Braneworlds without Z_2 symmetry in n dimensions

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Abstract

We consider a brane world in an arbitrary number of dimensions without Z_2 symmetry and derive the effective Einstein equation on the brane, where its right-hand side is given by the matter on the brane and the curvature in the bulk. This is achieved by first deriving the junction conditions for a non-Z_2 symmetric brane and second solving the Gauss equation, which relates the mean extrinsic curvature of the brane to the curvature in the bulk, with respect to the mean extrinsic curvature. The latter corresponds to formulating an explicit junction condition on the mean of the extrinsic curvature, analogue to the Israel junction condition for the jump of the extrinsic curvature. The derived equation is a basic equation for the study of Kaluza-Klein brane worlds in which some dimensions on the brane are compactified or for a regularization scheme for a higher codimension brane world, where the Kaluza-Klein compactification on the brane is regarded as a means to regularize the uncontrollable spacetime singularity created by the higher codimension brane.

1 Introduction

String theory suggests that our universe is not four dimensional but, rather, a submanifold (brane) embeded in a higher-dimensional spacetime (bulk). In particular, Randall and Sumdrum (RS) [2, 3] proposed an interesting brane world model. The RS model assumes a codimension-1 brane with Z_2 symmetry embeded in the bulk with a negative cosmological constant. However, to reconcile a higher-dimensional theory with the observed four-dimensional spacetime, the RS model is not sufficient. Since string theory suggests that the number of bulk dimensions is 10 or 11, the corresponding number of codimensions is 6 or 7. Therefore, we must consider a higher-codimension brane world. But a higher-codimension brane world has the serious problem that the brane becomes an uncontrollable spacetime singularity due to its self-gravity, except possibly for a codimension-2 brane world, which may give a reasonable cosmology. Thus it is necessary to develop a regularization method to realize a reasonable higher-codimension brane world.

For the above-stated purpose, we focus on a specific regularization scheme, which we now describe. Let us consider a codimenion-$(q+1)$ brane in an n-dimensional spacetime. We regularize this brane by expanding it into q-dimensions, so that it becomes a codimension-1 brane with q compact dimensions on the brane. Note that the resulting codimension-1 brane will not have the Z_2 symmetry. This regularization scheme is essentially the same as the Kaluza-Klein (KK) compactification of q spatial dimensions on the brane, which is called the KK brane world.

In this paper, partly to give a framework for the KK brane worlds and partly as a first step to formulate the above-mentioned regularization scheme for brane worlds of arbitrary codimension, we consider a codimension-1 brane world in an arbitrary number of spacetime dimensions without Z_2 symmetry and derive an effective Einstein equation on the brane, which is a generalization of the effective Einstein equation on the brane with Z_2 symmetry derived by Shiromizu, Maeda and Sasaki [4].

The work most relevant to the present one is that of Battye et al., [5] in which the non-Z_2 symmetric brane world is investigated. They study the junction condition in detail and point out that the effective Einstein equation has terms involving the mean of the extrinsic curvature across the brane which are not explicitly expressed in terms of either the matter on the brane or the curvature in the bulk. Then, they focused their investigation on a spatially homogeneous, isotropic brane. Our purpose here is to solve

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this problem and express the effective Einstein equation solely in terms of the matter on the brane and the curvature in the bulk, and also to present a straightforward generalization in which the number of spacetime dimensions of the bulk is extended from 5 to n.

Thoughout the paper, we use square brackets to denote the jump of a quantity across the brane and angled brackets to denote its mean. For an arbitrary tensor $\mathcal{A}$ (with tensor indices suppressed), we define $[\mathcal{A}] \equiv \mathcal{A}^+ - \mathcal{A}^-$, $\langle\mathcal{A}\rangle \equiv \frac{1}{2}(\mathcal{A}^+ + \mathcal{A}^-)$, where the superscript + denotes the side of the brane from which the normal vector n^A points toward the bulk.

