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RESEARCH ARTICLE

Michael Tsamparlis · Andronikos Paliathanasis

Lie and Noether symmetries of geodesic equations and collineations

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Abstract The Lie symmetries of the geodesic equations in a Riemannian space are computed in terms of the special projective group and its degenerates (affine vectors, homothetic vector and Killing vectors) of the metric. The Noether symmetries of the same equations are given in terms of the homothetic and the Killing vectors of the metric. It is shown that the geodesic equations in a Riemannian space admit three linear first integrals and two quadratic first integrals. We apply the results in the case of Einstein spaces, the Schwarzschild spacetime and the Friedman Robertson Walker spacetime. In each case the Lie and the Noether symmetries are computed explicitly together with the corresponding linear and quadratic first integrals.

Keywords Geodesics, General relativity, Classical mechanics, Collineations, Lie symmetries, Projective collineations, Noether symmetries, First integrals

1 Introduction

Geometrically the Lie symmetries of a system of differential equations are understood as automorphisms which preserve the set of the solution curves. In a space with a linear connection there is an inherent system of differential equations defined by the paths (or autoparallels) of the connection. It is well known that these curves (as a set) are preserved under the projective automorphisms of the space. Therefore it is reasonable one to expect a relation between the projective collineations of the space and the Lie symmetries of the system of differential equations of the paths.

A special case of the above scenario occurs in a Riemannian space in which case the paths are the (affinely or not) parameterized geodesics determined by the

M. Tsamparlis A. Paliathanasis Faculty of Physics, Department of Astronomy-Astrophysics-Mechanics, University of Athens Panepistemiopolis 157 83 Athens, Greece mtsampa@phys.uoa.gr · A. Paliathanasis paliathanasis@gmail.com

metric. In this case one expects a close relation to exist between the Lie symmetries of the geodesic equations and the projective collineations (PC) of the metric or their degenerates affine collineations (AC), homothetic Killing vector (HKV) and Killing vectors (KV).

Indeed this topic has been discussed extensively in the literature. Classic is the work of Katzin and Levin [1; 2; 3; 4]. Important contributions have also been done by Aminova [5; 6; 7; 8], Prince and Crampin [9] and many others. More recent is the work of Feroze et al. [10] where the case of maximally symmetric spaces of low dimension it is discussed.

Furthermore, because the geodesic equations follow from the variation of the geodesic Lagrangian defined by the metric and due to the fact that the Noether symmetries are a subgroup of the Lie group of Lie symmetries of these equations, one should expect a relation/identification of the Noether symmetries of this Lagrangian with the projective collineations of the metric or with its degenerates. Recent work in this direction has been by Bokhari et al. [11; 12] in which the relation of the Noether symmetries with the KVs of some special spacetimes it is discussed.

In the present paper we give a complete answer to both topics mentioned above. In Sect. 2 we give a brief introduction concerning the autoparallels of a symmetric connection. In Sect. 3 we determine the conditions for the Lie symmetries of the geodesic equations in covariant form and relate them with the special projective symmetries of the connection. A similar result has been obtained by Prince and Crampin in [9] using the bundle formulation of second order ordinary differential equations (ODE). In Sect. 4 we apply these conditions in the special case of Riemannian spaces and in Theorem 1 we give the Lie symmetry vectors in terms of the special projective collineations of the metric and their degenerates. In Sect. 5 we give the second result of this work, that is Theorem 2, which relates the Noether symmetries of the geodesic Lagrangian defined by the metric with the homothetic algebra of the metric and comment on the results obtained so far in the literature. Finally in Sect. 8 we apply the results to various cases and eventually we give the Noether symmetries and the associated conserved quantities of the Friedman Robertson Walker (FRW) spacetimes.

2 Preliminary results

Consider a C^{∞} manifold M of dimension n, endowed with a Γ^i_{jk} symmetric[1] connection. In a local coordinate system $\{x^i | i = 1,\dots,n\}$ the connection $\Gamma^i_{jk}\partial_i = \nabla_j \partial_k$ and the autoparallels of the connection are defined by the requirement:

$$\ddot{x}^i(t) + \Gamma^i_{jk}(x(t))\dot{x}^j(t)\dot{x}^k(t) = \phi(t)\dot{x}^i(t), \quad i = 1,\dots,n, \tag{1}$$

where t is a parameter along the paths. When ϕ vanishes, we say that the autoparallel is affinely parameterized and in this case t is called an affine parameter, that is one has:

$$\ddot{x}^i(t) + \Gamma^i_{jk}(x(t))\dot{x}^j(t)\dot{x}^k(t) = 0, \quad i = 1,\dots,n. \tag{2}$$

[1] The coefficients Γ^i_{jk} in general are not symmetric in the lower indices. In the autoparallel equation (1) the antisymmetric part of $\Gamma^i_{[jk]}$ (the torsion) does not play a role.

If $\mathbf{X} = X^a \partial_a$ is a vector field on the manifold the following identity holds (see Yano [13, eqn. (2.16)]):

$$\mathscr{L}_X \Gamma^i_{jk} = X^i_{,jk} + \Gamma^i_{jk,l} X^l + X^l_{,k} \Gamma^i_{lj} + X^l_{,j} \Gamma^i_{lk} - X^i_{,l} \Gamma^l_{jk}. \tag{3}$$

We say that the vector field $\mathbf{X}$ is an Affine Collineation (AC) iff:

$$\mathscr{L}_X \Gamma^i_{jk} = 0. \tag{4}$$

In flat space equation (4) implies the condition:

$$X_{a,bc} = 0 \tag{5}$$

the solution of which is:

$$X_a = B_{ab} x^b + C_a \tag{6}$$

where B_{ab} and C_a are constants. The geometric property/definition of affine collineations is that they preserve the set of autoparallels (i.e. paths) of the connection together with their affine parametrization (that is, by an affine symmetry an affinely parameterized autoparallel goes over to an affinely parameterized autoparallel of the same connection). From (6) we infer that in an n-dimensional space there are at most $n + n^2 = n(n+1)$ ACs and, when this is the case, it can be shown that the space is flat.

We say that a vector field $\mathbf{X}$ is a projective collineation of the connection if there exists a one form ω_i such that the following condition holds [13; 14]:

$$\mathscr{L}_X \Gamma^i_{jk} = \omega_j \delta^i_k + \omega_k \delta^i_j. \tag{7}$$

In a Riemannian space the form ω^i is closed, that is, there exists a function $f(x^i)$, called the projective function, such that:

$$\mathscr{L}_X \Gamma^i_{jk} = f_{,j} \delta^i_k + f_{,k} \delta^i_j. \tag{8}$$

In flat space condition (8) implies that:

$$X_{a,b} = B_{ab} + (A_c x^c) g_{ab} + C_b x_a. \tag{9}$$

which has the solution:

$$X_a = B_{ab} x^b + (A_b x^b) x_a + C_a \tag{10}$$

where again the various coefficients are constants. In an n-dimensional space there are at most $n^2 + n + n = n(n+2)$ projective collineations of the connection and when this is the case, it can be shown that the space is flat. This holds in any space irrespective of the signature of the metric and the (finite) dimension of space. In case that the function f satisfies the condition $f_{;ij} = 0$, that is, $f_{,i}$ is a gradient KV, the projective collineation is called special.

The geometric property/definition of the (proper) projective collineations is that they preserve the set of autoparallels, but they do not preserve their parametrization.

