

$[\omega]$ is called the Kähler class of ω .

Now, we can state Yau's theorem (Yau proved a conjecture made earlier by Calabi): **CY Theorem :** Given X a compact complex manifold *with trivial canonical bundle*, and given a Kähler form $\tilde{\omega}$ on X , there exist a unique Ricci flat metric in the Kähler class of $\tilde{\omega}$. That is, a unique Ricci-flat metric given by (B.29) for some $\omega \in [\tilde{\omega}]$.

On the other hand, it is “easy” to show that Ricci-flatness implies the triviality of the line bundle. For a non-compact manifold, the theorem does not hold (strictly speaking).

One can still find a Ricci-flat metric in general, but one must specify some boundary conditions at infinity.

Kähler moduli space. Given a Calabi-Yau manifold X , we see there are continuous families of Ricci-flat metrics, one for each cohomology class

$$[\omega] = \sum_{i=1}^{h^{1,1}} \lambda_i [\omega]^i. \tag{B.31}$$

These parameters λ are coordinates in a vector space $H^{1,1}(X)$ (here the $[\omega]^i$ are basis vectors). It is called the *Kähler moduli space* of X . Its dimension is denoted by $h^{1,1}$.

Complex moduli space. Given an algebraic variety, if one modifies the equation continuously, varying some parameters, the variety will be “deformed” accordingly. This is called a *variation of the complex structure*.

Consider the example of the torus of section B.1.2; we saw there are 10 parameters one can vary, but 9 of them do not change the complex structure, because they are just a linear reshuffling of the embedding space coordinates, so the complex moduli space of the torus is one dimensional.

Consider also the conifold, defined by $x_1x_2 - x_3x_4$. If I write, for instance,

$$x_1x_2 - bx_3x_4 + cx_4 = 0, \tag{B.32}$$

the constants b and c can obviously be absorbed in a redefinition of x_3 , with $x'_3 = bx_3 - c$. For an affine variety in \mathbb{C}^n , we can transform the variables by

$$Gl(n, \mathbb{C}) \times T_n, \tag{B.33}$$

with T_n the group of translations. In the case of \mathcal{C}_0 in \mathbb{C}^4 , we have 15 possible parameters for a generic polynomial of degree 2. However most of them can be removed by a $Gl(4, \mathbb{C}) \times T_4$ transformation. One can check that the only parameter which cannot be removed by such a transformation is the constant term,

$$x_1x_2 - x_3x_4 - a = 0. \tag{B.34}$$

Such a space is called the *deformed conifold*, and it is regular.

The space of all complex deformations of an algebraic variety X is called the *complex moduli space* of X . It is a rather complicated space. Its *linearisation* (the tangent space) is given by the cohomology group $H^{m-1,1}(X)$ (m the dimension of X) in the case of Calabi-Yau manifolds. In general, the question is much more complicated. In the particular case of the theory of complex deformations of toric Calabi-Yau singularities, there is some important results to be learned, as we will see.

B.2.3 Divisors and line bundles

Definition: A (Weyl) *divisor* D of a complex variety X is a linear combination (a formal sum with integer coefficients) of codimension one irreducible subvarieties,

$$D = \sum_i n_i V_i, \quad n_i \in \mathbb{Z}, \quad V_i \subset X. \tag{B.35}$$

If all $n_i \geq 0$, the divisor D is said to be *effective*.

To any line bundle L with a *regular section* s (which means that on any open set U_α , s_α is a polynomial in the local coordinates) we have an associated hypersurface Y in X defined by

$$Y = \{s(p) = 0, \quad p \in X\}. \quad (\text{B.36})$$

We can decompose Y into irreducible parts. On any affine patch, the polynomial s_α can be factorized in $\mathbb{C}[x_1, \dots, x_n]$. In fact, (s_α) is decomposed into prime ideals, and one keeps track of the multiplicity³ n_i of each distinct ideal P_i . The prime ideal P_i corresponds to the subvariety V_i in (B.35). More precisely, one should of course patch all the V_i^α together to construct $V_i \subset X$.

Going the other way around, an effective divisor $D = \sum_i n_i V_i$ defines a line bundle, denoted $\mathcal{O}_X(D)$. By definition its sections will vanish on each V_i with a zero of order n_i .

One can generalize this construction to any divisor, where now $n_i < 0$ corresponds to a pole of order n_i for the corresponding sections of $\mathcal{O}_X(D)$.

Example. On $X = \mathbb{C}P^n$, we can set $z_i = 0$ (z_i an homogeneous coordinate). It corresponds to the hyperplane H (any $H_i = \{z_i = 0\}$ is linearly equivalent to the others). A general divisor is then $D = nH$, $n \in \mathbb{Z}$. Its associated line bundle is usually denoted $\mathcal{O}(n)$. Note that $\mathcal{O}(-1)$, corresponding to $D = -H$, is really the dual of the hyperplane line bundle (i.e. its sections are in $\text{Hom}(\mathcal{O}(1), \mathbb{C})$). It is called the tautological line bundle of $\mathbb{C}P^n$.

B.3 Toric geometry 1: The algebraic story

We are now ready to discuss toric geometry. In this section we define a toric variety as a particular *holomorphic quotient* (or GIT quotient) of \mathbb{C}^n .

Definition: A *toric variety* X (of dimension m) is an algebraic variety containing the algebraic torus $\mathbb{T} = (\mathbb{C}^*)^m$ as a dense open subset, together with a natural action $\mathbb{T} \times X \rightarrow X$.

We can write X as

$$X_\Delta = \frac{\{\mathbb{C}^n \setminus Z_\Delta\}}{G}. \quad (\text{B.37})$$

Here, the group

$$G \cong (\mathbb{C}^*)^{n-m} \times \Gamma, \quad (\text{B.38})$$

is an algebraic torus times an abelian discrete group Γ . This construction generalizes the one for projective spaces. For it to make sense, we have to specify a set of points $Z_\Delta \subset \mathbb{C}^n$, and of course we must know how G acts on \mathbb{C}^n .

B.3.1 Cones and fan. Homogeneous coordinates

All this data defining a toric variety can be encoded in a simple auxiliary object called a *fan*. Hence the fan can be taken to define the toric variety. An equivalent definition will be in term of the gauged linear sigma-model of section B.5: the same data is present in both definitions, in particular the charge matrix to be defined momentarily. Moreover,

³There is a multiplicity because the ideal (s_α) is not radical in general.

this data is *combinatoric*, which means that it is given by discrete quantities. What makes toric geometry attractive is that complicated geometric problems can often be reduced to simpler combinatoric problems.

Let $N \cong \mathbb{Z}^m$ be a lattice, and $N_{\mathbb{R}} = N \otimes \mathbb{R}$ the vector space obtained by allowing real coefficients.

Definition: A strongly convex rational polyhedral cone $\sigma \subset N_{\mathbb{R}}$, or *cone* for short, is a set

$$\sigma = \left\{ \sum_i a_i v_i \mid a_i \geq 0 \right\}, \quad (\text{B.39})$$

generated by a finite set of vectors $\{v_i\}_{i=1}^n$ in N , and such that $\sigma \cap (-\sigma) = \{0\}$ (“strong convexity”).

Definition: A *fan* is a collection Δ of cones in $N_{\mathbb{R}}$ such that
 (i) each face of a cone is also a cone,
 (ii) the intersection of two cones is a face of each.

Let us call $\Delta(1)$ the set of one-dimensional cones in $N_{\mathbb{R}}$. The corresponding vectors in N are denoted (v_1, \dots, v_n) . To each v_i , one associates a *homogeneous coordinate* z_i . These are the coordinates on \mathbb{C}^n in the holomorphic quotient construction (B.37).

Remark that we always have $n \geq m$. The $(m \times n)$ matrix

$$(v_i^k) = (v_1^k, \dots, v_n^k) \quad (\text{B.40})$$

(with $k = 1, \dots, m$) induces a map

$$\phi : \mathbb{C}^n \rightarrow \mathbb{C}^m : (z_1, \dots, z_n) \mapsto \left(\prod_{i=1}^n z_i^{v_i^1}, \dots, \prod_{i=1}^n z_i^{v_i^m} \right). \quad (\text{B.41})$$

We define $\tilde{G} = (\mathbb{C}^*)^{n-m} \subset G$ to be the kernel of ϕ :

$$\tilde{G} = \text{Ker}(\phi). \quad (\text{B.42})$$

It is easily seen that \tilde{G} acts on \mathbb{C}^n as

$$\tilde{G} \supset (\mathbb{C}^*)_a : (z_1, \dots, z_n) \mapsto (\lambda^{Q_1^a} z_1, \dots, \lambda^{Q_n^a} z_n) \quad (\text{B.43})$$

for each a , where the charge vectors Q^a are in the kernel of the linear map (B.40), that is:

$$\sum_i (v_i^k) Q_i^a = 0. \quad (\text{B.44})$$

Hence, practically speaking, given a fan with n vectors in N we must find the $n - m$ linear relations among them. The coefficients are precisely the Q_i^a above.

The discrete group $\Gamma \subset G$ is defined as

$$\Gamma = N/N', \quad (\text{B.45})$$

where $N' \subset N$ is the sublattice generated over \mathbb{Z} by the vectors v_i . The quotient by this Γ gives rise to so-called *orbifold* singularities.

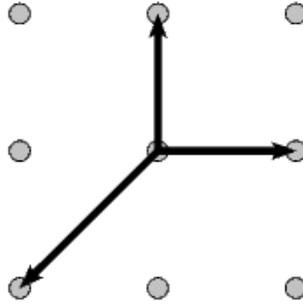


Figure B.1: The toric fan of \mathbb{CP}^2 . Notice that it contains 7 cones: three top-dimensional ones (of dimension 2), three 1-dimensional ones (generated by the vectors), and a single 0-dimensional one (the point in the center).

Last but not least piece of data in the construction, the zero set Z_Δ is found as follows: For any subset of $\Delta(1)$ (corresponding to vectors v_{i_1}, \dots, v_{i_l}) which do *not* generate a cone in Δ , associate an algebraic set V_{i_1, \dots, i_l} defined by $z_{i_1} = \dots = z_{i_l} = 0$. Then Z_Δ is the union of all these subsets of \mathbb{C}^n .

We'd better move on to examples.

- Consider the fan in figure B.1, generated by 3 vectors in $N \cong \mathbb{Z}^2$:

$$v_1 = (1, 0), \quad v_2 = (0, 1), \quad v_3 = (-1, -1). \quad (\text{B.46})$$

The one relation $v_1 + v_2 + v_3 = 0$ gives a single charge vector (see (B.44))

$$Q = (1, 1, 1), \quad (\text{B.47})$$

so we have the following group action of $G = \mathbb{C}^*$ on the homogeneous coordinates:

$$G : (z_1, z_2, z_3) \mapsto (\lambda z_1, \lambda z_2, \lambda z_3). \quad (\text{B.48})$$

Moreover, one sees that $Z_\Delta = \{(0, 0, 0)\}$. The construction obviously gives us \mathbb{CP}^2 as defined earlier.

- The (singular) conifold \mathcal{C}_0 is a 3-dimensional affine variety. It is not difficult to realize that a toric affine variety can only correspond to a single top-dimensional cone in the fan (see below). The fan for the conifold contains 10 cones (including the 0-dimensional one). It is generated by four lattice vectors in $N \cong \mathbb{Z}^3$:

$$v_1 = (0, 0, 1), \quad v_2 = (1, 0, 1), \quad v_3 = (1, 1, 1), \quad v_4 = (0, 1, 1). \quad (\text{B.49})$$

There is a single relation with charge vector $(1, -1, 1, -1)$, so G is one dimensional and acts as

$$G : (z_1, z_2, z_3, z_4) \mapsto (\lambda z_1, \lambda^{-1} z_2, \lambda z_3, \lambda^{-1} z_4). \quad (\text{B.50})$$

The zero set is

$$Z_\Delta = \{z_1 = z_3 = 0\} \cup \{z_2 = z_4 = 0\}. \quad (\text{B.51})$$

B.3.2 Coordinate rings and dual cones

The homogeneous coordinates are very useful for many purposes. However, it is natural to ask how we can describe a toric variety in *local* coordinates: as for manifolds, we would like to be able to cover our varieties with open sets equipped with local coordinates. The relevant notion of open sets is different here from the usual topology of differential geometry⁴, but this will not concern us here. We should say, however, that because we deal with singular spaces, the most “local” one can get is to affine varieties themselves. This is why it was so crucial to spend some time introducing them. Moreover, because the only non-singular affine variety is \mathbb{C}^m itself, for non-singular varieties the relevant open sets are simply \mathbb{C}^m and we recover the usual notions for complex manifolds, which we used in section B.2.

How do we find such local coordinates? The fan again provides the answer. To each top-dimensional cone we associate an affine variety (affine patch). The transition functions between these patches are also naturally encoded in the fan.

Given a single m -dimensional cone σ spanned by n vectors, we want to find the coordinate ring associated to it. Since a toric variety is defined as a quotient by G , local coordinates should be G -invariant polynomials⁵:

$$x = z_1^{n_1} \cdots z_n^{n_n}, \quad G : x \mapsto \lambda^{\sum_i Q_i^a n_i} x = x, \quad (\text{B.52})$$

which means that the positive integers n_i are such that $\sum_i Q_i^a n_i = 0$. Because of (B.44), this means that we can take

$$n_i = \langle w, v_i \rangle \quad (\text{B.53})$$

for any $w \in \text{Hom}(N, \mathbb{Z})$: The local coordinates are in one-to-one correspondence with elements in *the dual lattice* $M \cong \mathbb{Z}^m$,

$$M = \text{Hom}(N, \mathbb{Z}). \quad (\text{B.54})$$

In fact, the condition $n_i \geq 0$ defines the dual real cone $\sigma^\vee \in M_{\mathbb{R}}$,

$$\sigma^\vee = \{aw \in M_{\mathbb{R}} \mid a \in \mathbb{R}_{\geq 0}, \langle w, v_i \rangle \geq 0 \quad \forall v_i \in \sigma\}. \quad (\text{B.55})$$

Then, the coordinate ring we are looking for is simply

$$\mathcal{A}_\sigma = \mathbb{C}[\sigma^\vee \cap M]. \quad (\text{B.56})$$

Indeed $\sigma^\vee \cap M$ is a semi-group defining the monomials in the ring, and the addition in $\sigma^\vee \cap M$ becomes the multiplication in the ring. One can easily write this as the quotient of a polynomial ring by some ideals:

- First, find a *minimal* set of lattice vectors (w_1, \dots, w_r) generating $\sigma^\vee \cap M$; in general this is the most tricky part of the construction. We associate to this set the polynomial ring $\mathcal{R}_r = \mathbb{C}[x_1, \dots, x_r]$.

⁴The natural topology in algebraic geometry is called the Zariski topology. See any textbook such as [317].

⁵The reader should generalize the following considerations to the case when G has a non-trivial discrete subgroup Γ . See the examples below.

- Find all the relations between the w'_i 's, and associate to each relation an element of \mathcal{R}_r :

$$\sum_{i \in I} m_i w_i = \sum_{j \in J} m_j w_j, \quad m_i, m_j \in \mathbb{N} \quad \Rightarrow \quad p(x) = \prod_{i \in I} x_i^{m_i} - \prod_{i \in J} x_i^{m_i} \quad (\text{B.57})$$

where $I \cup J = \{1, \dots, r\}$ and $I \cap J = \emptyset$. This generates a prime ideal $P_\sigma = (p)$, and we then have

$$\mathcal{A}_\sigma = \mathbb{C}[\sigma^\vee \cap M] = \frac{\mathbb{C}[x_1, \dots, x_r]}{(p)}. \quad (\text{B.58})$$

It is not obvious but nonetheless true that this ideal is prime, and moreover it is such that the associated affine variety

$$U_\sigma = \text{Spec}\left(\frac{\mathbb{C}[x_1, \dots, x_r]}{P_\sigma}\right) \quad (\text{B.59})$$

has dimension m^6 . Here we used the notation of (B.21).

The affine varieties U_{σ_i} , $\sigma_i \in \Delta$, can be patched together to form a more general toric variety X_Δ . Suppose the cone τ is a *face* of both σ_i and σ_j . Then, we have that

$$\sigma_{i,j}^\vee \subset \tau^\vee \quad \Rightarrow \quad \mathbb{C}[\sigma_{i,j}^\vee \cap M] \subset \mathbb{C}[\tau^\vee \cap M] \quad \Rightarrow \quad U_\tau \subset U_{\sigma_i} \cap U_{\sigma_j}. \quad (\text{B.60})$$

In words, the affine set associated to the face is in the intersection of the affine sets of the two cones. Hence the relations between local coordinates in $x^{(i)}$ for U_{σ_i} and $x^{(j)}$ for U_{σ_j} can be read off from the relations between the generators of $\sigma_i^\vee \cap M$ and $\sigma_j^\vee \cap M$:

$$\sum_{l=1}^{r_i} q_l w_l^{(i)} = \sum_{l'=1}^{r_j} q_{l'} w_{l'}^{(j)}, \quad q_l, q_{l'} \in \mathbb{Z} \quad \Rightarrow \quad \prod_l (x_l^{(i)})^{q_l} = \prod_{l'} (x_{l'}^{(j)})^{q_{l'}} \quad (\text{B.61})$$

We see that the transition functions are always *rational functions*.

Examples:

- Consider again the fan for \mathbb{CP}^2 . There are three 2-dimensional cones, $\sigma_1, \sigma_2, \sigma_3$, and for each of them

$$U_{\sigma_i} = \text{Spec}(\mathbb{C}[\sigma_i^\vee \cap M]) \cong \mathbb{C}^2. \quad (\text{B.62})$$

Applying (B.61), we see that the transition functions between $U_{\sigma_1} = (x_1, x_2)$ and $U_{\sigma_2} = (y_1, y_2)$, for instance, are

$$x_1 = \frac{y_1}{y_2}, \quad x_2 = \frac{1}{y_2}. \quad (\text{B.63})$$

- Consider the simple fan in $N \cong \mathbb{Z}_2$ shown in the Fig.B.2(a). It has a single top dimensional cone, spanned by

$$v_1 = (1, 1), \quad \text{et} \quad v_2 = (1, -1). \quad (\text{B.64})$$

⁶This means that the *height* of the ideal P_σ is always $r - m$.

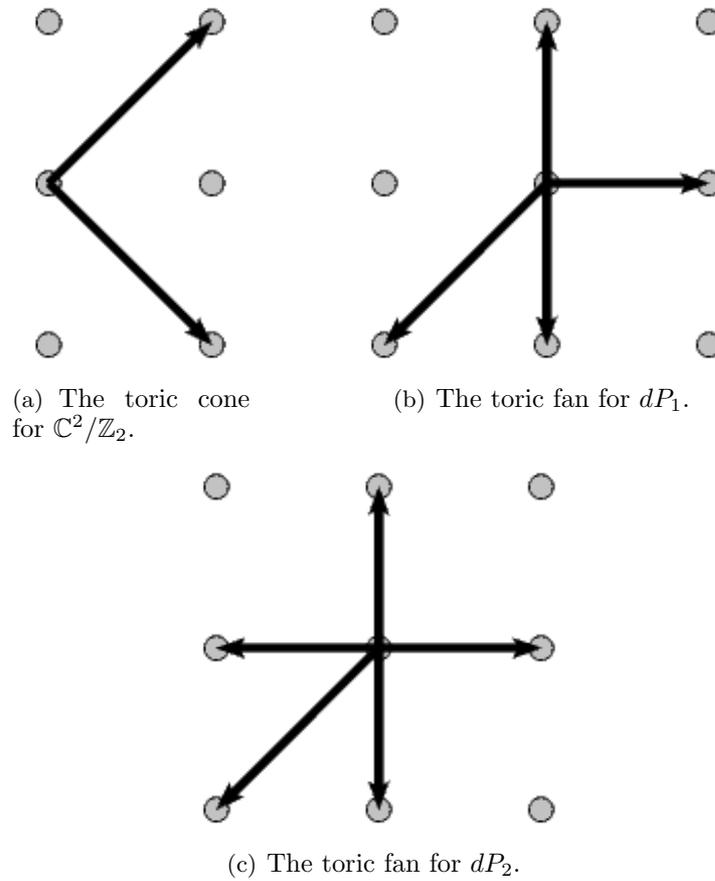


Figure B.2: Some examples of toric fans in dimension two.

