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Schwarzschild-de Sitter black hole in canonical quantization

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Abstract. We solve Wheeler-De Witt (WDW) metric probability wave equation on the apparent horizon hypersurface of the Schwarzschild de Sitter (SdS) black hole. To do so we choose radial dependent mass function $M(r)$ for its internal regions in the presence of a dynamical massless quantum matter scalar field $\psi(r)$ and calculate canonical super hamiltonian constraint on t -constant hypersurface near the horizon $r = M(r)$. In this case $M(r)$ become geometrical degrees of freedom while $\psi(r)$ is matter degrees of freedom of the apparent horizon. However our solution is obtained versus the quantum harmonic oscillator which defined against the well known hermit polynomials. In the latter case we obtain quantized mass of the SdS quantum black hole as $\sqrt{\Lambda}M(n) = \left(\frac{2n+1}{12\sqrt{2}}\right)^{\frac{1}{3}}$ in which Λ is the cosmological constant and $n = 0, 1, 2, \dots$ are quantum numbers of the hermit polynomials. This shows that a quantized SdS in its ground state has a nonzero value for the mass $M(0) = 0.38914/\sqrt{\Lambda}$. Thus one can infer that the latter result satisfies Penrose hypotheses for cosmic censorship where a causal singularity may be covered by a horizon surface.

1. Introduction

Quantum instability of the black holes was discovered firstly by Hawking [1,2]. Then some authors studied more details of the quantum black holes evaporation. Properly a detailed picture of the evaporation process and its final state can be given only within the framework of a complete and self-consistent pure quantum gravity theory, which has not yet to be found [3]. It will be valid for the Planck scale of the universe [4,5]¹. In absence of a pure quantum gravity theory, the evaporation process of quantum black holes and their final structure were studied in different approaches named as the semiclassical perturbative [1,2,6,7,8] and the non-perturbative Dirac's canonical quantization [9-22] methods respectively. At present a tractable way to non-perturbative quantum gravity is the WDW functional approach of canonical quantization [17]. Recent developments in quantum cosmology are based on the mini-super-space analysis of the WDW equation. To construct black hole mini-super-space models, we can assume spherically symmetric metrics. A full canonical formalism of spherically symmetric systems was extensively studied by Hajicek et al [14,15]. The difficult problem in quantization is how to treat the super-momentum constraint condition. Unlike in cosmological situations, we cannot find adequate mini-super-space variables for which the super-momentum constraint become trivial. For

¹ $M_p = (\hbar c/G)^{1/2} = 2.18 \times 10^{-8} kg;$
 $D_p = (\hbar G/c^3)^{1/2} = 1.62 \times 10^{-35} m,$
 $T_p = \hbar/M_p c^2 = 5.31 \times 10^{-44} s$



example, Rodrigues et al [22], proposed a black hole mini-super-space model to discuss the wave function of a decaying black hole. However the WDW wave functional equation derived from their mini-super-space variables turned out to be incompatible with the super-momentum constraint [23]. Their model is limited to a purely gravitational system. Then the dynamical behavior of a spherically symmetric metric is severally restricted by the super-hamiltonian and super-momentum constraints. In the classical dynamics the well-known Birkhoff theorem holds, which prohibits the existence of spherically symmetric gravitons. It is not certain whether even in quantum dynamics any interesting degrees of freedom for the exterior geometry of black holes can remain or not. However, we consider in this work, a spherically symmetric SdS black hole and require compatibility between the Hamiltonian and super-momentum constraints. We do not pursue the construction of mini-super-space models, which may be valid in the whole space-time. Our analysis is focused on quantization of the spherical hyper-surface ‘ $t = Constant$ ’, near the apparent horizon of a dynamical SdS black hole with a metric used in [24,25]. Details of the paper is as follows. In Sec.2 we derive WDW equation of a SdS quantum black hole boundary metric near its apparent horizon in time-independent regime. Then we use Taylor series expansion of the self interacting mass function potential and show that the WDW solution can be described via the quantum harmonic oscillator. We obtained quantized mass eigenvalues of the SdS quantum black hole. In Sec.4 we present a summary and concluding remarks of this work.