3 Lie point symmetries of the autoparallel equations

We write the system of ODEs (1) in the form $\ddot{x}^i = \omega^i(x,\dot{x},t)$ where:

$$\omega^i(x,\dot{x},t) = -\Gamma^i_{jk}(x)\dot{x}^j\dot{x}^k - \phi(t)\dot{x}^i. \tag{11}$$

The associated linear operator defined by this system of ODEs is:

$$\mathbf{A} = \frac{\partial}{\partial t} + \dot{x}^i\frac{\partial}{\partial x^i} + \omega^i(t,x^j,\dot{x}^j)\frac{\partial}{\partial \dot{x}^i}. \tag{12}$$

The condition for a Lie point symmetry for the system of equations is [15]:

$$[X^{[1]},\mathbf{A}] = \lambda(x^a)\mathbf{A} \tag{13}$$

where $X^{[1]}$ is the first prolongation of the symmetry vector $X = \xi(t,x)\partial_t + \eta^i(t,x)\partial_{x^i}$ defined as follows:

$$X^{[1]} = \xi(t,x,\dot{x})\partial_t + \eta^i(t,x,\dot{x})\partial_{x^i} + G^{[1]i}\partial_{\dot{x}^i} \tag{14}$$

and $G^{[1]i}$ are the coefficients of the first prolongation given by the formula:

$$G^{[1]i} = \frac{d}{dt}\eta^i - \dot{x}^i\frac{d}{dt}\xi = \eta^i_{,t} + \eta^i_{,j}\dot{x}^j - \xi_{,t}\dot{x}^i - \xi_{,j}\dot{x}^i\dot{x}^j\ .$$

It is a standard result [15] that (13) leads to the following three conditions:

$$-\mathbf{A}\xi = \lambda \tag{15}$$

$$G^{[1]i} = \mathbf{A}\eta^i - \dot{x}^i\mathbf{A}\xi \tag{16}$$

$$X^{[1]}(\omega^i) - \mathbf{A}(G^{[1]i}) = -\omega^i\mathbf{A}\xi. \tag{17}$$

For any function, $f(t,x^i), \mathbf{A}f = df/dt = f_{,t} + f_{,i}\dot{x}^i$ is the total derivative of f. Using this result we write the symmetry conditions as follows:

$$\lambda = -\frac{d\xi}{dt} \tag{18}$$

$$G^{[1]i} = \frac{d\eta^i}{dt} - \dot{x}^i\frac{d\xi}{dt} \tag{19}$$

$$X^{[1]}(\omega^i) - A(G^{[1]i}) = -\omega^i\frac{d\xi}{dt}. \tag{20}$$

We note that the second condition (19) defines the first prolongation $G^{[1]i}$. The first Eq. (18) gives the factor λ. Therefore the essential symmetry condition is Eq. (20).

To compute the symmetry condition we have to compute the quantities $X^{[1]}(\omega^i)$ and $A(G^{[1]i}) - \omega^i \frac{d\xi}{dt}$ taking into consideration (19) and (11). The result of this formal calculation is:

$$\begin{aligned}X^{[1]}(\omega^i) &= (\xi\partial_t + \eta^i\partial_{x^i} + G^{[1]i}\partial_{\dot{x}^i})(-\Gamma^i_{jk}(x)\dot{x}^j\dot{x}^k - \phi(x)\dot{x}^i)\\ &= -\eta^i_{,t}\phi + (-\xi\phi_{,t}\delta^i_j - \phi_{,k}\eta^k\delta^i_j - \eta^k_{,t}\Gamma^i_{kj} - \eta^k_{,t}\Gamma^i_{jk} - \phi\eta^i_{,j} + \phi\xi_{,t}\delta^i_j)\dot{x}^j\\ &\quad + (-\xi\Gamma^i_{(kj),t} - \eta^l\Gamma^i_{(kj),l} - \eta^l_{,k}\Gamma^i_{(lj)} - \eta^l_{,k}\Gamma^i_{(jl)} + \phi\xi_{,k}\delta^i_j + 2\xi_{,t}\Gamma^i_{(kj)})\dot{x}^j\dot{x}^k\\ &\quad + \xi_{,(k}\Gamma^i_{jl)}\dot{x}^j\dot{x}^k\dot{x}^l. \end{aligned} \tag{21}$$

$$\begin{aligned}A(G^{[1]i}) - \omega^i\frac{d\xi}{dt} &= \eta^i_{,tt} + (2\eta^i_{,tj} + \phi\eta^i_{,j} - 2\phi\xi_{,t}\delta^i_j - \xi_{,tt}\delta^i_j)\dot{x}^j\\ &\quad + (\eta^i_{,(jk)} - 2\xi_{,t(j}\delta^i_{k)} - \eta^i_{,l}\Gamma^l_{(jk)} + 2\xi_{,t}\Gamma^i_{(jk)} - 2\phi\xi_{(,j}\delta^i_{k)})\dot{x}^j\dot{x}^k\\ &\quad + (\xi_{,(j}\Gamma^i_{kl)} + \xi_{,m}\Gamma^m_{(kl}\delta^i_{j)} - \xi_{,(jk}\delta^i_{l)})\dot{x}^j\dot{x}^k\dot{x}^l.\end{aligned}$$

Substituting into the symmetry condition (20) and collecting terms of the same order in $\dot{x}^j$ we find the following equations: $(i = 1,\dots,n)$:
$(\dot{x})^0$ terms:

$$\eta^i_{,tt} + \eta^i_{,t}\phi = 0 \tag{22}$$

$(\dot{x})^1$ terms:

$$\xi_{,tt}\delta^i_j - \xi\phi_{,t}\delta^i_j - 2[\eta^i_{,tj} + \eta^k_{,t}\Gamma^i_{(kj)}] - [\phi\xi_{,t} + \phi_{,k}\eta^k]\delta^i_j = 0 \tag{23}$$

$(\dot{x})^2$ terms:

$$\begin{aligned}&(-\eta^i_{,(jk)} - \eta^l\Gamma^i_{(jk),l} - \eta^l_{,k}\Gamma^i_{lj} - \eta^l_{,k}\Gamma^i_{jl} + \eta^i_{,l}\Gamma^l_{jk}) + 2\xi_{,t(j}\delta^i_{k)} - 2\phi\xi_{(,j}\delta^i_{k)}\\ &\quad -\xi\Gamma^i_{(kj),t} = 0 \Rightarrow \mathscr{L}_\eta\Gamma^i_{(jk)} = -2\phi\xi_{(,j}\delta^i_{k)} + \xi\Gamma^i_{(kj),t} + 2\xi_{,t(j}\delta^i_{k)}\end{aligned} \tag{24}$$

$(\dot{x})^3$ terms:

$$(\xi_{,(jk} - \xi_{,|e|}\Gamma^e_{(jk})\delta^i_{l)} = 0\,. \tag{25}$$

Define the quantity:

$$\Phi = \xi_{,t} - \phi\xi. \tag{26}$$

Then condition (24) is written (note that $\phi_{,i} = 0$):

$$\mathscr{L}_\eta\Gamma^i_{(jk)} = 2\Phi_{(,j}\delta^i_{k)} - \xi\Gamma^i_{(kj),t}. \tag{27}$$

If we consider the vector $\xi = \xi\partial_t$ (which does not have components along ∂_i), we find that:

$$\mathscr{L}_\xi\Gamma^i_{(jk)} = \xi\Gamma^i_{(kj),t}.$$

Hence (27) is written as:

$$\mathscr{L}_{\mathbf{X}}\Gamma^i_{(jk)} = 2\Phi_{(,j}\delta^i_{k)}, \tag{28}$$

where $\mathbf{X}=\xi\partial_t+\eta^i(t,x)\partial_{x^i}$. We note that this condition is precisely condition (7) for a projective collineation of the connection $\Gamma^i_{(jk)}$ along the symmetry vector $\mathbf{X}$ with projective function Φ. Concerning the other conditions we note that (23) can be written in covariant form (relevant to the indices $a=1,2,\dots,n$) as follows:

$$\Phi_{,t}\delta^i_j-2\eta^i_{,t|j}=0 \tag{29}$$

where $\eta^i_{,t|j}=\eta^i_{,tj}+\eta^k_{,t}\Gamma^i_{(kj)}$ is the covariant derivative of the vector $\eta^i_{,t}$ with respect to $\Gamma^i_{(kj)}$. Similarly condition (25) can be written as

$$\xi_{|(jk}\delta^i_{l)}=0.$$

Contracting on the indices i and j we find the final form:

$$\xi_{|(jk)}=0. \tag{30}$$

This implies that $\xi_{,i}$ is a gradient KV of the metric of the space $\{x^i\}$.

Condition (22) is obviously in covariant form.

