On Lorentz-invariant spin-2 theories

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We construct Lorentz-invariant massless and massive spin-2 theories in flat spacetime. Starting from the most generic action of a rank-2 symmetric tensor field whose Lagrangian contains up to quadratic in first derivatives of a field, we investigate the possibility of new theories by using the Hamiltonian analysis. By imposing the degeneracy of the kinetic matrix and the existence of subsequent constraints, we classify theories based on the number of degrees of freedom and constraint structures and obtain a wider class of Fierz-Pauli theory as well as massless and partially massless theories, whose scalar and/or vector degrees of freedom are absent. We also discuss the relation between our theories and known massless and massive spin-2 theories.

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I. INTRODUCTION

The search for a theoretically consistent Lorentz-invariant massive graviton has been a challenging issue since 1939, when Fierz and Pauli (FP) proposed a linear theory of massive spin-2 field [1]. Once the FP mass term is taken into account in general relativity, one would naively expect that it recovers the results of general relativity as the mass of the graviton goes to zero. However, a nonvanishing degree of freedom (DOFs) in the massless limit would lead to discontinuity found by van Dam and Veltman [2] and Zakharov [3]. Although Vainshtein claimed that this problem can be cured by taking into account nonlinear interactions [4], Boulware and Deser pointed out the scalar d.o.f. responsible for Vainshtein's argument carries an extra ghost d.o.f., the so-called BD ghost [5]. Remarkably, this unwanted d.o.f. associated with higher derivatives can be eliminated by adding the fully nonlinear graviton's mass terms such that those higher derivative terms vanish due to the total derivative terms, and this theoretically consistent massive gravity theory is now known as de Rham-Gabadadze-Tolley (dRGT) theory [6,7].

In recent years, there have been a number of attempts for constructing a broad class of theories of massive gravity, e.g., mass-varying massive gravity [8], quasidilation theory [9], and massive bigravity [10]. The most interesting one would be new kinetic interactions for a massive graviton without introducing any extra d.o.f. Such a kinetic interaction embedded in dRGT theory has been first found in the context of a pseudolinear theory [11], however, its nonlinear completions cannot be included in dRGT theory in a consistent way due to the reappearance of BD ghost at the nonlinear level [12,13]. The crucial problem with these derivative interactions was the appearance of higher derivatives of the scalar mode in Euler-Lagrange equations, and this, in general, leads to the Ostrogradsky's ghost [14].

Meanwhile it has been recently argued that higher derivatives in Euler-Lagrange equations are not essentially problematic as long as the appropriate number of constraints exists in a theory. Such concrete examples are found in the context of point particles [15–20], their field theoretical application [21,22], scalar-tensor theories [15,23–26], and vector-tensor theories [27]. The key point of these theories is the degeneracy of the kinetic matrix, which provides an associated primary constraint (and subsequent constraint depending on the spin of a field), and an unwanted d.o.f. can be then successfully eliminated. In fact, in such a theory, higher time derivatives appearing in Euler-Lagrange equations should be removed by combining Euler-Lagrange equations and their time derivative;

therefore, initial conditions to solve the resultant differential equations are consistent with the number of DOFs in ghost-free theories.

This fact opens up a new direction of study for searching for new theories of a massive graviton, and it is therefore worth revisiting the pioneering attempt by Fierz and Pauli as a starting point of constructing a theoretically consistent massive gravity. To this end, in the present paper, we construct the most general quadratic theory of a massive spin-2 field and a massless spin-2 field with Lorentz invariance in flat spacetime, based on the Hamiltonian analysis. This paper is organized as follows. In Sec. II, we introduce the linear theory of a rank-2 symmetric tensor field and decompose it into tensors expressed by scalar quantities, those by vector quantities, and those by tensor quantities, where each quantity is defined based on transformation properties with respect to a three-dimensional rotation in Minkowski spacetime. The Hamiltonian formalism in Fourier space is also summarized. In Sec. III, we first derive the action for the transverse traceless tensor mode and a condition for avoiding instability. Then we find a condition to eliminate ghosty d.o.f. for the scalar and vector modes and classify theories based on the Hamiltonian analysis. In Sec. IV, we investigate the properties of the obtained theories under the field redefinition to see relations with the known theories. Sec. V is devoted to summary. In Appendix A, we perform the complete Hamiltonian analysis of the special cases. In Appendix B, the explicit proof of the existence of the ghost d.o.f. is given if a theory has 6 or more d.o.f. In Appendix C, we derive gauge transformation and construct gauge invariant variables for each case. We also derive conditions for avoiding ghost and gradient instabilities for the scalar mode from a reduced Lagrangian.

II. SETUP

In this section, we introduce the most general Lorentz-invariant action for a rank-2 symmetric tensor field, which contains up to the Lagrangian quadratic in the tensor field and two derivatives with respect to spacetime. Since a theory for the rank-2 symmetric tensor field in general contains 10 d.o.f., some of them might be ghost modes, which are unwanted d.o.f. in a theory. To this end, we then apply scalar, vector, and tensor decomposition to the rank-2 symmetric tensor field which is defined based on transformation properties with respect to a 3—dimensional rotation in Minkowski spacetime. We will also provide an overview of the Hamiltonian formalism in Fourier space.

A. Action

Let us consider a generic Lorentz-invariant action for a rank-2 symmetric tensor field $h_{\mu\nu}$ up to the quadratic order in Minkowski spacetime,

$$S[h_{\mu\nu}] = \int \mathrm{d}^4x(-\mathcal{K}^{\alpha\beta|\mu\nu\rho\sigma}h_{\mu\nu,\alpha}h_{\rho\sigma,\beta} - \mathcal{M}^{\mu\nu\rho\sigma}h_{\mu\nu}h_{\rho\sigma}), \quad (1)$$

where $\mathcal{K}^{\alpha\beta|\mu\nu\rho\sigma}$ and $\mathcal{M}^{\mu\nu\rho\sigma}$ are the most general combinations of the Minkowski metric $\eta_{\mu\nu}$,

$$\mathcal{K}^{\alpha\beta|\mu\nu\rho\sigma} = \kappa_1\eta^{\alpha\beta}\eta^{\mu\rho}\eta^{\nu\sigma} + \kappa_2\eta^{\mu\alpha}\eta^{\rho\beta}\eta^{\nu\sigma} + \kappa_3\eta^{\alpha\mu}\eta^{\nu\beta}\eta^{\rho\sigma} + \kappa_4\eta^{\alpha\beta}\eta^{\mu\nu}\eta^{\rho\sigma}, \quad (2)$$

$$\mathcal{M}^{\mu\nu\rho\sigma} = \mu_1\eta^{\mu\rho}\eta^{\nu\sigma} + \mu_2\eta^{\mu\nu}\eta^{\rho\sigma}, \quad (3)$$

and $\kappa_{1,2,3,4}$ and $\mu_{1,2}$ are constant parameters. Contracting all the Minkowski metric, the action can be rewritten, after integration by parts, as

$$S[h_{\mu\nu}] = -\int \mathrm{d}^4x[\kappa_1 h_{\mu\nu,\alpha}h^{\mu\nu,\alpha} + \kappa_2 h^{\alpha}{}_{\mu,\alpha}h^{\beta\mu}{}_{,\beta} + \kappa_3 h^{\alpha\beta}{}_{,\alpha}h_{,\beta} + \kappa_4 h_{,\alpha}h^{,\alpha} + \mu_1 h_{\mu\nu}h^{\mu\nu} + \mu_2 h^2], \quad (4)$$

where the indices of $h_{\mu\nu}$ are raised by $\eta^{\mu\nu}$ and h is the trace of $h_{\mu\nu}$ contracted with the Minkowski metric $\eta_{\mu\nu}$. A comma denotes a partial derivative with respect to spatial coordinates. The linearized Einstein-Hilbert action can be reproduced by setting $\kappa_2 = -\kappa_3 = 2\kappa_4 = -2\kappa_1$ up to an overall factor, and the kinetic term of Eq. (1) is then invariant under the gauge transformation $h_{\mu\nu} \to h_{\mu\nu} + \partial_\mu\xi_\nu + \partial_\nu\xi_\mu$, where ξ_μ is a gauge parameter. In addition to this choice of the parameters, when the mass parameters satisfy $\mu_1 = -\mu_2 \neq 0$, the Lagrangian (1) reproduces the Fierz-Pauli theory [1]. Although the Fierz-Pauli theory respects the gauge invariance in the kinetic term, it is not necessary for a generic massive spin-2 field that we consider in the present paper.

B. SVT decomposition

In order to simplify analysis, we decompose the rank-2 symmetric tensor field $h_{\mu\nu}$ into a transverse-traceless tensor, tensors expressed by transverse vectors, and tensors expressed by scalars where the scalar, vector, and tensors are defined based on transformation properties with respect to a three-dimensional rotation in Minkowski spacetime:

$$h_{00} = h^{00} = -2\alpha, \qquad h_{0i} = -h^{0i} = \hat{\beta}_{,i} + B_i \quad (B^i{}_{,i} = 0), \quad (5)$$

$$h_{ij} = h^{ij} = 2\mathcal{R}\delta_{ij} + 2\hat{\mathcal{E}}_{,ij} + F_{i,j} + F_{j,i} + 2H_{ij} \quad (F^i{}_{,i} = 0, H^i{}_i = H^{ij}{}_{,j} = 0). \quad (6)$$

Here the transverse-traceless tensor H_{ij}, two transverse vectors B_i and F_i, and four scalars α, $\hat{\beta}$, $\mathcal{R}$, and $\hat{\mathcal{E}}$, respectively, have two, four, and four components in total. Therefore, to obtain a theory whose number of d.o.f. is up to five including the d.o.f. of the transverse-traceless tensor, we need to eliminate two components in the vector sector and three components in the scalar sector, respectively. Otherwise, ghost d.o.f. appear as shown in Appendix B. Below we split the action into three parts and each of them is solely composed by single type of perturbations, that is, scalar, vector, and tensor perturbations, which is always possible at the level of linear perturbation,

$$S[h_{\mu\nu}] = S^S[\alpha, \hat{\beta}, \mathcal{R}, \hat{\mathcal{E}}] + S^V[B_i, F_i] + S^T[H_{ij}]. \quad (7)$$

In the next section, based on this separated action, we will look for theories with at most 5 d.o.f., finding the appropriate degeneracy conditions for the scalar and vector sectors, respectively.

C. Hamiltonian formalism in Fourier space

In this subsection, we briefly summarize the Hamiltonian formalism in the Fourier space. We, for convenience, work in the Fourier space, and the Fourier component of a field $A(t, \mathbf{k})$ is given by

$$A(t, \mathbf{k}) = \int \mathrm{d}^3 x A(t, \mathbf{x}) \mathrm{e}^{i\delta_{jk}k^j x^k}. \quad (8)$$

The Hamiltonian is defined by

$$H(t) = \int \mathrm{d}^3 k \mathcal{H}(t, \mathbf{k}), \quad (9)$$

where $\mathcal{H}$ is the Hamiltonian density in the Fourier space,

$$\mathcal{H}(t, \mathbf{k}) = \sum_I \dot{s}^I(t, \mathbf{k}) \pi_{s^I}(t, \mathbf{k}) - \mathcal{L}[s^I(t, \mathbf{k}), \dot{s}^I(t, \mathbf{k})], \quad (10)$$

where s^I and π_{s^I} are, respectively, sets of canonical fields and their conjugate momenta. If the system has n primary constraints $\mathcal{C}_i$, the total Hamiltonian and its density are given by

$$H_T(t) = \int \mathrm{d}^3 k \mathcal{H}_T(t, \mathbf{k}),$$
$$\mathcal{H}_T(t, \mathbf{k}) = \mathcal{H}(t, \mathbf{k}) + \sum_{a=1}^{n} \lambda_a(t, \mathbf{k}) \mathcal{C}_a(t, \mathbf{k}), \quad (11)$$

where λ_i are Lagrange multipliers associated with each primary constraint $\mathcal{C}_i$. The Poisson bracket between $\mathcal{A}$ and $\mathcal{B}$ is defined by

$$\{\mathcal{A}(t, \mathbf{k}), \mathcal{B}(t, \mathbf{k}')\} = \int d^3 k'' \sum_I \left[\frac{\delta \mathcal{A}(t, \mathbf{k})}{\delta s^I(t, \mathbf{k}'')} \frac{\delta \mathcal{B}(t, \mathbf{k}')}{\delta \pi_{s^I}(t, \mathbf{k}'')} - \frac{\delta \mathcal{A}(t, \mathbf{k})}{\delta \pi_{s^I}(t, \mathbf{k}'')} \frac{\delta \mathcal{B}(t, \mathbf{k}')}{\delta s^I(t, \mathbf{k}'')} \right]. \quad (12)$$

The time evolution of the function $\mathcal{A}(t, \mathbf{k})$ is given by

$$\begin{aligned} \dot{\mathcal{A}}(t, \mathbf{k}) &= \{\mathcal{A}(t, \mathbf{k}), H_T(t)\} \\ &= \int \mathrm{d}^3 k' \Big[\{\mathcal{A}(t, \mathbf{k}), \mathcal{H}(t, \mathbf{k}')\} \\ &\quad + \sum_{a=1}^{n} \lambda_a(t, \mathbf{k}') \{\mathcal{A}(t, \mathbf{k}), \mathcal{C}_a(t, \mathbf{k}')\} \Big]. \end{aligned} \quad (13)$$

III. HAMILTONIAN ANALYSIS AND CLASSIFICATION

In this section, we perform the Hamiltonian analysis for the theory (1) and classify the theory based on the number of d.o.f. and constraint structures. Since the number of d.o.f. in the theory is ten in general, one needs to adequately eliminate extra d.o.f. in each mode decomposed in the previous section. We first take a look at the tensor mode and derive the condition for avoiding a ghost mode. Then we seek conditions to eliminate unwanted modes for vector and scalar modes and conditions to have subsequent constraints. The existence of ghost d.o.f. in the scalar sector is proved in Appendix B when the total number of d.o.f. is more than five.

