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SSCL-Preprint-220
April 1993
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H.-J. Shih
J. Ellison
W. Schiesser

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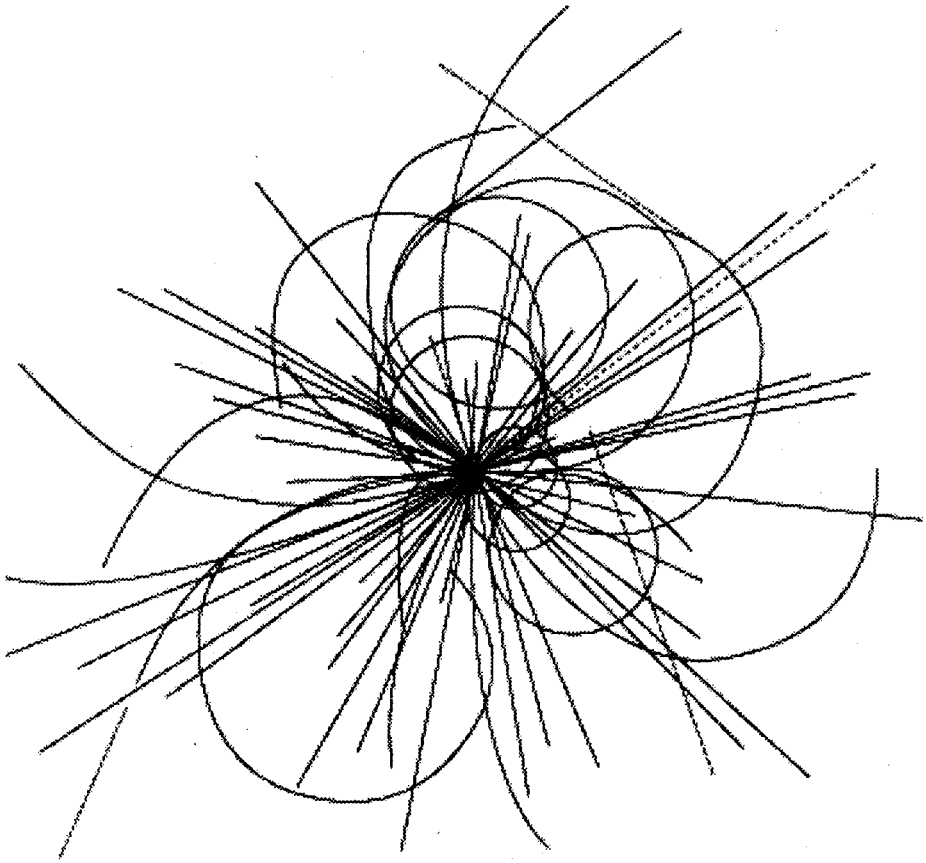

Superconducting Super Collider Laboratory

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DILUTE:
A Code for Studying Beam Evolution Under RF Noise*

H.-J. Shih

Superconducting Super Collider Laboratory†
2550 Beckleymeade Ave.
Dallas, TX 75237

J. Ellison

University of New Mexico
Albuquerque, NM 87131

W. Schiesser

Lehigh University
Bethlehem, PA 18015

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*Presented at the Computational Accelerator Physics Conference on February 22, 1993.

†Operated by the Universities Research Association, Inc., for the U.S. Department of Energy under Contract No. DE–AC35–89ER40486.

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H.-J. Shih
Superconducting Super Collider Laboratory, Dallas, TX 75237

J. A. Ellison
University of New Mexico, Albuquerque, NM 87131

W. E. Schiesser
Lehigh University, Bethlehem, PA 18015

ABSTRACT

Longitudinal beam dynamics under rf noise has been modeled by Dôme, Krinsky and Wang using a diffusion-in-action PDE. If the primary interest is the evolution of the beam in action, it is much simpler to integrate the model PDE than to undertake tracking simulations. Here we describe the code that we developed to solve the model PDE using the numerical Method of Lines. Features of the code include (1) computation of the distribution in action for the initial beam from a Gaussian or user-supplied distribution in longitudinal phase space, (2) computation of the diffusion coefficient for white noise or from a user-supplied spectral density for non-white noise, (3) discretization of the model PDE using finite-difference or Galerkin finite-element approximations with a uniform or non-uniform grid, and (4) integration of the system of ODEs in time by the solver RKF45 or a user-supplied ODE solver.

