LINEARIZATION OF PDEs

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Given a nonlinear scalar partial differential equation (PDE) or nonlinear system of PDEs one can do the following [1-4]:

(1) Determine whether or not it can be linearized by an invertible mapping without knowing the target linear PDE. Find such a mapping when it exists.

(2) Determine if it can be linearized by embedding the given PDE in a linear PDE through a non-invertible mapping without knowing the target linear PDE. Find such a mapping when it exists.

Let $R\{x,u\}$ denote a given system of PDEs with n independent variables $x = (x_1,\ldots,x_n)$ and m dependent variables $u = (u^1,\ldots,u^m)$. Let

$\mathbf{X} = \xi_i(x,u,\underset{1}{u},\ldots,\underset{k}{u})\frac{\partial}{\partial x_i} + \eta^\nu(x,u,\underset{1}{u},\ldots,\underset{k}{u})\frac{\partial}{\partial u^\nu}$ (summation over a repeated index) be an infinitesimal generator of local symmetries of $R\{x,u\}$ where $\underset{j}{u}$ denotes the set of coordinates corresponding to all j^{th} order partial derivatives of u with respect to x. Such local symmetries include point symmetries (k=0), contact symmetries ($\xi_i \equiv 0$, m=1, k=1), and Lie-Bäcklund symmetries.

Let $S\{z,w\}$ denote a target system of PDEs with n independent variables $z = (z_1,\ldots,z_n)$ and m dependent variables $w = (w^1,\ldots,w^m)$.

Let μ denote a mapping which transforms any solution $u = U(x)$ of $R\{x,u\}$ to a solution $w = W(z)$ of $S\{z,w\}$ given by mapping equations of the form

$$z = \phi(x,u,\underset{1}{u},\ldots,\underset{\ell}{u}),$$
$$w = \psi(x,u,\underset{1}{u},\ldots,\underset{\ell}{u}).$$

One can prove the following two theorems concerning invertible mappings μ.

Theorem 1. (scalar PDE, m=1, [5]): μ defines an invertible mapping from $(x,u,\underset{1}{u},\ldots,\underset{p}{u})$-space to $(z,w,\underset{1}{w},\ldots,\underset{p}{w})$-space if and only if μ is an invertible contact transformation

$$z = \phi(x,u,\underset{1}{u}),$$
$$w = \psi(x,u,\underset{1}{u}),$$
$$\underset{1}{w} = \underset{1}{\psi}(x,u,\underset{1}{u}).$$

Theorem 2. (system of PDEs, m≥2, [6]): μ defines an invertible mapping from $(x,u,\underset{1}{u},\ldots,\underset{p}{u})$-space to $(z,w,\underset{1}{w},\ldots,\underset{p}{w})$-space if and only if μ is an invertible point transformation

$$z = \phi(x,u),$$
$$w = \psi(x,u).$$

The following two theorems (cf. [2],[3]) hold for invertible mappings relating nonlinear systems of PDEs and linear systems of PDEs.

Theorem 3. (necessary conditions): If there exists an invertible mapping μ from a given nonlinear system of PDEs ($m>1$) to some linear system then $R\{x,u\}$ must admit an infinitesimal generator of the form

$$\mathbf{X} = \sum_{\sigma=1}^{m} (\alpha_i^\sigma F^\sigma(x,u) \frac{\partial}{\partial x_i} + \beta_\nu^\sigma F^\sigma(x,u) \frac{\partial}{\partial u^\nu})$$

where $\alpha_i^\sigma, \beta_\nu^\sigma$ are specific functions of (x,u) and $F = (F^1,\dots,F^m)$ is an arbitrary solution of some linear system of PDEs $L[X]F = 0$; $L[X]$ is a linear operator with respect to independent variables $X = (X_1(x,u),\dots,X_n(x,u))$.