2. Schwarzschild de Sitter quantum black hole

Let us we start with a dynamical massless scalar field ψ minimally coupled with the Einstein-Hilbert gravity action in 4D curved space time which is given in units $G = c = \hbar = 1$ as follows.

$$I = \frac{1}{16\pi} \int dx^4 \sqrt{g} (R - 2\Lambda + \zeta g^{\mu\nu} \partial_\mu \psi \partial_\nu \psi) \quad (1)$$

where ζ and ‘ Λ ’ are the coupling constant and the well defined cosmological constant. ‘ g ’ is absolute value of determinant of the background metric ‘ $g_{\mu\nu}$ ’ and ‘ R ’ is Ricci scalar. Varying the above action with respect to the metric components ‘ $g^{\mu\nu}$ ’, we can obtain metric field equation as follows.

$$G_{\mu\nu} + \Lambda g_{\mu\nu} = -8\pi T_{\mu\nu}[\psi] \quad (2)$$

in which

$$T_{\mu\nu}[\psi] = \frac{\zeta}{16\pi} \left[\partial_\mu \psi \partial_\nu \psi - \frac{g_{\mu\nu}}{2} g^{\alpha\beta} \partial_\alpha \psi \partial_\beta \psi \right] \quad (3)$$

is matter stress-tensor. Setting $\Lambda = 0$ and $\zeta = -16\pi$ the above equation has a static spherically symmetric asymptotically flat exact solution called as Janis-Newman-Winicour (JNW) solution as [26]

$$ds^2 = -\left(1 - \frac{b}{r}\right)^s dt^2 + \left(1 - \frac{b}{r}\right)^{-s} dr^2 + r^2 \left(1 - \frac{b}{r}\right)^{1-s} (d\theta^2 + \sin^2 \theta d\varphi^2) \quad (4)$$

with

$$\psi = \frac{q}{b\sqrt{4\pi}} \ln \left(1 - \frac{b}{r}\right) \quad (5)$$

in which

$$s = \frac{2M}{b}, \quad b = 2\sqrt{M^2 + q^2}. \quad (6)$$

In the above metric solution M and q are called as the black hole mass and the charge of the scalar field respectively. Setting $T_{\mu\nu} = 0$ the equation (2) has a spherically symmetric static

metric solution as follows [27] (see also [25]).

$$ds^2 = -\left(1 - \frac{2M}{r} - \frac{1}{3}\Lambda r^2\right)dt^2 + \frac{dr^2}{\left(1 - \frac{2M}{r} - \frac{1}{3}\Lambda r^2\right)} + r^2(d\theta^2 + \sin^2\theta d\varphi^2). \quad (7)$$

The above metric describes Schwarzschild static black hole in the presence of the cosmological event horizon. The constants defined by M and Λ are the black hole mass and the cosmological parameters respectively. Defining a dimensionless cosmological parameter

$$\xi = \frac{\Lambda(2M)^2}{3} \quad (8)$$

one can infer that for $0 < \xi < 1$ the above metric has two event horizon which are determined by solving $1 - \frac{2M}{r} - \frac{1}{3}\Lambda r^2 = 0$. They are obtained as $r_b \simeq 2M(1 + \xi)$ and $r_c \simeq \frac{2M}{\sqrt{\xi}}$ which are named as the radius of the black hole and the cosmological horizons respectively [7,8]. Size of the black hole horizon varies between zero and a large scale of the cosmological horizon. If the black hole horizon is much smaller than the cosmological horizon so that $r_b \sim r_c$ for $\xi \sim 0.46559$, the effects of the radiation coming from the cosmological horizon is negligible. In this case it is named as a degenerate SdS black hole. Degenerate solution in which the black hole has the maximum size is called the Nariai solution [27]. In this solution the two horizons have the same size and so the same temperature. Therefore it shall be in thermal equilibrium when we choose $0 \ll \xi < 1$. Intuitively one would expect any slight perturbation of the geometry which makes hotter the black hole with respect to its environment. Thus one may suspect the thermal equilibrium of the Nariai solution (Large scale black hole) to be unstable. We studied previously final state of an evaporating quantum perturbed SdS black hole by solving the backreaction equation in semiclassical quantum gravity (the perturbative approximation) approach [7,8]. Mathematical derivations of the backreaction equation in time-dependent and time-independent regimes, predicts a minimal remnant stable static SdS black hole final state. Instead of this perturbative approach we encourage to study stability of quantum evaporating SdS black hole in a non-perturbative canonical quantum gravity approach. To do so we should solve the WDW equation of the evaporating quantum SdS black hole as follows. Because it is very difficult to study the hamiltonian and super-momentum constraints in the whole space-time, here our analysis of the constraints is limited to the region near the apparent horizon. This is useful to study the black hole evaporation. Because we want to discuss how the two-dimensional spatial area $4\pi\varphi^2$ of the apparent horizon decreases. The apparent horizon location in the classical regime is determined by the following null equation.