THE DIFFUSION EQUATION

When there is noise in the rf system, the voltage across the gap of the rf cavity at time T is

$$V(T) = V_0(1 + a(T)) \sin(2\pi f_{rf} T + \psi(T)) \tag{1}$$

where $a(T)$ is the amplitude noise, $\psi(T)$ the phase noise, V_0 the rf peak voltage, and f_{rf} the rf frequency. The longitudinal dynamics of a single particle in the presence of rf noise is described by the Hamiltonian[1]

$$H(P, \phi, T) = H_0(P, \phi) + H_1(P, \phi, T) \tag{2a}$$

$$H_0(P, \phi) = \frac{P^2}{2} + U(\phi) \tag{2b}$$

* Work supported by the Superconducting Super Collider Laboratory which is operated by the Universities Research Association, Inc., for the U.S. Department of Energy under Contract No. DE-AC35-89ER40486.

$$H_1(P,\phi,T) = a(T)U(\phi) + P\dot{\psi}(T) \tag{2c}$$

where $U(\phi) = \Omega^2(1-\cos\phi)$, $P = 2\pi h\eta\delta/T_0$, ϕ is the phase deviation from the synchronous phase, Ω the angular frequency of the small amplitude synchrotron oscillation, h the harmonic number, η the slip factor (related to the momentum compaction factor α_c by $\eta = \alpha_c - \frac{1}{\gamma}$ where γ is the relativistic factor), δ the relative momentum deviation from the synchronous momentum p_s, and T_0 the revolution period. Without rf noise, particles move on the closed orbits in the P-ϕ phase space as shown in Fig. 1. Under the perturbation of rf noise, particles gradually move onto bigger closed orbits, eventually cross over the separatrix, and get lost.

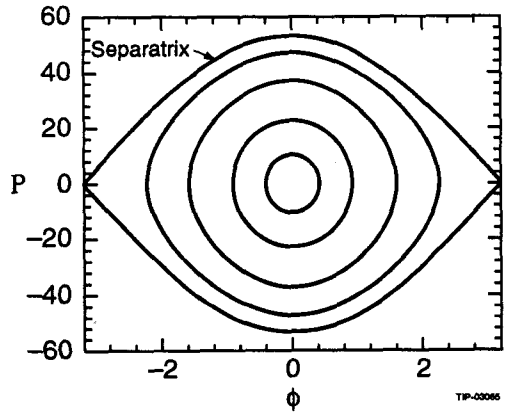

Fig. 1. Unperturbed particle orbits in longitudinal phase space.

If we define 2π times the action J to be the area in the P-ϕ phase space enclosed by the unperturbed closed orbit, i.e.,

$$J = \frac{1}{2\pi}\oint P d\phi \tag{3}$$

then the effect of rf noise is to induce a diffusion of particles in action. In the Dôme-Krinsky-Wang (DKW) theory,[1,2] this diffusion process is approximated by a Markov process, and the evolution of the probability density in action, $p(J,T)$, is governed by the following diffusion equation:

$$\frac{\partial p}{\partial T} = \frac{\partial}{\partial J}\left(\mathcal{D}(J)\frac{\partial p}{\partial J}\right). \tag{4}$$

Here $\mathcal{D}(J)$ is the diffusion coefficient determined from the noise spectral density. The solution of the diffusion equation (4) is uniquely defined by specifying

$$p(J_b,T) = 0, \tag{5}$$

$$p(J,0) = p_0(J), \tag{6}$$

where $p_0(J)$ is the initial action density. Equation (5) is an absorbing boundary condition at $J = J_b \lesssim J_s$ where $J_s = 8\Omega/\pi$ is the action at the separatrix. Thus particles are lost once they are diffused to the orbit at J_b near the separatrix. Since $\mathcal{D}(0) = 0$ according to the theory, no boundary condition at $J = 0$ is

needed. We have made extensive comparisons between the DKW diffusion theory and tracking simulations, and found good agreement. See Ref. 3 for details.