A. Tensor modes

The action in the tensor sector is given by

$$S^T[H_{ij}] = 4 \int \mathrm{d}t\, \mathrm{d}^3 k [\kappa_1 \dot{H}_{ij}^2 - (\kappa_1 k^2 + \mu_1) H_{ij}^2], \quad (14)$$

where a dot represents the derivative with respect to time t. As one can see from Eq. (14), the tensor modes are controlled by only two parameters κ_1 and μ_1, and the existence of tensor modes and the condition for avoiding the ghost instability demand

$$\text{“Condition 1”:}\ \kappa_1 > 0. \quad (15)$$

Throughout this paper, we always assume condition 1 [Eq. (15)], and then the number of the d.o.f. in the tensor sector is two. Furthermore, the parameter μ_1 should not be negative in order to avoid the tachyonic instability in the tensor sector.

B. Vector modes

In this subsection, we focus on the vector modes and find conditions to avoid extra ghost d.o.f. based on the Hamiltonian analysis. Before proceeding with the analysis, let us comment on counting the number of physical d.o.f. and constraints in the vector modes. Since a vector in vector modes V_i satisfies the transverse condition $k^i V_i = 0$, d.o.f. in V_i are two while there are three components. Hence when we have primary constraints of a vector type like $\pi_{V_i} = 0$, this should be understood as two primary constraints and not three since the transverse conditions put a constraint for one component in V_i and primary constraints should be obtained for the remaining 2 d.o.f.

The action for the vector modes can be written with the replacement $kF_i \to F_i$,

$$S^V[B_i, F_i] = \int \mathrm{d}t\, \mathrm{d}^3k[-(2\kappa_1 + \kappa_2)\dot{B}_i^2 + 2\kappa_1 \dot{F}_i^2 + 2\kappa_2 k B_i \dot{F}_i + 2(\kappa_1 k^2 + \mu_1)B_i^2 - (k^2(2\kappa_1 + \kappa_2) + 2\mu_1)F_i^2]. \quad (16)$$

One may immediately notice from Eq. (16) that there are the appearance of either ghost or gradient instabilities in B_i or F_i modes, depending on the sign of $2\kappa_1 + \kappa_2$ as well as κ_1. This concludes that one needs to at least eliminate either B_i or F_i in order to have 2 d.o.f. The existence of the tensor modes (15) leads to the unique option to have a primary constraint for B_i, that is,

$$\text{“Condition 2”}: \ 2\kappa_1 + \kappa_2 = 0 \Leftrightarrow \kappa_2 = -2\kappa_1. \quad (17)$$

With this condition in Eq. (17), the kinetic term of B_i vanishes, which implies B_i are manifestly nondynamical. Then the action for the vector mode can be recast as

$$S^V[B_i, F_i] = \int \mathrm{d}t\, \mathrm{d}^3k \mathcal{L}^V = \int \mathrm{d}t\, \mathrm{d}^3k[2\kappa_1 \dot{F}_i^2 - 4\kappa_1 k B_i \dot{F}_i + 2(\kappa_1 k^2 + \mu_1)B_i^2 - 2\mu_1 F_i^2]. \quad (18)$$

Apparently, the action for the vector modes depends on only two parameters, κ_1 and μ_1, as in the tensor modes. The conjugate momenta for B_i and F_i are given by

$$\pi_{B_i} \equiv \frac{\delta \mathcal{L}^V}{\delta \dot{B}_i} = 0, \quad (19)$$

$$\pi_{F_i} \equiv \frac{\delta \mathcal{L}^V}{\delta \dot{F}_i} = 4\kappa_1(\dot{F}_i - kB_i), \quad (20)$$

and we therefore have two primary constraints instead of three, as mentioned at the beginning of this subsection, which are defined by

$$\mathcal{C}_1^{B_i} = \pi_{B_i} = 0. \quad (21)$$

Then the Hamiltonian and the total Hamiltonian densities read

$$\mathcal{H}^V = \dot{B}_i \pi_{B_i} + \dot{F}_i \pi_{F_i} - \mathcal{L}^V \approx \frac{\pi_{F_i}^2}{8\kappa_1} + k\pi_{F_i} B_i - 2\mu_1 B_i^2 + 2\mu_1 F_i^2,$$
$$\mathcal{H}_T^V = \mathcal{H}^V + \lambda_{B_i}\pi_{B_i}, \quad (22)$$

where λ_{B_i} are Lagrange multipliers. We have suppressed the terms $\dot{B}_i \pi_{B_i}$ in the final expression of $\mathcal{H}^V$ since they vanish once the primary constraints (21) are imposed. One can easily check that the evolution of the primary constraints automatically yields secondary constraints,

$$\mathcal{C}_2^{B_i} \equiv \dot{\mathcal{C}}_1^{B_i} = \{\mathcal{C}_1^{B_i}, H_T^V\} = \{\mathcal{C}_1^{B_i}, H^V\} = k\pi_{F_i} + 4\mu_1 B_i \approx 0. \quad (23)$$

Then the time evolution of the secondary constraints are given by[1]

$$\dot{\mathcal{C}}_2^{B_i} = \{\mathcal{C}_2^{B_i}, H_T^V\} = \{\mathcal{C}_2^{B_i}, H^V\} + \{\mathcal{C}_2^{B_i}, \mathcal{C}_1^{B_j}\}\lambda_{B_j} \approx 0, \quad (24)$$

where the coefficients of λ_{B_i} are given by

$$\{\mathcal{C}_2^{B_i}, \mathcal{C}_1^{B_j}\} = 4\mu_1 \delta_{ij}. \quad (25)$$

Therefore, we have two cases:

Case V1: $\mu_1 = 0$.

In this case, in addition that the Poisson bracket Eq. (25) vanishes, $\{\mathcal{C}_2^{B_i}, H^V\}$ is trivially zero. Then, there is no more constraint. Thus there are two primary constraints $\mathcal{C}_1^{B_i}$ and two secondary constraints $\mathcal{C}_2^{B_i}$, and all of them are first class since all the Poisson brackets between these constraints vanish. Therefore,

$$\text{vector d.o.f.} = \frac{4 \times 2 - 4(2\,\text{primary} \,\&\, 2\,\text{secondary}) \times 2\,(\text{first class})}{2} = 0. \quad (26)$$

This case is exactly the same as the linearized Einstein's gravity, and thus the Lagrangian is invariant under the gauge transformation, $B_i \to B_i + \dot{\zeta}_i$ and $F_i \to F_i + \zeta_i$, where ζ_i is the 3-vector satisfying the transverse condition $\partial^i \zeta_i = 0$. Note that a theory with only 1 d.o.f. in the vector sector is prohibited by spatial covariance of the theory.

Case V2: $\mu_1 \neq 0$.

When $\mu_1 \neq 0$, the last equation (24) can be used to determine the Lagrange multipliers λ_{B_i},

[1]To be precise, one needs the integral over the Fourier space in front of the Lagrange multipliers λ_{B_j}, which can always be integrable because of the appearance of Dirac's delta function. For simplicity, we omit this integral and the arguments of each variable since the results do not change.

$$\lambda_{B_i} \approx \frac{1}{4\mu_1}\{\mathcal{C}_2^{B_i}, H^V\}. \tag{27}$$

Thus, there are two primary constraints $\mathcal{C}_1^{B_i}$ and two secondary constraints $\mathcal{C}_2^{B_i}$, and all of them are second-class since the Poisson brackets between these constraints are nonvanishing. Therefore, the number of d.o.f. for the vector modes is given by

$$\text{vector DOFs} = \frac{4\times 2 - 4(2\,\text{primary}\,\&\,2\,\text{secondary})}{2} = 2. \tag{28}$$

This case includes the FP theory.

C. Scalar modes

In this subsection, we investigate the scalar modes and classify theories by finding the condition to avoid the appearance of extra ghost d.o.f. In Appendix B, we see that there are dangerous d.o.f. if the system has 2 or more d.o.f. We also derive gauge transformation and conditions for avoiding instabilities of obtained theories in Appendix C.

As for scalar perturbations, by introducing dimensionless quantities β and $\mathcal{E}$, which are defined by $\beta \equiv k\hat{\beta}$ and $\mathcal{E} \equiv k^2\hat{\mathcal{E}}$, respectively, the action reduces to

$$S^S[\alpha,\beta,\mathcal{R},\mathcal{E}] = \int \mathrm{d}t\,\mathrm{d}^3k\mathcal{L}^S = \int \mathrm{d}t\,\mathrm{d}^3k(\mathcal{L}^S_{\text{kin}} + \mathcal{L}^S_{\text{cross}} + \mathcal{L}^S_{\text{mass}}), \tag{29}$$

where

$$\mathcal{L}^S_{\text{kin}} = 4(\kappa_1+\kappa_2+\kappa_3+\kappa_4)\dot{\alpha}^2 - (2\kappa_1+\kappa_2)\dot{\beta}^2 + 12(\kappa_1+3\kappa_4)\dot{\mathcal{R}}^2 + 4(\kappa_1+\kappa_4)\dot{\mathcal{E}}^2 - 4(\kappa_3+2\kappa_4)(-3\dot{\mathcal{R}}+\dot{\mathcal{E}})\dot{\alpha} - 8(\kappa_1+3\kappa_4)\dot{\mathcal{R}}\,\dot{\mathcal{E}}, \tag{30}$$

$$\mathcal{L}^S_{\text{cross}} = -4[(\kappa_2+\kappa_3)\dot{\alpha} + (\kappa_2+3\kappa_3)\dot{\mathcal{R}} - (\kappa_2+\kappa_3)\dot{\mathcal{E}}]k\beta, \tag{31}$$

$$\begin{aligned}\mathcal{L}^S_{\text{mass}} = {}& -4[k^2(\kappa_1+\kappa_4)+\mu_1+\mu_2]\alpha^2 + [k^2(2\kappa_1+\kappa_2)+2\mu_1]\beta^2 \\ & -4[k^2(3\kappa_1+\kappa_2+3\kappa_3+9\kappa_4)+3(\mu_1+3\mu_2)]\mathcal{R}^2 - 4[k^2(\kappa_1+\kappa_2+\kappa_3+\kappa_4)+\mu_1+\mu_2]\mathcal{E}^2 \\ & -4[(k^2(\kappa_3+6\kappa_4)+6\mu_2)\mathcal{R} - (k^2(\kappa_3+2\kappa_4)+2\mu_2)\mathcal{E}]\alpha \\ & +8[k^2(\kappa_1+\kappa_2+2\kappa_3+3\kappa_4)+(\mu_1+3\mu_2)]\mathcal{R}\mathcal{E}.\end{aligned} \tag{32}$$

Now under condition 2 [Eq. (17)], $2\kappa_1+\kappa_2=0$, the time derivative of β vanishes in the Lagrangian. Then β becomes nondynamical.

The canonical momenta for $Q=\{\alpha,\beta,\mathcal{R},\mathcal{E}\}$ are defined by $\pi_Q \equiv \delta\mathcal{L}^S/\delta\dot{Q}$ and read

$$\begin{pmatrix}\pi_\alpha\\ \pi_\beta\\ \pi_{\mathcal{R}}\\ \pi_{\mathcal{E}}\end{pmatrix} = 4\mathcal{K}^S\begin{pmatrix}\dot{\alpha}\\ \dot{\beta}\\ \dot{\mathcal{R}}\\ \dot{\mathcal{E}}\end{pmatrix} + 4\begin{pmatrix}2\kappa_1-\kappa_3\\ 0\\ 2\kappa_1-3\kappa_3\\ -2\kappa_1+\kappa_3\end{pmatrix}k\beta, \tag{33}$$

where

$$\mathcal{K}^S \equiv \begin{pmatrix}2(-\kappa_1+\kappa_3+\kappa_4) & 0 & 3(\kappa_3+2\kappa_4) & -(\kappa_3+2\kappa_4)\\ 0 & 0 & 0 & 0\\ 3(\kappa_3+2\kappa_4) & 0 & 6(\kappa_1+3\kappa_4) & -2(\kappa_1+3\kappa_4)\\ -(\kappa_3+2\kappa_4) & 0 & -2(\kappa_1+3\kappa_4) & 2(\kappa_1+\kappa_4)\end{pmatrix}. \tag{34}$$

Let us calculate the determinant of the kinetic matrix $\mathcal{K}^S$, which is given by

$$|\mathcal{K}^S| = -4\kappa_1(4\kappa_1^2 - 4\kappa_1\kappa_3 + 8\kappa_1\kappa_4 + 3\kappa_3^2). \tag{35}$$

Taking into account the condition for having the tensor mode $\kappa_1 \neq 0$ the determinant vanishes only when

$$\text{“Condition 3”}:\ 4\kappa_1^2 - 4\kappa_1\kappa_3 + 8\kappa_1\kappa_4 + 3\kappa_3^2 = 0 \iff \kappa_4 = -\frac{4\kappa_1^2 - 4\kappa_1\kappa_3 + 3\kappa_3^2}{8\kappa_1}. \tag{36}$$

Then, the degeneracy of the kinetic matrix (34) leads to an additional primary constraint in addition to the one for β, $\pi_\beta = 0$. Note that the linearized Einshtein-Hilbert kinetic term satisfies condition 3 [Eq. (36)]. When this degeneracy condition is satisfied, one of the eigenvalues of the kinetic matrix vanishes and the remaining eigenvalues λ will be a solution of the following eigenequation:

$$\lambda^2 + \left(10\kappa_1 - 26\kappa_3 + \frac{33\kappa_3^2}{2\kappa_1}\right)\lambda = 8[4(\kappa_1 - \kappa_3)^2 + \kappa_3^2] > 0. \tag{37}$$

As long as $\kappa_{1,3}$ are real, other eigenvalues will be nonvanishing.

