The Involutive Prolongation of the (Complex) Twisting, Type-N Vacuum Field Equations

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Abstract

We review the historical development of a mathematical description of twisting, Petrov-type N×N, complex, solutions of Einstein's vacuum field equations, emphasizing Jerzy Plebański's intimate involvement for many years. The 3 pde's needed are linear and second-order in a potential function, F, while they are non-linear and fourth-order in a projection function, x, that defines the projection onto 3 independent variables. We describe the use of Cartan's reduced characters and also Janet's theory of integrability for pde's, and then use each of these to determine involutive prolongations of the original system. The simplest approach to the involutive system has 6 pde's involved; however, one may also prolong the system so as to acquire explicitly all derivatives of F, in terms of 7th derivatives of x. In that approach, the final, *involutive*, system describes 5 distinct such derivatives of x, allowing the F-dependent system to appear as a 'linearization' of that system.

I. Introduction:

It was some 18 years ago when Jerzy Plebański and I began to work together on problems in general relativity, with a reasonably constant working relationship since that time. Although we did in fact meet 5 years earlier, on a boat in the Bosphorus, neither of us took much notice at that time; nonetheless, Jerzy seemed very pleased to have me join him here at the Centro de Investigaciónes y Estudios Avanzados, early in 1975. A few weeks after I arrived, Jerzy was invited to a conference, "The Riddle of Gravity," at Syracuse University, where he talked with both Ted Newman and Roger Penrose concerning their work on "heavens".[1] He came back to Mexico in an extremely excited state, claiming that he had heard very interesting things, but that he wanted to approach it in his way rather than what they were doing. During the next several days, he and I worked together all the days, and he worked alone all the nights as well. The result was his first paper[2] on $\mathfrak{h}$-spaces, originally titled "Heaven, Hell and Einstein Equations." This was the beginning of an extremely interesting, and fruitful, collaboration between the two of us, including many other people as time passed, on the behavior of complex-valued solutions to Einstein's field equations.

After he and Ivor Robinson generalized the original notions, to $\mathfrak{hh}$-spaces,[3] so that it was possible for the spaces being considered to allow real-valued, Minkowski-signature cross-sections, the size of the arena increased substantially, causing the review[4] in 1980, with Charles Boyer as well in a very important role, to be quite lengthy. About 2 years later, work with Gerardo Torres del Castillo[5] gave us a clear way to explicitly specify the (complex) Petrov Type of an $\mathfrak{hh}$-space that was being searched for. Jerzy and I wanted to achieve a wider dissemination of the power of the techniques for $\mathfrak{hh}$-spaces, deciding that a very reasonable way to do that would be to show clearly how they could be used to solve some very important, long-standing problem in the field. It took not too long to decide that the problem we wanted to use for this purpose would be the twisting, type-N, Einstein spaces. He and I had already solved completely the type-N problem for $\mathfrak{h}$-spaces[6], so this seemed like a natural extension. As well, there was only one known solution so far, due to Hauser[7]. Work began in 1981 and, unfortunately, continues to this day. In 1982, I spent my sabbatical semester in Mexico, working with Jerzy on this problem. We acquired considerable understanding of the problem, and found that it might in fact be fairly difficult. We achieved the solution to some extremely interesting systems of non-linear pde's, based on various seemingly-reasonable ansätze, each of which turned out to be a solution to the vacuum field equations, of type N on both the self-dual and anti-self-dual sides, but twisting **only** on one side.

In 1987 Jerzy spent his sabbatical semester with me in Albuquerque, where he worked almost non-stop on this problem, even though my participation was severely hindered by the very many academic duties that revolved around the fact that I was the chairperson of the department. By this time, we had a clear delineation of the problem that should be solved, more or less as reported in our paper[8] published last year, 1982. Mathematically, the problem was specified by saying that we wanted the solution of a triplet of pde's in 3 independent variables, two dependent ones, and also including two gauge functions, each of which depended only on two of the 3 variables. These last two functions are referred to as gauge functions since the equations, as obtained after the input of considerable geometrical intuition from Jerzy, had a very remarkable symmetry property that can be quickly summarized by saying that it was always possible to choose new independent variables so that these two gauge functions simply became two of the new independent variables, without in any way otherwise changing the form of the equations. The triplet of equations is *linear*, and of second order, in one of the unknown functions, which I will F, as well as being linear in the gauge functions, but quite nonlinear in the other unknown function. When several friends were shown the equations in that format, it was suggested that the equations might well be over-determined, and have no solutions, since there were three equations for only two unknown functions. Their comments were wrong; however, it was those comments that began my first serious inquiries into the questions surrounding the involutivity of systems of pde's.

Attempting to find interesting solutions of this triplet of equations, one of the important routes we were following centered around the fact that it was possible to differentiate the original equations, and manipulate them, so that one obtained new pde's that were *linear and first-order* in the variable F. Clearly first-order pde's are much nicer than second-order ones; therefore, we attempted to understand how this could happen, what it might mean, what was the number of them, etc. The results of the investigations on these questions through 1986, and indeed through 1991, are contained in Ref. 8. What can also be, more or less, seen in that article is the shape of the 4 independent, first-order pde's that had been obtained by mid-1986. While they do indeed have rather "messy" coefficients, Jerzy's work in "understanding" the meaning behind the equations allowed us to write them in a rather simple format, especially the first two of them. It is perhaps worth noting immediately that the existence of four such equations, in terms of $\{F_1, F_2, F_3, F\}$, allows us to consider the equations as homogeneous, algebraic equations, from whence we could conclude that the determinant of the coefficients must vanish. This, of course, was an equation which now had only one unknown function contained within it. Whether it was only the first of many such requirements on that single function, we had no way of knowing. What is not contained in that article is the fact that Jerzy worked for several solid months, while in Albuquerque, to obtain the last 2 of these 4 equations, and to obtain them in what seemed to us to be the best possible form. As well, of course, he worked very hard at finding new methods to obtain solutions. In May he suffered an extremely serious stroke, causing him to need to spend many months recovering and losing several years to that project. Therefore, I am very glad to now be able to announce that several of the questions underlying the structure of these equations have now been laid to rest, even though we still have no new, non-trivial solutions of them.

The complete set of integrability conditions for the vacuum, twisting, type-N problem can be formulated in as few as 6 equations, or it may require considerably more, since the size of the set of all integrability conditions, for a given system of pde's in more than one dependent variable, depends on how the equations are used to describe the submanifold in the corresponding jet space.