Notice that there is no relation between the two vectors, so \tilde{G} is trivial, however we do have a discrete group Γ in the quotient (B.37), $\Gamma = \mathbb{Z}_2$, since v_1 and v_2 only generates half of the lattice N . In term of local coordinates, we have the dual cone σ^\vee generated by $w_1 = (1, -1)$ and $w_2 = (1, 1)$. In order to generate the dual cone $\sigma^\vee \cap M$ (over \mathbb{Z}), we need to introduce a third vector $w_3 = (1, 0)$. Then, assigning homogeneous coordinates x, y, z to these three vectors, we have the relation

$$w_1 + w_2 = 2w_3 \quad \Leftrightarrow \quad xy = z^2. \tag{B.65}$$

The later equation is the algebraic definition of $\mathbb{C}^2/\mathbb{Z}_2$, seen as an affine variety.

- In Figures B.2(b) and B.2(c), we have drawn the toric fans for the first and second del Pezzo surfaces (denoted dP_1 and dP_2). As you can see from the fan, they are smooth surfaces (each dual cone corresponds to a \mathbb{C}^2 patch). You should be able to work out the transition functions between the patches as in the case of $\mathbb{C}P^2$.

Now comes an important proposition:

Proposition: A toric variety X_Δ is *compact* if and only if its fan Δ spans the whole $N_{\mathbb{R}}$.

See Chapter 2 of [314] for a proof. One sees in the above examples that $\mathbb{C}P^2$ and $dP_{1,2}$ are compact spaces, while $\mathbb{C}^2/\mathbb{Z}_2$ or the conifold are of course not.

B.3.3 Calabi-Yau toric varieties

In this subsection, we show how the Calabi-Yau condition is translated into a simple condition on the combinatoric data for X_Δ .

We saw in section B.2 that the Calabi-Yau condition for X is the triviality of the canonical bundle K_X . Here we show how one can express K_X in term of a simple set of divisors called toric divisors.

Definition: A *toric divisor* is a divisor invariant under the action of G .

Using the homogeneous coordinates (z_i) , we can easily define subvarieties that are G -invariant. Indeed, the simple algebraic sets

$$\{(z_1, \dots, z_n) \mid z_i = 0 \forall i \in I \subset \{1, \dots, n\}\}. \quad (\text{B.66})$$

are obviously G -invariant. In particular, the subvarieties

$$D_i \equiv \{z_i = 0\} \cap X_\Delta \quad (\text{B.67})$$

are toric divisors⁷. They actually generate the full group of divisors of X_Δ .

Consider X_Δ *smooth* with canonical bundle K_X . One can show that

$$K_X = \mathcal{O}_X(-\sum_i^n D_i). \quad (\text{B.68})$$

The argument goes as follows. Because X_Δ is regular, each coordinate ring \mathcal{A}_σ is freely generated:

$$U_\sigma \cong \mathbb{C}^k \times (\mathbb{C}^*)^{m-k}, \quad \Leftrightarrow \quad \mathcal{A}_\sigma = \mathbb{C}[x_1, \dots, x_k, x_{k+1}, x_{k+1}^{-1}, \dots, x_m, x_m^{-1}]. \quad (\text{B.69})$$

Consider for simplicity the case $k = m$, which means σ is of dimension m (the generalization is straightforward). A section of the canonical bundle is

$$\Omega = \frac{1}{x_1 \cdots x_m} dx_1 \wedge \cdots \wedge dx_m. \quad (\text{B.70})$$

This section corresponds to a divisor. Equivalently, the dual section in K_X^{-1} corresponds to an *effective* divisor, described locally by

$$\{x_1 \cdots x_m = 0\} \cap U_\sigma. \quad (\text{B.71})$$

It is called the anti-canonical divisor.

⁷This is because the ideal (z_i) has height one, which implies D_i is codimension one in X_Δ too.

On the other hand, a section of the line bundle $\mathcal{O}(\sum_i D_i)$ corresponds to the divisor

$$\{\{z_1 z_2 \cdots z_n = 0\} \cap X\} \subset X. \tag{B.72}$$

We know that

$$x_1 \cdots x_m = z_1^{\langle w, v_1 \rangle} \cdots z_n^{\langle w, v_n \rangle} \quad \text{with} \quad w = \sum_j^m w_j. \tag{B.73}$$

Suppose the first m vectors amongst the v_i 's span the cone σ . Since $U_\sigma \cong \mathbb{C}^m$, we have $\langle w, v_i \rangle = 1$ for $i = 1, \dots, m$. Hence the anti-canonical divisor corresponds to $\sum_i D_i$ on U_σ . This implies that $K_X^{-1} = \mathcal{O}(\sum_i^n D_i)$, which is what we wanted to show.

The important relation (B.68) allows us to state the Calabi-Yau condition (triviality of the canonical bundle) in a very simple way. Note that any G -invariant function, as defined in (B.52), is of course a section of the trivial bundle. We then see that $\mathcal{O}_X(\sum_i D_i)$ is trivial if and only if

$$G : z_1 \cdots z_n \mapsto \lambda^{\sum_i Q_i^a} (z_1 \cdots z_n) = z_1 \cdots z_n \quad \Leftrightarrow \quad \sum_i Q_i^a = 0, \tag{B.74}$$

or equivalently if there exist a dual vector $w \in M$ such that $\langle w, v_i \rangle = 1$ for all v_i in the fan. We then have shown the following:

Proposition: The toric variety X_Δ is Calabi-Yau if and only if all the vectors v_i in Δ end on the same hyperplane in N , which happens if and only if $\sum_i Q_i^a = 0 \forall a$.

Remark that we chose the v_i for the conifold in (B.49) especially to make the CY property explicit.

It also follows from the proposition at the end of the last subsection that a toric CY cannot be compact.

B.3.4 Toric diagrams and p-q webs

For toric Calabi-Yau varieties, the combinatoric information encoded in the fan can be expressed in term of a reduced lattice of dimension $m - 1$.

This is particularly convenient in order to describe toric CY threefolds (toric CY of dimension 3), which are the objects of main relevance to physics. Instead of drawing a 3-dimensional fan, we can simply project it on the special plane defined by $\langle w, v_i \rangle = 1$.

In the Figures are some examples of toric diagram. The one for the conifold is given in Fig.B.3(a), while Fig.B.3(b) corresponds to the complex cone over the dP_1 surface, which happens to be a Calabi-Yau singularity.

In Fig.B.3(c) is a singularity called the Suspended Pinch Point (SPP). One can easily show that, in local coordinates, the SPP is an affine variety in \mathbb{C}^4 defined by the ideal $(xy - z^2t)$ in $\mathbb{C}[x, y, z, t]$.

One can also draw the dual of the toric diagram, which is called the pq-web (simply, for each line in the toric diagram, you draw an orthogonal line in the pq-web). Such

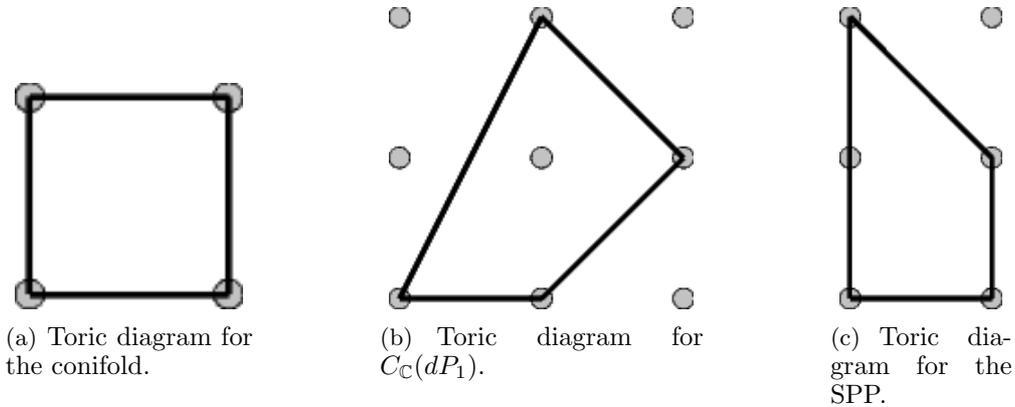


Figure B.3: Some examples of toric diagrams for local CY threefolds.

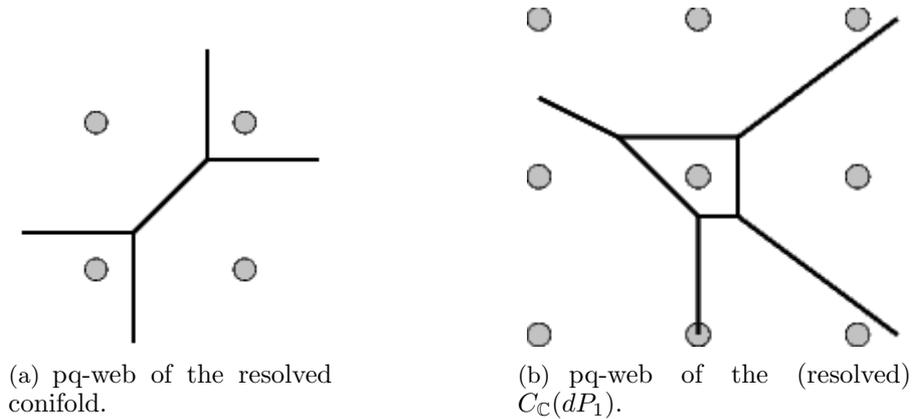


Figure B.4: pq-webs

webs have a nice physical interpretation as webs intersecting fivebranes [318, 319]: M-theory on a toric 3-fold is T-dual to type IIB with a web of D5-, NS5- and more generally (p, q) -5-branes reproducing the pq-web of the toric geometry.

Examples of pq-webs for the conifold and for the first del Pezzo cone are given in Figs B.4(a) and B.4(b). You first have to triangulate the diagram (see next Section), and then take the dual diagram.

B.4 Dealing with toric singularities

What is a singularity in algebraic geometry? Let X be an algebraic variety of dimension m . A point in X will be deemed singular if the tangent space at that point has dimension larger than $\dim X = m$.

Without loss of generality, we can define the tangent space $T_x X$ at the point x for affine varieties only:

Tangent space of X . If $X = Z(\mathcal{J})$, with \mathcal{J} a prime ideal of $\mathcal{R}_n = \mathbb{C}[x_1, \dots, x_n]$,

we can define the following ideal of \mathcal{R}_n , generated by degree one polynomials, for each point x :

$$\mathcal{J}_x = \left\{ \sum_i^n \frac{\partial p}{\partial x_i}(x) (x_i - x_i(x)) \in \mathcal{R}_n \mid p \in \mathcal{J} \right\}. \quad (\text{B.75})$$

This ideal generates a *linear* affine variety that we define to be the tangent space at $x \in X$,

$$T_x X \equiv Z(\mathcal{J}_x). \quad (\text{B.76})$$

This obviously generalizes the usual definition of a tangent space. Now, a point x in X is called non-singular if its tangent space has the same dimension as the variety X . Of course, X is said to be non-singular if it has no singular points. For singular points the dimension of $T_x X$ is larger than m . Practically speaking, when given an affine variety in terms of its defining polynomials $p(x)$ (i.e. in local coordinates), one finds the *singular locus* as the set of points x such that

$$p(x) = 0, \quad dp(x) = 0. \quad (\text{B.77})$$

For toric varieties, there is a straightforward **theorem** [314] which states that the affine variety X_σ associated to the cone σ is non-singular if and only if σ is generated by an integral basis of the lattice N .

Polytope and unit simplex. In m dimensions, we will call polytope the convex hull⁸ of k distinct points in $N \cong \mathbb{Z}^m$. Given a m -dimensional cone σ in a toric fan, the basic polytope is the polytope delimited by the origin and the vectors $v_i \in \sigma$. For instance, for the conifold we have $(0, 0, 0)$, $(0, 0, 1)$, $(1, 0, 1)$, $(1, 1, 1)$, $(0, 1, 1)$. In general we have n vectors v_i , so we have $k = n + 1$ points defining the basic polytope.

On the other hand, a simplex is the m -dimensional generalization of a triangle or tetrahedron: the convex hull of $m + 1$ points. We define the simplicial volume of a polytope as the number of simplexes it contains.

Indeed, any polytope can be subdivided into simplexes: this is called a simplicial decomposition. We can now reformulate the above theorem as:

Proposition: The affine variety X_σ associated to the cone σ is non-singular if and only if the basic polytope associated to $\sigma \cap N$ has unit simplicial volume.

B.4.1 Resolution of toric singularities and simplicial decomposition

We can then “desingularize” any toric variety by subdividing its associated fan further until every cone is based on a unit simplex.

For a toric CY threefold, simplicial decomposition is equivalent to a *triangulation* of its toric diagram. For instance, in Fig.B.3(a) one can see that the basic simplex of \mathcal{C}_0 has simplicial volume 2, while the cone over dP_1 in Fig.B.3(b) has simplicial volume 4, so they are singular. The two possible triangulations of the conifold diagram are shown in Figure B.5.

⁸As one can find in Wikipedia.org, for instance, a convex hull of k points is the minimal convex set containing these points. This is just the higher dimensional generalization of 2-dimensional polygons and 3-dimensional polyhedrons.

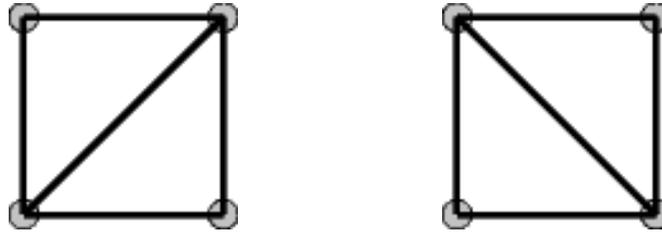


Figure B.5: The two possible resolutions of the conifold.

Example. Take the conifold again. Its basic simplex has volume 2. We can split it into a fan containing two cones, each of unit volume. This is called the *resolved* conifold. Now we have two 3-dimensional cones in the fan, σ_1 and σ_2 . The dual cones correspond to two copies of \mathbb{C}^3 :

$$\sigma_1^\vee : \{(1, 0, 0), (0, -1, 1), (-1, 1, 0)\} \rightarrow \mathbb{C}^3 = \{x_1, y_1, z_1\}, \quad (\text{B.78})$$

$$\sigma_2^\vee : \{(0, 1, 0), (-1, 0, 1), (1, -1, 0)\} \rightarrow \mathbb{C}^3 = \{x_2, y_2, z_2\}, \quad (\text{B.79})$$

We see that the relations between the vectors in the dual lattice give us the following transition functions between the two patches:

$$z_1 = \frac{1}{z_2}, \quad \frac{x_1}{y_1} = \frac{x_2}{y_2}, \quad x_1 z_1 = x_2, \quad \text{etc.} \quad (\text{B.80})$$

The second relation is actually the defining equation of the conifold singularity. Before the triangulation of the toric diagram, that was all what one would get. The triangulation procedure introduced new coordinates, z_1 and z_2 with $z_1 = 1/z_2$, which give the complex structure of a $\mathbb{C}P^1$. Away from the point $x_1 = y_1 = x_2 = y_2 = 0$, these coordinates are redundant, but at the former conifold singularity, we now have a full $\mathbb{C}P^1$.

Remark that in the homogeneous coordinate description, you still have the same four coordinates z_1, \dots, z_4 . What changes is that the zero set Z_Δ is now different when the fan is subdivided: $Z_\Delta = \{z_1 = z_2 = z_3 = z_4 = 0\}$, so that the singularity is effectively removed.

Such a procedure, which replaces an isolated singularity by a holomorphic cycle, is called a *resolution* of the singularity.

More precisely [313], a resolution (\tilde{X}, π) of the variety X is a non-singular variety \tilde{X} together with a surjective map $\pi : \tilde{X} \rightarrow X$ which is biholomorphic on open sets wherever π is also injective. In other words, π is a biholomorphism “away” from the singular points, while the singularities are replaced by some smooth spaces, for instance by means of a small resolution, or by blowing them up.

Blow up. A blow up is a procedure which replaces the singular locus of X by $\mathbb{C}P^{m-1}$. (Beware that in the physics literature the terms “blow up” is sometimes used to designate any kind of resolution.) Hence a blow up introduces new divisors, called *exceptional divisors* (these are defined as the prime divisors $E \subset \tilde{X}$ such that $\pi(E)$ has codimension 2 or more in X).

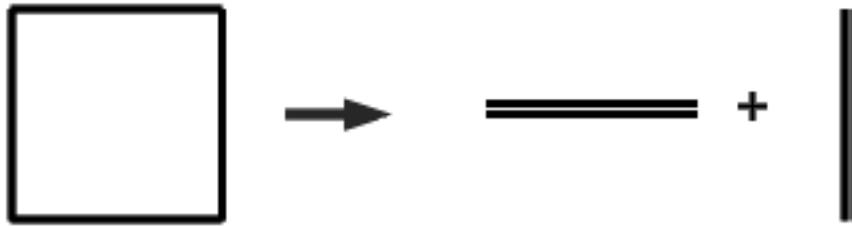


Figure B.6: Splitting of the conifold diagram into Minkowski summands.

Small resolution. On the other hand, a small resolution is a resolution such that \tilde{X} has no exceptional divisors. In particular, the resolution of the conifold is a small resolution.

The resolutions we usually deal with in string theory are actually *crepant resolutions*. The resolution (\tilde{X}, π) of X is said to be crepant when⁹

$$\pi^*(\mathcal{O}(K_X)) = \mathcal{O}(K_{\tilde{X}}). \quad (\text{B.81})$$

In particular, the Calabi-Yau condition is preserved by a crepant resolution.

For a toric CY threefold, a blow up consists in introducing a \mathbb{CP}^2 at the singularity, while a small resolution introduces a \mathbb{CP}^1 instead. We can convince ourselves that the blow up corresponds to adding an internal point in the toric diagram (see the pq-web Fig.B.4(b) for instance), while the small resolution corresponds to a triangulation which does not introduce new points (like for the conifold).

B.4.2 Deformation of toric singularities: the versal space

Another way to get rid of a singularity is to *deform* it: this modifies the complex structure. For instance, we saw that the conifold equation $x_1x_2 - x_3x_4 = 0$ admits a deformation to

$$x_1x_2 - x_3x_4 = e, \quad e \neq 0. \quad (\text{B.82})$$

This new space, called the *deformed conifold*, is non-singular. The complex structure has obviously changed, but it turns out that it is still a Calabi-Yau variety. However, it is *not* toric anymore, because the deformation has broken one of the \mathbb{C}^* action in the \mathbb{T}^3 acting on the singular conifold (as one can see from the equation). In this particular case, the Calabi-Yau metric is explicitly known [104].