The Lie symmetries of the autoparallel equations (11) (not necessarily affinely parameterized) for a general connection defined on a C^∞ manifold are given by the following covariant[2] equations:

$$\eta^i_{,tt}+\eta^i_{,t}\phi=0 \tag{31}$$

$$\xi_{|(jk)}=0 \tag{32}$$

$$\Phi_{,t}\delta^i_j-2\eta^i_{,t|j}=0 \tag{33}$$

$$\mathscr{L}_{\mathbf{X}}\Gamma^i_{(jk)}=2\Phi_{(,j}\delta^i_{k)}, \tag{34}$$

where the Lie symmetry vector $\mathbf{X}=\xi(t,x)\partial_t+\eta^i(t,x)\partial_{x^i}$.

In the following we restrict our considerations to the case of Riemannian connections, that is, the Γ^i_{jk} are symmetric and the covariant derivative of the metric vanishes.

4 Calculation of the Lie symmetry vectors for a Riemannian connection

We compute the Lie symmetry vectors for the case of affine parametrization ($\phi=0$) and the assumption $\Gamma^i_{jk,t}=0$ i.e. the Γ^i_{jk} are independent of the parameter t. The latter is a logical assumption because the Γ^i_{jk} are computed in terms of the metric which does not depend on the affine parameter t. Under these assumptions the symmetry conditions (31)–(34) read:

$$\eta^i_{,tt}=0 \tag{35}$$

$$\xi_{|(jk)}=0 \tag{36}$$

$$\xi_{,tt}\delta^i_j-2\eta^i_{,t|j}=0 \tag{37}$$

$$\mathscr{L}_\eta\Gamma^i_{jk}=2\xi_{,t(j}\delta^i_{k)}. \tag{38}$$

[2] These are covariant equations because, if we consider the connection in the augmented $n+1$ space $\{x^i,t\}$, all components of Γ which contain an index along the direction of t vanish. Therefore the partial derivatives wrt t can be replaced with a covariant derivative wrt t.

The solution of this system of equations is given in the following Theorem. The actual computations are given in the Appendix.

Theorem 1 *The Lie symmetry vector* $\mathbf{X} = \xi(t,x)\partial_t + \eta^i(t,x)\partial_{x^i}$ *of the equations of geodesics* (2) *in a Riemannian space involves KVs, HKVs, ACs and special PCs as follows:*

A. *The metric admits gradient KVs. Then*
  a. *The function*

$$\xi(t,x) = \frac{1}{2}\left(G_J S^J + M\right)t^2 + \left[E_J S^J + K\right]t + F_J S^J + L, \tag{39}$$

  where G_J, M, b, K, F_J *and* L *are constants and the index* J *runs along the number of gradient KVs*
  b. *The vector*

$$\eta^i(t,x) = A^i(x)t + B^i(x) + D^i(x) \tag{40}$$

  where the vector $A^i(x)$ *is a gradient HKV with conformal factor* $\psi = \frac{1}{2}(G_J S^J + M)$ *(if it exists),* $D^i(x)$ *is a non-gradient KV of the metric and* $B^i(x)$ *is either a special projective collineation with projective function* $E_J S^J(x)$ *or an AC and* $E_J = 0$ *in* (39).

B. *The metric does not admit gradient KVs. Then*
  a. *The function*

$$\xi(t,x) = \frac{1}{2}Mt^2 + Kt + L \tag{41}$$

  b. *The vector*

$$\eta^i(t,x) = A^i(x)t + B^i(x) + D^i(x), \tag{42}$$

  where $A^i(x)$ *is a gradient HKV with conformal factor* $\psi = \frac{1}{2}M, D^i(x)$ *is a non-gradient KV of the metric and* $B^i(x)$ *is an AC. If in addition the metric does not admit gradient HKV, then*

$$\xi(t) = Kt + L \tag{43}$$

$$\eta^i(x) = B^i(x) + D^i(x). \tag{44}$$

5 The Noether symmetries of the geodesic Lagrangian

In a Riemannian space the equations of geodesics (4) are produced from the geodesic Lagrangian:

$$L = \frac{1}{2}g_{ij}\dot{x}^i\dot{x}^j. \tag{45}$$

A vector field $\mathbf{X} = \xi(t,x^k)\partial_t + \eta^i(t,x^k)\partial_{x^i}$ is a Noether symmetry of this Lagrangian if there exists a smooth function $f(t,x^i)$ such that [15]

$$X^{[1]}L + \frac{d\xi}{dt}L = \frac{df}{dt}, \tag{46}$$

where $X^{[1]} = \xi(t,x^k)\partial_t + \eta^i(t,x^k)\partial_{x^i} + (\frac{d\eta^i}{dt} - \dot{x}^i\frac{d\xi}{dt})\partial_{\dot{x}^i}$ is the first prolongation of **X**. We compute:

$$X^{[1]}L = \frac{1}{2}\left(\eta^k g_{ij,k}\dot{x}^i\dot{x}^j + 2\frac{d\eta^k}{dt}g_{ik}\dot{x}^i - 2\dot{x}^i\dot{x}^j\frac{d\xi}{dt}g_{ij}\right).$$

Replacing the total derivatives in the rhs

$$\begin{aligned}\frac{d\xi}{dt} &= \xi_{,t} + \dot{x}^k\xi_{,k}\\ \frac{d\eta^i}{dt} &= \eta^i_{,t} + \dot{x}^k\eta^i_{,k}\end{aligned}$$

we find that

$$\begin{aligned}X^{[1]}L = \frac{1}{2}&\left(\eta^k g_{ij,k}\dot{x}^i\dot{x}^j + 2\eta^i_{,t}g_{ij}\dot{x}^j + \eta^i_{,r}g_{ik}\dot{x}^k\dot{x}^r + \eta^i_{,r}g_{kj}\dot{x}^k\dot{x}^r - 2\xi_{,t}g_{ij}\dot{x}^i\dot{x}^j\right.\\ &\left.-2\xi_{,k}g_{ij}\dot{x}^i\dot{x}^j\dot{x}^k\right).\end{aligned}$$

The term

$$\frac{d\xi}{dt}L = \frac{1}{2}\left(\xi_{,t} + \dot{x}^k\xi_{,k}\right)g_{ij}\dot{x}^i\dot{x}^j.$$

Finally the Noether symmetry condition (46) is

$$\begin{aligned}&-2f_{,t} + \left[2\eta^i_{,t}g_{ij} - 2f_{,i}\right]\dot{x}^j - \xi_{,k}g_{ij}\dot{x}^i\dot{x}^j\dot{x}^k\\ &+\left[\eta^k g_{ij,k} + \eta^k_{,i}g_{ik} + \eta^k_{,i}g_{kj} - g_{ij}\xi_{,t}\right]\dot{x}^i\dot{x}^j = 0.\end{aligned}$$

This relation is an identity hence the coefficient of each power of $\dot{x}^j$ must vanish. This results in the equations:

$$\dot{x}^i\dot{x}^j\dot{x}^k : \xi_{,k} = 0 \tag{47}$$

$$\dot{x}^i\dot{x}^j : L_\eta g_{ij} = 2\left(\frac{1}{2}\xi_{,t}\right)g_{ij} \tag{48}$$

$$\dot{x}^i : \eta^{;i}_{,t}g_{ij} = f_{,i} \tag{49}$$

$$\left(\dot{x}^i\right)^0 : f_{,t} = 0 \tag{50}$$

Condition (47) gives $\xi_{,k} = 0 \Rightarrow \xi = \xi(t)$.

Condition (50) implies $f(x^k)$ and then condition (47) gives that η^i is of the form:

$$\eta_i = f_{,i}t + K_i(x^j). \tag{51}$$

Then from (48) follows that $\xi_{,t}$ must be at most linear in t. Hence $\xi(t)$ must be at most a function of t^2. Furthermore from (48) follows that η^i is at most a CKV with conformal factor $\psi_H = \frac{1}{2}(At + B)$, where A, B are constants. We consider various cases.

Case 1: Suppose $\xi = \text{constant} = C_1$. Then η^i is a KV of the metric which is independent of t. This implies that either $f_{,i} = 0$ and $f = \text{constant} = A = 0$ or that $f_{,i}$ is a gradient KV. In this case the Noether symmetry vector is:

$$X^i = C_1\partial_t + g^{ij}\left(f_{,j}t + K_j(x^r)\right),$$

where K^i is a non-gradient KV of g_{ij}.