$$g^{\mu\nu}\partial_\mu\varphi\partial_\nu\varphi = 0. \quad (9)$$

This is a natural spatial light-like surface where we can impose the adequate boundary condition. In time independent regime we assume that the evaporating SdS quantum black hole is formed as

$$ds_{SdS}^2 = -\left(1 - \frac{2M(r)}{r} - \frac{\Lambda}{3}r^2\right)dt^2 + \frac{dr^2}{\left(1 - \frac{2M(r)}{r} - \frac{\Lambda}{3}r^2\right)} + r^2(d\theta^2 + \sin^2\theta d\varphi^2) \quad (10)$$

in which mass of the black hole $M(r)$ is changed vs r because of the presence of the quantum matter scalar field $\psi(r)$. Here we assume that the quantum matter scalar field dose not changed the cosmological parameter and so Λ is still maintain as a constant. Boundary counterpart of the above metric near the apparent horizon $r = 2M(r)$ reads

$$\lim_{r \rightarrow 2M(r)} ds_{SdS}^2 \approx \frac{4}{3}\Lambda M(r)^2 dt^2 - \frac{dr^2}{\frac{4}{3}\Lambda M^2(r)} + 4M^2(r)(d\theta^2 + \sin^2\theta d\varphi^2). \quad (11)$$

The above metric corresponds to internal region of the evaporating SdS black hole near the horizon because its signature is changed as $(+, -, +, +)$. This means that the time coordinate t behaves as spatial coordinate while r behaves as the time coordinate. Similar to the mass function the massless scalar field should be defined versus the radial coordinate r such that $\psi = \psi(r)$ in the time independent regime. Ricci scalar for the boundary metric (11) reads

$$R^\mu_\mu = R = -\frac{1}{2} \left(16\Lambda M M'' + 32\Lambda M'^2 + \frac{1}{M^2} \right). \quad (12)$$

Substituting (11) and (12) and integrating on the two-sphere region $\sin\theta d\theta d\varphi$ the action functional (1) reads

$$I = \int dt d\Omega \mathcal{L}(M, M', \Lambda, \psi') \quad (13)$$

where the lagrangian density \mathcal{L} is obtained by integrating by parts and eliminating divergenceless terms $(M^3 M')'$ as follows.

$$\mathcal{L}(m, \dot{m}, \dot{\sigma}) = 8m^2 \dot{m}^2 - \frac{3\zeta}{4} \dot{\sigma}^2 - 2m^2 - \frac{1}{2} \quad (14)$$

where the dot $\dot{}$ denotes to derivative with respect to dimensionless radial coordinate $\rho = r\sqrt{\Lambda}$ and we defined dimensionless mass function $m(\rho)$ and the field $\sigma(\rho)$ as follows.

$$m = \sqrt{\Lambda} M, \quad \sigma = \sqrt{\Lambda} \psi. \quad (15)$$

Applying the canonical momentum of the fields σ and m as

$$\pi_\sigma = \frac{\partial \mathcal{L}}{\partial \dot{\sigma}} = -\frac{3\zeta}{2} \dot{\sigma}, \quad \pi_m = \frac{\partial \mathcal{L}}{\partial \dot{m}} = 16m^2 \dot{m} \quad (16)$$

and definition of the hamiltonian density

$$\mathcal{H} = \pi_\sigma \dot{\sigma} + \pi_m \dot{m} - \mathcal{L} \quad (17)$$

we obtain

$$\mathcal{H} = \frac{\pi_m^2}{32m^2} - \frac{\pi_\sigma^2}{3\zeta} + 2m^2 + \frac{1}{2}. \quad (18)$$