1. One primary constraint: $4\kappa_1^2 - 4\kappa_1\kappa_3 + 8\kappa_1\kappa_4 + 3\kappa_3^2 \neq 0$

Let us consider the case with only one primary constraint, i.e., condition 3 [Eq. (36)] is not imposed. Thus, we have a primary constraint,

$$\mathcal{C}_1^\beta = \pi_\beta = 0. \tag{38}$$

The Hamiltonian and the total Hamiltonian densities in the scalar sector reads

$$\begin{aligned}\mathcal{H}^S &= \dot{\alpha}\pi_\alpha + \dot{\beta}\pi_\beta + \dot{\mathcal{R}}\pi_\mathcal{R} + \dot{\mathcal{E}}\pi_\mathcal{E} - \mathcal{L}^S \\ &\approx 4[\mu_1 + \mu_2 + k^2(\kappa_1 + \kappa_4)]\alpha^2 - 2\mu_1\beta^2 + 4[\mu_1 + \mu_2 - k^2(\kappa_1 - \kappa_3 - \kappa_4)]\mathcal{E}^2 \\ &\quad - 8[\mu_1 + 3\mu_2 - k^2(\kappa_1 - 2\kappa_3 - 3\kappa_4)]\mathcal{E}\mathcal{R} + 4[3\mu_1 + 9\mu_2 + k^2(\kappa_1 + 3\kappa_3 + 9\kappa_4)]\mathcal{R}^2 \\ &\quad - [4(2\mu_2 + k^2(\kappa_3 + 2\kappa_4))\mathcal{E} - 4(6\mu_2 + k^2(\kappa_3 + 6\kappa_4))\mathcal{R}]\alpha \\ &\quad + \frac{1}{32\kappa_1}(2\pi_\mathcal{R} + 3\pi_\mathcal{E})\pi_\mathcal{E} + (\pi_\alpha + \pi_\mathcal{E})k\beta - \frac{8\kappa_1(\kappa_1 + 3\kappa_4)\pi_\alpha^2 - 8\kappa_1(\kappa_3 + 2\kappa_4)\pi_\alpha\pi_\mathcal{R} - (2\kappa_1 - \kappa_3)^2\pi_{\mathcal{R}^2}}{32\kappa_1(4\kappa_1^2 - 4\kappa_1\kappa_3 + 3\kappa_3^2 + 8\kappa_1\kappa_4)},\end{aligned} \tag{39}$$

$$\mathcal{H}_T^S = \mathcal{H}^S + \lambda_\beta\pi_\beta. \tag{40}$$

We have suppressed the term $\dot{\beta}\pi_\beta$ in the final expression of $\mathcal{H}^S$ since it vanishes once the primary constraint (38) is imposed. The evolution of the primary constraint automatically yields a secondary constraint

$$\mathcal{C}_2^\beta \equiv \dot{\mathcal{C}}_1^\beta = \{\mathcal{C}_1^\beta, H_T^S\} = \{\mathcal{C}_1^\beta, H^S\} = -k\pi_\alpha - k\pi_\mathcal{E} + 4\mu_1\beta \approx 0. \tag{41}$$

Then, the Poisson bracket between the secondary and primary constraints of β is given by

$$\{\mathcal{C}_2^\beta, \mathcal{C}_1^\beta\} = 4\mu_1. \tag{42}$$

If $\mu_1 \neq 0$, then no more constraints will be generated and the last equation can be used to determine the value of λ_β, that is $\lambda_\beta \approx -\{\mathcal{C}_2^\beta, H^S\}/\{\mathcal{C}_2^\beta, \mathcal{C}_1^\beta\}$. In this case, we have one primary constraint $\mathcal{C}_1^\beta$, and one secondary constraint $\mathcal{C}_2^\beta$, all of which are second class. Therefore, the number of d.o.f. in the scalar sector is $(8-2)/2 = 3$, signaling the existence of an extra d.o.f. The explicit proof of the existence of a ghost d.o.f. is given in Appendix B. Therefore, one has to impose an extra condition $\mu_1 = 0$ in order to eliminate this extra d.o.f.

Case SI: $\mu_1 = 0$, $4\kappa_1^2 - 4\kappa_1\kappa_3 + 3\kappa_3^2 + 8\kappa_1\kappa_4 \neq 0$.

If $\mu_1 = 0$, the consistency of the secondary constraint $\mathcal{C}_2^\beta$ generates a tertiary constraint,

$$\begin{aligned}\mathcal{C}_3^\beta &\equiv \dot{\mathcal{C}}_2^\beta = \{\mathcal{C}_2^\beta, H_T^S\} = \{\mathcal{C}_2^\beta, H^S\} \\ &= 4k^3[(2\kappa_1 - \kappa_3)\alpha + (2\kappa_1 - 3\kappa_3)\mathcal{R} - (2\kappa_1 - \kappa_3)\mathcal{E}] \approx 0.\end{aligned} \tag{43}$$

Now one can check that all the constraints commute each other, i.e., $\{\mathcal{C}_i^\beta, \mathcal{C}_j^\beta\} = 0$ where $(i, j = 1, 2, 3)$. One can also check that $\dot{\mathcal{C}}_3^\beta = k^2\mathcal{C}_2^\beta$, implying no more constraint. There are one primary constraint $\mathcal{C}_1^\beta$, one secondary constraint $\mathcal{C}_2^\beta$, and one tertiary constraint $\mathcal{C}_3^\beta$, and all of them are first class. Therefore,

$$\text{scalar d.o.f.} = \frac{4 \times 2 - 3(1\,\text{primary}\,\&\,1\,\text{secondary}\,\&\,1\,\text{tertiary}) \times 2(\text{first class})}{2} = 1. \tag{44}$$

Since $\mu_1 = 0$ corresponds to case V1, the vector sector does not have any d.o.f., and the total number of d.o.f. in this case is three.

2. Two primary constraints: $4\kappa_1^2 - 4\kappa_1\kappa_3 + 8\kappa_1\kappa_4 + 3\kappa_3^2 = 0$

Hereafter we impose condition 3 [Eq. (36)], implying the existence of two primary constraints. As one can easily see, the canonical momenta π_α and $\pi_{\mathcal{R}}$ are not independent because of the degeneracy condition (36), i.e.,

$$(2\kappa_1 - 3\kappa_3)\pi_\alpha - (2\kappa_1 - \kappa_3)\pi_{\mathcal{R}} = 0. \tag{45}$$

This clearly shows that there are two primary constraints in this class of theories, namely, Eqs. (38) and (45). We classify theories depending on vanishing or nonvanishing of the coefficient of π_α, $2\kappa_1 - 3\kappa_3$. Hereafter, we will focus only on the case of $2\kappa_1 - 3\kappa_3 \neq 0$. The Hamiltonian analysis in the class of models with $2\kappa_1 - 3\kappa_3 = 0$ is presented in Appendix A.

In this case, we have the following two primary constraints:

$$\mathcal{C}_1^\alpha = \pi_\alpha - \frac{2\kappa_1 - \kappa_3}{2\kappa_1 - 3\kappa_3}\pi_{\mathcal{R}} = 0, \qquad \mathcal{C}_1^\beta = \pi_\beta = 0. \tag{46}$$

Here it is clear that $\mathcal{C}_1^\alpha$ and $\mathcal{C}_1^\beta$ commute each other. Then the Hamiltonian and the total Hamiltonian densities read

$$\begin{aligned}
\mathcal{H}^S &= \dot{\alpha}\pi_\alpha + \dot{\beta}\pi_\beta + \dot{\mathcal{R}}\pi_{\mathcal{R}} + \dot{\mathcal{E}}\pi_{\mathcal{E}} - \mathcal{L}^S, \\
&\approx \left[\frac{k^2(2\kappa_1 - \kappa_3)(2\kappa_1 + 3\kappa_3)}{2\kappa_1} + 4(\mu_1 + \mu_2)\right]\alpha^2 - 2\mu_1\beta^2 + \left[-\frac{3k^2(2\kappa_1 - \kappa_3)^2}{2\kappa_1} + 4(\mu_1 + \mu_2)\right]\mathcal{E}^2 \\
&\quad + \left[\frac{k^2(2\kappa_1 - \kappa_3)(10\kappa_1 - 9\kappa_3)}{\kappa_1} - 8(\mu_1 + 3\mu_2)\right]\mathcal{E}\mathcal{R} + \left[-\frac{k^2(14\kappa_1 - 9\kappa_3)(2\kappa_1 - 3\kappa_3)}{2\kappa_1} + 12(\mu_1 + 3\mu_2)\right]\mathcal{R}^2 \\
&\quad + \left[\frac{k^2(2\kappa_1 - \kappa_3)(2\kappa_1 - 3\kappa_3)}{\kappa_1} - 8\mu_2\right]\alpha\mathcal{E} + \left[-\frac{k^2(12\kappa_1^2 - 16\kappa_1\kappa_3 + 9\kappa_3^2)}{\kappa_1} + 24\mu_2\right]\alpha\mathcal{R} \\
&\quad + k\beta\pi_{\mathcal{E}} + \frac{3}{32\kappa_1}\pi_{\mathcal{E}}^2 + \frac{2\kappa_1 - \kappa_3}{2\kappa_1 - 3\kappa_3}k\beta\pi_{\mathcal{R}} + \frac{1}{16\kappa_1}\pi_{\mathcal{R}}\pi_{\mathcal{E}} - \frac{(2\kappa_1 - \kappa_3)(2\kappa_1 + 3\kappa_3)}{32\kappa_1(2\kappa_1 - 3\kappa_3)^2}\pi_{\mathcal{R}}^2,
\end{aligned} \tag{47}$$

and

$$\mathcal{H}_T^S = \mathcal{H}^S + \lambda_\alpha\mathcal{C}_1^\alpha + \lambda_\beta\mathcal{C}_1^\beta, \tag{48}$$

where we have suppressed the terms proportional to the primary constraints (46) in the final expression of $\mathcal{H}^S$.

The evolution of the primary constraints requires

$$\mathcal{C}_2^\alpha \equiv \dot{\mathcal{C}}_1^\alpha = \{\mathcal{C}_1^\alpha, H_T^S\} = \{\mathcal{C}_1^\alpha, H^S\} = c_2^\alpha\alpha + c_2^{\mathcal{R}}\mathcal{R} + c_2^{\mathcal{E}}\mathcal{E} \approx 0, \tag{49}$$

$$\begin{aligned}
\mathcal{C}_2^\beta &\equiv \dot{\mathcal{C}}_1^\beta = \{\mathcal{C}_1^\beta, H_T^S\} = \{\mathcal{C}_1^\beta, H^S\} \\
&= -\frac{2\kappa_1 - \kappa_3}{2\kappa_1 - 3\kappa_3}k\pi_{\mathcal{R}} - k\pi_{\mathcal{E}} + 4\mu_1\beta \approx 0,
\end{aligned} \tag{50}$$

where we have defined the coefficients,

$$c_2^\alpha = -\frac{8[\mu_1(2\kappa_1 - 3\kappa_3) - 4\mu_2\kappa_1 + 2k^2(2\kappa_1 - \kappa_3)(\kappa_1 - \kappa_3)]}{2\kappa_1 - 3\kappa_3}, \tag{51}$$

$$c_2^{\mathcal{R}} = -\frac{8[-3\mu_1(2\kappa_1 - \kappa_3) - 12\mu_2\kappa_1 + 2k^2(2\kappa_1 - 3\kappa_3)(\kappa_1 - \kappa_3)]}{2\kappa_1 - 3\kappa_3}, \tag{52}$$

$$c_2^{\mathcal{E}} = -\frac{8[\mu_1(2\kappa_1 - \kappa_3) + 4\mu_2\kappa_1 - 2k^2(2\kappa_1 - \kappa_3)(\kappa_1 - \kappa_3)]}{2\kappa_1 - 3\kappa_3}. \tag{53}$$

In general we will have two secondary constraints, but there exists an exceptional case where only one secondary constraint exists if $\kappa_1 = \kappa_3$. With this condition satisfied, the terms proportional to k^2 in the numerator of Eqs. (51)–(53) vanish and hence $\mathcal{C}_2^\alpha$ reduces to a trivial equation depending on the value of μ_1 and μ_2. This special case happens when either $\mu_1 = \mu_2 = 0$ or $\mu_1 + 4\mu_2 = 0$ is satisfied, which will be investigated at the end of this subsection. In the meantime, we consider the case with two nontrivial secondary constraints assuming $\kappa_1 \neq \kappa_3$.

The Poisson brackets between the primary and secondary constraints are given by

$$\{\mathcal{C}_2^\alpha, \mathcal{C}_1^\alpha\} = -\frac{32}{(2\kappa_1 - 3\kappa_3)^2}[\mu_1(4\kappa_1^2 - 6\kappa_1\kappa_3 + 3\kappa_3^2) + 4\mu_2\kappa_1^2], \tag{54}$$

$$\{\mathcal{C}_2^\beta, \mathcal{C}_1^\beta\} = 4\mu_1. \tag{55}$$

Thus, as long as these Poisson brackets are nonvanishing, no more constraints will be generated. In this case, we have two primary constraints $\mathcal{C}_1^\alpha$, $\mathcal{C}_1^\beta$ and two secondary constraints $\mathcal{C}_2^\alpha$, $\mathcal{C}_2^\beta$, and all the constraints are second class, which implies the number of d.o.f. is $(8-4)/2 = 2$. The explicit proof of the existence of a ghost d.o.f. is given in Appendix B. Therefore, one needs to eliminate an extra d.o.f. In order to have an extra constraint, there are three options: both μ_1 and μ_2 vanish (case SIIa), either (54) (case SIIc) or (55) (case SIIb) vanishes.

Case SIIa: $\mu_1 = \mu_2 = 0$.