Case 2: Suppose $\xi = 2t$. Then η^i is a HKV of the metric g_{ij} with homothetic factor 1. Then $\eta_i = H_i(x^j), f_{,i} = 0 \Rightarrow f = \text{constant} = 0$ where H^i is a HKV of g_{ij} with homothetic factor ψ, not necessarily a gradient HKV. In this case the Noether symmetry vector is:

$$X^i = 2\psi t\partial_t + H^i(x^r).$$

Case 3: $\xi(t) = t^2$. Then η^i is a HKV of the metric g_{ij} (the variable t cancels) with homothetic factor 1. Again $f_{,i}$ is a gradient HKV with homothetic factor ψ and the Noether symmetry vector is

$$X^i = \psi t^2\partial_t + g^{ij}f_{,j}t.$$

Therefore we have the result.

Theorem 2 *The Noether Symmetries of the geodesic Lagrangian follow from the KVs and the HKV of the metric g_{ij} as follows:*

$$\xi(t) = C_3\psi t^2 + 2C_2\psi t + C_1 \tag{52}$$

$$\eta^i = C_J S^{J,i} + C_I KV^{Ii} + C_{IJ}tS^{J,i} + C_2H^i(x^r) + C_3t(GHV)^i \tag{53}$$

$$f(x^i) = C_1 + C_2 + C_I + C_J + \left[C_{IJ}S^J\right] + C_3\left[GHV\right], \tag{54}$$

where $S^{J,i}$ are the C_J gradient KVs, KV^{Ii} are the C_I non-gradient KVs, H^i is a HKV not necessarily gradient and $(GHV)^i$ is the gradient HKV (if it exists) of the metric g_{ij}.

The importance of Theorems 1 and 2 is that one is able to compute the Lie symmetries and the Noether symmetries of the geodesic equations in a Riemannian space by computing the corresponding collineation vectors avoiding the cumbersome formulation of the Lie symmetry method. It is also possible to use the inverse approach and prove that a space does not admit KVs, HKVs, ACs and special PCs by using the calculational approach of the Lie symmetry method (assisted with algebraic manipulation programmes) and avoid the hard approach of Differential Geometric methods. In Sect. 8 we demonstrate the use of the above results.

Table 1 Collineations and corresponding First Integrals

Symmetry	Condition	First Integral
KV	$\xi_{(i;j)} = 0$	$\xi_i \dot{x}^i$
Gradient KV	$\xi_{;ij} = 0$	$\xi_{;i} \dot{x}^i$
HKV	$\xi_{i;j} = \psi g_{ij}$, $\psi_{,i} = 0$	$\xi_{i;j} \dot{x}^i \dot{x}^j$
Gradient HKV	$\xi_{;ij} = \psi g_{ij}, \psi_{,i} = 0$	$\xi_{;ij} \dot{x}^i \dot{x}^j$
AC	$L_\xi \Gamma^i_{jk} = 0, \xi_{(i;jk)} = 0$	$\xi_{i;j} \dot{x}^i \dot{x}^j$
PC	$L_\xi \Gamma^i_{jk} = 2\phi_{,(j}\delta^i_{k)}$	$(\xi_{i;j} - 4\phi g_{ij})\dot{x}^i \dot{x}^j$
Special PC	$L_\xi \Gamma^i_{jk} = 2\phi_{,(j}\delta^i_{k)}$, $\phi_{;ij} = 0$	$(\xi_{i;j} - 4\phi g_{ij})\dot{x}^i \dot{x}^j$

6 First Integrals of the geodesic equations and collineations

Consider a Riemannian space with metric g_{ij}. As we have shown in Theorem 1, the Lie symmetries of the geodesic equations of the metric coincide with the KVs, the HKV, the ACs and special PCs of the metric g_{ij} (if they are admitted). For each Noether symmetry one has the first integral

$$\mathbf{X}^{[1]} L + \frac{d\xi}{dt} L = \frac{df}{dt}, \tag{55}$$

where, $X^{[1]}$ is the first prolongation of $\mathbf{X}$. In this section we study the relationship between first integrals and corresponding conserved quantities of Noether symmetries.

We recall first some well-known definitions and results [1].

Consider the geodesic with tangent vector $\lambda^k = \frac{dx^i}{ds}$ where s is an affine parameter along the geodesic. An mth order First Integral of the geodesic is a tensor quantity $A_{r_1 \dots r_m}$ such that:

$$A_{r_1 \dots r_m} \lambda^{r_1} \dots \lambda^{r_m} = \text{constant} \tag{56}$$

or equivalently (because $\lambda_{i;j} = 0$):

$$P\left\{A_{r_1 \dots r_m;k}\right\} \lambda^k = 0, \tag{57}$$

where $P\{\}$ indicates cyclic sum over the indices enclosed. Without restriction of generality we may consider $A_{r_1 \dots r_m}$ to be totally symmetric; for example for two and three indices we have:

$$P\{A_{i;k}\} = (A_{i;k} + A_{k;i})$$
$$P\{A_{ij;k}\} = \left(A_{ij;k} + A_{kj;i} + A_{ik;j}\right).$$

Katzin and Levine [4] have proved the following results concerning First Integrals: Table 1

7 First Integrals of Noether symmetry vectors of geodesic equations

We know that, if $X=\xi(x^j,t)\partial_t+\eta^i(x^j,t)\partial_{x^i}$ is the generator of a Noether symmetry with Noether function f, then the quantity:

$$\phi=\xi\left(\dot{x}^i\frac{\partial L}{\partial\dot{x}^i}-L\right)-\eta^i\frac{\partial L}{\partial\dot{x}^i}+f \tag{58}$$

is a First Integral of L which satisfies $X\phi=0$. For the Lagrangian defined by the metric g_{ij}, i.e. $L=\frac{1}{2}g_{ij}\dot{x}^i\dot{x}^j$, we compute:

$$\phi=\frac{1}{2}\xi g_{ij}\dot{x}^i\dot{x}^j-g_{ij}\eta^i\dot{x}^j+f. \tag{59}$$

In (52), (53) and (54) we have computed the generic form of the Noether symmetry and the associated Noether function for this Lagrangian. Substituting into (59) we find the following expression for the generic First Integral:

$$\begin{aligned}\phi=&\frac{1}{2}\left[C_3\psi t^2+2C_2\psi t+C_1\right]g_{ij}\dot{x}^i\dot{x}^j\\&-\left[C_JS^{J,i}+C_IKV^{Ii}+C_{IJ}tS^{J,i}+C_2H^i(x^r)+C_3t(GHV)^{,i}\right]g_{ij}\dot{x}^j\\&+C_1+C_2+C_I+C_J+\left[C_{IJ}S^J\right]+C_3\left[GHV\right].\end{aligned} \tag{60}$$

From the generic expression we obtain the following First Integrals[3]:

$C_1\neq0$.

$$\phi_{C_1}=\frac{1}{2}g_{ij}\dot{x}^i\dot{x}^j \tag{61}$$

$C_2\neq0$.

$$\phi_{C_2}=t\psi g_{ij}\dot{x}^i\dot{x}^j-g_{ij}H^i\dot{x}^j+C_2 \tag{62}$$

$C_3\neq0$.

$$\phi_{C_3}=\frac{1}{2}t^2\psi g_{ij}\dot{x}^i\dot{x}^j-t(GHV)_{,i}\dot{x}^i+[GHV] \tag{63}$$

$C_I\neq0$.

$$\phi_{C_I}=KV_i^I\dot{x}^i-C_I \tag{64}$$

$C_J\neq0$.

$$\phi_{C_J}=g_{ij}S^{J,i}\dot{x}^j-C_J \tag{65}$$

$C_{IJ}\neq0$.

$$\phi_{IJ}=tg_{ij}S^{J,i}\dot{x}^j-S^J. \tag{66}$$

We conclude that the First Integrals of the Noether symmetry vectors of the geodesic equations are:

a. Linear, the ϕ_I,ϕ_J,ϕ_{IJ}
b. Quadratic, the $\phi_{c1},\phi_{c2},\phi_3$.