To obtain the WDW wave equation of the boundary action functional (11) near the black hole horizon we should substitute the Dirac's canonical quantization operators

$$\hat{\pi}_\sigma = \frac{1}{i} \frac{d}{d\sigma}, \quad \hat{\pi}_m = \frac{1}{i} \frac{d}{dm} \quad (19)$$

into the super-hamiltonian constraint (18) such that

$$\left[\frac{d^2}{dm^2} - \frac{32m^2}{3\zeta} \frac{d^2}{d\sigma^2} - 16m^2 - 64m^4 \right] \Psi(m, \sigma) = 0 \quad (20)$$

where $\Psi(m, \sigma)$ is called as the WD probability wave functional of the metric. It describes possible values of the metric components (11) near the horizon when it takes on particular values of the fields σ and m . We solve the WDW wave equation (20) by applying the separation of variables method as follows.

$$\Psi(\sigma, \eta) = e^{i\sqrt{3\zeta(1+\zeta)}\sigma} R(m) \quad (21)$$

the WDW wave equation (20) reads

$$\frac{d^2 R}{dm^2} + V(m)R(m) = 0 \quad (22)$$

where we defined the self interaction mass potential $V(m)$ as follows.

$$V(m) = 16(1 + 2\zeta)m^2 - 64m^4. \quad (23)$$

It is suitable to obtain Taylor series expansion of the above potential about its minimum point as follows. Its minimum point is obtained by solving the equation $\frac{dV(m)}{dm} = 0$ as

$$m_0^2 = \frac{1 + 2\zeta}{16} \quad (24)$$

for which one can infer

$$V(m_0) = 192m_0^4, \quad \left. \frac{d^2 V}{dm^2} \right|_{m=m_0} = -256m_0^2. \quad (25)$$

Up to higher order terms one can use (24) and (25) to obtain Taylor series expansion of the potential (23) as follows.

$$V(m) \approx 192m_0^4 - 126m_0^2(m - m_0)^2. \quad (26)$$

Substituting (26) and defining

$$m - m_0 = \kappa\chi, \quad (27)$$

with

$$\kappa^4 = \frac{1}{126m_0^2}, \quad 192\kappa^2 m_0^4 = 2n + 1, \quad n = 0, 1, 2, 3, \dots \quad (28)$$

the equation (22) reads

$$\frac{d^2 R(\chi)}{d\chi^2} + (1 + 2n - \chi^2)R(\chi) = 0 \quad (29)$$

which is similar to a quantum harmonic oscillator differential equation. Its solutions is given versus the Hermit polynomials as follows.

$$R_n(\chi) = \frac{e^{-\chi^2/2} H_n(\chi)}{\sqrt{2^n \pi^2 n!}}. \quad (30)$$

Its eigenvalues are obtained by solving (28) as follows.

$$m_0(n) = \left(\frac{2n + 1}{12\sqrt{2}} \right)^{\frac{1}{3}}. \quad (31)$$

Substituting (31) into the equations (15), (21) and (24) one can obtain

$$M(n) = \frac{1}{\sqrt{\Lambda}} \left(\frac{2n + 1}{12\sqrt{2}} \right)^{\frac{1}{3}}, \quad (32)$$

$$\Psi_n(\chi, \sigma) = e^{if(n)\sigma} R_n(\chi) \quad (33)$$

in which we defined

$$f(n) = \sqrt{\left(\frac{16(2n + 1)}{\sqrt{2}\sqrt{3}} \right)^{\frac{4}{3}} - \frac{1}{4}} \quad (34)$$

and

$$\zeta(n) = 8 \left(\frac{2n+1}{12\sqrt{2}} \right) \frac{2}{3} - \frac{1}{2}. \quad (35)$$

We know that imaginary part of the WDW wave solution (33) dose not have physical meaning while we can obtain maximal probability by removing this imaginary part. This restrict us to choose some quantized matter field as

$$\psi(n, j) = \frac{\sigma(n, j)}{\sqrt{\Lambda}} = \frac{2j\pi}{f(n)}, \quad j = 0, 1, 2, 3, \dots. \quad (36)$$

Numerical values of the SdS quantum black hole parameters $m_0(n)$, $f(n)$ and $\zeta(n)$ are plotted in figure 1 versus the quantum numbers n . This process of the black hole quantization may to be claim stability of an evaporating quantum perturbed SdS black hole in high energy Planck regimes. It is in accord to results of a perturbative approach of the problem which previously is studied by the author [6,7].