II. Description of the Problem

Recall that a (complex-valued) $\mathfrak{hh}$-space contains (at least) one congruence of null strings (completely null, totally geodesic, complex-valued, two-dimensional surfaces)[4], which, in the generic case, has a non-zero expansion that picks out a special direction on any given leaf of the congruence. This expansion form determines an affine parameter, ϕ^{-1}, which can be used as one of the 4 coordinates needed to specify the space. Since the non-zero value of this expansion is what causes the *real-valued twist* of any real section to be non-zero, its use as a coordinate "inhibits" the coordinate system from the beginning to have non-zero twist. As well, we recall[4] that an $\mathfrak{hh}$-space can be completely specified by a "Debye-type" potential function, W, that must satisfy a single non-linear pde, the hyperheavenly equation;i.e., W is a single scalar function whose second derivatives determine the metric, and the fourth derivatives the curvature. Such a space has a number of (local) geometrical invariants, two very important ones being the Petrov types of the self-dual and anti-self-dual parts of the curvature tensor. Since we are of course interested in this study as a method to eventually determine real-valued solutions of the vacuum field equations, it is important to cause these two invariants to agree. One side of the curvature is determined by the fourth derivatives of W, while the other side is determined by a derivative of a function, γ, constant on any given leaf of the congruence, that appears in the equation for W. To search for real-valued solutions of type N, we must append to the hyperheavenly equation the additional requirement that both parts of the curvature be of Petrov type N. This determines[5] explicitly the dependence of W on the affine parameter, ϕ^{-1}, as a third-order polynomial in inverse powers of ϕ^{-1}, i.e., in powers of ϕ, so that the problem is reduced to finding the coefficients of this polynomial, which now depend only on the 3 remaining coordinates, and the derivative of γ. The content of the original equation for W is now most efficiently stated in terms of a set of 2-forms,[8,9] adapted to the fact that this particular $\mathfrak{hh}$-space actually contains two null strings. The simpler half of this set of 2-forms is a realization of the Maurer-Cartan equations for $SL(2,\mathbb{C})$, which allows us to define some 4 quantities that "perform" like coordinates, $\{x, v, y, u\}$, **and** a single, unknown relation between them, that projects the problem down to the residual, underlying 3-dimensional space of independent variables, which we variously refer to as $x = x(v, y, u)$ or $v = v(x, y, u)$. The other half of the set of 2-forms specifies the existence of a particular potential function, F, and a gauge function, Δ, and the explicit equations that they must satisfy as functions of the above set(s) of coordinates. The resulting equations are actually "reasonable" and *linear* in F, as well as Δ and γ; however they are highly non-linear in this function that describes the projection into the space of 3 variables.

As worked out in detail in Ref. 7, the defining equations for the problem just described are most simply written in the format:

$$\begin{aligned} F_{33} - \gamma F = 0 \quad &, \\ x_1 F_{22} + 2x_{12}F_2 + (x_{122} - x_1\Delta)F = 0 \quad &, \\ x_{23}(F_{23} + F_{32}) + x_{223}F_3 + x_{233}F_2 + \tfrac{1}{2}x_{2233}F = 0 \quad &, \\ \Delta_3 = 0 \neq \Delta_1 \quad , \quad \gamma_2 = 0 \neq \gamma_1 \quad &, \end{aligned} \tag{2.1}$$

where we have caused the equations to appear linear in x by making an *anholonomic* choice for coordinates on the space of first derivatives. More precisely, taking M as the space of independent variables, with (local) coordinate chart $\{v, y, u\}$, and N as the space of dependent variables, $\{F, x, \Delta, \gamma\}$, we choose coordinates for the fibers of the first jet bundle, $J^1(M, N)$,

$$\{x_1, x_2, v_3, F_1, F_2, F_3, \Delta_1, \Delta_2, \gamma_1, \gamma_3\} \quad \text{as coordinates on } J^1, \tag{2.2}$$

where

$$F_1 \equiv \left(\frac{\partial F}{\partial v}\right)_{(y,u)} \quad , \quad F_2 \equiv \left(\frac{\partial F}{\partial y}\right)_{(v,u)} \quad , \quad F_3 \equiv \left(\frac{\partial F}{\partial u}\right)_{(x,y)} \quad , \tag{2.3}$$

and the subscripts on the parentheses indicate which variables are being held constant during the indicated differentiation. Therefore, subscripts 1 and 2 indicate partial derivatives with respect to one choice of coordinates, $\{v, y, u\}$, while subscript 3 indicates a partial derivative with respect to a different choice of coordinates, $\{x, y, u\}$, so that the subscripts do not commute. Instead, we have the commutator equations

$$G_{23} - G_{32} = \frac{x_{23}}{x_1}G_1 \quad , \quad G_{13} - G_{31} = \frac{x_{13}}{x_1}G_1 \quad , \quad G_{12} - G_{21} = 0 \quad , \tag{2.4}$$

for any function G.

We refer to the function Δ, and also γ, as gauge functions because of the remarkable symmetry the equations possess.[8] There are four sets of infinite-dimensional symmetries that generate *pure gauge* transformations.

I. If one chooses a new coordinate chart, $\{\tilde{x}, u, y\}$, with $\tilde{x} \equiv X(x,y)$, and lets Δ, γ, and $H \equiv x_1 F$ transform as scalars, then the three equations, Eq. (2.1), are left completely form-invariant. Since, however, Δ is a function only of x and y, i.e., $\Delta_3 = 0$, it follows that we may always choose Δ to be any function of its two arguments whatsoever, changing x accordingly as we go, but leaving invariant the form of the equations—surely the property of a gauge function. In particular, we often consider the situation where we simply choose Δ to have the value x, nothing more needing to be done to the equations, and losing no generality.