If $\mu_1 = \mu_2 = 0$, both Eqs. (54) and (55) vanish. In addition, one can see $\dot{\mathcal{C}}_2^\alpha \propto \mathcal{C}_2^\beta \approx 0$ and $\dot{\mathcal{C}}_2^\beta \propto \mathcal{C}_2^\alpha \approx 0$, which implies that no more constraint is therefore generated. Since all the constraints commute, all the primary and secondary constraints are first class. Therefore,

$$\text{scalar d.o.f.} = \frac{4\times 2 - 4(2\,\text{primary}\,\&\,2\ \text{secondary})\times 2(\text{first class})}{2} = 0. \tag{56}$$

In this case, the scalar mode as well as the vector mode do not have any d.o.f. (case V1), and only the tensor mode can propagate. Since the linearized Einstein-Hilbert term satisfies condition 3 [Eq. (36)], and the mass terms are absent, this class reduces to linearized general relativity when $2\kappa_1 - \kappa_3 = 0$. As we will see in the next section, the whole parameter family of this case SIIa can be mapped from linearized general relativity by a field redefinition.

Case SIIb: $\mu_1 = 0$, $\mu_2 \neq 0$.

In this case, since Eq. (54) is nonvanishing while Eq. (55) vanishes, λ_α is determined by $\dot{\mathcal{C}}_2^\alpha \approx 0$, that is $\lambda_\alpha = -\{\mathcal{C}_2^\alpha, H^S\}/\{\mathcal{C}_2^\alpha, \mathcal{C}_1^\alpha\}$. On the other hand as for $\mathcal{C}_2^\beta$, since $\mathcal{C}_2^\beta$ does not commute with $\mathcal{C}_2^\alpha$, we shall consider a linear combination of $\mathcal{C}_2^\beta$ and $\mathcal{C}_1^\alpha$ which commute with $\mathcal{C}_2^\alpha$ instead of the original $\mathcal{C}_2^\beta$:

$$\tilde{\mathcal{C}}_2^\beta \equiv \mathcal{C}_2^\beta - k\mathcal{C}_1^\alpha = -k(\pi_\alpha + \pi_\mathcal{E}). \tag{57}$$

The consistency of $\tilde{\mathcal{C}}_2^\beta$ yields a tertiary constraint,

$$\begin{aligned}\mathcal{C}_3^\beta \equiv \dot{\tilde{\mathcal{C}}}_2^\beta &= \{\tilde{\mathcal{C}}_2^\beta, H_T^S\} = \{\tilde{\mathcal{C}}_2^\beta, H^S\} \\ &= 4k^3[(2\kappa_1 - \kappa_3)\alpha + (2\kappa_1 - 3\kappa_3)\mathcal{R} - (2\kappa_1 - \kappa_3)\mathcal{E}] \approx 0.\end{aligned} \tag{58}$$

The Poisson brackets between this constraint and primary constraints vanish and one also find $\dot{\mathcal{C}}_3^\beta = k^2\mathcal{C}_2^\beta = k^2(\tilde{\mathcal{C}}_2^\beta + k\mathcal{C}_1^\alpha) \approx 0$, implying that no more constraint is generated. One can straightforwardly show that the constraints $\mathcal{C}_1^\beta$, $\tilde{\mathcal{C}}_2^\beta$, and $\mathcal{C}_3^\beta$ commute with all constraints including themselves. Therefore, there are two primary constraints $\mathcal{C}_1^\alpha$, $\mathcal{C}_1^\beta$, two secondary constraints $\mathcal{C}_2^\alpha$, $\tilde{\mathcal{C}}_2^\beta$, and one tertiary constraint $\mathcal{C}_3^\beta$. The constraints $\mathcal{C}_1^\beta$, $\tilde{\mathcal{C}}_2^\beta$, and $\mathcal{C}_3^\beta$ are first class, and the rest of them are second class. Therefore,

$$\text{scalar d.o.f.} = \frac{4\times 2 - 3(1\,\text{primary}\,\&\,1\,\text{secondary}\,\&\,1\,\text{tertiary})\times 2(\text{first class}) - 2(1\,\text{primary}\,\&\,1\,\text{secondary})}{2} = 0. \tag{59}$$

In this case, the vector mode does not propagate (case V1) and the total number of d.o.f. is two.

Case SIIc: $\mu_1(4\kappa_1^2 - 6\kappa_1\kappa_3 + 3\kappa_3^2) + 4\mu_2\kappa_1^2 = 0$, $\mu_1 \neq 0$.

In this case, since Eq. (55) does not vanish, λ_β can be determined by imposing $\dot{\mathcal{C}}_2^\beta \approx 0$, that is $\lambda_\beta = -\{\mathcal{C}_2^\beta, H^S\}/\{\mathcal{C}_2^\beta, \mathcal{C}_1^\beta\}$, and hence no more constraint will be generated as for $\mathcal{C}_1^\beta$. On the other hand, we solve the condition for μ_2 from $\{\mathcal{C}_2^\alpha, \mathcal{C}_1^\alpha\} = 0$, which is given by

$$\text{“Condition 4”}:\ \mu_2 = -\frac{4\kappa_1^2 - 6\kappa_1\kappa_3 + 3\kappa_3^2}{4\kappa_1^2}\mu_1. \tag{60}$$

Since $\mathcal{C}_2^\alpha$ does not commute with $\mathcal{C}_2^\beta$, we shall consider a linear combination of $\mathcal{C}_2^\alpha$ and $\mathcal{C}_1^\beta$ which commutes with $\mathcal{C}_2^\beta$ instead of the original $\mathcal{C}_2^\alpha$:

$$\tilde{\mathcal{C}}_2^\alpha \equiv \mathcal{C}_2^\alpha + k\frac{8(\kappa_1 - \kappa_3)}{2\kappa_1 - 3\kappa_3}\mathcal{C}_1^\beta. \tag{61}$$

Then, the time consistency of $\tilde{\mathcal{C}}_2^\alpha$ yields a tertiary constraint

$$\begin{aligned}\mathcal{C}_3^\alpha \equiv \dot{\tilde{\mathcal{C}}}_2^\alpha &= \{\tilde{\mathcal{C}}_2^\alpha, H_T^S\} = \{\tilde{\mathcal{C}}_2^\alpha, H^S\} \\ &= -\frac{4(\kappa_1 - \kappa_3)}{2\kappa_1 - 3\kappa_3}\left(\frac{k^2(2\kappa_1 - \kappa_3) - 2\mu_1}{2\kappa_1 - 3\kappa_3}\pi_\mathcal{R} + k^2\pi_\mathcal{E}\right) \approx 0.\end{aligned} \tag{62}$$

The Poisson bracket between this constraint and primary constraints vanishes and the time consistency of $\mathcal{C}_3^\alpha$ yields a quaternary constraint

$$\mathcal{C}_4^\alpha \equiv \dot{\mathcal{C}}_3^\alpha = \{\mathcal{C}_3^\alpha, H_T^S\} = \{\mathcal{C}_3^\alpha, H^S\} = c_4^\alpha\alpha + c_4^\mathcal{R}\mathcal{R} + c_4^\mathcal{E}\mathcal{E} \approx 0, \tag{63}$$

where

$$c_4^\alpha = -\frac{8(\kappa_1-\kappa_3)}{\kappa_1^2(2\kappa_1-3\kappa_3)^2}[2k^4\kappa_1^2(2\kappa_1-\kappa_3)^2 + k^2\kappa_1\mu_1(2\kappa_1-3\kappa_3)(2\kappa_1-\kappa_3) - 6\mu_1^2(4\kappa_1^2-6\kappa_1\kappa_3+3\kappa_3^2)], \quad (64)$$

$$c_4^\mathcal{R} = -\frac{8(\kappa_1-\kappa_3)}{\kappa_1^2(2\kappa_1-3\kappa_3)}[2k^4\kappa_1^2(2\kappa_1-\kappa_3) + k^2\kappa_1\mu_1(2\kappa_1-3\kappa_3) - 6\mu_1^2(4\kappa_1-3\kappa_3)], \quad (65)$$

$$c_4^\mathcal{E} = \frac{8(\kappa_1-\kappa_3)}{\kappa_1^2(2\kappa_1-3\kappa_3)^2}[2k^4\kappa_1^2(2\kappa_1-\kappa_3)^2 - k^2\kappa_1\mu_1(2\kappa_1+3\kappa_3)(2\kappa_1-\kappa_3) - 2\mu_1^2(2\kappa_1-3\kappa_3)(4\kappa_1-3\kappa_3)]. \quad (66)$$

The Poisson brackets between this constraint and primary constraints are

$$\{\mathcal{C}_4^\alpha, \mathcal{C}_1^\alpha\} = -\frac{192\mu_1^2(\kappa_1-\kappa_3)^2}{\kappa_1(2\kappa_1-3\kappa_3)^2}, \qquad \{\mathcal{C}_4^\alpha, \mathcal{C}_1^\beta\} = 0. \quad (67)$$

Then, the consistency of this constraint $\dot{\mathcal{C}}_4^\alpha \approx 0$ fixes the Lagrange multiplier $\lambda_\alpha = -\{\mathcal{C}_4^\alpha, H^S\}/\{\mathcal{C}_4^\alpha, \mathcal{C}_1^\alpha\}$. There are six second class constraints $\mathcal{C}_1^\alpha$, $\mathcal{C}_1^\beta$, $\tilde{\mathcal{C}}_2^\alpha$, $\mathcal{C}_2^\beta$, $\mathcal{C}_3^\alpha$, and $\mathcal{C}_4^\alpha$. Therefore,

$$\text{scalar d.o.f.} = \frac{4\times 2 - 6(2\,\text{primary}\,\&\,2\,\text{secondary}\,\&\,1\,\text{tertiary}\,\&\,1\,\text{quaternary})}{2} = 1. \quad (68)$$

In this case, the total number of d.o.f. is five (case V2). It should be noted that the Fierz-Pauli theory is included in this class since the linearized Einstein-Hilbert term satisfies condition 3 [Eq. (36)], and the condition $\mu_1 = -\mu_2$ is included in condition 4 [Eq. (60)]. Therefore, this is a wider class of Fierz-Pauli theory with 5 d.o.f. for a massive spin-2 field. The whole parameter family of this case can be mapped from Fierz-Pauli theory by a field redefinition as we will see in the next section.

Other case: $\kappa_1 = \kappa_3$, $\kappa_4 = -(4\kappa_1^2 - 4\kappa_1\kappa_3 + 3\kappa_3^2)/(8\kappa_1) = -(3/8)\kappa_1$.

Before the end of this section, we shall consider the case with $\kappa_1 = \kappa_3$ and $\mu_1 = \mu_2 = 0$ or $\mu_1 = -4\mu_2 \neq 0$ in which only one secondary constraint $\mathcal{C}_2^\beta$ (50) exists while $\mathcal{C}_2^\alpha$ (49) vanishes and hence no more constraint will be generated as for $\mathcal{C}_2^\alpha$.

First the condition for the mass parameter in case SIIc, condition 4, is equivalent to

$$\mu_2 = -\frac{4\kappa_1^2 - 6\kappa_1\kappa_3 + 3\kappa_3^2}{4\kappa_1^2}\mu_1 = -\frac{1}{4}\mu_1. \quad (69)$$

Since $\{\mathcal{C}_2^\beta, \mathcal{C}_1^\beta\} \propto \mu_1 \neq 0$, λ_β can be determined by imposing $\dot{\mathcal{C}}_2^\beta \approx 0$, that is $\lambda_\beta = -\{\mathcal{C}_2^\beta, H^S\}/\{\mathcal{C}_2^\beta, \mathcal{C}_1^\beta\}$. Since $\mathcal{C}_1^\alpha$ is first class while $\mathcal{C}_{1,2}^\beta$ are second class, we have

$$\text{scalar DOFs} = \frac{4\times 2 - 1(1\,\text{primary})\times 2(\text{first class}) - 2(1\,\text{primary}\,\&\,1\,\text{secondary})}{2} = 2. \quad (70)$$

As we see in Appendix B, one of the modes is a ghost and hence we will no longer consider this case.

Case SW: $\kappa_1 = \kappa_3$, $\kappa_4 = -(4\kappa_1^2 - 4\kappa_1\kappa_3 + 3\kappa_3^2)/(8\kappa_1) = -(3/8)\kappa_1$, $\mu_1 = \mu_2 = 0$.

In this case, which is similar to case SIIa except for the additional condition $\kappa_1 = \kappa_3$, the time consistency of $\mathcal{C}_2^\beta$ yields a tertiary constraint since $\{\mathcal{C}_2^\beta, \mathcal{C}_1^\beta\}$ vanishes,

$$\mathcal{C}_3^\beta \equiv \dot{\mathcal{C}}_2^\beta = \{\mathcal{C}_2^\beta, H_T^S\} = \{\mathcal{C}_2^\beta, H^S\} = 4k^3\kappa_1(\alpha - \mathcal{R} - \mathcal{E}) \approx 0. \quad (71)$$

The Poisson brackets with primary constraints are trivially satisfied and also the time evolution of $\mathcal{C}_3^\beta$ turns out to be $\dot{\mathcal{C}}_3^\beta \propto \mathcal{C}_2^\beta$. Since all the constraints commute each other, two primary, secondary, and tertiary constraints are first class. Therefore,

$$\text{scalar d.o.f.} = \frac{4\times 2 - 4(2\,\text{primary}\,\&\,1\,\text{secondary}\,\&\,1\,\text{tertiary})\times 2(\text{first class})}{2} = 0. \quad (72)$$

The total number of d.o.f. is two with 2 tensor modes and without the vector mode corresponding to case V1.