2 Pre-effective Einstein equation on the brane

We consider a family of $(n-1)$-dimensional timelike hypersurfaces (slicing) in an n-dimensional spacetime and identify one of them as a brane (i.e., a singular hypersurface). We denote the bulk metric by g_{MN} where $M = 0, 1, \cdots, n-1$. We denote the vector field unit normal to the hypersurfaces by n^M. Then the induced metric γ_{MN} on the hypersurfaces is given by $\gamma_{MN} = g_{MN} - n_M n_N$. The metric $\gamma^A{}_B = \gamma^{AC} g_{CB}$ acts as a projection operator, projecting bulk tensors onto the brane. Here, the Gauss equation gives

$$\bar{R}_{ab} \quad = \quad \mathcal{F}_{ab} + KK_{ab} - K_a{}^c K_{bc}\,, \tag{1}$$

where $\bar{R}_{abcd}$ is the $(n-1)$-dimensional Riemann curvauture, K_{ab} is the extrinsic curvature on the brane. For convenience, we introduce the tensor $\mathcal{F}_{ab}$ defined by

$$\mathcal{F}_{ab} \equiv \frac{n-3}{n-2}\mathcal{R}_{AB}\gamma^A_a\gamma^B_b + \frac{1}{n-2}\mathcal{R}_{CD}\gamma^{CD}\gamma_{ab} - \frac{1}{n-1}\mathcal{R}\gamma_{ab} + \mathcal{E}_{ab}\,, \tag{2}$$

where $\mathcal{R}_{ABCD}$ is the n-dimensional Riemann curvature and $\mathcal{E}_{ab}$ is the projected Weyl curvature on the brane, defined by $\mathcal{E}_{ab} \equiv \mathcal{C}_{ACBD} n^C n^D \gamma^A_a \gamma^B_b$. Using the fact that the brane induced metric satisfies the junction condition $[\gamma_{ab}] = 0$, the Gauss equation can be decomposed into two equations,

$$\langle\bar{R}_{ab}\rangle = \bar{R}_{ab} = \langle\mathcal{F}_{ab}\rangle + \frac{1}{4}([K][K_{ab}] - [K^c_a][K_{bc}]) + \langle K\rangle\langle K_{ab}\rangle - \langle K^c_a\rangle\langle K_{bc}\rangle\,, \tag{3}$$

$$[\bar{R}_{ab}] = 0 = [\mathcal{F}_{ab}] + \langle K\rangle[K_{ab}] + [K]\langle K_{ab}\rangle - 2\langle K_{(a}{}^c\rangle[K_{b)c}]\,, \tag{4}$$

where we use the relations between the jump and mean symbol$\langle\rangle$, $[]$. We can construct the effective Einstein equation on the brane without any symmetry. For convenience, we decompose the $\mathcal{F}_{ab}$ and K_{ab} into trace and traceless parts, $\mathcal{F}_{ab} = \frac{\mathcal{F}}{n-1}\gamma_{ab} + \omega_{ab}$, $K_{ab} = \frac{K}{n-1}\gamma_{ab} + \sigma_{ab}$, respectively. The effective Einstein equation is derived from the mean of the Gauss equation (3). Inserting the Israel junction condition [6] ($[K\gamma_{ab} - K_{ab}] = \kappa^2_{(n)}\bar{T}_{ab} = \kappa^2_{(n)}(-\lambda\gamma_{ab} + \tau_{ab})$) into it, we obtain

$$\bar{G}_{ab} = -\bar{\Lambda}\gamma_{ab} + \kappa^2_{(n-1)}\tau_{ab} + \frac{\kappa^2_{(n-1)}}{\lambda}S_{ab} + \langle\Omega_{ab}\rangle - \langle\sigma_a{}^c\rangle\langle\sigma_{bc}\rangle\,, \tag{5}$$

where the each terms is defined by

$$\kappa^2_{(n-1)} = \frac{n-3}{4(n-2)}\kappa^4_{(n)}\lambda\,, \tag{6}$$

$$\bar{\Lambda} = \frac{n-3}{2(n-1)}\langle\mathcal{F}\rangle + \frac{1}{2}\kappa^2_{(n-1)}\lambda + \frac{(n-2)(n-3)}{2(n-1)^2}\langle K\rangle^2 - \frac{1}{2}\langle\sigma^{cd}\rangle\langle\sigma_{cd}\rangle\,, \tag{7}$$