II. Similarly, the second symmetry allows one to choose a new coordinate chart, $\{\tilde{v}, u, y\}$, with $\tilde{v} \equiv V(v,u)$, and transform F, Δ, and γ as scalars, then the three equations are again left completely form-invariant. Since $\gamma_2 = 0$, this allows us to choose γ to have any (non-constant) value we might like, such as, for example, v. We have not chosen to permanently make any choices, since particular functional forms for them may simplify the finding of forms for F and x. Somewhat less important symmetries are

III. that any non-constant function of y may be used to replace y, with appropriate transformations of F and Δ, and

IV. that any non-constant function of u may be used to replace u, with again appropriate transformations[8] of F and γ.

Having given this introduction to the obtention of Eq. (2.1), we now want to note that the equations there are not involutive, rather, they possess non-trivial integrability conditions, as already suggested in the introduction. The "need" to be able to describe **all** of the integrability conditions has followed several different pathways. I want to describe two of those pathways here, and to use two different methods to describe them as well. To do this, we will need some superstructure concerning jet bundles and forms defined thereon. Therefore I note that we view the jet bundle, $J^k(M,N)$, for some integer k, as simply a "place" where the individual derivatives, through order k, have an independent life of their own. This necessitates our being concerned with whether particular functions on J^k could be construed as lifts of functions over M. As is standard,[10,11] we use the contact module, Ω^k, for this purpose, any such function having the property that $(j^k u)^*(\Omega^k) = 0$, where $j^k u$ is a local cross section of of the bundle, fibered over M, and $u : U \subseteq M \to N$. In the current, anholonomic coordinates, the contact module over J^1 takes the form

$$\Omega^1 : \begin{cases} \theta_x \equiv dx - x_1 dv - x_2 dy + x_1 v_3 du \quad , \\ \theta_F \equiv dF - F_1 dv - F_2 dy - (F_3 - F_1 v_3) du \quad , \\ \theta_\Delta \equiv d\Delta - \Delta_1 dv - \Delta_2 dy \quad , \\ \theta_\gamma \equiv d\gamma - \gamma_1 dv - (\gamma_3 - v_3 \gamma_1) du \end{cases} \tag{2.5}$$

Continuing this process upward, and making the choice that the indices on the coordinates should be in standard numerical order, introduces the following new coordinates on J^2/J^1:

$$\{x_{11}, x_{12}, x_{22}, x_{13}, x_{23}, v_{33}, F_{11}, F_{12}, F_{22}, F_{13}, F_{23}, F_{33}, \Delta_{11}, \Delta_{12}, \Delta_{22}, \gamma_{11}, \gamma_{13}, \gamma_{33}\}, \tag{2.6}$$

with the additional generators of the contact module given by

$$\Omega^2/\Omega^1 : \begin{cases} \theta_{x_1} \equiv dx_1 - x_{11}dv - x_{12}dy - (x_{13} - x_{11}v_3)du \quad , \\ \theta_{x_2} \equiv dx_2 - x_{12}dv - x_{22}dy - (x_{23} - x_{12}v_3)du \quad , \\ \theta_{v_3} \equiv dv_3 + (x_{13}/x_1)dv + (x_{23}/x_1)dy - [v_{33} + (x_{13}/x_1)v_1]du \quad , \\ \theta_{F_1} \equiv dF_1 - F_{11}dv - F_{12}dy - (F_{13} - F_{11}v_3)du \quad , \\ \theta_{F_2} \equiv dF_2 - F_{12}dv - F_{22}dy - (F_{23} - F_{12}v_3)du \quad , \\ \theta_{F_3} \equiv dF_3 - [F_{13} - (x_{13}/x_1)F_1]dv - [F_{23} - (x_{23}/x_1)F_1]dy \\ \qquad\qquad - [F_{33} - F_{13}v_3 + (x_{13}v_3/x_1)F_1]du \quad , \end{cases} \tag{2.7}$$

plus ones for Δ and γ. We continue this process to include derivatives through fourth for the function x, but only second derivatives in the other unknown functions, referring to this particular bundle as $J^{2,4}$.

We may then denote by $Y^{2,4}$ the submanifold of $J^{2,4}$ defined by the original system of pde's. Since the pde's are simply algebraic equations on the jet bundle, we now choose to solve that set of equations for a maximal set of variables, intending to use that as a way of "eliminating" them from future consideration. Referring to the variables so eliminated as *co-coordinates* for the submanifold, $Y^{2,4}$, we can give a coordinate presentation of the inclusion map, $i : Y \to J^{2,4}$, which is the identity on the remaining coordinates, and has our 3 equations solved for the remaining 3 *co-coordinates*, to describe the rest of the map. There are very many ways to make such a choice, but it turns out that such a choice has implications for succeeding choices, when one is prolonging the differential system. Therefore, we have looked at two distinct ones, each of which has considerable merit as a description of the entire problem.

An obvious choice is simply to eliminate as many as possible of the derivatives of the dependent variable F, since they appear linearly in the equations. This is in fact what Jerzy and I attempted to do, over a period of quite some time, all by hand, acquiring first-order pde's for F, of course at the expense of increasing the order of derivatives of x; however, the method we used did increase that order in a minimal way. At the beginning, then one would choose to eliminate the 3 variables, $\{F_{33}, F_{22}, F_{23}\}$. On the other hand, as it now turns out, there may well be a better approach, which uses Eqs. (2.1) to eliminate the 3 variables $\{F_{33}, F_{22}, x_{2233}\}$, which can, almost, be justified by saying that one is eliminating the highest-order derivatives available, in each equation. Either choice describes the same (co-dimension 3) submanifold,

$$Y \equiv Y^{2,4} \subset J^{2,4} \subseteq \mathbb{R}^{54} \quad ,$$

where the variables include the 3 independent variables, jet variables through the second derivatives of F, through the fourth derivatives of x, plus the non-zero first derivatives of Δ and γ. Our differential ideal is then the entire contact module, generated by the contact 1-forms for F and its first derivatives, derivatives from zero to three for x, and also for Δ and γ, all restricted back to Y:

$$\mathcal{I} \equiv \Omega^{(2,4)}\Big|_Y \subset Y^* \qquad \text{with } \dim(\mathcal{I}) = 26 \quad .$$

The "directions" for Cartan's test for involutivity, applied to linear Pfaffian systems by Ref. 10, require that we determine the reduced characters of the system, $\mathcal{I}$, named $(s'_1, s'_2, s'_3, \ldots)$ **and** the dimension of the space of integral elements, $V_n(\mathcal{I})$, where n is the number of independent variables, i.e., the dimension of M^*. Letting $\{\omega^i\}_1^3$ be a basis for M^*, denoting the generators of $\mathcal{I}$ by θ^α, and completing a basis for the rest of Y^* with a set of 1-forms, $\{\pi^\epsilon\}$, we may begins these tasks by writing

$$-d\theta^\alpha \equiv A_{\epsilon i}{}^\alpha\, \pi^\epsilon \wedge \omega^i + \tfrac{1}{2} C^\alpha_{ij}\, \omega^i \wedge \omega^j \qquad (\text{mod } \mathcal{I}) \quad . \tag{2.8}$$