These results are compatible with the corresponding results of Katzin and Levine [4].

[3] GHV stands for gradient HKV.

8 Applications

8.1 The Lie symmetries of geodesic equations in an Einstein space

Suppose X^a is a projective collineation with projection function $\phi(x^a)$, such that $\mathscr{L}_X \Gamma^a_{bc} = \phi_{,b}\delta^a_c + \phi_{,c}\delta^a_b$. For a proper Einstein space $(R \neq 0)$ we have $R_{ab} = \frac{R}{n} g_{ab}$ from which follows:

$$\mathscr{L}_X g_{ab} = \frac{n(1-n)}{R}\phi_{;ab} - \mathscr{L}_X(\ln R) g_{ab}. \tag{67}$$

Using the contracted Bianchi identity $[R^{ij} - \frac{1}{2}Rg^{ij}]_{;j} = 0$ it follows that in an Einstein space of dimension $n > 2$ the curvature scalar $R =$ constant and (67) reduces to:

$$\mathscr{L}_X g_{ab} = \frac{n(1-n)}{R}\phi_{;ab}.$$

It follows that if X^a generates either an affine or a special projective collineation, then $\phi_{;ab} = 0$. Hence X^a reduces to a KV. This means that proper Einstein spaces do not admit HKV, ACs, special PCs and gradient KVs [17; 18].

The above results and Theorem 1 lead to the following conclusion:

Theorem 3 *The Lie symmetries of the geodesic equations in a proper Einstein space of curvature scalar $R \neq 0$ are given by the vectors*

$$X = (Kt + L)\,\partial_t + D^i(x)\,\partial_i$$

where $D^i(x)$ is a nongradient KV and K, L are constants

Theorem 4 *The Noether symmetries of the geodesic equations in a proper Einstein space of curvature scalar $R \neq 0$ are given by the vectors:*

$$X = L\partial_t + D^i(x)\,\partial_i, f = constant$$

Theorem 3 extends and amends the conjecture of [10] to the more general case of Einstein spaces.

We apply the result to the case of the Euclidean 2 dimensional space. The metric g_{ij} of this space admits:

a. Two gradient KVs, the Y^1 and Y^2 generated by the functions $\phi_7, \phi_8,$
b. One nongradient KV, the Y^3,
c. One gradient HKV, the Y^4 (with homothetic factor 1).

Therefore the generic Noether symmetry vector is:

$$X = C_1\partial_t + C_2\left(2t\partial_t + Y^4\right) + C_3(t^2\partial_t + Y^4 t) + C_4 tY^1 + C_5 tY^2$$
$$+ C_6Y^1 + C_7Y^2 + C_8Y^3.$$

where (see Table 2)

$$Y^1 = \partial_x, \quad Y^2 = \partial_y, \quad Y^3 = y\partial_x - x\partial_y, \quad Y^4 = x\partial_x + y\partial_y.$$

Table 2 The conformal algebra of a flat n-dimensional metric

CKV	Components	#	$\psi(\xi)$	$F_{ab}(\xi)$	Comment
P_I	∂_I	n	0	0	Gradient KV
r_{AB}	$2\delta^d_{[A}x_{B]}\partial_d$	$\frac{n(n-1)}{2}$	0	η_{ABab}	Nongradient KV
H	$x^a\partial_a$	1	$(b=1)1$	0	Gradient HKV
K_I	$[2x_Ix^d - \delta^d_I(x_ax^a)]\partial_d$	n	$(b^i=\delta^i_I)2x_I$	$-4\eta_{I[a}x_{b]}$	Nongradient SCKV

8.2 The Noether symmetries of Schwarzschild metric

We consider the Schwarzschild metric

$$ds^2 = \left(1-\frac{2m}{r}\right)dt^2 - \frac{1}{\left(1-\frac{2m}{r}\right)}dr^2 - r^2\left(d\theta^2 + \sin^2\theta d\phi^2\right).$$

The geodesic Lagrangian is:

$$L = \left(1-\frac{2m}{r}\right)\dot{t}^2 - \frac{1}{\left(1-\frac{2m}{r}\right)}\dot{r}^2 - r^2\dot{\theta}^2 - r^2\sin^2\theta\dot{\phi}^2$$

The Noether symmetries have been computed in [11] as follows:

$$\begin{aligned} X_1 &= \partial_s, \quad X_2 = \partial_t \\ X_3 &= \cos\phi\,\partial_\theta - \cot\theta\sin\phi\,\partial_\phi \\ X_4 &= \sin\phi\,\partial_\theta + \cot\theta\cos\phi\,\partial_\phi \\ X_5 &= \partial_\theta. \end{aligned}$$

It easy to check that X_2, X_3, X_4 and X_5 are KVs of the Schwarzschild metric and, since this metric does not admit any gradient KVs or a HKV, these are the only Noether symmetries, a result compatible with Theorem 2 above.

The first integrals of the geodesic equations of the Schwarzschild metric are:

a. The metric integral:

$$\phi_s = g_{ij}\dot{x}^i\dot{x}^j = \left(1-\frac{2m}{r}\right)\dot{t}^2 - \frac{1}{\left(1-\frac{2m}{r}\right)}\dot{r}^2 - r^2\dot{\theta}^2 - r^2\sin^2\theta\dot{\phi}^2$$

b. The linear integrals defined by the KVs:

$$\begin{aligned} \phi_2 &= \dot{t} \\ \phi_3 &= (\cos\phi)\,\dot{\theta} - (\cot\theta\sin\phi)\,\dot{\phi} \\ \phi_4 &= (\sin\phi)\,\dot{\theta} + (\cot\theta\cos\phi)\,\dot{\phi} \\ \phi_5 &= \dot{\theta}. \end{aligned}$$

The extra Lie symmetry is $s\partial_s$.

Table 3 The projective algebra of the 2-d Euclidean sphere

Type	ϕ	Vector
K.V	0	$X^1 = \partial_x$
K.V	0	$X^2 = \cot y \cos x \partial_x + \sin x \partial_y$
K.V	0	$X^3 = -\cot y \sin x \partial_x + \cos x \partial_y$
Pr.Col	$-\sin y \sin y \cos x$	$X^4 = \sin y \sin x \cos y \partial_x - (2\cos^2 y - 1)\cos x \partial_y$
Pr.Col	$-\sin y \sin y \sin x$	$X^5 = -\sin y \cos x \cos y \partial_x - (2\cos^2 y - 1)\sin x \partial_y$
Pr.Col	$-\sin^2 y \cos^2 x$	$X^6 = 2\sin^2 y \cos x \sin x \partial_x - 2\cos^2 x \sin y \cos y \partial_y$
Pr.Col	$-\cos x \sin x \sin^2 y$	$X^7 = (-2\cos^2 x + 1)\sin^2 y \partial_x - 2\sin y \cos y \sin x \cos x \partial_y$
Pr.Col	$+\cos^2 x \sin^2 y$	$X^8 = -2\sin^2 y \cos x \sin x \partial_x - 2\sin^2 x \sin y \cos y \partial_y.$

8.3 The First Integrals of the Euclidean sphere of dimension 2

The metric of the Euclidean sphere of dimension 2 is:

$$ds_{SF}^2 = \sin^2 y dx^2 + dy^2 \tag{68}$$

The collineations admitted by the 2-d Euclidian sphere (space of constant curvature) are three KVs and five proper PCs as shown in Table 3 [16]:

Hence there exist three Line First Integrals due to the KVs (dot over a symbol indicates $\frac{d}{ds}$, s being an affine parameter)[4]:

[4] For example:

$$I_1 = g_{ij} X^i \dot{x}^j = g_{11} X^1 \dot{x}^1 + g_{22} X^2 \dot{x}^2 = \sin^2 y \cdot 1 \cdot \dot{x} + 1 \cdot 0 \cdot \dot{y} = \sin^2 y \cdot \dot{x}.$$