Theorem 4. (sufficient conditions): If the system of m first order PDEs

$$\alpha_i^\sigma \frac{\partial\Phi}{\partial x_i} + \beta_\nu^\sigma \frac{\partial\Phi}{\partial u^\nu} = 0,$$

has as n functionally independent solutions $X_1(x,u),\dots,X_n(x,u)$, and the system of m^2 first order PDEs

$$\alpha_i^\sigma \frac{\partial\psi^\gamma}{\partial x_i} + \beta_\nu^\sigma \frac{\partial\psi^\gamma}{\partial u^\nu} = \delta^{\gamma\sigma},$$

($\delta^{\gamma\sigma}$ is the Kronecker symbol) has a solution $\psi = (\psi^1(x,u),\dots,\psi^m(x,u))$, then the invertible mapping

$$z_j = \phi_j(x,u) = X_j(x,u), \ j=1,\dots,n,$$

$$w^\gamma = \psi^\gamma(x,u), \ \gamma=1,\dots,m,$$

transforms $R\{x,u\}$ to a linear system $S\{z,w\}$ given by $L[z]w = g(z)$ for some nonhomogeneous term $g(z)$.

The following two theorems (cf. [2],[3]) hold for invertible mappings relating nonlinear scalar PDEs and linear scalar PDEs.

Theorem 5. (necessary conditions): If there exists an invertible mapping μ from a given nonlinear scalar PDE to some linear scalar PDE then $R\{x,u\}$ must admit an infinitesimal generator of the form

$$\mathbf{X} = (\alpha_i F + \alpha_{ij} H_j) \frac{\partial}{\partial x_i} + (\beta F + \beta_j H_j) \frac{\partial}{\partial u} + (\lambda_i F + \lambda_{ij} H_j) \frac{\partial}{\partial u_i}$$

where $\alpha_i, \alpha_{ij}, \beta, \beta_j, \lambda_i, \lambda_{ij}$ are specific functions of $(x,u,\underset{1}{u})$ and $F = F(x,u,\underset{1}{u})$ is an arbitrary solution of some linear PDE $L[X]F = 0$; $L[X]$ is a linear operator with respect to independent variables $X = (X_1(x,u,\underset{1}{u}),\dots,X_n(x,u,\underset{1}{u}))$; $H_j = \frac{\partial F}{\partial X_j}, j=1,\dots,n$.

Theorem 6. (sufficient conditions): Suppose the system of n+1 first order PDEs

$$\alpha_i \frac{\partial \Phi}{\partial x_i} + \beta \frac{\partial \Phi}{\partial u} + \lambda_i \frac{\partial \Phi}{\partial u_i} = 0,$$

$$\alpha_{ij} \frac{\partial \Phi}{\partial x_i} + \beta_j \frac{\partial \Phi}{\partial u} + \lambda_{ij} \frac{\partial \Phi}{\partial u_i} = 0,$$

has as n functionally independent solutions $X_1(x,u,\underset{1}{u}),\ldots,X_n(x,u,\underset{1}{u})$; the system of n+1 first order PDEs

$$\alpha_i \frac{\partial \psi}{\partial x_i} + \beta \frac{\partial \psi}{\partial u} + \lambda_i \frac{\partial \psi}{\partial u_i} = 1,$$

$$\alpha_{ij} \frac{\partial \psi}{\partial x_i} + \beta_j \frac{\partial \psi}{\partial u} + \lambda_{ij} \frac{\partial \psi}{\partial u_i} = 0,$$

has a solution $\psi(x,u,\underset{1}{u})$; the system of n(n+1) first order PDEs

$$\alpha_i \frac{\partial \psi_j}{\partial x_i} + \beta \frac{\partial \psi_j}{\partial u} + \lambda_i \frac{\partial \psi_j}{\partial u_i} = 0,$$

$$\alpha_{ik} \frac{\partial \psi_j}{\partial x_i} + \beta_k \frac{\partial \psi_j}{\partial u} + \lambda_{ik} \frac{\partial \psi_j}{\partial u_i} = \delta_{kj},$$