The quantities $\beta^\alpha{}_i \equiv A_{\epsilon i}{}^\alpha\, \pi^\epsilon$ are referred to as the "tableau" matrix. The set of terms quadratic in ω^i are referred to as the torsion of the system. In principle, there could also be terms quadratic in the π^ϵ's. In the case when there are no such elements, the Pfaffian system that we have is designated *linear*; since our system is in fact linear, from now on we only describe the use of the theory for such systems. When the basis elements for M^* are ordered so that the number of algebraically-independent in column 1 is maximal, then the number of column 2 maximal—while maintaining those in column 1—and then the number in column i arranged maximal, while maintaining fixed in location all those in previous columns, these various maximal numbers are referred to as the reduced Cartan characters for the system, labeled by the symbols s'_i. However, in order for this to be a useful labeling, we must also know that there exists an affine transformation of the basis elements $\pi^\epsilon \longrightarrow Z^\epsilon{}_\delta\, \pi^\delta + W^\epsilon{}_i\, \omega^i$, i.e., a re-naming of the "extra" part of the basis of M^*, mixing in some of the basis of the rest as needed, so that the torsion transforms to zero. On the other hand, we must also determine the dimension of the space of (n-dimensional) integral elements, $V_n(\mathcal{I})$, i.e., the number of parameters needed to describe the space of n-planes tangent to the solution manifold. Such elements could be written in the form $\pi^\epsilon - p^\epsilon_i\, \omega^i$, where the p^ϵ_i are are coordinates in some exactly the paramter space needed to describe these n-planes. The general theory in Ref. 10 assures us that if such a parametrization exists, then it determines the transformation needed to transform away the torsion. Therefore, we next look at this problem, by creating another matrix, the rows of which are the exterior derivatives of all the algebraically-independent quantities, $\beta^\alpha{}_i$, just determined above, $\sum_{i=1} s'_i$ of them. Writing these derivatives modulo $\mathcal{I}$, we write the result as a matrix wedge-multiplying the basis of M^*. The origin of the various ρ^α_i ascribes various symmetry properties to these coefficients; provided there is a complete solution to this question, the algebraically-independent entities within this matrix are just the desired parameters that characterize the elements of $V_n(\mathcal{I})$, and their number is the dimension needed. The following inequality is then always true; however, in order for the system to be involutive, Cartan's criterion tells us that it must actually be an equality:

$$\sum_i i s'_i \geq \dim\left(V_3(\mathcal{I})\right) \quad , \tag{2.9}$$

In our case, the typical structure of the Cartan Tableau is of the form

Cartan Tableau for $\mathcal{I}$

$$
-\begin{pmatrix} d\theta_x \\ d\theta_F \\ d\theta_{x_1} \\ \vdots \\ d\theta_{F_1} \\ d\theta_{F_2} \\ d\theta_{F_3} \\ d\theta_{x_{111}} \\ \vdots \end{pmatrix} = \begin{pmatrix} 0 & 0 & 0 \\ 0 & 0 & 0 \\ 0 & 0 & 0 \\ 0 & 0 & 0 \\ d\,F_{11} & d\,F_{12} & --- \\ d\,F_{12} & d\,\widetilde{\mathbf{F}}_{\mathbf{22}} & --- \\ --- & --- & d\,\widetilde{\mathbf{F}}_{\mathbf{33}} \\ d\,x_{1111} & d\,x_{1112} & d\,x_{1113} \\ \vdots & \vdots & \vdots \end{pmatrix} \wedge \begin{pmatrix} dv \\ dy \\ du \end{pmatrix}
$$

where the symbol $---$ indicates that the value from Ω_2 should be inserted, while the **tilde** means that the value from the pde's should be inserted. The first 11 rows of the tableau matrix are identically zero, because of the later terms in the ideal. Inserting all the entries, we find that <u>every</u> non-zero entry in the first column is algebraically-independent, so $s_1' = 15$ is surely maximal. However, our first choice for a basis of M^* does not maximize s_2', since that would give us $s_2' = 6, s_3' = 1$, while the ordering $\{dv.\,dy + du, dy - du\}$ gives the maximal (correct) values $s_2' = 7, s_3' = 0$. With that ordering, we have $(s_1', s_2', s_3') = (15, 7, 0)$.

On the other hand, creating the matrix describing those 15+7 = 22 linearly-independent exterior derivatives just above, again modulo all the earlier 1-forms, we next find the number of algebraically-independent quantities that occur in that matrix. This is essentially a count of the independent "next-highest" derivatives of each dependent variable.

Integral Element Matrix for $\mathcal{I}$

$$
\begin{pmatrix} dF_{11} \\ dF_{12} \\ dF_{13} \\ dF_{23} \\ dx_{1111} \\ dx_{1112} \\ \vdots \end{pmatrix} - \begin{pmatrix} F_{111} & (F_{112}) & (\cdots) \\ F_{112} & \widetilde{\mathbf{F}}_{\mathbf{122}} & \underset{\sim}{(\cdots)} \\ F_{113} + @ & (F_{123} + @) & \cdots \\ F_{123} + @ & \widetilde{\mathbf{F}}_{\mathbf{223}} + @ & \widetilde{\mathbf{F}}_{\mathbf{233}} + @ \\ x_{11111} & (x_{11112}) & (x_{11113} + @) \\ x_{11112} & (x_{11122}) & (x_{11123} + @) \\ \vdots & \vdots & \vdots \end{pmatrix} \wedge \begin{pmatrix} dv \\ dy \\ du \end{pmatrix}
$$

where the entries in parentheses are not algebraically-independent from those in some prior column, and @ indicates lower-order terms, that arise from some commutator equation. This number is then $22 + 6 + 0 = 28$, so that the formula in Eq. (2.8) gives $15 + 2(7) = 29 > 28$, from which we conclude that the system is **not** involutive.

This fact is hardly surprising, since Jerzy and I had already found that we could obtain *clearly new*, first-order pde's for F by differentiating and manipulating the original pde's. Cartan and Kuranishi of course say that we should proceed in exactly that way, namely, to obtain an involutive system, we must prolong the original ideal to one higher jet and perform the test again, and perhaps yet again or maybe even again, but, at least, some finite number of times. The fact that this will be only a finite number of times is not necessarily "good news," since our calculation of the second pair of first-order pde's had required many months of effort. However, as it turns out, when the prolongation to $J^{3,5}$ is calculated, that differential system is actually involutive. The prolonged submanifold, $Y^{(1)}$, is of co-dimension 12, namely

$$
Y^{(1)} \equiv Y^{3,5} \subset J^{3,5} \subset \mathbb{R}^{79} \quad ,
$$

while the prolonged ideal acquires 4 more 1-forms for the independent second derivatives of F, 14 more for the independent fourth derivatives of x, and 4 more for the first derivatives of Δ and γ:

$$\mathcal{I}^{(1)} \subset (Y^{(1)})^* \quad \text{with} \quad \dim(\mathcal{I}^{(1)}) = 48 \quad .$$

The prolonged tableau is a 48×3 matrix, the first 26 rows of which are all zeros, and we find that $(s'_1, s'_2, s'_3) = (22, 6, 0)$. Looking for the space of integral elements for $\mathcal{I}^{(1)}$, the $22+6 = 28$ independent elements determine $28+6+0 = 34$ as the dimension of the space of integral elements, so that the equality in the sum is realized, and this first-prolonged system is involutive! As an aside, an involutive system can always be created as a differential system so that the (non-restricted) Cartan characters, $\{s_0, s_1, \ldots\}$ are equal to the restricted ones. The character s_0 is simply the total number of 1-forms, which in our (prolonged) case is 48; as well the total number of (algebraically- independent) variables in our prolonged space is 79, so that the Cartan genus of our system is $g \equiv n - \sum_{i=0}^{3}(s_i) = 79 - (48 + 22 + 6 + 0) = 3$.