Table 4 The conformal algebra of the 1 + 3 metric (72) for $K = \pm 1$

K	CKVs of ds_3^2	ψ_3	#CKVs of ds_{1+3}^2	ψ_{1+3}
1	$H = x^a \partial_a$	$\psi_+(H) = U(1 - \frac{1}{4} x^\alpha x_\alpha)$	1 $H_1^+ = -\psi_+(H) \cos\tau \partial_\tau + H \sin\tau$	$\psi_+(H) \sin\tau$
1	$H = x^a \partial_a$	$\psi_+(H) = U(1 - \frac{1}{4} x^\alpha x_\alpha)$	1 $H_2^+ = \psi_+(H) \sin\tau \partial_\tau + H \cos\tau$	$\psi_+(H) \cos\tau$
1	$C_\mu = (\delta^a_\mu - \frac{1}{2} U x_\mu x^a) \partial_\alpha$	$\psi_+(C_\mu) = -U x^\mu$	3 $Q_\mu^+ = -\psi_+(C_\mu) \cos\tau \partial_\tau + C_\mu \sin\tau$	$\psi_+(C_\mu) \sin\tau$
1	$C_\mu = (\delta^\alpha_\mu - \frac{1}{2} U x_\mu x^a) \partial_\alpha$	$\psi_+(C_\mu) = -U x^\mu$	3 $Q_{\mu+3}^+ = \psi_+(C_\mu) \sin\tau \partial_\tau + C_\mu \cos\tau$	$\psi_+(C_\mu) \cos\tau$
-1	$H = x^\alpha \partial_\alpha$	$\psi_-(H) = U(1 + \frac{1}{4} x^\alpha x_\alpha)$	1 $H_1^- = \psi_-(H) \cosh\tau \partial_\tau + H \sinh\tau$	$\psi_-(H) \sinh\tau$
-1	$H = x^\alpha \partial_\alpha$	$\psi_-(H) = U(1 + \frac{1}{4} x^\alpha x_\alpha)$	1 $H_2^- = \psi_-(H) \sinh\tau \partial_\tau + H \cosh\tau$	$\psi_-(H) \cosh\tau$
-1	$C_\mu = (\delta^\alpha_\mu + \frac{1}{2} U x_\mu x^a) \partial_\alpha$	$\psi_-(C_\mu) = U x^\mu$	3 $Q_\mu^- = \psi_-(C_\mu) \cosh\tau \partial_\tau + C_\mu \sinh\tau$	$\psi_-(C_\mu) \sinh\tau$
-1	$C_\mu = (\delta^\alpha_\mu + \frac{1}{2} U x_\mu x^a) \partial_\alpha$	$\psi_-(C_\mu) = U x^\mu$	3 $Q_{\mu+3}^- = \psi_-(C_\mu) \sinh\tau \partial_\tau + C_\mu \cosh\tau$	$\psi_-(C_\mu) \cosh\tau$

$$I_1 = \dot{x} \sin^2 y \tag{69}$$

$$I_2 = \dot{x} \sin y \cos y \cos x + \dot{y} \sin x \tag{70}$$

$$I_3 = -\dot{x} \sin y \cos y \sin x + \dot{y} \cos x \tag{71}$$

and five Quadratic First Integrals corresponding to the special PCs:

$$Q_1 = 4\dot{y} \sin x \sin^2 y \,(\dot{y} \cot x \cot y - \dot{x})$$

$$Q_2 = 4\dot{y} \cos x \sin^2 y \,(\dot{x} + \dot{y} \tan x \tan y)$$

$$Q_3 = 4\,(\dot{x} \sin y \cos x + \dot{y} \cos y \sin x)^2 - 4\dot{x}^2 \sin^2 y - 4\dot{y}^2 \cos^2 y$$

$$Q_4 = 4 \sin x \cos x \left(\dot{x}^2 \sin^2 y - \dot{y}^2 \cos^2 y\right) + 4\dot{x}\dot{y} \sin y \cos y \left(1 - 2\cos^2 x\right)$$

$$Q_5 = -4\,(\dot{x} \cos x \sin y + \dot{y} \sin x \cos y)^2$$

8.4 The 1 + 3 decomposable metric

We consider next the metric which is a 1 + 3 decomposable metric:

$$ds_4 = -d\tau^2 + U^2 \delta_{\alpha\beta} dx^\alpha dx^\beta \tag{72}$$

where Greek indices take the values 1, 2, 3. It is well known [16] that this metric admits 15 CKVs (it is conformally flat). Seven of these vectors are KVs (the six nongradient KVs of the 3-metric $\mathbf{r}_{\mu\nu}, \mathbf{I}_\mu$ plus the gradient KV ∂_τ) and nine proper CKVs. The vectors of this conformal algebra are shown in Table 4

According to Theorem 2 this metric admits the following Noether symmetries (see also [4]):

$$\partial_s \;, \mathbf{r}_{\mu\nu}, \mathbf{I}_\mu, \partial_\tau : f = \text{constant}$$

$$s\partial_\tau : f = \tau$$

with Noether conserved quantities:

$$\phi_s = \frac{1}{2} g_{ij} \dot{x}^i \dot{x}^j \tag{73}$$

$$\phi_\tau = \dot{\tau} \tag{74}$$

$$\phi_{\tau+1} = s\dot{\tau} - \tau \tag{75}$$

$$\phi_I = \mathbf{I}^I_i \dot{x}^i \tag{76}$$

$$\phi_r = \mathbf{r}_{(AB)_i} \dot{x}^j . \tag{77}$$

8.5 The Noether symmetries of the FRW metrics

In a recent paper Bokhari and Kara [12] studied the Lie symmetries of the conformally flat Friedman Robertson Walker (FRW) metric with the view to understand how Noether symmetries compare with conformal Killing vectors. More specifically they considered the conformally flat FRW metric[5]:

$$ds^2 = -dt^2 + t^{\frac{4}{3}}\left(dx^2 + dy^2 + dz^2\right)$$

and found that the Noether symmetries are the eight vectors:

$$\partial_s, S^J, r_{AB}, \mathbf{H}$$

where S_J are the gradient KVs $\partial_x \partial_y, \partial_z$ and r_{AB} are the three nongradient KVs (generating SO(3)) whereas the vector ∂_s counts for the gauge freedom in the affine parametrization of the geodesics. The vector

$$\mathbf{H} = 6s\partial_s + 3t\partial_t + x\partial_x + y\partial_y + z\partial_z$$

where $3t\partial_t + x\partial_x + y\partial_y + z\partial_z$ is a non-gradient HV. Therefore they confirm our Theorem 2 that the Noether vectors coincide with the KVs and the HKV of the metric. Furthermore their claim that '…the conformally transformed Friedman model admits additional conservation laws not given by the Killing or conformal Killing vectors' is not correct.

In the following we compute all the Noether symmetries of the FRW spacetimes. To do that we have to have the homothetic algebra of these models [19]. There are two cases to consider, the conformally flat models $(K = 0)$ and the non-conformally flat models $(K \neq 0)$.

We need the conformal algebra of the flat metric, which in Cartesian coordinates is given in Table 2.
Case A: $K \neq 0$
The metric is:

$$ds = R^2(\tau)\left[-d\tau^2 + \frac{1}{\left(1+\frac{1}{4}Kx^i x_i\right)^2}\left(dx^2 + dy^2 + dz^2\right)\right]. \quad (78)$$

For a general $R(\tau)$ this metric admits the nongradient KVs $\mathbf{P}_I, \mathbf{r}_{\mu\nu}$ (see Table 2) and does not admit an HKV. Therefore the Noether symmetries of the geodesic Lagrangian

$$L = -\frac{1}{2}R^2(\tau)\dot{\tau}^2 + \frac{1}{2}\frac{R^2(\tau)}{\left(1+\frac{1}{4}Kx^i x_i\right)^2}\left(\dot{x}^2 + \dot{y}^2 + \dot{z}^2\right)$$

of the FRW metric (78) are:

$$\partial_s, \mathbf{P}_I, \mathbf{r}_{\mu\nu}$$

[5] The second metric $ds^2 = -t^{-\frac{4}{3}}dt^2 + dx^2 + dy^2 + dz^2$ they consider is the Minkowski metric whose Lie and Noether symmetries are well known.