It is convenient to introduce the non-dimensional variables,

$$x = \frac{J}{J_b}, \qquad t = \frac{T}{T_1}$$

where T_1 is some convenient time scale. In terms of these variables, the probability density in x, $\rho(x,t) = J_b p(J,t)$, satisfies

$$\frac{\partial}{\partial x}\rho(x,t) = \frac{\partial}{\partial x}\left(D(x)\frac{\partial}{\partial x}\rho(x,t)\right), \tag{7}$$

$$\rho(1,t) = 0, \tag{8}$$

$$\rho(x,0) = J_b\, p_0(J_b x). \tag{9}$$

Here $D(x)$ is the scaled diffusion coefficient related to $\mathcal{D}(J)$ by

$$D(x) = \frac{T_1}{J_b^2}\mathcal{D}(J_b x). \tag{10}$$

The code DILUTE was developed to solve the well posed PDE problem formed by Eqs. (7), (8) and (9). Before we solve it using the Method of Lines, we must compute the diffusion coefficients and the initial action density.

THE DIFFUSION COEFFICIENTS

In general, the diffusion coefficient is given by $\mathcal{D}(J) = \mathcal{D}_a(J) + \mathcal{D}_\psi(J)$ where[1]

$$\mathcal{D}_a(J) = 4\sum_{m=2,4,\ldots}^{\infty} \frac{(m\omega_s)^4}{\sinh^2(mv)} S_a(m\omega_s), \tag{11}$$

$$\mathcal{D}_\psi(J) = 4\sum_{m=1,3,\ldots}^{\infty} \frac{(m\omega_s)^4}{\cosh^2(mv)} S_\psi(m\omega_s). \tag{12}$$

Here S_a and S_ψ are the spectral densities for amplitude and phase noise, respectively. The quantities J, $\omega_s(J)$, and $v(J)$ are easily defined through the intermediate variable k, $0 \le k < 1$, by

$$J = \left(\frac{8\Omega}{\pi}\right) k^2 B(k), \tag{13}$$

$\omega_s = \Omega(\pi/2K(k))$, and $v = (\pi/2)K(\sqrt{1-k^2})/K(k)$, where K is the complete elliptic integral of the first kind and $B(k) = \int_0^{\pi/2} \cos^2 x dx/\sqrt{1-k^2\sin^2 x}$. The variable k is related to the unperturbed energy $h = H_0(p,\phi)$ by $h = 2k^2\Omega^2$. It is also the relative momentum amplitude of the unperturbed orbit with respect to the separatrix, i.e., $k = \delta_{\max}/\delta_h$ where δ_h is the bucket half-height.

For white phase and amplitude noise, the following expressions for $\mathcal{D}$ can be derived:[1]

$$\mathcal{D}_a(J) = \frac{32}{\pi^2}\Omega^4 S_a k^4 K(k) B_1(k), \tag{14}$$

$$\mathcal{D}_\psi(J) = \frac{8}{\pi^2}\Omega^4 S_\psi k^2 K(k)(B(k) - 4k^2 B_1(k)), \tag{15}$$

where $B_1(k) = \int_0^{\pi/2} \sin^2 x \cos^2 x \sqrt{1 - k^2 \sin^2 x}\ dx$. The functions $B(k)$ and $B_1(k)$ are evaluated by quadrature. As we compute $\mathcal{D}$ and J as functions of k, we spline fit $\mathcal{D}(k)$ vs. $J(k)$ to obtain $\mathcal{D}(J)$.

THE INITIAL ACTION DENSITY

If the initial distribution of the beam in the P-ϕ phase space is bi-Gaussian, i.e.,

$$p(P,\phi) = \frac{1}{\sqrt{2\pi}\sigma_p} e^{-P^2/2\sigma_p^2} \frac{1}{\sqrt{2\pi}\sigma_\phi} e^{-\phi^2/2\sigma_\phi^2}, \tag{16}$$

the initial action density $p_0(J)$ is calculated as follows. Let $p_H(h)$ be the probability density in energy $h = 2k^2\Omega^2$, then

$$\int_0^h p_H(x)dx = 4\int_0^{\phi_h} \left(\int_0^{\sqrt{2(h-U(\phi))}} p(P,\phi)dP \right) d\phi,$$