III. An Understanding of the Answer

The complete prolongation of our 3 (original) pde's onto the jet, $J^{3,5}$ provides us with 9 new equations, the 3 derivatives of each one, so one expects the prolongation, $i^{(1)} : Y^{(1)} \to J^{3,5}$ to have co-dimension $\leq$ 12. It is useful to describe how this works following the two distinct lines of coordinatization for Y already mentioned. I will refer to the choice that uses $\{F_{33}, F_{22}, x_{2233}\}$ as co-coordinates as Option 2, and the alternative option, to eliminate $\{F_{33}, F_{22}, F_{23}\}$ as co-coordinates, as Option 1, where the numbers are historically motivated. The two sets correspond to different choices of coordinates on the same submanifold, Y. However, the prolonged equations, created by differentiating the original equations and then restricting them to $Y^{(1)}$, i.e., by re-expressing them without co-coordinates, appear different in these two options. In both options, five of the new co-coordinates are (essentially) the same, $\{F_{133}, F_{233}, F_{333}, F_{122}, F_{222}\}$. However, for Option 2, the other 4 co-coordinates (on J^1) are $\{F_{223}, x_{12233}, x_{22233}, x_{22333}\}$, while for Option 1, the choices are, $\{F_{223}, F_{123}, F_{12}, F_{13}\}$. The last two choices in Option 1 appear since there are insufficiently many third derivatives among the 9 new equations to solve them all for third derivatives of F. It is this occurrence that will cause us to be able to determine all second derivatives, and to determine first-order pde's that F must satisfy.

Both sets of 12 equations do determine 12 co-coordinates for the *same* submanifold, $Y^{(1)} \subset J^{3,5}$, but they present different pictures concerning the involutivity of the equations. Understanding how this happens is, I believe, useful. We first ask whether there are any integrability conditions remaining. To answer this question completely, we must count the dimension of the space of integral elements, which "amounts to" a second prolongation of the system, to $J^{4,6}$. The resulting 3+9+17 equations completely specify the submanifold, $Y^{(2)} \subset J^{4,6}$. The submanifold so determined is identical in both coordinate presentations, as one would hope. Therefore, as a (coordinate-free) differential system, the two options are the same. However, from the point of view of pde's and the explicit determination of co-coordinates, they give different answers concerning involutivity. Better language might be that the choice of coordinate presentation in Option 1 "hides" the involutivity of the differential system? For Option 2, the important question is whether the different methods of obtaining a value for F_{2233}, via $\partial_2 F_{233}$ or via $\partial_3 F_{223}$, give the same answer. As it turns out, they do indeed give the same answer, modulo the set of 12 equations already agreed to, which explains why the co-dimension mentioned above is 17, rather than 18, the number of independent second derivatives of 3 quantities. In point of fact, to answer this single question we do not need all the 3+9 equations; it is actually sufficient to simply know the original 3 equations, plus the 3 additional equations $\partial_2(F_{33}$ equation), $\partial_3(F_{22}$ equation), and $\partial_1(x_{2233}$ equation), so I believe that this set of 6 equations is a minimal set of equations that defines the involutive system. The other 3 equations will be presented below, in Eqs. (3.1-3).

Before proceeding to discuss the virtues of Option 1, I want to now return to a very important point, connected with the various Cartan characters. The original purpose of finding an involutive prolongation, as I read the history books, was so that one could know exactly what sort of initial data could be consistently given, to determine a unique solution of the given system of pde's. Cartan's characters have already told us a portion of that answer; however, I found an alternative approach to be considerably more informative. Since I have also used that approach to interpret the meaning of the 9 equations, and of the several more that follow below, I want now to comment on it, and to answer very explicitly just what is the free initial data available. There is an alternative method to that of Cartan, originally created by Riquier[12], Vessiot[13], and Janet[14], at about the same time as Cartan's work was done. Pommaret[15] and others have written fairly recent books on it, which have a co-homological approach. However, people involved with computers, attempting to teach them to solve overconstrained systems of pde's, appear to have found it quite useful; therefore, the readable references come from fairly unexpected quarters. Schwartz[16] in Germany and co-workers of Vinogradov in Russia[17] have been automating this approach. Work of Stormark,[18] in Stockholm, has been the most useful to me. He describes Janet's theory of integration, via Hilbert bases for monomials for the coordinates for submanifold of the jet bundle that describes the pde. Beginning with a given set of pde's, and a particular choice of ordering for the coordinates on J^∞, he allows one to decide which equations, for a given prolongation, must actually be calculated to see if they will generate new information such as integrability conditions, and which ones are simply there because they are involved in the prolongation. The members of this last set are those in which there is no ambiguity concerning how to determine that particular prolonged equation, such as, for example, the quantity F_{333} in our problem which may only be determined by taking ∂_3 on the equation for F_{33}. An example that he works out in detail begins with the 2-equation system, $p_{33} = x_2 p_{11}, p_{22} = 0$. This simple example requires 5 successive prolongation steps, adding along the way the other pde's, $p_{322} = 0$, $p_{3211} = 0$, $p_{31111} = 0$, $p_{211} = 0$, $p_{1111} = 0$, before it achieves involutivity, whereupon one is then able to show that the general solution to the original problem depends (only) on 12 constants.