IV. THEORETICAL PROPERTIES

In this section, we investigate theoretical properties of the obtained theories in the previous section in detail. We first consider a field redefinition linear in $h_{\mu\nu}$, which helps us to understand structures and classification of the theories. Furthermore, we clarify the crucial differences between cases SI and SIIb, which cannot be obtained from the known theories (linearized general relativity and Fierz-Pauli theory) by any invertible field redefinition.

A. Linear field redefinition

In this subsection, we consider a linear field redefinition of the rank-2 tensor $h_{\mu\nu}$, which respects Lorentz invariance. A possible field redefinition as studied in Refs. [28,29] is

$$h_{\mu\nu} = \Omega^2 \bar{h}_{\mu\nu} + \Gamma \bar{h} \eta_{\mu\nu}, \tag{73}$$

where Ω and Γ are constants, $\bar{h}$ is the trace of $\bar{h}_{\mu\nu}$ contracted by $\eta_{\mu\nu}$. Hereafter we set $\Omega = 1$ without loss of generality since it only changes the normalization of the Lagrangian. The inverse transformation is given by

$$\bar{h}_{\mu\nu} = h_{\mu\nu} - \frac{\Gamma}{1+4\Gamma} h \eta_{\mu\nu}. \tag{74}$$

When $\Gamma = -1/4$, this transformation is not invertible since its determinant vanishes.

Now we apply this transformation to our theories. After the field redefinition, the Lagrangian (1) reads

$$\begin{aligned} S[h_{\mu\nu} &= \bar{h}_{\mu\nu} + \Gamma \bar{h} \eta_{\mu\nu}] \\ &= \int \mathrm{d}^4 x (-\bar{\mathcal{K}}^{\alpha\beta|\mu\nu\rho\sigma} \bar{h}_{\mu\nu,\alpha} \bar{h}_{\rho\sigma,\beta} - \bar{\mathcal{M}}^{\mu\nu\rho\sigma} \bar{h}_{\mu\nu} \bar{h}_{\rho\sigma}), \end{aligned} \tag{75}$$

where

$$\begin{aligned} \bar{\mathcal{K}}^{\alpha\beta|\mu\nu\rho\sigma} &= \bar{\kappa}_1 \eta^{\alpha\beta}\eta^{\mu\rho}\eta^{\nu\sigma} + \bar{\kappa}_2 \eta^{\mu\alpha}\eta^{\rho\beta}\eta^{\nu\sigma} \\ &\quad + \bar{\kappa}_3 \eta^{\alpha\mu}\eta^{\nu\beta}\eta^{\rho\sigma} + \bar{\kappa}_4 \eta^{\alpha\beta}\eta^{\mu\nu}\eta^{\rho\sigma}, \end{aligned} \tag{76}$$

$$\bar{\mathcal{M}}^{\mu\nu\rho\sigma} = \bar{\mu}_1 \eta^{\mu\rho}\eta^{\nu\sigma} + \bar{\mu}_2 \eta^{\mu\nu}\eta^{\rho\sigma}. \tag{77}$$

The straightforward calculation shows the relation between κ_i, μ_i and $\bar{\kappa}_i, \bar{\mu}_i$:

$$\bar{\kappa}_1 = \kappa_1, \quad \bar{\kappa}_2 = \kappa_2, \quad \bar{\kappa}_3 = 2\Gamma\kappa_2 + (1+4\Gamma)\kappa_3, \tag{78}$$

$$\bar{\kappa}_4 = 2\Gamma(1+2\Gamma)\kappa_1 + \Gamma^2\kappa_2 + \Gamma(1+4\Gamma)\kappa_3 + (1+4\Gamma)^2\kappa_4, \tag{79}$$

$$\bar{\mu}_1 = \mu_1, \qquad \bar{\mu}_2 = 2\mu_1\Gamma(1+2\Gamma) + \mu_2(1+4\Gamma)^2. \tag{80}$$

It should be noted that coefficients $\kappa_{1,2}$ and μ_1 are invariant under this transformation as long as $\Omega = 1$. We also find that the degeneracy conditions for the scalar and vector modes, Eqs. (17) and (36), are invariant under the field redefinition Eq. (73). Namely, when κ_i and μ_i satisfy condition 2 and condition 3, the transformed parameters still satisfy the same relations; that is,

$$\begin{aligned} &\text{“Condition 2”}:\ 2\bar{\kappa}_1 + \bar{\kappa}_2 = 0, \\ &\text{“Condition 3”}:\ 4\bar{\kappa}_1^2 - 4\bar{\kappa}_1\bar{\kappa}_3 + 3\bar{\kappa}_3^2 + 8\bar{\kappa}_1\bar{\kappa}_4 = 0. \end{aligned} \tag{81}$$

In addition to these, the additional condition to remove the extra d.o.f. for the scalar modes, $\mu_1 = 0$ for cases SI/SIIa($\overline{\text{SIIa}}$)/SIIb($\overline{\text{SIIb}}$) or condition 4 [Eq. (60)] for case SIIc is also an invariant quantity. It immediately follows that linear degenerate theories can be transformed to different linear degenerate theories with a different parameter set through the field redefinition Eq. (73). In particular, when one chooses

$$\Gamma = -\frac{2\kappa_1 - 3\kappa_3}{12(\kappa_1 - \kappa_3)} \tag{82}$$

one can obtain the degenerate theories with $\bar{\kappa}_3 = 2\bar{\kappa}_1/3$, as in case $\overline{\text{SIIa}}$ from the degenerate theories with $\kappa_3 \neq 2\kappa_1/3$ like in case SIIa. We also note that the same kinetic structure as the linearized Einstein-Hilbert term can be obtained for all the theories in case SII by setting $\bar{\kappa}_3 = 2\bar{\kappa}_1$ via the field redefinition with

$$\Gamma = -\frac{2\kappa_1 - \kappa_3}{4(\kappa_1 - \kappa_3)}. \tag{83}$$

Hence, upon the field redefinition (73), one can show that

$$\text{Cases SIIa and } \overline{\text{SIIa}} \longleftrightarrow \text{Linearized general relativity}, \tag{84}$$

$$\text{Case SIIb} \longleftrightarrow \text{Case } \overline{\text{SIIb}}, \tag{85}$$

$$\text{Cases SIIc and } \overline{\text{SIIc}} \longleftrightarrow \text{Fierz-Pauli theory}. \tag{86}$$

B. Symmetry of the Lagrangian

It is interesting to see the symmetry of the Lagrangian as also studied in Refs. [28,29]. When $2\kappa_1 + \kappa_2 = 0$ is satisfied, Lagrangian (4) can be rewritten as assuming $\kappa_1 \neq \kappa_3$,

$$\begin{aligned} S[h_{\mu\nu}] &= \kappa_1 S^{\mathrm{LGR}}[\tilde{h}_{\mu\nu}] \\ &\quad - \int \mathrm{d}^4 x \kappa_1 \frac{4\kappa_1^2 - 4\kappa_1\kappa_3 + 3\kappa_3^2 + 8\kappa_1\kappa_4}{8(\kappa_1 - \kappa_3)^2} \tilde{h}_{,\alpha}\tilde{h}^{,\alpha} \\ &\quad - \int \mathrm{d}^4 x \left[\mu_1 \tilde{h}_{\mu\nu}\tilde{h}^{\mu\nu} + \frac{(2\kappa_1 - \kappa_3)\kappa_3\mu_1 + 4\kappa_1^2\mu_2}{4(\kappa_1 - \kappa_3)^2} \tilde{h}^2 \right], \end{aligned} \tag{87}$$

where

$$\tilde{h}_{\mu\nu} \equiv h_{\mu\nu} + \frac{\kappa_3 - 2\kappa_1}{4\kappa_1} h\eta_{\mu\nu}, \tag{88}$$

and S^{LGR} stands for the action of linearized general relativity:

$$S^{\mathrm{LGR}}[h_{\mu\nu}] \equiv -\int \mathrm{d}^4x (h_{\mu\nu,\alpha}h^{\mu\nu,\alpha} - 2h^{\alpha}{}_{\mu,\alpha}h^{\beta\mu}{}_{,\beta} + 2h^{\alpha\beta}{}_{,\alpha}h_{,\beta} - h_{,\alpha}h^{,\alpha}). \tag{89}$$

First, we should note that the kinetic term of tensorial parts in $\tilde{h}_{\mu\nu}$ is nicely summarized in a single term, that is S^{LGR}. This is not possible with a generic κ_2. The difference between $\tilde{h}_{\mu\nu}$ and $h_{\mu\nu}$ is subtle since $\tilde{h}_{\mu\nu}$ reduces to $h_{\mu\nu}$ once $\kappa_3 = 2\kappa_1$ holds, which can be realized by a suitable field redefinition as studied in the last subsection. Hence its difference is not essential in the discussion below at least in the absence of matter.

Now let us study the symmetry of the Lagrangian. The first term S^{LGR} is invariant under a transformation with diffeomorphisms for $\tilde{h}_{\mu\nu}$:

$$\tilde{h}_{\mu\nu} \to \tilde{h}_{\mu\nu} + \partial_\mu \xi_\nu + \partial_\nu \xi_\mu, \tag{90}$$

while other terms do not in general. However, once the gauge parameters satisfy a condition such that $\partial_\mu \xi^\mu = 0$, the kinetic term enjoys this restricted gauge symmetry since $\tilde{h}$ is invariant under this gauge transformation. Moreover, if μ_1 also disappears, this symmetry becomes the symmetry of the whole Lagrangian. This subgroup of the diffeomorphisms is known as transverse diffeomorphisms [28].

It is now clear that cases SI and SIIb ($\mathrm{S\overline{IIb}}$) possess such the transverse gauge symmetry with 3 gauge parameters, 1 for the scalar sector and 2 for the vector sector, since μ_1 is vanishing as studied in detail in Appendix C. On the other hand, case SIIa ($\mathrm{S\overline{IIa}}$) respects diffeomorphisms with 4 gauge parameters since $\mu_1 = \mu_2 = 0$ as in linearized general relativity. There is no gauge symmetry in case SIIc ($\mathrm{S\overline{IIc}}$) since both μ_1 and μ_2 are present. As for the remaining case SW, the Lagrangian can be written as

$$\begin{aligned} S^{\mathrm{SW}}[h_{\mu\nu}] &= -\kappa_1 \int \mathrm{d}^4x \left[h_{\mu\nu,\alpha}h^{\mu\nu,\alpha} - 2h^{\alpha}{}_{\mu,\alpha}h^{\beta\mu}{}_{,\beta} + h^{\alpha\beta}{}_{,\alpha}h_{,\beta} - \frac{3}{8}h_{,\alpha}h^{,\alpha}\right] \\ &= -\kappa_1 \int \mathrm{d}^4x [\hat{h}_{\mu\nu,\alpha}\hat{h}^{\mu\nu,\alpha} - 2\hat{h}^{\alpha}{}_{\mu,\alpha}\hat{h}^{\beta\mu}{}_{,\beta}] = \kappa_1 S^{\mathrm{LGR}}\left[\hat{h}_{\mu\nu} \equiv h_{\mu\nu} - \frac{1}{4}h\eta_{\mu\nu}\right]. \end{aligned} \tag{91}$$

Since the Lagrangian can be described in the form of linearized general relativity with traceless tensor $\hat{h}_{\mu\nu} \equiv h_{\mu\nu} - (1/4)h\eta_{\mu\nu}$, this theory enjoys diffeomorphism invariance as well as the invariance under a field redefinition (73). This enhanced symmetry is called as Weyl-invariant transverse diffeomorphism invariance [28].

C. New theories

We have seen that the cases SI, SIIb, and $\mathrm{S\overline{IIb}}$ are the new theories of a spin-2 field, which cannot be mapped into linearized general relativity and Fierz-Pauli theory by the field redefinition as studied in the previous subsection IV A. In this subsection, we take a closer look at the Lagrangian of these theories particularly focusing on its scalar sectors.

Now let us first consider linearized general relativity to understand the structure of theories, which can be simply obtained by setting $2\kappa_1 - \kappa_3 = 0$ in the case SIIa or equivalently performing a field redefinition with a special Γ (83). Rewriting the Lagrangian in the scalar sector in terms of the gauge invariant variables (C13), we obtain up to total derivative terms

$$\mathcal{L}^S_{\mathrm{LGR}} = -3\dot{\mathcal{R}}^2 + k^2\mathcal{R}^2 + 2k^2\tilde{\alpha}\mathcal{R}, \qquad \tilde{\alpha} = \alpha + \frac{\dot{\beta}}{k} - \frac{\ddot{\mathcal{E}}}{k^2}, \tag{92}$$

where we have set the overall factor κ_1 to be $1/8$ for simplicity. Then the variation with respect to $\tilde{\alpha}$ gives the constraint $\mathcal{R} = 0$, and it is manifest that the Lagrangian becomes zero after substituting the constraint. Thus we have confirmed that the number of d.o.f. in the scalar sector is zero in the Lagrangian formalism, that is consistent with the Hamiltonian analysis in the previous section.

Cases SIIb & $\mathrm{S\overline{IIb}}$: Now we would like to perform the same analysis for the case IIa. Again, we can set $2\kappa_1 - \kappa_3 = 0$ and $\kappa_1 = 1/8$ without loss of generality. Then the Lagrangian in the scalar sector in terms of the gauge invariant variables (C5) is given by

$$\begin{aligned} &\mathcal{L}^S_{\mathrm{IIb}} = -3\dot{\mathcal{R}}^2 + k^2\mathcal{R}^2 + 2k^2\tilde{\alpha}\mathcal{R} - 4\mu_2\tilde{\mathcal{E}}^2, \\ &\tilde{\mathcal{E}} = \mathcal{E} - \alpha - 3\mathcal{R}. \end{aligned} \tag{93}$$

One can clearly see that the first three terms are exactly the same as in the case of SIIa ($\mathrm{S\overline{IIa}}$). However, in this case,

TABLE I. The number of the d.o.f., the conditions, free parameters, and comments for each case is shown. For any case, condition 1 [Eq. (15)] and condition 2 [Eq. (17)] are always imposed. Conditions 3,4,5 are shown in Eqs. (36), (60), and (A1) respectively. Among free parameters, κ_1 is not included since it only changes the normalization of the Lagrangian if its sign is appropriately chosen.