⁹The canonical bundle for a singular variety is itself tricky to define. A straightforward generalisation of the idea of holomorphic line bundle is what is called an *invertible sheaf* (which is a sheaf of modules locally isomorphic to the structure sheaf \mathcal{O}_X). Then one works with the sheaf $\mathcal{O}(K_X)$, the sheaf of regular sections of K_X , which is assumed to be invertible. You can pull-back this sheaf using π , but in general $\pi^*(\mathcal{O}(K_X))$ is not equal to $\mathcal{O}(K_{\tilde{X}})$. It turns out that the discrepancy can come from exceptional divisors only, and if there is no discrepancy the resolution is said to be *crepant* (so we see that small resolutions are crepant by definition).

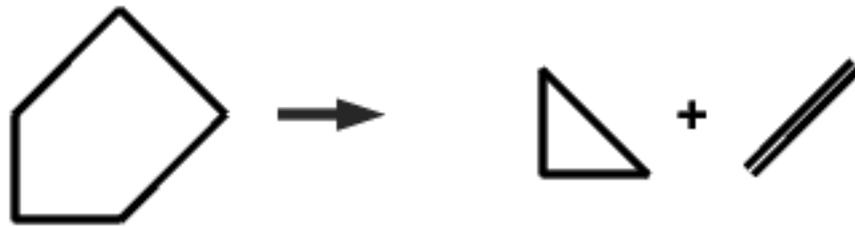


Figure B.7: Splitting of the $C_{\mathbb{C}}(dP_2)$ diagram into Minkowsky summands. The triangle corresponds to a remaining singularity which admits no complex deformation.

It turns out that for any deformation of the defining polynomials which is of degree lower or equal to these same polynomials, the resulting deformed variety is still Calabi-Yau. Of course in general we don't know the Ricci-flat metric on it, but the CY theorem guarantees its existence.

Since we are dealing here with non-compact CY varieties, we also should not modify the boundary conditions at infinity. This means that we focus on *normalizable deformations*, which are those which do not change the defining polynomials at infinity.

For a single intersection variety like the one above, it is easy to work out by hand all the possible deformations. For more complicated varieties, however, it becomes tedious. Also, for non-complete intersection varieties¹⁰, it may happen that there is no consistent modification of the defining equations.

For toric varieties, there exists a very useful algorithm, due to Altmann [250], which gives the number of normalizable deformations of the singularity for any *isolated toric CY singularity* (and also their explicit form, see [250], or [320, 321] for some physics papers which use it in detail).¹¹ We will focus on CY threefolds, that we can draw as toric diagrams on a sheet of paper, and where all the interesting phenomena occur.

The various complex deformations of an isolated CY singularity correspond to the possible “Minkowski decompositions” of the toric diagram. This means that we deform the toric diagram into closed sub-diagrams (called Minkowski summands). See Figures B.6 and B.7. What we are really looking for are the “breathing modes” of the toric diagram. We do it in the following way:

- Consider an affine toric Calabi-Yau threefold, with its toric diagram D containing n points and n edges. First, assign to each edge of $D \subset \mathbb{Z}^2$ a lattice vector

$$d^i = p_h - p_t, \quad (\text{B.83})$$

given by the difference between the head and the tail of the corresponding edge of D , when going in the counterclockwise direction.

¹⁰One talks of non-complete intersection when the dimension of the embedding space \mathbb{C}^n minus the number of defining polynomials is smaller than the dimension of X . It is the general case. (In algebraic language, it means that the height of the defining ideal is smaller than the number of generating polynomials.)

¹¹Notice that the SPP in the example above is not an isolated singularity: it has a full \mathbb{C} worth of singularities, a singularity line.

- Define the vector space

$$V(D) = \{(t_1, \dots, t_n) \mid \sum_i^n t_i d^i = 0\}. \tag{B.84}$$

This vector space, including the trivial (t, t, t, t) component, is obviously of dimension $n - 2$. Ignoring the trivial rescaling, this is the linearized space of deformations of X , of dimension $n - 3$. The deformation could be obstructed at second order, however.

- The *versal*¹² space of complex deformations of X is defined by the following ideal of $\mathbb{C}[t_1, \dots, t_n]$:

$$\mathcal{J} = \left(p_k \equiv \sum_i^n (t_i)^k d^i \mid k \in \mathbb{Z}_{>0} \right). \tag{B.85}$$

Actually this ideal is generated by the finite set of polynomials p_1, \dots, p_K , where K is the maximum of the lattice width of the minimal pair of strips containing D [250].

This whole procedure amounts to find the Minkowski summands of the diagram D . In term of the dual pq-web, it corresponds to splitting the pq-web into sub-webs in equilibrium (i.e. the external legs must still sum to zero). For instance, you can see that the diagram in Fig.B.3(b) admits no Minkowski decomposition. This means that the dP_1 singularity cannot be deformed: although its linear space of deformations (B.84) has dimension one, there is an obstruction at second order.

Example. Consider the conifold, whose diagram is just a square. We have the following edge vectors:

$$d^1 = (1, 0), \quad d^2 = (0, 1), \quad d^3 = (-1, 0), \quad d^4 = (0, -1). \tag{B.86}$$

The linear space of deformation is simply generated by (t_1, t_2, t_1, t_2) . There is no higher order obstruction so the versal space boils down to the linear space

$$\text{Spec}(\mathbb{C}[t]) = \mathbb{C}, \tag{B.87}$$

corresponding to the freedom of adding a constant term e in (B.82).

B.5 Toric geometry 2: Gauged linear sigma-model

There is an alternative, complementary approach to toric varieties, which does not directly rely on algebraic geometry, but rather deals with the *symplectic* or (more precisely) *Kähler* properties of our varieties.

The idea is to split the quotient by $(\mathbb{C}^*)^{n-m}$ in (B.37) into two steps. Since

$$\mathbb{C}^* \cong U(1) \times \mathbb{R}_{>0}, \tag{B.88}$$

¹²“Versal space” means that all the possible deformations are there, but that the same deformation might appear several times (if it appear only once we would have a “universal” space of deformation, that is what happens for compact Calabi-Yau varieties, whose complex moduli space has a simpler topology. See [313].).

we will first fix some “point” $t \in \mathbb{R}_{>0}$ (and $t \rightarrow 0$ will correspond to a singular limit for the toric variety), and secondly we will divide by the $U(1)$ action (which is the gauge group, in the physics parlance). Such a procedure is well defined because we have a well defined Kähler form on the parent space \mathbb{C}^n . It is called a *Kähler quotient* of \mathbb{C}^n .

B.5.1 Kähler quotient and moment maps

Before exploring the “physics”, let us briefly explain what is a Kähler quotient mathematically. We will focus on the quotient of \mathbb{C}^n by an abelian group. The group $G = U(1)^r$ ($r = n - m$) acts on \mathbb{C}^n as (compare to (B.43))

$$U(1)^r : \mathbb{C}^n \rightarrow \mathbb{C}^n \quad : \quad (z_i) \mapsto (e^{i \sum_a \xi_a Q_i^a} z_i), \quad (\text{B.89})$$

where $\xi = (\xi_1, \dots, \xi_r)$ are element of the Lie algebra $\mathfrak{g} = \mathfrak{u}(1)^r \cong \mathbb{R}^r$ of the gauge group. The action of \mathfrak{g} is then

$$\xi \cdot F(z) = (i \sum_i \sum_a \xi_a Q_i^a z_i \frac{\partial}{\partial z_i} + c.c.) F(z). \quad (\text{B.90})$$

The complex conjugate is necessary to make it a real action.

Definition: Given a Kähler manifold with Kähler 2-form ω , a *moment map* μ for the group action of G on \mathbb{C}^n is an element of the dual Lie algebra, \mathfrak{g}^* , such that

$$d\langle \mu, \xi \rangle \equiv d(\mu^a \xi_a) = i_\xi \omega, \quad (\text{B.91})$$

where here i_ξ denote the interior product with the vector ξ appearing on the r.h.s. of (B.90).

You can easily show that, in our case, the Kähler manifold being simply \mathbb{C}^n with the canonical Kähler form

$$\omega = -i \sum_i dz_i \wedge d\bar{z}_i, \quad (\text{B.92})$$

the moment maps are

$$\mu^a = \sum_i Q_i^a |z_i|^2 - t^a, \quad (\text{B.93})$$

where the t^a are integration constants. Then, the Kähler quotient proceeds as follows:

- Set $\mu^a = 0$, i.e.

$$\sum_i Q_i^a |z_i|^2 = t^a \quad \forall a, \quad (\text{B.94})$$

This is called a restriction to a level set at level t . The parameters t^a could be set to zero, as we will see.

- The second step is to quotient by the compact gauge group $U(1)^r$, whose action was defined in (B.89).

The first step defines a lower-dimensional *real* algebraic submanifold in the space $\mathbb{R}_{\geq 0}^n$ spanned by the $|z_i|^2$'s. Then the second step tells us which subgroup $U(1)^m$ of the $U(1)^n \subset \mathbb{C}^n$ torus must be fibered at each point to produce the final m -dimensional variety.

B.5.2 The GLSM story

If you are familiar with supersymmetric theories (or you have read the introduction carefully), the above must have looked like known territory. The restriction to a level set is simply the imposition of the D-term constraints in some abelian gauge theory, while the second steps corresponds to fixing the gauge freedom (restricting to gauge orbits).

Hence, we can see toric varieties as the *moduli space of vacua* of a “gauged linear sigma-model” (GLSM). We have n chiral fields whose scalar component are the z_i ’s, and they are charged under the gauge group as

$$\begin{array}{c|ccc|c}
 & z_1 & \cdots & z_n & \text{FI} \\
 \hline
 U(1)_1 & Q_1^1 & \cdots & Q_n^1 & t^1 \\
 \vdots & & \ddots & & \vdots \\
 U(1)_r & Q_1^r & \cdots & Q_n^r & t^r
 \end{array}$$

Because the gauge group is $U(1)^r$, there are possible Fayet-Iliopoulos (FI) parameters t^a in the D-terms conditions (B.94). This was first realised by Witten in [322], where he used a 2-dimensional GLSM as an auxiliary device to find 2-dimensional CFTs. Here the auxiliary theory is four dimensional (the main difference with respect to [322] being that the FI parameters are real), and its infrared corresponds to a Calabi-Yau “as probed by D3-branes”¹³.

Examples. Consider the GLSM with a single $U(1)$ and four fields with the following charges:

$$\begin{array}{c|cccc}
 & z_1 & z_2 & z_3 & z_4 \\
 \hline
 U(1) & p & p & -p + q & -p - q
 \end{array}$$

and no FI term. The resulting toric Calabi-Yau singularity is a real cone over a real 5-dimensional Sasaki-Einstein¹⁴ space called $Y^{p,q}$ [323]. This family of toric CY singularities has received a lot a attention in the physics litterature during the last years, because the corresponding Ricci-flat metrics are known explicitly [105], which is a rather spectacular feat and allowed for some new checks of the AdS/CFT correspondence.

B.5.3 Toric varieties as torus fibration of polytopes

An affine toric variety X can be visualized quite simply as a torus fibration over a *polytope* Δ :

$$U(1)^m \rightarrow X \xrightarrow{\mu_H} \Delta \tag{B.95}$$

Indeed, the toric variety X has an isometry group

$$H = U(1)^m = \frac{U(1)^n}{U(1)^r}, \tag{B.96}$$

and there is a moment map μ_H on X associated to this H . This moment map is precisely the map which projects out the $U(1)^m$ fibers in the exact short sequence (B.95) [323].

¹³You should not take this analogy too seriously: the GLSM is an auxiliary construction, like the fan, there is *a priori* no real physics there.

¹⁴Sasaki basically means that the real 6-dimensional cone is Kähler, while the Einstein condition on the 5-dimensional base metric implies the Ricci-flatness of the cone. Hence a SE manifold of real dimension $2n - 1$ is the real base of a CY cone of complex dimension n .

For cones ($t^a = 0$), the polytope is precisely the toric cone σ for X . Given the charge vectors Q^a , one can construct several σ 's such that

$$\sigma = \left\{ \text{Span}((v_i) \in N \cong \mathbb{Z}^m) \mid \sum_i Q_i^a v_i \forall a \right\}. \quad (\text{B.97})$$

All these σ are related to each other by $Sl(m, \mathbb{Z})$ transformations. Consider, for instance, taking an orthogonal basis of \mathbb{Z}^m for the first m lattice vectors v_i (corresponding to the first m homogeneous coordinates). The remaining vectors of σ follow from (B.44). This choice of lattice basis vectors corresponds to a choice of subgroup for

$$U(1)^m \subset U(1)^n. \quad (\text{B.98})$$

This is simply because we made a choice about which of the homogeneous coordinates (z_i) are the “dependent” ones. Here we chose the variables (z_{m+1}, \dots, z_n) to be functions of the (z_1, \dots, z_m). More precisely, the modulus are fixed by the D-terms (B.94), while the phases of (z_{m+1}, \dots, z_n) are redundant $U(1)^r$ degrees of freedom that we can gauge fix.

Then we see explicitly that the affine toric variety is realized as a $U(1)^m$ fibration of $\sigma \subset N_{\mathbb{R}}$. In the bulk, the real torus $T^m \cong U(1)^m$ is non-degenerate, while on the intersection of the hyperplane ($v_j = 0, \forall j \in I \subset (1, \dots, m)$) with the cone σ , there is a degeneration of the ($\#I$)-torus ($T_{j_1}^1 \times \dots \times T_{(\#I)}^1$). At the tip of σ the whole torus shrinks to zero, and we have a singularity.

Example. Take the conifold again. We have $Q = (1, -1, 1, -1)$. If we take a basis of \mathbb{Z}^3 as $(0, 0, 1)$, $(0, 1, 1)$ and $(1, 1, 1)$, we must have that the fourth vector in σ be $(1, 0, 1)$. On the other hand, if we take the orthogonal basis for the lattice, σ is generated by

$$(1, 0, 0), \quad (0, 1, 0), \quad (0, 0, 1), \quad (1, -1, 1). \quad (\text{B.99})$$

We see that the first σ is obtained from the second by the $Sl(3, \mathbb{Z})$ transformation

$$\begin{pmatrix} 0 & 1 & 1 \\ 0 & 0 & 1 \\ 1 & 1 & 1 \end{pmatrix}. \quad (\text{B.100})$$

When $t^a \neq 0$ for some a 's, we have a (possibly partially) *resolved* singularity. From the polytope point of view, the resolution amounts to “chopping off” the tip of σ , since we cannot reach the point $z_1 = \dots = z_n = 0$ anymore. As an aside, let us note the interesting relation between the parameter t^a and the period of the Kähler form on the corresponding 2-cycle C^a (in the case of a small resolution by a $\mathbb{C}P^1$) [288] :

$$\int_{C^a} \omega = t^a. \quad (\text{B.101})$$

So the FI parameters in the GLSM really map to the “Kähler volumes” of the resolving cycles.

The GLSM perspective is very interesting in order to explore the topology of toric varieties, and it is “easier” because more explicit. One can easily visualize toric divisors and compute their intersections using the GLSM. Nice reviews exist in the physics literature on this part of the story. See in particular: [288], [324], and the chapter 7 of [325].

Appendix C

$\mathcal{N} = 1$ renormalization group and Seiberg duality

C.1 RG equations for $\mathcal{N} = 1$ gauge theories

Let us briefly recall some results on the renormalization of $\mathcal{N} = 1$ gauge theories. Consider a theory with a gauge coupling g and some superpotential couplings h_n . We have a superpotential $W = \sum_n h_n \mathcal{O}_n$ for some operators \mathcal{O}_n polynomial in the UV chiral superfields of the theory.

C.1.1 Superpotential couplings

It is well known that the holomorphic superpotential couplings h_n are not renormalized [131]. However, the physical coupling do suffer wave-function renormalisation. Consider for instance the superpotential term

$$W \supset h \mathcal{O} = h \prod_i (\Phi_i)^{d_i} \quad (\text{C.1})$$

where the operator \mathcal{O} has a classical dimension $d_{\mathcal{O}} = \sum_i d_i$. We define the dimensionless physical coupling as

$$\tilde{h} = \mu^{d_{\mathcal{O}}-3} \left(\prod_i Z_i^{-\frac{d_i}{2}} \right) h \quad (\text{C.2})$$

where μ is the renormalisation scale and Z_i is the wave-renormalisation factor of the field Φ_i ¹. Consequently, the quantum beta function for \tilde{h} is

$$\beta(\tilde{h}) = \mu \frac{\partial \tilde{h}}{\partial \mu} = \left(d_{\mathcal{O}} + \frac{1}{2} \sum_i d_i \gamma_i - 3 \right) \tilde{h}. \quad (\text{C.3})$$

We use the following definition for the anomalous dimension of a field ϕ :

$$\gamma_{\phi} = -\mu \frac{\partial}{\partial \mu} \ln Z_{\phi}. \quad (\text{C.4})$$

¹Note that $\Phi = \sqrt{Z} \Phi_0$, with Φ and Φ_0 the renormalised and bare quantities, respectively.

The quantum dimension for a scalar ϕ is $1 + \frac{1}{2}\gamma_\phi$. In general, the beta function for any superpotential coupling is

$$\beta(\tilde{h}_n) = (\dim \mathcal{O}_n - 3)\tilde{h}_n, \quad (\text{C.5})$$

with $\dim \mathcal{O}_n$ the quantum dimension of the operator \mathcal{O}_n . Note that anomalous dimensions are functions of all the coupling constants of the theory, and also of the renormalisation scale μ , $\gamma_i = \gamma_i(h_n, g; \mu)$. At a quantum fixed point, the dependence on μ of course disappears.

C.1.2 Gauge coupling

One usually defines the holomorphic gauge coupling

$$\tau = \frac{\theta}{2\pi} + i\frac{4\pi}{g^2}. \quad (\text{C.6})$$

The angle θ is the coupling of the topological term $F \wedge F$. This holomorphic coupling is only renormalised at one loop, according to

$$\mu \frac{\partial \tau}{\partial \mu} = i\frac{b_0}{2\pi}, \quad \text{with} \quad b_0 = \frac{3}{2}T(\text{adj}) - \frac{1}{2}\sum_i T(r_i). \quad (\text{C.7})$$

Here $T(r)$ is the index of the representation r . The representations index are normalized such that the fundamental of $SU(N)$ has index $T(\text{fond}) = 1$ (then $T(\text{adj}) = 2N$ for $SU(N)$). In particular, for $SU(N)$ SQCD with N_f flavor, we have $b_0 = 3N - N_f$.

On the other hand the physical gauge coupling does suffer higher loop corrections, again from wave-function renormalisation of the chiral fields. The all-order formula is called the NSVZ beta function [209]. It is also often called the “exact beta function”, but of course this beta function again depends on the anomalous dimensions, which are unknown functions of the gauge coupling, $\gamma_i(g)$. The NSVZ beta function can also be derived by arguments similar to the ones just presented for the case of the superpotential couplings [326]. It reads

$$\beta\left(\frac{8\pi^2}{g^2}\right) = b_0 + \frac{1}{2}\sum_i T(r_i)\gamma_i. \quad (\text{C.8})$$

Remark that we actually only wrote the numerator of the NSVZ function. The physical meaning of the denominator (which has a pole) is unclear. The way we will mainly use the “exact” formulae (C.3) and (C.8) is as tools to find non-trivial exact fixed points of the renormalisation group flow, as explained in the main text.