Stormark's method (or Janet's) is of course very explicitly dependent on one's choice of coordinates, but coordinate-covariant in that it works easily in any set of coordinates desired. His method also makes it very easy to prolong from $Y^{(1)} \subset J^{3,5}$ to simply some subspace of $J^{3,6}$, instead of having to go all the way to $J^{4,6}$. Beginning with a given set of pde's one first solves each one for the highest derivative of some unknown function that appears within. Beginning with those given co-coordinates, one must specify a unique method to calculate all higher derivatives of each unknown function appearing. This need for uniqueness requires an ordering of the independent variables, which we make, somewhat arbitrarily, as $3 > 2 > 1$, which specifies an ordering in the algorithm that computes the unique value of all higher derivatives. Following Stormark, we use a series of tables to describe the choices made and the co-coordinates chosen, as pictured below. The entries across the top indicate the independent variables; the entries down the first column indicate the co-coordinates chosen to be eliminated, and the entries in the other columns indicate "legal" monomials that may be pre-pended, denoted with "$\cdot$", to the co-coordinates to create monomials for higher derivatives

Applied to our first prolongation, via Option 2, we use our equations to tell us the values, on $J^{2,4}$, of F_{33}, F_{22}, and x_{2233}, viewed as co-coordinates defining Y. Since 3 is the largest index, the equation with the largest number of 3's is dealt with first. Any derivative of F of the form $F_{33\sigma}$ is to be obtained by calculating it from the σ-th derivative of the F_{33} equation, where σ stands for any number of derivatives. On the other hand, beginning with the F_{22} equation, we may determine all derivatives of the form F_{22i}, *but* with i made from 1's and/or 2's. Therefore F_{332} is to be determined as $F_2 \cdot F_{33}$, from F_{33}, rather than from F_{22}. There is then no way to compute F_{223}; the system is *incomplete*, as Stormark defines it. We will use a $\bullet$ to indicate a difficulty in any location where there is a dis-allowed (or illegal) monomial multiplication. Such entries will denote either an *incomplete* system, caused by the lack of a legal way to calculate that quantity, or a possible *integrability condition* caused by the existence of more than one way to calculate a quantity. In the former case, the needed equation will be calculated, violating the ordering rules but then appended to the system of co-coordinates; in the latter case, the two methods will be compared and the resulting equation, if non-trivial, will be solved for an additional co-coordinate and appended to the system.

Stormark Table
for Option 2

	3	2	1
${F_3}^2$	F_3	F_2	F_1
${F_2}^2$	$\bullet$	F_2	F_1
${x_3}^2{x_2}^2$	x_3	x_2	x_1

Original pde's

The $\bullet$ in the table indicates a place where a question must be answered. In this case, the system is *incomplete* since there is no legal way to calculate $F_3{F_2}^2$. We compute it from the $\partial_3(F_{22}$-equation), and add it to the system, and re-order the entries in the new table presented below.

With these, now, 4 equations on $J^{3,5}$, the $\bullet$ indicates a possible integrability question, while the $\surd$ indicates a previous $\bullet$ that has already been resolved.

	3	2	1
$F_3{}^2$	F_3	F_2	F_1
$F_3F_2{}^2$	•	F_2	F_1
$F_2{}^2$	$\surd$	F_2	F_1
$x_3{}^2x_2{}^2$	x_3	x_2	x_1

First S-prolongation

The question arises: Is $F_3{\cdot}F_3F_2{}^2 = F_2{}^2{\cdot}F_3{}^2$? Since the answer is NO, the equation must be solved for some highest derivative and added to the system. After eliminating F_{233}, using the legal definition, $F_2{\cdot}F_3{}^2$, it still contains F_{123} and x_{12233}; we elect to solve for x_{12233}.

We then present the latest results in yet a new Stormark table:

	3	2	1
$F_3{}^2$	F_3	F_2	F_1
$F_3F_2{}^2$	$\surd$	F_2	F_1
$F_2{}^2$	$\surd$	F_2	F_1
$x_1x_3{}^2x_2{}^2$	x_3	x_2	x_1
$x_3{}^2x_2{}^2$	x_3	x_2	•

Second S-prolongation

The question arises: Is $x_1{\cdot}x_3{}^2x_2{}^2 = x_1x_3{}^2x_2{}^2$? When computed, the answer to the question is YES! The system is now involutive, with these equations. The 5 listed above, and the F_{233} that was needed "along the way." Since this answer is simply $0 = 0$, we may look at this last "S-prolongation" as simply the calculation of integral elements that had to be done, in Cartan's approach to this problem.

These extra three equations, for F_{223}, F_{233}, and x_{12233} are given below. It It is interesting that in none of the 6 equations do Δ or γ appear differentiated, since the only derivatives of them that occur are those that must vanish.

$$F_{233} = \frac{F_1\,x_{233}}{x_1} - \frac{2\,F_1\,x_{13}\,x_{23}}{x_1^2} + \frac{2\,F_{13}\,x_{23}}{x_1} + F_2\,\gamma \quad , \tag{3.1}$$

$$\begin{aligned} F_{223} = {} & \frac{F\,x_{122}\,x_{13}}{x_1^2} + \frac{2\,F_2\,x_{12}\,x_{13}}{x_1^2} - \frac{2\,F_2\,x_{123}}{x_1} \\ & - \frac{F\,x_{1223}}{x_1} - \frac{F_3\,x_{122}}{x_1} - \frac{2\,F_{23}\,x_{12}}{x_1} + \Delta\,F_3 \quad , \end{aligned} \tag{3.2}$$

$$\begin{aligned} x_{12233} = {} & \frac{2\,F_1\,F_2\,x_{233}}{F^2} - \frac{2\,F_{12}\,x_{233}}{F} - \frac{2\,F_1\,x_{11}\,x_{23}{}^2}{F\,x_1^2} + \frac{2\,F_{11}\,x_{23}{}^2}{F\,x_1} - \frac{2\,F_1{}^2\,x_{23}{}^2}{F^2\,x_1} \\ & - \frac{4\,F_1\,x_{12}\,x_{13}\,x_{23}}{F\,x_1^2} + \frac{4\,F_{12}\,x_{13}\,x_{23}}{F\,x_1} + \frac{4\,F_1\,x_{123}\,x_{23}}{F\,x_1} + \frac{4\,F_1\,F_{23}\,x_{23}}{F^2} + \frac{2\,F_1\,x_{13}\,x_{223}}{F\,x_1} \\ & + \frac{2\,F_1\,F_3\,x_{223}}{F^2} - \frac{2\,F_{13}\,x_{223}}{F} + \frac{2\,F_2\,x_{12}\,x_{133}}{F\,x_1} - \frac{4\,F_2\,x_{12}\,x_{13}{}^2}{F\,x_1^2} + \frac{4\,F_2\,x_{123}\,x_{13}}{F\,x_1} \\ & + \frac{4\,F_{23}\,x_{12}\,x_{13}}{F\,x_1} - \frac{2\,F_2\,x_{1233}}{F} - \frac{4\,F_{23}\,x_{123}}{F} - \frac{2\,F_3\,x_{1223}}{F} - \frac{4\,F_{123}\,x_{23}}{F} + \frac{2\,F_3\,x_{122}\,x_{13}}{F\,x_1} \\ & + \frac{x_{122}\,x_{133}}{x_1} - \frac{2\,x_{122}\,x_{13}{}^2}{x_1^2} + \frac{2\,x_{1223}\,x_{13}}{x_1} \quad . \end{aligned} \tag{3.3}$$

