INDICES FOR PLETHYSMS OF REPRESENTATIONS OF LIE SUPERALGEBRAS

Joris Van der Jeugt
Faculty of Mathematical Studies
The University
Southampton SO9 5NH (U.K.)

1. INDICES FOR REPRESENTATIONS OF LIE ALGEBRAS

Let G be a simple Lie algebra over the complex numbers $\mathbb{C}$, and $H \subset G$ a Cartan subalgebra. Let ρ be a representation of G in a finite-dimensional vector space V. Then we have the weight space decomposition for V:

$$V = \bigoplus_{\lambda} V_{\lambda} \quad (\lambda \in H^{*}), \tag{1}$$

where

$$V_{\lambda} = \{ v \in V \mid \rho(h)v = \lambda(h)v \ , \ \forall h \in H^{*} \}. \tag{2}$$

If $V \neq \{0\}$ then λ is called a weight of ρ and V_{λ} is the weight space associated to λ. For the Lie algebra G there exists a non-degenerate invariant symmetric bilinear form <,> , which induces a non-degenerate form <,> on H^{*}. This form is positive definite on H^{*}_{R}, the real span of the simple roots of G. The index of order 2n ($n \in N$) of ρ is then defined by [1,2]

$$I_{\rho}^{(2n)} = \sum_{\lambda} \langle \lambda, \lambda \rangle^{n} , \tag{3}$$

where the summation is over all weights of ρ (including multiplicity). If V is an irreducible highest weight module $V(\Lambda)$, then the index (3) is denoted by $I_{\Lambda}^{(2n)}$. Note that

$$\begin{aligned} I_{\Lambda}^{(0)} &= \dim V(\Lambda) = N_{\Lambda}, \\ I_{\Lambda}^{(2)} &= N_{\Lambda} \langle C_2(\Lambda) \rangle , \end{aligned} \tag{4}$$

where the latter symbol is the eigenvalue of the second-order Casimir operator on $V(\Lambda)$. The second-order index is in fact proportional to Dynkin's index.[3,4] Indices for highest weight representations $V(\Lambda)$ can be calculated explicitly and are polynomial expressions in terms of the components of the highest weight Λ.[1,2] These indices satisfy some very elegant properties that are useful in e.g. decomposing tensor products or determining a branching to a subalgebra. For example, if the tensor product of $V(\Lambda_1)$ and $V(\Lambda_2)$ decomposes as follows:

$$V(\Lambda_1) \otimes V(\Lambda_2) = \bigoplus_{\Lambda} V(\Lambda) \tag{5}$$

then

$$N_2\, I_{\Lambda_1} + N_1\, I_{\Lambda_2} = \sum_{\Lambda} I_{\Lambda}\ , \tag{6}$$

where $N_i = \dim V(\Lambda_i)$ and I_Λ is the second-order index (we drop the superscript (2n) if n=1). Equation (6), and eventually some higher order equations, can actually help to determine the decomposition (5) once Λ_1 and Λ_2 are given.[1]

It is the purpose of this paper to show that similar quantities can be defined for Lie superalgebras, with equally useful properties.

2. LIE SUPERALGEBRAS

Let $L = L_{\bar{0}} \oplus L_{\bar{1}}$ be a basic classical Lie superalgebra, i.e. one of the series $A(m,n)$, $B(m,n)$, $C(n)$, $D(m,n)$, $D(2,1;\alpha)$, $G(3)$ or $F(4)$,[5] and H a Cartan subalgebra of L. For a finite-dimensional representation ρ of L in $V = V_{\bar{0}} \oplus V_{\bar{1}}$, we have again the weight space decomposition (1), where now λ is called an even (resp. odd) weight if $V_\lambda \subset V_{\bar{0}}$ (resp. $V_{\bar{1}}$).

For L there exists again a non-degenerate invariant supersymmetric bilinear form $\langle,\rangle$, which induces a non-degenerate form $\langle,\rangle$ on $H^\star$.[6] However, this form is not positive definite on $H^\star_R$, and would therefore violate all the useful properties needed for a meaningful theory of indices.[7] Hence, for this purpose we define a different form $(,)$ on $H^\star$. Since H is also the Cartan subalgebra of $L_{\bar{0}}$, and $L_{\bar{0}}$ is a reductive Lie algebra, we can take for $(,)$ the form associated to the non-degenerate invariant bilinear form on $L_{\bar{0}}$.[7] This new inner product on $H^\star$ is now positive definite on $H^\star_R$ and can be used in order to define indices for Lie superalgebras.

3. INDICES FOR LIE SUPERALGEBRAS

For a representation V of L with weights λ we define the (second-order) index and superindex as follows:

$$I = \sum_{\lambda} (\lambda,\lambda) \ , \tag{7}$$

$$S = \sum_{\lambda} (-1)^{\sigma(\lambda)} (\lambda,\lambda) \ , \tag{8}$$

where $\sigma(\lambda)=\bar{0}$ ($\bar{1}$) if λ is even (odd). If V is an irreducible highest weight module $V(\Lambda)$, the index and superindex are denoted by I_{Λ} and S_{Λ}. When L is of type I ($A(m,n)$, $m \neq n$, or $C(n)$), then $L_{\bar{0}}$ contains a one-dimensional center z. We define the anomaly and superanomaly of V by

$$A = \sum_{\lambda} \lambda(z), \qquad A^S = \sum_{\lambda} (-1)^{\sigma(\lambda)} \lambda(z) \ . \tag{9}$$

In (7)-(9), the summation is always over all weights λ of V. When $L_{\bar{0}}$ contains no center z, A and A^S are equal to zero.