C.2 Seiberg duality in the conformal window

Consider $\mathcal{N} = 1$ SQCD with $SU(N_c)$ gauge group and N_f flavors Q_i, \tilde{Q}^j , $i, j = 1, \dots, N_f$, often called the “electric” theory. The flavor symmetry of this theory is $SU(N_f)_L \times SU(N_f)_R$, acting independently on Q and \tilde{Q} . It can be shown that for $3N_c - N_f > \varepsilon$ with ε positive and very small, there exists a perturbative fixed point at two loops, at

a critical value $g = g^* \sim \frac{\epsilon}{N}$ [83]. It was famously conjectured by Seiberg [83] that this fixed point is in fact exact and that it exist for any N_f in the range

$$\frac{3}{2}N_c < N_f < 3N_c. \quad (\text{C.9})$$

This range is called the *conformal window* of SQCD. Using the NSVZ beta function, we see that the anomalous dimension γ_0 of the quarks Q, \tilde{Q} must be $\gamma_0 = (N_f - 3N_c)/N_f$ at the fixed point, and that the superconformal R-charge is

$$R(Q) = R(\tilde{Q}) = \frac{N_f - N_c}{N_f}. \quad (\text{C.10})$$

As we decrease N_f , the anomalous dimensions get large and the interacting fixed point becomes strongly coupled. Let us call this SCFT the “electric SCFT”.

Consider also another $\mathcal{N} = 1$ theory with $SU(N_f - N_c)$ gauge group and N_f flavors $q_i, \tilde{q}^j, j = 1, \dots, N_f$, together with a gauge singlet field M_j^i transforming in the (N_f, \bar{N}_f) of $SU(N_f)_L \times SU(N_f)_R$, and a superpotential

$$W = y q_i M_j^i \tilde{q}^j, \quad (\text{C.11})$$

which preserves the $SU(N_f)_L \times SU(N_f)_R$ flavor group. This is called the “magnetic” theory. Using the exact beta functions for the gauge coupling g and the superpotential coupling y , one can show that such theory might indeed posses an exact fixed point as long as the parameters N_f and N_c satisfy (C.9). First notice that if $y = 0$ we are at the SQCD fixed point with $\tilde{N}_c = N_f - N_c$, with $\gamma_0 = (3N_c - 2N_f)/N_f$ for the quarks q, \tilde{q} , while the gauge singlet M is decoupled and must have dimension 1. Hence the operator $M\tilde{q}q$ is a relevant deformation of the SQCD fixed point. For any small y the theory then flows away from this fixed point, and can reach another fixed point where the anomalous dimension for M is increased to $\gamma_M = (4N_f - 6N_c)/N_f$. This isolated fixed point at non-trivial values of g and y is called the “magnetic SCFT”.

The Seiberg duality conjecture is that the electric and magnetic SCFT are precisely equivalent. For reviews of the beautiful consistency checks one can make of this conjecture, we refer to [131]. It is interesting to note that at the magnetic fixed point the fields M_j^i can be identified with the SQCD mesons $\tilde{Q}^i Q_j$ in the electric SCFT; for instance it is easy to check they both have the same quantum dimension, $\Delta(M) = \Delta(\tilde{Q}Q) = 3(N_f - N_c)/N_f$. For this reason the field M is often called the “dual meson”.

Seiberg duality is also often stated as “the IR limits of the electric and magnetic theories are equivalent”; this formulation is then correct even for $N_f < \frac{3}{2}N_c$, when SQCD does not reach a fixed point but becomes dual to an IR free magnetic theory in the IR [83].

C.3 Seiberg duality with a quartic superpotential

The $\mathcal{N} = 1$ Seiberg duality as presented in the previous section is a duality between two isolated fixed points. In that respect it is quite different from the S-duality of $\mathcal{N} = 4$,

for instance, which acts on a continuous line of fixed points. An interesting extension of Seiberg duality is obtained by adding a quartic superpotential

$$W = h(Q_i \tilde{Q}^j)(Q_j \tilde{Q}^i) \quad (\text{C.12})$$

to the electric SCFT. For h small, we can compute the beta function using the anomalous dimensions of the electric SCFT. We have three cases:

- $N_f = 2N_c$. This case was discussed in the main text, in section 4.1.4. The beta function for g and for h are proportional, so there is a line of fixed points.
- $N_f > 2N_c$. We have $R(Q) > \frac{1}{2}$ at the electric SCFT. The quartic superpotential (C.12) is then irrelevant, as it would be classically.
- $N_f < 2N_c$. In this case we have $R(Q) < \frac{1}{2}$ in the electric SCFT, so the quartic superpotential is *relevant*. We can then use it to perturb the fixed point and start a non-trivial RG flow.

The third case is the most interesting: we see an instance of a superpotential which is classically irrelevant but becomes *relevant* at strong coupling (near the electric SCFT fixed point).

The Seiberg dual theory has a $SU(N_f - N_c)$ gauge group, N_f dual quarks and the dual mesons, with superpotential

$$W = y q_i M_j^i \tilde{q}^j + h M_j^i M_i^j. \quad (\text{C.13})$$

Note that the coupling h looks like a mass term. However, we must check its RG running as in the case of the electric theory. For $N_f = 2N_c$ the coupling h is marginal, hence we have a line of fixed points, as expected from the electric theory. When $N_f > 2N_c$ the coupling h is irrelevant, while for $N_f < 2N_c$ it is relevant.

If we integrate out the field M , we obtain the effective superpotential

$$W = h' (q_i \tilde{q}^j)(q_j \tilde{q}^i), \quad (\text{C.14})$$

with $h' \sim y^2/m$, so we see that the electric and magnetic theories have the same superpotential. However, when h is a relevant coupling, the dual h' is an irrelevant coupling, and vice-versa. As carefully explained in [85], when $N_f < 2N_c$ Seiberg duality relates the RG flow *away from* the electric SCFT, as h grows from 0 to infinity, to the dual RG flow from $h' = \infty$ to 0, *towards* the magnetic SCFT.

In the case $N_f = 2N_c$, both electric and magnetic theories have an $SU(N_c)$ gauge group and a marginal quartic superpotential, so they are really self-similar, and Seiberg duality is most similar to an S-duality, relating strong to weak coupling h on the line of fixed points.

Seiberg duality for SQCD with a quartic superpotential can be derived from a careful analysis of $\mathcal{N} = 2$ SQCD, as was done in [155]; in that early reference, however, they missed the fact that the quartic superpotential can actually turn out to be relevant in the far IR, which is quite important to understand the famous duality cascade of [118].

Appendix D

Spherical coordinates on \mathbb{R}^6

In this Appendix we discuss some coordinates on S^5 and the associated harmonic functions (which we present for completeness although we will not need them in any detail). We then go on to derive the equation (4.46) of Chapter 4.

D.1 Spherical polar coordinates on \mathbb{R}^6

We consider the following spherical polar coordinates on \mathbb{R}^6 :

$$\begin{aligned}
 x^1 &= r \cos \theta_1 \\
 x^2 &= r \sin \theta_1 \cos \theta_2 \\
 x^3 &= r \sin \theta_1 \sin \theta_2 \cos \theta_3 \\
 x^4 &= r \sin \theta_1 \sin \theta_2 \sin \theta_3 \cos \theta_4 \\
 x^5 &= r \sin \theta_1 \sin \theta_2 \sin \theta_3 \sin \theta_4 \cos \varphi \\
 x^6 &= r \sin \theta_1 \sin \theta_2 \sin \theta_3 \sin \theta_4 \sin \varphi
 \end{aligned} \tag{D.1}$$

where $\theta_j \in [0, \pi]$ and $\varphi \in [0, 2\pi)$. The volume form is then

$$r^5 dr \wedge d\text{vol}(S^5) = r^5 dr \wedge (\sin \theta_1)^4 (\sin \theta_2)^3 (\sin \theta_3)^2 \sin \theta_4 d\theta_1 \wedge d\theta_2 \wedge d\theta_3 \wedge d\theta_4 \wedge d\varphi. \tag{D.2}$$

The Laplacian on \mathbb{R}^6 is

$$\Delta_{\mathbb{R}^6} = \frac{1}{r^5} \partial_r r^5 \partial_r + \frac{1}{r^2} \Delta_{S^5}, \tag{D.3}$$

with the angular Laplacian on S^5 that reads

$$\begin{aligned}
 \Delta_{S^5} &= (\partial_{\theta_1} + 4 \cot \theta_1) \partial_{\theta_1} + \frac{1}{(\sin \theta_1)^2} (\partial_{\theta_2} + 3 \cot \theta_2) \partial_{\theta_2} + \frac{1}{(\sin \theta_1 \sin \theta_2)^2} (\partial_{\theta_3} + 2 \cot \theta_3) \partial_{\theta_3} + \\
 &+ \frac{1}{(\sin \theta_1 \sin \theta_2 \sin \theta_3)^2} (\partial_{\theta_4} + \cot \theta_4) \partial_{\theta_4} + \frac{1}{(\sin \theta_1 \sin \theta_2 \sin \theta_3 \sin \theta_4)^2} \partial_{\varphi}^2
 \end{aligned} \tag{D.4}$$

D.2 Spherical harmonics of S^5 and Gegenbauer polynomials

Let m_0, m_1, m_2, m_3, m_4 be integers such that $m_0 \geq m_1 \geq \dots \geq m_4 \geq 0$. The eigenvalue equation is

$$\Delta_{S^5} Y_m(\theta_i; \pm\varphi) = -m_0(m_0 + 4)Y_m(\theta_i; \pm\varphi). \quad (\text{D.5})$$

A complete set of spherical harmonics of degree m_0 is given by:

$$Y_m(\theta_i; \pm\varphi) = e^{\pm im_4\varphi} \cdot (\sin \theta_1)^{m_1} C_{m_0-m_1}^{m_1+2}(\cos \theta_1) \cdot (\sin \theta_2)^{m_2} C_{m_1-m_2}^{m_2+\frac{3}{2}}(\cos \theta_2) \cdot (\sin \theta_3)^{m_3} C_{m_2-m_3}^{m_3+1}(\cos \theta_3) \cdot (\sin \theta_4)^{m_4} C_{m_3-m_4}^{m_4+\frac{1}{2}}(\cos \theta_4). \quad (\text{D.6})$$

Here $C_n^m(x)$ is the Gegenbauer polynomial of degree n and order m . Gegenbauer polynomials, for $m > 0$, can be defined by the generating function [327, 328]

$$G(x, t) \equiv \sum_{n=0}^{\infty} C_n^m(x) t^n = \frac{1}{(1 - 2xt + t^2)^m}. \quad (\text{D.7})$$

A useful explicit expression for the Gegenbauer polynomials can be easily derived by writing $G(\cos \theta, t)$ as a function of the complex variable $w = te^{i\theta}$:

$$G(\cos \theta, t) = G(w) = \frac{1}{(w - 1)^m (\bar{w} - 1)^m}. \quad (\text{D.8})$$

It follows that

$$C_n^m(\cos \theta) = \sum_{k=0}^n \frac{\Gamma(m + n - k)}{(n - k)! \Gamma(m)} \frac{\Gamma(m - k)}{k! \Gamma(m)} e^{i(n-2k)\theta}. \quad (\text{D.9})$$

For completeness, let us write down the differential equation whose solutions are the Gegenbauer polynomials [327]:

$$(x^2 - 1) \frac{d^2}{dx^2} C_n^m(x) + (2m + 1)x \frac{d}{dx} C_n^m(x) - n(n + 2m) C_n^m(x) = 0. \quad (\text{D.10})$$

They form an orthogonal set on $[-1, 1]$:

$$\int_{-1}^1 dx (1 - x^2)^{m-\frac{1}{2}} C_n^m(x) C_r^m(x) = \frac{2^{1-2m} \pi \Gamma(n + 2m)}{n!(n + m) [\Gamma(m)]^2} \delta_{nr}, \quad (\text{D.11})$$

and this latter formula implies

$$\int_0^\pi d\theta (\sin \theta)^{2m} C_n^m(\cos \theta) = 2^{-2m} \pi \Gamma(2m + 1) [\Gamma(1 + m)]^2 \delta_{n0}. \quad (\text{D.12})$$

We could also define an orthonormalised set of harmonics $\{\mathcal{Y}_m\}$,

$$\int_{S^5} \mathcal{Y}_m \mathcal{Y}_n^* = \delta_{nm}. \quad (\text{D.13})$$

By using the above properties of the C_n^m 's, we find the normalization constant a_m , defined as

$$\mathcal{Y}_m = a_m Y_m, \quad (\text{D.14})$$

to be given by

$$a_m = \left(\frac{(m_0 - m_1)! \cdots (m_3 - m_4)! [\Gamma(2 + m_1) \cdots \Gamma(\frac{1}{2} + m_4)]^2 (2 + m_0) \cdots (\frac{1}{2} + m_3)}{2\pi^5 4^{-3-(m_1+\cdots+m_4)} \Gamma(4 + m_0 + m_1) \cdots \Gamma(1 + m_3 + m_4)} \right)^{\frac{1}{2}}. \quad (\text{D.15})$$

D.3 Solving for the warp factor of a 2 stacks system

Consider a stack of $N - N'$ D3-branes at $(x^i) = 0$, and a stack of N' D3-branes at $(x^i) = (s, 0, \cdots, 0)$. The warp factor equation,

$$\Delta_6 h = -2\kappa^2 \tau_3 \left((N - N') \delta^6(x) + N' \delta^6(x - s) \right), \quad (\text{D.16})$$

can be expanded in harmonics. In this particular case, we have

$$\delta^6(x - s) = \delta(x^1 - s) \delta(x^2) \cdots \delta(x^6) = \frac{\delta(r - s)}{r^5} \sum_{m_0} \frac{(m_0 + 2)}{2\pi^3} Y_{m_0}(\theta_1), \quad (\text{D.17})$$

with $Y_{m_0}(\theta_1) = C_{m_0}^2(\cos \theta_1)$. Then, expanding as

$$h(r, \theta_1) = \sum_{m_0} \tilde{H}_{m_0}(r) Y_{m_0}(\theta_1), \quad (\text{D.18})$$

we have the radial equations

$$\left[\Delta_r - \frac{m_0(m_0 + 4)}{r^2} \right] \tilde{H}_{m_0}(r) = -4R_{(N-N')}^4 \frac{\delta(r)}{r^5} \delta_{m_0,0} - 4R_{N'}^4 \frac{(m_0 + 2)}{2} \frac{\delta(r - s)}{r^5}, \quad (\text{D.19})$$

where $R_N^4 = 4\pi N \alpha'^2 g_s$. It is then straightforward to derive (for $r > s$)

$$h(r, \theta_1) = \frac{R_N^4}{r^4} \left(1 + \frac{N'}{N} \sum_{m_0=1}^{\infty} \left(\frac{s}{r} \right)^{m_0} Y_{m_0}(\theta_1) \right). \quad (\text{D.20})$$

This is equation (4.46).

Appendix E

Seiberg-Witten theory

Seiberg-Witten theory is the theory which provides the full non-perturbative solution for $\mathcal{N} = 2$ supersymmetric gauge theories in four dimensions, in the sense that it gives the exact low energy Wilsonian action. In this Appendix we will introduce the SW formalism from scratch, emphasizing the aspects which are important to understand the work presented in Chapter 6. We first introduce the SW formalism in the field theory language, and later mention the beautiful relationship between SW theory and M-theory. We focus on the case of the special unitary gauge group $SU(N)$.

E.1 $\mathcal{N} = 2$ vector multiplet and the effective action

The $\mathcal{N} = 2$ four dimensional vector multiplet contains a vector, two Weyl fermions and one complex scalar, all in the adjoint representation of the gauge group. In order to construct a supersymmetric action, it is useful to introduce an $\mathcal{N} = 2$ superspace $(x, \theta, \bar{\theta}, \tilde{\theta}, \bar{\tilde{\theta}})$, where the Grassmannians $\tilde{\theta}, \bar{\tilde{\theta}}$ extend the usual $\mathcal{N} = 1$ superspace. The $\mathcal{N} = 2$ vector superfield can be expanded in terms of $\mathcal{N} = 1$ *chiral* superfields Φ and $\mathcal{W}_\alpha = -\frac{1}{4}\bar{D}^2 D_\alpha V$, as [329, 330]

$$\Psi(y, \theta, \tilde{\theta}) = \Phi(y, \theta) + \sqrt{2}\tilde{\theta}^\alpha \mathcal{W}_\alpha(y, \theta) + \tilde{\theta}\tilde{\theta}\Psi^{(D)}(y, \theta), \quad (\text{E.1})$$

where we have used the chiral coordinate $y^\mu = x^\mu + i\theta\sigma^\mu\bar{\theta} + i\tilde{\theta}\sigma^\mu\bar{\tilde{\theta}}$, and

$$\Psi^{(D)}(y, \theta) = \Phi^\dagger(y - i\theta\sigma^\mu\bar{\theta}, \theta, \bar{\theta})e^{2V(y - i\theta\sigma^\mu\bar{\theta}, \theta, \bar{\theta})}\Big|_{\theta\bar{\theta}}. \quad (\text{E.2})$$

The most general action for the $\mathcal{N} = 2$ vector superfield is

$$\mathcal{L} = \frac{1}{4\pi} \text{Im} \int d^2\theta d^2\tilde{\theta} \mathcal{F}(\Psi), \quad (\text{E.3})$$

where \mathcal{F} is any *holomorphic* function of Ψ , called the *prepotential*. It is the holomorphicity property of the prepotential which ultimately allows to compute the effective action exactly in any vacuum. In the UV, the theory is defined by

$$\mathcal{F}(\Psi) = \frac{\tau}{2}\Psi^2, \quad (\text{E.4})$$

with τ the usual holomorphic gauge coupling, as in (C.6). Expanding in $\mathcal{N} = 1$ components, this gives the renormalizable action

$$S = \int d^4x \left\{ \mathbb{I}m \int d^2\theta \frac{\tau}{8\pi} \mathcal{W}^2 + \int d^2\theta d^2\bar{\theta} \frac{1}{g^2} \Phi^\dagger e^{2V} \Phi \right\}. \quad (\text{E.5})$$

The factor of $1/g^2$ in front of the kinetic term for Φ might look unconventional, but it is natural in a $\mathcal{N} = 2$ invariant action. Classically, the moduli space of vacua is the so-called Coulomb branch, for any ϕ (ϕ the complex scalar) such that $[\phi, \phi^\dagger] = 0$. A convenient parametrization of the classical moduli space is by the eigenvalues of ϕ ,

$$\phi = \text{diag}(\varphi_1, \dots, \varphi_N), \quad \sum_i \varphi_i = 0, \quad (\text{E.6})$$

and up to permutations of the φ_i 's (which are residual gauge symmetries). In a generic vacuum, the gauge group is broken as

$$SU(N) \rightarrow U(1)^{N-1}. \quad (\text{E.7})$$

Note that quantum mechanically the parameters φ_i are not good coordinates on the moduli space. Instead one should in principle parametrize the moduli space by a set of gauge invariant polynomials.

In a generic vacuum the residual gauge symmetry is abelian. Expanding (E.3), we see that the low energy effective action (LEEA) should take the form¹

$$S = \int d^4x \frac{1}{4\pi} \mathbb{I}m \left\{ \int d^2\theta \frac{1}{2} \mathcal{F}_{ab}(\Phi) \mathcal{W}^{a\alpha} \mathcal{W}_\alpha^b + \int d^2\theta d^2\bar{\theta} \Phi^{\dagger a} \mathcal{F}_a(\Phi) \right\}, \quad (\text{E.8})$$

where $a, b = 1, \dots, N-1$ are Lie algebra indices in the Cartan of $SU(N)$, and

$$\mathcal{F}_a = \frac{\partial \mathcal{F}}{\partial \Phi^a}, \quad \mathcal{F}_{ab} = \frac{\partial^2 \mathcal{F}}{\partial \Phi^a \partial \Phi^b} \equiv \tau_{ab} \quad (\text{E.9})$$

Remark that the effective gauge coupling $\mathbb{I}m \tau_{ab}$ is a metric on field space, since we have the scalar kinetic term

$$\mathcal{L} \supset -\frac{1}{4\pi} \mathbb{I}m (\tau_{ab}(\phi)) \partial_\mu \bar{\phi}^a \partial^\mu \phi^b. \quad (\text{E.10})$$

We then have the metric on field space (and henceforth on moduli space)

$$ds^2 = \mathbb{I}m (\tau_{ab}) d\phi^a d\bar{\phi}^b. \quad (\text{E.11})$$

On physical ground, this metric should be positive definite. However, since τ is an holomorphic function $\mathbb{I}m \tau$ is harmonic, so it cannot be positive everywhere. We will see that the way out of this conundrum is that ϕ^a are not good coordinates everywhere: we will need several patches to consistently map the full moduli space.