$$S_{ab} = \frac{\tau}{n-3}\tau_{ab} - \frac{\tau^2}{2(n-3)}\gamma_{ab} - \frac{n-2}{n-3}\tau^c_a\tau_{bc} + \frac{n-2}{2(n-3)}\tau^{cd}\tau_{cd}\gamma_{ab}\,, \tag{8}$$

$$\langle\Omega_{ab}\rangle = \langle\omega_{ab}\rangle + \frac{n-3}{n-1}\langle K\rangle\langle\sigma_{ab}\rangle\,. \tag{9}$$

The first term $\bar{\Lambda}$ on the right-hand side of Eq. (5) represents the effective cosmological constant. We note, however, that this quantity may not be constant in general, as is clear from its expression in Eq. (7). The second and third terms are contributions from the energy-momentum tensor on the brane and its quadratic term, which are the same as in the Z_2 symmetric case [4]. The fourth traceless term, $\langle\Omega_{ab}\rangle$, which is traceless by definition, is an extension of the $\mathcal{E}_{ab}$ term in the Z_2 symmetric case. Finally, the last term is a new term, which has no analog in the Z_2 symmetric case. As is clear from its definition, this term arises from the square of the mean extrinsic curvature $\langle K_{ab}\rangle$, and it vanishes only if the traceless part of $\langle K_{ab}\rangle$ is zero. The above effective Einstein equation, (5), is completely general in the sense that no symmetry has been imposed. However, it is useless, except in the Z_2 symmetric case, because it depends strongly on the unknown mean extrinsic curvature $\langle K_{ab}\rangle$. In order to make it meaningful, it is necessary to express $\langle K_{ab}\rangle$ in terms of geometrical quantities in the bulk (i.e., the bulk metric and curvature) and the brane energy-momentum tensor.

3 The mean extrinsic curvature $\langle K_{ab}\rangle$

The equation we solve is (4),

$$-[\mathcal{F}_{ab}] = 2\kappa_{(n)}^2 \hat{\tau}_{(a}{}^{c}\langle K_{b)c}\rangle + \langle K\rangle [K_{ab}] \,. \tag{10}$$

where we use the Israel junction condition and for convenience we introduce the “hatted” energy-momentum tensor:

$$\hat{\tau}_{ab} \equiv \tau_{ab} - \frac{(n-3)\lambda - \tau}{2(n-2)}\gamma_{ab} = \bar{T}_{ab} - \frac{1}{2(n-2)}\bar{T}\gamma_{ab} \,. \tag{11}$$

Here we seek a general solution without particular assumptions concerning the brane energy-momentum tensor. Our method consists of two parts. First, we obtain the trace of the mean extrinsic curvature $\langle K\rangle$ by introducing the inverse of the hatted tensor $\hat{\tau}_{ab}$. Second, with $\langle K\rangle$ known, we rewrite the second Gauss equation as a matrix equation for $\langle K_{ab}\rangle$. This matrix equation can be solved by using the tetrad (more precisely, the vielbein) decomposition of $\hat{\tau}_{ab}$. Using this strategy, we obtain the general solution for the mean extrinsic curvature.

$$\begin{aligned}\kappa_{(n)}^2\langle K_{ab}\rangle &= -\frac{1}{2}(\hat{\tau}^{-1})_a{}^c[\mathcal{F}_{bc}] + \frac{(n-3)(\hat{\tau}^{-1})^{de}[\mathcal{F}_{de}]}{2((n-3)^2 - \hat{\tau}^m{}_m(\hat{\tau}^{-1})^n{}_n)}\left(\gamma_{ab} - \frac{\hat{\tau}^c{}_c}{n-3}(\hat{\tau}^{-1})_{ab}\right) \\ &\quad - \sum_{i\neq j}\frac{1}{\hat{\tau}_{(i)} + \hat{\tau}_{(j)}}\bar{e}_a^{(i)}\bar{e}_b^{(j)}[\omega_{(i)(j)}] \,. \end{aligned} \tag{12}$$