where ϕ_h is the maximum phase of the unperturbed orbit, i.e., $U(\phi_h) = h$. Differentiating the equation above with respect to h gives

$$p_H(h) = 4\int_0^{\phi_h} [p(P_m(\phi),\phi)/P_m(\phi)]\, d\phi, \tag{17}$$

where $P_m(\phi) = \sqrt{2(h - U(\phi))}$. The action density $p_0(J)$ is then related to $p_H(h)$ by

$$p_0(J) = p_H(h)\frac{dh}{dJ} = p_H(h)\omega_s. \tag{18}$$

The standard deviations σ_p and σ_ϕ in Eq. (16) are determined by specifying the longitudinal emittance $\epsilon_L = \sigma_{\Delta E}\sigma_{\Delta t}$ and using the relationship $\sigma_{\Delta t} = (\eta/\Omega\beta^2 E_s)\sigma_{\Delta E}$, where ΔE is the energy deviation from the synchronous energy E_s, $\Delta t = T_0\,(\phi/2\pi h)$, and $\beta = p_s c/E_s$. The integral in Eq. (17) is evaluated by quadrature. Again, p_H, ω_s and J are computed as functions of k, and the spline fit between $\rho_H \cdot \omega_s$ and J is made to obtain $p_0(J)$.

THE METHOD OF LINES SOLUTIONS

Our PDE problem is solved numerically using the Method of Lines. That is, we 1) set up a uniform or variable grid: $x_1 = 0, x_1, x_2, \ldots x_{N-1}, x_N = 1$, 2) obtain from the PDE an ODE in time at each grid point by some discretization method, and 3) integrate the system of ODEs by an ODE solver. The finite-difference and Galerkin finite-element methods have been used for discretization. Reliability of our numerical solutions has been checked. See Ref. 4 for details.

THE FINITE-DIFFERENCE METHOD

In this method, some finite-difference approximation is employed to evaluate the right-hand side of Eq. (7) (and thus to obtain an ODE in time) at each grid point.

In the case of a uniform grid, the following three-point, second-order approximation is used

$$f_{x,i\neq 1,N} = \frac{1}{2\Delta x}(f_{i+1} - f_{i-1}) \tag{19a}$$

$$f_{x,i=1} = \frac{1}{2\Delta x}(-3f_1 + 4f_2 - f_3) \tag{19b}$$

$$f_{x,i=N} = \frac{1}{2\Delta x}(f_{N-2} - 4f_{N-1} + 3f_N) \tag{19c}$$

in evaluating $\frac{\partial \rho}{\partial x}$ and $\frac{\partial}{\partial x}\left(D\frac{\partial \rho}{\partial x}\right)$. Here $\Delta x = 1/(N-1)$ is the grid spacing.

In the case of a variable grid, the following approximation is used

$$\rho_{t,i\neq 1,N} \cong \frac{2}{\Delta x_i + \Delta x_{i-1}}\left(D_{i+\frac{1}{2}}\frac{\rho_{i+1} - \rho_i}{\Delta x_i} - D_{i-\frac{1}{2}}\frac{\rho_i - \rho_{i-1}}{\Delta x_{i-1}}\right) \tag{20a}$$

$$\rho_{t,i=1} \cong \frac{2}{\Delta x_1}\left(D_{\frac{3}{2}}\frac{\rho_2 - \rho_1}{\Delta x_1} - 0\right) \tag{20b}$$

$$\rho_{t,i=N} = 0 \tag{20c}$$

where $\Delta x_i = x_{i+1} - x_i$ and $D_{i+\frac{1}{2}} = (D_i + D_{i+1})/2$.

THE GALERKIN FINITE-ELEMENT METHOD

We approximate $\rho(x,t)$ by a sum of basis functions $\Phi_i(x)$:

$$\rho_G(x,t) = \sum_{i=1}^{N} C_i(t)\Phi_i(x). \tag{21}$$

The basis functions $\Phi_i(x)$ are piecewise linear with $\Phi_i(x_i) = 1$ and $\Phi_i(x_{i-1}) = \Phi_i(x_{i+1}) = 0$, as shown in Fig. 2. Thus $\rho_G(x_j,t) = C_j(t)$. Since $\rho(1,t) = 0$, $C_N(t) = 0$ and we only need to determine $C_j(t)$, $j = 1,2,\ldots,N-1$.