¹This does not mean that any vacuum has a Lagrangian description, however: in some vacua we have to include extra hypermultiplets to describe massless monopoles and dyons, and we can do that only for mutually local excitations.

E.1.1 R-symmetry and the perturbative prepotential

Classically, the UV theory (E.5) has a R-symmetry $U(1)_R$ which assigns charge 0, 1 and 2 to the vector, the fermions and the complex scalar, respectively. There is however a chiral anomaly,

$$\phi \rightarrow e^{i2\alpha}\phi, \quad \theta \rightarrow \theta - 4N\alpha, \quad (\text{E.12})$$

which breaks $U(1)_R$ to \mathbb{Z}_{4N} . Moreover, the actual symmetry acting on the Coulomb branch is \mathbb{Z}_{2M} since ϕ has R-charge 2. We can easily compute the *perturbative* running coupling, by the following trick [331, 332]: since $\mathcal{F}'' = \tau$, the anomalous shift in the theta angle can be written as

$$\mathcal{F}''(e^{2i\alpha}\phi) - \mathcal{F}''(\phi) = -\frac{4N\alpha}{2\pi} \quad \Rightarrow \quad \frac{\partial^3 \mathcal{F}}{\partial \phi^3} = \frac{i}{\pi} \frac{N}{\phi}, \quad (\text{E.13})$$

where the implication is for α infinitesimal. Integrating the expression on the right, we have

$$\tau_{\text{pert}} = \mathcal{F}''_{\text{pert}} = -\frac{2N}{2\pi i} \ln \frac{\phi}{\Lambda}, \quad \text{with} \quad \Lambda = \mu e^{2\pi i \tau(\mu)}, \quad (\text{E.14})$$

while the prepotential can be schematically written as

$$\mathcal{F}_{\text{pert}} = -\frac{N}{2\pi i} \phi^2 \ln \frac{\phi}{\Lambda}. \quad (\text{E.15})$$

This result is actually exact in perturbation theory [331]. However, non-perturbative effects will drastically change the result at strong coupling.

E.2 Electric-magnetic duality

Classical electric-magnetic duality for abelian gauge fields exchanges the field strength with its Hodge dual, $F \leftrightarrow *F$. Consequently, it exchanges electric and magnetic sources, or equation of motion (EOM) and Bianchi identity (BI)

$$d * F = *j_e, \quad dF = *j_m \quad \text{with} \quad j_e \leftrightarrow j_m. \quad (\text{E.16})$$

In the absence of sources, this is also a symmetry of the quantum theory. Formally, it can be seen as a change of variable in the path integral. Suppose that we take F instead of the gauge field A to be the dynamical field (the one we path integrate over). We can do that provided we also introduce a Lagrange multiplier A_D which imposes the BI,

$$S = \int_{\mathbb{R}^{3,1}} \left\{ -\frac{1}{4} F \wedge *F + \frac{1}{2} A_D \wedge dF \right\} \quad (\text{E.17})$$

We can then perform the path integral over F . The equation of motion for F tells us that $*F = F_D$; plugging into the action, we have a new action² which only depends on A_D , through $F_D = dA_D$,

$$S_{\text{dual}} = \int_{\mathbb{R}^{3,1}} \left\{ -\frac{1}{4} F_D \wedge *F_D \right\}. \quad (\text{E.18})$$

²Recall that in Lorentzian signature the operator $*$: $\Lambda^2 \rightarrow \Lambda^2$ squares to -1 in four dimensions, so $F = -*F_D$.

This generalizes easily to the free abelian action for the $\mathcal{N} = 1$ vector multiplet. We can write it in term of \mathcal{W}_α and the auxiliary vector superfield V_D imposing the Bianchi identity $D\mathcal{W} - \bar{D}\bar{\mathcal{W}} = 0$,

$$S = \frac{1}{8\pi} \int d^4x \text{Im} \left\{ \int d^2\theta \tau \mathcal{W}^2 + \frac{1}{2} \int d^2\theta d^2\bar{\theta} V_D D\mathcal{W} \right\} \tag{E.19}$$

$$= \frac{1}{8\pi} \int d^4x \text{Im} \int d^2\theta \{ \tau \mathcal{W}^2 - 2\mathcal{W}_D \mathcal{W} \}. \tag{E.20}$$

Solving for \mathcal{W} in term of $\mathcal{W}_D = -\frac{1}{4}\bar{D}^2 D V$, we have $\mathcal{W} = \frac{1}{\tau}\mathcal{W}_D$ which gives the S-dual action

$$S = \frac{1}{8\pi} \int d^4x \text{Im} \int d^2\theta \tau_D \mathcal{W}_D^2 \quad \text{with} \quad \tau_D = -\frac{1}{\tau}. \tag{E.21}$$

Moreover, the action (E.19) is also invariant under $\tau \rightarrow \tau + 1$, which shifts to theta angle by 2π . In total, we have a $Sl(2, \mathbb{Z})$ invariance,

$$\tau \rightarrow \frac{a\tau + b}{c\tau + d}, \quad \begin{pmatrix} a & b \\ c & d \end{pmatrix} \in Sl(2, \mathbb{Z}). \tag{E.22}$$

This invariance also extends to the $\mathcal{N} = 2$ free vector multiplet. Let us define the dual field

$$\Phi_D = \frac{\partial \mathcal{F}}{\partial \Phi}. \tag{E.23}$$

The S-duality can be understood as a Legendre transformation on the prepotential [330],

$$\mathcal{F}_D(\Phi_D) = \mathcal{F}(\Phi) - \Phi \Phi_D, \quad \text{with} \quad \Phi = -\frac{\partial \mathcal{F}_D}{\partial \Phi_D} \tag{E.24}$$

Obviously,

$$\tau_D = \frac{\partial^2 \mathcal{F}_D}{\partial \Phi_D^2} = -\left(\frac{\partial \Phi_D}{\partial \Phi} \right)^{-1} = -\frac{1}{\tau}. \tag{E.25}$$

The second term in equation (E.8) is also invariant under this duality transformation. Under a generic $Sl(2, \mathbb{Z})$ transformation, we have

$$\begin{pmatrix} \Phi_D \\ \Phi \end{pmatrix} \rightarrow \begin{pmatrix} a & b \\ c & d \end{pmatrix} \begin{pmatrix} \Phi_D \\ \Phi \end{pmatrix} \tag{E.26}$$

Note that for $\tau = \frac{\partial \phi_D}{\partial \phi}$ the transformation (E.22) follows from this transformation. When we have r abelian fields, the duality group generalizes to the symplectic group $Sp(2r, \mathbb{Z})$. In particular, the fields Φ, Φ_D have the invariant Lagrangian

$$\int d^2\theta d^2\bar{\theta} \left(\Phi_D^{\dagger a} \quad \Phi^{\dagger a} \right) \delta_{ab} \begin{pmatrix} 0 & 1 \\ -1 & 0 \end{pmatrix} \begin{pmatrix} \Phi_D^b \\ \Phi^b \end{pmatrix}, \tag{E.27}$$

with $a, b = 1, \dots, r$. Since $\tau_{ab} = \mathcal{F}_{ab} = \partial_a \Phi_{D b}$, we can write the scalar metric (E.11) in term of ϕ and ϕ_D simply as

$$ds^2 = d\phi_D^a d\bar{\phi}_a. \tag{E.28}$$

The actual moduli space \mathcal{M} is some complicated submanifold of complex dimension r in $\mathbb{C}^{2r} \cong \{\phi_D, \phi\}$. Let us call z the coordinates on \mathcal{M} , on some patch. The metric on the moduli space in local coordinates reads

$$ds^2 = \frac{\partial \phi_D^c}{\partial z^a} \frac{\partial \bar{\phi}_c^{\bar{b}}}{\partial \bar{z}^b} dz^a d\bar{z}^b. \tag{E.29}$$

We have uncovered an important geometric structure, thanks to S-duality: the space $\mathbb{C}^{2r} \cong \{\phi_D, \phi\}$ is naturally seen as a $Sp(2r, \mathbb{Z})$ bundle³,

$$\mathbb{C}^r \longrightarrow \mathbb{C}^{2r} \xrightarrow{\pi} \mathcal{M}. \tag{E.30}$$

We can interpret the fields $(\phi(z), \phi_D(z))$ as a non-trivial *section* of this bundle: the functions (ϕ_D^a, ϕ^a) on different patches are related by non-trivial $Sp(2r, \mathbb{Z})$ duality transformations on overlaps.

E.3 Singularities and massless monopoles

The Wilsonian action for $\mathcal{N} = 2$ $SU(N)$ SYM,

$$S = \int d^4x \frac{1}{4\pi} \mathbb{I}m \left\{ \int d^2\theta \frac{1}{2} \tau_{ab}(\Phi) \mathcal{W}^{a\alpha} \mathcal{W}_\alpha^b + \int d^2\theta d^2\bar{\theta} \Phi_a^\dagger \Phi_D^a \right\}, \tag{E.31}$$

describes $N - 1$ massless photons and their associated $\mathcal{N} = 2$ superpartners. In the IR limit any such vacuum is a trivial SCFT. This action is valid at low energy up to the scale of the first massive excitations. The lowest mass states will be the ones which are protected by the $\mathcal{N} = 2$ supersymmetry, saturating the BPS bound $M \geq \sqrt{2}|Z|$ which follows from the $\mathcal{N} = 2$ SUSY algebra. Remark that the possible massive excitation we are talking about can be thought as 't Hooft-Polyakov monopoles [293, 294] in the semi-classical limit; they are some solitonic excitations of the non-linear $SU(N)$ theory. Moreover, they fall into $\mathcal{N} = 2$ hypermultiplets. Perturbatively, it is well known that $|Z| = h\phi/g^2$ for a 't Hooft-Polyakov monopole of magnetic charge h when the θ angle vanishes⁴, while for generic complex coupling τ we have $Z = h\phi\tau$, due to the Witten effect [333], and $Z = \phi(h\tau + q)$ for a dyon. The non-perturbative expression for Z in the $\mathcal{N} = 2$ low energy theory can be deduced from the requirement of duality invariance; it is given by [154]

$$Z = h_a \phi_D^a + q^a \phi_a, \tag{E.32}$$

with $(\phi, \phi_D) = (\phi(z_0), \phi_D(z_0))$ in the particular vacuum $z = z_0$. The charges q and h are the electric and magnetic charge of the state. Moreover, only the excitations with h and q relatively prime are stable [154]. The $Sl(2N - 2, \mathbb{Z})$ duality symmetry acts as

$$\begin{pmatrix} \phi_D \\ \phi \end{pmatrix} \rightarrow M \begin{pmatrix} \phi_D \\ \phi \end{pmatrix}, \quad \begin{pmatrix} h \\ q \end{pmatrix} \rightarrow (M^{-1})^t \begin{pmatrix} h \\ q \end{pmatrix}. \tag{E.33}$$

³This is similar to the way symplectic bundles appear in classical mechanics. (ϕ_D, ϕ) are the analog of the canonical variables (p, q) . The form $d\phi \wedge d\bar{\phi}_D - d\phi_D \wedge \bar{d}\phi$ is a symplectic form, and (E.28) a symplectic metric.

⁴In our context this form of the mass is forced by the $\mathcal{N} = 2$ -preserving superpotential term $\Phi \tilde{Q} Q$, when coupling an hypermultiplet to the vector multiplet.

At isolated points on the moduli space, some BPS monopoles (generically some dyons of a given charge (q, h)) might have a vanishing mass, $M = \sqrt{2}|Z| = 0$. For instance if $\phi_D = 0$ in a vacuum the monopole of charge $(1, 0)$ would be massless. In general, for $m, n \in \mathbb{Z}$ relatively prime, if the particular combination $m\phi_D - n\phi$ vanishes, the dyon of charge $(h, q) = (m, n)$ becomes massless, as we can see from (E.32). At such a point the metric (E.29) is singular: we should include an hypermultiplet in the LEEA to account for the new massless degrees of freedom.

Singularities on the moduli space correspond to *monodromies* of the section (ϕ_D, ϕ) . Suppose we have a hypermultiplet (written Q, \tilde{Q} in term of two $\mathcal{N} = 1$ chiral superfields) with charge $(h, q) = (0, 1)$ (electrically coupled) to a $U(1)$ vector multiplet, which becomes massless at $z = z_0$. Suppose we can choose some local coordinates on the moduli space as $z - z_0 = \phi$. Since the $U(1)$ gauge group is IR free in the presence of a massless hypermultiplet, we can apply perturbation theory near $\phi = 0$. One can easily show (for instance, use (C.7) in Appendix C with $b_0 = -2N_f = -2$ for a gauge group $U(1)$) that the gauge coupling goes like

$$\tau(\phi) = \frac{2}{2\pi i} \ln \frac{\phi}{\Lambda}. \quad (\text{E.34})$$

We then have

$$\phi = z - z_0, \quad \phi_D = \frac{1}{\pi i} (z - z_0) \ln \frac{(z - z_0)}{\Lambda}. \quad (\text{E.35})$$

When performing a rotation around z_0 , the function ϕ is invariant by assumption but ϕ_D changes as $\phi_D \rightarrow \phi_D + 2\phi$. This is a monodromy, which we can also write as

$$\begin{pmatrix} \Phi_D \\ \Phi \end{pmatrix} \rightarrow M_{(0,1)} \begin{pmatrix} \Phi_D \\ \Phi \end{pmatrix}, \quad \text{with} \quad M_{(0,1)} = \begin{pmatrix} 1 & 2 \\ 0 & 1 \end{pmatrix}. \quad (\text{E.36})$$

From this result we can easily work out the monodromy matrix around a point where any dyon of charge (m, n) becomes massless. If m and n are relatively prime, there exist an $Sl(2, \mathbb{Z})$ matrix S such that $(m, n) = S(0, 1)$:

$$(m \ n) = S(0 \ 1), \quad S = \begin{pmatrix} a & b \\ m & n \end{pmatrix}, \quad \text{such that} \quad an - bm = 1. \quad (\text{E.37})$$

The later equation in term of a and b is called Bezout's identity, and it always has a solution. It follows that the monodromy matrix around the generic dyon is

$$M_{(m,n)} = S^{-1} M_{(0,1)} S = \begin{pmatrix} 1 + 2mn & 2n^2 \\ -2m^2 & 1 - 2nm \end{pmatrix}. \quad (\text{E.38})$$

This result straightforwardly generalizes to a gauge group $U(1)^r$ and $Sp(2r, \mathbb{Z})$ -valued monodromies.

There is also a weak coupling monodromies at infinity on the moduli space, which is not of this type. A path circling around infinity is homotopic to (minus) the product of all the paths going once around a single strong coupling singularity. Hence the weak coupling monodromy at infinity can be found by taking the product of all the strong coupling monodromies. More details can be found e.g. in [166].

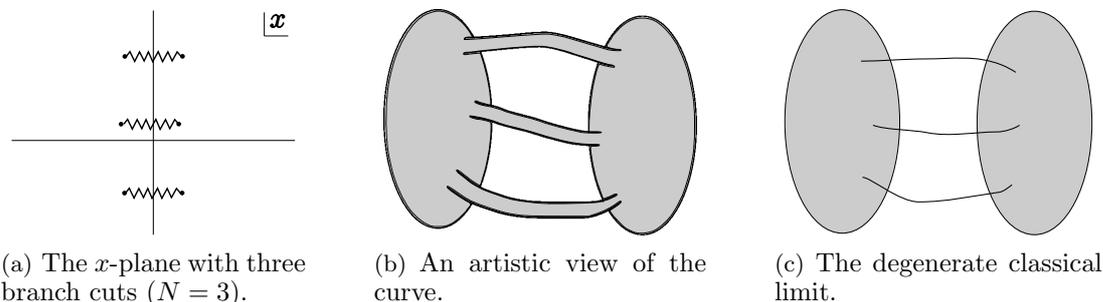


Figure E.1: A hyperelliptic curve of genus 2, corresponding to a typical $SU(3)$ vacuum. On the left hand side we show the x -plane with $2N = 6$ singular points, and a choice of branch cuts connecting the upper and lower sheet. The branch cuts correspond to the handles connecting the two Riemann spheres on the middle figure. The right hand side is a degenerate case, corresponding to a classical limit, where the branch cuts have shrunk to double point singularities.

E.4 Solution through the Seiberg-Witten curve

Solving for the Wilsonian action of $\mathcal{N} = 2$ SYM is equivalent to finding the correct section (ϕ_D, ϕ) which enters into the metric (E.29) on the moduli space. This was done by Seiberg and Witten [154] for $SU(2)$, and further generalized to $SU(N)$ [165, 166], by using physical insight as to which dyons might become massless at strong coupling. The corresponding monodromies gives enough information to find (ϕ_D, ϕ) , in principle.

Nevertheless, it is a very hard mathematical problem. As often the case with the great achievements in mathematical physics, some further insight could be obtained by geometrizing the problem. The key point is to notice that the coupling matrix τ_{ab} , which is positive definite and transforms under $Sp(2N - 2, \mathbb{Z})$, can be interpreted as the period matrix of a Riemann surface of genus $g = N - 1$. Actually, the complex moduli space of a generic Riemann surface is too large. It turns out the special class of genus $N - 1$ Riemann surfaces described by

$$y^2 = \prod_{k=1}^{2N} (x - x_k^{(0)}) \quad (\text{E.39})$$

have the right properties; here $x_k^{(0)}$ are complex structure parameters. These Riemann surfaces are called *hyperelliptic curves*. They are represented here as a double-sheeted cover of the Riemann sphere (with coordinate x) with $2N$ branch points; see Figure E.1 for a graphical view.

The Seiberg-Witten solution for any Coulomb branch vacuum $z = z_0$ is given in term of a particular hyperelliptic curve $\Sigma(z_0)$, where the roots $x_k^{(0)}$ are related to the coordinates on the moduli space and to the strong coupling scale Λ . We also need a particular meromorphic one-form $\lambda \in H^1(\Sigma(z_0), \mathbb{C})$. Then, the local value of the section (ϕ_D, ϕ) is given by

$$\phi_{Da} = \oint_{\alpha_a} \lambda, \quad \phi_b = \oint_{\beta_b} \lambda, \quad (\text{E.40})$$

with (α_a, β_b) a symplectic basis of 1-cycles on the curve Σ . Of course, a change of basis

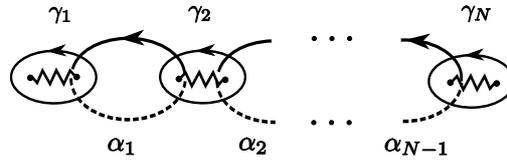


Figure E.2: Representations of the α and γ cycles of the genus $N - 1$ curve.

corresponds to a duality transformation on (ϕ_D, ϕ) . Moreover, the monodromies on the moduli space beautifully translate to monodromies of the curve. A singularity on the moduli space associated to a massless dyon of charge (m, n) corresponds to the 1-cycle $\nu = m\alpha + n\beta$ pinching off. The monodromy (in the complex moduli space of the curve) of any other 1-cycle $\gamma = (h, q)$ around $\nu = (m, n)$ is given by a simple Picard-Lefschetz formula [334]

$$M_\nu : \gamma \longrightarrow \gamma - 2\langle \gamma, \nu \rangle \nu, \tag{E.41}$$

with the intersection number $\langle \gamma, \nu \rangle = hn - qm$. It automatically reproduces the monodromy (E.33)-(E.38).