Table 5 The special forms of the scale factor for $K = \pm 1$

K	Proper CKV	#	Conformal Factor	$R(\tau)$ for KVs	$R(\tau)$ for HKV
± 1	$\mathbf{P}_\tau = \partial_\tau$	1	$(\ln R(\tau))_{,\tau}$	c	$\exp(\tau)$
1	$\mathbf{H}_1^+$	1	$-\frac{\psi_+(\mathbf{H})}{R(\tau)}(R(\tau)\cos\tau)_{,\tau}$	$\frac{c}{\cos\tau}$	$\nexists$
1	$\mathbf{H}_2^+$	1	$\frac{\psi_+(\mathbf{H})}{R(\tau)}(R(\tau)\sin\tau)_{,\tau}$	$\frac{c}{\sin\tau}$	$\nexists$
1	$\mathbf{Q}_\mu^+$	3	$-\frac{\psi_+(\mathbf{C}_\mu)}{R(\tau)}(R(\tau)\cos\tau)_{,\tau}$	$\frac{c}{\cos\tau}$	$\nexists$
1	$\mathbf{Q}_{\mu+3}^+$	3	$\frac{\psi_+(\mathbf{C}_\mu)}{R(\tau)}(R(\tau)\sin\tau)_{,\tau}$	$\frac{c}{\sin\tau}$	$\nexists$
-1	$\mathbf{H}_1^-$	1	$\frac{\psi_-(\mathbf{H})}{R(\tau)}(R(\tau)\cosh\tau)_{,\tau}$	$\frac{c}{\cosh\tau}$	$\nexists$
-1	$\mathbf{H}_2^-$	1	$\frac{\psi_-(\mathbf{H})}{R(\tau)}(R(\tau)\sinh\tau)_{,\tau}$	$\frac{c}{\sinh\tau}$	$\nexists$
-1	$\mathbf{Q}_\mu^-$	3	$\frac{\psi_-(\mathbf{C}_\mu)}{R(\tau)}(R(\tau)\cosh\tau)_{,\tau}$	$\frac{c}{\cosh\tau}$	$\nexists$
-1	$\mathbf{Q}_{\mu+3}^-$	3	$\frac{\psi_-(\mathbf{C}_\mu)}{R(\tau)}(R(\tau)\sinh\tau)_{,\tau}$	$\frac{c}{\sinh\tau}$	$\nexists$

with Noether integrals

$$\phi_s = \frac{1}{2} g_{ij}\dot{x}^i\dot{x}^j, \quad \phi_I = \mathbf{P}_i^I \dot{x}^i, \quad \phi_r = \mathbf{r}_{(AB)_i}\dot{x}^j. \tag{79}$$

Concerning the Lie symmetries we note that the FRW spacetimes do not admit ACs [20] and furthermore do not admit gradient KVs. Therefore they do not admit special PCs. The Lie symmetries of these spacetimes are:

$$\partial_s, s\partial_s, \mathbf{P}_I, \mathbf{r}_{\mu\nu}.$$

For special functions $R(\tau)$ it is possible to have more KVs and HKV. In Table 5 we give the special forms of the scale factor $R(t)$ and the corresponding extra KVs and HKV for $K = \pm 1$.

From Table 5 we infer the following additional Noether symmetries of the FRW-like Lagrangian for special forms of the scale factor:

Case A(1): $R(t) = c$ =constant, the space is the 1+3 decomposable.

Case A(2) $K = 1, R(t) = \exp(\tau)$. In this case we have the additional gradient HKV $\mathbf{P}_\tau$ generated by the function $\frac{1}{2}\exp 2\tau$. Therefore for this scale factor the Noether symmetry vectors, the Noether function and the conserved Noether quantities are:

$$\partial_s, \mathbf{P}_I, \mathbf{r}_{\mu\nu}, 2s\partial_s + \mathbf{P}_\tau : f = \text{constant}$$
$$s^2\partial_s + s\mathbf{P}_\tau : f = \frac{1}{2}\exp(2\tau)$$

with Noether Integrals

$$\phi_s, \phi_I, \phi_r, \phi_\tau = s g_{ij}\dot{x}^i\dot{x}^j - g_{ij}^i \mathbf{P}_\tau \dot{x}^j \quad \text{and} \quad \phi_{\tau+\mathbf{1}} = \frac{1}{2}s^2 g_{ij}\dot{x}^i\dot{x}^j$$
$$-s\mathbf{P}_\tau^i \dot{x}_i + \frac{1}{2}\exp 2\tau.$$

The Lie symmetries are:

$$\partial_s, s\partial_s, \mathbf{P}_I, \mathbf{r}_{\mu\nu}, \mathbf{P}_\tau, s^2\partial_s + s\mathbf{P}_\tau.$$

Case A(3a) $K=1, R(\tau)=\frac{c}{\cos\tau}$. In this case we have the additional non-gradient KVs H_1^+, Q_μ^+. Therefore the Noether symmetries are:

$$\partial_s, \mathbf{P}_I, \mathbf{r}_{\mu\nu}, H_1^+, Q_\mu^+ : f = \text{constant}$$

with Noether Integrals

$$\phi_s, \phi_I, \phi_r, \phi_{H_1^+} = \left(H_1^+\right)_i \dot{x}^i \quad \text{and} \quad \phi_{Q_\mu^+} = \left(Q_\mu^+\right)_i \dot{x}^i.$$

The Lie symmetries are:

$$\partial_s, s\partial_s, \mathbf{P}_I, \mathbf{r}_{\mu\nu}, H_1^+, Q_\mu^+$$

Case A(3b) $K=1, R(\tau)=\frac{c}{\sin\tau}$. In this case we have the two nongradient KVs $H_2^+, Q_{\mu+3}^+$.

The Noether Symmetries are:

$$\partial_s, \mathbf{P}_I, \mathbf{r}_{\mu\nu}, H_2^+, Q_{\mu+3}^+ : f = \text{constant}$$

with Noether Integrals

$$\phi_s, \phi_I, \phi_r, \phi_{H_2^+} = \left(H_2^+\right)_i \dot{x}^i \quad \text{and} \quad \phi_{Q_{\mu+3}^+} = \left(Q_{\mu+3}^+\right)_i \dot{x}^i.$$

The Lie symmetries are:

$$\partial_s, s\partial_s, \mathbf{P}_I, \mathbf{r}_{\mu\nu}, H_2^+, Q_{\mu+3}^+.$$

Case A(4a) $K=-1, R(\tau)=\frac{c}{\cosh\tau}$. In this case we have the two additional non-gradient KVs H_1^-, Q_μ^-.
The Noether Symmetries are:

$$\partial_s, \mathbf{P}_I, \mathbf{r}_{\mu\nu}, H_1^-,\ Q_\mu^- : f = \text{constant}$$

with Noether Integrals

$$\phi_s, \phi_I, \phi_r, \phi_{H_1^-} = \left(H_1^-\right)_i \dot{x}^i \quad \text{and} \quad \phi_{Q_\mu^-} = \left(Q_\mu^-\right)_i \dot{x}^i.$$

The Lie symmetries are:

$$\partial_s, s\partial_s, \mathbf{P}_I, \mathbf{r}_{\mu\nu}, H_1^-, Q_\mu^-.$$

Case A(4b) $K=-1, R(\tau)=\frac{c}{\sin\tau}$, we have the nongradient KV $H_2^-,\ Q_{\mu+3}^-$.
The Noether Symmetries are:

$$\partial_s, \mathbf{P}_I, \mathbf{r}_{\mu\nu}, H_2^-, Q_{\mu+3}^- : f = \text{constant}$$

with Noether Integrals

$$\phi_s, \phi_I, \phi_r, \phi_{H_1^-} = \left(H_2^-\right)_i \dot{x}^i \quad \text{and} \quad \phi_{Q_{\mu+3}^-} = \left(Q_{\mu+3}^-\right)_i \dot{x}^i.$$