The other thing we receive from Stormark's approach is an explicit method for determining the "complementary variables," by which he means those functions on the jet bundle which are "free" and available for arbitrary choice to be used as initial conditions for the pde's being considered. One might hope to use this information to create information to sum a Taylor series—at least in some special cases—although I have yet to see how. Following his straight-forward, if complicated, procedure, the entire list of these "complementary variables," which of course agrees with the count á la Cartan, is given by

$$
\begin{aligned}
&\text{4 functions of 2 variables}: \begin{cases} x(v,y,0) \\ v_3(v,y,0) \\ v_{33}(v,0,u) \\ x_{233}(v,0,u) \end{cases}, \\
&\text{4 functions of 1 variable}: \begin{cases} F(v,0,0) \\ F_2(v,0,0) \\ F_3(v,0,0) \\ F_{23}(v,0,0) \end{cases}, \\
&\text{2 ``gauge'' functions}: \begin{cases} \Delta(x,y) \\ \gamma(v,u) \end{cases}.
\end{aligned}
\tag{3.4}
$$

There are considerably more free functions than might have been expected. Some of them are of course gauge freedoms, irrelevant to the final, distinct manifolds; some of the extra freedom, on the other hand, may well have to do with the fact that we are still allowed complex transformations, i.e., we must eventually choose real slices, which may cut down the final amount of freedom available. We can also note that the way that the choice of initial variables oscillates between $\{v,y\}$ and $\{v,u\}$ surely is simply the mirror, in this approach, of the difficulty that arose with the choice of the variables in the Cartan calculation, that one must actually use $\{v, y+u, y-u\}$, instead of the original $\{v,y,u\}$.

IV. How Many First-Order Pde's are There?

I now want to recall that in Jerzy's original approach, which followed Option 1, he was finding first-order pde's for F, by the use of differential concomitants. Stormark's and Janet's approach is of course deliberately intended to make the most efficient use of such calculations. Therefore, we now retreat to Option 1 and the first prolongation, where one has 3+9 equations to consider. Before we may ask questions concerning the higher derivatives, we should stay on $J^{3,6}$ and ask questions about the other second derivatives. This is most easily seen with respect to the following Stormark table:

Stormark Table
for Option 1

	3	2	1
$F_3{}^2$	F_3	F_2	F_1
F_2F_3	•	F_2	F_1
$F_2{}^2$	•	F_2	F_1

Original pde's

This system is *complete*, but two questions arise:

(1) Is $F_3 \cdot F_3{}^2 = F_2 \cdot F_3{}^2$?

(2) Is $F_3 \cdot F_2{}^2 = F_2 \cdot F_2F_3$?

Both answers are no. Both new equations must be appended to the system; we resolve them for their highest F-derivatives, finding F_{13} from (1) and F_{12} from (2).

We then present the results of these calculations, having expanded the system to 5 pde's, in the next table:

	3	2	1
$F_3{}^2$	F_3	F_2	F_1
F_2F_3	$\surd$	F_2	F_1
F_1F_3	•	•	F_1
$F_2{}^2$	$\surd$	F_2	F_1
F_1F_2	•	•	F_1

First S-prolongation

Four questions arise from this table:

(3) Is $F_3 \cdot F_1F_3 = F_1 \cdot F_3{}^2$?

(4) Is $F_2 \cdot F_1F_3 = F_1 \cdot F_2F_3$?

(5) Is $F_3 \cdot F_1F_2 = F_1 \cdot F_2F_3$?

(6) Is $F_2 \cdot F_1F_2 = F_1 \cdot F_2{}^2$?

The answers are no for (3), (4), and (6), but yes for (5), which is identical to (4). We solve (4) for F_{11}, and treat (3) and (6) as equations to be solved for F_2 and F_3, noting that their highest x-derivatives are x_{223333} and x_{222233}, respectively.

The above calculations are already rather lengthy; the equality of (5) and (4) was shown by calculations performed by the symbolic algebra system, Macsyma. Also note that this equality of those two equations maintains the number of new equations at 17, instead of 18, for both options. At this point we have explicit determinations of all the second derivatives of F, and two first-order pde's for F. These last two constitute the first pair of first-order pde's that Jerzy and I found, in the early 1980's. At this point, however, there still remain some questions concerning integrability. In 1986-7, the question was whether one can actually perform the prolongation to $J^{(3,7)}$, to ask whatever integrability conditions are being put forth by the system, viewed in this way. Based on a good understanding of the problem, Jerzy developed a notation that allowed the manual derivation of an additional two first-order pde's, also allowing all 4 to be written[8] on half a page.

The symbolic computer performs the job of calculation much faster, if the questions are asked correctly; however, the answers provide exactly **no** insight into the problem. In addition, if the questions are not asked in a very well-formed way, the computer enters into calculations so large that it is unable to do anything useful with them at all. The computer simply determines these 4 first-order pde's as polynomials in all the variables, with 20, 54, 59, and 121 terms. An approach is to consider selecting 3 of the equations for the first derivatives, and inserted that information into the fourth one. Conceptually simpler is to ask that the determinant of the four pde's be zero, since they constitute a homogeneous, algebraic system of equations for the 4 unknown functions, F_1, F_2, F_3, and F. The result is an equation involving only x and its derivatives, plus, of course, the gauge functions, Δ and γ. Although this equation was written out, it is of very high degree, and generally not useful. A more serious problem was that we were now very unsure how many more such first-order pde's there might be, nor could we see any clean way to determine the rest of them in any reasonable time span.