In this geometric language, the full solution we are looking for is a family of elliptic curves $\Sigma(z)$ fibered over the quantum moduli space parametrized by the coordinates z . For simplicity, let us just state the solution [165, 166, 159], and briefly argue that it is physically sensible. It is convenient to parametrize the Coulomb branch as in (E.6). However, it is important to keep in mind that quantum mechanically we cannot identify φ_i with ϕ_i . The $SU(N)$ curve and the associated 1-form are

$$y^2 = P(x)^2 + 4\Lambda^{2N}, \quad \lambda = \frac{1}{2\pi i} \frac{x}{y} dP(x), \quad \text{with} \quad P(x) = \prod_{i=1}^N (x - \varphi_i). \tag{E.42}$$

At weak coupling, $\Lambda \ll \varphi_i, \forall i$, we have double point singularities, instead of branch cuts. The Riemann surface it then highly degenerate, as shown in Figure E.1(c), with all the β -cycles shrinking to zero. Moreover, when k different φ_i coincide, some $k - 1$ α cycles will shrink too. This corresponds to the singularities on the classical moduli space due to the presence of massless W-bosons, where the gauge group is only partially broken to $SU(k) \times U(1)^{N-k}$. Note that in the quantum moduli space, as described by this curve, there is no such non-abelian point with interacting massless vector fields.

It is interesting to check the decoupling limit from $SU(N)$ to $SU(N - 1)$. Suppose $\varphi_N \gg \varphi_{i \neq N}$. Then we can factorize it from the curve, and consider the effective curve for $SU(N - 1)$

$$\tilde{y}^2 = \prod_{i=1}^{N-1} (x - \varphi_i) + 4\tilde{\Lambda}^{2N-2}, \quad \text{with} \quad \tilde{\Lambda}^{2N-2} = \frac{\Lambda^{2N}}{\varphi_N^2}. \tag{E.43}$$

This relation between the dynamical scales of $SU(N)$ and $SU(N - 1)$ is the one expected from the standard decoupling argument.

Let us order the $2N$ branch points $x_k^{(0)}$ of (E.42) in some way. As long as $\varphi_i \gg \Lambda$ and that the φ_i are well separated, the branch points come in pairs of almost double point, so that it is natural to consider branch cuts between each pair $x_{2i-1}^0 = \varphi_i - \epsilon$,

$x_{2i}^0 = \varphi_i + \epsilon$; this semi-classical understanding was used to draw Figure E.1(b). Let us call γ_i the 1-cycle which circles around the almost double point at $x = \varphi_i$ (note that $\sum_{i=1}^N \gamma_i = 0$). One can check that, in the classical limit,

$$\phi_i = \oint_{\gamma_i} \lambda = \frac{1}{2\pi i} \oint_{\gamma_i} \frac{xP'(x)}{P} dx + o(\Lambda) = \varphi_i + o(\Lambda). \quad (\text{E.44})$$

Let us call α_a the 1-cycle which goes from x_{2a+1}^0 to x_{2a}^0 on the upper sheet, and back on the lower sheet, for $a = 1, \dots, N-1$. These cycles are represented in Figure E.2. Let us also define the $N-1$ independent cycles

$$\beta_a = \sum_{i \leq a} \gamma_i. \quad (\text{E.45})$$

These are the cycles appearing in (E.40). One can check that $\langle \alpha_a, \beta_b \rangle = \delta_{ab}$.

E.5 SW curves for $\mathcal{N} = 2$ SQCD with N_f flavors

Consider $\mathcal{N} = 2$ SQCD, namely the $SU(N)$ theory coupled to N_f hypermultiplets (flavors). In $\mathcal{N} = 1$ notation, the hypermultiplets (Q, \tilde{Q}^\dagger) are coupled to the adjoint chiral field Φ through the superpotential

$$W = \sqrt{2} \tilde{Q}_j \Phi Q^j + \sum_j m_j \tilde{Q}_j Q^j, \quad (\text{E.46})$$

with $j = 1, \dots, N_f$. We allow for arbitrary $\mathcal{N} = 2$ preserving masses for the hypermultiplets. The various branches of the moduli space of this theory were studied in detail in [155]. We can distinguish between the Higgs branches and the Coulomb branch. On the former the hypermultiplets get a VEV and the gauge group is fully broken. The classical metric on the Higgs branch is not renormalized in a $\mathcal{N} = 2$ theory. On the other hand the Coulomb branch looks classically similar to the one of the pure $SU(N)$ SYM theory, since a generic non-zero VEV for Φ gives a mass to all the hypermultiplets. The crucial new phenomenon can appear at places where the Coulomb branch meets a Higgs branch. In particular, the locus of intersection can change non-perturbatively, giving rise to a dual interpretation of the low energy physics. We review the relevant such facts when we need them in Chapter 6.

In the case of non-zero masses m_j , the non-perturbative central charge formula becomes [80]

$$Z = h_a \phi_D^a + q^a \phi_a + n^j m_j, \quad (\text{E.47})$$

where n_j is the charge under the $U(1)$ flavor symmetry acting on the hypermultiplet of mass m_j . The duality group leaving Z invariant is a larger group $Sl(2, 2N-2, \mathbb{Z}) \times \mathbb{Z}^{N_f}$, so that the fields (a_D, a) characterizing the Coulomb branch are a section of a more complicated bundle. We are mostly interested in the case $m_j = 0$ anyway. The Seiberg-Witten curve for $\mathcal{N} = 2$ SQCD with $N_f < 2N$ is [80, 169, 171]

$$y^2 = P(x)^2 + 4\Lambda^{2N-N_f} \prod_{j=1}^{N_f} (x + m_j), \quad \text{with} \quad P(x) = \prod_{i=1}^N (x - \varphi_i). \quad (\text{E.48})$$

For $N_f > 2N$ the theory is IR free, and the description in term of the curve (E.48) breaks down above some scale of order Λ . The case $N_f = 2N$ is special, because the exact beta function identically vanishes and the theory is conformal. The determination of the Seiberg-Witten curve in that case [80, 169] is a rather subtle issue, related to the possibility of having a S-duality invariant theory (similarly to $\mathcal{N} = 4$ SYM). We remark that the SW curve for the $N_f = 2N$ elliptic quiver of Chapter 6 might be better motivated than the similar curve in the usual $N_f = 2N_c$ theory, due to the clearer string theory embedding of the elliptic quiver.

E.6 SW curves from M-theory

A suprisingly simple relationship between the Seiberg-Witten theory approach to $\mathcal{N} = 2$ gauge theories on the one hand and string theory/M-theory on the other hand was uncovered by Witten [157] using the type IIA/M-theory duality. Consider two parallel type IIA NS5-branes separated by a distance $L = \Delta x^6$ in the x^6 direction, and a bunch of N D4-branes stretching between them:

	0	1	2	3	4	5	6	7	8	9	
NS5	○	○	○	○	○	○					
NS5'	○	○	○	○	○	○					
D4	○	○	○	○			○				

(E.49)

This setup brane-engineers $\mathcal{N} = 2$ $U(N)$ SYM. The boundary condition for the D4-brane ending on the NS5-brane sets to zero the A_6 component of the gauge field ⁵, as well as the scalar fields corresponding to x^7, x^8, x^9 . At energies smaller than $1/L$, the low energy theory is four dimensional $\mathcal{N} = 2$ SYM. The complex scalar in the $\mathcal{N} = 2$ multiplet corresponds to motion of the D4-branes along the NS5-branes, in the $v = x^4 + ix^5$ direction. The gauge coupling of the low energy theory is

$$\frac{8\pi^2}{g^2} = \frac{\Delta x^6}{\sqrt{\alpha'} g_s}. \quad (\text{E.50})$$

The perturbative beta function for g can be accounted for by the bending of the NS5-branes due to the D4-branes tension: the D4-brane sources the field responsible for the x^6 profile of each NS5-brane. The $\log v$ solution of the resulting Laplace equation reproduces the perturbative running of g [157], because the radius $|v|$ can be related to the VEV of a (probe) eigenvalue of the adjoint field, which is in turn related to the RG scale in a $\mathcal{N} = 2$ theory.

The crucial observation is that the non-perturbative corrections to the low energy theory can be captured by uplifting the IIA setup to M-theory. This means that we take $R_{10} = \sqrt{\alpha'} g_s$ large. We want to do that while keeping the gauge coupling fixed, so we rescale the distance L to be large too. Remark that this limit $R_{10}, L \rightarrow \infty$ seems opposite to the field theory limit. However, what makes it useful is that we have $\mathcal{N} = 2$ supersymmetry, in which case the LEEA only depends on holomorphic quantities. In M-theory, the IIA fivebranes and D4-branes all lift to a *single smooth M5-brane*. The

⁵It follows by T-duality from the type IIB setup of [54].

M5-brane seen as an algebraic curve can be identified with the Seiberg-Witten curve of the SYM theory! As an illustration, consider Figure E.1: Fig. E.1(c) can be seen as the type IIA description, where the lines stand for the D4-branes stretching between the two NS5-branes; Fig. E.1(b) can be interpreted as a smooth M5-brane. We see that the D4-branes ticken into “tubes” as we increase g_s .

Define the complex coordinates

$$u = i \frac{x^6 + ix^{10}}{2\pi R_{10}}, \quad t = e^{2\pi i u}. \tag{E.51}$$

The M5-brane embedding is described in term of an equation $F(v, t)$. At fixed v , the roots of $F(t, v) = 0$ give the t positions of the NS5-branes; since we have two NS5-branes, $F(t, v)$ is quadratic in t . If we fix a value of t “in between” the two NS5-branes, the roots of F would classically correspond to positions of D4-branes on the v -plane. There are N D4-branes so F is of degree N in v . The M5-brane curve $F = 0$ is of the form [157]

$$A(v)t^2 + B(v)t + C(v) = 0. \tag{E.52}$$

In our example, we actually have $A = -C = 1$, and the roots of B are the positions of the D4-branes in the classical setup. We can identify this curve with (E.42) by the identifications

$$x = v, \quad y = 2\Lambda^N \left(t + \frac{B}{2} \right), \quad P = \Lambda^N B. \tag{E.53}$$

We can similarly include flavors, in the form of semi-infinite D4-branes ending on the left hand side NS5-brane from the left or on the right hand side one from the right. The position of such semi-infinite D4-branes (corresponding to hypermultiplet masses) is encoded in the roots of $A(v)$ and $B(v)$ in (E.52).

In Chapter 6 we study a so-called elliptic model: we put two NS5-branes on a compact circle in the x^6 -direction, together with D4-branes wrapping the circle. The M-theory uplift of that configuration [157, 168] is discussed in that chapter.

E.7 Effective field theory approach to the cascading SW curve

Let us check the statements of section 6.4.2 in Chapter 6 concerning the RG flow and the double points (6.39)-(6.40), using an effective field theory approach for the Seiberg-Witten curve between two strong coupling transitions. Defining $\xi \equiv v^M$ and $\alpha \equiv z_0^M$, we have the following Seiberg-Witten curve,

$$\frac{\xi \prod_{j=0}^{h-1} (\xi^2 + q^{\frac{1}{2}+2j} \alpha^2)}{(\xi - \alpha) \prod_{j=0}^{h-1} (\xi^2 + q^{\frac{3}{2}+2j} \alpha^2)} = g(t|q) = q^{\frac{1}{4}} \left(t + \frac{1}{t} \right) + \mathcal{O}(q^{\frac{5}{4}}) \tag{E.54}$$

Now, defining $\Lambda_j^{2M} \equiv q^{j+\frac{1}{2}} z_0^{2M}$, we can look at the curve in the range $\Lambda_{2n} < v < \Lambda_{2n-1}$, where, at small q ,

$$\frac{R}{S} \approx -\frac{q^{\frac{1}{4}} (\xi^2 + \Lambda_{2n}^{2M})}{\Lambda_{2n}^M \xi} = g(t, q) \approx q^{\frac{1}{4}} \left(t + \frac{1}{t} \right), \tag{E.55}$$

which gives

$$\xi\Lambda_{2n}^M t^2 + (\xi^2 + \Lambda_{2n}^{2M})t + \xi\Lambda_{2n}^M = 0, \quad (\text{E.56})$$

This is a SW curve for a $SU(2M)$ gauge group with $2M$ massless flavors [157], at the baryonic root (hence it has exact double points). Extracting the roots for t (and neglecting Λ_n/v because of large M), one finds

$$u_1 = -u_2 = -\frac{M}{2\pi i} \log \left(\frac{v}{\Lambda_{2n}} e^{-\frac{2\pi i k}{2M}} \right) = -\frac{M}{2\pi i} \log \frac{v}{\Lambda_{2n}} + \frac{1}{2}k, \quad (\text{E.57})$$

where $k = 0, 1$. We see that at $v = \Lambda_{2n}$, $u_1 = u_2 = 0, \frac{1}{2}$ (that is, the two NS5's intersect at $x^6 = 0$, but in fact the corresponding M5 brane also self-intersects at two distinct points on the torus). Since $\tau_1 = u_2 - u_1 = \tau - \tau_2$, we have reproduced the correct perturbative running of the gauge couplings. Also notice that this effective field theory for the first node is valid only up to $v = \Lambda_{2n-1}$, where according to (E.57) $u_1 = u_2 = \frac{\tau}{2}, \frac{\tau}{2} + \frac{1}{2}$ (that is when the coupling of the second gauge group hits a Landau pole).

One can perform the same analysis for the second gauge group, i.e. for the double points at $u = \frac{\tau}{2}, \frac{\tau}{2} + \frac{1}{2}$, obtaining (6.40).

Appendix F

The conifold and a \mathbb{Z}_2 orbifold thereof

F.1 Generalities on the conifold geometry

The singular conifold C_0 can be defined as an affine variety in $\mathbb{C}^4 \cong \{z_1, z_2, z_3, z_4\}$,

$$z_1 z_2 - z_3 z_4 = 0 . \quad (\text{F.1})$$

By a linear change of coordinates, this can also be written as: $w_1^2 + w_2^2 + w_3^2 + w_4^2 = 0$. The conifold is a CY cone, whose base is a Sasaki-Einstein manifold called $T^{1,1}$ [104]. The latter is described algebraically by the intersection of the cone with a unit sphere in \mathbb{C}^4 : $\sum_{i=1}^4 |w_i|^2 = 1$. In terms of real coordinates, $w_i = x_i + iy_i$, one gets $\vec{x} \cdot \vec{x} = 1/2$, $\vec{y} \cdot \vec{y} = 1/2$, $\vec{x} \cdot \vec{y} = 0$, which can be seen as an S^2 fibration over S^3 . However such a fibration is trivial¹, so that topologically $T^{1,1} \cong S^2 \times S^3$. The following coordinate system on the cone will be useful²

$$z_1 = r^{3/2} e^{\frac{i}{2}(\psi + \phi_1 + \phi_2)} \sin \frac{\theta_1}{2} \sin \frac{\theta_2}{2}, \quad (\text{F.2})$$

$$z_2 = r^{3/2} e^{\frac{i}{2}(\psi - \phi_1 - \phi_2)} \cos \frac{\theta_1}{2} \cos \frac{\theta_2}{2}, \quad (\text{F.3})$$

$$z_3 = r^{3/2} e^{\frac{i}{2}(\psi - \phi_1 + \phi_2)} \cos \frac{\theta_1}{2} \sin \frac{\theta_2}{2}, \quad (\text{F.4})$$

$$z_4 = r^{3/2} e^{\frac{i}{2}(\psi + \phi_1 - \phi_2)} \sin \frac{\theta_1}{2} \cos \frac{\theta_2}{2}. \quad (\text{F.5})$$

Here, $0 \leq \psi \leq 4\pi$, $0 \leq \phi_i \leq 2\pi$, $0 \leq \theta_i \leq \pi$, and we have the following angular periodicities

$$\begin{pmatrix} \psi \\ \phi_1 \\ \phi_2 \end{pmatrix} \simeq \begin{pmatrix} \psi + 4\pi \\ \phi_1 \\ \phi_2 \end{pmatrix} \simeq \begin{pmatrix} \psi + 2\pi \\ \phi_1 + 2\pi \\ \phi_2 \end{pmatrix} \simeq \begin{pmatrix} \psi + 2\pi \\ \phi_1 \\ \phi_2 + 2\pi \end{pmatrix}. \quad (\text{F.6})$$

¹We can cover S^3 with two patches, intersecting at the equator. The bundle is constructed by specifying a transition function on this equator (itself an S^2), which is a map from S^2 to $SO(3)$, the structure group of the fiber. Such maps are always trivial ($\pi_2(SO(3)) = 0$), so the bundle is trivial.

²Remark that we differ from the conventions of [118] by a flip in the orientation of the angles ϕ_i .