For two representations $V(\Lambda_1)$ and $V(\Lambda_2)$ with dimensions N_1, N_2, superdimensions[8] N_1^S, N_2^S, and decomposition rule

$$V(\Lambda_1) \otimes V(\Lambda_2) = \sum_{\lambda} V(\Lambda) \ , \tag{10}$$

we have the following properties:

$$\sum_{\Lambda} I_{\Lambda} = N_2 \, I_{\Lambda_1} + N_1 \, I_{\Lambda_2} + 2A_{\Lambda_1} A_{\Lambda_2} \ , \tag{11}$$

$$\sum_{\Lambda} S_{\Lambda} = N_2^S \, S_{\Lambda_1} + N_1^S \, S_{\Lambda_2} + 2A_{\Lambda_1}^S A_{\Lambda_2}^S \ . \tag{12}$$

These equations, the counterparts of (6), are also very useful in studying decompositions.[7]

4. INDICES FOR PLETHYSMS

A plethysm[2] for a Lie algebra G is the component of the direct product of n copies of some representation ρ of G in V whose permutation symmetry is described by a Young tableau. For example, if the representation ρ is denoted by $\square$, then the product of two copies of ρ allow the following plethysm:

$$\square\otimes\square = \square\square + \begin{array}{c}\square\\\square\end{array} \tag{13}$$

(note that these Young tableau are not describing any particular representation; $\square$ can be any representation ρ). The elements of (13) are determined by

$$\begin{array}{ll}\square\square : & x\otimes y + y\otimes x\ , \\ \begin{array}{c}\square\\\square\end{array} : & x\otimes y - y\otimes x\ .\end{array} \qquad (\,x,y\in V) \tag{14}$$

Clearly, the elements in (14) describe invariant subspaces of V under the action of ρ(but if $\square$ is irreducible, $\square\square$ and $\begin{array}{c}\square\\\square\end{array}$ are not necessarily irreducible).

When ρ is a representation of a Lie superalgebra L in some vector space $V = V_{\bar{0}} \oplus V_{\bar{1}}$, it is very easy to check that the following subspaces of $V\otimes V$ are invariant under the action of ρ:

$$\begin{array}{ll}\square\square : & x\otimes y + (-1)^{\xi\eta}\, y\otimes x\ , \\ \begin{array}{c}\square\\\square\end{array} : & x\otimes y - (-1)^{\xi\eta}\, y\otimes x\ .\end{array} \qquad (x\in V_{\xi},\ y\in V_{\eta}) \tag{15}$$

One notes from (14) that the 'supersymmetry' class $\square\square$ is in fact equal to:

$$\square\square = \boxed{0}\boxed{0}\times 1 + \boxed{0}\times\boxed{1} + 1\times\begin{array}{c}\boxed{1}\\\boxed{1}\end{array}\ , \tag{16}$$

where 0 or 1 refers to even or odd vectors of V. Then a plethysm or supersymmetry class of n copies of a representation ρ of L is again described by a Young tableau, but the permutation symmetry for odd vectors is conjugate to that for even vectors. For example, the Young tableau $\begin{array}{l}\square\square\\\square\end{array}$ for $V = V_{\bar{0}} \oplus V_{\bar{1}}$, where $V_{\bar{0}}$ and $V_{\bar{1}}$ are considered as two separate $L_{\bar{0}}$-modules, is then equal to

$$\begin{array}{l}\square\square\\\square\end{array} = \begin{array}{l}\boxed{0}\boxed{0}\\\boxed{0}\end{array} + \begin{array}{c}\boxed{0}\\\boxed{0}\end{array}.\boxed{1} + \boxed{0}\boxed{0}.\boxed{1} + \boxed{0}.\boxed{1}\boxed{1} + \boxed{0}.\begin{array}{c}\boxed{1}\\\boxed{1}\end{array} + \begin{array}{l}\boxed{1}\boxed{1}\\\boxed{1}\end{array}\ . \tag{17}$$

Hence, for the supersymmetry class corresponding to $\begin{array}{l}\square\square\\\square\end{array}$, we need to take the conjugate of every tableau with labels 1 in (17) :

$$\begin{array}{l}\square\square\\\square\end{array} = \begin{array}{l}\boxed{0}\boxed{0}\\\boxed{0}\end{array} + \begin{array}{c}\boxed{0}\\\boxed{0}\end{array}.\boxed{1} + \boxed{0}\boxed{0}.\boxed{1} + \boxed{0}.\begin{array}{c}\boxed{1}\\\boxed{1}\end{array} + \boxed{0}.\boxed{1}\boxed{1} + \begin{array}{l}\boxed{1}\boxed{1}\\\boxed{1}\end{array}\ . \tag{18}$$

Every plethysm describes an invariant subspace in the direct product of n copies of some representation $\square$ of L.

Now there exist several formulae[7)] relating (super)dimension, (super)index and (super)anomaly of a plethysm to that of the original representation $\square$, e.g.

$$
\begin{aligned}
I_{\square\square} &= N_{\square} I_{\square} + 2\, S_{\square} + A_{\square}^{2} \,, \\
I_{\begin{smallmatrix}\square\\ \square\end{smallmatrix}} &= N_{\square} I_{\square} - 2\, S_{\square} + A_{\square}^{2} \,.
\end{aligned}
\qquad (19)
$$

These expressions enable one to calculate the previously mentioned quantities for a plethysm. The knowledge of such quantities for plethysms are then very useful in the study of plethysms, in particular in order to recognize their constituents.

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