Case	d.o.f.	Conditions	Free parameters	Comments
SI & V1	$3 = 2 + 0 + 1$	$\mu_1 = 0$	κ_3, κ_4, μ_2	New theories
SIIa & V1	$2 = 2 + 0 + 0$	Condition 3 & $\mu_1 = \mu_2 = 0$	κ_3	General relativity is included
SIIb & V1	$2 = 2 + 0 + 0$	Condition 3 & $\mu_1 = 0$ & $\mu_2 \neq 0$	κ_3, μ_2	New theories
SIIc & V2	$5 = 2 + 2 + 1$	Condition 3 & 4 & $\mu_1 \neq 0$	κ_3, μ_1	Fierz-Pauli is included
S$\overline{\text{IIa}}$ & V1	$2 = 2 + 0 + 0$	Condition 5 & $\mu_1 = \mu_2 = 0$	None	$-2\kappa_1 + 3\kappa_3 = 0$ limit of SIIa
S$\overline{\text{IIb}}$ & V1	$2 = 2 + 0 + 0$	Condition 5 & $\mu_1 = 0$ & $\mu_2 \neq 0$	μ_2	$-2\kappa_1 + 3\kappa_3 = 0$ limit of SIIb
S$\overline{\text{IIc}}$ & V2	$5 = 2 + 2 + 1$	Condition 5 & $\mu_1 + 3\mu_2 = 0$ & $\mu_1 \neq 0$	μ_1	$-2\kappa_1 + 3\kappa_3 = 0$ limit of SIIc

there is an extra term, $\mu_2\tilde{\mathcal{E}}^2 = \mu_2(\mathrm{Tr}h_{\mu\nu})^2$, where $-\tilde{\mathcal{E}}$ is the trace of a spin-2 field. Since it is completely decoupled from $\mathcal{R}$ and $\tilde{\alpha}$ and has no kinetic term, it is actually a nondynamical d.o.f. The case SIIb (S$\overline{\text{IIb}}$) cannot be obtained from any gauge fixing of the case SIIa (S$\overline{\text{IIa}}$), and these theories are therefore independent of each other. Since $\tilde{\alpha}$ and $\tilde{\mathcal{E}}$ are nondynamical, the Lagrangian becomes zero after integrating out these variables just as in the case SIIa (S$\overline{\text{IIa}}$).

Case SI: Finally, we consider the case SI, whose number of scalar d.o.f. is one. In this case, we can also choose $2\kappa_1 - \kappa_3 = 0$ and $\kappa_1 = 1/8$ without loss of generality. Furthermore, we set $\kappa_4 = 1/8$ to simplify the coefficient of the kinetic term for $\tilde{\mathcal{E}}$. Then the Lagrangian in terms of the gauge invariant variables (C5) is given by

$$\mathcal{L}_{\mathrm{I}}^S = -3\dot{\mathcal{R}}^2 + k^2\mathcal{R}^2 + 2k^2\tilde{\alpha}\mathcal{R} + \dot{\tilde{\mathcal{E}}}^2 - (k^2 + 4\mu_2)\tilde{\mathcal{E}}^2. \tag{94}$$

In this case, the trace of $h_{\mu\nu}$, namely, $-\tilde{\mathcal{E}}$, which again decouples from other variables $\mathcal{R}$ and $\tilde{\alpha}$, becomes dynamical since it has now the kinetic term, differently from the case SIIb. Thus after plugging the constraint of $\tilde{\alpha}$, 1 scalar d.o.f. remains in the case SI, which is also consistent with the Hamiltonian analysis.

V. SUMMARY AND DISCUSSION

Summary.—In the present paper, we constructed the most generic spin-2 field theories in a flat spacetime with at most 5 d.o.f. without the ghost mode, whose Lagrangian consists of the quadratic terms of the field and its first derivatives. By decomposing the spin-2 field $h_{\mu\nu}$ into the transverse-traceless tensor, the tensor composed by transverse vectors, and the tensor composed by scalar components, we classified theoretically consistent theories based on the Hamiltonian analysis in a systematic manner.

We found that the existence of the tensor d.o.f. is always controlled by one parameter κ_1, which is assumed to be nonzero while we imposed the degeneracy conditions in order to eliminate extra problematic d.o.f. for the vector and scalar modes. Under the degeneracy conditions, we found two classes in the vector sector: 2 propagating vector d.o.f. and no d.o.f. As in the vector sector, we have also classified theories in the scalar sector based on the Hamiltonian analysis. The classification of the obtained theories is summarized in Table. I. The case SIIa and SIIc are a wider class of the known theories: linearized general relativity and Fierz-Pauli theory, and we have shown that the cases SIIa and SIIc can be mapped from these known theories by field redefinition. On the other hand, the cases SI and SIIb are new theories, which cannot be mapped from the known theories. The case SIIb has the same number of d.o.f. in linearized general relativity, however, it has less gauge d.o.f., and it contains the trace of the spin-2 field, which is nondynamical. On the other hand, the case I has the dynamical d.o.f. coming from the trace of the spin-2 field while the vector d.o.f. is absent. The other remaining three cases: S$\overline{\text{IIa}}$, S$\overline{\text{IIb}}$, and S$\overline{\text{IIc}}$, are the subset of the cases SIIa, SIIb, and SIIc and hence the case S$\overline{\text{IIb}}$ is also a new theory. We provided the conditions for avoiding the ghost, gradient, and tachyonic instabilities by calculating the reduced Lagrangian in Appendix C.

Nonlinear extension to gravity theory.—It is interesting to consider a nonlinear extension to (massive) gravity theory of the obtained theories for a spin-2 field. Let us consider the following Lagrangian,[2]

$$S = \int \mathrm{d}^4x\sqrt{-g}[a_1R + a_2hR + a_3h_{\mu\nu}R^{\mu\nu} + \mathfrak{K}^{\alpha\beta|\mu\nu\rho\sigma}\nabla_\alpha h_{\mu\nu}\nabla_\beta h_{\rho\sigma} + \mathfrak{M}^{\mu\nu\rho\sigma}h_{\mu\nu}h_{\rho\sigma} + \mathcal{O}(h^2R, h^3)], \tag{95}$$

[2]The effect of the background curvature in the Lagrangian has been discussed in the literature, e.g., Ref. [30], which is beyond the scope of the present paper.

where the fluctuation tensor is defined by $h_{\mu\nu} = g_{\mu\nu} - \eta_{\mu\nu}$. The coefficients of the kinetic and mass terms, $\mathfrak{K}^{\alpha\beta|\mu\nu\rho\sigma}$ and $\mathfrak{M}^{\mu\nu\rho\sigma}$, are functions of the metric $g_{\mu\nu}$,

$$\mathfrak{K}^{\alpha\beta|\mu\nu\rho\sigma} = b_1 g^{\alpha\beta} g^{\mu\rho} g^{\nu\sigma} + b_2 g^{\mu\alpha} g^{\rho\beta} g^{\nu\sigma} + b_3 g^{\alpha\mu} g^{\nu\beta} g^{\rho\sigma} + b_4 g^{\alpha\beta} g^{\mu\nu} g^{\rho\sigma}, \tag{96}$$

$$\mathfrak{M}^{\mu\nu\rho\sigma} = \mu_1 g^{\mu\rho} g^{\nu\sigma} + \mu_2 g^{\mu\nu} g^{\rho\sigma}, \tag{97}$$

and $a_{1,2,3}$ and $b_{1,2,3,4}$ are constant parameters. By expanding the action up to the quadratic terms in $h_{\mu\nu}$ and identifying this field as the (massive) spin-2 field introduced in this paper, one can find that the relations between these constant parameters and the κ parameters introduced in the Lagrangian (1) are given by

$$\kappa_1 = b_1 - \frac{1}{2} a_1 + \frac{1}{2} a_3, \qquad \kappa_2 = b_2 + a_1 - a_3,$$
$$\kappa_3 = b_3 - a_1 - a_2 - \frac{1}{2} a_3, \qquad \kappa_4 = b_4 + \frac{1}{2} a_1 + a_2. \tag{98}$$

Then the gravitational action (95) perturbed around Minkowski space-time reproduces the Lagrangian (1) at the linear level. Therefore it is for sure that this Lagrangian (95) with the conditions that we have obtained does not have an extra d.o.f. at linear order. However, one might need some extra tuning of the parameters a_i and b_i for avoiding the appearance of Boulware-Deser ghost at nonlinear level, and it has to be carefully examined just as in the construction of the dRGT theory.

Matter coupling.—Although we have obtained the interesting theories of the spin-2 field, one carefully needs to introduce a coupling to matter fields. To clarify this, let us consider the case SIIc with the matter coupling $h_{\mu\nu} T^{\mu\nu}/M$, where $T^{\mu\nu}$ is the energy-momentum tensor of an external source, and M is a mass parameter, which corresponds to Planck's mass in general relativity. Ignoring the vector modes,[3] the Λ_3 decoupling limit with the Stuckelberg decomposition, $h_{\mu\nu} = \hat{h}_{\mu\nu} + 2m^2 \partial_\mu \partial_\nu \pi + b m^2 \eta_{\mu\nu} \Box \pi$, and the conformal transformation, $\hat{h}_{\mu\nu} = \mathfrak{h}_{\mu\nu} - (c_1/\kappa_1 - \kappa_3) \pi \eta_{\mu\nu}$ yields the following Lagrangian,

$$\mathcal{L}^{(\mathrm{DL})} = \mathcal{L}^{(\mathrm{DL})}_{\mathrm{tensor}}[\mathfrak{h}] - \frac{6c_1}{\kappa_1} (\partial_\mu \pi)^2 + \frac{1}{M} \left(\mathfrak{h}_{\mu\nu} T^{\mu\nu} - \frac{c_1}{\kappa_1 - \kappa_3} \pi T \right) + \frac{1}{\Lambda_3^3} [2(\partial_\mu \partial_\nu \pi) T^{\mu\nu} + b(\Box \pi) T], \tag{99}$$

where $\mathcal{L}^{(\mathrm{DL})}_{\mathrm{tensor}}[\tilde{h}]$ is the kinetic Lagrangian of the case IIc, $\mu_1 = c_1 m^2$, m is a mass parameter, $T = \eta_{\mu\nu} T^{\mu\nu}$ and $\Lambda_3 = (M m^2)^{1/3}$. Here, the new interaction appears if $b \neq 0$ and $T \neq 0$ which cannot be shown in the Fierz-Pauli theory with minimally coupled matter. The equation of motion for the scalar mode π naively contains the second derivatives of the energy-momentum tensor, which might lead to higher derivatives, depending on matter fields. Whether this matter coupling introduce an extra d.o.f. associated with higher derivatives or not will be reported in future work (see also Ref. [31]).

[3]A careful analysis shows that the decoupling limit Lagrangian of the vector field becomes a massless $U(1)$ field, which completely decouples from the other modes.

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APPENDIX A: OTHER DEGENERATE THEORIES IN THE SCALAR SECTOR

In this Appendix we consider special cases with $2\kappa_1 - 3\kappa_3 = 0$,

$$\text{"Condition 5"}: \ \kappa_3 = \frac{2}{3} \kappa_1, \tag{A1}$$

which also satisfies condition 3 [Eq. (36)] as well as condition 1 [Eq. (15)] and condition 2 [Eq. (17)]. To summarize, we have

$$\kappa_1 > 0, \qquad \kappa_2 = -2\kappa_1, \qquad \kappa_3 = \frac{2}{3} \kappa_1,$$
$$\kappa_4 = -\frac{4\kappa_1^2 - 4\kappa_1 \kappa_3 + 3\kappa_3^2}{8\kappa_1} = -\frac{\kappa_1}{3}. \tag{A2}$$

In this case, the momenta are given by

$$\pi_\alpha = \frac{16}{3} \kappa_1 (k\beta - \dot{\alpha}), \qquad \pi_\mathcal{E} = -\frac{16}{3} \kappa_1 (k\beta - \dot{\mathcal{E}}), \tag{A3}$$

and other momenta gives two primary constraints,

$$\mathcal{C}_1^\beta = \pi_\beta = 0, \qquad \mathcal{C}_1^\mathcal{R} = \pi_\mathcal{R} = 0. \tag{A4}$$

The Hamiltonian and the total Hamiltonian densities are given by

$$\begin{aligned}\mathcal{H}^S &= \dot{\alpha}\pi_\alpha + \dot{\beta}\pi_\beta + \dot{\mathcal{R}}\pi_\mathcal{R} + \dot{\mathcal{E}}\pi_\mathcal{E} - \mathcal{L}^S,\\ &\approx 4\left[\frac{2k^2\kappa_1}{3} + (\mu_1+\mu_2)\right]\alpha^2 - 2\mu_1\beta^2 - 4\left[\frac{2k^2\kappa_1}{3} - (\mu_1+\mu_2)\right]\mathcal{E}^2 + 12(\mu_1+3\mu_2)\mathcal{R}^2\\ &\quad - \frac{8}{3}(3\mu_1+9\mu_2-2k^2\kappa_1)\mathcal{E}\mathcal{R} - 8\left[\mu_2\mathcal{E} + \frac{1}{3}(2k^2\kappa_1 - 9\mu_2)\mathcal{R}\right]\alpha - \frac{3}{32\kappa_1}(\pi_\alpha^2 - \pi_\mathcal{E}^2) + k(\pi_\alpha + \pi_\mathcal{E})\beta,\end{aligned} \tag{A5}$$

$$\mathcal{H}_T^S = \mathcal{H}^S + \lambda_\beta\pi_\beta + \lambda_\mathcal{R}\pi_\mathcal{R}, \tag{A6}$$

where we have suppressed the terms proportional to the primary constraints (A4) in the final expression of $\mathcal{H}^S$. Then one finds

$$\begin{aligned}\mathcal{C}_2^\mathcal{R} \equiv \dot{\mathcal{C}}_1^\mathcal{R} = \{\mathcal{C}_1^\mathcal{R}, H_T^S\} = \{\mathcal{C}_1^\mathcal{R}, H^S\}\\ = -\frac{8}{3}[(9\mu_2 - 2k^2\kappa_1)\alpha + (2k^2\kappa_1 - 3\mu_1 - 9\mu_2)\mathcal{E}\\ + 9(\mu_1+3\mu_2)\mathcal{R}],\end{aligned} \tag{A7}$$

$$\mathcal{C}_2^\beta = \dot{\mathcal{C}}_1^\beta = \{\mathcal{C}_1^\beta, H_T^S\} = \{\mathcal{C}_1^\beta, H^S\} = -k\pi_\alpha - k\pi_\mathcal{E} + 4\mu_1\beta. \tag{A8}$$