where $(\hat{\tau}^{-1})_{ab}$ is a inverse matrix of the “hatted” energy-momentum tensor $\hat{\tau}_{ab}$ and $\bar{e}_a^{(i)}$ is a local Lorentz frame in which the hatted tensor $\hat{\tau}_{ab}$ is diagonalized. We refer to this as the junction condition for the mean of the extrinsic curvature, which is a counterpart to the conventional junction condition for the jump of the extrinsic curvature, Israel junction condition [6]. We also note that this result is valid only if we have $\hat{\tau}_{(i)} + \hat{\tau}_{(j)} \neq 0$ for all possible pairs of (i) and (j). We need a special treatment in the case that any of the demonimators are zero.

4 Effect of $\langle K_{ab}\rangle$ on the brane

Low energy limit

We conjecture that the low energy regime, where $|\tau_{ab}| \ll \lambda$, Einstein gravity is recovered on the brane. For this reason, we believe that the contributions of the ω_{ab} and S_{ab} terms become negligibly small. To examine this, let us consider the solution for the mean extrinsic curvature up to $O(\tau_{ab}^2, \omega_{ab})$. In this case, the inverse of the hatted energy-momentum tensor is given by

$$(\hat{\tau}^{-1})^{ab} = \frac{2(n-2)}{n-3}\lambda^{-1}\left(\gamma^{ab} + \frac{2(n-2)}{n-3}\lambda^{-1}\left(\tau^{ab} - \frac{\tau}{2(n-2)}\gamma^{ab}\right) + \cdots\right) . \tag{13}$$

Then $\langle K_{ab}\rangle$ can be readily obtained as

$$\kappa_{(n)}^2\langle K_{ab}\rangle = \frac{[\mathcal{F}]}{2(n-1)\lambda}\left\{\gamma_{ab} + \frac{1}{\lambda}\left(\tau\gamma_{ab} - (n-2)\tau_{ab}\right)\right\} + \mathcal{O}(\tau_{ab}^2, \omega_{ab}) \,. \tag{14}$$

Using this, the effective Einstein equation becomes

$$\bar{G}_{ab} = -\bar{\Lambda}^{LE}\gamma_{ab} + {\kappa^2_{(n-1)}}^{LE}\tau_{ab} + O(\tau^2_{ab}, \omega_{ab}), \tag{15}$$

where

$${\kappa^2_{(n-1)}}^{LE} = \frac{n-3}{4(n-2)}\kappa^4_{(n)}\lambda - \frac{(n-2)(n-3)[\mathcal{F}]^2}{4(n-1)^2\kappa^4_{(n)}\lambda^3}\,, \tag{16}$$

$$\bar{\Lambda}^{LE} = \frac{n-3}{2(n-1)}\langle\mathcal{F}\rangle + \frac{n-3}{8(n-2)}(\kappa^2_{(n)}\lambda)^2 + \frac{(n-2)(n-3)[\mathcal{F}]^2}{8\kappa^4_{(n)}(n-1)^2\lambda^2}\,. \tag{17}$$

Thus, Einstein gravity is recovered. However, in contrast to our naive expectation, the contribution of the mean extrinsic curvature gives rise to new correction terms from the bulk, both to the gravitational constant and to the cosmological constant, which are not necessarily constant.

5 Conclusion

We considered a general codimension-1 brane in an arbitrary number of dimensions without Z_2 symmetry and we obtained expressions for both the jump and the mean of the extrinsic curvature in terms of the bulk curvature tensor and the brane energy-momentum tensor. With this result, we derived the effective Einstein equation on the brane in its most general form, which is a generalization of the Shiromizu-Maeda-Sasaki equation [4] to the case in which Z_2 symmetry does not exist. The derived effective Einstein equation has a new term arising from the mean extrinsic curvature, and this new term leads to the appearance effectively anisotropic matter on the brane.

Thus, our result is a basic equation for the hybrid brane world scenario, in which some spatial dimensions on the brane are Kaluza-Klein compactified. Also, it provides a basis for higher codimension brane worlds in which a higher codimension brane is regularized by a codimension-1 brane with extra dimensions on the brane compactified to an infinitesimally small size.

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