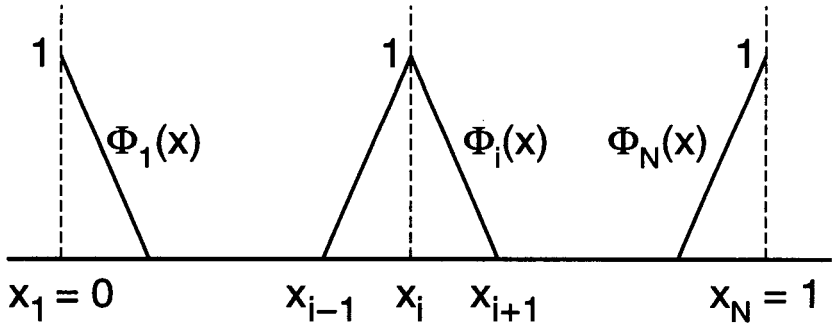

Fig. 2. Basis functions in the Galerkin approximation.

To obtain the conditions on $C_j(t)$, we define the residual functions $R(x,t)$,

$$R(x,t) = \frac{\partial \rho_G}{\partial t} - \frac{\partial}{\partial x}\left(D(x)\frac{\partial \rho_G}{\partial x}\right) \tag{22}$$

and apply the Galerkin condition,

$$\int_0^1 \Phi_j(x) R(x,t)dx = 0, \quad j = 1,2,\ldots,N-1. \tag{23}$$

The conditions on $C_j(t)$ are given by

$$\mathbf{C}'(t) = (\mathbf{A}^{-1}\mathbf{B})\mathbf{C}(t), \tag{24}$$

where $\mathbf{C}(t)$ is the column vector $(C_1(t), C_2(t), \ldots, C_{N-1}(t))$ and the $N-1$ by $N-1$ matrices $\mathbf{A}$ and $\mathbf{B}$ are

$$\mathbf{A} = \begin{pmatrix} \frac{\Delta x_1}{3} & \frac{\Delta x_1}{6} & 0 & 0 & \ldots & 0 \\ \frac{\Delta x_1}{6} & \frac{\Delta x_1+\Delta x_2}{3} & \frac{\Delta x_2}{6} & 0 & \ldots & 0 \\ 0 & \frac{\Delta x_2}{6} & \frac{\Delta x_2+\Delta x_3}{3} & \frac{\Delta x_3}{6} & \ldots & 0 \\ \vdots & & & & & \vdots \\ 0 & \ldots & 0 & \frac{\Delta x_{N-3}}{6} & \frac{\Delta x_{N-3}+\Delta x_{N-2}}{3} & \frac{\Delta x_{N-2}}{6} \\ 0 & \ldots & 0 & 0 & \frac{\Delta x_{N-2}}{6} & \frac{\Delta x_{N-2}+\Delta x_{N-1}}{3} \end{pmatrix} \tag{25}$$

$$\mathbf{B} = \begin{pmatrix} -\frac{d_1}{\Delta x_1^2} & \frac{d_1}{\Delta x_1^2} & 0 & 0 & \ldots & 0 \\ \frac{d_1}{\Delta x_1^2} & -\frac{d_1}{\Delta x_1^2} - \frac{d_2}{\Delta x_2^2} & \frac{d_2}{\Delta x_2^2} & 0 & \ldots & 0 \\ 0 & \frac{d_2}{\Delta x_2^2} & -\frac{d_2}{\Delta x_2^2} - \frac{d_3}{\Delta x_3^2} & \frac{d_3}{\Delta x_3^2} & \ldots & 0 \\ \vdots & & & & & \vdots \\ 0 & \ldots & 0 & \frac{d_{N-3}}{\Delta x_{N-3}^2} & -\frac{d_{N-3}}{\Delta x_{N-3}^2} - \frac{d_{N-2}}{\Delta x_{N-2}^2} & \frac{d_{N-2}}{\Delta x_{N-2}^2} \\ 0 & \ldots & 0 & 0 & \frac{d_{N-2}}{\Delta x_{N-2}^2} & -\frac{d_{N-2}}{\Delta x_{N-2}^2} - \frac{d_{N-1}}{\Delta x_{N-1}^2} \end{pmatrix} \tag{26}$$

Here $d_i = \int_{x_i}^{x_{i+1}} D(x)dx$. See Ref. 4 for the derivation of Eq. (24) for the case of a uniform grid. The extension to the case of a variable grid is straightforward.