Further progress wandered until we were able to utilize techniques to answer the questions underlying the involutivity of the problem. Using the current techniques, the next prolongation table could be explicitly calculated:

	3	2	1
$F_3{}^2$	F_3	F_2	F_1
F_2F_3	$\surd$	F_2	F_1
F_1F_3	$\surd$	$\surd$	F_1
$F_2{}^2$	$\surd$	F_2	F_1
F_1F_2	$\surd$	$\surd$	F_1
$F_1{}^2$	$\bullet$	$\bullet$	F_1
F_3	$\bullet$	$\bullet$	$\bullet$
F_2	$\bullet$	$\bullet$	$\bullet$

Second S-prolongation

Now eight questions arise:

(7) Is $F_3{\cdot}F_1{}^2 \;=\; F_1{\cdot}F_1F_3$?
(8) Is $F_2{\cdot}F_1{}^2 \;=\; F_1{\cdot}F_1F_2$?
(9) Is $F_3{\cdot}F_3 = F_3{}^2$?
(10) Is $F_3{\cdot}F_2 = F_2F_3$?
(11) Is $F_2{\cdot}F_3 = F_2F_3$?
(12) Is $F_2{\cdot}F_2 = F_2{}^2$?
(13) Is $F_1{\cdot}F_3 = F_1F_3$?
(14) Is $F_1{\cdot}F_2 = F_1F_2$?

The answers are that (7) and (8) are simply linear combinations of (9)-(14), but those 6 are new equations in F_1, F, and x-derivatives. Of these 6 equations, (7) and (8) are simply the second pair of first-order pde's known for several years. However, the other 4 equations are new, and could be said to increase the number of first-order pde's to *eight*! However, at this point, we still do not know if there may not be yet more; the number might in fact seem to be mounting rapidly. Noting that each of these 6 pde's is linear in a distinct 7th derivative of x, the approach, then, is to (1) eliminate F_2 and F_3 via the two resolutions above, then (2) solve an arbitrary one, say (14), for F_1, and then (3) eliminate F_1 from the remaining 5 equations. Since the equations have always been homogeneous in the F-derivatives, this means that those 5 may be divided by F, giving us 5 remaining equations for x only. The problem thus presented, however, sounds rather difficult, since we are asking that one unknown function, $x = x(v, y, u)$ should satisfy 5 different equations for 5 of its seventh derivatives. Creating a new Stormark table for these equations—presented below—we find exactly 5 non-trivial integrability questions to be asked, on $Y^{(4)} \subset J^{3,8} \subset J^{6,8}$. The calculations involved to ascertain whether these are yet 5 new conditions that must be appended to the system might never have been accomplished had it not been for the power of the symbolic algebra system and computer involved.

	3	2	1
$x_2{}^3x_3{}^4$	x_3	x_2	x_1
$x_1x_2{}^2x_3{}^4$	x_3	•	x_1
$x_2{}^4x_3{}^3$	•	x_2	x_1
$x_2{}^5x_3{}^2$	•	x_2	x_1
$x_1x_2{}^4x_3{}^2$	•	•	x_1

Third S-prolongation

Five questions arise:

(15) Is $x_2{\cdot}x_1x_2{}^2x_3{}^4 = x_1{\cdot}x_2{}^3x_3{}^4$?
(16) Is $x_3{\cdot}x_2{}^4x_3{}^3 = x_2{\cdot}x_2{}^3x_3{}^4$?
(17) Is $x_3{\cdot}x_2{}^5x_3{}^2 = x_2{\cdot}x_2{}^4x_3{}^3$?
(18) Is $x_3{\cdot}x_1x_2{}^4x_3{}^2 = x_1{\cdot}x_2{}^4x_3{}^3$?
(19) Is $x_2{\cdot}x_1x_2{}^4x_3{}^2 = x_1{\cdot}x_2{}^5x_3{}^2$?

The computer assures us that every one of the answers is **"Yes"**; when proper substitutions are made, all 5 of the above equations are identically satisfied. Finally, the system is found to be involutive in this choice of coordinate presentation as well. From the point of view of Option 1, this prolongation is of course also involutive since every prolongation of an involutive system is also (still) involutive.[10] In addition, we have finally answered the interesting question (that Jerzy and I had discussed at length), namely, how many first-order, linear pde's for F are there in some "complete" set? In 1987 we had 4, on $J^{1,7}$. The answer is now seen to be that the complete set is 8 such first-order, linear pde's, but the 4 others still involve only derivatives of x through the seventh. (From the point of view of the computer, the other 4 are also polynomials, with 137, 194, 311, and 472 terms.) We certainly welcome that answer to a question long sought for.

Many other people have worried about questions concerning the type N problem, of course, certainly including Ted Newman and Ivor Robinson, speakers here at this conference. About 6 years ago Alan Held speculated, at the triennial Jena Relativitätstheorie Conference, that no new solution to this problem would ever be found, on the grounds that the number of steps needed to create that solution was in fact infinite. Our calculation indeed refutes that surmise.

In addition to being a long-sought answer, is there any other value to these 8 first-order pde's? We hope so, but are not yet certain. One approach clearly is to use this as a hunting-ground for reasonable ansätze, looking for solutions. Whether these equations, basically without F's are more useful for that purpose than those 6 equations that come from looking at the problem from the viewpoint of Option 2, we do not know. A couple of years ago, I was already involved in this "game," and presented results at the Marcel Grossmann Conference in Japan.[19] At that time, I had hoped that there were some solutions "close" to Hauser's solution. In the current notation, Hauser's solution may be described as having

$$\begin{gathered} x = -v + \tfrac{1}{2}(y+u)^2 \quad , \\ F = F(v, y+u) \quad , \\ \Delta = \frac{3}{8x} \quad , \quad \gamma = \frac{3}{8v} \quad . \end{gathered} \tag{4.1}$$

The particular fact that F is some hypergeometric function also comes out of the equations, but is a straight-forward detail that follows if you require solutions to our equations that satisfy Eqs. (4.1). My calculation from two years ago assumed only Eq. (4.1a), the functional form of x, leaving explicitly open the possibility that $F_y \neq F_u$. If that had been true, then the new solution would not have the (true) Killing vector that Hauser's solution has, namely ∂_u in these coordinates. Beginning with that form for x, I was able to use the computer to find additional equations beyond the original four first-order pde's—without, of course, any particular knowledge as to whether or not there were yet more. These additional equations were sufficient to prove that Eq. (4.1a) implied the rest of those constraints; i.e., given that form of x, there were no other solutions. Not very helpful, certainly, but at least interesting.

A different way to view this problem is now the following. Suppose one is given the 5 different 7th order, quasi-linear pde's for 5 derivatives of x. For some given choice of Δ and γ, these involve only derivatives of x; therefore, the system is very overconstrained. Nonetheless, we have shown that the system is in fact compatible. It is quite common, these days, to begin with a nonlinear pde and to "find the linear problem associated with it," in the language of, say, Neugebauer. Perhaps these 8 first-order, linear pde's have some relation to such a question. I do not yet know how, but there does seem some hope.

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