3. Concluding remarks

Applying the canonical quantum gravity approach we solved the WDW equation of the quantum SdS black hole metric near its apparent horizon in presence of a massless quantum scalar matter field. WD wave solution is obtained against the hermit polynomials and its quantum numbers makes quantized mass of the quantum SdS black hole. The most important message of this article is the credibility of Penrose's cosmic censorship where causal singularity of the SdS quantum evaporating black hole $r = 0$ is covered by the horizon. Its final state reaches to a minimal remnant stable black hole where the positive cosmological constant has important role for quantization of the SdS black hole. As a future work one can use our method given in the present paper to quantize other forms of the black holes.

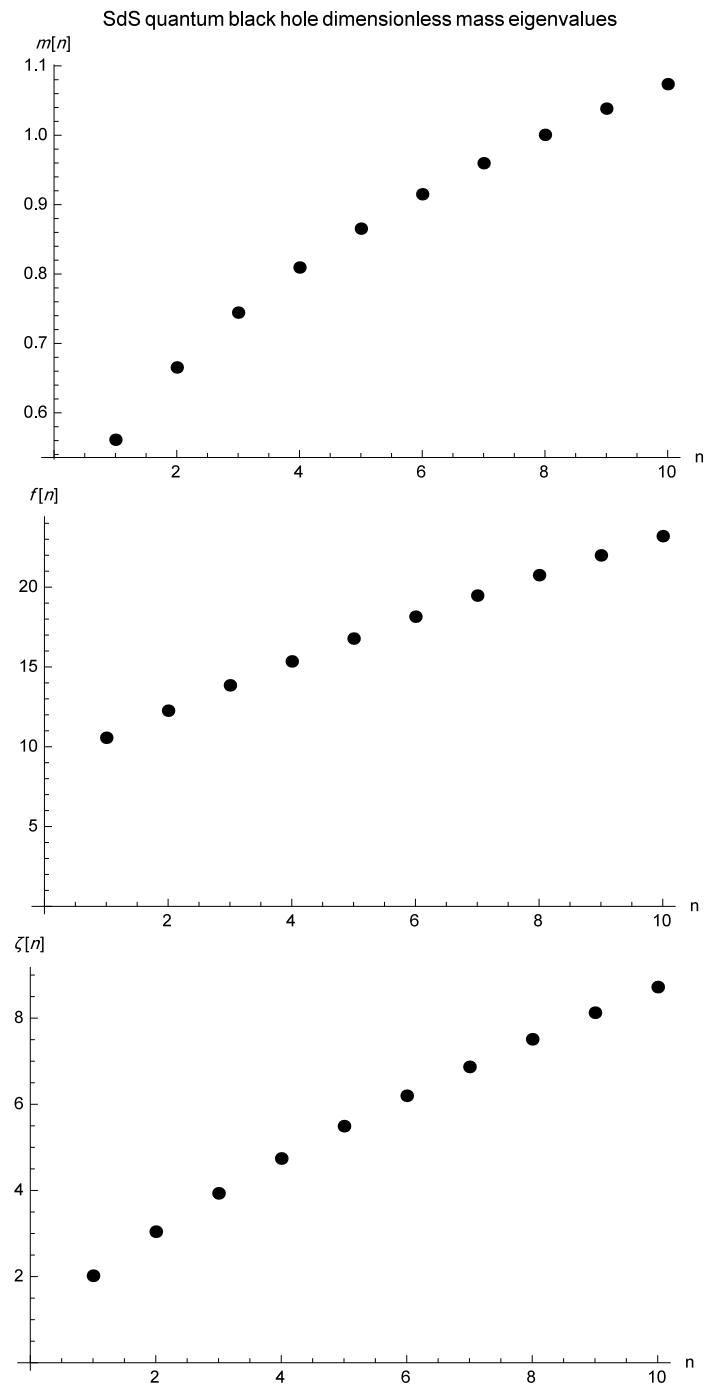


Figure 1. Diagram of the SdS quantum black hole parameters $m_0(n)$, $f(n)$ and $\zeta(n)$ vs the quantum numbers n .

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