INTEGRATION

The resulting system of ODEs in time is integrated by RKF45, a non-stiff ODE solver written by H.A. Watts and L.F. Shampine.[5] However, when the diffusion coefficient or the action density varies rapidly, the integration with RKF45 can be very slow. In this case, a stiff ODE solver, for example, LSODES written by A.C. Hindmarsh,[6] is recommended to speed up the integration.

THE CODE USAGE

The code is written in Fortran 77. It makes use of the IMSL library for evaluating some special functions and integration by quadrature.

INPUT PARAMETERS

The input parameters are defined in the subroutine `INIT`. They are listed below in four categories:

1) Beam/machine parameters:
   - `CS` C_s, the machine circumference (m).
   - `ALC` α_c, the momentum compaction factor.
   - `EVO` V_0, the rf peak voltage (GeV).
   - `PO` E_s, the total energy of synchronous particle (GeV).
   - `HNO` h, the harmonic number.
   - `AM` m_s, the rest energy of synchronous particle (GeV).
   - `EPSL` ϵ_L, the longitudinal rms emittance (GeV-sec).
   - `AKB` k_b, the critical boundary in k. It is defined by $J_b = J_s k_b^2 B(k_b)$.
   - `IB` flag for beam type: 0 for a Gaussian beam and 1 for a user-supplied beam.
2) RF noise parameters:
   - `IW` flag for noise type: 0 for non-white noise and 1 for white noise.
   - `IF` flag for including phase noise: 0 for no and 1 for yes.
   - `IA` flag for including amplitude noise: 0 for no and 1 for yes.
   - `T1` T_1, the convenient time scale (sec).
   - `SPHI` spectral density for white phase noise to be specified when `IW` = 1 and `IF` = 1.
   - `SA` spectral density for white amplitude noise to be specified when `IW` = 1 and `IA` = 1.
3) Discretization parameters:
   - `N` total number of grid points.
   - `IV` flag for grid type: 0 for uniform grid and 1 for variable grid.
   - `ID` flag for discretization method: 0 for finite-difference method and 1 for Galerkin finite-element method.
4) Integration parameters:
   - `TO` initial time of integration.
   - `TF` final time of integration.
   - `TP` integration time step.
   - `IS` flag for integrator type: 0 for the integrator RKF45 and 1 for user-supplied integrator.
   - `ABSERR` absolute error tolerance in integration.
   - `RELERR` relative error tolerance in integration.

USER-SUPPLIED FUNCTIONS OR SUBROUTINES

In addition to the subroutine `INIT`, the following functions or subroutines must be supplied when the indicated flag is selected:

1) Subroutine BEAM:
supplied when IB = 1. Here the user defines the longitudinal variables δ and l for up to 5000 particles, where l is the longitudinal displacement from the center of the bunch (related to ϕ by $\phi = 2\pi hl/C_s$).
2) Function SPECA:
supplied when IW = 0 and IA = 1. Here the user defines the spectral density for amplitude noise as a function of angular frequency.
3) Function SPECF:
supplied when IW = 0 and IF = 1. Here the user defines the spectral density for phase noise as a function of angular frequency.
4) Subroutine VGRID:
supplied when IV = 1. Here the user defines the number and coordinates of the grid points for a variable grid.
5) Subroutine SOLVE:
supplied when IS = 1. Here the user calls his or her own ODE integrator.

OUTPUT

The integration results are written out to the file DILUTE.OUT at the end of each integration step in the following format:

```
      WRITE(1,9000)T
      WRITE(1,9000)(U(I),I=1,N)
9000  FORMAT(1X,11E12.5)
```

where T is the final time of the integration step, N the number of grid points and U(I) the density at grid point i, $\rho(x_i, t)$.

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