The Poisson brackets between the secondary and the primary constraints are given by

$$\{\mathcal{C}_2^\beta, \mathcal{C}_1^\beta\} = 4\mu_1, \tag{A9}$$

$$\{\mathcal{C}_2^\mathcal{R}, \mathcal{C}_1^\mathcal{R}\} = -24(\mu_1+3\mu_2). \tag{A10}$$

If both are nonvanishing, the Lagrange multipliers λ_β and $\lambda_\mathcal{R}$ are determined, then it has two primary constraints $\mathcal{C}_1^\beta$, $\mathcal{C}_1^\mathcal{R}$ and two secondary constraints $\mathcal{C}_2^\beta$, $\mathcal{C}_2^\mathcal{R}$. Since all the constraints are second class, the total number of d.o.f. in the scalar sector is $(8-4)/2 = 2$. As shown in Appendix B, we have a ghost d.o.f. in this case. To remove an extra d.o.f., we have three options: $\mu_1 = \mu_2 = 0$ (case $\mathrm{S\overline{II}a}$), $\mu_1 = 0$ (case $\mathrm{S\overline{II}b}$), or $\mu_1 + 3\mu_2 = 0$ (case $\mathrm{S\overline{II}c}$).

Case $\mathrm{S\overline{II}a}$: $\mu_1 = \mu_2 = 0$.

If $\mu_1 = \mu_2 = 0$, both Eqs. (A9) and (A10) vanish. In this case, all the primary constraints $\mathcal{C}_1^\beta$, $\mathcal{C}_1^\mathcal{R}$ and secondary constraints $\mathcal{C}_2^\beta$, $\mathcal{C}_2^\mathcal{R}$ are first class since all the constraints commute with each other. Therefore,

$$\text{scalar d.o.f.} = \frac{4\times 2 - 4(2\,\text{primary}\,\&\,2\,\text{secondary})\times 2(\text{first-class})}{2} = 0. \tag{A11}$$

Since the vector mode does not have a d.o.f. (case V1), only the tensor mode propagate. This case corresponds to the case SIIb with an additional condition $-2\kappa_1 + 3\kappa_3 = 0$ as one can see from the structure of the constraint algebra as well as the Lagrangian more explicitly.

Case $\mathrm{S\overline{II}b}$: $\mu_1 = 0$. If $\mu_1 = 0$, the Lagrange multiplier $\lambda_\mathcal{R}$ can be determined by $\dot{\mathcal{C}}_2^\mathcal{R} \approx 0$, that is, $\lambda_\mathcal{R} = -\{\mathcal{C}_2^\mathcal{R}, H^S\}/\{\mathcal{C}_2^\mathcal{R}, \mathcal{C}_1^\mathcal{R}\}$. On the other hand, the consistency of $\mathcal{C}_2^\beta$ generates a tertiary constraint

$$\mathcal{C}_3^\beta \equiv \dot{\mathcal{C}}_2^\beta = \{\mathcal{C}^\beta, H_T^S\} = \{\mathcal{C}^\beta, H^S\} = \frac{16}{3}k^3\kappa_1(\alpha - \mathcal{E}) \approx 0. \tag{A12}$$

One also finds $\dot{\mathcal{C}}_3^\beta \propto \mathcal{C}_2^\beta$ and hence no more constraint is generated. $\mathcal{C}_1^\beta$, $\mathcal{C}_2^\beta$, and $\mathcal{C}_3^\beta$ are first-class since these constraints commute with all other constraints, i.e., $\{\mathcal{C}_i^\beta, \mathcal{C}_j^\beta\} = \{\mathcal{C}_i^\beta, \mathcal{C}_k^\mathcal{R}\} = 0$, where $i, j = 1, 2, 3$ and $k = 1, 2$ while $\mathcal{C}_1^\mathcal{R}$ and $\mathcal{C}_2^\mathcal{R}$ are second class. Therefore,

$$\text{scalar d.o.f.} = \frac{4\times 2 - 3(1\,\text{primary}\,\&\,1\,\text{secondary}\,\&\,1\,\text{tertiary})\times 2(\text{first class}) - 2(1\,\text{primary}\,\&\,1\,\text{secondary})}{2} = 0. \tag{A13}$$

Then the total number of d.o.f. is two since the vector mode does not propagate for $\mu_1 = 0$ (case V1) while tensor modes are present.

Case $\mathrm{S\overline{II}c}$: $\mu_1 + 3\mu_2 = 0$, $\mu_1 \neq 0$.

If $\mu_1 + 3\mu_2 = 0$, the Lagrange multiplier λ_β can be determined by $\dot{\mathcal{C}}_2^\beta \approx 0$, i.e., $\lambda_\beta = -\{\mathcal{C}_2^\beta, H^S\}/\{\mathcal{C}_2^\beta, \mathcal{C}_1^\beta\}$. As for $\mathcal{C}_2^\mathcal{R}$, since it does not commute with $\mathcal{C}_2^\beta$, we shall consider a linear combination of constraints:

$$\tilde{\mathcal{C}}_2^{\mathcal{R}} \equiv \mathcal{C}_2^{\mathcal{R}} - 2k\mathcal{C}_1^{\beta}. \tag{A14}$$

The consistency of $\tilde{\mathcal{C}}_2^{\mathcal{R}}$ generates a tertiary constraint

$$\mathcal{C}_3^{\mathcal{R}} \equiv \dot{\tilde{\mathcal{C}}}_2^{\mathcal{R}} = \{\tilde{\mathcal{C}}_2^{\mathcal{R}}, H_T^S\} = \{\tilde{\mathcal{C}}_2^{\mathcal{R}}, H^S\} = \left(k^2 - \frac{3\mu_1}{2\kappa_1}\right)\pi_\alpha + k^2\pi_{\mathcal{E}} \approx 0. \tag{A15}$$

Since the Poisson brackets between this constraint and primary constraints are vanishing the time evolution of the tertiary constraint $\mathcal{C}_3^{\mathcal{R}}$ yields a quaternary constraint and, in fact,

$$\begin{aligned}\mathcal{C}_4^{\mathcal{R}} \equiv \dot{\mathcal{C}}_3^{\mathcal{R}} = \{\mathcal{C}_3^{\mathcal{R}}, H_T^S\} = \{\mathcal{C}_3^{\mathcal{R}}, H^S\} &= -8\left(\frac{2\kappa_1 k^4}{3} - \frac{\mu_1^2}{\kappa_1}\right)\alpha - \frac{12\mu_1^2}{\kappa_1}\mathcal{R} \\ &\quad + 4\left(\frac{4\kappa_1 k^4}{3} - 2k^2\mu_1 + \frac{\mu_1^2}{\kappa_1}\right)\mathcal{E} \approx 0.\end{aligned} \tag{A16}$$

Since this does not commute with $\mathcal{C}_1^{\mathcal{R}}$, no more constraint is generated and the time consistency of $\mathcal{C}_4^{\mathcal{R}}$ determines the Lagrange multiplier as $\lambda_{\mathcal{R}} = -\{\mathcal{C}_4^{\mathcal{R}}, H^S\}/\{\mathcal{C}_4^{\mathcal{R}}, \mathcal{C}_1^{\mathcal{R}}\}$. There are six second-class constraints $\mathcal{C}_1^{\mathcal{R}}, \mathcal{C}_1^{\beta}, \tilde{\mathcal{C}}_2^{\mathcal{R}}, \mathcal{C}_2^{\beta}, \mathcal{C}_3^{\mathcal{R}}$, and $\mathcal{C}_4^{\mathcal{R}}$. Therefore,

$$\text{scalar d.o.f.} = \frac{4\times 2 - 6(2\text{ primary \& 2 secondary \& 1 tertiary \& 1 quaternary})}{2} = 1. \tag{A17}$$

In this case, the total number of degrees of freedom is five combining two vector modes (case V2) and two tensor modes. One may notice that the structure of the constraints and the number of the degrees of freedom are the same as the case SIIc. This implies that this case S$\overline{\text{IIc}}$ is the special case of the case SIIc with $-2\kappa_1 + 3\kappa_3 = 0$.

APPENDIX B: APPEARANCE OF UNWANTED GHOST

In this Appendix, we show that there is ghost if a system has two or more d.o.f. in the scalar sector. More clearly, we will show that at least one of the ghost modes originates from the structure of the Einstein-Hilbert action. Because of the general covariance of the Einstein-Hilbert action, the signs of the kinetic terms for scalar and tensor modes are opposite. Assuming Newton's constant is positive, the kinetic term for a scalar mode is wrong, as found in Eq. (92). In the case of general relativity, thanks to diffeomorphism invariance, there is no intrinsic d.o.f. in the scalar sector and hence this ghost mode is not activated. But once the diffeomorphism invariance is lost, this ghost revives, in general, which is explicitly shown below.

The conditions to have 2 or more d.o.f. are summarized as follows:

$$3\text{d.o.f.}: 4\kappa_1^2 - 4\kappa_1\kappa_3 + 3\kappa_3^2 + 8\kappa_1\kappa_4 \neq 0, \qquad \mu_1 \neq 0, \tag{B1}$$

$$2\text{d.o.f.}: 4\kappa_1^2 - 4\kappa_1\kappa_3 + 3\kappa_3^2 + 8\kappa_1\kappa_4 = 0, \qquad \mu_1 \neq 0, \qquad \mu_1(4\kappa_1^2 - 6\kappa_1\kappa_3 + 3\kappa_3^2) + 4\mu_2\kappa_1^2 \neq 0, \tag{B2}$$

$$2\text{d.o.f.}: 3\kappa_1 = 3\kappa_3 = -8\kappa_4, \qquad \mu_1 = -4\mu_2 \neq 0. \tag{B3}$$

In addition to these conditions, we further assume $2\kappa_1 + \kappa_2 = 0$ and $\kappa_1 \neq 0$ to evade the ghost in the vector sector. First we focus on the first two cases. Then we find the basic structure of the Lagrangian in the last case is essentially the same as in the first two cases, and conclude the existence of the ghost in the last case too. It should be noticed that we have $4\kappa_1^2 - 4\kappa_1\kappa_3 + 3\kappa_3^2 + 8\kappa_1\kappa_4 = \kappa_1(3\kappa_1 + 8\kappa_4)$ and $\mu_1(4\kappa_1^2 - 6\kappa_1\kappa_3 + 3\kappa_3^2) + 4\mu_2\kappa_1^2 = \kappa_1^2(\mu_1 + 4\mu_2)$, respectively, thanks to the first condition in Eq. (B3).

In order to simplify discussion, we further assume $\kappa_3 = 2\kappa_1$, which is always realized by performing a field redefinition as studied in IV A. In order to show the existence of a ghost, we shall take a look at Lagrangian (29), which now reduces to

$$\begin{aligned}\mathcal{L}^S[\alpha, \beta, \mathcal{R}, \mathcal{E}] = 8\kappa_1(&-3\dot{\mathcal{R}}^2 + k^2\mathcal{R}^2 + 2k^2\mathcal{R}\tilde{\alpha}) \\ &+ 4(\kappa_1 + \kappa_4)(\dot{\tilde{\mathcal{E}}}^2 - k^2\tilde{\mathcal{E}}^2) \\ &+ 2\mu_1(-4\alpha^2 + \beta^2 - 12\mathcal{R}^2 - 4\alpha\tilde{\mathcal{E}} \\ &- 8\alpha\mathcal{R} - 8\mathcal{R}\tilde{\mathcal{E}}) - 4(\mu_1 + \mu_2)\tilde{\mathcal{E}}^2.\end{aligned} \tag{B4}$$

Here, $\tilde{\alpha}$ and $\tilde{\mathcal{E}}$ stand for

$$\tilde{\alpha} = \alpha + \frac{\dot{\beta}}{k} - \frac{\ddot{\mathcal{E}}}{k^2}, \qquad \tilde{\mathcal{E}} = -\mathrm{tr}[h_{\mu\nu}] = \mathcal{E} - \alpha - 3\mathcal{R}. \tag{B5}$$