Table 6 The special forms of the scale factor for $K=0$

#	Proper CKV	Conformal factor ψ	$R(\tau)$ for KVs	$R(\tau)$ for HKV
1	$\mathbf{P}_\tau=\partial_\tau$	$(\ln R(\tau))_{,\tau}$	c	$\exp(\tau)$
3	$\mathbf{M}_{\tau\alpha}=x_\alpha\partial_\tau+\tau\partial_\alpha$	$x_\alpha(\ln R(\tau))_{,\tau}$	c	$\nexists$
1	$\mathbf{H}=\mathbf{P}_\tau+x^a\partial_a$	$\tau(\ln R(\tau))+1$	c/τ	$\nexists$
1	$\mathbf{K}_\tau=2\tau\mathbf{H}+(x^cx_c-\tau^2)\partial_\tau$	$-(\ln R(\tau))_{,\tau}(-\tau^2+x^2+y^2+z^2)+2\varepsilon\tau$	$\nexists$	$\nexists$
3	$\mathbf{K}_\mu=2x_\mu\mathbf{H}-(x^cx_c-\tau^2)\partial_\mu$	$2x_\mu[\tau(\ln R(\tau))_{,\tau}+1]$	c/τ	$\nexists$

The Lie symmetries are:

$$\partial_s, s\partial_s, \mathbf{P}_I, \mathbf{r}_{\mu\nu}, H_2^-, Q_{\mu+3}^-.$$

Case B: $K=0$

In this case the metric is:

$$ds=R^2(t)\left(-dt^2+dx^2+dy^2+dz^2\right)$$

and admits three nongradient KVs $\mathbf{P}_I$ and three nongradient KVs $\mathbf{r}_{AB}$. Therefore the Noether symmetries are:

$$\partial_s, \mathbf{P}_I, \mathbf{r}_{AB} : f=\text{constant}$$

with Noether Integrals:

$$\phi_s=\frac{1}{2}g_{ij}\dot{x}^i\dot{x}^j\ ,\ \phi_{P_I}=\mathbf{P}_i^I\dot{x}^i \quad \text{and} \quad \phi_F=\ (\mathbf{r}_{AB})_i\dot{x}^j.$$

The Lie symmetries are:

$$\partial_s, s\partial_s, \mathbf{P}_I, \mathbf{r}_{AB}.$$

Again for special forms of the scale factor one obtains extra KVs and HKV as shown in Table 6.

From Table 6 we have the following special cases.

Case B(1): $R(t)=c=$constant. Then the space is the Minkowski space.
Case B(2): $R(t)=\exp(\tau)$. Then $\mathbf{P}_\tau$ becomes a gradient HKV ($\psi=1$, gradient function $\frac{1}{2}\exp(2\tau)$). Hence the Noether symmetries are

$$\partial_s, \mathbf{P}_I, \mathbf{r}_{AB}, 2s\partial_s+\mathbf{P}_\tau : f=\text{constant}$$
$$s^2\partial_s+s\mathbf{P}_\tau : f=\frac{1}{2}\exp(2\tau)$$

with Noether Integrals

$$\phi_s, \phi_{P_I}, \phi_F, \phi_{\mathbf{P}_\tau}=sg_{ij}\dot{x}^i\dot{x}^j-g_{ij}(\mathbf{P}_\tau)^i\dot{x}^j \text{ and}$$
$$\phi_{\mathbf{Y+1}}=\frac{1}{2}s^2g_{ij}\dot{x}^i\dot{x}^j-s(\mathbf{P}_\tau)_i\dot{x}^i+\mathbf{P}_\tau.$$

The Lie symmetries are:

$$\partial_s, s\partial_s, \mathbf{P}_I, \mathbf{r}_{AB}, \mathbf{P}_\tau, s^2\partial_s+s\mathbf{P}_\tau.$$

Case B(3): $R(t) = \tau^{-1}$. Then we have four additional nongradient KVs, the $\mathbf{H}$, and $\mathbf{K}_\mu$, and the Noether symmetries are:

$$\partial_s, \mathbf{P}_I, \mathbf{r}_{AB}, \mathbf{H}, \mathbf{K}_\mu : f = \text{constant}$$

with Noether Integrals

$$\phi_s, \phi_{P_I}, \phi_F, \phi_{\mathbf{H}} = (\mathbf{H})_i \dot{x}^j \quad \text{and} \quad \phi_{\mathbf{K}_\mu} = \left(\mathbf{K}_\mu\right)_i \dot{x}^i.$$

The Lie symmetries are:

$$\partial_s, s\partial_s, \mathbf{P}_I, \mathbf{r}_{AB}, \mathbf{H}, \mathbf{K}_\mu.$$

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Appendix

Proof of Theorem 1

Equation (35) implies:

$$\eta^i(t,x) = A^i(x)t + B^i(x) \tag{80}$$

where $A^i(x), B^i(x)$ are arbitrary differentiable vector fields.

The solution of Eq. (36) is:

$$\xi(t,x) = C_J(t)S^J(x) + D(t) \tag{81}$$

where $C_J(t), D(t)$ are arbitrary functions of the affine parameter t and $S^J(x)$ is a function the gradient of which is a gradient KV, i.e.:

$$S^J(x)_{|(i,j)} = 0. \tag{82}$$

The index J runs through the number of gradient KVs of the metric. Condition (37) gives

$$2A(x)^i_{|j} = \left[C_J(t)_{,tt} S^J(x) + D(t)_{,tt}\right] \delta^i_j. \tag{83}$$

Because the *lhs* is a function of x only we must have:

$$D(t)_{,tt} = M \Rightarrow D(t) = \frac{1}{2}Mt^2 + Kt + L, \text{ where } M, K, L \text{ are constants} \tag{84}$$

$$C_J(t)_{,tt} = G_J = \text{constant} \Rightarrow C_J(t) = \frac{1}{2}G_J t^2 + E_J t + F_J, \text{ where } G_J, E_J, F_J \text{ are constants.} \tag{85}$$

Substituting into (83) we find that

$$2A(x)^i_{;j} = \left(G_J S^J(x) + M\right)\delta^i_j \Rightarrow A(x)_{i;j} = \frac{1}{2}\left(G_J S^J(x) + M\right) g_{ij}, \tag{86}$$

where we have lowered the index because the connection is metric (i.e. $g_{ij|k} = 0$). The last equation implies that the vector $A(x)^i$ is a conformal Killing vector with conformal factor $\psi = \frac{1}{2}(G_J S^J(x) + M)$. Because $A(x)_{[i;j]} = 0$, this vector is a gradient CKV.

We continue with condition (38) and replace $\eta^i(t,x)$ from (80):

$$\begin{aligned}\mathscr{L}_{\mathbf{A}}\Gamma^i_{jk}t + \mathscr{L}_{\mathbf{B}}\Gamma^i_{jk} &= 2\xi_{,t\,(,j}\delta^i_{k)} = 2\left[(G_J t + E_J)\,S^J(x) + Mt + K\right]_{|(j}\,\delta^i_{k)}\\ &= 2\,(G_J t + E_J)\,S^J(x)_{|(j}\delta^i_{k)} \Rightarrow\\ \mathscr{L}_{\mathbf{A}}\Gamma^i_{jk} &= 2G_J S^J(x)_{,(j}\delta^i_{k)} \qquad (87)\\ \mathscr{L}_{\mathbf{B}}\Gamma^i_{jk} &= 2E_J S^J(x)_{,(j}\delta^i_{k)}. \qquad (88)\end{aligned}$$

The last two equations imply that the vectors $A^i(x)$ and $B^i(x)$ are special projective collineations or affine collineations of the metric (or one of their specializations) with projective functions $G_J S^J(x)$ and $E_J S^J(x)$ or zero respectively. Because A^i is a projective collineation and a CKV it must be a gradient HKV with homothetic factor $\rho + \frac{1}{2}M$. Furthermore (86) implies that

$$A^i = (\rho + \frac{1}{2}M)x^i + L^i,$$

where L^i is a nongradient KV.

Using the above results we find for $\xi(t,x)$:

$$\xi(t,x) = C_J(t)S^J + D(t) = \frac{1}{2}(G_J S^J + M)t^2 + (E_J S^J + K)t + F_J S^J + L.$$

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