In these coordinates, the Calabi-Yau metric reads: $ds_{C_0}^2 = dr^2 + r^2 ds_{T^{1,1}}^2$, with the Sasaki-Einstein metric of $T^{1,1}$

$$ds_{T^{1,1}}^2 = \sum_{i=1,2} \frac{1}{6} (d\theta_i^2 + \sin^2 \theta_i d\phi_i^2) + \frac{1}{9} (d\psi - \sum_{i=1,1} \cos \theta_i d\phi_i)^2. \quad (\text{F.7})$$

It describes a circle bundle, where the circle ψ is fibered over $S^2 \times S^2$. In terms of the natural vielbein for the two 2-spheres, $u_i = d\theta_i$, $v_i = \sin \theta_i d\phi_i$ ($i = 1, 2$), it is useful to define rotated vielbein for the 2-spheres [335]

$$\begin{pmatrix} \sigma_1 \\ \sigma_2 \end{pmatrix} = \begin{pmatrix} \cos \frac{\psi}{2} & -\sin \frac{\psi}{2} \\ \sin \frac{\psi}{2} & \cos \frac{\psi}{2} \end{pmatrix} \begin{pmatrix} u_1 \\ v_1 \end{pmatrix} \quad \begin{pmatrix} \Sigma_1 \\ \Sigma_2 \end{pmatrix} = \begin{pmatrix} \cos \frac{\psi}{2} & -\sin \frac{\psi}{2} \\ \sin \frac{\psi}{2} & \cos \frac{\psi}{2} \end{pmatrix} \begin{pmatrix} u_2 \\ v_2 \end{pmatrix}. \quad (\text{F.8})$$

Let us also define $\zeta = d\psi - \sum_{i=1,2} \cos \theta_i d\phi_i$. For the singular conifold, we will use the following ordered vielbein

$$\left\{ e^r = dr, e^\psi = \frac{r}{3} \zeta, e^1 = \frac{r}{\sqrt{6}} \sigma_1, e^2 = \frac{r}{\sqrt{6}} \sigma_2, e^3 = \frac{r}{\sqrt{6}} \Sigma_1, e^4 = \frac{r}{\sqrt{6}} \Sigma_2 \right\}. \quad (\text{F.9})$$

The metric of the conifold then reads $ds_{C_0}^2 = \sum_{n=1}^6 (e^n)^2$, and the volume form is

$$d\text{vol}_{C_0} = e^r \wedge e^\psi \wedge e^1 \wedge e^2 \wedge e^3 \wedge e^4 = \frac{1}{108} r^5 dr \wedge d\psi \wedge d\theta_1 \wedge \sin \theta_1 d\phi_1 \wedge d\theta_2 \wedge \sin \theta_2 d\phi_2. \quad (\text{F.10})$$

A complex vielbein can be defined as

$$\left\{ E^1 = e^1 + ie^2, E^2 = e^3 + ie^4, E^3 = e^r + ie^\psi \right\}. \quad (\text{F.11})$$

In terms of this complex structure, the Kähler form is

$$J \equiv \frac{i}{2} \left(E^1 \wedge \overline{E^1} + E^2 \wedge \overline{E^2} + E^3 \wedge \overline{E^3} \right) = d \left(\frac{r^2}{6} \zeta \right), \quad (\text{F.12})$$

which is $(1, 1)$, closed and satisfies $J \wedge J \wedge J = 6 d\text{vol}_{C_0}$. It is exact, since we are at the zero resolution point in Kähler moduli space where the cohomology class of J is trivial. The holomorphic top form is

$$\Omega^{(3,0)} \equiv E^1 \wedge E^2 \wedge E^3 = -\frac{4}{9} \frac{dz_1 \wedge dz_2 \wedge dz_3}{z_3}. \quad (\text{F.13})$$

Let us now review 2- and 3-(co)cycles for the conifold. We have the closed $(1, 1)$ -form

$$\begin{aligned} \omega_2^{CF} &\equiv \frac{3i}{2r^2} \left(E^1 \wedge \overline{E^1} - E^2 \wedge \overline{E^2} \right) = \frac{1}{2} (\sigma_1 \wedge \sigma_2 - \Sigma_1 \wedge \Sigma_2) = \\ &= \frac{1}{2} (\sin \theta_1 d\theta_1 \wedge d\phi_1 - \sin \theta_2 d\theta_2 \wedge d\phi_2). \end{aligned} \quad (\text{F.14})$$

The 2-cycle in $T^{1,1}$ is topologically a 2-sphere \mathcal{C}_{CF} . It can be represented by

$$\mathcal{C}_{CF} : \quad \theta_1 = \theta_2 \equiv \theta, \quad \phi_1 = 2\pi - \phi_2 \equiv \phi, \quad \psi = 0, \quad \phi \in [0, 2\pi), \theta \in (0, \pi). \quad (\text{F.15})$$

It turns out that $\int_{C_{CF}} \omega_2^{CF} = 4\pi$. In addition, one usually defines the real closed 3-form

$$\omega_3^{CF} \equiv \zeta \wedge \omega_2^{CF} , \tag{F.16}$$

which is the real part of the imaginary-self-dual (ISD) primitive (2,1)-form

$$\omega^{(2,1)} \equiv \frac{9}{2r^3} E^3 \wedge \left(E^1 \wedge \overline{E^1} - E^2 \wedge \overline{E^2} \right) = \left(\zeta - 3i \frac{dr}{r} \right) \wedge \omega_2^{CF} , \tag{F.17}$$

defined on the whole conifold. Imaginary self-duality means that $*_6 \omega^{(2,1)} = i \omega^{(2,1)}$. The 3-cycle in $T^{1,1}$ has the topology of a 3-sphere. We call it A_{CF} . It can be represented by

$$A_{CF} : \quad \theta_2 = \phi_2 = 0 . \tag{F.18}$$

Its orientation is such that $\int_{A_{CF}} \omega_3^{CF} = 8\pi^2$.

F.2 The orbifolded conifold geometry

In this appendix, we derive the results presented in section 7.2 concerning the relation between the ranks in the quiver, the cycles wrapped by the different fractional branes, and the fluxes present in the supergravity solution. In order to do this, we need first to discuss in detail the compact 2-cycles of the geometry, on which the branes can wrap. Then we discuss the compact 3-cycles of the geometry, which support the RR fluxes sourced by the branes, and their intersections with the 2-cycles (in the base of the singular cone). This will allow us to write the 3-form fluxes directly in terms of the ranks of the gauge groups in the quiver.

The CY singularity on which our gauge theory is engineered is a non-chiral \mathbb{Z}_2 orbifold of the conifold (F.1), obtained considering the following action on the coordinates z_i in \mathbb{C}^4

$$\Theta : \quad (z_1, z_2, z_3, z_4) \rightarrow (z_1, z_2, -z_3, -z_4) . \tag{F.19}$$

The orbifold geometry is still an algebraic variety. To describe it one can introduce a complete set of invariants: $x \equiv z_3^2$, $y \equiv z_4^2$ and $t \equiv z_3 z_4$, which satisfy the constraint $xy = t^2$. The conifold equation is rewritten as $t = z_1 z_2$ so that t can be eliminated and we are left with

$$f = (z_1 z_2)^2 - xy = 0 . \tag{F.20}$$

The singular locus $f = df = 0$ consists of two complex lines that meet at the tip of the geometry $\{z_1 = z_2 = x = y = 0\}$, and corresponds to the fixed point locus of the orbifold action Θ .

One can use real coordinates as well, those already defined in appendix F.1. The orbifold action (F.19), which is an identification in the covering space, where we will work, reads

$$\Theta : \quad (\phi_1, \phi_2) \rightarrow (\phi_1 - \pi, \phi_2 + \pi) . \tag{F.21}$$

The two complex lines, that we call the p and q line respectively, are defined, in complex and real coordinates, as

$$p = \{z_1 = x = y = 0, \forall z_2\} = \{\theta_1 = \theta_2 = 0, \forall r, \psi'\} \tag{F.22}$$

$$q = \{z_2 = x = y = 0, \forall z_1\} = \{\theta_1 = \theta_2 = \pi, \forall r, \psi''\} , \tag{F.23}$$

where $\psi' = \psi - \phi_1 - \phi_2$ and $\psi'' = \psi + \phi_1 + \phi_2$ are (well defined) angular coordinates along the singularity lines. In a neighborhood of the singular lines (and outside the tip) the geometry looks locally like the A_1 -singularity $\mathbb{C} \times \mathbb{C}^2/\mathbb{Z}_2$. The fixed point curve p sits at the north poles of both S^2 's while the curve q sits at the south poles.

2-cycles and resolutions

From the above analysis it follows that the singular geometry has three vanishing 2-cycles. Two of these three cycles arise due to the orbifold action; such exceptional 2-cycles are located all along the $\mathbb{C}^2/\mathbb{Z}_2$ singular lines p and q (F.22), and we call them \mathcal{C}_2 and \mathcal{C}_4 , respectively. Locally, one could resolve the space into an ALE space fibered over \mathbb{C}^* . The third relevant 2-cycle descends from the 2-cycle of the double covering conifold geometry, whose base $T^{1,1}$ is topologically $S^2 \times S^3$.

Our goal in what follows is to pinpoint the precise map between vanishing 2-cycles, wrapped D5-branes, 3-form RR fluxes and quiver rank assignments. To this end, it will prove useful to take advantage of our CY cone being a toric variety³, since in this case one can use standard techniques to understand the structure of 2-cycles and their intersections. Let us sketch how this comes about.

A toric variety can be described as the moduli space of an associated supersymmetric gauged linear σ -model (GLSM). Consider n chiral superfields $t_i, i = 1 \dots n$ charged under a product of abelian gauge groups $U(1)^s$, with charges $Q_a^i, a = 1 \dots s$. In the absence of a superpotential, the potential for the scalar components is

$$V(t_i) = \sum_{a=1}^s \left(\sum_{i=1}^n Q_a^i |t_i|^2 - \xi_a \right)^2 . \tag{F.24}$$

where ξ_a are Fayet-Iliopoulos parameters (FI). The moduli space of vacua \mathcal{M} is given by the D-flatness equations modulo $U(1)^s$ gauge transformations

$$\mathcal{M} = \left\{ t_i \in \mathbb{C}^n \mid \sum_{i=1}^n Q_a^i |t_i|^2 = \xi_a \quad \forall a = 1, \dots, s \right\} / U(1)^s , \tag{F.25}$$

where $U(1)^s$ acts as $t_i \rightarrow e^{i Q_a^i \phi^a} t_i$. When the FI's are such that $\dim \mathcal{M} = n - s, \mathcal{M}$ is the desired toric variety (and $n - s = r$ is just the number of isometry abelian factors). Putting the FI's to zero the variety, if admissible, is scale invariant: this corresponds to a cone. As the FI's change, the Kähler moduli of \mathcal{M} also change and one gets resolutions or blow-ups. Generically, different regions in the parameter space of the FI parameters correspond to different resolutions, delimited by flop transition curves.

In our case the GLSM has six fields t_i whose charges Q_a^i are reported in the table below

t_1	t_2	t_3	t_4	t_5	t_6	
0	0	1	-2	1	0	ξ_2
1	-1	0	1	-1	0	ξ_β
-2	1	0	0	0	1	ξ_4

(F.26)

³A toric manifold is a manifold of complex dimension r which admits an isometry group (at least as big as) $U(1)^r$. A toric CY threefold is then a CY threefold whose isometry group is at least $U(1)^3$. For a recent introduction, see e.g. [288].

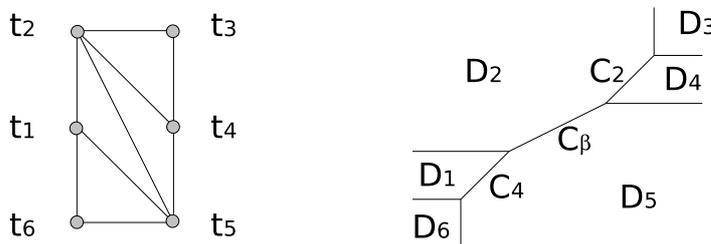


Figure F.1: The toric diagram and the dual (p, q) -web. The specific toric diagram triangulation is the one related to having all $\xi_a > 0$ in the associated GLSM.

We can parameterize the toric variety with the gauge invariants

$$t_3 t_4 t_5 = z_1 \quad t_1 t_2 t_6 = z_2 \quad t_1 t_2^2 t_3^2 t_4 = x \quad t_1 t_4 t_5^2 t_6^2 = y \quad (\text{F.27})$$

which, consistently, satisfy the defining equation (F.20). We can also give a parametrization for the so-called toric divisors, which are the four-dimensional hypersurfaces in the toric CY defined by $D_i = \{t_i = 0\}$. We recognize $D_4 = \{z_1 = x = y = 0\}$ as the p line and $D_1 = \{z_2 = x = y = 0\}$ as the q -line.

The toric diagram and the related (p, q) -web corresponding to choosing all $\xi_a > 0$ (which amounts to a given triangulation of the toric diagram) are depicted in Figure F.1. For the particular resolution corresponding to $\xi_2, \xi_\beta, \xi_4 > 0$ the three holomorphic 2-cycles can be directly read from the (p, q) -web. They can be explicitly constructed as intersections of toric divisors

$$\mathcal{C}_2 = D_2 \cdot D_4 \quad \mathcal{C}_\beta = D_2 \cdot D_5 \quad \mathcal{C}_4 = D_1 \cdot D_5 . \quad (\text{F.28})$$

This can be explicitly checked using D-term equations, which for the intersections of interest are

$$D_2 D_4 : \quad |t_3|^2 + |t_5|^2 = \xi_2 \quad |t_6|^2 = 2|t_1|^2 + \xi_4 \quad |t_1|^2 = |t_5|^2 + \xi_\beta \quad (\text{F.29})$$

$$D_2 D_5 : \quad |t_4|^2 + |t_1|^2 = \xi_\beta \quad |t_3|^2 = 2|t_4|^2 + \xi_2 \quad |t_6|^2 = 2|t_1|^2 + \xi_4 \quad (\text{F.30})$$

$$D_1 D_5 : \quad |t_2|^2 + |t_6|^2 = \xi_4 \quad |t_3|^2 = 2|t_4|^2 + \xi_2 \quad |t_4|^2 = |t_2|^2 + \xi_\beta . \quad (\text{F.31})$$

As one can see, each \mathcal{C}_i topologically is a $\mathbb{C}P^1$ (parameterized by the first two variables in each row) of volume ξ_i .

Let us consider also another basis of 2-cycles, which arises in a different resolution of the singular conical geometry (corresponding to a different triangulation of the toric diagram). Consider the region in the space of FI parameters where $\xi_\beta < 0$ with $\xi_2 + \xi_\beta > 0$ and $\xi_4 + \xi_\beta > 0$. We can introduce

$$\xi_1 = \xi_4 + \xi_\beta > 0 \quad \xi_3 = \xi_2 + \xi_\beta > 0 \quad \xi_\alpha = -\xi_\beta > 0 . \quad (\text{F.32})$$

This new resolution can be obtained from the one in Figure F.1 with a flop transition on $\mathcal{C}_\beta \leftrightarrow \mathcal{C}_\alpha$. The toric diagram triangulation and the corresponding dual (p, q) -web for the new geometry are sketched in Figure F.2. In order to have a nice presentation of the

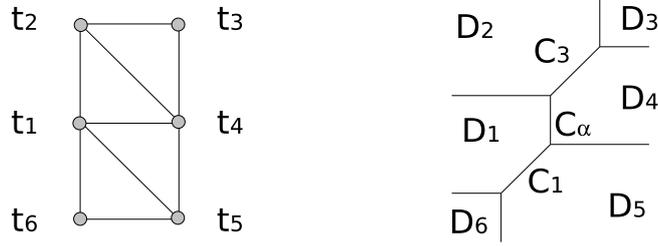


Figure F.2: The toric diagram and the dual (p, q) -web in the region of the FI parameter space where $\xi_\beta < 0$.

GLSM charges in terms of the new positive FI's, we can linearly re-shuffle Table (F.26) getting

$$\begin{array}{cccccc|c}
 t_1 & t_2 & t_3 & t_4 & t_5 & t_6 & \\
 \hline
 -1 & 0 & 0 & 1 & -1 & 1 & \xi_1 \\
 -1 & 1 & 0 & -1 & 1 & 0 & \xi_\alpha \\
 1 & -1 & 1 & -1 & 0 & 0 & \xi_3
 \end{array} \tag{F.33}$$

Repeating the same analysis as before one finds the holomorphic⁴ 2-cycles in this new resolution in terms of toric divisors

$$\mathcal{C}_3 = D_2 \cdot D_4 \quad \mathcal{C}_\alpha = D_1 \cdot D_4 \quad \mathcal{C}_1 = D_1 \cdot D_5 . \tag{F.34}$$

Again the FI parameters are the positive volumes of the corresponding 2-cycles \mathcal{C}_i . From the relations among FI parameters we read the relations

$$\mathcal{C}_1 = \mathcal{C}_4 + \mathcal{C}_\beta \quad \mathcal{C}_3 = \mathcal{C}_2 + \mathcal{C}_\beta , \tag{F.35}$$

which can be thought of as relations in homology between vanishing cycles.

A comment is in order at this point. In this non-chiral case, vanishing 2-cycles are in one-to-one correspondence with possible fractional branes. All the divisors are non compact 4-cycles. This implies that all dual 2-cycles support non-anomalous fractional branes. This does not hold in general, as only 2-cycles dual to non-compact 4-cycles give anomaly-free fractional branes, their number being equal to the number of 3-cycles in the real base of the CY cone (which in turn corresponds to the number of baryonic charges). This is the geometric counterpart of the dual gauge theory being non-chiral. Conversely, chiral theories are related to CY cones where there are compact 4-cycles around. The latter put constraints on the allowed fractional D3-branes configurations, because of the RR tadpole cancellation condition.

Once we wrap a D5-brane on a 2-cycle, it will thus source a 3-form RR flux. We turn to consider the compact 3-cycles of the geometry which can support this flux, and their dual non-compact 3-cycles.

⁴Notice that generically if an homology class \mathcal{C} has a holomorphic representative, $-\mathcal{C}$ does not because the representative becomes antiholomorphic and one should look for a different one. In particular, in different resolutions the rôle of homology classes with a holomorphic representative is exchanged.

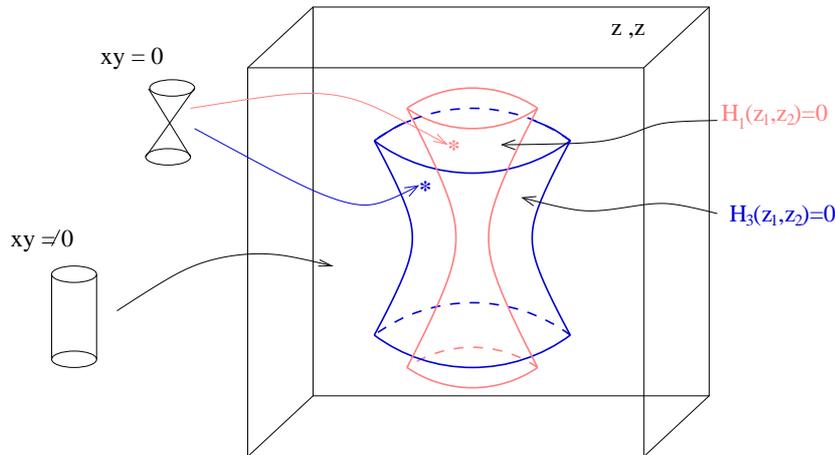


Figure F.3: The 6-dimensional manifold seen as a singular \mathbb{C}^* fibration over the (z_1, z_2) space. The surfaces $H_k(z_1, z_2) = z_1 z_2 - \epsilon_k = 0$, $k = 1, 3$, are the loci where the \mathbb{C}^* fiber degenerates to a cone $xy = 0$ and a non-trivial S^1 shrinks.

3-cycles and deformations

The study of compact and non-compact 3-cycles is best performed in a regular geometry obtained by complex deformation of the singular space, rather than by resolution (which is a Kähler deformation).

The algebraic variety (F.20) admits two normalizable complex deformations parameterized by ϵ_1 and ϵ_3 [182]

$$f = (z_1 z_2 - \epsilon_1)(z_1 z_2 - \epsilon_3) - xy = 0 . \tag{F.36}$$

The deformed geometry is regular for $\epsilon_1 \neq \epsilon_3$, provided $\epsilon_1 \epsilon_3 \neq 0$. For $\epsilon_1 = \epsilon_3 \neq 0$ it still has a \mathbb{C}^* line of A_1 singularities (locally $\mathbb{C} \times \mathbb{C}^2/\mathbb{Z}_2$) and corresponds to a \mathbb{Z}_2 orbifold of the deformed conifold. For $\epsilon_3 = 0$ it has a conifold singularity at the tip.

A convenient way to visualize the geometry is to regard (F.36) as a singular \mathbb{C}^* fibration over $\mathbb{C}^2 \simeq (z_1, z_2)$

$$xy = H_1(z_1, z_2) H_3(z_1, z_2) \quad \text{with} \quad H_k(z_1, z_2) = z_1 z_2 - \epsilon_k . \tag{F.37}$$

At any point (z_1, z_2) where $H_1(z_1, z_2)H_3(z_1, z_2) \neq 0$ the fiber has equation $xy = c \neq 0$ and is a copy of \mathbb{C}^* . On each surface $H_k(z_1, z_2) = 0$ the fiber degenerates to a cone $xy = 0$ and an S^1 shrinks. On the other hand, each surface $H_k(z_1, z_2) = 0$ is an hyperboloid in \mathbb{C}^2 and has the topology of \mathbb{C}^* . For a general deformation, $\epsilon_1 \neq \epsilon_3$, they are disjoint and never touch. When $\epsilon_1 = \epsilon_3$ they degenerate one on top of the other, while when one deformation parameter vanishes the corresponding hyperboloid degenerates into a cone. See Figure F.3 for a picture of the geometry.