Thanks to the condition $2\kappa_1 + \kappa_2 = 0$, it is clear that the Lagrangian consists of three parts, namely, linearized general relativity (the first three terms), the kinetic term for the trace of $h_{\mu\nu}$, $-\tilde{\mathcal{E}}$, and the mass terms as shown in Eq. (32). A crucial term for the existence of a ghost is the one proportional to $\ddot{\mathcal{E}}$ in $\tilde{\alpha}$, which gives a cross term $\dot{\mathcal{R}}\dot{\mathcal{E}}$ after integration by part. In the presence of μ_1, the gauge invariance of the Lagrangian is totally lost and hence $\mathcal{R}$, $\mathcal{E}$, and also possibly $\tilde{\mathcal{E}} \sim \alpha$ are physical d.o.f. while β is nondynamical since it has no kinetic term. If $\kappa_1 + \kappa_4$ vanishes, the kinetic term for $\tilde{\mathcal{E}}$ is lost, which ends up with 2 d.o.f. Since β is not a dynamical d.o.f. in any case, the constraint equation for β yields

$$\beta = \frac{4k\kappa_1}{\mu_1}\dot{\mathcal{R}}. \tag{B6}$$

Then after eliminating the dependence of β in the Lagrangian (B4), the coefficient of $\dot{\mathcal{R}}^2$ can be changed. But still one finds

$$\begin{aligned} \mathcal{L}^S[\mathcal{R},\mathcal{E},\tilde{\mathcal{E}}] &= -8r\kappa_1\dot{\mathcal{R}}^2 + 16\kappa_1\dot{\mathcal{R}}\dot{\mathcal{E}} + 4(\kappa_1+\kappa_4)\dot{\tilde{\mathcal{E}}}^2 \\ &\quad + [\text{no time derivative terms}] \\ &= -8r\kappa_1\left(\dot{\mathcal{R}} - \frac{1}{r}\dot{\mathcal{E}}\right)^2 + \frac{8}{r}\kappa_1\dot{\mathcal{E}}^2 + 4(\kappa_1+\kappa_4)\dot{\tilde{\mathcal{E}}}^2 \\ &\quad + [\text{no time derivative terms}], \end{aligned} \tag{B7}$$

where $r = 3 + 4\kappa_1 k^2/\mu_1$. Now it is clear that no matter what the signature of r is, the kinetic terms for $\mathcal{R}$ and $\mathcal{E}$ have opposite sign. Then at least either $\mathcal{R}$ or $\mathcal{E}$ will be a ghosty mode while the nature of $\tilde{\mathcal{E}} \sim \alpha$ will be determined by the coefficient of $\kappa_1 + \kappa_4$; that is, it is a ghost or not or even nondynamical. For Eq. (B2), i.e., $\kappa_1 + \kappa_4 = 0$, the appearance of a ghost mode is still inevitable since $\tilde{\mathcal{E}} \sim \alpha$ constraint does not involve any time derivatives and the kinetic part of the Lagrangian remains the same as Eq. (B7).

Finally, we shall consider the last case (B3), whose Lagrangian reads

$$\begin{aligned} \mathcal{L}^S = 8\kappa_1\left[-3\dot{\tilde{\mathcal{R}}}^2 - 5k^2\tilde{\mathcal{R}}^2 + 2k^2\tilde{\mathcal{R}}\left(\mathcal{E} + \frac{\dot{\beta}}{k} - \frac{\ddot{\mathcal{E}}}{k^2}\right)\right] \\ + 2\mu_1(\beta^2 - 4\mathcal{E}^2 - 24\tilde{\mathcal{R}}^2 + 16\mathcal{E}\tilde{\mathcal{R}}), \end{aligned} \tag{B8}$$

where

$$\tilde{\mathcal{R}} \equiv \frac{1}{4}(-\alpha + \mathcal{R} + \mathcal{E}). \tag{B9}$$

We note that the essential structure of the Lagrangian is quite similar to that in the cases studied above, Eq. (B4). Because of the cross term $\tilde{\mathcal{R}}\ddot{\mathcal{E}} \sim \dot{\tilde{\mathcal{R}}}\dot{\mathcal{E}}$, one will find a ghost in a similar manner while the constraint equation for β yields $\beta = (4k\kappa_1/\mu_1)\dot{\tilde{\mathcal{R}}}$.

APPENDIX C: GAUGE TRANSFORMATIONS AND REDUCED LAGRANGIANS

In this Appendix, we derive a gauge transformation for the cases which have first-class constraints. By using gauge invariant quantities, we derive reduced Lagrangians and the conditions for avoiding ghost and gradient instabilities. The details of finding gauge transformations from a system with first-class constraints can be referred to Refs. [32–35]. Since the cases S$\overline{\text{II}}$a, S$\overline{\text{II}}$b, and S$\overline{\text{II}}$c are equivalent to SIIa, SIIb, and SIIc modulo the invertible field transformation (73), we will omit the detailed analysis for those cases.

1. Case SI

Let us consider the scalar sector in the case SI. The generating function is given by the linear combination of the first-class constraints, $G = \epsilon_i(t)\mathcal{C}_i^\beta$. Then $\dot{G} = \partial G/\partial t + \{G, H^S\} = 0$ gives

$$\epsilon_1 + k^2\epsilon_3 + \dot{\epsilon}_2 = 0, \qquad \epsilon_2 + \dot{\epsilon}_3 = 0. \tag{C1}$$

Introducing the gauge parameter $\epsilon(t)$, we obtain the following relations:

$$\epsilon_1 = \ddot{\epsilon} - k^2\epsilon, \qquad \epsilon_2 = -\dot{\epsilon}, \qquad \epsilon_3 = \epsilon. \tag{C2}$$

Then the gauge transformation of the scalar components of $h_{\mu\nu}$ can be obtained from $\delta h_{\mu\nu} = \{h_{\mu\nu}, G\}$,

$$\delta\alpha = k\dot{\epsilon}, \qquad \delta\beta = \ddot{\epsilon} - k^2\epsilon, \qquad \delta\mathcal{R} = 0, \qquad \delta\mathcal{E} = k\dot{\epsilon}. \tag{C3}$$

One can easily check that the Lagrangian for the scalar mode is invariant under this gauge transformation. Taking into account the gauge transformation of the vector mode, which is identical to the one in diffeomorphisms, the gauge transformation in the case SI can be covariantly written as

$$h_{\mu\nu} \to h_{\mu\nu} + \partial_\mu\xi_\nu + \partial_\nu\xi_\mu, \quad \text{with} \quad \partial^\mu\xi_\mu = 0, \tag{C4}$$

which is transverse diffeomorphisms [28].

To find the reduced Lagrangian, let us now define the gauge invariant variables,

$$\tilde{\alpha} = \alpha + \frac{\dot{\beta}}{k} - \frac{\ddot{\mathcal{E}}}{k^2}, \qquad \tilde{\mathcal{E}} = \mathcal{E} - \alpha - 3\mathcal{R}. \tag{C5}$$

Note that the gauge invariant variable $\tilde{\mathcal{E}}$ is nothing but the trace of $-h_{\mu\nu}$. Using the gauge invariant variables introduced in the above, we rewrite the Lagrangian in terms of $\tilde{\alpha}$, $\tilde{\mathcal{E}}$, and $\mathcal{R}$ by eliminating β. And then after solving the constraint generated by the equation of motion for $\tilde{\alpha}$, we can integrate out $\mathcal{R}$. Finally, we obtain the reduced action,

$$\mathcal{L}^S = \frac{4\kappa_1^2 - 4\kappa_1\kappa_3 + 3\kappa_3^2 + 8\kappa_1\kappa_4}{2\kappa_1} \times \left[\dot{\tilde{\mathcal{E}}}^2 - \left(k^2 + \frac{8\mu_2\kappa_1}{4\kappa_1^2 - 4\kappa_1\kappa_3 + 3\kappa_3^2 + 8\kappa_1\kappa_4}\right)\tilde{\mathcal{E}}^2\right]. \tag{C6}$$

Therefore, the conditions for avoiding ghost and tachyonic instabilities are

$$4\kappa_1^2 - 4\kappa_1\kappa_3 + 3\kappa_3^2 + 8\kappa_1\kappa_4 > 0, \qquad \mu_2 > 0. \tag{C7}$$

2. Case SIIa

In this case, the generating function is given by $G = \sum_{i,j}^2 \epsilon_{ij}\mathcal{C}_{ij}$, where ϵ_{ij} is the gauge parameter, and $\mathcal{C}_{11} = \mathcal{C}_1^\alpha$, $\mathcal{C}_{12} = \mathcal{C}_2^\alpha$, $\mathcal{C}_{21} = \mathcal{C}_1^\beta$ and $\mathcal{C}_{22} = \mathcal{C}_2^\beta$. Then, the condition $\dot{G} = 0$ gives two equations,

$$\epsilon_{11} + \dot{\epsilon}_{12} + \frac{2\kappa_1 - \kappa_3}{4(\kappa_1 - \kappa_3)} k\epsilon_{22} = 0, \tag{C8}$$

$$\epsilon_{21} + \dot{\epsilon}_{22} - \frac{4(\kappa_1 - \kappa_3)}{2\kappa_1 - 3\kappa_3} k\epsilon_{12} = 0. \tag{C9}$$

These equations can be recasted into

$$\epsilon_{11} = -\dot{\epsilon}_1 - \frac{2\kappa_1 - \kappa_3}{4(\kappa_1 - \kappa_3)} k\epsilon_2, \qquad \epsilon_{12} = \epsilon_1,$$
$$\epsilon_{21} = \frac{4(\kappa_1 - \kappa_3)}{2\kappa_1 - 3\kappa_3} k\epsilon_1 - \dot{\epsilon}_2, \qquad \epsilon_{22} = \epsilon_2, \tag{C10}$$

where we have introduced two gauge parameters $\epsilon_1(t)$ and $\epsilon_2(t)$. The gauge transformations are given by

$$\delta\alpha = -\dot{\epsilon}_1 - \frac{2\kappa_1 - \kappa_3}{4(\kappa_1 - \kappa_3)} k\epsilon_2, \qquad \delta\beta = \frac{4(\kappa_1 - \kappa_3)}{2\kappa_1 - 3\kappa_3} k\epsilon_1 - \dot{\epsilon}_2, \tag{C11}$$

$$\delta\mathcal{R} = (2\kappa_1 - \kappa_3)\left[\frac{\dot{\epsilon}_1}{2\kappa_1 - 3\kappa_3} - \frac{k\epsilon_2}{4(\kappa_1 - \kappa_3)}\right], \qquad \delta\mathcal{E} = -k\epsilon_2. \tag{C12}$$

One can construct gauge invariant variables as follows:

$$\alpha_{\mathrm{IIa}} = \alpha - \frac{2\kappa_1 - \kappa_3}{4(\kappa_1 - \kappa_3)}\mathcal{E} + \frac{2\kappa_1 - 3\kappa_3}{4(\kappa_1 - \kappa_3)}\left(\frac{\dot{\beta}}{k} - \frac{\ddot{\mathcal{E}}}{k^2}\right),$$
$$\mathcal{E}_{\mathrm{IIa}} = \mathcal{E} - \alpha - \frac{2\kappa_1 - 3\kappa_3}{2\kappa_1 - \kappa_3}\mathcal{R}. \tag{C13}$$

In the Einstein-Hilbert limit, $2\kappa_1 - \kappa_3 = 0$, $\mathcal{R}$ itself is gauge invariant just as in general relativity, and α_{IIa} coincides with $\tilde{\alpha}$ defined in the cases SI and also SIIb. Using these gauge invariant variables, one can show that the reduced Lagrangian becomes zero as expected.

Taking into account the gauge transformation of the vector mode, the gauge transformation in the case SIIa can be covariantly written as

$$h_{\mu\nu} \to h_{\mu\nu} + \partial_\mu\xi_\nu + \partial_\nu\xi_\mu + b\partial_\rho\xi^\rho\eta_{\mu\nu}, \tag{C14}$$

where the constant b is given by

$$b = -\frac{2\kappa_1 - \kappa_3}{2(\kappa_1 - \kappa_3)}, \tag{C15}$$

Note that the transformation in the case $b = 0$, equivalently $2\kappa_1 = \kappa_3$, reduces to the standard gauge transformation in linearized general relativity.

3. Case SIIb

In the case SIIb, we first arrange the first-class and second-class constraints. To do so, we define the following Hamiltonian including only the second-class constraint $\mathcal{C}_1^\alpha$:

$$H_T^S = H^S + \lambda_\alpha\mathcal{C}_1^\alpha. \tag{C16}$$

Then the first-class constraints can be written as

$$\tilde{\mathcal{C}}_2^\beta = \{\mathcal{C}_1^\beta, H_T^S\}, \qquad \tilde{\mathcal{C}}_3^\beta = \{\tilde{\mathcal{C}}_2^\beta, H_T^S\},$$
$$\dot{\tilde{\mathcal{C}}}_3^\beta = \{\tilde{\mathcal{C}}_3^\beta, H_T^S\} = k^2\tilde{\mathcal{C}}_2^\beta. \tag{C17}$$

Then, as in the previous case, we define the generating function as $G = \epsilon_i(t)\mathcal{C}_i^\beta$, and $\dot{G} = \partial G/\partial t + \{G, H^S\} = 0$ yields

$$\epsilon_1 + k^2\epsilon_3 + \dot{\epsilon}_2 = 0, \qquad \epsilon_2 + \dot{\epsilon}_3 = 0. \tag{C18}$$

In this case, one obtains the same equations as in the case SI. Thus the gauge transformations of α, β, $\mathcal{R}$, and $\mathcal{E}$ are also the same as well as the gauge transformation in a covariant form, which is given by Eq. (C4). Using the gauge invariant variables introduced in the case SI, one can show that the reduced Lagrangian after solving all the constraints becomes zero, implying no scalar d.o.f.

4. Case SIIc

In this case, all the constraints are second class, which implies that there is no gauge d.o.f. After integrating out the variables α, β, and $\mathcal{R}$, we finally obtain the reduced Lagrangian for $\mathcal{E}$:

$$\mathcal{L}^S = \frac{24\mu_1^2}{(2\kappa_1 k^2 + 3\mu_1)^2}[\kappa_1\dot{\mathcal{E}}^2 - (\kappa_1 k^2 + \mu_1)\mathcal{E}^2]. \tag{C19}$$

One can immediately see that the remaining scalar d.o.f. is always ghost-free as long as $\kappa_1 > 0$, and the conditions for avoiding tachyonic instability are given by $\mu_1 > 0$.

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