has n functionally independent solutions $\psi = (\psi_1(x,u,\underset{1}{u}),\ldots,\psi_n(x,u,\underset{1}{u}))$; and $(z,w,\underset{1}{w}) = (X(x,u,\underset{1}{u}),\psi(x,u,\underset{1}{u}),\underset{1}{\psi}(x,u,\underset{1}{u}))$ defines a contact transformation. Then the invertible mapping $z_j = \phi_j(x,u,\underset{1}{u}) = X_j(x,u,\underset{1}{u})$, $w = \psi(x,u,\underset{1}{u})$, $w_j = \psi_j(x,u,\underset{1}{u})$, transforms $R\{x,u\}$ to $S\{z,w\}$ given by $L[z]w = g(x)$ for some nonhomogeneous term $g(z)$.

Numerous examples illustrating Theorems 3 to 6 are given in [2-4]. A symbolic manipulation algorithm [7] exists which <u>automatically</u> determines whether or not a given PDE admits an infinitesimal generator $\mathbf{X}$ satisfying the necessary conditions of Theorems 3 or 5.

The algorithms presented in Theorems 3 to 6 can be extended to <u>non-invertible mappings</u> by extending the classes of symmetries admitted by PDEs to <u>nonlocal symmetries</u> realized as <u>potential symmetries</u> [2,4,8]. Here let $Q\{x,u\}$ denote a given nonlinear system of r PDEs with independent variables $x = (x_1,\ldots,x_n)$ and dependent variables $u = (u^1,\ldots,u^k)$. A symmetry admitted by $Q\{x,u\}$ is nonlocal if its infinitesimals at any point x depend on the global behaviour of $u(x)$. Suppose $Q\{x,u\}$ has one PDE of order ℓ written in conserved form:

$$D_i\, f^i(x,u,\underset{1}{u},\ldots,\underset{\ell-1}{u}) = 0. \tag{1}$$

Through (1) introduce (uniquely to within a gauge) n-1 auxiliary dependent variables (<u>potentials</u>) $v = (v^1,\ldots,v^{n-1})$ defined by

$$f^1 = \frac{\partial v^1}{\partial x_2},$$

$$f^j = (-1)^{j-1}\left[\frac{\partial v^j}{\partial x_{j+1}} + \frac{\partial v^{j-1}}{\partial x_{j-1}}\right] \quad 1<j<n, \tag{2}$$
$$f^n = (-1)^{n-1}\frac{\partial v^{n-1}}{\partial x_{n-1}}.$$

Now define an auxiliary system $R\{x,u,v\}$ of $r+n-1$ PDEs obtained by replacing (1) of $Q\{x,u\}$ by (2). Most importantly $Q\{x,u\}$ is embedded in $R\{x,u,v\}$: If $(u(x),v(x))$ solves $R\{x,u,v\}$ then $u(x)$ solves $Q\{x,u\}$; if $u(x)$ solves $Q\{x,u\}$ then there is some $v(x)$ such that $(u(x),v(x))$ solves $R\{x,u,v\}$. Clearly the relationship between $Q\{x,u\}$ and $R\{x,u,v\}$ is non-invertible.

Suppose $R\{x,u,v\}$ admits an infinitesimal generator of the form

$$\mathbf{X} = \xi_i(x,u,v)\frac{\partial}{\partial x_i} + \eta^\sigma(x,u,v)\frac{\partial}{\partial u^\sigma} + \zeta^\mu(x,u,v)\frac{\partial}{\partial v^\mu} \tag{3}$$

such that (i) $(\xi(x,u,v),\eta(x,u,v))$ depends explicitly on v; (ii) the criteria of Theorems 3 and 4 are satisfied. Then one can construct an invertible mapping μ which transforms $R\{x,u,v\}$ to a linear system of PDEs $S\{z,w\}$. Consequently the composition of μ and the non-invertible mapping which relates $Q\{x,u\}$ and $R\{x,u,v\}$ yields a non-invertible mapping which embeds $Q\{x,u\}$ in the linear system $S\{x,w\}$. Numerous examples are given in [2], [4].

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