Figure F.3 is very useful to visualize compact and non-compact 3-cycles as well as 2-cycles in the deformed geometry. Any line segment of real dimension one in the \mathbb{C}^2 space (z_1, z_2) which begins and ends on the locus $xy = 0$ represents a closed submanifold of real dimension two, obtained by fibering on that segment an S^1 which lives in the $\mathbb{C}_{x,y}^*$

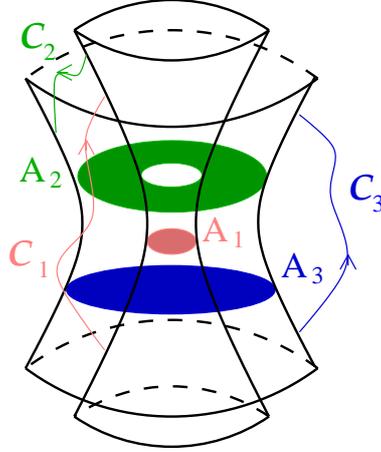


Figure F.4: The projection of the A and C cycles in the (x, y) space. The non-compact B -cycles are obtained as C -cycles fibers over r .

cylinder and shrinks to zero at the endpoints. When the line segment is non-contractible (keeping the endpoints on the $xy = 0$ locus), it represents a non-trivial element in the homology group $H_2(\mathcal{M}, \mathbb{Z})$. In the same way, a real dimension two surface with boundary on the $xy = 0$ locus gives rise to a closed dimension three submanifold after the S^1 has been fibered on it. When the surface is non-contractible (keeping the boundary on the $xy = 0$ locus), it gives rise to a non-trivial 3-cycle. Compact 3-cycles A_i arise from compact surfaces while non-compact 3-cycles B_i arise from non-compact surfaces.

In Figure F.4 we depicted the various 2-cycles C_i and compact 3-cycles A_i for the deformed orbifolded conifold. We have used the basis which is most natural when complex deformations are concerned. Non-compact 3-cycles B_i are easily obtained as well: the real dimension two base surfaces are non-compact “vertical” foils with one or two boundaries on the degeneration loci, and are related to the line segment supporting the 2-cycles C_i .

In the regular deformed geometry, a canonical symplectic basis for the third homology group $H_3(\mathcal{M}, \mathbb{Z})$ is given by $\{A_1, A_3, B_1, B_3\}$ with intersection numbers $A_i \cdot B_j = \delta_{ij}$. A_1 and A_3 have topology S^3 while B_1 and B_3 have topology \mathbb{R}^3 . One can also consider a linear combination of them, $A_2 = A_1 - A_3$ (see Figure F.4) and its dual $B_2 = -B_1 + B_3$: they have intersection number $A_2 \cdot B_2 = -2$.

The asymptotic behavior of supergravity solutions based on these spaces is fixed, among other parameters, by the D5-charges at infinity. These are constructed by integrating suitable currents on the 3-cycles in radial sections of the asymptotically conical geometry. This is equivalent to considering any radial section in the singular conical geometry ($\epsilon_1 = \epsilon_3 = 0$). The latter perspective is useful because from any 3-cycle in a radial section we can construct a non-compact conical 4-cycle having the 3-cycle as its radial section: this allows us to introduce a concept of holomorphy and to use toric divisors instead of 3-cycles in radial sections.

From the GLSM description we know that the number of compact 3-cycles in radial sections (which equals the number of baryonic charges and the number of non-anomalous fractional branes) is three. For concreteness we choose the following basis: A_2, A_4 and

A_{CF} . A_2 is the radial section of the toric divisor D_4 , and corresponds to the product of the exceptional 2-cycle \mathcal{C}_2 along the p -line (which is $\cong \mathbb{C}^*$) with S^1 in the latter; in the same way, A_4 is the radial section of the toric divisor D_1 , and is the product of the exceptional \mathcal{C}_4 along the q -line times S^1 . A_{CF} is the compact 3-cycle of the covering space conifold⁵: under the orbifold action it has an image, and no fixed points. In particular, the representative 3-cycle at $\theta_2 = \pi/2$ and $\phi_2 = 0$ is mapped to the divisor $\{x = z_1^2, y = z_2^2\}$ which has the GLSM description $t_1 t_2^2 = t_4 t_5^2$. Comparing the charges we find that A_{CF} corresponds to the toric divisor $D_1 + 2D_2 = D_4 + 2D_5$. Summarizing, our basis of 3-cycles and the corresponding toric divisors are

$$A_2 \simeq D_4 \qquad A_4 \simeq D_1 \qquad A_{CF} \simeq D_1 + 2D_2 = D_4 + 2D_5 . \qquad (\text{F.38})$$

Notice that in the deformed geometry $A_2 = -A_4$ in homology. Nevertheless they can give rise to different charges when explicit sources are present in the geometry and this is in fact the case of $\mathcal{N} = 2$ branes which do not undergo complete geometric transition.

In order to compute the 3-form fluxes generated by D5-branes wrapped on 2-cycles, we will need the intersection matrix between divisors and 2-cycles. In our basis we find

	$A_2 \simeq D_4$	$A_4 \simeq D_1$	A_{CF}	
\mathcal{C}_2	-2	0	0	(F.39)
\mathcal{C}_4	0	-2	0	
\mathcal{C}_β	1	1	-1	

This table is computed from the charges in Table (F.26): in the GLSM construction each gauge field gives rise to an element \mathcal{C}_a of the homology group $H_2(\mathcal{M}, \mathbb{Z})$, and the intersection between it and a toric divisor D_i is the charge Q_a^i .

The fractional branes/ranks correspondence

We have now all the ingredients to finally figure out the precise correspondence between fractional branes (that is wrapped D5-branes) and quiver rank assignments.

Consider a D5-brane wrapped on a 2-cycle \mathcal{C}_i of our CY_3 . The Bianchi identity for F_3 is violated by the source

$$dF_3 = -2\kappa^2 \tau_5 \Omega_4 , \qquad (\text{F.40})$$

where Ω_4 is a 4-form with δ -function support on the D5 world-volume. We are interested in the flux generated on a 3-cycle A_j in the radial section. First we have to resolve the geometry, switching on the FI parameters of the associated GLSM. This does not change the holomorphic data nor the quantized charges. Then we identify a non-compact divisor D_j which has A_j as radial section. Being the geometry smooth, A_j turns out to be the boundary of D_j

$$\int_{A_j} F_3 = - \int_{D_j} dF_3 = 2\kappa^2 \tau_5 \int_{D_j} \Omega_4 = 2\kappa^2 \tau_5 (D_j, \mathcal{C}_i) , \qquad (\text{F.41})$$

where (D_j, \mathcal{C}_i) is the intersection number as in Table (F.39), and we fixed the orientation ambiguity requiring consistency with known cases, such as the conifold and the \mathbb{Z}_2 orbifold

⁵Actually $A_{CF} = A_1 + A_3$.

of \mathbb{R}^6 . If there is a holomorphic representative for \mathcal{C}_i , we can then directly compute the intersection from the GLSM data.

The last thing to determine are the quiver rank assignments corresponding to each fractional brane. A D5-brane wrapped on the exceptional 2-cycles \mathcal{C}_2 and \mathcal{C}_4 along the $\mathbb{C}^2/\mathbb{Z}_2$ lines p and q gives rise to an $\mathcal{N} = 2$ fractional brane, and we conventionally choose the rank assignments to be, respectively, $(0, 1, 1, 0)$ and $(1, 1, 0, 0)$. The rank assignment for a D5-brane wrapped on \mathcal{C}_β can be defined by observing that the combination $\mathcal{C}_{CF} = 2\mathcal{C}_\beta + \mathcal{C}_2 + \mathcal{C}_4$ does not couple to twisted fields and gives rise to the orbifold of the Klebanov-Tseytlin theory [127], see Table (F.39). This implies that the corresponding gauge theory is the orbifold of the KT theory. We can say that the ranks for one D5 on \mathcal{C}_β are (a, b, c, d) . Requiring that $2\mathcal{C}_\beta + \mathcal{C}_2 + \mathcal{C}_4$ is in the class $(N + 1, N, N + 1, N)$ or $(N, N + 1, N, N + 1)$, which do correspond to the orbifold of the KT theory, singles out two possibilities for \mathcal{C}_β : either $(1, 0, 1, 1)$ or $(0, 0, 0, 1)$. To select the correct option we should consider the induced D3-charge on the fractional D3 probe.

The induced D3-charge is proportional to the integral of B_2 (or more generally of $\mathcal{F} = B_2 + 2\pi\alpha'F_2$) on the corresponding 2-cycle \mathcal{C} :

$$Q_3 = \tau_5 \int_{\mathcal{C}} \mathcal{F} = \tau_3 \frac{1}{4\pi^2\alpha'} \int_{\mathcal{C}} (B_2 + 2\pi\alpha'F) . \tag{F.42}$$

The actual value depends on the background value of B_2 . This is arbitrary at this level (and it is related to the UV cut-off values of the gauge couplings in the dual gauge theory). We only require these background values to be positive (so as to describe mutually BPS objects) and less than one (in order to describe non-composite, that is elementary, objects). Along the p and q lines the physics is locally $\mathbb{C}^2/\mathbb{Z}_2$, thus we can naturally set [39]: $\int_{\mathcal{C}_2} B_2 = \int_{\mathcal{C}_4} B_2 = (4\pi^2\alpha')/2$. If we consider the KT theory and set also [127] $\int_{\mathcal{C}_{CF}} B_2 = (4\pi^2\alpha')/2$, then using the previous relation $\mathcal{C}_{CF} = 2\mathcal{C}_\beta + \mathcal{C}_2 + \mathcal{C}_4$, we get $\int_{\mathcal{C}_\beta} B_2 = -(4\pi^2\alpha')/4$.

This implies that while the $\mathcal{N} = 2$ branes have positive D3-charge, a D5-brane wrapped on \mathcal{C}_β has negative D3-brane charge and it is not mutually BPS. Putting one unit of worldvolume flux on the wrapped D5 we get positive D3-charge: $3/4$. The total D3-charge for $\mathcal{C}_{CF} = 2\mathcal{C}_\beta + \mathcal{C}_2 + \mathcal{C}_4$ (with two units of flux on \mathcal{C}_β) is $5/2$. This is exactly the D3-charge of the configuration $(3, 2, 3, 2)$, which implies that one D5-brane wrapped on \mathcal{C}_β with one unit of worldvolume flux gives rise to the theory $(1, 0, 1, 1)$. A similar analysis shows that a D5-brane wrapped on $\mathcal{C}_\alpha = -\mathcal{C}_\beta$ (with no background world-volume flux) corresponds to a rank assignment $(0, 1, 0, 0)$. Finally, direct application of Table (F.39) tells us what the fluxes sourced by D5-branes wrapped on any 2-cycles are.

Our findings are summarized in the Table below

	$-\int_{A_2} F_3$	$-\int_{A_4} F_3$	$-\int_{A_{CF}} F_3$	D3-charge	gauge theory
D5 on \mathcal{C}_2	2	0	0	1/2	(0,1,1,0)
D5 on \mathcal{C}_4	0	2	0	1/2	(1,1,0,0)
D5 on \mathcal{C}_β	-1	-1	1	3/4	(1,0,1,1)
D5 on \mathcal{C}_α	1	1	-1	1/4	(0,1,0,0)

(F.43)

where fluxes are in units of $4\pi^2\alpha'g_s$.

As anticipated, we will use D5 branes wrapped on \mathcal{C}_2 , \mathcal{C}_4 and $\mathcal{C}_\alpha = -\mathcal{C}_\beta$ without worldvolume flux as a basis for fractional branes to discuss our gauge/gravity duality.

This is the most natural basis for discussing rank assignments parametrized as in Figure 7.1, where fractional branes modify the ranks of the first three quiver nodes only.

F.3 Poisson equation on the singular conifold

The Poisson equation for the warp factor on the conifold reads

$$\left[\frac{1}{r^5} \partial_r r^5 \partial_r + \frac{1}{r^2} \sum_{i=1}^2 \left[\frac{6}{\sin \theta_i} \partial_{\theta_i} \sin \theta_i \partial_{\theta_i} + 6 \left(\frac{1}{\sin \theta_i} \partial_{\phi_i} - \cot \theta_i \partial_{\psi} \right)^2 \right] + \frac{9}{r^2} \partial_{\psi}^2 \right] h = \frac{C}{r^6} \delta' s \quad (\text{F.44})$$

where the RHS is the same as in (7.18). Due to the symmetries of the configuration with $\mathcal{N} = 2$ branes at the tip, the ansatz for the warp factor does not depend of ψ and ϕ_i . Then we are left with

$$\left[\frac{1}{r^5} \partial_r r^5 \partial_r + \frac{1}{r^2} \sum_{i=1}^2 \frac{6}{\sin \theta_i} \partial_{\theta_i} \sin \theta_i \partial_{\theta_i} \right] h = \frac{C}{r^6} \delta' s . \quad (\text{F.45})$$

Following [147], we propose an ansatz

$$h = \frac{1}{r^4} g(t, \theta_1, \theta_2) \quad t = \log \frac{r}{r_0} \quad (\text{F.46})$$

with which the Laplacian simplifies to

$$\Delta h = \frac{1}{r^6} \left\{ -4 \partial_t g + \partial_t^2 g + \sum_{i=1}^2 \frac{6}{\sin \theta_i} \partial_{\theta_i} \sin \theta_i \partial_{\theta_i} g \right\} . \quad (\text{F.47})$$

Some solutions are $g = Q + At - Cf(\theta_1, \theta_2)$ and the equation reduces to

$$-C \sum_{i=1}^2 \frac{6}{\sin \theta_i} \partial_{\theta_i} \sin \theta_i \partial_{\theta_i} f = 4A + C \delta' s . \quad (\text{F.48})$$

The constant Q is related to a $\delta(r)$ that is the number of D3-branes at the tip. In [147] a constraint relation between A and C is found, which amounts to charge cancellation on the compact angular sections. We will not care about it here, and simply try to find solutions.

It will prove useful to introduce Legendre polynomials, which are eigenfunctions of the angular Laplacian⁶

$$\Delta_{ang} = \sum_{i=1}^2 \frac{6}{\sin \theta_i} \partial_{\theta_i} \sin \theta_i \partial_{\theta_i} = 6 \sum_{i=1}^2 \partial_{\cos \theta_i} (1 - \cos^2 \theta_i) \partial_{\cos \theta_i} \quad (\text{F.49})$$

$$\Delta_{ang} P_n(\cos \theta_i) = -6n(n+1) P_n(\cos \theta_i) \quad (i = 1, 2) . \quad (\text{F.50})$$

The last formula follows from the differential equation

$$(1 - x^2) P_n''(x) - 2x P_n'(x) + n(n+1) P_n(x) = 0 . \quad (\text{F.51})$$

⁶We only write the relevant part including derivatives with respect to θ_i .

The eigenfunctions of the angular Laplacian on the conifold are products of Legendre polynomials

$$\Delta_{ang} P_{l_1}(\cos \theta_1) P_{l_2}(\cos \theta_2) = -6[l_1(l_1 + 1) + l_2(l_2 + 1)] P_{l_1}(\cos \theta_1) P_{l_2}(\cos \theta_2). \quad (\text{F.52})$$

The product of δ -functions is easily written as

$$4\delta(1 - \cos \theta_1)\delta(1 - \cos \theta_2) = \sum_{l_1=0}^{\infty} (2l_1 + 1) P_{l_1}(\cos \theta_1) \sum_{l_2=0}^{\infty} (2l_2 + 1) P_{l_2}(\cos \theta_2). \quad (\text{F.53})$$

Then the solution we are looking for is

$$f = \frac{1}{24} \sum_{l_1, l_2 \neq (0,0)}^{\infty} \frac{(2l_1 + 1)(2l_2 + 1)}{l_1(l_1 + 1) + l_2(l_2 + 1)} P_{l_1}(\cos \theta_1) P_{l_2}(\cos \theta_2), \quad (\text{F.54})$$

where this last sum excludes $(l_1, l_2) = (0, 0)$. One gets

$$\Delta_{ang} f = -\delta(1 - \cos \theta_1)\delta(1 - \cos \theta_2) + \frac{1}{4}. \quad (\text{F.55})$$

F.4 Periods of Ω

Here we provide some details on the computation of the periods of Ω in the deformed orbifolded conifold. A general expression for the holomorphic 3-form is given by

$$\Omega \propto \frac{1}{2\pi i} \oint_{P=0} \frac{dw_1 \wedge dw_2 \wedge dw_3 \wedge dw_4}{P} = \frac{dw_1 \wedge dw_2 \wedge dw_3}{\partial P / \partial w_4}, \quad (\text{F.56})$$

where $P[w]$ is the polynomial equation defining the geometry. We take

$$P = xy - (u^2 - v^2 + \epsilon_1)(u^2 - v^2 + \epsilon_3) = 0. \quad (\text{F.57})$$

The geometry is described as in appendix F.2: the cylinder $xy = \text{const.}$ is fibered over $\mathbb{C}^2 \cong \{u, v\}$. The fibration degenerates at the loci

$$u_1^2 = v^2 - \epsilon_1, \quad \text{and} \quad u_2^2 = v^2 - \epsilon_3, \quad (\text{F.58})$$

and the 2- and 3-cycles are visualised as in Fig.F.4.

Choosing a convenient normalisation, we have

$$\Omega = \frac{1}{2\pi^2} \frac{du \wedge dv \wedge dx}{x}. \quad (\text{F.59})$$

Then, for any 3-chain Π_3

$$\int_{\Pi_3} \Omega = \frac{i}{\pi} \int_{C_j} du \wedge dv = \frac{i}{\pi} \int_{\gamma_j} u dv. \quad (\text{F.60})$$

Here C_j is a 2-chain over which an S^1 is fibered according to (F.57), giving us the 3-chain, and γ_j is its boundary. The geometry is then visualized as a double-sheeted v -plane, with the upper and lower sheets connected through the cuts at $u_1^2 = 0$ and $u_2^2 = 0$ (see F.58).

Then the 3-cycle A_i corresponds to γ_i circling around the corresponding cut on the v -plane, while for B_i one goes from the upper sheet to the lower one through the cut. Using the indefinite integral

$$F(v, \epsilon) \equiv \int \sqrt{v^2 - \epsilon} \, dv = \frac{1}{2} \left[v \sqrt{v^2 - \epsilon} - \epsilon \log(v + \sqrt{v^2 - \epsilon}) \right], \quad (\text{F.61})$$

whose expansion for $v^2 \gg \epsilon$ goes as

$$F(v, \epsilon) = \frac{1}{2}v^2 - \frac{1}{4}\epsilon \log(4v^2e) + \mathcal{O}\left(\frac{\epsilon^2}{v^4}\right), \quad (\text{F.62})$$

we obtain

$$\int_{A_j} \Omega = \epsilon_j, \quad \text{and} \quad \int_{B_j} \Omega = \frac{\epsilon_j}{2\pi i} \log\left(\frac{\epsilon_j}{4ev_0^2}\right) + \text{regular}, \quad (\text{F.63})$$

where $v = v_0$ is a cut-off for the non-compact cycle.

Similarly, we can consider a 3-chain Ξ_3 that begins on a representative of \mathcal{C}_2 stretching between $u_1 = \xi$ and $u_2 = \xi$ in $\mathbb{C}^2 = \{u, v\}$, and goes to infinity at $v = v_0$. For $|v_0|^2 \gg |\epsilon_k|$, the integral of Ω over Ξ_3 is (notice that contrarily to what happens for the B -cycle we do not integrate past the cut)

$$\int_{\Xi_3} \Omega = \frac{1}{2\pi i} \left[F(\xi, \epsilon_1) - F(\xi, \epsilon_3) + (\epsilon_1 - \epsilon_3) \log(2e^{1/4} v_0) \right] + \text{regular}. \quad (\text{F.64})$$

In the limit $|\xi|^2 \gg |\epsilon_k|$, we get the simpler result (7.66).

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